EYNTKA Partial Differential Equations

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Contents

Summary

These notes will be the basis for futures notes on PDE's. At the moment, there is a strong focus on solving Wave equations, since they were the subject of my summer research project. The notes start off with an introduction on PDEs and general methods of solving PDEs. They then focus on wave quations, and progresively generalize the context in which we solve the wave equation, from increasing in dimension, to allowing weak solutions, to solving non-linear wave equations, to solving wave equations in more general geometrical settings.

1

Introduction to PDEs

When working with continuously differential functions (or more specifically smooth functions), we may take the derivative of the function (in the smooth case, we may take it infinitely many times). Doing the inverse is also simple due to the fundamental theorem of calculus. It is also simple to take some composition or algebraic combination of function's and differentiate. The converse, however, is much more difficult, that is finding a function that satisfies an algebraic combination of functions and their derivatives. This becomes even harder if multiple variables are involved. In this chatper, I'll go over what I've intuited of this process, and how mathematicians (and physicists) approach this problem.

On a high level will give you how certain parameters relate to each other over time. As a simple example, take:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \qquad \text{or} \qquad \frac{\partial}{\partial t} - \frac{\partial}{\partial x} = 0 \tag{1.1}$$

What this equation is giving us is a restriction we require a function $f(t, x)$ to obey, that is:

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} = 0$$

For example $f(x, t) = x + t$ would satisfy this equation. one key skill to learn when solving problems in PDEs is to be able to "read" such an equation. For example, equation (1.1) tells us that the speed at which our "system" is evolving over time is identically propostional to the speed at whichour "system" is moving. Sometimes, it is quite subtle what an equation may represent. For example:

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = 0$$

is very similar to equation (1.1), however it represents how a *wave* evolves over time, whith behavior that is very differnet from equation (1.1).

Also, after seeing how heat equations and diffusion equations are described I have a new motivation for learning PDE's: I wonder if at some point these techniques of relating "local information" in describing how it's dynamics evolve (in physics, it would be how it changes in time, in more general scenarios it can be a recursive function or a sequence) will at some point lend itself useful in solving more general porblems. For example, let's say we have a particularly complicated mathematical object. Let's say we can solve it in some nice special cases, perhaps with some restricting assumptions. Can we then step-by-step evolve the problem (or model an evolution of the problem) to eliminate the restrictions? Can we localize in some way to simplify it, and glue that information together? These feel like modelling something like a PDE.

As a final motivaor, every differnetial form has a "cannonical representative" in that there always exists representative ω such that ω vanishes under the Laplacian: $\Delta\omega = 0$ (functions, and more generally forms, that vanishes under the laplacian are called *harmonic*). This is the beginning of *Hodge Theory*, which is central in studying the cohomology groups of smooth manifolds. In the same vein as geometry but from antoher perspective, PDEs are also used to model *ricci flow*, which is about the flow of a manifold given certain parameters as it changes across time.

1.1 Intuition

Given any function or combination of functions via operations or composition, we can always find it's derivative, which is a function. Conversely, if we are given $f'(x)$, we can find $f(x)$ [up to a constant] by taking the reverse of the derivative, i.e. the integral. Thus, these two operators are inverses of each other! It is then natural to ask if we can perhaps take more complicated combinations of derivative's and see if there is a function that satisfies them. As an easy example, let's say:

$$f'(x) = 0 \qquad \text{for all } x$$

then we know that $f(x) = c$ for any constant c. If $f''(x) = 0$, then we know that $f'(x) = c$ and $f(x) = bx + c$ (i.e. any function for which $f'' \equiv 0$ must be linear). Now, what if we mix single and double derivaties, for example:

$$f''(x) + f'(x) + f(x) = 0 \qquad \text{for all } x$$

is there a function f that satisfies this? What if we do $f' + g' = 0$: are there functions f and g that satisfy this? Perhaps the easiest example of two functions satisfying an equation would be $fg' + f'g = 2$. Then the LHS is the product rule, so $fg = 2x + c$, giving us what we need to know to find equations f and g[1].

The reason why finding the more complicated version of this "converse" or "surjectivity" is interseting to me is because the derivative can give lot's of local information about a system, and sometimes that's all the information we have. For example, it is much easier to describe the force of gravity as

$$F = ma$$

where $a = dv/dt$, and $v = d(\text{displacement})/dt = dD/dt$. We can use this equation to construct a gravitational field to how any object at any point will have a certain force on them represented by a

[1] at least some relatively nice f and g, ex. being smooth, analytic, or polynomials

vector pointined downwards times the mass of the object, but this "global solution" has a lot more information than is needed. More generally, it is often easier to come up with the "localizations", since they are easier to define or control most of the time.

We thus have formed a set of tools to allow us to work with some very common equations involving derivatives, called *[ordinary] differential equations.* The word "ordinary" is to distinguish it from the case where we can take the derivative of muliple variables, which we call *partial differential equations.* Since the study of differential equation's where we differentiate one variable came first, it is called the "ordinary" differential equation. Most of the time, the deriative signifies a single quantity that changes smoothly or can be approximated smoothly (ex. time passing, population increase, fluctuation of income etc).

We focus on solving *linear differential equations* with the hope that we may generalize to the non-linear case. For example:

$$5y'' + 1y' - 3y = 0$$

is a linear differential equation. Since it is equal to 0, it is called a *homogeneous* linear ODE. Since trigonometric functions have very simple derivative properties, they are also studied regularly. Though this might seem limiting, these in fact cover a satisfying large sample of scenario's which we may encounter in physics. If your motivation is within pure mathematics, then having a standard repertoire to solve linear and trionometric ODE's allow's us to use these as "local tools" to solve non-linear ODE's or ODE's approximated by a trigonometric sequence (think of that result in 357). Here is small a list of types of equations and techuniques used to solve ODE's:

1. First order ODE:
   (a) Separable: if the equation can be written in the form: $\frac{dy(x)}{dx} = \frac{m(x)}{n(y)}$, or equivalently:
   $$\frac{dy(x)}{dx} + \frac{m(x)}{n(y)} = 0$$
   Then we can solve this ODE
   (b) Linear (Using an integral factor): If the equation can be written in the form:
   $$\frac{dy(x)}{dx} + p(x)y(x) = g(x)$$
   then we can solve this ODE
2. Second order ODE:
   (a) If the ODE has constant coefficients, is linear, and homogeneous:
   $$ay'' + by' + cy = 0$$
   then we can solve this ode, with $y(x) = Ae^{rx}$ for some $r \in \mathbb{C}$ and $A \in \mathbb{C}$.
   (b) If the ODE is Equidimensional (Cauchy-Euler), linear, homogeneous:
   $$ax^2y'' + bxy' + cy = 0$$
   then the solution is of the form $y(x) = Ax^p$ for some constant $p \in \mathbb{C}$.

There are many more that I've not listed here, I just wanted to give a list to give an idea of what was done.

1.2 Intuition behind Partial Differential Equations

Partial differential equations, often abbreviated as PDEs, let's us *relate* quantities. Another way of seeing how PDE's are different is that ODE's usually only model one variable (ex. angle of a pendulum), while a PDE can angle a whole set of points at once (ex. all the temperatures of a rod, as they change over time). For example, we may hope that the position of a particle (locally) relates to the force put on that particle (locally), and we want to do this for every particle in a system (more generally a continuum). For a more mathematical motivation, they show up more and more often in differential geometry (think of lee bracket's, lie derivative, and geometric analysis), they let us study D-modules (modules over the ring of differential operators, which tries to find aglebraic properties of differential operators), and geometry topology (PDE's were important in defining Ricci flow to show that any loop on a locally $\mathbb{R}^3$ space which can contract to a point is in fact a 3-sphere.

There is no unified study of PDE's, there is an amalgamation of "models" or "problems" that were of interest that are represented as a PDE. Example include:

- transport equation: Models how something move's across time
- wave equation: Models how a "wave" moves across time. Exapmles of wave are "literal" water waves, gravitaional waves, sound waves, physical waves (ex. a string or membrane vibrating), and so forth. What characterizes a wave is that the more "curved" any part of the wave is (i.e. the more concave it is), the more it accelerrates. To capture the curve-ness, the term $\frac{\partial^2}{\partial x^2}$ is used (which you can think of as representing the average), while the acceleration is captured by $\frac{\partial^2}{\partial t^2}$. Hence put together, we get:

  $$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \quad \Leftrightarrow \quad \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = 0$$

  if we add more dimensions, we would write:

  $$\frac{\partial^2}{\partial t^2} - \sum_i^n \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial t^2} - \nabla = 0$$

  Due to the proclivity of wave equations, we usually write:

  $$\square = \frac{\partial^2}{\partial t^2} - \nabla$$

  Hence, a function $u(t, \mathbf{x})$ would represent a [homogeneous] wave equation if it satisfies:

  $$\square u = 0$$

- heat equation: The heat equation, usually written (in 1 dimension) as $\frac{\partial}{\partial t} = \frac{\partial^2}{\partial x^2}$ represents how heat naturally averages over time
- maxwell's equation: These are a more complicated version of the wave equations to take into account some interaction of the eletromagnetic force.

Though these each have their own technics and precedents on how to approach them, there is a sort of unifying theme in PDE's, which is to ask whether a certain solution to a PDE is "well-posed":

1. Given certain conditions (usually the domain and free choice, like a disk or $\mathbb{R}^2 - \{0\}$ or $\mathbb{R}^2$ as examples of domain and as free choice, it can be any constant, any funciton, any polynmial function, any two functions, etc.) there exists a unique solution to the PDE
2. By "continuously" changing the free choices (i.e. the dependence on the given data and parameters), one continuously changes the corresponding solutions

What "continuously" means here can depend: we may want a condition as strong as smoothness, to being continuous.

2

Wave equations

The goal of this chatper is to be able to solve basic homogenous wave equations of the form $\Box u = 0$.

2.1 one Dimension

2.1.1 wave equations

I found an excellent intuition on why the 2nd order derivative appears!

`https://youtu.be/ly4S0oi3Yz8?t=428`

I also this current opinion on how to think about $\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2}$: Recall that $\frac{\partial^2}{\partial x^2}$ represents the average of the points around it (think of the 3b1b veido on the heat equation). The $\frac{\partial^2}{\partial t^2}$ represents how points accelerate over time. Think of:

$$F = m\frac{d^2}{dt^2} = ma$$

where the a term represents how an object would accelerate (simply take $u(x,t)$ where $\frac{\partial^2}{\partial t} = c$ for some constant $c \in \mathbb{R}$ and see how it represents the "graph" uniformally accelerating (or moving at a constant speed if $\frac{\partial^2}{\partial t^2} = 0$)). Now, the fact that $\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2}$ means that the acceleration of the point represents the average around any point, i.e. the concavity!

In terms of actually solving an wave equation, we will start by focusing on the case of:

$$0 = \frac{\partial^2}{\partial t^2} - \sum_i^n \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial t^2} - \nabla = \Box$$

2.1.2 The d'Alembert's Formula

We'll do the 1d case first:

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = 0 \tag{2.1}$$

with initival value problem (IVP), also known as *Cauchy problem* of:

$$u(0,x) = f(x) \qquad \partial_t u(0,x) = g(x)$$

These two functions are the given data to find the existence and uniqueness of a solution to equation 2.1. What extra restriction that need to be addedto f and g will be discussed later. It might seem obvious that some differentiation condition needs to be added, however after the answer is revealed, we'll see that we can get away with weaker assumptions.

With some clever manipulation, there is a solution $u(t,x)$ to this equation with respect to f and g. First, let

$$r := x + t \qquad s := x - t$$

Then changing the operators, we get:

$$2\partial_r = \partial_t - \partial_x \qquad -2\partial_s = \partial_t + \partial x$$

Thus, substituting this into equation 2.1, we get:

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = -4\partial_s\partial_r u = 0$$

which implies that $\partial_r u$ is independent of s, that is $\partial_r u = v(r)$. Thus, if $W(s)$ is our integration constant, integarting with respect to r, we get:

$$u(r,s) = \int^r v(r')dr' + W(s) := V(r) + W(s)$$

Thus, substituting back our original varaibles, we get:

$$u(t,x) = V(x+t) + W(x-t) \tag{2.2}$$

which can be thought of as decomposed u into a left-moving component and a right-moving component. We next want to express V an T in terms of our initial data. From equation 2.2, we can express the initial data at as:

$$u(0,x) = V(x) + W(x) = f(x)$$
$$\partial_t u(0,x) = V'(x) - W'(x) = g(x)$$

Differentiating the first line with respect to x and combining the two equations to solve for V' and W', we get:

$$V'(x) = \frac{1}{2}f'(x) + \frac{1}{2}g(x) \qquad W'(x) = \frac{1}{2}f'(x) - \frac{1}{2}g(x)$$

Integrating with respect to x, we get:

$$V(x) = \frac{1}{2}f(x) + \frac{1}{2}\int_0^x g(x')dx' + c_1$$
$$W(x) = \frac{1}{2}f(x) - \frac{1}{2}\int_0^x g(x')dx' + c_2$$

Since $c_1 + c_2 = 0$ so that we get equality with equation 2.2, we get that:

$$u(t,x) = V(x+t) + W(x-t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(x')dx' \tag{2.3}$$

This final form is known as *d'Alembert's formula* for solutions of the one-dimensional wave equation. Notice hat as we've defiend equation 2.3, we only requite that $f, g \in C^0(\mathbb{R})$ (i.e. that they be continuous), or really $f, g \in L^1_{\text{loc}}$. This gives rise to a $u(t,x)$ that is *not* necessarily C^2. However, in order for $u(x,t)$ to be a solution, it must be twice differentiable, which would imply any slice would have to be twice differentiable, while $f, g \in L^1_{\text{loc}}$ would surely fail in general. All is not lost though, we will show later that such asolution is known as a *weak solution*, which we'll be getting to in chapter 3.

The next important remark we must make about this solution is that it only depends on some time-limited amount of data: the solution of $u(t,x)$ depends upon $(f(x'), g(x'))$ only over the interval $x - t \leq x' \leq x + t$ and on f at the points $x \pm t$. The set:

$$\{x + t = x'\} \cup \{x - t = x'\}$$

is called the *light cone*, and conversely given some x, the set

$$\{x' + t = x\} \cup \{x' - t = x\}$$

is called the *backwards light cone*:

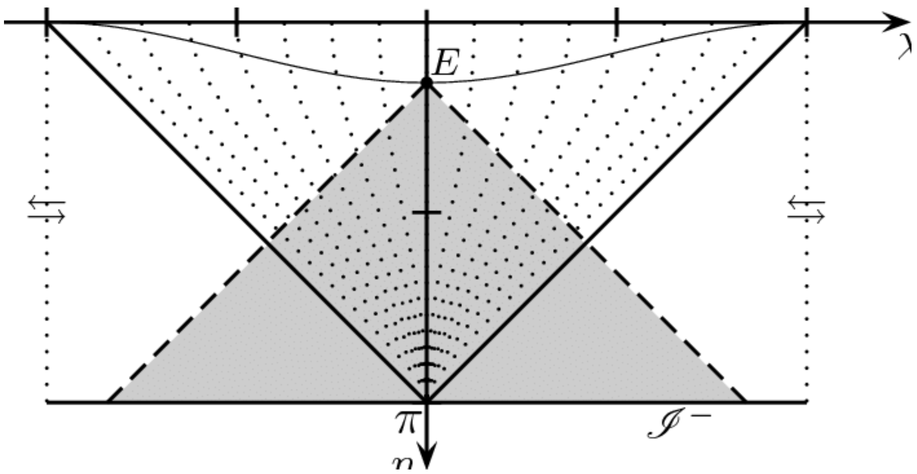

Figure 2.1: light cone and reverse light cone

Notice that given d'Alembert's formula, the solution $u(t,x)$ given above depnds on the initial data $(f(x), g(x))$ only for values x' that is inside the backwards light cone, or more precisely the interior of the light cone with boundary defined by the set above and the line $\{t = 0\}$:

PICTURE HERE

Thus, we say such a solution satisfies the principle of *finite propagation speed.* Thus, if f, g have compact support within some interval $[-R.R]$,

$$\text{supp}(u(t,x)) \subseteq [-R - |t|, R + |t|]$$

as seen in figure BOOK IMAGE. if we consider the slope of the above to be c (notice that $c = 1$, then the information given by the initial data in the interval $[-R, R]$ does not travel faster then $c = 1$, and so at tiem $t > 0$, the information has not propogated further than $[-(R+t), (R+t)]$. This property is called *Huygen's principle*, and will appear often when we talk about solutions to wave equations.

(A word on the strong Huygen priniple)

2.2 Duhamel's Principle

We next look into the *inhomogeneous* 1d wave equation:

$$\partial_t^2 u - \partial_x^2 u = h(t,x) \tag{2.4}$$

with initial data

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \quad h(t,x) \in C^2$$

If we have $h(t,x) = 0$, then we can find a solution $u_1(t,x)$ using d'Alembert's formula, and if $f = g = 0$, it is easy to get antoher [linear] solution $u_2(t,x)$, even if $h(t,x)$ is nonzero (as we'll explore momentarily). Thus, since equation (2.4) is a linear equation, we get a solution $u(t,x) = u_1(t,x) + u_2(t,x)$, which in fact by the nature of u_1 and u_2 in fact gives the general solution! It remains to show how we can find u_2, which will be used using *Duhamel's principle*.

Start by solving the "standard form" of the IVP, namely with:

$$f = h = 0$$

Using d'Alembert's formula, we get:

$$W(t)(g)(x) = u(t,x) = \int_{x-t}^{x+t} g(s)ds$$

where we'll call $W(t)(g)(x)$, or just $W(t)$ the *propagation operator* or *solution opeartor*. Using this operator, we can show that the general solution is in fact:

$$u(t,x) = \partial_t(W(t)(f)(x)) = W(t)(g)(x)$$

We simply verify that this satisfies the initial conditions. For $u(0,x)$:

$$u(0,x) = \partial_t(W(0)(f)(x)) + W(0)(g)(x) = f(x) + 0$$

$$\partial_t u(0,x) = \partial_t\left(\partial_t(W(0)(f)(x))\right) + \partial_t(W(0)(g)(x)) = \nabla W(0)(f) + g(x) = 0 + g(x)$$

Now that we expressed the d'Alembert's solution when the IVP is in standard form, we will use this to show how to solve the inhomogenous case. Letting $h \neq 0$, we'll show that:

$$u_2(t,x) = \int_0^t W(t-s)(h(s,\cdot))(x)ds$$

Indeed, verifying directly that $u_2(t,x)$ satisfies the IVP:

$$\begin{aligned}
u_2(t,x) &= 0 \\
\partial_t u_2(t,x) &= \partial_t \int_0^t W(t-s)(h(s,\cdot))(x)ds\bigg|_{t=0} = W(0)(h(t,\cdot))(x) = 0 \\
\partial_t^2 u_2(t,x) &= \int_0^t \partial_t^2 W(t-s)(h(s,\cdot))(x)ds + \partial_t \left(W(0)(h(t,\cdot))(x)\right) \\
&= \nabla u_2(t,x) + h(t,x)
\end{aligned}$$

Combining the result to get $u = u_1 + u_2$ and decomosing $W(t)(g)(x)$, we get:

$$\begin{aligned}
u(t,x) = u)1(t,x) + u_2(t,x) \\
= \frac{1}{2}\left(f(t+x) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(x')dx' \\
\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} h(s,x')dx'ds
\end{aligned}$$

The double integral in the above takes into account the copnonent $u_2(t,x)$ of the solutions over the backwards lightcone.

2.2.1 Method of images

Many times, a wave equation must also specify a boundary condition, thus we need to consider an *initial boundary problem*, or *IBP* for short. An example of a boundary problem would be :

$$u(t,0) = 0$$

which is known as the *Dirchelet condition*, and

$$\partial_x u(t,0) = 0$$

which is known as the *Neumann condition*. We will start by solving wave equations with the Dirchelet condition. We will take advantage of the fact that odd functions ($h(-x) = -h(x)$) that are continuous $x = 0$ must satisfy $h(0) = 0$. Let's assuem the sapcial domain is only $\mathbb{R}_{\geq 0}$, so that the inital values $f(x), g(x)$ are only definedon $\mathbb{R}_{\geq 0}$. Naturally, to be compatible with the IBP, we need that $f(0) = g(0) = 0$. We first define extension of f, g onto all of $\mathbb{R}$ by taking $f(-x) = -f(x)$, $g(-x) = -g(x)$. Using d'Alembert formula, we get the solution to the cauchy problem:

$$u(t,x) = \frac{1}{2}\left(f(x+t) - f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(x')dx'$$

Clearly, this expression is an odd fucntion with respect to x, since f and g are odd. Note that for any $x > t$, this expression only involves the arguments of f, g for positive x; however, for $x < t$, the expression involves negative argument even though $x > 0$. To remedy this we use the fact that he extended f, g are odd to rewrite the expression as:

$$u(t,x) = \frac{1}{2}\left(f(t+x) - f(t-x)\right) + \frac{1}{2}\int_{t-x}^{t+x} g(x')dx'$$

Thus, this solution is only dependnt upon $(f(x), g(x))$, $x > 0$. Due to the odd extension, the result looks like this acccross time:

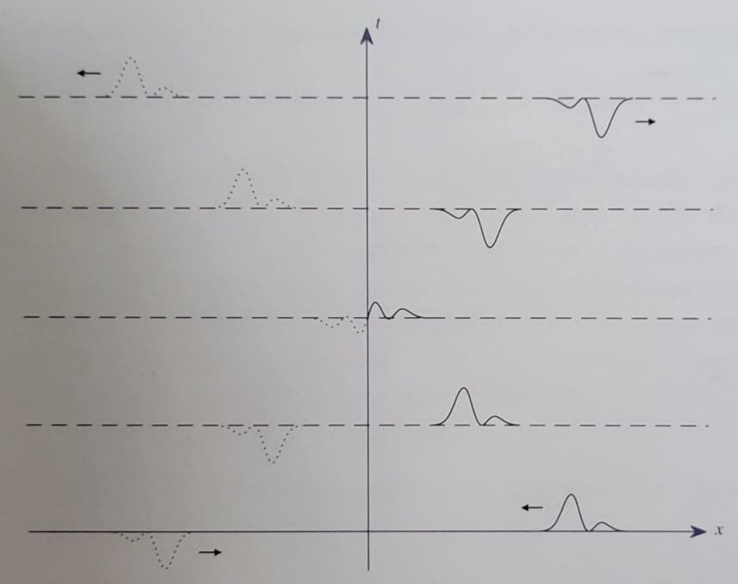

Figure 6. The method of images for Dirichlet boundary conditions.

(Then there is solving the Neumann condition, which since we are taking the derivative will end up being the same type of extension except it is an even extension instead of an odd one)

2.2.2 Separation of Variables

Let:

$$\partial_t^2 - \partial_x^2 u = 0 \qquad x \in (0, 2\pi)$$

with Dircihlet boundary conditions:

$$u(t, 0) = 0 = u(t, 2\pi)$$

we will seek a special solution in the form of the product $u(t, x) = v(t)w(x)$ (i.e. the relation between the time and space variables can be seperated to "independent" function which interact via multiplication) so that:

$$\partial_t^2 vw = v\partial_x^2 u$$

Assuming the measure of points where $v, w = 0$ is zero, dividing both sides by vw, we get that a.e.:

$$\frac{\partial_t^2 v}{v} = \frac{\partial_x^2 w}{w}$$

Since the left hand side only dependts on t while the right hand side only depends on t, the value of this equation is necessarily a constant:

$$\frac{\partial_t^2 v}{v} = \frac{\partial_x^2 w}{w} = \lambda$$

This makes the problem into 2 ODE's: an *evolution problem* for $v(t)$ in order to satisfy the initial conditions, and an eigenvalue problem for an ODE for $w(x)$ so that the solution satisfie sthe boundary condtiions. The eigenvalue probelm is for the pari $(w(x), \lambda)$, wher the function $w(x)$ satisfie the Dirichlet boundary conditions on $x \in (0, 2\pi)$:

$$\partial_x^2 w = \lambda w \qquad w(0) = 0 = w(2\pi)$$

Given an eigenfunctio nand an eiganvalue from this problem, the evolution equation for v is:

$$\frac{d^2}{dt^2} v = \lambda v$$

with initia ldat a $v(0) = a$, $\dot{v}(0) = b$ that has to be specified. The eigenvalue problem has a countable number of solutions, which are of the form:

$$w_k(x) = \frac{1}{\sqrt{\pi}} \sin\left(\frac{1}{2} kx\right) \qquad \lambda_k = \frac{-k^2}{4}, \; k \in \mathbb{N}_{>0}$$

Thus, we can choose countably many constants (a_k, b_k), $k \in \mathbb{N}_{>0}$ such that

$$v_k(t) = a_k \cos(w_k t) + b_k \frac{}{\sin(w_k t)} w_k$$

with temporal fequencies $w_k = \sqrt{-\lambda_k} = \frac{1}{2} k$. Combining these give us the general solution:

Theorem 2.2.1: General Solution To Seperation Of Variables

The genral solution of the initial-boundary value problem given at the beginning of this section is a lienar super position of these special solutiosn:

$$u(t, x) = \sum_{k=1}^{\infty} \left(a_k \cos(w_k t) + b_k \frac{\sin(w_k t)}{w_k} \right) \frac{1}{\sqrt{\pi}} \sin\left(\frac{1}{2} kx\right)$$

where the initail data is specified by:

$$u(0, x) = f(x) = \sum_{k=1}^{\infty} a_k \frac{1}{\sqrt{\pi}} \sin\left(\frac{1}{2} kx\right) = \sum_{k=1}^{\infty} a_k w_k(x)$$

and

$$\partial_t u(0, x) = g(x) = \sum_{k=1}^{\infty} b_k \frac{1}{\sqrt{\pi}} \sin\left(\frac{1}{2} kx\right) = \sum_{k=1}^{\infty} a_k w_k(x)$$

Furthermore, the sine series used to describe $f(x), g(x)$ is compelte as an orthonormal basis for the Hilbert space $L^2(0, 2\pi)$; thus any initial $f(x), g(x) \in L^2$ can be represented by their Fourier sine series cofefficeints $(a_k, b_k)_{k \in]N}$ giving rise to a solution $u(t, x)$.

Proof
Just take everything we talked about earlier. For the L^2 basis, recall 457 and 357.

To Calculate the coefficeints, we do the usual trick in Hilbert spaces given an orthonormal basis, recalling that the inner prodcut is:

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g}(x)dx$$

(Comments on how it applies to quantum mechanics and music here)

2.3 Wave Equation in Higher Dimensions

we now study equations of the form:

$$\partial_t^2 u - \nabla u = 0$$

where $u \in \mathbb{R}^{1+n}$ and $\nabla u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. This is often represented as:

$$\Box u = 0$$

2.3.1 Fourier Series

We first solve this using Fourier series. Assuming $f, g \in S$ is in the Schwartz class, we can take the Fourier Transform and get:

$$\partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0$$

making this into an ODE. Solutions of this ODE are composed of linear combinations of $e^{\pm i|\xi|t}$. Takiung into account that $\hat{u}(0,.\xi) = \hat{f}(\xi)$ and $\partial_t \hat{u}(0,\xi) = \hat{g}(\xi)$, we get the expression:

$$\hat{u}(t,\xi) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi)$$

And so the inverse fourier transform gives the solution:

$$u(t,x) = \frac{1}{\sqrt{2\pi}^n}\int e^{i\xi x}\left(\cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)l}{|\xi|}\hat{g}(\xi)\right)d\xi$$

Thus, if f, g are of Schwartz class, we get a solution. However, this is by observation not necessary, we only need that:

$$\hat{f}(\xi), \frac{\hat{g}(\xi)}{|\xi|} \in L^2(\mathbb{R}_x^n)$$

to have a solution. We will soon explore this when we find weak solutions.

(a word on Lorenz Transfomration's leaving the solution independent, this is a prelude to chapter 6)

2.3.2 Method of Spherical Means

Definition 2.3.1: Spherical Mean

Given a function $g \in C(\mathbb{R}^n)$, it's *spherical mean* centered about the point $x \in \mathbb{R}^n$ is

$$M(h)(x,r) = \frac{1}{w_n r^{n-1}} \int_{|x-y|=r} h(y) dS_y$$

that is, it is the average of $h(x)$ over the sphere S^{n-1} centered at x of radius r.

the constant in front is the normalization term: $w_n r^{n-1}$ is the surface area of the sphere S^{n-1} of radius r of dimension n. When $n = 1$, the sphere mean of a function $f(x)$ is:

$$M(f)(x,r) = \frac{1}{2}(f(x+r) + f(x-r))$$

which is the first term the d'Alembert formula. If we do a change of variables in the equation in the definition to $y \mapsto x + r\xi$, where $\xi \in S_1(0)$, there is another expression for the spherical mean:

$$M(h)(x,r) = \frac{1}{w_n} \int_{|\xi|=1} h(x+r\xi) dS_\xi \tag{2.5}$$

Though we define if for $r \geq 0$, it is easy to see that equation 2.5 shows that this function is even in r:

$$M(h)(x,-r) = M(h)(x,r)$$

Lemma 2.3.1: Darboux Equation

Given $h \in C^2(\mathbb{R}^n)$,

$$\triangle_x M(h)(x,r) = (\partial_r^2 + \frac{n-1}{r}\partial_R) M(h)(x,r)$$

a formula that realtes the Laplace operator of $M(h)$ in the n-dimensional x-variables to an operator involving only tone-dimeniwdimensional r derivatives.

Proof
This follows form computations using multivariable calculus. First, take one derivative:

$$\begin{aligned}\partial_r M(h)(x,r) &= \frac{1}{w_n} int_{|\xi|=1} \partial_r h(x+r\xi) dS_\xi \\ &= \frac{1}{w_n} \int_{|\xi|=1} \sum_{j=1}^{n} \partial_{x_j} h(x+r\xi)\xi_j dS_\xi\end{aligned}$$

Notice that $\sum_{j=1}^n \partial_{x_j} h(x+r\xi)\xi_j = \nabla h \cdot N$,the outwards noraml derivative of h on the unit sphere.

Thus, we use Gren's theorem and continue:

$$\begin{aligned}
\frac{1}{w_n}\int_{|\xi|=1}\sum_{j=1}^{n}\partial_{x_j}h(x+r\xi)\xi_j dS_\xi &= \frac{1}{w_n}\int_{|\xi|<1} r\triangle_x h(x+r\xi)d\xi \\
&= \frac{1}{w_n r^{n-1}}\triangle_x\int_{|x-y|<r} h(y)dy \\
&= \frac{1}{w_n r^{n-1}}\triangle_x\left(\int_0^r\int_{|x-y|=\rho} h(y)dS_y d\rho\right) \\
&= \frac{1}{r^{n-1}}\triangle_x\left(\int_0^r \rho^{n-1}M(h)(x,\rho)d\rho\right)
\end{aligned}$$

the next second is to take a second derivagie after multiplying through by r^{n-1}:

$$\begin{aligned}
\partial_r\left(r^{n-1}\partial_r M(h)(x,r)\right) &= \partial_r\left(\triangle_x\int_0^r \rho^{n-1}M(h)(x,\rho)d\rho\right) \\
&= \triangle_x\left(r^{n-1}M(h)(x,r)\right)
\end{aligned}$$

Thus:

$$\begin{aligned}
\triangle_x M(h)(x,r) &= \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r M(h)(x,r)\right) \\
&= \left(\partial_r^2+\frac{n-1}{r}\partial_r\right)M(h)(x,r)
\end{aligned}$$

as we sought to show.

A couple of analysis facts worth reminding ourselve:

Proposition 2.3.1: Analysis Facts

1. For $h(x)\in C(\mathbb{R}^n)$, the value of $h(x)$ at any $x\in\mathbb{R}^n$ can be recovered from tis spherical mean:

   $$h(x)=\lim_{r\to 0}M(h)(x,r)=M(h)(x,0)$$

   For $h(x)\in C^2(\mathbb{R}^n)$,

   $$\partial_r M(h)(x,0)=\lim_{r\to 0}\frac{r}{w_n}\int_{|\xi|<1}h(x+r\xi)d\xi=0$$

The first point will allow us to recoer the solution to wave equations posed in terms of the spherical mean ,while the other will help in computaion.

Proof
See Folland chapter 3 section 4 to outline how to get this result, or your MAT457 notes

We can now use the spherical means formula to find solutions to the wave equations, in particular solutions that are not as strong as being of Schwartz class. Next section, we will show how to find even weaker solutions. Firs,t suppose that $u(t,x)$ is a solution of the Cauch yproblem for the question equaiton. Then its spherical mean $M(u)(t,x,r)$ can be defiend from the equation for the spherical means, and it satisfies an auxiliary equation in the reduced space-time variables $(t,r) \in \mathbb{R}^2$. Specificaly, define:

$$M(u)(t,x,r) = \frac{1}{w_n} \int_{|\xi|=1} u(t, x + r\xi) dS_\xi$$

Taking time derivatves and using the fact that $u(t,x)$ is a solution of the wave equation, we obtain:

$$\begin{aligned} \partial_t^2 M(u)(t,x,r) &= \frac{1}{w_n} \int_{|\xi|=1} \partial_t^2 u(t, x + r\xi) dS_\xi \\ &= \frac{1}{w_n} \int_{|\xi|=1} \triangle_x u(t, x + r\xi) dS_\xi \\ &= \triangle_x M(u)(t,x,r) \end{aligned}$$

Now using the Darbous equation, we get:

$$\partial_t^2 M(u)(t,x,r) = \left(\partial_r^2 + \frac{n-1}{r} \partial_r \right) M(u)(t,x,r) \tag{2.6}$$

a PDE in the two variables $(t,r) \in \mathbb{R}^2$. This is known as the *Euler-Poisson-Darboux* equation.

2.3.3 Spherical Mean in 3 Dimension

We now show how to find solutiosn given the spherical Equation (2.6) is normally posed as an IVP:

$$\begin{aligned} \partial_t^2 M(u)(t,x,r) &= \left(\partial_r^2 + \frac{n-1}{r} \partial_r \right) M(u)(t,x,r) \\ M(u)(0,x,r) &= M(f)(x,r) \\ \partial_t M(u)(0,x,r) &= M(g)(x,r) \end{aligned}$$

The solution of this will vary quite a bit on the dimension we choose. The case of $n = 3$ is the simplest, and so we start with it. First, we do the substituion $M(u)(t,x,r) \mapsto rM(u)(t,x,r)$, giving:

$$\partial_t^2 \left(rM(u)(t,x,r) \right) = r \left(\partial_r^2 + \frac{2}{r} \partial_r \right) M(u)(t,x,r) = \partial_r^2 \left(rM(u)(t,x,r) \right)$$

Thus, the function $v(t,r) = rM(u)(t,x,r)$ is a solution of the wave equation in one dimension, and the variable x is relegated to the role of a parameter. The solution is ginve by d'Alembert formula

$$\begin{aligned} v(t,r) &= rM(u)(t,x,r) \\ &= \frac{1}{2} \left((r+t) M(f)(x, r+t) + (r-t) M(f)(x, r-t) \right) \\ &\quad + \frac{1}{2} \int_{r-t}^{r+t} \rho M(g)(x,\rho) d\rho \end{aligned}$$

Since $M(f)(x,r)$ and $M(g)(x,r)$ are even functions under $r \mapsto -r$, the n$rM(f)(x,r)$ and $rM(g)(x,r)$ are odd; therefore we may re-write the above expression, dividng through by r:

$$\begin{aligned} M(u)(t,x,r) &= \frac{1}{2r}\left((r+t)M(f)(x,r+t) + (r-t)M(f)(x,r-t)\right) \\ &+ \frac{1}{2r}\int_{t-r}^{t+r} \rho M(g)(x,\rho)d\rho \end{aligned}$$

Note that we used the fact that $\int_{t-r}^{r-t} \rho M(g)(x,\rho)d\rho - 0$, which hodls since $rM(g)(x,r)$ is odd. Taking the limit as $r \to 0$ and and using poposition 2.3.1, we see that we recover a reprsetnation for the solution $u(t,x)$.

Theorem 2.3.1: Kirchhoff's Formula

When $n = 3$, the solution to the wave equation is given by the expression

$$u(t,x) = \partial_t\left(tM(f)(x,t)\right) + tM(g)(x,t) \tag{2.7}$$

which is well defined as long as $f(x) \in C^1(\mathbb{R}^3)$ and $g(x) \in C(\mathbb{R}^3)$

Proof

Piece together all that was put above

Note that in terms of pointwise regularity, the solution is genreally less smooth than the initial data, for the Kirchhoff formula depends upon the derivativge of $f(x)$. In particular, carrying out the differentiation in equation (2.7), we obtain

$$u(t,x) = \frac{1}{4\pi t^2}\int_{|x-y|=t}\left(tg(y) = f(y) + t\nabla f(y)\cdot\frac{x-y}{|x-y|}\right)dS_y$$

Corollary 2.3.1: Solution Space

Givien initial data $g(x) \in C^2(\mathbb{R}^3)$ and $f(x) \in C^3(\mathbb{R}^3)$, the spherical means solutions given in equation (2.7) is a solution $u(t,x) \in C^2(\mathbb{R}^1_t \times \mathbb{R}^3_x)$

Proof

immediate.

(comment on loss of differentiability, and how this doesn't happen in Sobelev space)

2.3.4 Spherical Means for odd Dimension

Let's first recall the Euler-Poisson Daboux equation for the n-dimensional wave equation with IVP:

$$
\begin{aligned}
\partial_t^2 M(u)(t,x,r) &= \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) M(u)(t,x,r) \\
M(u)(0,x,r) &= M(f)(x,r) \\
\partial_t M(u)(0,x,r) &= M(g)(x,r)
\end{aligned}
$$

The goal is to have an algebraic reduction of the wave equation to two dimensions in the varaibvles (t,r). This is carrid out in the odd-dimenisonal case as follows:

Lemma 2.3.2: Useful Relations

Suppose that $k \geq 1$ is an integer and that $h = h(r) \in C^{k+1}(R_+^1)$. Then:

$$
\partial_r^2 \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}h\right) = \left(\frac{1}{r}\partial_r\right)^{k} \left(r^{2k}\partial_r h\right)
$$

$$
\left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}h\right) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1}\partial_r^j h
$$

with the combinatorial coefficients being $\beta_0^k = 1 \cdot 3 \cdot \ldots \cdot (2k-1) := (2k-1)!!$.

Proof

Simply do an argument by induction

The ruslt is now useful since we can set $n = 2k+1$ and take:

$$
v(t,x,r) := \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}M(u)\right)(t,x,r)
$$

For a give solution $u(t,x)$ of the wave equation. Then using the identies from corrollary 2.3.2

$$
\begin{aligned}
\partial_r^2 v &= \partial_r^2 \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}M(u)\right) \\
&= \frac{1}{r}\partial_r^k \left(r^{2k}\partial_r M(u)\right) \\
&= \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}\partial_r^2 M(u) + 2kr^{2k-2}\partial_r M(u)\right) \\
&= \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}\left(\partial_r^2 M(u) + \frac{2k}{r}\partial_r M(u)\right)\right) \\
&= \left(\frac{1}{r}\partial_r\right)^{k-1} \left(r^{2k-1}\partial_t^2 M(u)\right) \\
&= \partial_t^2 v(t,r)
\end{aligned}
$$

Thus, $v(t,x,r)$ is the quantity that satisfies the one-dimensional wave equation in the variables (t,r), for which we can applyh the d'Alembert formula:

$$v(t, x, r) = \frac{1}{2}\left(v(0, r+t) + v(0, r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \partial_t v(0, x, \rho) d\rho$$

with initial data:

$$\begin{aligned} v(0, x, r) &= \left(\frac{1}{4}\partial_r\right)^{k-1} \left(r^{2k-1} M(f)\right)(x, r) \\ \partial_t v(0, x, r) &= \left(\frac{1}{4}\partial_r\right)^{k-1} \left(r^{2k-1} M(g)\right)(x, r) \end{aligned}$$

Then we can recover the solution as a limit:

$$\begin{aligned} u(t, x) &= \lim_{r\to 0} M(u)(t, x, r) = \lim_{r\to 0} \frac{v(t, x, r)}{\beta_0^k r} \\ &= \frac{1}{(n-2)!!}\left[\partial_t \left(\frac{1}{t}\partial_t\right)^{(n-3)/2} \left(t^{n-2} M(f)(t, x)\right) + \left(\frac{1}{t}\partial_t\right)^{(n-3)/2} \left(t^{n-2} M(g)(t, x)\right)\right] \end{aligned}$$

Piecing it all together:

Theorem 2.3.2: Solution For Odd Dimension

For n odd, $f \in C^{(n-1)/2}(\mathbb{R}^n)$ and $g \in C^{(n-3/2}(\mathbb{R}^n)$, the solution formula given above gives a classical solution to the wave equation in $\mathbb{R}^n$

Proof
piece the above together

It is intereting that we can quantify the possibler loss of smoothness of the solution voer the initial data, made claer by the formula above the theorem. Indeed, the reduction process from $M(u)$ to v involves $k-1 = (n-3)/2$ derivatives, and the limit involve sone derivative; therefore in general the solution will be less regular than the initial data by $k = (n-1)/2$ many derivatives.

2.3.5 Huygen's Principle

Just like in the one-dimenisonal case, the information of the wave equation only propagates finitely far. We will see that the strong Huygen principle will hold for odd dimensions but not for even. In the follwoing, we will show it for $n = 3$:

Theorem 2.3.3: Huygen's Principle And Strong Huygen's Principle

Consider solutions to the wave equatio nfor $x \in \mathbb{R}^3$ and assume that the Cauchy data $f(x)$ and $g(x)$ are compactlysupported such that $\mathrm{supp}(f) \cup \mathrm{supp}(g) \subseteq B_R(0)$. Then:

1. **(Huygen's principle)**: The solution $u(t, x)$ has its support within the bounded region $B_{R+|t|}(0)$:
$$\mathrm{supp}(u(t, \cdot)) \subseteq B_{R+|t|}(0)$$
2. **(Strong Huygen's principle)**: Additionally, for $|t| > R$, for any space-time point (t, x) in the region inside the light cones given by $\{(t, x) \mid |x| \leq |t| - R\}$, again the solution vainshes; $u(t, x) = 0$

Proof

1. The result follows from kirchhoff's Formula
2. When $|t| > R$, and $|x| < |t| - R$, again the backwards light cone emanating from the point (t, x) does not intersect the support of the initial dta, in this case because the sphere $\{y \mid |x = y| = |t|\}$ is too big and has passed outside of $B_R(0)$.

   This proof is geometric, only depending upon the fact that the solution is represented as a spherical mean. Therefore the result holds for solutions of the wave equation in arbitrary odd dimension, as shown in the equation above theorem 2.3.2.

(something slightly stronger is in fact true. p. 133 of "A Course on PDE's" by Walter Craig.

2.3.6 Duhamel's Principle in Higher Dimensions

here

3

Weak Solutions and Distributions

We know deal with the case where solutions can be "weaker". As we saw, d'Alembert formula works perfectly well if $f, g \in L^1_{\text{loc}}(\mathbb{R})$. However, the initial conditions *require* that f, g be differentiable (afterall, they are slices of the function $u(t,x)$ and $\partial_t u(t,x)$, which must be at least twice and once differentiable respectively). So is there some way to circumnavigate this? The content of the first section of this chapater covers precisely this. In the next section, we shall generlize the results and define a whole new object that will allow us to grealy broaden our notion of function that will allow us to differentiate function that are simply locally itnegrable!

Let's start with the usual solution $u \in C^2$ on a time strip:

$$S_T = [0,T] \times \mathbb{R}^n$$

and assume that:

$$\Box u = F \qquad u|_{t=0} = f \qquad \partial_t u|_{t=0} = g$$

We are going to use the idea of a test function φ. If you want to think of it in a Folland way, $\varphi \in C_c^\infty(S_T)$, and we are defining a functional on $C_c^\infty(S_T)$ taking $\varphi \mapsto \int \Box u \varphi d(x,t)$. We will take advantage of integration by parts to get the following:

$$\begin{aligned}
\int_{S_T} F\varphi dt dx &= \int_{S_T} (\Box u)\varphi \, \mathrm{d}t\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(-\int_0^T \partial_t u \partial_t \varphi \, \mathrm{d}t - g(x)\varphi(0,x) \right) \mathrm{d}x - \int_{S_T} u \triangle \varphi \, \mathrm{d}t\mathrm{d}x \\
&= \int_{S_T} u \Box \varphi \mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^n} f(x)\partial_t \varphi(0,x) \, \mathrm{d}x - \int_{\mathbb{R}^n} g(x)\varphi(0,x)\mathrm{d}x
\end{aligned}$$

Which leads us to the following definition:

Definition 3.0.1: Weak Solution

Let $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $F \in L_{\text{loc}}(S_T)$. We say $u \in L_{\text{loc}}(S_T)$ is a *weak solution* of the wave equation if:

$$\int_{S_T} (\Box u)\varphi \,\mathrm{d}t\mathrm{d}x = \int_{S_T} u\Box\varphi \mathrm{d}t\mathrm{d}x - \int_{\mathbb{R}^n} f(x)\partial_t\varphi(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^n} g(x)\varphi(0,x)\mathrm{d}x \tag{3.1}$$

for all $\varphi \in C_c^\infty$ supported in $(-\infty, T) \times \mathbb{R}^n$

The next shows how this is indeed a generlization:

Proposition 3.0.1: When Weak Solutoin Is Classical Solution

A weak solution that belongs to $C^2(S_T)$ is a classical solution

Proof

Taking equation (3.1), we see first that:

$$\int u\Box\varphi \, dtdx = \int F\varphi \, dtdx$$

and integratin by parts gives us that $\int u\Box\varphi = \int(\Box u)\varphi$. For the next part, recall that if $h \in L^1_{\text{loc}}$ and $\int h\varphi = 0$ for all $\varphi \in C_c^\infty$, then $h = 0$ almost everywhere. Thus since:

$$\int (\Box u - F)\varphi dtdx = 0$$

and φ was arbitrary, we conclude that $\Box u = F$ on S_T up to zero-measure.

Next, we need to show that u takes on the initial values of f and g. Let's first show that:

$$\partial_t u(0, x) = g(x)$$

First, the above equation is equivalent to:

$$\int_{\mathbb{R}^n} \partial_t u(0,x)\psi(x)dx = \int_{\mathbb{R}^n} g(x)\psi(x)dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n) \tag{3.2}$$

Fix any such ψ. we will do some clever manipulation to "eliminate" the integral. Let $a(t)$ be the smooth step function going from 1 to zero smoothly:

$$a(t) = \begin{cases} 1 & \text{for } 0 \geq t \\ \text{smoooth} & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t \geq 1 \end{cases}$$

Then $\theta_k(t) = a(kt)$ will squish the function so that it's closer and closer to being a step function. Importantly for us, notice that θ_k has the following properties:

$$\theta_k'(0) = 0 \qquad \theta_k(t) = 0 \quad \text{for } t \geq 1/k$$

Now, letting $\varphi = \varphi_k = \theta_k(t)\psi)$ from equation (3.1), then the right hand side of equation (3.1) reads as (taking k large enough so that $1/k < T$):

$$\int_{\mathbb{R}^n} \left(\int_0^{1/k} F(t,x)(akt)\,\mathrm{d}t \right) \psi(x)dx + \int_{\mathbb{R}^n} g(x)\psi(x)dx \tag{3.3}$$

Since $F \in C(S_T)$ (recall $\Box u = F$), the first term is $O(1/k)$ as $k \to \infty$. Now, I claim that the left hand side of equation (3.1) is equal to:

$$\int_{\mathbb{R}^n} \partial_t u(0,x)\psi(x)\,dx + O(1/k) \tag{3.4}$$

Notice that if (3.3) equals to (3.4), then as we take the limit as $k \to \infty$, we get equation (3.2). To show this is the case, notice the left hand side of (3.1) is:

$$\int_{S_T} u(t,x)\theta_k''(t)\psi(x)dtdx - \int_{\mathbb{R}^n} \left(\int_0^{1/k} u(t,x)a(kt)dt \right) \triangle psi(x)dx$$

The second term in the above is $O(1/k)$, and after integration by parts the first term becomes:

$$-\int_{S_T} \partial_t u(t,x)\theta_k'(t)\psi(x)dtdx$$

(Note that there are no boundary terms since $\theta_k'(t) = 0$ for $t = T = 0$). Doing integration by parts a second time, we get:

$$\int_{\mathbb{R}^n} \left(\int_0^{1/k} \partial_t^2 u(t,x)a(kt)dt \right) \psi(x)dx + \int_{\mathbb{R}^n} \partial_t u(0,x)\psi(x)dx$$

where again the first term is $O(1/k)$, which proves the claim, and thus completes the proof

(exercise doing the same for showing that $u(0,x) = f(x)$. It proceeds like the proof above, but this time choosing $\theta_k(t) = \frac{1}{k}b(kt)$ where b is some smooth compactly suported function such that $b(0) = 0$ and $b'(0) = 1$, thus $\theta_k(0) = 0$ and $\theta_k'(0) = 1$ and the support shrinks to the origin as $k \to \infty$ (see that the error term is not $O(1/k^2)$))

Example 3.1: Weak Solution

1. I think putting some examples here is a good idea

3.1 Distributions

Distributions a generlization of functions that allow us to give more general solution than classical solutions, similar to how we found a weak solution. Locally integrable functions will be distributions (in the appropriate interpretation), but we'll see that there are general solution's that don't even fall under the notion of "function".

3.1.1 Intuition

For now, let $f \in L^p$. Then as we know, $(L^p)^* \cong L^q$ is an isometry where $1 \leq p < \infty$. In particular, we have that

$$g \mapsto \int fg$$

is mapped to by f. Using the Lebesgue differentiation theorem, by choosing $\varphi_k = m(B_r)^{-1}\chi_{B_r}$, where B_r is a ball of ardius r around x, will allow us to recevoer the value of $f(x)$ for a.e. x, as $\lim_{r\to 0} \int f\varphi_r$. In this way, we can think of L^p as being embedded into $(L^q)^*$ (since we really only care what happens a.e.).

Now, let us modify the above by letting f be merely locally integrable (in L^1_{loc}), but $\varphi \in C_c^\infty$. Then it is clear that $\varphi \mapsto \int f\varphi$ is still a well-defined functional on C_c^∞ , and the pointwise values of f can be rcecoved a.e. from it, by extending theroem ref: HERE (8.15 in Folland). Unlike in the case for L^p spaces, there are in fact *many* linear functionals on C_c^∞ that are *not* of the form $\varphi \mapsto \int f\varphi$. These, subject ot some mild continuity conditions we will soon specify, will be our "generlized functions".

3.1.2 Definitions

We will first rapid-fire some important definitions to be up to speed with terminology. Recall that $C_c^\infty(E)$ where $E \subseteq \mathbb{R}^n$ is the set of all smooth functions whose support is compact and contained in E. If $U \subseteq \mathbb{R}^n$ is open, $C_c^\infty(U)$ is the union of space $C_c^\infty(K)$ where K ranges oer all compact subsets of U. Each $C_c^\infty(K)$ is a Fréchet space with the topology defined by the norms

$$\varphi \mapsto \|\partial^\alpha\|_u \qquad (\alpha \in \{0, 1, 2, ...\}^n)$$

Thus ,the sequence $\{\varphi_i\}$ converges if and only if $\partial_i^\varphi \to \partial^\alpha\varphi$ uniformly for all α. The completness of $C_c^\infty(K)$ is shown through exercise 9 in folland in section 5.1. With these, we take the following definitions:

1. A sequence $\{\varphi_i\}$ in $C_c^\infty(U)$ converges in C_c^∞ to φ is $\{\varphi_i\} \subseteq C_c^\infty(K)$ for some compact set $K \subseteq U$ and $\varphi_i \to \varphi$ in the topology of $C_c^\infty(K)$, that is, $\partial^\alpha\varphi_i \to \partial^\alpha\varphi$ uniformally for all α
2. If X is a locally convex topologicla vector space and $T : C_c^\infty(U) \to X$ is a linaer map, T is *continuous* if $T|_{C_c^\infty(K)}$ is continous for each compact $K \subseteq U$, that is, if $T(\varphi_i) \to T(\varphi)$ whenever $\varphi_i \to \varphi$ in $C_c^\infty(K)$ and $K \subseteq U$ is compact
3. A linear map $T : C_c^\infty(U) \to C_c^\infty(U')$ is *continuous* if for each compact $K \subseteq U$, there is a compacty $K' \subseteq U'$ such that $T(C_c^\infty(K)) \subseteq C_c^\infty(K')$ and T is continuous from $C_c^\infty(K)$ to $C_c^\infty(K')$

With these, we will define a distribution:

Definition 3.1.1: Distribution

A *distribution* on U is a continous lihear functional on $C_c^\infty(U)$. The space of all distributiosn on U is denoted by $\mathcal{D}'(U)$, and we set $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$. We impose the weak* topology on $\mathcal{D}'(U)$, that is, the topology of pointwise converge on $C_c^\infty(U)$.

A couple of remarks are in order: first, the $\mathcal{D}'$ notation comes from the fact that Schwart'z used the notation of $\mathcal{D}$ for C_c^∞ (probably because such functions are *D*ifferentiable), and so $\mathcal{D}'$ was the funcitonals on C_c^∞. Next, there is a locally convex topology on C_c^∞ with respect to which sequential convergence in C_c^∞ is given by the convergence defined earlier in C_c^∞, and the continuity of a linear map $T : C_c^\infty \to X$ and $T : C_c^\infty \to C_c^\infty$ is given by the other two points. However, to define it is relatively complicated and doesn't contribute much to the current exposition of distributions , and so will be ommited.

Before giving some examples of distributions, we will remind the reader how of how to check for continuity in a TVS:

Proposition 3.1.1: Distribution Continuity Check

Let $u : C_c^\infty(U) \to \mathbb{R}$ (or $\mathbb{C}$) be linear. Then $u \in \mathcal{D}'(U)$ if and only if for every compact set $K \subseteq U$, there exists a $C_K > 0$ and $N_K \in \mathbb{N}$ such that

$$\|\langle u, \varphi\rangle\| \leq C_k \sum_{\|\alpha\| \leq N_K} \|\partial^\alpha \varphi\|_L^\infty$$

Proof

Given the above equation, then it is easy to check that if $\varphi_i \to \varphi$, then $\langle u, \varphi_i\rangle \to \langle u, \varphi\rangle$. (see chapter 4 for details)

Example 3.2: Distributions

1. For every $f \in L^1_{\text{loc}}(U)$, we can define a distribution on U< namely the funcitonal $\varphi \to \int f\varphi$, and two functions define the same distributiosn precisely when they are equal a.e.
2. Every Radon measure μ on U defines a distributiosn $\varphi \mapsto \int \varphi d\mu$.
3. If $x_0 \in U$ and α is a multi-index, the map $\varphi \mapsto \partial^\alpha(x_0)$ is a distribution that does not arise from a function; it arises from a measure μ prcisely when $\alpha = 0$, in which case μ is the point mass at x_0

As mentioned earlier, we often will associate the a function f with the distribution it has, and even label the distribution by f. For this reason, it might be confusing to write $f(\varphi)$ to mean the distribution defined by f taking in φ, and so for historical reasons we use the notation $\langle f, \varphi\rangle$.

Let's next establish some common notation:

1. First, it is often enough that we will reflect function. Thus, if we put a tilde over a function, we mean:
$$\tilde{\varphi}(x) = \varphi(-x)$$
2. Second, the point-mass is a very common distribution, which we will denote by δ:
$$\langle \delta, \varphi\rangle = \varphi(0)$$

As an example of the use of the detla funciton:

Proposition 3.1.2: Convergence To Delta Function

Let $f \in L^1(\mathbb{R}^n)$ and $\int f = a$, and for $t > 0$, let $f_t(x) = t^{-n} f(t^{-1}x)$. Then$f_t \to a\delta$ in $\mathcal{D}'$ as $t \to 0$

Proof
If $\varphi \in C_c^\infty$, by theroem 8.15, we have:

$$\langle f_t, \varphi \rangle = \int f_t \varphi = f_t * \tilde{\varphi}(0) \to a\tilde{\varphi}(0) = a\varphi(0) = a\langle \delta, \varphi \rangle$$

completing the proof.

For equality of distributions, note that it makes to say that $F = G$ on an open set V if and only if $\langle F, \varphi \rangle = \langle G, \varphi \rangle$ for all $\varphi \in C_c^\infty(V)$. If F and G are continuous functions, this will imply they are pointwise equal, and if F and G are only locally integrable,this means they are equal a.e. on V.

Since a function in $C_c^\infty(V_1 \cup V_2)$ need not be supported in either V_1 or V_2, it is not immediately obvious that if $F = G$ on V_1 and on V_2 then $F = G$, then $F = G$ on $V_1 \cup V_2$. However, this is the case:

Proposition 3.1.3: Equality On Support

Let $\{V_\alpha\}$ be a collectio nof open subsets of U and let $V = \bigcup_\alpha V_\alpha$. If $F, G \in \mathcal{D}'(U)$ and $F = G$ on each V_α, then $F = G$ on V

Proof
Folland p. 284

As a consequence of proposition 3.1.3, if $F \in \mathcal{D}'(U)$, there is a maximal open subset U on which $F = 0$, namely the union of all open subset on which $F = 0$. Its compliment in U is called the *support* of F. If it's compliment is compact, it is called the *compact support*.

One of amazing facts about distributions is that even though they are not "functions", we will be able to do many functional operations like diferentation, multiplication by a "constant", convolution, and translation!! To achieve this, we need some theoretical build-up. Suppose U and V are open sets in $\mathbb{R}^n$ and T is a liner map from some subsapce V of $L^1_{\text{loc}}(U)$ into $L^1_{\text{loc}}(V)$. Suppose that there is antoher linaer map $T' : C_c^\infty(V) \to C_c^\infty(U)$ such that

$$\int (Tf)\varphi = \int f(T'\varphi) \qquad (f \in V,\ \varphi \in C_c^\infty(V))$$

suppose that T' is continuous in the sense defined earlier. Then T can be extended to a map from $\mathcal{D}'(U)$ to $\mathcal{D}'(V)$, still dentoed by T, by:

$$\langle TF, \varphi \rangle = \langle F, T'\varphi \rangle \qquad (F \in \mathcal{D}'(U),\ \varphi \in C_c^\infty(V))$$

The continuity of T' guarantees that the original map T, as well as the extension to distributions, is continuous with respect to the weak* topology on distributions, namely if $F_\alpha \to F \in \mathcal{D}'(U)$, then $Tf_1a \to TF$ in $\mathcal{D}'(V)$. With this, we can define the following on distributions

1. (Differentiation) Let $Tf = \partial^\alpha f$, defined on $C^{|\alpha|}(U)$. If$\varphi \in C_c^\infty(U)$, integration by parts gives $\int(\partial^\alpha f)\varphi = (-1)^{|\alpha|}\int f(\partial^\alpha \varphi)$; there are no boundary terms since φ has compact suppor.t. Henc $T' = (-1)^{|\alpha|}T|_{C_c^\infty(U)}$, and we can define the *derivative* $\partial^\alpha F \in \mathcal{D}'(U)$ of any $F \in \mathcal{D}'(U)$ by:

$$\langle \partial^\alpha F, \varphi\rangle := (-1)^{|\alpha|}\langle F, \partial^\alpha, \varphi\rangle$$

   By this procedure, we can define the derivatives of arbirary locally integrabl funtions, even when they are not differentiable in the classical sense! We will give the derivative of certain locally integrable functions soon!!

2. (multiplication by a smooth function, a "scalar") Let $\psi \in C^\infty(U)$ and define $Tf = \psi f$. Then evidently, $T' = T|_{C_c^\infty(U)}$, and so we can define $\psi F \in \mathcal{D}'(U)$ for any $F \in \mathcal{D}'(U)$ to be:

$$\langle \psi F, \varphi\rangle := \langle F, \psi\varphi\rangle$$

3. (Translation) Let $y \in \mathbb{R}^n$ and let $V = U + y = \{x + y \mid x \in U\}$. Take $T = \tau_y$ (recall that $\tau_y f(x) = f(x-y)$). Via u-substituion, we get that $\int f(x-y)\varphi(x)dx = \int f(x)\varphi(x+y)dx$, and so $T' = \tau_{-y}|_{C_c^\infty(U)}$. For a distribution $F \in \mathcal{D}'(U)$, we define:

$$\langle \tau_y F, \varphi\rangle := \langle F, \tau_{-y}\varphi\rangle$$

4. (convolution with linear maps) Let S be an invertible linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Letting $V = S^{-1}(U)$ and $Tf = f \circ S$, Then $T'\varphi = |\det(S)|^{-1}\varphi \circ S^{-1}$ (by change of variables, see theorem ref:HERE). Thus, for $F \in \mathcal{D}'(U)$, define:

$$\langle F \circ S, \varphi\rangle := |\det S|^{-1}\langle F, \varphi \circ S^{-1}\rangle$$

5. (convolution I) Let $\psi \in C_c^\infty$, and let

$$V = \{x \mid x - y \in U \text{ for } y \in \operatorname{supp}(\psi)\ \}$$

   Note that V is open but may be empty. If $f \in L^1_{\text{loc}}(U)$, we get that the integral:

$$f * \psi(x) = \int f(x-y)\psi(y)dy = \int f(y)\psi(x-y)dy = \int f(\tau_x\tilde{\psi})$$

   is well-defined for all $x \in V$. Since we transfered over from f to ψ, we can define for $F \in \mathcal{D}'(U)$:

$$F * \psi(x) = \langle F, \tau_x\tilde{\psi}\rangle$$

   Since $\tau_x\tilde{\psi} \to \tau_{x_0}\tilde{\psi}$ as $x \to x_0$, $F * \psi$ is a continuous function (in fact, C^∞, as we'll show in a moment). A pertinent example of this is the following:

$$\delta * \psi(x) = \langle \delta, \tau_x\tilde{\psi}\rangle = \tau_x\tilde{\psi(0)} = \psi(x)$$

   so δ is the multiplicative identity of convolution!

6. (convolution II) Let $\psi, \tilde{\psi}$ and V be like the last point. If $f \in L^1_{\text{loc}}(U)$ and $\varphi \in C_c^\infty(U)$, then

$$\int (f * \psi)\varphi = \int\int f(y)\psi(x-y)\varphi(y)dydx = \int f(\varphi * \tilde{\psi})$$

   Thus $Tf = f * \psi$ maps $L^1_{\text{loc}}(U)$ to $L^1_{\text{loc}}(V)$ and $T'\varphi = \varphi * \tilde{\psi}$. Thus, we get:

$$\langle F * \psi, \varphi\rangle = \langle F, \varphi * \tilde{\psi}\rangle$$

(theorem on how the two convolutions are actually the same thing)

Next, we will show that smooth compact functions $C_c^\infty(U)$ are dense in $\mathcal{D}'(U)$

Lemma 3.1.1: Build-Up For Density

Suppose that $\varphi \in C_c^\infty, \psi C_c^\infty$ and $\int \psi = 1$, and let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. Then

1. Given any neighborhood U of $\text{supp}(\varphi)$, we have $\text{supp}(\varphi * \psi_t) \subseteq U$ for t
2. $\varphi * \psi_t \to \varphi$ in C_c^∞ as $t \to 0$

Proof
Folland p. 287

Proposition 3.1.4: Density Of Smooth Compact Functions In Distributions

Let $U \subseteq \mathbb{R}^n$ be open. Then $C_c^\infty(U)$ is dense in $\mathcal{D}'U)$ in the topology of $\mathcal{D}'(U)$

Proof
Folland p. 287

3.1.3 More continuity properties

Suppose

$$u_i \to u \text{ in } \mathcal{D}'(U) \qquad \varphi_i \to \varphi \text{ in } C_c^\infty(U)$$

Then is it true that:

$$\langle u_i, \varphi_i \rangle \to \langle u, \varphi \rangle$$

The answer is yes, though it is not immediate. First, using bilinearity, we can re-write the left hand side as the right hand side in the following:

$$\langle u_i, \varphi_i \rangle - \langle u, \varphi \rangle = \langle u_i, \varphi_i - \varphi \rangle + \langle u_i - u, \varphi \rangle$$

Since we have that $u_i \to u$, the second term on the right hand side certainly goes to zero. However, as u_i varies, it is not clear that the first term on the right hand side goes to zero. However, if the continuity estimate given in proposition 3.1.1 would hold for all u_i with some constant C_k and N_k (i.e. *independent* of i), then the first term would also converge to zero, since $\varphi_i \to \varphi$. In fact, it turns out that such estimates do indeed exist, which is very non-trivial since the convergence in $\mathcal{D}'(U)$ has the topology given by pointwise convergence! To show this, we requires some important build-up results:

Theorem 3.1.1: Uniform Boundedness Principle

Consider an indexed family $\{u_\lambda\}_{\lambda\in I} \subseteq \mathcal{D}'(U)$. If

$$\sup_{\lambda\in I} |\langle u_\lambda, \varphi| < \infty \qquad \text{for all } \varphi \in C_c^\infty(U)$$

then for every compact $K \subseteq U$, there exists C_K and N_K such that

$$\sup_{\lambda\in I} |\langle u_\lambda, \varphi| \leq C_k \sum_{\|\alpha\|\leq N_K} \|\partial^\alpha \varphi\|L^\infty \qquad \text{for all } \varphi \in C_c^\infty(K)$$

Proof

See *Analysis Now* by Peterson, Chapter 2, Section 2, end of the section.

Corollary 3.1.1: Completeness Of Distributions

Let $\{u_i\}$ be a sequence in $\mathcal{D}'(U)$. Suppose that for every $\varphi \in C_c^\infty(U)$,

$$\lim_{i\to\infty} \langle u_i, \varphi\rangle$$

exists, and denote it's limit by $\langle u, \varphi\rangle$. Then the u as defined is an element of $\mathcal{D}'(U)$. As a consequence, $\mathcal{D}'(U)$ is complete.

Proof

The fact that $\mathcal{D}'(U)$ is complete comes from the fact that if $\{u_i\}$ is a cauchy sequence, $\langle u_i, \varphi\rangle$ is a cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) for every φ, and hence converges in $\mathbb{R}$ (or $\mathbb{C}$), and then we apply the first result of the corollary.

3.1.4 Time-dependent Distributions

Consider a map

$$\mathbb{R} \to \mathcal{D}'(\mathbb{R}^n), \qquad t \mapsto u_t$$

Since $\mathcal{D}'(\mathbb{R}^n)$ has a topology, the toplogy of point-wise convergence, the notion of continuity and differentiability can be defined. Thus, the above function is continuous at t_0 if and only if $t \to \langle u_t, \varphi\rangle$ is continuous for every $\varphi \in C_c^\infty(U)$, and u is differentiable at t_0 if and only if there exists a $v \in \mathcal{D}'(\mathbb{R}^n)$ such that for every $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\frac{d}{dt} \langle u_t, \varphi\rangle|_{t=t_0} = \langle v, \varphi\rangle$$

and we write $u_t' = v$. This can naturally be repeated to form k-times differentiables maps in $C^k(\mathbb{R}, \mathcal{D}')$.

Proposition 3.1.5: Continuous Distributions Form Distributions

Every $u \in C(\mathbb{R}, \mathcal{D}')$ defines a distributions on $\mathbb{R}^{n+1}$ by

$$\langle u, \psi \rangle = \int_{\mathbb{R}} \langle u_t, \psi(t, \cdot) \rangle \qquad \text{for } \psi \in C_c^\infty(\mathbb{R}^n)$$

namely, $u \in \mathcal{D}'(\mathbb{R}^{1+n})$

Proof

First, the integrand is continuous and compactly supported as a function of t, and so the integral exists. The functional u is in $\mathcal{D}'(\mathbb{R}^{1+n})$ if and only if the all derivatives are properly bounded for all compact sets $K \subseteq \mathbb{R}^{1+n}$. Since the compact sets of the form $I \times K'$ are dense in the set of compact sets, it suffices to take $I \times K'$. But then, by theorem 3.1.1, there exist $C_{I,K'} > 0$ and $N_{I,K'} \in \mathbb{N}$ such that

$$\sup_{t \in I} |\langle u(T), \varphi \rangle| \leq C_{I,K'} \sum_{|\alpha| \leq N_{I,K'}} \|\partial^\alpha \varphi\|_{L^\infty} \qquad \forall \varphi \in C_c^\infty(K')$$

Applying this to each $t \in I$ with $\varphi = \psi(t, \cdot)$, and then integrating with respect to t, we get the needed estimate to show it is bounded, showing it is indeed continuous, and so completing the proof.

With this build-up, we are now able to ask the following question: if $u, u' \in C^1(\mathbb{R}, \mathcal{D}')$, is it possible that:

$$\partial_t u = u'$$

with the derivative defined distributionally. The answer is yes! Unwarpping the definition, the above equation is equivalent to:

$$\int_{\mathbb{R}} \langle u(t), (-1)\partial_t \psi(t, \cdot) \rangle dt = \int_{\mathbb{R}} \langle u'(t), \psi(t, \cdot) \rangle dt$$

and since $u \in C^1$ (i.e. the derivative is continuous), then the derivative can be written in terms of distributions like so:

$$\frac{d}{dt} \langle u(t), \psi(t, \cdot) \rangle = \langle u'(t), \psi(t, \cdot) \rangle + \langle u(t), \partial_t \psi(t, \cdot) \rangle$$

To see this, note that if you take $f(t) = \langle u(t), \psi(t, \cdot) \rangle$, then

$$\begin{aligned} &\frac{1}{h}\{f(t+h) - f(t)\} \\ &= \left\langle \frac{1}{h}\left\{u(t+h) - u(t)\right\}, \psi(t, \cdot) \right\rangle \left\langle u(t+h), \frac{1}{h}\{\psi(t+h, \cdot) - \psi(t, \cdot)\} \right\rangle \end{aligned}$$

and then apply the uniform boundedness principle on the last term.

3.1.5 Distributional Solution for $\Box u = 0$

We can now consider using distribution's to solve problems, namely:

$$\Box u = 0 \qquad \Leftrightarrow \qquad \langle u, \Box \varphi \rangle = 0 \qquad \forall \varphi \in C_c^\infty(U)$$

with the Cauchy problem:

$$u|_{t=0} = f \qquad \partial_t u|_{t=0} = g$$

In general, it does not make sense to restrict a distribution u to the plane at $t = 0$ (what would this mean for a measure?). However, this *does* make sense for a time-dependent distribution as discussed in the last section. In fact, given *any* two distributions f, g, we can find a solution to the wave equation using time-dependent distributions!!

Theorem 3.1.2: General Weak Solution To The Wave Equation

For all $f, g \in \mathcal{D}'(\mathbb{R}^n)$, there exists a time-dependent distribution $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^n))$ which solves the Cauchy problem

Note that we can replace $\mathbb{R}^n$ with U given we propertly seperate U to be $U = I \times U'$. Furthermore, this solution is unique, which will be covered in the next section. The solution to u with respect to f and g is in fact extremely similar to D'Alembert's formula: we will have a *wave operator* $W(t)$ and will get the solution to be:

$$u(t) = W'(t) * f + W(t) * g \tag{3.5}$$

where

$$W \in C^\infty(\mathbb{R}, \mathcal{E}'(\mathbb{R}^n)) \qquad W(0) = W''(0) = 0 \qquad W'(0) = \delta$$

And furthermore, the Wave operator respects Huygen's principle:

$$\operatorname{supp} W(t) \subseteq \{x \mid |x| \leq |t|\}$$

and if the dimension n is odd and $n \geq 3$:

$$\operatorname{supp} W(t) \subseteq \{x \mid |x| = |t|\}$$

Before describing $W(t)$, let's show that an operator $W(t)$ satisfying the above condition does give a [weak] solution to the wave equation. First, since $W(t)$ is compactly supported, the convolution is well-defined for each t for all $f, g \in \mathcal{D}'(\mathbb{R}^n)$. Thus, equation (3.5) defines a smooth time-dependent distribution (check this). Next, the initial condition's are satisfied since:

$$u(0) = W'(0) * f + W(0) * g = \delta * f = f$$
$$u'(0) = W''(0) * f + W'(0) * g = \delta * g = g$$

Finally, to see that $\square u = 0$, we use an approximation argument. Since $C_c(\mathbb{R}^n)$ is dense in distributions, choose $f_i \to f$ and $g_i \to g$ in $\mathcal{D}'(U)$. Let u_i be the solution to the cauchy problem with f_i and h_i. Then:

$$u_i(t) = W'(t) * f_i + W(t) * g_i$$

Now, in $\mathcal{D}'(\mathbb{R}^{1+n})$:

$$u_i \to u$$

but then in $\mathcal{D}'(\mathbb{R}^{1+n})$:

$$\square u_i \to \square u$$

since $\square u_i = 0$ for all i, $\square u = 0$, completing the proof.

(there are a lot of exercises here going through the important details to show how what is above is correct)

We next tackle what the wave operator $W(t)$ actually is. (Selberg basically just shows how the d'Alembert forumla and sphere mean formula is the wave propegator, p. 29)

3.2 More on Distributional Solutions

We now move on to non-homogeneous solutions: we want to find a $u \in C^1([0,T], \mathbb{R}^n)$ that sovles

$$\begin{cases} \Box u = F & \text{on} \qquad S_T = (0,T) \times \mathbb{R}^n \\ u|_{t=0} = f & \partial_t u|_{t=0} = g \end{cases}$$

Theorem 3.2.1: Unique Solution to Distributional Solution

The solution of the above equation is unique and of class $C^1([0,T], \mathcal{D}'(\mathbb{R}^n))$

Proof

Guided exercise in selberg p. 35

Proposition 3.2.1: Further Result about Weak Derivative I

Let $u, v \in C([0,T], \mathcal{D}'(\mathbb{R}^n))$ and set $S_T = (0,T) \times \mathbb{R}^n$. If

$$\partial_t u = v$$

i.e. u is the weak derivative of v, then $u \in C^1([0,T], \mathcal{D}'(\mathbb{R}^n))$ and $u' = v$

Proof

Selberg p. 36

Proposition 3.2.2: Further Results About Weak Derivatives II

Let $\Omega \subseteq \mathbb{R}^n$ be open. If $u \in C(\Omega)$ and the distributional derivative $\partial^\alpha u$ are also in $C(\Omega)$, for all $|\alpha| < k$, then $u \in C^k(\Omega)$

Proof

Selberg p.37

Proposition 3.2.3: Consant Weak Derivative

Let $u \in \mathcal{D}'(I)$, where $I \subseteq \mathbb{R}$ is an open interval. Then:

$$u' = 0 \Rightarrow u \text{ is constant}$$

Proof

p.37

3.2.1 A Fundamental Solution for the Square Operator

(Not sure what this is)

3.2.2 Energy Identity

The energy identity gives a limits to how much the solution to a wave equation, or more generally a solution who is always on compact support which includes *inhomogeneous wave equations*, could have grown in a finite amount of time. In particular, we will bound $\|\nabla u(t,\cdot)\|_{L^2}$, by the "total" amount of potential energy that could have accumulated in the system. We first need to establish some notation and a lemma. Recall that the gradient of a differentiable function $u \in \mathbb{R}^{1+n}$ is:

$$\nabla u = (u_t, \nabla_x u)$$

which gives a basis for the tangent space of the codomain of u given the standard coordinates.

Lemma 3.2.1: Energy Identity in Homogeneous Case

Suppose $u \in C^2([0,T], \mathbb{R}^n)$ solves $\Box u = 0$. and that $u(t,\cdot)$ is compactly supported for every $t \in [0,T]$. Then:

$$\|\partial u(t,\cdot)\|_L^2 = \|\partial u(0,\cdot)\|_L^2$$

Proof

Define the energy identity:

$$e(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial u(t,x)|^2 \, \mathrm{d}x = \frac{1}{2}\|u(t,\cdot)\|_{L^2}^2$$

Differentiating with respect to t and integrating by parts, we get:

$$e'(t) = \int_{\mathbb{R}^n} u_t \Box u dx$$

since $\Box u = 0$, we get that $e'(t) = 0$, and so $e(t) = c$ is a constant, and so $e(0) = e(t)$, which let's us conclude that:

$$\|\nabla u(0,\cdot)\|_{L^2} = \|\nabla u(t,\cdot)\|_{L^2}$$

as we sought to show.

We now move on to u which can be inhomogeneous. In fact, this proof works even more generally than solutiosn to wave equations, though as we will see it will naturally be applicable for wave equations

Theorem 3.2.2: Energy Inequality (Identity)

Let $u \in C^2([0,T], \mathbb{R}^n)$ and $u(t,\cdot)$ be compactly supported for every $t \in [0,T]$. Then:

$$\|\partial u(t,\cdot)\|_{L^2} \leq \|\partial u(0,\cdot)\|_{L^2} + \int_0^t \|\Box u(s,\cdot)\|_{L^2} ds$$

Proof

We start like how we've done in lemma 3.2.1: define the energy identity

$$e(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\partial u(t,x)|^2 \,\mathrm{d}x = \frac{1}{2}\|u(t,\cdot)\|_{L^2}^2 \tag{3.6}$$

Differentiating with respect to t and integrating by parts, we get:

$$e'(t) = \int_{\mathbb{R}^n} u_t \Box u \, dx \tag{3.7}$$

Now, apply the Cauchy-Schwartz identity(equivalently Hölders equality) to equation (3.7), we get:

$$e'(t) \leq \|\partial u(t,\cdot)\|_{L^2} \|\Box u(t,\cdot)\|_{L^2} = \sqrt{2e(t)}\|\Box u(t,\cdot)\|_{L^2}$$

Thus, whenver $e'(t) \neq 0$, we get:

$$\frac{d}{dt}\sqrt{e(t)} = \frac{e'(t)}{\sqrt{e(t)}} \leq \frac{1}{\sqrt{2}}\|\Box u(t,\cdot)\|_{L^2}$$

Thus, integrating with $\int_0^t$, on boht sides, we get:

$$\sqrt{e(t)} \leq \sqrt{e(0)} = \int_0^t \|\Box u(s,\cdot)\|_{L^2} ds$$

which, when substituing in what $e(t)$ equals to from equation (3.6), we get the inequality form the theorem, completing the proof.

3.2.3 Fourier Transform and Tempered Distributions

We now generlize earlier results of solving PDE's using Fourier transformations but use fourier transforms of distributions. Let $\mathcal{S}$ be the Schwartz space and $\mathcal{S}'$ is the dual of $\mathcal{S}$. Then naturally,

$$E' \subseteq \mathcal{S}' \subseteq \mathcal{D}'$$

where E' is the set of all distributions on compact support. Since we will be using $\mathcal{S}'$ often, we will give it a name:

Definition 3.2.1: Tempered Distribution

Let $\mathcal{S}$ be the schwartz space. Then $\mathcal{S}'$, the dual of $\mathcal{S}$, is called the set of *tempered distributions*.

If $f \in \mathcal{S}'$, it has a *Fourier Transformations*:

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle$$

If $u \in E'$, then $\hat{u}$ is a C^∞ function given by:

$$\hat{u}(\xi) = (2\pi)^{-n/2} \langle u, E_\xi \rangle$$

where $E_\xi \in C^\infty(\mathbb{R}^n)$ is $E_\xi(x) = e^{-ix\xi}$. To make sure that function multiplication still remains of schwartz class, we will need a $\psi \in C^\infty(\mathbb{R}^n)$ to be slowly increasing, that is for all partial derivative's to have a growth at most:

$$|\partial^\alpha \psi(x)| \leq C_\alpha (1 + |x|)^{N(\alpha)} \qquad \forall \alpha$$

Then $\psi\varphi \in \mathcal{S}$ whenver $\varphi \in \mathcal{S}$. Thus, in this case, we can define for $u \in \mathcal{S}'$:

$$\langle \psi u, \varphi \rangle = \langle u, \psi\varphi \rangle$$

With this, we are ready to use Fourier Transfroms to solve the wave equation: let u be the solution of:

$$\Box u = 0 \quad \text{on } \mathbb{R}^{1+n}, \quad u|_{t=0} = f \quad \partial_t u|_{t=0} = g$$

with $f, g \in \mathcal{S}(\mathbb{R}^n)$. Applying the Fourier transform to the space of variables:

$$\hat{u}(t, \xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(t, x) dx$$

Then $\Box u = 0$ transforms into:

$$\partial_t \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0 \qquad \hat{u}(0, \xi) = \hat{f}(\xi) \qquad \partial_t \hat{u}(u, \xi) = \hat{g}(\xi)$$

But as before, this is just a second order ODE with respect to t with solution:

$$\hat{u}(t, \xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi) \tag{3.8}$$

Now, we've seen earlier that:

$$u(t, \cdot) = W'(t) * f + W(t) * g$$

solve the wave equation. Since $W(t) \in E'$, we can apply the Fourier transform to the above equation and get:

$$\hat{u}(t, \xi) = (2\pi)^{n/2} \widehat{W'(t)}(\xi)\hat{f}(\xi) + (2\pi)^{n/2} \widehat{W(t)}(\xi)\hat{g}(\xi)$$

Comparing this with equation (3.8), we see that:

$$\widehat{W(t)}(\xi) = (2\pi)^{-n/2} \frac{\sin(t|\xi|)}{|\xi|} \qquad \widehat{W'(t)}(2\pi)^{-n/2} \cos(t|\xi|) \tag{3.9}$$

Giving us the full solution when $\Box u = 0$. For the inhomogeneous case, $\Box u = F$, with vanishing data at $t = 0$, we have:

$$u(t,x) = \int_0^t W(t-s) * F(s,x)dx$$

Applying the Fourier transform and applying equation (3.9), we get:

$$\hat{u}(t,\xi) = \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\hat{F}(s,\xi)ds$$

Thus, the full solution to the inhomogeneous case:

$$\Box u = 0 \quad \text{on } \mathbb{R}^{1+n}, \quad u|_{t=0} = f \quad \partial_t u|_{t=0} = g$$

is:

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi) + \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\hat{F}(s,\xi)ds$$

3.2.4 Sobelev Space H^s

So far, we found solutions to PDE's lie in $\mathcal{D}'$. This is great in finding general solutions, however the space $\mathcal{D}'$ doens't give much in the way of control of the regularity of functions (i.e. how much the derivative varries). In later studies of solutions to PDE's it will be quite important that we have some control over the derivative so that the solution "stays" nice. So, we can think of Sobelev spaces as spaces that

1. have sufficinetly many derivatives
2. the functions have sufficiently nice regularity properties.

For any fixed $s \in \mathbb{R}$, consider $\xi \to (1+|\xi|^2)^{s/2}$ in C^∞. This function is slowly increasing, and thus we can define $\Lambda^s : \mathcal{S}' \to \mathcal{S}'$

$$\Lambda^s f = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f$$

Note that Λ^s is continuous, being a composition of continuous functions. Morever, it is an isomrophism with inverse Λ^{-s}. Similaly, this makes Λ^s when restricted to $\mathcal{S}$ an isomrophism from $\mathcal{S}$ to itself. Now, set $H^s(\mathbb{R}^n) = \Lambda^{-s}(L^2(\mathbb{R}^n))$ with norm:

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}$$

or written another way, $H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' \;\middle|\; \Lambda^s f \in L^2 \right\}$. By Plancherel's formula:

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{f}\|_{L^2}$$

Note that $H^0 = L^2$ and $s < t \Rightarrow H^t \subseteq H^s$. Thus, $\Lambda^2 : H^s \to L^2$ is an isometric isomorphism, and hence H^s is a Hilbert space. Since $\mathcal{S}$ is dense in L^2, $\Lambda^s(\mathcal{S}) = \mathcal{S}$, and so $\mathcal{S}$ is dense in H^s. Furthermore, $\Lambda^s : H^t \to H^{s-t}$ is also an isometric isomorphism for all $s,t \in \mathbb{R}$.

Finally, there is another way of looking at Sobelev spaces H^s: if $s \in \mathbb{N}$, then:

$$H^s = \left\{ f \in L^2 \;\middle|\; \partial^\alpha f \in L^2 \text{ for } |\alpha| \leq s \right\}$$

and the norm $\|f\|_{H^s}$ is equivalent to:

$$\left(\sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2}^2 \right)^{1/2}$$

which follows from Plancherel's formula

3.2.5 L^2 estimates for solutions of $\Box u = 0$

Selberg p. 43

4

Non-Linear Equations

The goal of this chapter is to now generalize the conditions on which we find wave equations in *inhomogeneous* and *non-linear* space. We will first we will first allow f, g to be in H^s and H^{s-1} respectively instead of $\mathcal{D}'$, which wil guarantee enough regularity to find a solution in the *inhomogeneous* case. We will then generalize and try to find solution for the *non-linear* case, which we will show is in fact a much harder problem, and in contrast to the linear case there will not always exist a global solution.

The goal is to show that we can find *local solutions*. The space of functions that will be nice enough to find these solutions will turn out to be Sobelev spaces, and so they will play a prominent role in this chapter.

4.1 Existence and Uniqueness Inhomogeneous PDE

We first solve the Cauchy problem in the nicest of Sobelev spaces. Take the inhomogenous PDE:

$$\Box u = F \tag{4.1}$$

$$u|_{t=0} = f \qquad \partial_t u|_{t=0} = g \tag{4.2}$$

Theorem 4.1.1: Unique Solution In Sobelev Space

Let $s \in \mathbb{R}$. Let $f \in H^s$ and $g \in H^{s-1}$ and $F \in L^1([0,T], H^{s-1})$. Then for every $T > 0$, there exists a unique solution u which belongs to

$$C([0,T], H^s) \cap C^1([0,1], H^{s-1}) \tag{4.3}$$

and solves equation (4.1) on $S_T = (0,T) \times \mathbb{R}^n$. Moreover, u satisfies:

$$\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \leq C_T \left(\|f\|_{H^s} + \|g\|_{H^s} + \int_0^1 \|F(t')\|_{H^{s-1}} dt' \right)$$

for all $0 \leq t \leq T$ where $C_T = C(1+T)$ and C only depends on s

Proof

This proof was broken down into the following exercises:

1. First, $F \in L^1([0,T], H^{s-1})$ means that

$$[0,T] \to \mathbb{R} \qquad t \mapsto \|F(t)\|_{H^{s-1}}$$

is in $L^1([0,T])$. Given such an F, it defines a distributionon S_T by:

$$\langle F, \psi \rangle = \int_0^T \langle F(t), \psi(t, \cdot) \rangle dt \qquad \psi \in C_c^\infty(S_T)$$

The key to see that the above integral converges, and defines an element of $\mathcal{D}'(S_T)$ is the following: we have the inequality:

$$|\langle u, \varphi \rangle| \leq \|u\|_{H^s} \|u\|_{H^{-s}} \qquad \forall s \in \mathbb{R},\ u \in H^s,\ \varphi \in \mathcal{S}$$

next, we have the inclusion:

$$C_c^\infty \subseteq \mathcal{S} \subseteq H^s \subseteq \mathcal{S}' \subseteq \mathcal{D}'$$

which are all sequentially continuous

2. Next, for any interval $I \subseteq \mathbb{R}$, we get the continuous inclusion:

$$C(I, H^s) \subseteq C(I, \mathcal{S}') \subseteq C(I, \mathcal{D}')$$

and similarly for every C^k. Thus, if u is in the solution space given in equation (4.3), then $u \in C^1([0,T], \mathcal{D}')$. This in fact proves uniqueness.

3. the rest is elaboarted in Selberg p.46

For existence, (it is the standard approximation method, so for any $f, g \in \mathcal{S}$ and $F \in C^\infty([0,T], \mathcal{S})$, we have a solution. We will construct an appropraite Banach space and show that these will converge to the values in our solution space)

4.2 Non-linear Cauchy Problem

We now start looking at non-linear equations. We start by showing that it is *not* always the case that there is global existence!! Let

$$\Box u = (\partial_t u)^2 \tag{4.4}$$

$$u|_{t=0} = f \qquad \partial_t u|_{t=0} g \tag{4.5}$$

Then:

Theorem 4.2.1: Non-Global Solution to Wave Equation

For the above Cauchy problem, there exists $g \in C_c^\infty(\mathbb{R}^n)$ such that the above does *not* admit a C^2 solution past time T.

Proof

Take g to be a constant $c > 0$. Then equation (4.4) reduces to the ODE $u_{tt} = u_t^2$, or if we set $y = u_t$, we get:

$$y' = y^2 \qquad y(0) = c$$

Then the solution to this separable ODE IVP is:

$$y(t) = \frac{c}{1 - ct}$$

which blows up as $t \to 1/c$. Thus, $u(t, x) = -\log(1 - ct)$ sovles equation (4.4) with condition (4.5), but $u \to \infty$ as $t \to 1/c$

In order to know that this is indeed the only possible solution, we show uniqueness for solutions to equation (4.4):

Theorem 4.2.2: Uniqueness Of Above Wave Equation

Let $u \in C^2(\Omega)$ solve $\Box u = (\partial_t u)^2$ in the solid backwards light cone

$$\Omega = \{(t, x) \mid 0 \le t < T, |x - x_0| < T_t\}$$

with base $B_0 = \{x \mid |x - x_0| < T - t\}$, then u is uniquely determined by its data:

$$u|_{B_0} \qquad \partial_t u|_{B_0}$$

Thus, if u, v are two solutions to equation (4.4) with the same inital data in Ω, then $u = v$. To see this, notice that $u - v$ solves the *linear* equation

$$\Box(u - v) = a(t, x)\partial_t(u - v) \qquad \text{in } \Omega$$

where $a = \partial_t u + \partial_t v \in C^1(\Omega)$, and that $u - v$ and $\partial_t(uv)$ vanish in B_0.

Proof
The above decomposition is important for the proof of this theorem will be done in the next section, see theorem ref:HERE

4.3 Local Existence and Uniqueness

We now study the general non-linear case. As we've seen, global solutions need not occur, however local existence solutions can occur. We will study exactly how these local existence solutions can happen in this section. Consider the Cauchy Problem:

$$\Box u = F(u, \partial u) \qquad F(0) = 0 \tag{4.6}$$

$$u|_{t=0} = f \qquad \partial_t u|_{t=0} = g \tag{4.7}$$

In the following theorems, we will consider u and F as real-valued, but the results generalise easily to system valued functions (i.e. $\mathbb{R}^n$-valued), which we ommit for presentation clarity

We will present many theorems and show where they lead to: these present the core of what is needed for solving a non-linaer PDE. However, these theorems are quite complex, and so we will state the narrative here, and in the next section we will solve them for a special but very important case. Then, in the next chapter, we will proceed with solving these theorems.

Theorem 4.3.1: Local Existence And Uniqueness

Let $s > \frac{n}{2} + 1$. Then for all $(f, g) \in H^s \times H^{s-1}$, there exists $T > 0$ and a unique

$$u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$$

Solving the IVP in equation (4.6) on $S_T = (0, T) \times \mathbb{R}^n$. Furthermore, the time T can be chosen to depend continuously on $\|f\|_{H^s} + \|g\|_{H^{s-1}}$

We next might want to see when we can extend a solution to a time $T' > T$:

Theorem 4.3.2: Continuation Theorem

Let $s > \frac{n}{2} + 1$. Fix f, g as in theorem 4.3.1. Let $T^* = T^*(f, g)$ be the supremum of all $T > 0$ such that equation (4.6) has a solution on S_T satisfying equation (4.7). Then either $T^* = \infty$ or:

$$\partial u \in L^\infty(S_T)$$

Using these two, we can restrict the IVP functions to smooth compact functions and always find a solution:

Theorem 4.3.3: Smooth Solution

If $f, g \in C_c^\infty(\mathbb{R}^n)$. Then there exists a $T > 0$ and a unique

$$u \in C^\infty([0,T], \mathbb{R}^n)$$

solving equation (4.6) on S_T

By the uniqueness of a solution in the backwards light cone, we can generlize the previosu result to the following:

Corollary 4.3.1: Global Smooth Solution

Let $f, g \in C^\infty(\mathbb{R}^n)$. Then there exists a set

$$A = \{(t,x) \mid 0 \le t \le T(x)\}$$

where T is a continuous and stricitly positive function on $\mathbb{R}^n$, and a unique solution $u \in C^\infty(A)$ of equation (4.6).

Proof
Selberg p. 53

4.4 The model equation $\Box u = (\partial_t u)^2$

We now prove these theorems in the special case where:

$$\Box u = (\partial_t u)^2$$

We will need the following identities and inequalities:

1. *Energy Identity* presented in section 3.2.2
2. *Sobelev's Lemma* on $\mathbb{R}^n$, in particular the inequality:
$$\|f\|_{L^\infty} \le C_s \|f\|_{H^s} \qquad \forall f \in H^s \quad s > \frac{n}{2}$$
3. A *calculus identity* (not sure why Selberg put the article "a" instead of "the")
$$\|fg\|_{H^s} \le C_s(\|f\|_{L^\infty}\|g\|_{L^\infty} + \|f\|_{H^s}\|\|g\|_{L^\infty})$$
for all $s \ge 0$ and all $f, g \in H^s \cap L^\infty$. This is a spcial case of *Moser's Identity* (which would be needed for the general case of solving a non-linear PDE)
4. A special case of *Gronwall's Lemma*: If $C_1, C_2 \ge 0$ are given constants, and A is a continuous non-negative function on $[0,T]$ such that
$$A(t) \le C_1 + C_2 \int_0^t A(\tau)d\tau \qquad 0 \le t \le T$$

Then:

$$A(t) \leq C_1 e^{C_2 t} \qquad 0 \leq t \leq T$$

This theorem is easy to see if you recall in EYNTKA 457 that we have classified all translation invariant function with amplitude 1 in the section on fourier transforms.

We will prove these identities later in the book. One more identity we will use and prove now is Sobelev's lemma:

Theorem 4.4.1: Sobelev's Lemma

Let $s > k + \frac{n}{2}$ where k is a non-negative integer. Then:

$$H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

where the inclusion is continuous. Even better:

$$\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty} \leq C_s \|f\|_{H^s}$$

where C_s is independent of f

Proof

First, consider the case when $k = 0$. The key is to use the Cauchy-Schwartz inequality: if $f \in H^s$ where $s > \frac{n}{2}$, then:

$$\int |\hat{f}(\xi)| d\xi = \int (1 + |\xi|^2)^{s/2} (1 + |\xi|^2)^{-s/2} |\hat{f}(\xi)| d\xi \leq C_s \|f\|_{H^s}$$

where

$$C_s^2 = \int (1 + |\xi|^2)^{-s} d\xi < \infty$$

Thus, by the Fourier inversion formula and Riemann-Lebesuge formula:

$$f \in C_0 \cap L^\infty$$

and furthermore:

$$\|f\|_\infty \leq \int |\hat{f}(\xi)| d\xi \leq C_s \|f\|_{H^s}$$

which establishes the case for $k = 0$. For $k > 0$, consider $f \in H^s$ where $s > k + \frac{n}{2}$. Then apply the special case we've just proved to:

$$\partial^\alpha f \in H^{s-|\alpha|}$$

so that $\partial^\alpha f \in C_0 \cap L^\infty$ and:

$$\|\partial^\alpha f\|_\infty \leq C_{s-|\alpha|} \|\partial^\alpha f\|_{H^{s-|\alpha|}} \leq C_s \|f\|_{H^s}$$

and so, from proposition 3.2.1 that $f \in C^k$.

As a consequence:

$$|f(x)| \leq C_s \|f\|_{H^s} \qquad s > \frac{n}{2}$$

which by the definition of the Sobelev norm means:

$$|f(x)| \leq C \sum_{|\alpha| \leq \frac{n+1}{1}} \|\partial^\alpha f\|_{L^2}$$

We now prove these theorems in the special case where:

$$\Box u = (\partial_t u)^2 \tag{4.8}$$

$$u|_{t=0} = f \qquad \partial_t u|_{t=0} = g$$

Theorem 4.4.2: Local Existence And Uniqueness for Model non-linear PDE

Let $s > \frac{n}{2} + 1$. Then for all $(f, g) \in H^s \times H^{s-1}$, there exists $T > 0$ and a unique

$$u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

Solving the IVP in equation (4.8) on $S_T = (0,T) \times \mathbb{R}^n$. Furthermore, the time T can be chosen to depend continuously on $\|f\|_{H^s} + \|g\|_{H^{s-1}}$

Proof

Existence: The technique used in this proof is very important and will be used often. We will define a clever Cauchy sequence that will converge to our result. Set:

$$u_{-1} = 0$$

Then define $u_0, u_1, ...$ inductively by:

$$\Box u_i = (\partial_t u_{i-1})^2 \quad u_i|_{t=0} = f, \ \partial_t u_i|_{t=0} = g \tag{4.9}$$

Then for $T > 0$ define $X_T = C([0,T], H^s) \cap C^1([0,T], H^{s-1})$. Note that X_T is a Banach space with the norm:

$$\|u\|_{X_T} = \sup_{0 \leq t \leq T} (\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}})$$

Now, notice that:

$$x_{i-1} \in X_T \Rightarrow x_i \in X_T$$

Indeed, since $s - 1 > \frac{n}{2}$, Sobelev's lemma and the calculus inequality give us:

$$\|vw\|_{H^{s-1}} \leq C_s \|v\|_{H^{s-1}} \|w\|_{H^{s-1}} \tag{4.10}$$

for all $v, w \in H^{s-1}(\mathbb{R}^n)$. Thus:

$$u_{i-1} \in X_T \Rightarrow (\partial_{u_{i-1}})^2 \in C([0,T], H^{s-1})$$

which by the very first theorem in Selberg's notes (the one giving solutions in the best case scenario), we get that each iteration in equation (4.9) has a unique solution in X_T. Thus, the sequence of iterates is well-defined in X_T for any $T > 0$. Our next goal is to show this sequence is Cauchy, provided a sufficiently small $T > 0$. The limit of this sequence will be the solution to equation (4.8) on $S_T = (0,T) \times \mathbb{R}^n$ since

$$\begin{aligned} u_i \to u \in X_T \quad &\Rightarrow \quad u_i \to u \quad \text{in } \mathcal{D}'(S_t) \\ &\Rightarrow \Box u_i \to \Box u \quad \text{in } \mathcal{D}'(S_T) \end{aligned}$$

and by equation (4.10):

$$\begin{aligned} u_i \to u \in X_T \quad &\Rightarrow \quad (\partial_t u_i)^2 \to (\partial_t u)^2 \quad \text{in } C([0,T], H^{s-1}) \\ &\Rightarrow \Box(\partial_t u_i)^2 \to (\partial_t u)^2 \quad \text{in } \mathcal{D}'(S_T) \end{aligned}$$

which shows the IVP is satisfied and so u will solve the Cauchy Problem.

We now proceed to show that this is indeed a cauchy sequence. We will break this down in two steps:

(**Step 1**) The sequence is bounded:

$$\|u_i\|_{X_T} \leq 2CE_s \qquad \text{for} \qquad i = 0,1,2,... \tag{4.11}$$

provided that T is so small so that:

$$T \leq \frac{1}{8C^2 E_s}$$

where

$$E_s = \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

and $C > 0$ is a constant which only depends on s and n. Let's say the bound in equation (4.11) holds for $j-1$. Then if we apply the energy inequality (theorem 3.2.2) to the iterations (4.9) and use the estimates derivated in equation (4.10), we get:

$$\|u_i\|_{X_T} \leq Ce_s + CT\|u_{i-1}\|^2_{X_T} \leq Ce_s + CT(2CE_s)^2$$

where we get that C only depends on s and n if $T \leq 1$ (C comes from the energy inequlaity and the esimate form equation (4.10)). If then $T \leq \frac{1}{8C^2 E_s}$ holds, then the right hand side is bounde by $2CE_s$, and so we obtain the bound in equation (4.11) fo rall j by induciton. Since the case $j = -1$ holds trivially.

Step 2 We will show the sequence satisfies:

$$\|u_{i+1} - u_i\|_{X_T} \leq \frac{1}{2}\|u_i - u_{i-1}\|X_T$$

which will show it's Cauchy. To see this, note that:

$$\Box(u_{i+1} - u_i) = \partial_t(u_i - u_{i-1})(u_i + u_{i+1})$$

with vanishing initial data at $t = 0$. Thus, if we apply the energy inequality and use equation (4.10) and the uniform bound of equation (4.11) we get:

$$\begin{aligned} \|u_{i+1} - u_i\|_{X_T} &\leq CT\left(\|u_i\|_{X_T} + \|u_{i-1}\|_{X_T}\right)\|u_i - u_{i-1}\|_{X_T} \\ &\leq CT4CE_s\|u_i - u_{i-1}\|_{X_T} \end{aligned}$$

Thus, w eget the desried bound using the fact that $T \leq \frac{1}{8C^2 E_s}$, completing the proof for existence

Uniqueness Now suppose that $u, v \in X_T$ for soem $T > 0$ solve the Cauchy problem on $S_T = (0, T) \times \mathbb{R}^n$ with the identical data at $t = 0$. Then:

$$\Box(u - v) = \partial_t(u + v)\partial_t(u - v)$$

and thensetting

$$A(t) = \|(u - v)(t)\|_{H^{s-1}} + \|\partial_t(u - v(t))\|_{H^{s-1}}$$

we get by the energy inequality and the result form equation (4.10):

$$A(t) \leq C \int_0^t A(t')\, dt' \qquad \text{for} \qquad 0 \leq t \leq T$$

for some constant C independent of t (note that C depends on the norms $\|u\|_{X_T}$ and $\|v\|_{X_T}$, but this is not a problem since u and v are considered fixed for this argument). Then by Gronwall's Lemma, $E(t) = 0$ for $0 \leq t \leq T$, and so $u = v$ in S_T, showing the solution is also unique and so completing the proof.

Theorem 4.4.3: Continuation Theorem for Model non-linear PDE

Let $s > \frac{n}{2} + 1$. Fix f, g as in theorem 4.3.1. Let $T^* = T^*(f, g)$ be the supremum of all $T > 0$ such that equation (4.6) has a solution on S_T satisfying equation (4.7). Then either $T^* = \infty$ or:

$$\partial u \in L^\infty(S_T)$$

Proof
Let $s > \frac{n}{2} + 1$, $f\ inH^s$, and $g \in H^{s-1}$. Then it is enough to prove that if $0 < T < \infty$ and

$$u \in C([0, T), H^s) \cap C^1([0, T), H^{s-1})$$

solves the model non-linear PDE opn $S_T = (0, T) \times \mathbb{R}^n$ and satisfies

$$\nabla u \in L^\infty(S_T)$$

then

$$\sup_{0 \leq t < T} (\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^s-1}) < \infty$$

Since then, by theorem 4.4.2, we can etend the solution to a time strip $[0, T + \epsilon] \times \mathbb{R}^n$, for some $\epsilon > 0$.

To prove this, define:

$$A(t) = \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \tag{4.12}$$

By the energy inequality and the calculus inequality, we get:

$$\begin{aligned} A(t) &\leq C_{s,T} \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \int_0^t \|\partial_t u(t')\|_{L^\infty} \|\partial_t u(t')\|_{H^{s-1}} dt' \right) \\ &\leq C_{s,T} \left(\|f\|_{H^s} + \|g\|_{H^{s-1}} + \|\nabla u\|_{L^\infty(S_T)} \int_0^t \|A(t')dt' \right) \end{aligned}$$

and now using Gronwall's lemma gives equation (4.12), completing the proof.

Theorem 4.4.4: Smooth Solution for Model non-linear PDE

If $f, g \in C_c^\infty(\mathbb{R}^n)$. Then there exists a $T > 0$ and a unique

$$u \in C^\infty([0, T], \mathbb{R}^n)$$

solving equation (4.6) on S_T

Proof

Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then *a priori* we havethat $f, g \in H^s$ for every $s \in \mathbb{R}$. Fix $s_0 > \frac{n}{2} + 1$. Then by theorem 4.4.2, there exists a $T > 0$ and a unique solution:

$$u \in C([0, T], H^{s_0}) \cap C^1([0, T], H^{s_0 - 1})$$

Solving the general non-linear PDE problem on $S_T = (0, T) \times \mathbb{R}^n$. We also have from the same theorem that for every $s > s_0$ there exists a $T_s > 0$ such that

$$u \in C([0, T_s], H^s) \cap C^1([0, T_s], H^{s-1})$$

I claim that we can take $T_s = T$. Indeed, this follows from theorem 4.4.3 since by Sobelev's lemma and by the fact that u is in $C([0, T], H^{s_0}) \cap C^1([0, T], H^{s_0 - 1})$:

$$\nabla u \in L^\infty(S_T)$$

Thus

$$u \in C(0, T], H^{s_0}) \cap C^1([0, T], H^{s_0 - 1})$$

for every $s \geq s_0$. By Sobelev's lemma again, we get:

$$\partial_t^j \partial_x^\alpha u \in C([0, T], \mathbb{R}^n) \tag{4.13}$$

for $j \in \{0, 1\}$ and all α. Since u solves the non-linear PDE, we have that:

$$\partial_t^2 u = \triangle u + (\partial_t u)^2 \tag{4.14}$$

Thus, using this and equation (4.13), we have:

$$\partial_t^2 \partial_x^\alpha u \in C([0, T], \mathbb{R}^n)$$

so equation (4.13) holds for $j \in \{0, 1, 2\}$ and all α. Applying ∂_t to both sides of equation (4.14), we get:

$$\partial_t^3 u = \triangle \partial_t u + 2\partial_t u \partial_t^2 u$$

and now from this we see that equation (4.13) holds for $j \in \{0, 1, 2, 3\}$. If we keep taking successive time derivatives of the equation, we obtain equation (4.13) for all j by induciton. Thus, we have that:

$$u \in C^\infty([0, T], \mathbb{R}^n)$$

as we sought to show.

5

Littlewood-Paley Theory

As we know, L^2 is a very natural generaliation of a space with an inner product when working over spaces of functions that are measurable under a given measure. In this chapter, we will learn how we can generlize the results of L^2 spaces into L^p spaces. In particular, we will figure out how to decompose a function f into "localized" frequences (f_p), which we then can use results of L^2 theory.

5.1 Little-Paley Decomposition

the first thing we'll do is find a way to decompose an L^p function (really any function) so that each piece will be L^2 Let $\chi \in C_c^\infty(\mathbb{R}^n)$ such that

$$0 \leq \chi \leq 1 \qquad \chi(\xi) = \begin{cases} 1 & \text{if } |\xi| < 1 \\ 0 & \text{if } |\xi| \geq 1 \end{cases}$$

Then define $S_j : \mathcal{S}' \to \mathcal{S}$ by

$$\widehat{S_j f} = \chi(2^{-j}\xi)\hat{f}$$

for $j \in \mathbb{N}$, that is S_j takes only a certain ranges of frequencies. Equivalently, we can write

$$S_j = \mathcal{F}^{-1}\chi(2^{-j})\mathcal{F}$$

Lemma 5.1.1: decomposing Schwartz Functions

For every $\varphi \in \mathcal{S}$, $\chi(2^{-j})\varphi \to \varphi$ in $\mathcal{S}$ as $j \to \infty$

Proof
was left as eercise

Since $\mathcal{F}$ is an automorphism on $\mathcal{S}$, we get:

$$S_j f \to f \qquad \text{in } \mathcal{S}$$

for every $f \in \mathcal{S}$. this then implies that $S_j f \to f$ in $\mathcal{S}'$ for every $f \in \mathcal{S}'$. Next, we define the following decomposition similar to martingale-difference representation used in the proof of Azuma's inequality: define

$$\triangle_0 = S_0 \qquad \triangle_j = S_j - S_{j-1} \qquad j \in \mathbb{N}$$

Thus, we get $f = \sum_0^\infty \triangle_j f$ in $\mathcal{S}$ (resp. $\mathcal{S}'$) for every $f \in \mathcal{S}$ (resp. $f \in \mathcal{S}'$)

Definition 5.1.1: Littlewood-Paley Decomposition

The decomposition $\{\triangle_j f\}_0^\infty$ is called the *Littlewood-Paley decomposition* of f.

Notice that $\widehat{\triangle_j f} = \beta_j(\epsilon)\hat{f}$ where

$$\beta_0(\xi) = \chi(\xi) \qquad \beta_j(\xi) = \chi(2^{-j}\xi) - \chi(2^{1-j}\xi) \qquad j \in \mathbb{N} \tag{5.1}$$

Notice that by definition of χ, we get tha $0- \leq B_j \leq 1$ for all j, and so

$$\sum_0^\infty \beta_j^2 \leq \sum_0^\infty \beta_j = 1$$

Proposition 5.1.1: Littlewood-Paley Decomposition

Let $\{\triangle_j f\}_0^\infty$ be the littlewood-Paley decomposition of f. Then:

1. $\operatorname{supp} \widehat{S_j f} \subseteq \{|\xi| \leq 2^j\}$
2. $\operatorname{supp} \widehat{\triangle_j f} \subseteq \{2^{j-2} \leq |\xi| \leq 2^j\}$
3. $S_j f = 2^{jn}\psi(2^j \cdot) * f$ where $\hat{\psi} = \chi$
4. $\triangle_i f = 2^{jn}\varphi(2^j \cdot) * f$ where $\hat{\varphi} = \chi(\xi) - \chi(2\xi)$
5. $\|S_j f\|_{L^p} \leq C\|f\|_{L^p}$ where $C = \|\psi\|_{L^1}$
6. $\|\triangle_i f\|_{L^p} \leq C\|f\|_{L^p}$ where $C = \|\varphi\|_{L^1}$
7. $S_j f \in C^\infty$ if $f \in L^p$ for $1 \leq p \leq \infty$
8. $\triangle_j f \in C^\infty$ if $f \in L^p$ for $1 \leq p \leq \infty$

Proof

1. here
2. here
3. here
4. here
5. Follows (3) and Young's inequlaity

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p} \qquad 1 \leq p \leq \infty$$

6. follows from (4) and Young's inequality
7. here
8. here

We will apply this theory to Sobelev spaces H^s. Recall that

$$H^s = \left\{ f \in \mathcal{S}' \;\middle|\; \Lambda^s f \in L^2 \right\}$$

where $\Lambda^s = (I - \triangle)^{s/2}$. We shows that H^s is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^s} = \langle \Lambda^s f, \Lambda^s g \rangle_{L^s} = \int (1 + |\xi|^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$$

giving us norm $\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}$, and $\Lambda : H^s \to L^2$ is a unitary isomorphism. Being a hilbert space, a few easy observations can be done: if $\hat{f}$ and $\hat{g}$ are almost disjiont, then $f \perp g$ in H^s. H^s is self-dual via the map $g \mapsto \langle \cdot, g \rangle_{H^s}$. By the property of H^s, we can see that H^s is unitarily isomorphic to H^{-s} via the pairing $\langle \cdot, \cdot \langle : H^s \times \S \to \mathbb{C}$ through

$$|\langle f, g \rangle| = |\Lambda^s f, \Lambda^{-s} g| = \left|\Lambda^s f \Lambda^{-s} g\right| \leq \|f\|_{H^s} \|g\|_{H^{-s}}$$

for all $f \in H^s$ and $g \in \mathcal{S}$. Thus, the map $g \mapsto \langle \cdot, g \rangle$ extends to a linear map from H^{-s} into $(H^s)^*$, which is a surjective isometry in view of the self-duality of L^2.

With this overview, we now present some quick observations relating H^s and the theory developped at the beginning

1. $S_j f \to f$ in H^s for everfy $f \in H^S$, since:

$$\|S_j f - f\|_{H^s}^2 = \int [1 - \chi(2^{-j})]^2 (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \to 0$$

by the DCT

2. If $2^{j-2} \leq |\xi| \leq 2^j$, then

$$(1 + |\xi|^2)^{s/2} \approx 2^{js}$$

in the senes that there is a constant $C_s > 0$, indepennet of j, such that

$$C_s^{-1} 2^{js} \leq (1 = |\xi|^2)^{s/2} \leq C_s 2^{js}$$

we then get that

$$\|\triangle_i f\|_{H^s} \approx 2^{js}\|\nabla_j f\|_{L^2}$$

3. In view of the above, we have

$$\triangle_j f \perp \triangle_k f \qquad \text{if} \qquad |j-k| \geq 2$$

relative to the inner product on H^s

With these, we have the following estimate:

Proposition 5.1.2: Estimating Sobelev Norm

Let $s \in \mathbb{R}$. Then for every $f \in \|H^s\|$,

$$\|f\|_{H^s}^2 \approx \sum_0^\infty 2^{2js}\|\triangle_j f\|_{L^2}^2$$

Proof
By the 2nd observation, we already have that:

$$\|f\|_{H^s}^2 \approx \sum_0^\infty \|\nabla_j f\|_{H^s}^2$$

Then, using orthogonality, our 3rd observation, and the Cauchy-Schwarz ineqluaity, we get

$$\begin{aligned}
\|f\|_{H^s}^2 &= \langle f, f\rangle_{H^s} = \left\langle \sum \Delta_j f, \sum \Delta_d f \right\rangle_{H^s} \\
&= \sum\sum \langle \Delta_j f, \Delta_k f\rangle_{H^s} \\
&= \sum_{l=-1}^{1}\sum_{j=0}^{\infty} \langle \Delta_j f, \Delta_{j+l} f\rangle_{H^s} \\
&\leq \sum_{l=-1}^{1}\sum_{j=0}^{\infty} \|\Delta_j f\|_{H^s}\|\Delta_{j+l} f\|_{H^s} \\
&\leq \sum_{l=-1}^{1}\left(\sum_{j=0}^{\infty}\|\Delta_j f\|_{Hs}^2\right)^{1/2}\left(\sum_{j=0}^{\infty}\|\Delta_{j+l} f\|_{H^s}^2\right)^{1/2} \\
&\leq 3\sum_{j=0}^{\infty}\|\Delta_j f\|_{H^s}^2
\end{aligned}$$

On the other hand, by the observations of what is the Fourier transform on $\Delta_j f$ in equation (5.1),

$$\begin{aligned}\sum_0^\infty \|\delta_j f\|_{H^s}^2 &= \sum_0^\infty \int (1+|\xi|)^2 |\beta_j(\xi)\hat{f}(\xi)|^2 d\xi \\ &= \int \left[\sum_i^\infty \beta_j^2(\xi)\right] (1+|\xi|)^2 |\hat{f}(\xi)|^2 d\xi \\ &\le \int (1+|\xi|)^2 |\hat{f}(\xi)|^2 d\xi \\ &= \|f\|_{H^2}^2\end{aligned}$$

completing the proof.

Using this result, we can find what types of sequences in L^2 would converge in H^s. Let's say $\{f_i\}$ is a sequence in L^2 such that for soem $R > 0$,

$$\operatorname{supp} \hat{f}_i \subseteq \{R^{-1}2^j \le |\xi| R2^j\}$$

and $\sum 2^{2js}\|f)i\|_{L^2}^2 < \infty$. Then $f = \sum f_i$ converges in H^s, and

$$\|f\|_{H^s}^2 \approx \sum 2^{2js}\|f_i\|_{L^2}^2$$

If $s > 0$, we can get essentially the same result if we only assume

$$\operatorname{supp} \hat{f}_i \subseteq \{|\xi| \le R2^j\}$$

Proposition 5.1.3: Better Bounding Condition

Let $s > 0$. Suppose $f_i \in L^2$ satisfie s

$$\operatorname{supp} \hat{f}_i \subseteq \{|\xi| \le R2^j\}$$

for some $R \ge 1$, and that

$$\sum_0^\infty 2^{2js}\|f_j\|_{L^2}^2, \infty$$

Then $f = \sum f_j$ converges in H^s, and

$$\|f\|_{H^s}^2 \le C_{s,R} \sum_0^\infty 2^{2js}\|f_j\|_{L^2}^2$$

For the proof of this proposition, we need a quick lemma

Lemma 5.1.2: build up lemma

Let $s < 0$ and $R \geq 1$. If $f \in H^s$ and

$$\operatorname{supp} \hat{f} \subseteq \{|\xi| \leq R\}$$

then

$$\|f\|_{H^s} \leq 2^s R^s \|f\|_{L^2}$$

Proof
Simply notie that:

$$|\xi| \leq R \Rightarrow (1 + |\xi|^2)^s \leq (2R^2)^s$$

Proof
of proposition 5.1.3: Selberg p.65

Note that if $R = 2^q$, then the constant $C_{s,R}$ is of the form $C_s 2^{2qs}$. This will be important for the proof of Mosers inequality.

5.2 Calculus Inequality

We know used what we've learned to prove the inequalities form section 4.4.

Selberg p.66-67

5.3 Moser Inequality

Theorem 5.3.1: Moser Inequality

Assume that $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞ and $F(0) = 0$. Then for all $s \geq 0$, there isa continuous function $\gamma = \gamma_s : \mathbb{R} \to \mathbb{R}$ such that

$$\|F(f))\|_{H^s} \leq \gamma(\|f\|_{L^\infty})\|f\|_{H^s}$$

for all $\mathbb{R}^n$-valued $f \in H^s \cap L^\infty$

this will have an intersting consequence:

Corollary 5.3.1: Moser Inequality Consequence

If $s > \frac{n}{2}$, the map $f \mapsto F(f)$, $H^s \to H^s$ is smooth

Proof
after we prove Moser inequality

To prove the Mosers, we will need a generlization of lemma 5.1.2. First, recall that the *spectrum* of f is the supprot of its Fourier transform.

Lemma 5.3.1: Bernstein's Lemma

Assuem that the spectrum of $f \in L^p$, $1 \leq p \leq \infty$, is conatined in the ball $|\xi| \leq 2^j$. Then:

$$\|\partial^\alpha f\|_{L^p} \leq C_\alpha 2^{j|\alpha|}\|f\|_{L^p}$$

for any multi-index α. Moreover, if the spectrum is contained in $2^{j-2} \leq |\xi| \leq 2^j$, then

$$C_k^{-1} 2^{jk}\|f\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C_k 2^{jk}\|f\|_{L^p}$$

for any $k \in \mathbb{N}$

Proof
this is essentially an application of Youngs inequlaity

Proof
of Moser Inequality (Selberb p. 68)

5.4 Further Applications of Littlewood-Paley Theory

This is cool because it builds up to $H^{n/2}$ not being an algebra, and how it embeds into L^p or L^∞ under appropriate circumstances.

We first state without proving the following:

Theorem 5.4.1: Generalizing to L^p

If $2 \leq p < \infty$, then

$$\|f_{L^p}\|^2 \leq C \sum_0^\infty \|\triangle_i f\|_{L^p}^2$$

and if $1 < p \leq 2$, then:

$$\sum_0^\infty \|\triangle_i f\|_{L^p}^2 \leq C\|f\|_{L^p}^2$$

Proof
not proven.

With this, we will show the following:

Theorem 5.4.2: Embeddgins For H^s

If H^s is the a sobelev space, then:

$$H^s \hookrightarrow L^{\frac{2n}{n-2s}} \qquad 0 \leq s < \frac{n}{2} \tag{5.2}$$

$$H^s \hookrightarrow L^{\frac{2n}{n-2s}} s > \frac{n}{2} \tag{5.3}$$

Proof

FIrst, recall the poin mass δ belongs to H^{-s} when $s > n/s$. Thus:

$$|f(x)| = |\langle \delta(\cdot - x), f \rangle| \leq \|\delta(\cdot - x)\|_{H^s} \|f\|_{H^s} = \|\delta_{H^{-s}}\|f\|_{H^s}$$

proving equation (5.3). For equation (5.2), by Young's inequality we get:

$$\|\triangle_i f\|_{L^{\frac{2n}{n-2s}}} \leq C 2^{js} \|f\|_{L^2}$$

and since $\triangle_i f = \triangle_i \left(\sum_{k=i-1}^{i+1} \delta_k f \right)$, we get:

$$\|\triangle_i f\|_{L^{\frac{2n}{n-2s}}} \leq C \sum_{k=i-1}^{i+1} 2^{ks} \|\triangle_k f\|_{L^2}$$

Now to complete the proof, squrae both sides, sum over j ,and use theorem 5.4.1 on the left hand side.

The embedding in equation (5.2) is precise

Example 5.1: Fails If $S \leq \frac{n}{2}$

We want ot show that $H^{n/2}$ is not a subset of L^∞, so assume $H^{n/2} \subseteq L^\infty$> Then by the closed graph theorem, we have $H^{n/2} \hookrightarrow L^\infty$< so

$$f\|_{L^\infty} \leq C \|f\|_{H^{n/2}}$$

but hteen, this implies that $\delta \in (H^{n/2})^* = H^{-n/2}$, which is false

Corollary 5.4.1: Not All Sobelev Spaces Form Algebras

$H^{n/2}$ is not an algebra

Proof

Assume that it is. Then $\|f^k\|_{H^{n/2}} \leq C^k \|f\|_{H^{n/2}}^k$ for all $K \in \mathbb{N}$. Thus, if $\|f\|_{H^{n/2}} < 1/C$, then $f^k \to 0$ in $H^{n/2}$ as $k \to \infty$. But this implies that some subsequence converges to zero a.e. on $\mathbb{R}^n$, so we must have $|f| < 1$ a.e. But this means that the ball of radius $1/C$ in $H^{n/2}$ is conatined in $L^\infty <$ whihc implies that H^s is a subset of L^∞, which contradicts example 5.1.

6

Global Existence for Non-Linear Wave Equation

We are now in a position to start undersatnding the general theorem for the global existence of a non-linear wave equation. It will turn out that being a dimension less than 4 makes it more complicated! As we've shown, a global solution doesn't always exist, however we can find an "almost global" solution by being able to scale the initial f and g by a chosen ϵ to extend the solution to as far as we'd like, with the extra condition that the non-linear term F starts out at the origin with no momentum.

The Klainerman-Sobelev inequality can be thought of what has to be "sacrificed" in order for the vector fields to commute. We can just simply commute them without changing the value of the norm, but we can bound how much it would change the value.

6.1 Goal

We are going to solve the following non-linear cauchy problem in full generality. In $\mathbb{R}^{1+n}$, let:

$$\Box u = F(\partial u) \tag{6.1}$$

$$u|_{t=0} = \epsilon f \qquad \partial_t u|_{t=0} = \epsilon g \tag{6.2}$$

where $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is a given smooth functions which to second order at the origin:

$$F(0) = 0 \qquad DF(0) = 0$$

Our goal is to prove the following theorem:

Theorem 6.1.1: Global Existence Of Non-Linaer Wave Equation

Let $n \geq 4$. Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then there exists $\epsilon_0 > 0$ such that the two above equations has a solution

$$u \in C^\infty([0,\infty), \mathbb{R}^n)$$

provided $\epsilon \leq \epsilon_0$.

Note that ϵ_0 depends on f and g. Furthermore, the solution is unique due to the material in chapter 4.

For dimension $n \in \{1,2,3\}$, we will get (form the proof above) an asymptotic lower bounds on the lifespan:

$$T_\epsilon = T^*(\epsilon f, \epsilon g)$$

as $\epsilon \to 0$. Recall that the lifespan is the supremum of $T > 0$ such that the two equations at the begning has a solutoin $u \in C^\infty([0,T] \times \mathbb{R}^n)$. Specifically, we shall see that there exists a $c > 0$ st

$$\begin{aligned} T_\epsilon &\geq e^{c/\epsilon} && \text{if } n = 3 \\ T_\epsilon &\geq e^{c/\epsilon^2} && \text{if } n = 2 \\ T_\epsilon &\geq c/\epsilon && \text{if } n = 1 \end{aligned}$$

6.2 The Invariant Vector Fields

The Sovelev inequality states that:

$$|f(x)| \leq C_s \|f\|_{H^s} \qquad s > \frac{n}{2}$$

which by the definition of the Sobelev norm means:

$$|f(x)| \leq C \sum_{|\alpha| \leq \frac{n+1}{1}} \|\partial^\alpha f\|_{L^2}$$

This limit on the size of f with respect to to the growth of it's partial was important for the solution in section 4.4. We wil need a similar estimate on $u(t,x)$ in terms of the L^2 norm for derivatives of u multiplied by a decay factor of t.

Now, I'm not exactly sure what's going on here, but the space-time derivatives invovled are the following vector-fields:

$$\partial_t, \partial_1, ..., \partial_n \tag{6.3}$$

$$\Omega_{ij} = x_j \partial_i - x_i \partial_j \tag{6.4}$$

$$\Omega_{0j} = t\partial_i - x_i \partial_t \tag{6.5}$$

$$L_0 = t\partial_t \sum_1^n x_i \partial_i \tag{6.6}$$

Letting $x_0 := t$, we get consistent notation for all the vector-fields. The Ω_{ij} vector-fields have lot's of overlap by skew-symmetry, meaning we in fact have all these vector-fields if we only consider indexes $1 \leq i < j \leq n$. Thus, we have a total of :

$$(n+1) + \frac{n(n-1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} + 1$$

different vector-fields. We can enumerate all these vecot-fields as :

$$\Gamma = (\Gamma_0, \Gamma_1, ..., \Gamma_m) \qquad m = \frac{(n+1)(n+2)}{2}$$

(more here)

6.3 The Klainerman-Sobelev Inequality

The following will replace the Sobelev inequality:

Theorem 6.3.1: Klainerman-Sobelev Inequality

There is a constant C such that

$$(1+t+|x|)^{\frac{n+1}{2}} |u(t,x)| \leq C \sum_{|\alpha| \leq \frac{n+1}{2}} \|\Gamma^\alpha u(t,\cdot)\|_{L^2} \qquad \text{for} \qquad t > 0, x \in \mathbb{R}^n$$

whenever $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ and $\text{supp}(u(t,\cdot))$ is compact for every t

Proof
left for page 64

Note that this theorem implies the result locally, that is for $0 < t < T$ with the same constant C if $u \in C^\infty([0,T) \times \mathbb{R}^n)$. To prove this reuslt, we will requires some lemmas. The first of these express that the *homogeneous* vector fields (6.4) (6.4) (6.4) span the tangent space of $\mathbb{R}^{1+n}$ outside the lightcone:

$$\Omega_{ij} = x_j \partial_i - x_i \partial_j$$
$$\Omega_{0j} = t\partial_i - x_i \partial_t$$
$$L_0 = t\partial_t \sum_1^n x_i \partial_i$$

Lemma 6.3.1: Klainerman-Sobelev Inequality Lemma I

For any multi-index $\alpha \neq 0$ and $(t, x) \notin \Lambda = \{(t, x) \mid |t| = |x|\}$ (i.e. not in the light-cone):

$$\partial^\alpha = \sum_{1 \leq |\beta| \leq |\alpha|} c_{\alpha\beta}(t, x)\Gamma^\beta$$

where $c_{\alpha\beta}$ are C^∞ and homogeneous of degree -$|\alpha|$ outside the ligthcone Λ. In fact, the sum of the right hand side only involves the homogeneous vector-fields.

Proof
Selberg p. 76

The following is a localized Sobelev Inequality

Lemma 6.3.2: Klainerman-Sobelev Inequality Lemma II

Given $\delta > 0$, there exists a constant C_δ such that

$$|f(0)|^2 \leq C_\delta \sum_{|\alpha| \leq \frac{n+2}{2}} \int_{|y|<\delta} |\partial^\alpha f(y)|^2 dy$$

for all $f \in C^\infty(\mathbb{R}^n)$. Moreover,

$$\sup_{0<\delta_0 \leq \delta} C_\delta < \infty$$

Proof
Fix a cut of $\chi \in C_c^\infty(\mathbb{R}^n)$ which is 1 in the unit ball at the origin. Applying

$$|f(x)| \leq C \sum_{|\alpha| \leq \frac{n+1}{1}} \|\partial^\alpha f\|_{L^2}$$

to the function

$$\chi(y/\delta)f(y)$$

yields the desired inequality with $C_\delta \leq C(1 + \delta^{-n-2})$. The final statemtn naturally follows.

Finally, the third lemma requires a Sobelev inequaltiy on the spherical means, that is we need a Sobolev inequality for a smooth function $v(q, \omega)$ where $q \in \mathbb{R}$ and $\omega \in S^{n-1}$. First, notice that the vectorfields Ω_{ij}, $1 \leq i < j \leq n$, can be seen as vector-fields on S^{n-1}. We can accordingly write for any point $\omega \in S^{n-1}$:

$$\partial_\omega^\alpha = \Omega_{1,2}^{\alpha_1} \cdots \Omega_{n-1,n}^{\alpha_m}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ and $m = n(n-1)/2$

Lemma 6.3.3: Klainerman-Sobelev Inequality Lemma III

Given $\delta > 0$, there exists a constant C_δ such that

$$|v(q,\omega)|^2 \leq C_\delta \sum_{j+|\alpha| \leq \frac{n+2}{2}} \int_{|p|<\delta} \int_{\epsilon \in S^{n-1}} |\partial_q^j \partial_\omega^\alpha v(q+p,\epsilon) f(y)|^2 d\sigma(\eta) dp$$

for all $v \in C^\infty(\mathbb{R} \times S^{n-1})$. Moreover,

$$\sup_{0<\delta_0\leq\delta} C_\delta < \infty$$

Proof
This follows from lemma 6.3.2 if we cover the sphere S^{n-1} by finitely many coordinate charts and choose a parition of unity. The key observation is that the vectorfields

$$\Omega_{1,2}, \Omega_{2,3}, ..., \Omega_{n-1,n}$$

spans $T_\omega S^{n-1}$ for all $\omega \in S^{n-1}$

With these, we can proceed with the proof of the Klainerman-Sobelev inequality:

Proof
of the Klainerman-Sobelev inequality If we have $t + |x| \leq 1$, then the inequality follows from the stanardd Sobelev inequality

$$|f(x)| \leq C \sum_{|\alpha| \leq \frac{n+1}{1}} \|\partial^\alpha f\|_{L^2}$$

and so we will assume $t + |x| > 1$. The argument will deped on whether (t, x) is lose to the lightcone or not.

(**Case 1**) Assume that:

$$|x| \leq \frac{t}{2} \qquad \text{or} \qquad |x| \geq \frac{3t}{2} \tag{6.7}$$

and define R to be:

$$R = t + |x| > 1$$

I claim that the coefficeints in lemmma 6.3.1 satisfy:

$$|y| \leq \frac{R}{8} \Rightarrow |c_{\alpha\beta}(t, x+y)| \leq CR^{-|\alpha|} \qquad \text{for} \qquad 1 \leq |\beta| \leq |\alpha| \leq \frac{n+2}{2}$$

Assuming this values holds, we can apply lemma 6.3.2 to the function $z \mapsto u(t, x+Rz)$ and get

$$\begin{aligned} |u(t,x)|^2 &\leq C \sum_{|\alpha| \leq \frac{n+2}{2}} \int_{|z|\leq\frac{1}{8}} \left| R^{|\alpha|} \partial_x^\alpha u(t, x+Rz) \right|^2 dz \\ &= CR^{-n} \sum_{|\alpha| \leq \frac{n+2}{2}} \int_{|z|\leq\frac{R}{8}} \left| R^{|\alpha|} \partial_x^\alpha u(t, x+Rz) \right|^2 dz \end{aligned}$$

where we did a change of varialbes to $y = Rz$. Now, using lemma 6.3.1 we have

$$\alpha \neq 0 - -.\partial_x^\alpha u(t, x+y) = \sum_{1 \leq |\beta| \leq |\alpha|} c_{\alpha\beta}(t, x+y)(\Gamma^\beta u)(t, x+y)$$

we can get the result (using the assumption we made) that

$$R^n |u(t,x)|^2 \leq C \sum_{|\alpha| \leq \frac{n+2}{2}} \|\Gamma^\alpha u(t, \cdot)\|_{L^2}^2$$

showing that the inequality holds in the regions defined in equations (6.7) and by R. It now remains to show these are valid assumptions. We follows by using equation (6.7) and that

$$|y| \leq \frac{R}{8} \Rightarrow |t - |x+y|| \geq cR \quad \text{for some } c > 0$$

Rearranging, we get:

$$\left| \frac{t + |x+y|}{R} - 1 \right| \leq \frac{1}{8}$$

so the point $(t/R, (x+y)/R$ is in a compact set disjoint for mthe light cone Λ. Thus, our assumptions follow from the continuity and homogeneity of $c_{\alpha\beta}$

(**Case 2**) We know assume

$$\frac{t}{2} \leq |x| \leq \frac{3t}{2} \tag{6.8}$$

and $t + |x| > 1$. We will solve this by switching to polar coordiantes. Let

$$x = r\omega \qquad \text{where} \qquad r > 0, \ \omega \in S^{n-1}$$

Then we re-write the solutions as:

$$u(t,x) = u(t, r\omega) = v(t, r-t, w) = v(t, q, w)$$

where $q = r - t$, and $v(q, w)$ is a smooth function on S^{n-1}. Re-writing u in terms of v, we have that:

$$v(t, q, w) = u(t, (t+q)w)$$

This effects the derivative $\partial_q v$ which results in:

$$\partial_q v = \sum_1^n \omega_j \partial_j u = \partial_r u$$

furthermore, for $1 < i < j \leq n$, we have:

$$\Omega_{ij} u = \Omega_{ij} v$$

where on the left hand side we consider Ω_{ij} to act in ω as a vectorfield on S^{n-1}.

Thus, with the restriction imposed in equation (6.8) ($\frac{t}{2} \leq |x| \leq \frac{3t}{2}$) and the third Klainerman-Sobelev lemma (lemma 6.3.3), when applied to $v(t,q,\omega)$, we get:

$$|u(t,w)|^2 = |v(t,q,\omega)|^2 \tag{6.9}$$

$$\leq C_t \sum_{j+|\alpha|\leq\frac{n+2}{2}} \int_{|p|<\frac{t}{4}} \int_{\eta\in S^{n-1}} |\partial_q^j \partial_\omega^\alpha v(t,q+p,\eta)|^2 d\sigma(\eta)dp \tag{6.10}$$

$$\overset{!}{\leq} C_t \sum_{j+|\alpha|\leq\frac{n+2}{2}} \int_{|p|<\frac{t}{4}} \int_{\eta\in S^{n-1}} \left|\partial_r^j \Gamma^\alpha u(t,(t+q+p)\epsilon)\right|^2 d\sigma(\eta)dp \tag{6.11}$$

where we used the fact that $\Omega_{ij}u = \Omega_{ij}v$ and $\partial_q v = \sum_1^n \omega_j \partial_j u = \partial_r u$ for the $\overset{!}{\leq}$ inequality. Next, notice we can in fact replace C_t by the C we found earlier, that is $C_t \leq C$, meaning was can find a constant C independent of t. Since:

$$|x| \leq \frac{3t}{2} \Rightarrow 1 < t+|x| \leq \frac{5t}{2} \Rightarrow t > \frac{2}{5}$$

we have that t is bounded away from 0, and so C_t is bounded above given the last statement of lemma 6.3.3.
Next, by the restrictin given for case 2, we have that $|q| = |r-t| \leq t/2<$ and $|p| \leq t/4$ implies that:

$$\frac{t}{4} \leq t+p+q \leq 2t$$

Thus, since $\partial_r u = \sum_1^n \eta_i \partial_i u$, and doing a chagne of variables to $r = t+p+q$ in equation 6.9, we get:

$$\begin{aligned}
|u(t,x)|^2 &\leq C \sum_{|\beta|\leq\frac{n+2}{2}} \sum_{\frac{t}{4}\leq r\leq 2t} \int_{\eta\in S^{n-1}} \left|\Gamma^\beta u(t,r\eta)\right|^2 d\sigma(\eta)dr \\
&\leq Ct^{1-n} \sum_{|\beta|\leq\frac{n+2}{2}} \int_0^\infty \int_{\eta\in S^{n-1}} \left|\Gamma^\beta u(t,r\eta)\right|^2 d\sigma(\eta) r^{n-1}dr \\
&= Ct^{1-n} \sum_{|\beta|\leq\frac{n+2}{2}} \left\|\Gamma^\beta u(t,\cdot)\right\|_{L^2}^2
\end{aligned}$$

Since $t \approx |x|$ and $t+|x| > 1<$ we have that this proves the Klainerman-Sobelev inequaltiy in the revious for case 2, completing the proof.

6.4 Proof of main theorem

We know go towards proving theorem 6.1.1, which as a reminder is:

Theorem 6.4.1: Global Existence Of Non-Linaer Wave Equation Reminder

Let $n \geq 4$. Let $f, g \in C_c^\infty(\mathbb{R}^n)$. Then there exists $\epsilon_0 > 0$ such that the two equations

$$\Box u = F(\partial u) \tag{6.12}$$

$$u|_{t=0} = \epsilon f \qquad \partial_t u|_{t=0} = \epsilon g \tag{6.13}$$

where $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is a given smooth functions which to second order at the origin:

$$F(0) = 0 \qquad DF(0) = 0$$

has a solution

$$u \in C^\infty([0, \infty), \mathbb{R}^n)$$

provided $\epsilon \leq \epsilon_0$.

We will make a few observations wrapped into a lemma to make the proof easier:

Lemma 6.4.1: Build-Up For Existence Theorem

1. $|F(z)| \leq G_2(|z|)|z|^2$, $|DF(z)| \leq G_2(|z|)|z|$, $|D^m F(z)| \leq G_m(|z|)$ for $m \geq 2$ for all $z \in \mathbb{R}^{1+n}$, where $G_2, G_3, ...$ are continuous, increasing functions and $D^m F = \partial^\alpha F$ with $|\alpha| = m$
2. $\Box \Gamma^\alpha = \sum_{|\beta| \leq |\alpha|} c_{\alpha\beta} \Gamma^\beta \Box$ where $c_{\alpha\beta}$ are constants
3. For $\alpha \neq 0$, $\Gamma^\alpha[F(\partial u)]$ is a lieanr combination of terms:
$$[D^m F](\partial u) \Gamma^{\beta_1} \partial u \cdots \Gamma^{\beta_m} \partial u \tag{6.14}$$
where $1 \leq m \leq |\alpha|$ and $\sum_1^m |\beta_i| = |\alpha|$
4. In the above equation (equation 6.14), at most one β_i can have order $|\beta_i| > |\alpha|/2$. If we order the β_i so that $|\beta_m| = \max_{1 \leq i \leq m} |\beta_i|$, then
$$|\beta_i| \leq \frac{|\alpha|}{2} \qquad \text{for} \qquad 1 \leq i \leq m-1$$
5. Let $N = n + 4$. If $|\alpha| \leq N$ and $|\beta_i| \leq |\alpha|/2$, then
$$|\beta_i| + 1 + \frac{n+2}{2} \leq N$$

Proof

1. Since F vanishes in the first order, we get lots of information. Set
$$G_m(r) = \sup_{|z| \leq r} |D^m F(z)|$$

Then G_m is continuous and increasing, and we get the 3rd equation holds. Now write

$$\partial_j F(z) = \partial_j f(z) - \partial_j F(0) = \int_0^1 \frac{d}{dt}\left[\partial_j F(tz)\right] dt = \left[\int_0^1 \nabla \partial_j F(tz) dt\right] \cdot z$$

if we take the absolute value, we get the second equation. Finally, applying the same argumetn to $F(z)$ gives:

$$|F(z)| \leq \sup_{0\leq t\leq 1} |DF(tz)||z| \leq G_2(|z|)|z|^2$$

giving us the first equation, completing the proof.

2. This follows from the conditions we have on these vector fields, namely:

$$[\Box, \partial_i] = 0 \qquad 0 \leq i \leq n$$

$$[\Box, \Omega_{ij}] = 0 \qquad 0 \leq i < j \leq n$$

$$[\Box, L)0] = 2\Box$$

3. this is a matter of induction

4. this result is immedaite

5. $N/2 + 1 + (n + 2/2) \leq N$ if and only if $N + 4 \leq N$

With all of this done, we are ready to tackle the main theorem

Proof
of the main theorem: We startwith some reductions. Set $N = n + 4>$ Define

$$A(t) = \sum_{|\alpha|\leq N} \|\Gamma^\alpha u(t, \cdot)\|_{L^2} \qquad 0 \leq t < T$$

whenever $u \in C^\infty([0,T) \times \mathbb{R}^n)$ solves the equations in theorem 6.4.1 on $[0,T) \times \mathbb{R}^n$ for $T > 0$. By the seonc dequation ,we get

$$A(0) \leq \frac{A\epsilon}{2} \tag{6.15}$$

where A depends only on f and g and their derivatives. I lcaim that there exists a $\epsilon_0 > 0$ such that if $T > 0$ and $C^\infty([0,T) \times \mathbb{R}^n)$ sovles the equations on $[0,T) \times \mathbb{R}^n$ with $\epsilon \leq \epsilon_0 <$ then $A(t) \leq A\epsilon$ for all $0 \leq t < T$. To see this, observe that by the Sovelev inequality:

$$\|\partial u\|_{L^\infty([0,T)\times\mathbb{R}^n)} \leq C \sup_{0\leq t<T} A(t)$$

Therefore, if the claim holds, it follows (see theorem ref:HERE) that the lifespan $T_\epsilon = \infty$ whenever $\epsilon \leq \epsilon_0$.

Our next reduction is to prove the claim on the set

$$E = \{t \in [0,T) \mid A(s) \leq A\epsilon \text{ for all } 0 \leq s \leq t\}$$

Since $A(0) \leq A\epsilon/2$, E is nonempty. Since $A(t)$ is continuous with respect to t, E is relatively closed in $[0,T)$ (i.e. closed in the subspace topology induced by $[0,T)$). If we can show that E Is relatively open (open in the subspace topology of $[0,T)$), then it will follow that $E = [0,T)$, and so we proceed with proving E is relatively open.

Fix $t_0 \in E$ wit h$T_0 < T>$ Since $A(t)$ is continuous, there eixtss a $t_1 > t_0$ such that

$$A(t) \leq A\epsilon \qquad \text{for} \qquad 0 \leq t \leq t_1$$

This is known as the *boot-strap assumption.* We will prove this implies

$$A(t) \leq A\epsilon \qquad \text{for} \qquad 0 \leq t \leq t_0$$

for ϵ sufficiently small. IT suffices to prove that

$$A(t) \leq \frac{A\epsilon}{2} + C_A\epsilon \int_0^t \frac{A(s)}{(1+s)^{\frac{n-1}{2}}} ds$$

Then by GRonwall's Lemma we have:

$$A(t) \leq \frac{A\epsilon}{2} \exp\left[C_A\epsilon \int_0^t \frac{ds}{(1+s)^{\frac{n-1}{2}}}\right]$$

Sincethe integral in the above equation is finite when $n \geq 4<$ we only needto choose $\epsilon > 0$ so so small that

$$\exp\left[C_A\epsilon \int_0^t \frac{ds}{(1+s)^{\frac{n-1}{2}}}\right] \leq 2$$

which completes the proof

With these simplification, we are ready to prove the main result. Given that

$$[\Gamma_i, \partial_j] = \sum_{k=0}^{n} a_{ijk}\partial_k$$

we can apply the energy inequatliy, obtainign

$$\begin{aligned} A(t) &\leq A(0) + C_N \int_0^t \sum_{|\alpha|\leq N} \|\Box\Gamma^\alpha u(s,\cdot)\|_{L^2} ds \\ &\leq \frac{A\epsilon}{2} + C_N \int_0^t \sum_{|\alpha|\leq N} \|\Gamma^\alpha\Box u(s,\cdot)\|_{L^2} ds \\ &\overset{!}{=} \frac{A\epsilon}{2} + C_N \int_0^t \sum_{|\alpha|\leq N} \|\Gamma^\alpha[F(\partial u)](s,\cdot)\|_{L^2} ds \end{aligned}$$

where $\overset{!}{=}$ comes from the fact that $A(0) \leq \frac{A\epsilon}{2}$ and lemma 6.4.1(2), and C_N denotes a generic constant which can change form line to line. Next ,we estimate the value inside the integral and summand, that is:

$$\|\Gamma^\alpha[F(\partial u)](s,\cdot)\|_{L^2} \qquad |\alpha| \leq N$$

If $\alpha = 0$, we use the fact that $|F(z)| \le G_2(|z|)|z|^2$ to get

$$\|F(\partial u)(t,\cdot)\|_{L^2} \le G_2(\|\partial u(t,\cdot)\|_{L^\infty})\|\partial u(t,\cdot)\|_{L^\infty}\|\partial u(t,\cdot)\|_{L^2} \tag{6.16}$$

The first facto on the right hand side is bounded by a continuous function of A since by Sobelev's inequality and the fact that $A(t) \le 2A\epsilon$ (where we can assume $\epsilon \le 1$):

$$\|\partial u(t,\cdot)\|_{L^\infty} \le CA(t) \le 2CA\epsilon \tag{6.17}$$

By the Klainerman-Sobelev inequality, the second factor in equation (6.16) is bounded by

$$C\frac{A(t)}{(1+t)^{(n-1)/2}}$$

and by the fact that $A(t)|le2A\epsilon$, the third factor is bounded by $2A\epsilon$.

If $\alpha \neq 0$, then by lemma 6.4.1(3) we can rewrite $\Gamma^\alpha u(t,\cdot)$ as:

$$[D^m F](\partial u)\Gamma^{\beta_1}\partial u \cdots \Gamma^{\beta_m}\partial u$$

where we bound the L^2 norms in the space by

$$\|[D^m F]\,(\partial u(t,\cdot))\|_{L^\infty}\prod_{i=1}^{m-1}\|\Gamma^{\beta_i}\partial u(t,\cdot)\|_{L^\infty}\|\Gamma^{\beta_m}\partial u(t,\cdot)\|_{L^2} \tag{6.18}$$

We will now get our result when we consider the case of $m \ge 2$ and $m = 1$. If $m \ge 2$, then since $|D^m F(z)| \le G_m(|z|)$ equation (6.17) we just proved, the first factor is bounded by a continuous function A. Then by the fact that $A(t) \le 2A\epsilon$ and the fact that $|\beta_m| \le |\alpha| \le N$, we see that the last factor is bounded by $2A\epsilon$. Since we may assume that lemma 6.4.1(4) and (5) holds, we have that $|\beta_i| + 1|(n+2)/2 \le N$ for $1 \le i \le m-1$. Hence, by the Klainerman=Sobelev inequality and $A(t) \le 2A\epsilon$,

$$\prod_{i=1}^{m-1}\|\Gamma^{\beta_i}\partial u(t,\cdot)\|_{L^\infty} \le C\left[\frac{A(t)}{(1+t)^{\frac{n-1}{2}}}\right]^{m-1} \le CA^{m-2}\frac{A(t)}{(1+t)^{\frac{n-1}{2}}}$$

and so, we conclude that equation (6.18) is bounded by

$$C_A\epsilon A(t)(1+t)^{-(n-1)/2}$$

whne $m \ge 2$. When $m = 1<$ we get the same bound if we use $|DF(z)| \le G_2(|z|)|z|$ insead, completing the proof.

6.5 Looking at Low Dimensions Solutions

We have limited ourselves to the case where $n \ge 4$. The case for lower dimensions is harder, and we start addressing it in this section.

First of, if $n \in \{1, 2, 3\}$, then the global existence fails in general (we'll show some later). Fortunately, the proof used for the $n \ge 4$ case gives asymptotic lower bounds on the lifespan: if T is the supremum of all lifespance, we have:

$$T_\epsilon = T^*(\epsilon f, \epsilon g)$$

as $\epsilon \to 0$. Specifically, we'll see that there exists a $c > 0$ such that

$$T_\epsilon \geq \begin{cases} e^{c/\epsilon} & n = 3 \\ c/\epsilon^2 & n = 2 \\ c/\epsilon & n = 1 \end{cases}$$

for sufficinetly small ϵ, with c depending on f and g.

6.5.1 Proof of lower bounds

For some $T > 0$, Let $u \in C^\infty([0,T) \times \mathbb{R}^n)$ solve

$$\Box u = F(\partial u)$$

$$u|_{t=0} = \epsilon f, \quad \partial_t u|_{t=0} = \epsilon g$$

where $F(0) = DF(0) = 0$. Recall from the proof of global existence for $n \geq 4$ that if we let $N = n+4$ and

$$A(t) = \sum_{|\alpha| \leq N} \|\Gamma^\alpha \partial u(t, \cdot)\|_{L^2}, \quad 0 \leq t < T$$

then there is a constant $A = A(f, g)$ such that the boot-strap assumption

$$A(t) \leq 2A\epsilon, \quad 0 \leq t \leq T' < T$$

oimplies

$$A(t) \leq A\epsilon/2 + C_A\epsilon \int_0^t \frac{A(s)}{(1+s)^{(n-1)/2}} ds \quad 0 \leq t \leq T'$$

and so by Gronwall's lemma:

$$A(t) \leq A\epsilon/2 + \exp\left[C_A\epsilon \int_0^t \frac{ds}{(1+s)^{(n-1)/2}}\right] \quad 0 \leq t \leq T'$$

when $n \geq 5$, we have that $\int_0^t \frac{ds}{(1+s)^{(n-1)/2}} < \infty$, and so we get a bound $A(t) \leq A\epsilon$ on $[0, T']$ provided that $0 < \epsilon < \epsilon_0$ where ϵ_0 is determined by the condition

$$\exp\left[C_A\epsilon_0 \int_0^| infty \frac{ds}{(1+s)^{(n-1)/2}}\right] = 2$$

showing us that ϵ_0 depends on A, and so on (f, g) (but not on T). Combining this boot-strap argument with the continuity method then gives the *a priori* bound

$$A(t) \leq A\epsilon \quad 0 \leq t < T$$

(finish here, I want to see the counter-example)

6.5.2 The Null Condition And Global Existence For Dimension 3

As we mentioned at the beginning, in general the existence of global smooth solutiosn for small data *fails* in dimension $n = 3$ for equations of type $\Box u = F(\partial u)$. The following was found by F. John, we will not yet show it doesn't have a solution

Example 6.1: Non-Existence For Third Dimension

We'll show that every smooth solution of

$$\Box u = (\partial_t u)^2 \quad t \geq 0,\ x \in \mathbb{R}^3$$

with non zero data in $C_c^\infty(\mathbb{R}^3)$ blows up in finite time.

Interestingly, or frustatingly depending on how you see it, there are very similar looking equations that do have a oslution. The following due to Nirenberg:

$$\Box u = (\partial_t u)^2 - \sum_1^3 (\partial_i u)^2 \quad t \geq 0,\ x \in \mathbb{R}^3$$

have global solutions for small data

$$u|_{t=0} = \epsilon f \quad \partial_t u|_{t=0} \epsilon g$$

The key is to notice that

$$v(t,x) = 1 - e^{-u(t,x)}$$

sovles the *linear* Cauchy problem:

$$\Box v = 0 \quad v|_{t=0} 1 - e^{\epsilon f},\ \partial_t v|_{t=0} \epsilon g e^{-\epsilon f} \tag{6.19}$$

which naturally has a global smooth solution. The inverse transformation of $u \to v$ is

$$u(t,x) = -\log(1 - v(t,x))$$

This is well-defined as long as $|v| < 1$, at which point u solves the Cauchy problem introdued by Nirenberg. To ensure that v is *globally* small, we do

$$\|v(t,\cdot)\|_{L^\infty} < 1 \qquad \text{for all} \qquad t \geq 0$$

we only have to take $\epsilon > 0$ sufficinetly small, depending on f and g. Indeed, remember that in the linear case when $n = 3$ we have the decay estimate

$$\|v(t,\cdot)\|_{L^\infty} \leq \frac{A}{1+t} \qquad \text{for all} \qquad t \geq 0$$

where A is a constant which depends *linaerly* on L^∞ norm of $v|_{t=0}$, $\nabla_x v|_{t=0}$ and $\partial_t v|_{t=0}$. In view of the Cauchy problem in equation (6.19), therefor, $A < 1$ if $\epsilon > 0$ is sufficinetly small. Then the transformation $v \to u$ is globally defined, giving a global smooth solution to the original cauchy problem.

These examples suggest that in dimension $n = 3$, the question of global existence of smooth solutions of systems of the type $\Box u = B(\partial u, \partial u)$, where each vector component of B is a bilinear form in ∂u, depends strongly on the algebraic structure of B. More generally, for a system of the form $\Box u = F(\partial u)$, where F vanishes to second order at the origin, it is the quadratic part of F that determines the global regularity properties of the equation. The higher order terms are not important. [Recall from the remark at the end of the previous section that we always have global existence for nonlinearities $F(\partial u)$ which vanish to third order at 0.]

6.5.3 Statement Of Null Condition

We will now consider a system of N equations of the form:

$$\Box u^I = F^I(u\partial u) \qquad (t,x) \in R^{1+3}$$

where the unkonwn u and the given C^∞ functio nF are $\mathbb{R}^n$-valued:

$$u = (u^1, ..., u^N) \qquad F = (F^1, ..., F^N)$$

Definition 6.5.1: Null Vectors

A vector $\xi = (\epsilon_0, \epsilon_1, ..., \epsilon_4)$ is *null* if ξ and $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. In other words, $|xi$ lies in the ligh cone (or null cone) in Minkowski space $\mathbb{R}^{1+3}$

Definition 6.5.2: Quadratic Part

Let F^I be as in:

$$\Box u^I = F^I(u\partial u) \qquad (t,x) \in R^{1+3}$$

then the *quadratic part* of F^I is

$$F^I_{(2)}(z) = \sum_{|\alpha|=2} \frac{1}{\alpha!}\partial^\alpha F^I(0)z^\alpha$$

where $z \in \mathbb{R}^{N+(n+1)N}$ correponds to $(u, \partial u)$

As we've seen in the previous section, some conditions on the quadratic part of F need to be imposed to insure the existence of a global smooth solution for small data. The relevant priciple is the so called *null conditon of Klainerman and Christodoulou.*

Definition 6.5.3: Null Condition

Let F be as in

$$\Box u^I = F^I(u\partial u) \qquad (t,x) \in R^{1+3}$$

Then we say that F satisfies the *null condition* if

1. F vanishes to second order at the origin:

$$F(0)0 = \qquad DF(0) = 0$$

   Thus, by Taylor's theorem, $F(z) = F_{(2)}(z) + R(z)$ where R is C^∞ and vanishes to third order at 0.

2. The quadratic part of F is of the form

$$F^I_{(2)}(u, \partial u) = \sum_{J,K=1}^{N} \sum_{\mu,\nu=0}^{3} a^{I_{\mu\nu}}_{JK} \partial_\mu u^J \partial_\nu u^K$$

   where the a's are real constants satisfying, fo all $I, J, K = 1, ..., N$

$$\sum_{\mu,\nu=0}^{3} a^{I_{\mu\nu}}_{JK} \xi_\mu \xi_\nu = 0$$

   for all null vectors ξ.

Notice that $F_{(2)}$ is only allowd to depend on ppu, not on u.

Example 6.2: Null Condition

Notice that the equation introduced by Nirenberg earlier:

$$\Box u = (\partial_t u)^2 - \sum_{1}^{3} (\partial_i u)^2 \quad t \geq 0,\ x \in \mathbb{R}^3$$

satisfies this condition, while the equation found by F. John:

$$\Box u = (\partial_t u)^2 \quad t \geq 0,\ x \in \mathbb{R}^3$$

does not satisfy the null condition

Lemma 6.5.1: Null Form

If B is a real bilinear form on $\mathbb{R}^n \times \mathbb{R}^4$ such that

$$B(\xi, \xi) = 0$$

for all null vectors $\xi<$ then B is a linear comination, with real coefficients, of the so-called *null forms*:

$$Q_0(\xi, \eta) = \xi_0\eta_0 - \sum_1^3 \xi_i\eta_i$$

$$Q_{\mu\nu}(\xi, \eta) = \xi_\mu\eta_n u - \xi_\nu\eta\mu \qquad 0 \leq \mu < \nu \leq 3$$

Note that the converse of this lemma is obvious, and so this is the interseting dierction to prove.

Proof
BY the given conditions, we can deduce that

$$B(\xi, \xi) = \xi^T A\xi = \sum a^{\mu\nu}\xi_\mu\xi\nu$$

where $A = (a^{\mu\nu})$ is a real 4×4 matrix and we condier ξ as a column vector with transpoe $\xi^T = (\xi_0, ..., \xi_3)$. Now decompseo A Into its symetric and skew-symmetric parts:

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = A_s + A_a$$

Since $\xi^T A\xi = (\xi^T A\xi)^T = \xi^T A^T\xi$, we see that:

$$\xi^T A_s\xi = \xi^T A\xi = 0$$

for nul ξ. Using this condition with the null vectors:

$$\xi^T = (\pm 1, 1, 0, 0) \quad (\pm 1, 0, 1, 0) \quad (\pm 1, 0, 0, 1)$$

and then with:

$$\xi^T = (\sqrt{2}, 1, 1, 0) \quad (\sqrt{2}, 1, 0, 1) \quad (\sqrt{2}, 0, 1, 1)$$

is not hard to see through some elementary linear algebra that A_s must be of the form:

$$A_s = a^{00}\,\mathrm{diag}(1, -1, -1, -1)$$

which naturally corresponds to Q_0. On the other hand,the skew-symetric part A_a gives a combination of $Q_{|mu\nu}$ in the obvious way. Summing it all up, we have

$$B = a^{00}Q_0 + \sum_{0\leq\mu<\nu\leq 3} \frac{1}{2}(a^{\mu\nu} - a^{\nu\mu})Q_{\mu\nu}$$

completing the proof.

Corollary 6.5.1: Further Decomposition

Let F be as in equation

$$\Box u^I = F^I(u\partial u) \qquad (t,x) \in R^{1+3}$$

Then F satisfies the null condition if and only if $F^I(u, \partial u)$ is of the form:

$$\sum_{J,K} a^I_{JK} Q_)(\partial u^J, \partial u^K) + \sum_{J,K} \sum_{0 \leq \mu < \nu \leq 3} b^{I\mu\nu}_{JK} Q_{\mu\nu}(\partial u^J, \partial u^K) + R^I(u, \partial u)$$

where the a's and b's are real constants and R^I is smooth and vanishes to third order at 0

Consider:

$$\Box u^I = F^I(u\partial u) \qquad (t,x) \in R^{1+3}$$

with initial data

$$u|_{t=0}\epsilon f \qquad \partial_t u|_{t=0}\epsilon g$$

where $f = (f^1, f^2, ..., f^N)$ and $g = (g^1, g^2, ..., g^N)$ belond to $C_c^\infty(\mathbb{R}^3)$ and $\epsilon > 0$> Then:

Theorem 6.5.1: Global Existence Under Right Conditions For Low Dimension

Assume that F in the above equation satisfies the null condition. Then there exists $\epsilon_0 = \epsilon_0(f, g) > 0$ such that the above has a smooth global solution provided $\epsilon < \epsilon_0$

6.5.4 Improved Decay

We next tkae a closer look at null forms. In particular, they have better decay proeprties than generic bilinear forms. In order to state the resutl, we will need some more notaiton.

(p. 91-4 Selberg)

6.5.5 An energy inequality and Hörmander's estimate

We need two more results (Selberg called them ingredients) for the main proof. The first is a generlization of the energy inequality

$$\|\partial u(t,\cdot)\|_{L^2} \leq \|\partial u(0,\cdot)\|_{L^2} + \int_0^t \|\Box u(s,\cdot)\|_{L^2} ds \tag{6.20}$$

Recall that the original proof was done by noticing that $u_t \Box u$ is a spacetime divergence:

$$u_t \Box u = \text{div}_{t,x}(e_0, e') \tag{6.21}$$

where $e_0 = \frac{1}{2}|\partial u|^2$ and $e' = -u_t \nabla_x u$. Integrating this idenitty for fixed t gives:

$$\int u_t \Box u dx = \frac{d}{dt} \int e_0 dx - \int \text{div}_x(u_t \nabla_x u) dx$$

and the last term vanishes by the divergence theorem, if we assume that u decays sufficenlty fast as $|x| \to \infty$. For the *energy* $E(t) = \int e_0(t,x)dx$ one of then obtains ,after applying the Cauchy-Schwarz inequality to the left ahdn side of the above identity:

$$E'(t) \leq \sqrt{E(t)} \|\Box u(t,\cdot)\|_{L^2}$$

and then the equation at the top of this seciton follows.

We now will want to generalize this method by replacing equation (6.21) with

$$X(\partial)u \cdot \Box u = \mathrm{div}_{t,x}(e_0, e')$$

where $X(\partial)$ is some first order differential operator and (e_0, e') is some spacetime vector invovling u, such that the associated energy is non-negative:

$$E(t) = \int e_0(t,x)dx \geq 0$$

Then, by integrating $X(\partial)u \cdot \Box u = \mathrm{div}_{t,x}(e_0, e')$ one obtains a generalized energy inequality! Set

$$X(\partial)\overline{X} \cdot \partial + 2t \tag{6.22}$$

hwhere as usual ∂ is the spacetime gradient and

$$\overline{X} = (1 + t^2 + |x|^2, 2tx_1, 2tx_2, 2tx_3)$$

Let m be the matrix $\mathrm{diag}(1,-1,-1,-1)$ (i.e. represent the Minkowski metric), and let $\overline{1} = (1,0,0,0)$. It then turns out that equation (6.22) holds with

$$(e_0, e') = X(\partial)u \cdot m(\partial u) - \frac{1}{3}(\partial u)^T m(\partial u)\overline{X} - v^2\overline{1}$$

where we consier ∂u as a column vector for the purposes of matrix multiplication. Integrating equation (6.22) then yeilds us the identity

$$\frac{d}{dt}E(t) = \int X(\partial)u \cdot \Box u dx$$

for the energy $E(t) = \int e_0 dx$, and the right hand side side is bounded by

$$\|(1 + t + |\cdot|)^{-1} X(\partial)u(t,\cdot)\|_{L^2} \|(1 + t + |\cdot|)\Box u(t,\cdot)\|_{L^2}$$

Then we can show that we can bound the left term being multiplied by:

$$\|(1 + t + |\cdot|)^{-1} X(\partial)u(t,\cdot)\|_{L^2} \leq C\sqrt{E(t)}$$

Putting this all together, we obtain:

$$\sqrt{E(t)} \leq C\sqrt{E(0)} + C\int_0^t \|(1 + s + |\cdot|)\Box u(s,\cdot)\|_{L^2} ds$$

Furthermore, Selberg claims that

$$E(t) \approx \sum_{|\alpha| \leq 1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2$$

Finally, recalling that the ommutaiotn relations between $\Box$ and the invariant vector-fields (see Selberg), we finally obtain:

Proposition 6.5.1: Updated Energy Inequality

For any integer $M \geq 0$, there eixsts a constant C such that

$$\sum_{|\alpha|\leq M+1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}$$
$$\leq$$
$$\sum_{|\alpha|\leq M+1} \|\Gamma^\alpha u(0,\cdot)\|_{L^2} + \sum_{|\alpha|\leq M} \int_0^t \|(1+s+|\cdot|)\Gamma^\alpha \Box u(s,\cdot)\|_{L^2} ds$$

for all $t > 0$ and all $u \in C^\infty([0,\infty)\times\mathbb{R}^3)$ with compact support in x for each t.

Proof
See Sogge's book for details says Selberg

Using these observations, we can state the second result we need (also proven in Sogge's book).

Theorem 6.5.2: Hörmander's Inequality

There exists a C such that if $F \in C^2([0,\infty)\times\mathbb{R}^3)$ and $\Box u = F$ with vanishing initial data at $t = 0<$ then

$$(1+t+|x|) \leq C \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} |\gamma^\alpha F(s,y)| \frac{dyds}{1+s+|y|}$$

Proof
see Sogge's book.

6.5.6 Proof of the main Theorem

Since F is assumed to satisfy the null condition ,we konw that the system in the above equaiton satisfies

$$\Box u^I = \sum_{J,k} a^I_{JK} Q_0(\partial u^J, \partial u^K) + \sum_{J,K<\mu,\nu} b^{I_{\mu\nu}}_{JK} Q_{\mu\nu}(\partial u^J, \partial u^K) + R^I(u, \partial u)$$

where R^I vanishes to the third order at 0. We also specify intial data

$$u|_{t=0} = \epsilon f \qquad \partial_t u|_{t=0}\epsilon g$$

For simplicity, we will ignore the higher order term R^I. We shall apply the Hörmander inequality to $\Gamma^\alpha u$ where u solves the above cauchy prbolem, but in order to do thsi we must substract off the solutoin $w_\alpha(w^1_\alpha, ..., w^N_\alpha)$ of the linear Cauchy problem

$$\Box w_\alpha = 0 \quad w_\alpha|_{t=0} = (\Gamma^\alpha u)|_{t=0} \quad \partial_t w_\alpha|_{t=0} = (\partial_t \Gamma^\alpha u)|_{t=0}$$

Thus, $\Gamma^\alpha u - w_\alpha$ has vanishing initial data, so we may apply Hörmander's inequality. But then we also need to esatime $|w_\alpha|$. We start the proof by looking at this estiamte

Proof
for low dimension non-linear wave equations: We'll first show that if u solves

$$\Box u^I = \sum_{J,k} a^I_{JK} Q_0(\partial u^J, \partial u^K) + \sum_{J,K<\mu,\nu} b^{I\mu\nu}_{JK} Q_{\mu\nu}(\partial u^J, \partial u^K) + R^I(u, \partial u) \tag{6.23}$$

with initial cauchy data

$$u|_{t=0} = \epsilon f \qquad \partial_t u|_{t=0} \epsilon g$$

and w_α solves

$$\Box w_\alpha = 0 \quad w_\alpha|_{t=0} = (\Gamma^\alpha u)|_{t=0} \quad \partial_t w_\alpha|_{t=0} = (\partial_t \Gamma^\alpha u)|_{t=0}$$

then we get

$$|w_\alpha(t,x)| \leq \frac{C_\alpha \epsilon}{1+t} \qquad \forall t \geq 0,\ x \in \mathbb{R}^3$$

where C_α is independnet of t and ϵ.

Proof. Let u_0 sovle $\Box u_0 = 0$ with the above initial data. Then $\Box \Gamma^\alpha u = 0$, and so by the decay estimate for solutiosn of the homogeneous wave equation (remember the result from the chapter)

$$|\Gamma^\alpha u_0(t,x)| \leq \frac{C_\alpha}{1+t} \qquad d\forall t \geq 0,\ x \in \mathbb{R}^3$$

Thus, to get the desired estimate (inequality), it is enoug hto show that

$$|\Gamma^\alpha u_0(t,x) \leq \frac{C_\alpha \epsilon}{1+t} \qquad \forall t \geq 0,\ x \in \mathbb{R}^3$$

Since $\Box(w_\alpha - \Gamma^\alpha u_0) = 0$, this estimate again follows from the decay property we used above, if we just observe that the initial data are $O(\epsilon^2)$. In fact, the data here is

$$\Gamma^\alpha u(0,x) - \Gamma^\alpha u_0(0,x), \qquad \partial_t \Gamma^\alpha u(0,x) - \partial_t \Gamma^\alpha u_0(0,x)$$

To express $\Gamma^\alpha u(0,x)$ in terms of ϵ, f, g, we use equation (6.23) and the cauchy data (whenever there are two or more time derivatives we need to use the equation). If we do the same for $\Gamma^\alpha u(0,x)$ and subract, all terms which are linera in ϵ cancel out, and we are left with terms arising from the nonlinearity in equation (6.23), and which therefore are at least quadratic in ϵ, completing the proof. $\square$

With this result, we now continue onto the main theorem. We break down the proof in 5 steps:

1. Let $0 < T_0 < \infty$ and set $S_{T_0} = [0, T_0) \times \mathbb{R}^3$. Suppose $u \in C^\infty(S_{T_0})$ solves equation (6.23) and the cauchy data (setting $R^I = 0$ for simplicity). We shall prove the existence of $\epsilon_0 > 0<$ independnet of T_0 such that

$$0 < \epsilon < \epsilon_0 \Rightarrow u, \partial u \in L^\infty(S_{T_0})$$

   Once we know this, it follows from local existence that we've done in previous section that the lifespan $T_\epsilon = \infty$

2. The implication given in setp one will follows if we can prove the *a priori* estimate

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot)\|_{L^\infty} \le \frac{A\epsilon}{1+t} \tag{6.24}$$

for $0 \le t < T_0 <$ provided $\epsilon < \epsilon_0$. Here A is a constant independent of T_0 and ϵ (A will depend on f and g), and k is a sufficiently large integer $k = 4$ works)

3. The plan to prove the estimate in step 2 is to use the continuity method. Thus, we define

$$E = \{T \in [0, T_0) \mid \text{step 2 holds for all } 0 \le t \le T\ \}$$

Clearly $0 \in E$ if we take A sufficinetely large, and E is evidently closed Ifwe can show that E is open in $[0, T_0)$, it will therefore follows that $E = [0, T_0)$, finishing the proof.

To that end, fix $T \in E >$ By continuity of the left hand side of step 2 (note that $\Gamma^\alpha u$ is smooth and compactly supported in x On $[0, T'] <$ by Huygen's principle), there certainly exists $T' > T$ such that

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot)\|_{L^\infty} \le \frac{2A\epsilon}{1+T} \qquad \text{for} \qquad 0 \le t \le T' \tag{6.25}$$

The idea now is to use a boot-strap argument to show that we have the stronger estimate shown in step 2 on $[0, T']$. It then will follow that $T' \in E$, proving that E is open. To prove the above estimate implies the estaimte in step 2 when ϵ is sufficinelty small, we first show that the abveo estiamte implies

$$\sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(t,\cdot)\|_{L^2} \le C_0(1+t)^{C_1\epsilon} \sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(0,\cdot)\|_{L^2} \tag{6.26}$$

for $0 \le t \le T'$ where C_0 and C_1 are absoltue constants. Then we prove that the above implies the estimate in step 1 if ϵ is sufficeintly small.

4. We know prove that (6.25) $\Rightarrow$ (6.26). Define $A(t)$ to be the left hand side of equation (6.26). Then by the above proposition:

$$A(t) \le CA(0) + C \sum_{|\alpha \le k+2} \int_0^t \|(1+s+|\cdot|)\Gamma^\alpha \Box u(s,\cdot)\|_{L^2} ds$$

if we ignroe R^I, then $\Box u^I$ is just a lianer combinatio nof $Q(\partial u^J, \partial u^K)$ for the null forms Q, and so form propositin ref:HERE we get, takign the higher order derivatives in $QL^2 <$ and the lower order in L^∞,

$$A(t) \le CA(0) + C \int_0^t A(s) \left(\sum_{|\alpha| \le \frac{k+2}{2}+1} \|\Gamma^\alpha u(s,\cdot)_{L^\infty} \right) ds$$

Since $\frac{k+2}{2} + 1 = \frac{k}{2} + 2$ if $k \ge 4$, the second factor in the integrand is bounded by $\frac{2A\epsilon}{1+s}$ according to equation (6.25), and so we get

$$A(t) \le CA(0) = C'\epsilon \int_0^t \frac{A(s)}{1+s} ds \qquad 0 \le t \le T'$$

Then by Gronwall's lemma:

$$A(t) \leq CA(0)\exp\left[C'\epsilon\int_0^t (1+s)^{-1}ds\right] = CA(0)(1+t)^{C'\epsilon}$$

proving equation (6.26)

5. We prove (6.26) $\Rightarrow$ (6.24), i.e. estimate from step 2. We can naturally choose A so large so that our result from the observation at the begnning of the proof:

$$|w_\alpha(t,x)| \leq \frac{C_\alpha\epsilon}{1+t} \qquad \forall t \geq 0,\ x \in \mathbb{R}^3$$

implies:

$$\sum_{|\alpha|\leq k} \|w_\alpha(t,\cdot)\|_{L^\infty} \leq \frac{A\epsilon/2}{1+t}$$

for all $t \geq 0$> Thus, equation (6.24) follows if we can show that

$$\sum_{|\alpha|\leq k} \|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} \leq \frac{A\epsilon/2}{1+t} \tag{6.27}$$

for $0 \leq t \leq T'$> TO prove this we apply Hörmander's inequality, which give

$$(1+t)\sum_{|\alpha|\leq k} \|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} \leq C\sum_{|\beta|\leq 2}\int_0^t\int_{\mathbb{R}^3}\sum_{|\alpha|\leq k}|\Gamma^\beta\Box\Gamma^\alpha u(s,y)|\frac{dyds}{1+s}$$

Using the commutation relation betwen $\Box$ and the invariant vector fields, we may boudn the right hand side by

$$C\sum_{|\alpha|\leq k+2}\int_0^t\int_{\mathbb{R}^3}|\Gamma^\alpha\Box u(s,y)|\frac{dyds}{1+s}$$

and ignore again the R^I it suffices to estimate this with $\Box u$ replaced by $Q(\partial u^J, \partial u^k)$ for ea hof the null fomrs Q. Then if we apply proposition ref:HERE (8 in Selberg. p. 98) we can bound the last expression by

$$C\sum_{|a|\leq k+3}\int_0^t\int_{\mathbb{R}^3}|\gamma^\alpha u(s,y)|^2\frac{dyds}{(1+s)^2}$$

this equals

$$C\sum_{|a|\leq k+3}\int_0^t|\gamma^\alpha u(s,\cdot)|^2_{L^2}\frac{ds}{(1+s)^2}$$

and using equation (6.26) we bound this by

$$CA(0)^2\int_0^t(1+s)^{2C_1\epsilon-2}ds$$

if $2C_1\epsilon < 1$< the integral is uniformally bounded in t, and since $A(0) = O(\epsilon)$< we finally obtain the bound

$$(1+t)\sum_{|\alpha|\leq k}\|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} \leq C\epsilon^2$$

for $0 \leq t \leq T'$, hwere C is an absoltue constant. if $C\epsilon \leq A/2$, then equation (6.27) follows, completing the proof.

7

Future Reading

future reading (based off of Selberg's chapters):

1. Strichartz type estimates
2. Applicaiton to Maxell-Klein-Gordon
3. More on well-posedness