Pontryagin Duality and Fourier Transform

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Pontryagin Duality

In the 19th century, Joseph Fourier was trying to find out how to model the movement of heat through solid bodies, in particular model how heat seems to dissipate and homogenise within its medium. His efforts lead to the development of *Fourier series* and the *Fourier Transfrom* of periodic function with sufficient regularity properties, namely that $f \in L^2(\mathbb{T}^n)$ so that $\mathcal{F}(f) = \widehat{f} \in \ell^2(\mathbb{Z}^n)$. Under the right conditions, we may generalize the Fourier transform to functions on $\mathbb{R}^n$ (usually, $f \in L^1(\mathbb{R}^n)$ or $f \in \mathcal{S}(\mathbb{R}^n)$). In this paper, we will extend the Fourier transform to locally compact abelian groups and show that many classical results of Fourier analysis hold.

In order to define the Fourier transformation on groups, we require the notion of *topological groups* which will allow us to define a Borel σ-algebra on which we will define a measure known as the *Haar measure*. With a measure, we may define the notion of *integrable function on a group G*. We next require the generalization of the domain of $\widehat{f}$. When working with $\mathbb{T}^n$, the answer was "obvious" in that we had $\widehat{f} : \mathbb{Z}^n \to \mathbb{R}$ since $\widehat{f}$ enumerated the orthonormal basis of $L^2(\mathbb{T}^n)$. When we generlized the Fourier transform to functions $f \in L^p(\mathbb{R}^n)$ (for $1 \leq p \leq 2$), we had that the domain of $\widehat{f}$ was $\mathbb{R}^n$, and in general the domain of $\widehat{f}$ transformed like *functionals*. We will explore this concept when we define *Pontryagin Duality*.

After having introduced the relevant concepts, we shall introduce Fourier transforms on locally compact (Hausdorff) abelian group. Due to the Pontryagin duality, many classical results of Fourier Analysis are readily recovered. We shall not go over too much detail as this is more an essay than the beginning of a course in Harmonic Analysis, however some interesting highlights will be added.

The paper will end with some interesting uses and consequence of the Fourier transform and of the Pontryagin duality, including how it demonstrates compact abelian groups are just as hard to classify as abelian groups, how the Fourier Transform shows up in Representation Theory of finite groups, how Pontryagin Duality allows us to classify locally compact abelian groups through their characters.

A quick disclaimer that I will be assuming some 1st-year graduate analysis knowledge for this paper as I will have to touch on topics covered in those courses.

1.1 Topological Group Haar Measure

Let G be a group. G is a *topological group* if the there is a topology endowed in G such that the binary operation of G is continuous ($(x, y) \mapsto xy$) and the inverse map is continuous $x \mapsto x^{-1}$. We shall require that G have nice enough properties so that we may define a measure on it. It will turn out that we will require that G be Hausdorff and locally compact abelian. We shall not prove in detail how to define such a measure, but we shall show some examples to get comfortable with the notion. We first show some properties of topological groups:

Lemma 1.1.1: Translation Is Homeomorphism

Let G be a topological group with (continuous) binary operation $\cdot$. Then the map $x_g : G \to G$ defined by

$$x \mapsto g \cdot x$$

is a homeomorphism

Proof **:**
For notational simplicity, let ρ represent the (continuous) binary operation. To show x_g is bijective, notice that $x_{g^{-1}}$ is a two-sided inverse:

$$x_g \circ x_{g^{-1}}(x) = x_g(g^{-1}x) = gg^{-1}x = x = g^{-1}gx = x_{g^{-1}}(gx) = x_{g^{-1}} \circ x_g(x)$$

Hence, for any $V \subseteq G$, there exists a U such that $gU = V$. To show x_g is continuous for any g, we take advantage of continuity of ρ. Since ρ is continuous, so is $\rho_g : \{g\} \times M \to M$ with the subspace topology. Taking any open $V \subseteq M$, we know that $\rho_g^{-1}(V)$ is open. By the bijectivity argument we've just done, there exists a U such that $V = gU$. Hence,

$$\rho_g^{-1}(V) = \rho_g^{-1}(gU) = \{g\} \times U$$

Since the set $\{g\} \times U$ is open in the product topology, U is open in G, hence $x_g^{-1}(V) = U$, showing the pre-image of open sets is open and hence x_g is continuous. The argument holds also for $x_{g^{-1}}$ and hence x_g is a homeomorphism. In particular, gU is open for all $g \in G$, completing the lemma.

We will be dealing with Hausdroff groups, though this is not too much of a restriction as the following proposition demonstrates:

Proposition 1.1.1: Topological Groups And Hausdorffness

Let G be a topological group and let H be the closure of $\{e\}$. Then G/H is Hausdorff (with respect to the quotient topology), G is Hausdorff if and only if $H = \{e\}$

Proof **:**
Let $\overline{H}$ denote the closure of $\{e\}$. Note that the closure of a subgroup is a group: Let $x, y \in \overline{H}$ so that there is nets $\langle x_\alpha \rangle_{\alpha \in A}$ and $\langle y_\beta \rangle_{\beta \in B}$ in H that converges to x and y. Then $x_\alpha^{-1} \to x^{-1}$ and $x_\alpha y_\beta \to xy$ (with the usual product ordering on $A \times B$), so x^{-1} and xy belong to $\overline{H}$. Hence, we may ask whether H is normal so that G/H is a well-defined topological group. Evidently, $\overline{H}$ is

the smallest (or minimal) closed subgroup of G (since $\{e\}$ is the smallest subgroup of G). Then $\overline{H}$ has to be normal, since if it weren't then if $H' = gHg^{-1}$ was some conjugate such that $H \neq H'$[a], then $H' \cap H$ would be a closed subgroup smaller than H. Hence, G/H is well-defined, and it is a simple to check that it indeed is a topological group with respect to the quotient topology. Then $\{\overline{e}\}$ is closed since its pre-image of $\overline{H}$. Finally, by lemma 1.1.1, the map $x_{\overline{g}}$ is a homeomorphism for all $g \in G$, and so the image $\overline{e} \mapsto \overline{g}$ maps a closed set to a closed set, namely all single point sets are closed, completing the proof.

[a]Note that H' is closed since left and right multiplication is a homeomorphism

Using the above, we can see that any Borel measurable function $f : G \to \mathbb{C}$ would be constant on cosets of H, and so naturally defined on G/H, and so we loose little with the imposition of Hausdorfness (i.e. Hausdrofness is sort of a natural condition to put on topological groups when working with measurable functions)

Example 1.1: Topological Group

1. For any group G, if G is equipped with the discrete topology it is a topological group, and all homomorphisms are continuous between any other topological group. In this way, the theory of topological groups can be thought as subsuming group theory. Any group with the indiscrete topology (also known as trivial topology) is also a topological group.

2. If $H \leqslant G$ is also a topological space, then H is a topological group with the subspace topology.

3. The real numbers $\mathbb{R}$ and $\mathbb{R}^\times$ under multiplication is a topological group. More generally, $\mathbb{R}^n$ is a topological group, an any topological *topological vector-space* (a vector-space where the action is continuous as well) is a topological group

4. The circle $\mathbb{T}$, and the n-torus $(\mathbb{T})^n$ are topological groups. Since $\mathbb{R}/\mathbb{Z}$ is homeomorphic homeomorphic and isomorphic to $\mathbb{T}$, $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}$ are isomorphic as topological groups. If we take $\mathbb{T}$ as a subset of $\mathbb{C}$, then we $\mathbb{T} = U(1) = \{x \in \mathbb{C} \mid |x| = 1\}$.

5. The general linear group over the reals $\mathrm{GL}_n(\mathbb{R})$ is a topological group (multiplication and inverse are both polynomials). Similarly, the special linear group,
$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n \mid \det(A) = 1\}$$
is a topological group.

6. Take $O(n) \subseteq \mathrm{GL}(n)$, where $O(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^t = I\}$. This is a topological group known as the *orthogonal group*, since it consists of all orthonormal basis of $\mathbb{R}^n$. Similarly, we can define $U(n) = \left\{A \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; A\overline{A}^t = I\right\}$ is a topological group known as the *unitary group*. Restricting to determinant one matrix, we also get:
$$\mathrm{SO}(n) = \mathrm{SL}_n(\mathbb{R}) \cap O(n) \qquad \mathrm{SU}(n) = \mathrm{SL}_n(\mathbb{R}) \cap U(n)$$
Notice that some of these groups are compact (ex. $\mathrm{SU}(n)$) while others are locally compact (like $\mathrm{GL}_n(\mathbb{R})$).

7. Any Lie group is naturally a topological group. An example of a topological group that is not a Lie group is $\mathbb{Q}$ with the subspace topology induced by $\mathbb{R}$.

8. The p-adic integers, $\mathbb{Z}_p$ with the induced inverse-limit topology (in this case, the p-adic topology) is a topology group. This group is compact, homeomorphic to the Cantor set, but is totally disconnected (hence cannot be a Lie Group)

9. Similarly to how the topology $\mathbb{Z}_p$ can be defined using the p-adic metric, we may give $\mathbb{Q}$ different p-adic metric, and taking the completion of $\mathbb{Q}$ with respect to different p-adic metrics forms $\mathbb{Q}_p$, which is also a Lie Group.

We shall focus on locally compact abelian groups. One of the most remarkable facts about them is that there is a natural measure we may define on them called the *Haar measure*

Definition 1.1.1: Haar Measure

Let G be a locally compact topological group. Then a *left* (resp. *right*) *Haar measure* on G is a nonzero left-invariant (resp. right-invariant) Radon measure μ on G.

We shall state, but not prove, the following very interesting result:

Theorem 1.1.1: Haar Measure

Let G be a locally compact topological group. Then there exists a left Haar measure on G.

Proof **:**
See Foland's *Real Analysis* p.344

Example 1.2: Haar Measure

1. Most famously, the Lebesgue measure is a (left- and right-invariant) Haar measure on $\mathbb{R}^n$

2. The counting measure is a (left and right) Haar measure on any group (not necessarily locally compact) with the discrete topology

3. Let $\mathbb{T}$ be the circle group, and consider $f : [0, 2\pi] \to \mathbb{T}$ defined by

$$f(t) = (\cos(t), \sin(t))$$

then we may define the Haar measure by:

$$\mu(S) = \frac{1}{2\pi} m(f^{-1}(S))$$

where m is the Lebesgue measure on $[0, 2\pi]$. We multiplied by $(2\pi)^{-1}$ so that $\mu(T) = 1$

4. Let $G = \mathbb{R}^\times$. Then a Haar measure μ can be given by

$$\mu(S) = \int_S \frac{1}{t} dt$$

For any Borel subset $S \subseteq \mathbb{R}^\times$. For example, if $S = [a, b]$, then $\mu(S) = \log(b/a)$. This measure is indeed translation invariant. Since μ is the supremum over the integral of character functions on $[a, b]$, it suffices to check the measure of $[a, b] = S$ is invariant under left (resp. right) translation. Indeed:

$$\mu(gS) = \mu([ga, gb]) = \log(gb/ga) = \log(b/a) = \mu(S)$$

and similarly for Sg. We may define a similar measure by taking

$$\mu(S) = \int_S \frac{1}{|t|} dt$$

5. Let $G = \mathrm{GL}_n(\mathbb{R})$. Then one may define a (left and right) Haar measure to be

$$\mu(S) = \int_S \frac{1}{|\det(X)|^n} dX$$

where dX is the Lebesgue measure on $\mathbb{R}^{n^2}$ identified with the set of all $n \times n$ matrices (this follows from the change of variables formula).

From here on out, when we shall drop the term "topological", assuming all our future groups will be topological groups. Since we have defined a measure, we may define measurable functions, and in particular integrable functions. Since Haar measures are Radon measures, we may define the various L^p-spaces associated to G:

$$L^p(G, \mu) = \left\{ f : G \to \mathbb{C} \;\middle|\; \int_G |f(x)|^p d\mu(x) < \infty \right\}$$

A classical result of Haar measures is that two Haar measures on G are equal up to some scalar λ, and hence we may if we want drop μ and write $L^p(G)$ if information like isometry between spaces is not of interest. In our case, we shall want to keep track of which measure we use when we generalize Plancherel's Theorem.

For future reference, for any $f \in L^1(G)$, define $L_z f$ to be

$$L_z(f)(x) = f(z^{-1}x)$$

Where we take the inverse so that $L_{yz} = L_y L_z$.

1.2 Pontryagin Duality

We now define the next relevant concept. Let G be a locally compact (Hausdorff) abelian group. If G was also a topological vector space (i.e. a topological field k acts continuously on G so that G becomes a topological vector space), then we would have a natural notion of duality since G would be a vector space over some underlying field k. For general group, no such "natural dualizing" object exists. Instead, we may try to find some interesting group so that the contravariant functor between locally compact abelian groups (which we'll denote **LCA**):

$$\mathrm{Hom}(-, H) : \mathbf{LCA} \to \mathbf{LCA} \tag{1.1}$$

has properties similar to what we are used to in the dual theory we have in Functional Analysis (ex. reflexivity or properties of adjoints) and that we recover some classical results from Fourier Theory.

Thanks to work by Lev Pontryagin, we have that a natural choice for H in equation (1.1) is the circle group with the usual subspace topology of $\mathbb{R}^2$. We shall henceforth label this group as $\mathbb{T}$.

Definition 1.2.1: Pontryagin Dual

Let G be a locally compact abelian group. Then

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$$

where $\mathrm{Hom}(G, \mathbb{T})$ is the collection of all continuous homomorphisms, is the *Pontryagin dual* of G endowed with the topology of uniform convergence of compact sets[a]. An element of $\widehat{G}$ is called a *character* of G.

[a] If you are familiar with complex analysis, this topology is commonly used for the convergence of holomorphic functions

For a better understanding of the topology on $\widehat{G}$, consider The collection of sets

$$W_K^V := \left\{ \chi \in \widehat{G} \;\middle|\; \chi(K) \subseteq V \right\}$$

where $K \subseteq G$ is a compact subset and $V \subseteq \mathbb{T}$ is a neighborhood containing the identity. Then the basis for the topology of $\widehat{G}$ is the collection of all W_K^V along with their translations. For examples of such convergence, recall power series centered at 0 ([1]) on $\mathbb{C}$ with radius R converge uniformly in $\overline{B_r(0)}$ when $r < R$, but it does not necessarily converge uniformly on $r = R$. Hence, power series technically converge in the *compact-open* topology[2].

Example 1.3: Pontryagin Dual

1. Let $G = \mathbb{Z}/n\mathbb{Z}$ with the discrete topology. Then:

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{T}) = \mathbb{Z}/n\mathbb{Z}$$

   To see this, notice that any $f \in \widehat{G}$ is determined by where it maps 1. Since G has finite order, $f(1)$ must have finite order. Necessarily, $f(1)$ maps to a root of unity $e^{\frac{2\pi i}{k}}$ where $1 \leq k \leq n$. Now it is easy to finish the computations to see that $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$

2. Let $G = \mathbb{R}$. Then:

$$\widehat{\mathbb{R}} = \mathrm{Hom}(\mathbb{R}, \mathbb{T}) = \mathbb{R}$$

   To see this, we will slightly modify a proof presented in Foland's *Real Analysis* Theorem 8.19. Let $\chi \in \widehat{\mathbb{R}}$. Let $a \in \mathbb{R}$ such that $\int_0^a \chi(t)dt \neq 0$. Such an a exists, or else by the Lebesgue Differentiation Theorem $\chi = 0$ a.e. Setting $A = \left(\int_0^a \chi(t)dt\right)^{-1}$, we get:

$$\chi(x) = \chi(x)AA^{-1} = A\int_0^a \chi(x)\chi(t)dx = A\int_0^a \chi(x+t)dt = A\int_x^{x+a} \chi(t)dt$$

   Since the function inside the integral continuous, χ is in fact C^1 (in fact C^∞, but we will not need this fact). Now, notice that:

$$\chi'(x) = A[\chi(x+a) - \chi(x)] = B\chi(x)$$

[1] they may be centered anywhere, but we will center them at zero for simplicity

[2] To be precise, they are locally uniformly convergent, which since $\mathbb{C}$ is locally compact is equivalent to uniform convergence on compact sets

where $B = A(\chi(a) - 1)$. This is now a simple ODE, which can be solved to show we get an exponential (and is unique by the uniqueness of a solution to an ODE). For an alternative approach, notice that

$$\frac{d}{dx}(e^{-Bx}\chi(x)) = 0$$

so $e^{-Bx}\chi(x)$ is constant. Since $\chi(0) = 1$, we have $\chi(x) = e^{Bx}$, and since $|\chi(x)| = 1$, B must be purely imaginary meaning $B = 2\pi i\xi$ for some $\xi \in \mathbb{R}$, completing the proof.

We may easily generalize this proof for when $G = \mathbb{R}^n$. In this case, the character's will be of the form $\chi(x) = e^{2\pi i\xi\cdot x}$ where $\xi \cdot x$ is the dot product.

3. Let $G = \mathbb{Z}$. Then:

$$\widehat{\mathbb{T}} = \mathrm{Hom}(\mathbb{T}, \mathbb{T}) = \mathbb{Z}$$

and similarly:

$$\widehat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) = \mathbb{T}$$

For the first, recall that $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. Then looking at the above proof, we would consider χ to be periodic with period 1 and $e^{2\pi i\xi} = 1$ if and only if $\xi \in \mathbb{Z}$. A similar argument works for the second case.

Looking at this from a second perspective, from usual Fourier Analysis we know that the Fourier transform is a map $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$, and so it should be expected that $\mathbb{Z}$ and $\mathbb{T}$ are Pontryagin dual to each other.

You may ask if it is always the case that that $\widehat{\widehat{\mathbb{Z}}} = \mathbb{Z}$ just like for vector-spaces. We know that in general duality theory, this is not the case (the dual of the dual may be much larger than the original object, think of $C_b(X)$). If it is the case, that means that the dual $\widehat{G}$ of G contains all the required information about G. For Pontryagin duality, it is always reflexive:

Theorem 1.2.1: Pontryagin Duality Theorem

Let G be a locally compact abelian group. Then there is a natural isomorphism:

$$G \mapsto \widehat{\widehat{G}}$$

mapping $g \mapsto \mathrm{ev}_g$ where $\mathrm{ev}_g : \widehat{G} \to G$ is defined by $\mathrm{ev}_g(\chi) = \chi(g)$.

We shall delay the proof until we have the notion of Fourier Transform for groups, and due to its complexity only present some interesting parts of it. Note that in general, $G \ncong \widehat{G}$ (think of $L^p(X) \ncong L^q(X) = (L^p(X))^*$), however if G is finite and abelian then $G \cong \widehat{G}$ (though not canonically). The essential take away from Pontryagin duality is that $\widehat{G}$ contains enough information on G to recover G.

Remark The map in theorem 1.2.1 can in fact be augmented to be a covariant functor (even an exact functor, i.e. it preserves short exact sequences). Hence, we may look at the study of extending group or the homology of groups in the Pontryagin dual.

1.3 Generalizing the Fourier Transform

We may now combine all we have defined into the following

Definition 1.3.1: L^1 Fourier Transform On Group

Let $f \in L^1(G)$. Then define the *Fourier Transform* of f to be $\widehat{f} : \widehat{G} \to \mathbb{C}$ via

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\mu(x)$$

This definition is the proper generalization of Fourier transform. Let's say $G = \mathbb{R}$ with the usual Lebesgue measure. Then as we saw in example 1.3, χ is the map $\xi \mapsto e^{-2\pi i\xi}$ so that

$$\widehat{f}(\chi) = \int_{\mathbb{R}} f(x)e^{2\pi i\xi}dx$$

Remark the Fourier transform $f \mapsto \widehat{f}$ is the *Galfand Transformation* of f. If we take $A(\widehat{G}) = \left\{\widehat{f} \;\middle|\; f \in L^1(G)\right\}$, then $A(\widehat{G})$ is a separating sub-algebra of $C_0(\widehat{G})$. It is also not to hard to see that it is self-adjoint, and so by the Stone-Weiestrass theorem it is dense in $C_0(\widehat{G})$.

By our usual Fourier theory, we recover some important properties, namely:

Proposition 1.3.1: Properties Of Abstract Fourier Transform

Let $f \in L^1(G)$ where G is a locally compact (Hausdroff) abelian group with a Haar measure μ. Then:

1. $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$
2. $\widehat{L_x f}(\chi) = \widehat{f}(\chi)$
3. The Riemann-Lebesgue lemma holds

Proof **:**

1. This is from direct computation and applying Fubini's Theorem and properties of the character:

$$\begin{aligned}
\widehat{(f * g)}(\xi) &= \int\int f(x-y)g(y)\overline{\chi(x)}dydx \\
&= \int\int f(x-y)\overline{\chi(x-y)}g(y)\overline{\chi(y)}dxdy \\
&= \widehat{f}(\xi)\int g(y)\overline{(\chi(y))}dy \\
&= \widehat{f}(\xi)\widehat{g}(\xi)
\end{aligned}$$

2. Usually, $\mathcal{F}(L_y f)(x) = e^{-2\pi i y x} f(x)$. For the Fourier transform on groups, we get that its translation invariant since

$$\widehat{L_x f(\chi)} = \int_G f(x^{-1}y)\overline{\chi}(y)dy = \int_G f(y)\overline{\chi(y)}dy = \widehat{f}(\chi)$$

3. (Riemann-Lebesgue Lemma) Notice that the map $\mathcal{F}$ mapping f to $\widehat{f}$ maps $L^1(G)$ to bounded continuous functions. Since the integral is finite and the image of the characters is $\mathbb{T}$ (namely we may bound it with $\sup_{x\in G}(\chi(x))$), $\widehat{f}$ will vanish at infinity. In fact, $\mathcal{F}$ is norm decreasing and hence is continuous. Hence, we may write

$$\mathcal{F} : L^1(G) \to BC_0(G)$$

where $BG_0(G)$ is the collection of bounded continuous functions that vanish at infinity.

One of the most important results in Fourier Analysis is the ability to recover f from $\widehat{f}$ under suitable conditions, that is the *Fourier Inversion Theorem*. This result naturally generalizes. Note that since $f \in L^1(G)$, we may treat f as being define μ-a.e., and so we may only recover f μ-a.e. in the following theorem unless f has more regularity:

Theorem 1.3.1: Fourier Inversion Theorem For Groups

Let G be a locally compact abelian group with Haar measure μ. Then there is a unique Haar measure $\widehat{\mu}$ on $\widehat{G}$ such that whenever $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G})$, then μ-a.e. we have:

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)d\widehat{\mu}(x)$$

If f is continuous, then is is true for all $x \in G$

Proof **:**
See:

D. Ramakrishnan and R. J. Valenza, Fourier analysis on number fields (Graduate Texts in Mathematics, vol. 186, Springer, 1999), chapter 3.

Another very important result is Fourier Analysis on L^2;

Theorem 1.3.2: Plancherel Inversion Theorem For Groups

Let G be a locally compact abelian group, μ a Haar measure on G, and $\widehat{\mu}$ the dual measure on $\widehat{G}$ given in theorem 1.3.1. If $f : G \to \mathbb{C}$ is continuous with compact support, then $\widehat{f} \in L^2(\widehat{G})$ and

$$\int_G |f(x)|^2 d\mu(x) = \int_{\widehat{G}} \left|\widehat{f}(\chi)\right|^2 d\widehat{\mu}(\chi)$$

In particular, The Fourier transform is an L^2_μ isometry from $C_c(G)$ to $L^2_{\widehat{\mu}}(\widehat{G})$

Proof **:**
See *Fourier Analysis on Groups* by Walter Rudin, Chapter 3.

The Plancherel formula allows us to generalize the space on which the Fourier transform is defined. To see this, first note that $C_c(G)$ and $L^2(G)$ are Banach spaces. By a classical result in Real analysis, $C_c(G)$ is L^2-dense in $L^2(G)$. By proposition 2.1.11 in *Analysis Now*, there is a unique extension from $C_c(G)$ to $L^2(G)$ of the Fourier transform so that we get a *unitary operator*

$$\mathcal{F} : L^2_\mu(G) \to L^2_{\widehat{\mu}}(\widehat{G})$$

If G is compact, then $L^2(G) \subseteq L^1(G)$, and hence we get this result for general L^2-functions. If G is *not* locally compact, then to define the L^2 Fourier transform, we would start on some dense subspace like $C_c(G)$ and then extend isometrically to the whole space using Plancherel's Theorem.

Finally, we can recover the inverse Fourier transform of the dual group for the L^2 Fourier transform

Theorem 1.3.3: Dual Group Fourier Transform

Let G be a locally compact abelian group with Haar measure μ and let $\mathcal{F}$ be the L^2 Fourier transform. Then the adjoint of the Fourier Transform restricted to continuous functions on compact support is the Inverse Fourier Transform

$$L^2_{\widehat{\mu}}(\widehat{G}) \to L^2_\mu(G)$$

Proof **:**
See *Fourier Analysis on Groups* by Walter Rudin, Chapter 3.

Notable example would be when $G = \mathbb{T}$ so that $\widehat{G} = \mathbb{Z}$ and the Fourier transform computes the Fourier series of periodic functions. If G is finite, then we will get the *discrete Fourier Transform* from this process.

1.4 Proof Pontryagin Dual

We come back now to Pontryagin duality and prove a special case of it:

Proof **:**
(of Pontryagin Duality)

This theorem is rather difficult to prove. We shall prove that α is injective, where $\alpha : G \to \widehat{\widehat{G}}$ is defined by $g \mapsto \text{ev}_g$ (the evaluation map at g). The proof can be completed by showing that α is in fact a dense embedding onto it's image, which since α is over a locally compact Hausdorff spaces will make α an open map, and hence must map to the entire codomain, completing the proof.

Let $f \in L^1(G)$ and $\chi \in \widehat{G}$. Recall that $\widehat{L_x f} = \widehat{f}$, and so by the Fourier inversion theorem, $L_x f = f$ for all $f \in C_c^+(G)$ (the set of all continuous positive operators with compact support). Let $x \in G \setminus \{e\}$. Suppose for the sake of contradiction that $\alpha(x) = 1$, that is $\chi(x) = 1$ for all $\chi \in \widehat{G}$. We shall show that this contradicts $L_x f = f$. Since G is Hausdorff, there exists an open

neighborhood U of e such that
$$U \cap (x^{-1}U) = \emptyset$$
Without loss of generality we may choose U so that it lies in a compact neighborhood of the identity. Then BY Urysohn's lemma, there exists a continuous positive definition function $f \neq 0$ with support in U. Since $f \in L^1(G)$ an f is compactly supported and positive, $f \in C_c^+(G)$. Then the support of f and $L_x f$ are disjoint, but that contradict $L_x f = f$, completing the proof.

For more information, see `https://www.math.uh.edu/~haynes/files/topgps8.pdf`

1.5 Interesting Uses

Pontryagin has a couple of very interesting consequences, perhaps a good starting point is that the classification of compact abelian groups is just as difficult as the classification of general abelian groups using the following result:

Theorem 1.5.1: Compactness Criterion For LCA Group

Let G be a locally compact abelian group. Then G is compact if and only if $\widehat{G}$ is discrete. Conversely, G is discrete if and only if $\widehat{G}$ is compact.

Proof **:**
The fact that G is compact implies $\widehat{G}$ is discrete and that G is discrete implies $\widehat{G}$ is compact comes straight from the topology of G and $\widehat{G}$ (namely that it is the compact-open topology). To get the converse of both, we simply apply Pontryagin duality.

Proposition 1.5.1: Unit In $L^1(G)$

Let G be a locally compact abelian group with Haar measure μ. Then $L^1(G)$ has a unit if and only if G is discrete.

Proof **:**
We know that $L^1(G)$ always has an approximate identity (think of good kernels). The identity of $L^1(G)$ would be an element such that $f * \mathrm{id} = \mathrm{id} * f = f$, so $\widehat{f} \cdot \widehat{\mathrm{id}} = \widehat{f}$. We see through this formula that $\widehat{\mathrm{id}}$ would have to be the Dirac-delta distribution at 1 with norm 1. This distribution is a function if and only if G is discrete.

Representation Theory

Another place this comes in is within representation theory. Recall in [1, chapter 24.4] there was a quick word on Fourier transforms in representation theory. In classical representation theory of finite groups, we define the character to be $\chi : G \to \mathbb{C}^*$. Since G is finite, G maps to a root the roots of unity in $\mathbb{T}$, and hence the notion of character in classical representation theory aligns with our definitions. Then we may define the inner product of characters to be:

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2}(g)$$

which will give a way to decompose representations into irreducible components via irreducible characters. If G is abelian, then may translate the decomposition theorem for representations of finite abelian groups to the decomposition of$L^2(G)$ via

$$L^2(G) = \bigoplus_{i=1}^{n} E_n$$

and in the infinite case we would have take the closure

$$L^2(G) = \widehat{\bigoplus_{i \in \mathbb{N}} E_n}$$

where E_n are the subspaces given by the usual orthonormal basis for L^2 when doing Fourier analysis. Hence, representation theory and Fourier analysis are linked!!

Class Field Theory

For those how have seen some class field theory, they have seen the Artin Reciprocity map. Recall that the idele group is a locally compact group, and so it is precisely the context in which we expect Pontryagian duality to be applied! In the local case, we have the Artin map $K^* \cong \mathrm{Gal}_K(K^{\mathrm{ab}})$ between two locally compact groups; taking the Pontryagin dual will preserve this mapping and let us work with the characters of these groups! In the global case using the Adèle ring and Idele group, we shall not get an isomorphism in the Pontryagin dual but a perfect pairing.

Bibliography

[1] Nathanael Chwojko-Srawley. *Everything You Need To Know About Undergraduate Algebra*. 1st ed. NA. URL: https://nathanaelsrawley.com/assets/pdfs/notes/EYNTKA_algebra.pdf.