Everything You Need To Know About MAT436

Nathanael Chwojko-Srawley

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Contents

Abstract The course will survey the branch of mathematics developed (in its abstract form) primarily in the twentieth century and referred to variously as functional analysis, linear operators in Hilbert space, and operator algebras, among other names (for instance, more recently, to reflect the rapidly increasing scope of the subject, the phrase non-commutative geometry has been introduced). The intention will be to discuss a number of the topics in Pedersen's textbook Analysis Now. Students will be encouraged to lecture on some of the material, and also to work through some of the exercises in the textbook (or in the suggested reference books).

These notes compromise my notes on Banach spaces, Hilbert spaces, and the Spectral theory.

A blanket assumption will be that all vector-spaces will be over fields of characteristic 0. Finite characteristic fields are a different thing entirely which I'm not well aware of yet, but from what I've seen in even galois theory, it acts quite differently. As a consequence of this, all fields k we will work with will be between $\mathbb{Q} \subseteq k \subseteq \mathbb{C}$ (up to ring isomorphism). In particular, we have the nice fact that for any vector-space V, we have a nice dual V^* which will map to a space k which (after inhereting the subspace topology of $\mathbb{C}$) has all the nice topological notion we may desire.

The first thing to do is to introduce a new way of looking at vector-spaces beyond the universal property of free modules. Recall that the space $L^p(\mathbb{R}^n)$ (for $n \in \mathbb{N}_{>0}$) is a vector-space over $\mathbb{R}$. As an exercise, it should be shown that it is an infinite dimensional vector-space[1]. Since it is a real vector-space, it is isomorphic to $\mathbb{R}^N$ for appropriate cardinality N, $N \geq |\mathbb{N}|$. However, this space is *certainly* not $\mathbb{R}^{\mathbb{N}}$, since $\mathbb{R}^n$ compromises all real-sequences that are unbounded. So we would imagine some much larger cardinality, perhaps $\mathbb{R}^{\mathbb{R}}$. However, we run into many problems:

1. $\mathbb{R}^{\mathbb{R}}$ would be indistinguishable to L^1, L^2, L^∞, and so on by what we've done.
2. If you think of a basis as a "super-dense"' set, where instead of sequenences converging to any point via the dense set, we only require a *finite* sequence (i.e. a function $f : X \to \mathbb{R}^{\mathbb{R}}$ where $|X| < \infty$), and not only that but also require the converge be to be done *linearly.* Even more powerfully, it requires that each element in the dense set be [linearly] *independent* of each other. Over-all, this is a huge huge ask. How would you even find a basis so that every element of $L^2(\mathbb{R})$ is represented by a finite linaer combination of linearly independent basis elements??

Overall, it seems that the free-module intuition doesn't work anymore[2]. Instead, when we think that a space is a vector-space, we should think of it in terms of the binary operation being defined and the scalar multiplication being defined. For example, if V is some infinite dimensional vector-space, that means that we know we can piece together elements inside it quite easily to get another element via $+$, and any element can be scalad by some constant. This ability to manipulate functions in this way should be what we should think of when thinking about infinite dimensional vector-spaces.

So how then do we intuite infintie dimensoinal vector-spaces if not the universal property of free module? Is there another universal property? Well, it turns out that in infinite dimensions, the situation is much more complicated. We will essentially find out how we deal with different types of infinite-dimensional vector-spaces by adding extra structure (in particular, a norm or an inner

[1] answer: the functions $\chi_{[n,n+1)}$ are linearly independent for all $n \in \mathbb{N}$

[2] We will show how we can get something as close as possible to this intuition when we study Hilbert Spaces in chapter ref:HERE

product), that will give us size or "shape" information for the vectors, which introduces a new notion of "finiteness" into our analysis (or even better a new way of comparing vectors to work with). Notice that since we are introducing a norm/inner-product, we are working with topologies. Unlike the finite dimensional case, where given a fixed dimension all norms are the same[3], in infinite dimensions we will have vector-space with *different norms* or *different inner products* (and hence, that give us *different* topologies, unlike in the finite-dimensional case where there is only one "reasonable" topology which we may define that keeps our intuition of size or angle the same). This has many consequences we shall explore, for example in the finite dimensional case all linear maps are continuous, but not so in the infinite dimensional case (there will always be non-continuous linear maps). Thus, we may properly think of infinite dimensional vector-spaces as different than what we study in linear algebra, which doesn't offer the right tools to study such spaces. In fact, the place that does give us the right tools to figure out how to estimate or approximate things to limit the problem of infinities is *analysis*. Hence, we will essentially always require that all our spaces are *complete* in order to work with limits without any issues.

[3]up to scaling, however this does not matter as a scaled norm produced the same topology

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Banach Spaces

One of the key problems a norm tries to fix is making sure there is still a reasonable "finiteness" to your space. Though a normed vector-space doesn't have an ordering, there is something simlar to it: in the same way the distance between any two real numbers if a real numbers (i.e. finite) the *norm* of any element in a Banach space is *finite*. Thus, though a space might be infinite like L^1, there is still some notion of finiteness allowing us to bring in many of the concepts we have worked with before and avoid the pathologies of the infinite.

1.1 Basic Definitions and Properties

Let k be either $\mathbb{R}$ or $\mathbb{C}$, let V or X denote a vector-space over k, with 0 representing the zero vector. Let kx denote the subspace of X spanned by x. If $M, N \subseteq X$, then $M + N = \{m + n \mid m \in M, n \in N\}$.

Definition 1.1.1: Semi-Norm And Norm

Let V be a vector-space. A *semi-norm* on V is a function $x \mapsto \|x$ from V to $[0, \infty)$ satisfying the propeties:

1. $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality)
2. $\|\lambda x\| = |\lambda|\|x\|$ (homogeneity of scalars)

If furthermore:

(3) $\|x\| = 0$ if and only if $x = 0$

then the function is called a *norm*

Definition 1.1.2: Normed Vector-Space

Let V be a vector space. Then if there exists a function $\|\cdot\|$ that is defined on V, then $(V, \|\cdot\|)$ is called a *normed vector-space.*

If V is a normed vector-space, we can induce a metric $\rho(x, y) = \|x - y\|$, namely since:

$$\|x - z \leq \|x - y\| + \|y - z\| \qquad \|x - y\| = \|(-1)(x - y)\| = \|y - x\|$$

Since any metric induces a topology, the induced topology by the metric induced by a norm $\|\cdot\|$ is called the *norm topology* on V. A nice consequence of the induced norm topology is that addition and scalar multiplication is continuous. From the axioms of a semi-norm, $\|0\| = 0$. There are norms for which $x \neq 0$ but $\|x\| = 0$, as we'll soon show. Note that the norm $\|\cdot\| : V \to \mathbb{F}$ is evidently continuous in the topology it induces[1]. If $x_n \to x$ in X, then for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|x_n - x\| < \epsilon$. By the reverse triangle inequality:

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| < \epsilon$$

showing that $|\|x_n\| - \|x\||$ get's arbitrarily close, and so $\|x_n\| \to \|x\|$, i.e. $\|\cdot\|$ is continuous. This means that the pre-image of $\{0\}$ of $\|\cdot\|$ is closed. Furthermore, it is a vector-space. Denoting

$$K = \{x \in V \mid \|x\| = 0\}$$

Then if $x, y \in K$, $0 \leq \|x + y\| \leq \|x\| + \|y\| = 0$, and $\|cx\| = |c|\|x\| = |c|0 = 0$. Hence, we may consider V/K. What norm to put on V/K that is natural with respect to the projection map $\pi : V \to V/K$ will be addressed soon.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_1$ are said to be *equivalent* if there exits some $C_1, C_2 > 0$ st

$$C_1\|\cdot\|_1 \leq \|\cdot\|_2 \leq C_2\|\cdot\|_1$$

Equivalent norms define equivalent metrics and the same topology, and so the same Cauchy sequences. Recall that in finite dimensions, all norms are equivalent, meaning there is only one notiont of distance. As we'll see, this is not the case in infinite dimensions.

Since we have a metric, we have a notion of completeness. As is the case when working with analysis, we will require our spaces to be complete:

Definition 1.1.3: Banach Space

Let V be a normed vector space. Then V is a *Banach space* if it is complete with respect to it's induced metric.

Example 1.1: Banach Spaces

1. Every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ is a banach space. In this space, all linear map are continuous.

2. Let K be a compact space, and take the set of all continuous function that vanish at infinity: $C_0(K)$. Then $C_0(K)$ is a Banach space with the operator norm (see definition 1.1.6). We

[1] Note that in the metric space topology, continuous if and only if sequentially continuous

will later show (Banach–Mazur theorem) that all real separable banach spaces are isometric (the isomrophisms in the category of banach spaces) to a subspace of $C_0([0,1])$, giving us a universal way of thinking about Banach spaces. If the space is not separable, if the smallest dense set D of a Banach space X has density of cardinality α, then X is isometric to $C_0([0,1]^\alpha, \mathbb{R})$[a]

3. If X is a topological space, then $B(X,k)$ and $BC(X,k)$ (all bounded functionals and bounded continuous funcitonals) form a Banach space with the uniform norm $\|f\|_u = \sup_{x\in X}|f(x)|$.
4. For p where $1 \leq p \leq \infty$, the space $L^p(X)$ for some measurable space X is a Banach space with the L^p-norm (recall that $L^p(\mu)$ is the set of equivalence classes functions that agree a.e.; if $L^p(\mu)$ represented only functions, then $\|\cdot\|_p$ would be a semi-norm)
5. Two other common examples of Banach spaces we will encounter later will be special spaces of differentiable functions, Banach Algebras, and Hilbert Spaces.

There are also examples of spaces that are not Banach Spaces:

1. Let X be a locally compact Hausdorff space and consider $C_c(X)$ with the ∞-norm. Then it is not a Banach space, not being complete. It is easy to se that $C_c(X)$, the set of continuous bounded function, is a banach space and contains $C_c(X)$, and so it suffices to find the closure of $C_c(X)$ in $C_b(X)$. The closure turns out to be $C_0(X)$.
2. Let $U \subseteq \mathbb{R}^n$ be open and take $C_0(U, \mathbb{C})$. Then This space is *not* a Banch space
3. The space $H(U, \mathbb{C})$ of all holomorphic functions from U to $\mathbb{C}$ is *not* a Banach Space
4. The space $C_c^\infty(\mathbb{R}^n)$ is *not* a Banach space
5. (If you know some PDE's) The space of distributions $\mathcal{D}'$ is *not* a Banach Space

These are all examples of a weaker structure known as a *topological vector space* which we shall cover in section 1.4.

[a] An interesting result using Banach-Mazur's theorem by Luis Rodríguez-Piazza shows that the space of smooth functions with the uniform norm is isometric to the space of nowhere differentiable functions

There is a nice way to check whether a vector space V is a Banach space which requires the following definition (this essentially follows from the fact that continuity if and only if sequential continuity in a metric space and that $\|\cdot\|$ is continuous)

Definition 1.1.4: Absolutely Convergent

Let V be a normed vector space. and $\{x_k\}_k^\infty \subseteq V$ a sequence. Then the sequence is said to *converge absolutely* if

$$\sum_k^\infty \|x_k\| < \infty$$

Lemma 1.1.1: Cauchy and Absolute Convergence

Let V be a normed vector space. Then V is complete if and only if every absolutely convergent series in V converges.

The idea is that $\mathbb{R}$ has an order, making it easier to produce many arguments about sequences.

Proof **:**
Let's first say that X is complete and that $\sum_{n=1}^{\infty} \|x\| < \infty$. To show it converges, we take advantage of completeness and show there is a Cauchy sequence. In particular, let $S_n = \sum_{k=1}^{n} x_k$ be the partial sum. Then since the series is absolutely convergent, there for all $n > m$,

$$\|S_n - S_m\| \leq \sum_{m+1}^{n} \|x_k\| \to 0 \quad \text{as } m, n \to \infty$$

so the sequence $\{S_n\}$ is Cauchy, and so converges, so $\{x_k\}$ converges.

Conversely, let's say an absolutely convergent series converges, and choose any Cauchy sequence $\{x_k\}$. Since it's a Cauchy sequence, choose a subsequence such that

$$\left\|x_{k_{i+1}} - x_{k_i}\right\| < 2^{-i}$$

Take $y_1 = x_{k_1}$ and $y_i = x_{i+1} - x_i$ for $i > 1$ so that $\sum_{i=1}^{n} y_i = x_{k_n}$. Then notice that $\{y_i\}$ is in fact absolutely convergent:

$$\sum_{i=1}^{\infty} \|y_i\| \leq \|y_1\| + \sum_{i=1}^{\infty} 2^{-i} = \|y_1\| + 1 < \infty$$

So $\sum_{i=1}^{\infty} y_i = \lim x_{k_n}$ exists. Since the subsequence of every Cauchy sequence approaches the same "point" (just like convergent sequences), we see that $\{x_n\}$ also converges to the same point, showing that Cauchy sequences are in fact convergent sequences, as we sought to show.

Next, we would want to study which linear functions between two vector-spaces are *continuous*. It will *not* be the case that all linear functions will be continuous. The following builds up to the classification of continuous linear functions:

Definition 1.1.5: Bounded Linear Map

Let $T : X \to Y$ be a linear map between two normed vector space. Then if there exists some C such that for all $x \in X$

$$\|T(x)\| \leq C\|x\|$$

Then T is said to be *bounded*, or is a *bounded linear operator*.

Note how this is different then boundedness of a set (ex. the range is bounded if $\|T(x)\| \leq C$). This definition is would be useless to us since no nonzero linear map would satisfy it. All linear map over finite dimensional vector spaces clearly bounded (can you find the C?), so this concept gets interesting in infinite dimesion

Proposition 1.1.1: Bounded ⇔ Continuous

Let $T : X \to Y$ be a linear map between two normed vector spaces. Then the following are equivalent:

1. T is continuous
2. T is continuous at 0
3. T is bounded

Proof **:**
If T is the zero map, then we're done, so assume $T \neq 0$

(1) implies (2) by definition. For (2) implies (3), Since T is continuous at 0, for for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x - 0\| < \|x\| < \delta$, then $\|T(x) - T(0)\| = \|T(x) - 0\| = \|T(x)\| < \epsilon$. Choosing $\epsilon = 1$, we get that there exists a δ_1 such that

$$\|x\| < \delta_1 \to \|T(x)\| < 1$$

we know want to show there exists a C such that $\|T(y)\| \leq C\|y\|$ for all y in the domain. We will do this by taking advantage of linearity and shrinking element so that we can take advantage of continuity.

Define $x(y) = \frac{\delta_1}{2\|y\|} y$ so that

$$\|x(y)\| = \left\| \frac{\delta_1}{2\|y\|} y \right\| = \frac{\delta_1}{2} \frac{1}{\|y\|} \|y\| = \frac{\delta_1}{2} < \delta_1$$

Therefore, by continuity and linearity:

$$\|T(x(y))\| = \left\| T\left(\frac{\delta_1}{2\|y\|} y \right) \right\| = \frac{\delta_1}{2\|y\|} \|T(y)\| < 1$$

since this is true for all y, we if we re-arrange we have:

$$\|T(y)\| < \frac{2}{\delta_1} \|y\|$$

showing that T is indeed bounded.

Finally, if $\|T(x)\| \leq C\|x\|$, then for all $\epsilon > 0$, choose $\delta = C^{-1}\epsilon$ so that

$$\|T(y) - T(x)\| = \|T(y - x)\| \leq \epsilon$$

completing the proof.

Note that for $T \in B(X, Y)$ to be an isomorphism, we require that T^{-1} also be bounded[2]. If $\|T(x)\| = \|x\|$ for all x, then T is called an *isometry*.

Since the topology is that of norms, then continuous linear functions between normed vector-spaces

[2]We will show using the open mapping theorem that it suffices to show T is continuous and bijective

are the homomorphisms, meaning they preserve the norm (to the degree of boundedness mentioned in the above proposition) and the vector-space structure (to the degree the map is injective). We will denote the set of bounded linear functions (which we will call linear operators) as $B(X,Y)$, $\mathcal{L}(X,Y)$, or $\mathrm{Hom}_k(X,Y)$, where we understand that since X and Y are normed spaces $\mathrm{Hom}_k(X,Y)$ refers to continuous lienar maps.

Like with most fields of mathematics, we are more often more interested in the functions between spaces rather than the spaces itself, for it is function that ultimatly give us information about the structure we are working with. Thus, if we can use the technics we will be developing on $B(X,Y)$, we may gain many interesting results. For this to be the case, we require to introduce a norm on $B(X,Y)$. Naturally, we want that the norm on $B(X,Y)$ to be compatible with X and Y in some way, especially the codomain Y (as we usually investigate the structure of the image of a map). The following norm on $B(X,Y)$ will be shown to be exactly that:

Definition 1.1.6: Operator Norm

$B(X,Y)$ is a normed vector space with norm[a]

$$\begin{aligned}\|T\| &= \sup_{\|x\|=1} \|T(x)\| \\ &= \inf\{C \mid \|T(x)\| \leq C\|x\| \text{ for all } x\} \\ &= \sup_{\|x\|\neq 0} \frac{\|T(x)\|}{\|x\|}\end{aligned}$$

If $\|T(x)\| = \|x\|$, then T is called an *isometry*

[a] the third equality holds if $V \neq \{0\}$

In the spacial case where we have $B(X,k)$, we write V^* and call it the dual space with the *norm topology*. It is useful to establish when is $B(X,Y)$ a Banach space

Proposition 1.1.2: Completness Of $B(X,Y)$

Let X,Y be normed vector-space and $B(X,Y)$ the normed vector-space of all bounded linear functions from X to Y. If Y is complete, so is $B(X,Y)$.

***Proof* :**
Let (T_n) be a cauchy sequence in $B(X,Y)$. For each $x \in X$, we have a cauchy sequence $(T_n(x)) \subseteq Y$, and thus a limit vector which we will denote $T(x)$. This defines a function $T : X \to Y$ which is clearly linear. The inequality

$$\begin{aligned}\|T(x) - T_n(x)\| &= \lim_m \|T_m(x) - T_n(x)\| \\ &\leq (\limsup_m \|T_m - T_n\|)\|x\|\end{aligned}$$

shows that T is bounded and that $T_n \to T$.

Hence, V^* is always a banach space, since $\mathbb{R}$ and $\mathbb{C}$ are complete. Next, we see what happens with common vector-space constructions like products and quotients.

Subspaces, Product spaces, Quotient Spaces

There are 3 equivalent norms we can put on the product of two normed vector-spaces:

1. $\|(x, y)\| = \max(\|x\|, \|y\|)$
2. $\|(x, y)\| = \|x\| + \|y\|$
3. $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$

The key to these being equivalent is that we are only multiplying by a finite number of spaces, a different strategy would have to be implemented for norms on infinite dimensional vector spaces (recall the different ℓ^p-norms put on the set of all sequences). For subspace, it is clear that if $W \subseteq V$ is a subspace of a normed vector space V, then W has an induced norm with respect to V. What is more interesting is if V is a banach space and we want W to be a Banach space. We then require all sequences to converge. Since V is Hausdorff (even metrizable since there is a norm on V), we see that W is complete if W is a *closed subspace* with respect to the topology of V. For quotients V/W, we would like to define a new norm on V/W so that the natural projeciton map $\pi : V \to V/W$ is continuous. This leads us to the following definition:

Proposition 1.1.3: Banach Quotient Space

Let X be a normed vector space and $Y \subseteq X$ a subspace of X. Then $\pi : X \to X/Y$ is the quotient map. This induces the topology on X/Y by defining:

$$\|\pi(x)\| = \|x + Y\| = \inf_{y \in Y} \|x - y\|$$

which is a *seminorm* on X/Y. If Y is closed, then this is in fact a norm, and if furthermer X is Banach, then X/Y is a banach space.

The intuition in my mind for this definition of quotient norm is to "eliminate" any effect of the quotienting space Y on any element of $x + Y$. Take for example $\mathbb{R}^2/\mathbb{R}$. Then the the norm of $\|(x, y) + \mathbb{R}\|$ should really be $|x|$. Furthermore, π is naturally a continuous map since it is norm-decreasing, meaning $\|\pi(x)\| \leq \|x\|$ (simply choose $C = 1$).

Proof **:**
For subadditivity, we take advantage of the definition of infimum: for every $x_1, x_2 \in X$, there exists an $\epsilon > 0$ and elements $y_1, y_2 \in Y$ such that

$$\begin{aligned}\|\pi(x_1)\| + \|\pi(x_2)\| + \epsilon &\geq \|x_1 - y_2\| + \|x_2 - y_2\| \\ &\geq \|x_1 + x_2 - (y_1 + y_2)\| \\ &\geq \|\pi(x_1 + x_2)\|\end{aligned}$$

Since ϵ was arbitrary, we have subaditivity. Homogeneity is immediate since π is linear, and hence we have a seminorm.

If $\|\pi(x)\| = \|x + Y\| = 0$, there is a sequence $(y_n) \subseteq Y$ such that $\|x - y_n\| \to 0$. If Y is closed,

$\lim_n(y_n) = x \in Y$, meaning $\pi(x) = 0$. It follows that X/Y is a normed space if and only if Y is closed in X.

Now, say $(z_n) \subseteq X/Y$ is a cauchy sequence. Then we can find a subsequence (z'_n) were

$$\|z'_{n+1} - z'_n\| < 2^{-n}$$

By induction we can choose $x_n \in X$ such that $\pi(x_n) = z'_n$ and $\|x_{n+1} - x_n\| < 2^{-n}$. If X is banach, (x_n) converges to an element $x \in X$, and since π is "norm decreasing" (the norm the same or smaller), then (z'_n) converges to $\pi(x)$. Hence, (z_n) converges to $\pi(x)$, and so X/Y is complete, as we sought to show.

Proposition 1.1.4: Universal Property of Quotient Maps

If $T \in B(X, Y)$ where X and Y are normed spaces, and if $W \subseteq X$ is a closed subspace of X containing $\ker(T)$, then then there exists an operator $\tilde{T} \in B(X/W, Y)$ defined pointwise as $\tilde{T}(\pi(x)) := T(x)$ such that

$$\|\tilde{T}\| = \|T\|$$

in particular, the following diagram commutes in **Ban**:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & Y \\ X/W \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

Proof **:**
First, by elementary linear algebra we have that $\tilde{T}$ is an operator on X/W, which is a normed space by the earlier proposition. Since

$$\|\tilde{T}(\pi(x))\| = \|T(x)\| = \|T(x - z)\| \leq \|T\|\|x - z\|$$

for every $z \in W$, it follows form the definition of the quotient norm that

$$\|\tilde{T}(\pi(x)) \leq \|T\|\|\pi(x)\|$$

hence, $\|\tilde{T}\| \leq \|T\|$. The reverse inequality $\|\tilde{T}\| \geq \|T\|$ is clear since Q is norm decreasing.

Proposition 1.1.5: Extending Banach Spaces

Let X be a normed space and Y a closed subspace. Then if Y and X/Y are Banach spaces, then X is a Banach space.

Proof **:**
Let $(x_n) \subseteq X$ be a cauchy sequence. Since $\pi : X \to X/Y$ is norm-decreasing, $(\pi(x_n)) = (y_n) \subseteq X/Y$ is a cauchy sequence. Since X/Y is complete, it converges to a point $\overline{x} \in X/Y$. Next, by

the definition of the quotient norm, for every x_n, there exists an n st

$$\|(x - x_n) - y_n\| \leq \frac{1}{n} + \|\pi(x - x_n)\|$$

Then we get:

$$\begin{aligned} \|y_n - y_m\| &= \|(y_n + x - x_n) - (y_m + x - x_m) + x_n - x_m\| \\ &\geq \frac{1}{n}\|\pi(x - x_n)\| + \frac{1}{m}\|\pi(x - x_m)\| + \|x_n + x_m\| \end{aligned}$$

which is clearly cauchy, and hence since Y is banach it is complete and so this sequence converges to y. Finally, it is easy to see that

$$\|x_n - (x + y)\| \leq \|x_n - (x - y_n)\| + \|y - y_n\|$$

showing that $x - n \to x + y$, as we sought to show.

Finite space naturally have none of the problems introduced in infinite dimensions and preserve all of their nice properties even when subspace of infinite dimensinoal vectorspace:

Proposition 1.1.6: Finite Dimensional Normed Space

Every finite-dimensional subspace Y of a normed space X is a Banach space, and consequently is closed in X. Moreover, if $\dim(Y) = n$, every linear isomrophism of $\mathbb{F}^n$ onto Y is a homeomorphism

***Proof* :**
p. 46 Analysis now.

Working with infinite dimensional space dense subsets become of key interest. As a simple example, the smooth bump functions form a dense subset of L^p for all $1 \leq p < \infty$. The following simplifies for us how we may define functions from spaces given a known dense set:

Proposition 1.1.7: Defining Operator using Dense Sets

Let X and Y be banach spaces and X_0 is a dense subspace of X. Then every operator $T_0 \in B(X_0, Y)$ has a unique extension to an operator $T \in B(X, Y)$ and $\|T\| = \|T_0\|$

***Proof* :**
For each $x \in X$, there is a sequence $(x_0) \subseteq X_0$ that converges to x: $x_n \to x$. Since T_0 is a bounded operator ($\|T_0(x_n)\| \leq C\|x_n\|$), it is clear that $(T_0(x_0))$ is a convergent sequence in Y; let's say it converges to $T(x)$. It should be clear that if we choose another sequence converging to x and do this procedure, then we will get the same value of $T(x)$. Thus, the map $x \to T(x)$ is a operator that is linear, extends T_0, and by construciton

$$\|T\| = \|T_0\|$$

an excellent example would be polynomial over a compact set are dense over continuous functions over compact sets, which themselves are dense over all L^p spaces. Hence in the compact case, for real or complex functions, it suffices to define our function over a the polynomials. For the case of locally compact spaces, Note that smooth bump functions are dense in L^p, and since they have compact support a polynomial sequnce uniformally converges to them. Hence, we may define a lot of operators on different real/complex functions spaces over locally compact spaces using polynomals.

For future reference, it will be important to remember that all normed spaces can be completed and that the norm between a normed space X and its completion $\overline{X}$ is isometric:

Proposition 1.1.8: Unique Extension To Banach Space

For each normed space X, there is a Banach space $\overline{X}$, uniquely determiend up to isometric isomoprhism, such that $\overline{X}$ contains X as a dense subspace

***Proof* :**
p. 47 Analysis now

An important concept in any infinite dimensional space is to figure out which sets are compact. For example, you may recall in real analysis that we proved the Stone-Weistrass theorem, or even simpler that in $\mathbb{R}^n$ a set is compact if it is closed and bounded. The following shows that subsets that are "almost finite-dimensional" are compact.

Proposition 1.1.9: Compact Sets In Banach Spaces

Let $K \subseteq V$ be a subset of a Banach Space V. Then the following are equivalent:

1. K is compact
2. K is sequentially compact
3. K is closed and bounded, and for every $\epsilon > 0$, K lies in the ϵ-neighborhood

$$\{x \in V \mid \|x - y\| < \epsilon \text{ for some } y \in W\}$$

of a finite dimensional subspace W of V.

Suppose further that there is a nested sequence $V_1 \subseteq V_2 \subseteq \cdots$ of finite dimensional subspaces of V such that $\bigcup_{n=1}^{\infty} V_n$ is dense. Then the following statement is equivalent to the first three:

4. K is closed and bounded, and for every $\epsilon > 0$, there exits an n such that K lies in the ϵ-neighborhood of V_n

***Proof* :**
exercise

Example 1.2: Compact Sets In Banach Spaces

Let $1 \leq p < \infty$ and take a set $K \subseteq \ell^p(\mathbb{N})$. In order for it to be compact in the norm topology, it needs to be closed, bounded, and also uniformly p^{th}-power integrable at spatial infinity, that is for every $\epsilon > 0$, there exists an $n > 0$ such that

$$\left(\sum_{m>n} |f(m)|^p \right)^{1/p} \leq \epsilon$$

for all $f \in K$. This shows that the moving bump set is not uniformly p^{th} power integrale and thus not a compact subset, even though it's closed and bounded.

This condition doesn't translate to the continuous equivalent $L^p(\mathbb{R})$. We also need some uniform integrability at a "finte scale", which would be described using harmonic analysis tools like Fourier transforms, and hence we simply mention it here without developing it further.

Given the Banach space $\text{End}_k(V)$, it is natural have a second binary operation of composition and ask if it is also continuous (hence bounded). More generally, if $T \in B(X, Y)$, and $S \in L(Y, Z)$, then

$$\|ST(x)\| \leq \|S\|\|T(x)\| \leq \|S\|\|T\|\|x\|$$

and so $ST \in B(X, Z)$. In particular, that makes $B(X, X) = \text{End}_k(X)$ into an algebra, and when X is complete, $B(X, X)$ is a *Banach Algebra.* More generally:

Definition 1.1.7: Banach Algebra

Let X be a normed k-algebra where addition, multiplication, and scalar multiplication are continuous with respect to the induced topology. Then if X is complete, X is a *Banach Algebra.*

For the multiplication to be continuous, it suffices to check that $\|xy\| \leq \|x\|\|y\|$ (can you see why?). If X has a multiplicative unit, it is called *unital.* Not all Banach algebra's have a multiplicative unit, $C_0(X)$ where X is a locally compact (hausdroff) space and $C_0(X)$ is the set of functions from X to $\mathbb{R}$ (or $\mathbb{C}$) that vanish at infinity is *not* unital. On the otherhand, $C_b(X)$, the set of bounded functions with pointwise operations with the supremum norm *is* unital. We will work more with Banach Algebras later.

Exercise 1.1.1

1. Let $A, B \subseteq V$ are two closed subspaces. Show that $A + B$ need not be a closed subspace! However, if one of A or B is finite dimensional, then $A + B$ is closed.

2. Show that the open unit ball in X maps onto (by π) the open unit ball of X/Y, but a similar statement about closed unit balls holds only in special cases.

3. For the categorically inclined, it should be verified that the topology on X/Y is the quotient topology corresponding to the equivalent relation $\sim$ where $x_1 \sim x_2$ if and only if $x_1 - x_2 \in Y$.

1.2 Baire Category Theorem

In this section, we introduce some of the basic tools that allow us to work with functions between Banach spaces. Of keen interest to us is the *open mapping theorem*, the *closed graph theorem*, and the *uniform boundedness theorem*. The open mapping theorem make's it sufficient that a surjective continuous linear operator is open, which in particular means that if it is also bijective it is an isomorphism. The closed graph theorem allows us to show that a linear operator $T : X \to Y$ is continuous if the graph $\Gamma(T)$ is closed, i.e. if all sequences in the graph converge. This will lead to great simplifications in showing a function is continuous. Finally, the uniform boundedness theorem will give a powerful criterion a family of operators is uniformally bounded when their images or each seperately (i.e. "point-wise") bounded.

The underpinning results we need are about dense sets:

Proposition 1.2.1: Baire Category Theorem I

Let (A_n) be a sequence of open dense subset of a complete metric space (X, d). Then $\bigcap_n A_n$ is dense in X

***Proof* :**
Let B_0 be any closed ball in X with radius $r > 0$. Then since A_1 is dense in X and is open, the set $A_1 \cap \text{int}(B_0)$ contains a closed ball B_1 with radius $2^{-1}r$. Since A_2 is an open set and B_1 is a closed ball, we can repeat this with $A_2 \cap \text{int}(B_1)$ aind find a closed ball B_2 with radius $r^{-1}r$. By induction, we get a sequence of closed balls (B_n) i nX such that $B_n \subseteq A_n \cap \text{int}(B_{n-1})$, each with radius $< 2^{-n}r$ for every n. Since X is complete (and Hausdorff, since metric topological are normal), there exists a unique point $\{x\}$ such that

$$\{x\} = \bigcap_n B_n \subseteq B_0 \cap \left(\bigcap A_n\right)$$

This shows that $\bigcap A_n$ intersects every nontrivial ball X, which implies density, completing the proof.

Corollary 1.2.1: Countable union of Empty Interior Sets

The countable union of closed sets with empty interior has empty interior

***Proof* :**
exercise

Proposition 1.2.2: Baire Category Theorem II

Let $T : V \to W$ be a bounded linear operator between two Banach spaces, $T \in B(V, W)$. Then if the image of the unit ball $B_1(0)$ is dense in some ball $B_r(0)$ with $r > 0$ ($T(B_1(0))$ dense in $B_r(0)$), then

$$B_{(1-\epsilon)r}(0) \subseteq T(B_1(0))$$

for every $\epsilon > 0$

In a sense, If $T(B_1(0))$ is dense in some ball $B_r(0)$, then it is the "supremum" of $B_r(0)$

Proof **:**
Let $U = T(B_1(0))$. Since $T(B_1(0))$ is dense in $B_r(0)$, for any y and $\epsilon > 0$, there is a $y_1 \in U$ such that $\|y - y_1\| < \epsilon r$. Consider now ϵU. This ball is dense in $B_{\epsilon r}(0)$. Hence, there exists a y_2 such that $\|(y - y_1) - y_2\| < \epsilon^2 r$. Continuing by inductiong, we get a sequence (y_n) where $y_n \in \epsilon^{n-1} U$ and

$$\left\| y - \sum_{k=1}^{n} y_k \right\| < \epsilon^n r$$

Choose now and $x_n \in V$ such that $T(x_n) = y_n$ and $\|x_n\| \leq \epsilon^{n-1}$. Then $\sum x_N = x \in X$ converges, and by definition $T(x) = y$. Since $\|x\| \leq \sum \epsilon^{n-1} = (1-\epsilon)^{-1}$ [a] we have

$$(1-\epsilon)^{-1} U \supseteq B_r(0)$$

completing the proof.

[a] recall that $1 - b^n = (1-b)(1 + b + b^2 + \cdots + b^{n-1})$

Theorem 1.2.1: Open Mapping Theorem

Let X and Y be Banach spaces and $T \in B(X, Y)$ be a bounded linear operator with $T(X) = Y$. Then T is an open map (i.e. it is a surjective bicontinuous map)

Proof **:**
Since T is surjective:

$$Y = T(X) = \bigcup_n \overline{T(_n(0))}$$

Then by corollary 1.2.1, not all closed sets $\overline{TB_n(0)}$ have an empty interior. Thus, there is an n and a ball $B_\epsilon(y)$ such that

$$B_\epsilon(y) \subseteq \overline{T(B_n(0))}$$

Thus, $T(B_1(0))$ is dense in $B_{n^{-1}\epsilon}(y)$. Since

$$2B_{n^{-1}\epsilon}(0) \subseteq B_{n^{-1}\epsilon}(y) - B_{n^{-1}\epsilon}(y)(???)$$

$T(B_1(0))$ is also dense in $B_{n^{-1}\epsilon}(0)$. By Bare Category Theorem II, $B_\delta(0)$ is contained in $T(B_1(0))$ for every $\delta < n^{-1}\epsilon$. Since every open set $U \subseteq X$ is a union of balls, we conclude from lineartiy of T that $T(U)$ conain a neighborhood[a] around each of its points, thus $T(U)$ is open, completing the proof.

[a] in fact closed balls

Corollary 1.2.2: Bijective And Continuous

Let $T : X \to Y$ be a bijective bounded operator. Then it has a bounded inverse, that is T is in fact a Banach-space Isomorphism

Corollary 1.2.3: Equivalent Banach Spaces

Let X be a vector-space, and a Banach space under two norms $\|\cdot\|_a$ and $\|\cdot\|_b$. Then if $\|\cdot\|_a \leq \alpha\|\cdot\|_b$ for some $\alpha > 0$, then there is a $\beta > 0$ such that $\|\cdot\|_b \leq \beta\|\cdot\|_a$

Proof **:**
$\|\cdot\|_a$ is a surjective bounded operator, and hence open, and so has a boundd inverse

The next powerful result we get is a quick way of proving a function is bounded:

Theorem 1.2.2: Closed Graph Theorem

Let $T : X \to Y$ be an operator between Banach spaces. Then if the graph

$$\Gamma(T) = \{(x, y) \in X \times Y \mid T(x) = y\}$$

is closed in $X \times Y$,then T is bounded

Proof **:**
Using the ∞-norm on $X \times Y$, and assume that $\Gamma(T)$ is closed in $X \times Y$. Then the projection map $\pi_1 : \Gamma(T) \to X$ $\pi(x, T(x)) = x$ is a norm decreasing bijective map, and hence has a bounded inverse π_1^{-1}. The other projection $\pi_2 : \Gamma(T) \to D$ given by $\pi(x, T(x)) = T(x)$ is also norm decreasing giving us π_2^{-1}. Hence, we have:

$$T = \pi_2 \circ \pi^{-1}$$

showing that T is bounded, completing the proof.

Finally, the following result is a way from passing from pointwise boundedness to uniform boundedness, only requiring the completness of our domain.

Theorem 1.2.3: Principle Of Uniform Boundedness

Consider a fimaly $\{T_\Lambda \mid \lambda \in \Lambda\} \subseteq B(X, Y)$ where X and Y are Banach spaces. Then if each set $\{T_\lambda(x) \mid \lambda \in \Lambda,\ x \in X\}$ is bounded in Y, then

$$\{\|T_\lambda\| \mid \lambda \in \Lambda\}$$

is bounded

Proof **:**
Analysis now, p. 54

Corollary 1.2.4: Net Convergence

Let $(T_\lambda)_{\lambda\in\Lambda} \subseteq B(X,Y)$ be a net set $(T_\lambda(x))_{\lambda\in\Lambda}$ is bounded and converges in Y or every $x \in X$. Then there is a $T \in B(X,Y)$ such that $T_\lambda(x) \to T(x)$ for every $x \in X$.

Proof **:**
Analysis now p. 54

If our net is a sequence, then convergence of eac hsequence $(T_n(x))$ for all $x \in X$ automatically implies boundedness. It also suffices in the principle of uniform boundedness to know that each net is bounded and that the net is convergent for a dense set of elements.

1.3 Dual Spaces

Let V be a Banach space over k (k being either $\mathbb{R}$ or $\mathbb{C}$). Since we are in the category of Banach spaces, we'll consider $X := \text{Hom}(V,k)$ to the collection of all continuous linear functionals (i.e. bounded linear functionals). We will start by proving the Hahn-Banach extension theorem, which shows that there exists "enough" linear functions in X^* to make their study intersting (this theorem is the equivalent of the Uryohn's Theorem from topology).

Definition 1.3.1: Minkowski Functional

Let $m : V \to \mathbb{R}$ be a subatitive function $m(x+y) \le m(x)+m(y)$ that is positive homogeneous $m(tx) = tm(x)$ for all $t \ge 0$ in $\mathbb{R}$. Then m is called a *Minkowski functional.*

Note that it is not a semi-norm since homogeneity only needs to hold for positive scalars.

Lemma 1.3.1: Fundamental Lemma

If m is a Minkowski functional on a real vector-space V and φ si a funcitonal on a linear subspace $Y \subseteq X$ that is dominated by m (so $\varphi(y) \le m(y)$ for all $y \in Y$), then there exists a function $\tilde{\varphi}$ on X dominated by M such that $\tilde{\varphi}|_Y = \varphi$

Proof **:**
Analyasis now p. 57. Also in Foland in the section on Linear Functionals.

Theorem 1.3.1: Hahn-Banach Extension theorem

If m is a seminrom on a vector space V and φ is a functional on a subspace $Y \subseteq Y$ such that $|\varphi| \le m$, then there is a functional $\tilde{\varphi}$ on X such that $|\tilde{\varphi}| \le m$ and $\tilde{\varphi}|_Y = \varphi$

Proof **:**
Analysis now p. 57

Corollary 1.3.1: Hahn-Banch Corollary I

For every $x \neq 0$ in a normed space X, there is a $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$

Proof **:**
Define φ on $\mathbb{F}x$ by $\varphi(\alpha x) = \alpha\|x\|$. Then $\|\varphi\| = 1$. Applying Hahn-Banach on $Y = \mathbb{F}x$ and $m = \|\cdot\|$ yeilds the desired result.

Corollary 1.3.2: Hahn-Banach Corollary II

For every closed subspace $Y \subseteq X$ of a normed space X and $x \in X \setminus Y$, there iss a $\varphi \in X^*$ with $\|\varphi\| = 1$, $\varphi|_Y = 0$, and $\varphi(x) = \inf\{\|x-y\| \mid y \in Y\}$

Proof **:**
Apply the first corollary on the normed space X/Y, then note that functionals in $(X/Y)^*$ may be regarded as elements in X^* that annihilate Y

Definition 1.3.2: Annihilator

For a subspace $Y \subseteq X$ of a normed space X, the *annihilator* of Y is defined as:

$$Y^\perp = \{\varphi \in X^* \mid \varphi(y) = 0,\ \forall y \in Y\}$$

Similarly, we define the annihilator of a subspace $Y^* \subseteq X^*$ to be

$$(Y^*)^\perp = \{x \in X \mid \varphi(x) = 0,\ \forall \varphi \in Y^*\}$$

It is immedaite that $Y \subseteq (Y^\perp)^\perp$, and it follows by corollary 1.3.2 that in fact $Y = (Y^\perp)^\perp$. for every norm closed subspace $Y^* \subseteq X^*$.

(More here)

1.4 Weak Topologies and TVS's

(This post seems like an excellent post to follow! link here)

There are many topologies on which the operations are continuous (i.e. + and scalar multiplication is continuous) that do not arrise from norms. The topology on a vector-space induced by a norm is in fact quite strong. In general, such a space (in infinite dimensions) doesn't have many compact set, and so we loose many of the nice results of working on comapct or pre-compact sets. This is naturally unfavourable in analysis, since many arguments rely on properties of sequences converging. Looking at functional analysis as the study of sequences, norms induce a particular definition of convergence: $\|x - x_n\|_V \to 0$. However, there are many other notions of convergence that are weaker that are quite useful, for example:

1. pointwise convergence
2. convergence in measure
3. smooth convergence
4. convergence on compact sets

None of these types of convergence are induced by a norm. We will instead be induced new types of topologies on our vector-spaces. The two types of topologies we will usually impose on V will be known as the *weak topology* and the *weak* topology*. We will define analogous topologies on $\mathrm{Hom}_k(V, W)$ which somewhat confusingly will be named *strong operator topology* and *weak operator topology*.

Definition 1.4.1: Topological Vector Space (TVS)

Let V be a vector-space over $\mathbb{R}$ or $\mathbb{C}$. Then V is a *topological vector space* (TVS) if the operation of addition and scalar multiplication are continuous. A topological vector-space is said to be *locally convex* if there is a base for the topology consisting of convex sets.

It should be verified that the translation map $x \mapsto x_0 + x$ and the dilation map $x \mapsto \lambda x$ are both homeomorphissm for any scalar λ and vector x_0.

Example 1.3: Topological Vector Spaces

1. Let V be a normed vector-space with the topology induced by the norm. Then $\|c \cdot x\| \leq |c|\|x\|$ (in fact $=$) and $\| + (x, y)\| \overset{!}{\leq} 2(\|x\| + \|y\|)$ where $\overset{!}{\leq}$ comes from the triangle inequality and the fact that $\|(x, y)\| = \max(\|x\|, \|y\|)$. Hence, V is a topologicla vector space. If $\|\cdot\|$ was a quasi-norm so that we replace the triangle inequality with $\|x + y\| \leq K(\|x\| + \|y\|)$ then a quasi-normed space V is still a topological vector space.
2. Let V be a semi-normed space so that $\|x\| = 0$ does not necessarily imply $x = 0$. Then V is a topological vector space with the topology induced by the semi-norm. V is Hausdroff if and only if the semi-norm is a norm.
3. Any subbspace $W \subseteq V$ of a topological vector space V is a topological vector space with the subspace topology. Similarly for products and quotients.
4. Let V be a vector-space and $(\mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topologies on V that makes V a topological vector space. Let $\mathcal{T} = \langle \mathcal{T}_\alpha \rangle_{\alpha \in A}$ be the topology generated by $\bigcup_{\alpha \in A} \mathcal{T}_\alpha$ (this is the weakest topology that contains each $\mathcal{T}_\alpha$). Then $(V, \mathcal{T})$ is also a topological vector space. Importantly, a sequence (resp. net) $x_n \to x$ in $\mathcal{T}$ if and only if $x_n \to x$ in $\mathcal{T}_\alpha$ for all $\alpha \in A$ (recall that $x_n \to x$ in a general topology space if for any neighborhood U of x, there exists an N such that for all $n \geq N$, $x_n \in U$). Commonly, we will have $(\mathcal{T}_\alpha)_{i=1}^\infty$ be generated by countable collection of semi-norms. A complete space generated by a countable collection of semi-norms is called a *Fréchet space*.
5. Let $T : V \to W$ be a linear map between two vector spaces. Let V be a topology induced by

a family of semi-nrosm $(\|\cdot\|_{V_\alpha})_{\alpha\in A}$ and W the topology induced by a family of semi-norm $(\|\cdot\|_{W_\beta})_{\beta\in B}$. Then T is continuous if and only if for each $\beta \in B$, there exists a finite subsets $A_\beta \subseteq A$ and a constnat C_β such that

$$\|T(x)\|_{W_\beta} \leq C_\beta \sum_{\alpha\in A_\beta} \|x\|_{V_\alpha}$$

for all $x \in V$.

6. (Pointwise Convergence) Let X be a set and $\mathbb{C}^X$ the collection of all complex valued functions $f : X \to \mathbb{C}$. Then $\mathbb{C}^X$ is a complex-vector-space. For each $x \in X$, define the seminorm:

$$\|f\|_x := |f(x)|$$

Then the topology generated by all of these seminroms is the *topology of pointwise convergence* on $\mathbb{C}^X$ (it is also the product topology on this space). A sequence $(f_n) \subseteq \mathbb{C}^X$ converges to f in this topology if and only if it converges pointwise. Note that if X has more than one point, then none of the semi-norms individually generate a Hausdroff topology, but taken all together the induced topology is Haudsorff.

7. (Uniform Convergence) Let X be a topological space and $C(X)$ the set of complex-valued continuous function $f : X \to \mathbb{C}$. If X is not compact, then in general we would not expect functions in $C(X)$ to be bounded, so the sup-norm will not necessarily make $C(X)$ into a normed vector-space. Nonetheless, we can still define "balls" $B_r(f) \subseteq C(X)$ by:

$$B_r(f) := \left\{ g \in C(X) \;\middle|\; \sup_{x\in X} |f(x) - g(x)| \leq r \right\}$$

these do form a topological structure on $C(X)$, but not a topological vector space structure since scalar multiplication is no longer continuous. Nonetheless, we have that $(f_n) \subseteq C(X)$ converges in this topology to a limit $f \in C(X)$ if and only if f_n converge uniformaly to f. Thus $\sup_{x\in X} |f_n(x) - f(x)|$ is finite for sufficinetly large n and converges to 0 as $n \to \infty$.

8. (Uniform Convergence on Compact sets) Let X and $C(X)$ be as in the above example. For every compact subset $K \subseteq X$, define a seminorm:

$$\|f\|_{C(K)} = \sup_{x\in K} |f(x)|$$

The topology generated by all these seminorms as k rangeds over all compact subsets of X is called the *topology of uniform convergence on compact sets*. It is stronger than the topology of pointwise convergence but weaker than the topology of uniform convergence, and is frequently used in complex analysis. A sequence $(f_n) \subseteq C(X)$ converges to $f \in C(X)$ in this toplogy if and only if f_n converges uniformly to f on each compact set. Is the resulting space a topological vector space?

9. (Smooth Convergence) Let $C^\infty([0,1])$ be the space of smooth functions $f : [0,1] \to \mathbb{C}$. Then we can define the C^k-norm on this space for any non-negative integer k by the formula:

$$\|f\|_{C^k} := \sum_{i=1}^{k} \sup_{x\in[0,1]} |f^{(i)}(x)|$$

where $f^{(i)}$ is the ith derivative of f. The topology generated by all C^k norms $k = 0, 1, 2, \ldots$, is the *smooth topology.* A sequence f_n converges in this topology to a limit f if $f_n^{(i)}$ converges uniformally to $f^{(i)}$ for each $i \geq 0$. Is this a topological vector space?

10. (Convergence in measure) Let $(X, \mathcal{M}, \mu)$ be a measure space and $L(X)$ the set of measurable functions $f : X \to \mathbb{C}$. Then the sets

$$B_{r,\epsilon}(f) := \{g \in L(X) \mid \mu(\{x \mid |f(x) - g(x)| \geq r\} < \epsilon)\}$$

for $f \in L(X)$, $\epsilon, r > 0$ forms the basis for a topology that makes $L(X)$ into a topological vector space. A sequence $(f_n) \subseteq L(X)$ converges to f in this topology if and only if it converges in measure.

11. Let $[0, 1]$ be given the usual Lebesgue measure. Then show that $L^\infty([0, 1])$ cannot be given a topological vector space structure in which a sequence $(f_n) \subseteq L^\infty([0, 1])$ converges to f in this topology if and only if it converges a.e.

12. Let V be an $\mathbb{R}$-vector space. a subset $U \subseteq V$ is called *algebraically open* if

$$\{t \in \mathbb{R} \mid x + tv \in U\}$$

is open for all $x, v \in V$. Then any open set in a topological vector space is algebraically open. Note that there are algebraically open sets that are not open (take any dense subset of S^1, say D, and $\mathbb{R} \setminus \{D\}$.) This larger set of algebraically open sets forms a topology, but this topology does not give the $\mathbb{R}^2$ structure of a topological vector space.

13. Let X be a LCH space. On $\mathbb{C}^X$, the topology of uniform convergence on compact sets is defined by the seminorms $p_K(f) = \sup_{x\in K} |f(x)|$ as K rangers over compact subsets of X. If X is σ-compact and $\{U_n\}$ are as in proposition ref;HERE (folland p. 168), this topology is defined by the seminorms $p_n(f) = \sup_{x\in\overline{U_n}} |f(x)|$. In this case, $\mathbb{C}^X$ is easly seen to be complete, so it is a Fréchet space. By proposition ref:HERE, so is $C(X)$.

14. The space $L^1_{\text{loc}}(\mathbb{R}^n)$ is a Fréchet space with the topology defined by the seminroms $p_k(f) = \int_{|x|\leq k} |f(x)|dx$ (completeness follows from completness of L^1).

15. Here is another useful place where TVS are defined. Let's say we want to study a space in which d/dx is continuous (and hence bounded) on $C^\infty([0, 1])$. However, this is essentially impossible to define the topology through a norm. Indeed, if $f_\lambda(x) = e^{\lambda x}$, then $(d/dx)f_\lambda = \lambda f_\lambda$, and so $\|d/dx\| \geq \lambda$ for all λ, no matter what norm is used on $C^\infty([0, 1])$. There are a few ways to rectify this:

    (a) One can consider differentiation as an unbounded operator form V to W where W is a suitable banace space and X is a dense subpsac eof W (exercise 30 in Folland)

    (b) One can consider differentiation as a bounded linear map from one Banach space V to a different one W such that $V = C^k([0, 10])$ and $W = C^{k-1}([0, 1])$ (execise 9)

    (c) One can consider differentiation as a continuous operator on a locally convex space V whose topology is not given by a norm. It is easy to construct families of seminorms on the space of smooth functions st differentiation becomes continuous almost by definition. For example, the seminorms:

$$p_k(f) = \sup_{0\leq x\leq 1} |f^{(k)}(x)|$$

where $k \in \{0, 1, 2, ...\}$ makes $C^\infty([0,1])$ into a Fréchet space (the completeness is proved in exercise 9), and d/dx is continuous by proposition 5.15, since $p_k(f') = p_{k+1}(f)$. Other examples are in exer exercsie 45 in chatper 9 of Folland.

Proposition 1.4.1: Topological Vector Space And Hausdorff

Let V be a topological vector space. Then V is Hausdroff if and only if $\{0\}$ is closed

Proof **:**
exercise

We may ask what properties of Banach spaces extend to topological vector space's. Some nice properties of Banach spaces include the uniform boundedness principle and the open mapping theorem. These can be extended to the slighty larger class of *Fréchet spaces* that we mentioned in the examples above. We will single them out and talk about them in the next section.

If V is a Topological vector space, we can take V^* to be the set of all continuous functionals from V to the base field $k \in \{\mathbb{R}, \mathbb{C}\}$. Since V has no norm, V^* has no norm topology. We can however still define interesting topologies on it. This topology is dual to a weaker topology we can put on V:

Definition 1.4.2: Weak And Weak* Topologies

Let V be a topological vector space and V^* its dual. THen

1. the *weak topology* on V is the topology generated by the seminorms $\|x\|_\lambda = |\lambda(x)|$ for all $\lambda \in V^*$
2. the *weak* topology* on V^* is the topology generated by the seminorms $\|\lambda\|_x := |\lambda(x)|$ for all $x \in V$

If V is also a normed vector-space, then the weak topology on V is weaker than the norm topology. If V is a normed space, then so is V^* and $(V^*)^*$. the weak* topology on V^* makes V^* into a topological vector space, and is a topology weaker than the weak topology on V^* (which is deifned using the double dual $(V^*)^*$). When V is reflexive ($V \cong V^{**}$), then the weak and weak* topologies on V^* are in fact equivalent.

Definition 1.4.3: Weak Convergence

Let V a vector-space endowed with the weak topology. Then a sequence $(x_n) \subseteq V$ that converges in V with the weak topology is called *weakly convergent* and is said to *weakly converge* to a point x. A sequence weakly converges $x_n \to x$ if and only if

$$\lambda(x_n) \to \lambda(x) \qquad \forall \lambda \in V^*$$

and is usually denoted as $x_n \rightharpoonup x$. Similarly, $\lambda_n \rightharpoonup \lambda$ if $\lambda_n(x) \to \lambda(x)$ for all $x \in V$ (λ_n, as a function of V, converges pointwise to λ)

Proposition 1.4.2: Normed Vector Spaces Weak Topology

Let V be a normed vector-space. Then the weak topology on V and the weak* topology on V^* are both Hausdorff. In particular, we conclude that in the weak and weak* limits, when they exist they are unique

***Proof* :**
Use the Hahn-Banach Theorem

Example 1.4: Norm, Weak, Weak* Convergence
Let $V = c_0(\mathbb{N})$, $V^* = \ell^1(\mathbb{N})$, and $V^{**} = \ell^\infty(\mathbb{N})$. Let $e_1, e_2, \ldots$ be the standard basis of any of these vetor-spaces.

1. The sequence (e_i) converges weakly in V to zero, but not not converge in norm in V
2.

On $B(X, Y)$, we can usually add some more topologies that allow for more sequences to converge:

Definition 1.4.4: Strong And Weak Operator Topology

Let $B(X, Y)$ be a Banach space. Then

1. The strong operator otopology is the topology induced by the seminorms $T \mapsto \|T(x)\|_Y$ for all $x \in X$
2. the *weak operator topology* is the topology induce dby the seminorm $T \mapsto |\lambda(T(x))|$ for all $x \in X$ and $\lambda \in Y^*$

Hence, a sequence $(T_n) \subseteq B(X, Y)$ converges in the strong operator topology to a limit T if and only if $T_n(x) \to T(x)$ strongly in Y for all $x \in X$, and T_n converges also in the weak operator topology. T_n converges to T in the operator norm topology if and only if $T_n(x)$ converges *uniformally* to $T(x)$ on bounded sets. Hence, we see that the weaker operator topology is weaker tha nthe strong operator topology, which in turn is (somewaht confusingly) weaker than the operator norm topology.

We can reprhase the uniform boundedness principle for convergence now as follows:

Proposition 1.4.3: Uniform Boundedness Principle Reformulation

Let $(T_N) \subseteq B(X, Y)$ be a sequence of bounded linear operators from a Banach space X to a normed space Y, and let $T \in B(X, Y)$ be another bounded lienar operator, and let D be a dense subspace of X. Then the following are equivalent:

1. T_n converges in the strong operator topology fo $B(X, Y)$ to T
2. T_n is bounded in the operator nomr (i.e. $\|T_n\|_{op}$ is bounded) and the restriction of T_n to D converges in the strong operator topology of $B(D, Y)$ to the restriction of T to D

One important result is that in the weak operator topology, unit balls are always compact:

Theorem 1.4.1: Alaoglu's Theorem

Let X be a normed space, and B^* the unit ball of X^*. Then B^* is compact in the w^*-topology (the weak operator topology)

Proof **:**
You can check analysis now, or for a cool measure theoertic way look at Folland.

1.4.1 Fréchet Spaces

Most types of convergences are accomplished by a family of semi-norms. For example, convergence on compact sets, pointwise convergnce, uniform convergence, and smooth convergence are all convergence in Fréchet spaces. In analysis, there will be many spaces that will Fréchet spaces, notably when you encounter distributions and require to define a topology on the dual space. Hence, a moment to get a better undersatnding of them is worth it.

Theorem 1.4.2: Semi-Norms Generate TVS

Let $\{p_\alpha\}_{\alpha \in A}$ be a family of norms on a vector space V. If $x \in X$, $\alpha \in A$, and $\epsilon > 0$, let

$$U_{x,\alpha,\epsilon} = \{y \in V \mid p_\alpha(y - x) < \epsilon\}$$

and let $\mathcal{T}$ be the topology generated by the sets $U_{x,\alpha,\epsilon}$. Then:

1. For each $x \in X<$ the inite intersections of the sets $U_{x,\alpha,\epsilon}$ $(\alpha \in A, \epsilon > 0)$ form a neigborhood base at x
2. if $\langle x_i \rangle_{i \in I}$ is a net in V ,then $x_i \to x$ if and only if $p_\alpha(x_i - x) \to 0$ for all $\alpha \in A$.
3. $(V, \mathcal{T})$ is a locally convex topological vector-space

Proof **:**
Folland p. 166

Definition 1.4.5: Fréchet Space

A complet eHausdorff topological vector-space whose topology is defined by a countable family of seminroms is called a *Fréchet space.*

(theorems on how to characterize continuous maps and when a spac is hausdorff, and also if it is hausdorff, it is equivalent to define a translation-invariant metric)

2

Hilbert Spaces

(Another intuition to incorporate: Since Hilbert spaces are self-dual due to Riesz represnetation theorem, they have a very simple duality theory. Think of how C^∞ does is not self-dual, and that's why we have "non-trivial"' distributions, or how $(L^p)^*$ has a very easy isomorphism to L^q. In the case of a hilbert space H, was have $H^* \cong H$, and so any extra information we might try to get form a dual (ex. notion of size, boundedness, oscillation, regulartiy, decay, etc.) can be represented by an element of the Hilbert space! This makes the dual theory easier to study!!)

(Another important property of Hilbert spaces is that we can always find a dense set that is *linear*. That is, any element $x \in H$ is equal to a convegent sequence $x = \sum_i^\infty x_i$. This is in a sense the best type of dense set!)

(Some geometric properties that are lost in Banach spaces is that we may have unit balls with corners, or may have closed convex sets with no elements of minimal norm. There is also no notion of "perpendicularity" and no good notion of a basis, at best having dense sets)

When dealing with Euclidean Geometry, we have many useful conceptions such as the triangle inequality, the notion of perpendicularity, the idea notion of "area" and the translation invariance of said area. We generalized these notion to vector spaces over $\mathbb{R}$ or $\mathbb{C}$, which preservd many of these intuition's in the finite-dimensional case. However, moving up to the infinite dimensional case, these intuitions are no longer necessarily preserved (examples here?). Furthermore, even in the finite dimensional case over $\mathbb{R}$ and $\mathbb{C}$, without a topology that permits the notion of completeness, we cannot do many of the constructions we've bene donig thus far that are dependent upon completion (essentially, evertyhing we have been working with).

The goal of this chapter is to be able to talk about infinite dimensional vector-spaces which still have some geometric notions, as well as some topological notions. Perhaps the most common such example would be functions spaces (some of which we will explore in this section), where many are infinite dimensional. If would be nice if our notion of geometry alos applied to them, and so the goal of this section is to see whether they do.

Note that the notion of a metric and a norm is not sufficiently strong enough to capture the geometrical ideas we will present: the inner product really is necessary to induce these ideas, as we shall soon see.

2.1 Basic definition and properties

Definition 2.1.1: Inner Product Space

Let H be a complex vector space with an inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ where

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle x, z \rangle$
3. $\langle ax, y \rangle = a\langle x, y \rangle$, $a \in \mathbb{C}$
4. $\langle x, x \rangle \in (0, \infty)$ for all nonzero $x \in H$
5. $\langle x, x \rangle = 0$ if and only if $x = 0$

We'll define $\|x\| = \sqrt{\langle, x, x \rangle}$. This is naturally a norm since:

1. $\|x\| = \sqrt{\langle x, x \rangle} = 0$ if and only if $x = 0$
2. $\|ax\| = \sqrt{\langle ax, ax \rangle} = \sqrt{a\overline{a}\langle x, x \rangle} = \sqrt{a^2\langle x, x \rangle} = |a|\sqrt{\langle x, x \rangle} = |a|\|x\|$

Proving the triangle inequality is a bit more difficult: If we square both sides, we it is equivalent to show:

$$\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

expanding out the left hand side, we get:

$$\|x + y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2$$

so we have to show that $\operatorname{Re}\langle x, y \rangle \leq \|x\|\|y\|$. In fact, a bit more is true: $|\langle x, y \rangle| \leq \|x\|\|y\|$. Though we are using this inequality to prove the triangle inequality, notice that this in fact gives us a general bound on the value of the inner product! This inequality is one of the central tools used in the study of Hilbert spaces:

Theorem 2.1.1: Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\|\|y\|$$

Proof :
If $\|y\| = 0$ then $y = 0$, and then $|\langle x, 0\rangle| = 0$, so assume $\|y\| \neq 0$. Then for all $t \in \mathbb{R}$

$$\begin{aligned}
0 &\leq \langle x - ty, x - ty\rangle \\
&= \|x\|^2 + 2t \operatorname{Re}\langle x, y\rangle + t^2\|y\|^2 \\
&= \|x\|^2 + \frac{2\operatorname{Re}\langle x, y\rangle^2}{\|y\|^2} + \frac{\operatorname{Re}\langle x, y\rangle^2}{\|y\|^2} \qquad\qquad t = \frac{\operatorname{Re}\langle x, y\rangle}{\|y\|^2} \\
\Rightarrow |\operatorname{Re}\langle x, y\rangle| &\leq \|x\|\|y\|
\end{aligned}$$

To finish the proof, pick $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that

$$|\operatorname{Re}(\alpha\langle x, y\rangle)| = |\langle x, y\rangle|$$

Notice this does not break the inequality, and so we can repeat the entire process but with α taken into account with t, and so:

$$|\langle x, y\rangle \leq \|x\|\|y\|$$

as we sought to show

With this, the rest of the proof that $\|\cdot\|$ inducd by $\langle\cdot,\cdot\rangle$ is a norm follows

Theorem 2.1.2: Triangle Inequality For Induced Norm

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof :

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&\leq \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2 i
\end{aligned}$$

The Cauchy-Shwartz inequality is the tool we use to bring in geometry as we know it into Hilbert space. For example, in real inner product spaces, we will define the angle between two vectors to be θ where θ satisfies:

$$\langle u, v\rangle = \cos(\theta)\|u\|\|v\|$$

Many other cool properties of the Cauchy-Shwartz inequality can be found here: (commented)

Now, notice we can define a metric by doing $\rho(x, y) = \|x - y\|$:

1. $\rho(x, x) = \|x - x\| = 0$ if and only if $x - x = 0$ so $x = x$
2. $\rho(x, y) = \|x - y\| \geq 0$
3. $\rho(x, y) = \|x - y\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = \rho(x, y) = \rho(y, z)$

Thus, we can define a natural topology on any pre-hilbert space. If that topology is complete, we give it a name:

Definition 2.1.2: Hilbert Space

Let H be an inner product space. Then H is a Hilbert space if it is *complete* with respect to the induced metric.

Example 2.1: Hilbert Spaces

1. $L^2(\mu)$ for a complex measure μ is a Hilbert space with the inner product being:

$$\langle f, g \rangle_{L^2} = \int_X f(x)\overline{g(x)}d\mu(x)$$

each axiom is simple to verify:

2. $\mathbb{R}^n$ is a Hilbert space with the inner product being:

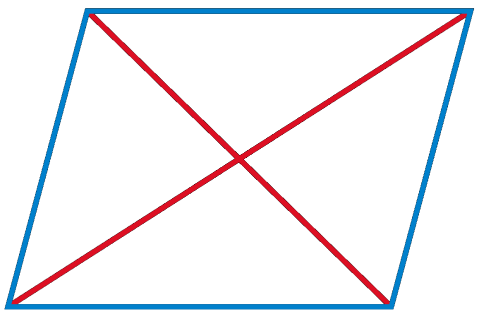
$$\langle x, y \rangle = \sum_i x_i y_i$$

Hilbert spaces are very euclidean in nature, for example:

Theorem 2.1.3: Parralelogram

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Visually, this menas that the "area" of a parralelogram is can be measured in two ways:



Proof **:**

$$\begin{aligned}\|x+y\|^2+\|x-y\|^2 &= \langle x+y,x+y\rangle+\langle x-y,x-y\rangle \\ &= 2\langle x,x\rangle+\langle x,y\rangle-\langle x,y\rangle+\langle y,x\rangle-\langle y,x\rangle+2\langle y,y\rangle \\ &= 2\langle x,x\rangle+2\langle y,y\rangle \\ &= 2\|x\|^2+2\|y\|^2\end{aligned}$$

It turns out that a norm satisfying the parralelogram law will in fact define an inner product. This shows how norms define geometries that are more general than euclidean:

Example 2.2: Exploring Parralelogram Law

1. (The polarization Identity) For any $x, y \in \mathcal{H}$,

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2)$$

Proof. This is simply a matter of computation:

$$\begin{aligned}&\frac{1}{4}(\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2) \\ &= \frac{1}{4}(\langle x+y,x+y\rangle-\langle x-y,x-y\rangle+i\langle x+iy\rangle-i\langle x-iy,x-iy\rangle) \\ &= \frac{1}{4}(\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle-(\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle) \\ &\quad+i(\langle x,x\rangle+\langle x,iy\rangle+\langle iy,x\rangle+\langle iy,iy\rangle-(\langle x,x\rangle-\langle x,iy\rangle-\langle iy,x\rangle+\langle iy,iy\rangle)) \\ &= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle+i(2\langle x,iy\rangle+2\langle iy,x\rangle)) \\ &= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle+i(-i)2\langle x,y\rangle+2ii\langle y,x\rangle) \\ &= \frac{1}{4}(2\langle x,y\rangle+2\langle y,x\rangle+2\langle x,y\rangle-2\langle y,x\rangle) \\ &= \frac{1}{4}4\langle x,y\rangle \\ &= \langle x,y\rangle\end{aligned}$$

□

2. If $\mathcal{H}'$ is another Hilbert space, a linear map from $\mathcal{H}$ to $\mathcal{H}'$ is unitary if and only if it is isometric and surjective

Proof. Let U be unitary so that $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$ for any $x, y \in \mathcal{H}$. Letting $x = y$ gives us $\|Ux\|_2 = \|x\|_1$, giving us that it's an isometry. Since U is invertible, it is surjective, and so U is a surjective isometry.

Conversely, let U be a surjective isometry. We need to show its invertible (it suffices to show it's injective), an isometry, and that U and U^{-1} are bounded. Clearly, U is bounded since

$\|U(x)\| \leq 1\|x\|$. For injectivity, let $Ux = Uy$, so $Ux - Uy = 0$. Taking the norm on both sides, we get:

$$0 = \|0\|_2 = \|Ux - Uy\|_2 = \|U(x-y)\|_2 = \|x-y\|_1$$

where the last equality is due to U being an isometry. Since $\|z\| = 0$ if and only if $z = 0$, we have $x - y = 0$, i.e. $x = y$, proving that U is injective. Along with surjectivity, that makes U invertible. With this, we can also show that U^{-1} is bounded: for any $y \in \mathcal{H}'$, there exists an $x \in \mathcal{H}$ such that $U(x) = y$. Thus:

$$\|U^{-1}(y)\| = \|x\| \stackrel{!}{=} \|U(x)\| = \|y\|$$

where $\stackrel{!}{=}$ comes from U being an isometry, showing that $\|U^{-1}(y)\| \leq 1 \cdot \|y\|$, making it also bounded.

Finally, to show that $\langle Ux, Uy\rangle_2 = \langle x, y\rangle_1$, we will use part (a):

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \\
&\stackrel{!}{=} \frac{1}{4}(\|U(x+y)\|^2 - \|U(x-y)\|^2 + i\|U(x+iy)\|^2 - i\|U(x-iy)\|^2) \\
&= \frac{1}{4}(\|U(x)+U(y)\|^2 - \|U(x)-U(y)\|^2 + i\|U(x)+iU(y))\|^2 - i\|U(x)-iU(y)\|^2) \\
&= \langle U(x), U(y)\rangle
\end{aligned}$$

where $\stackrel{!}{=}$ comes from U being an isometry. Thus:

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

As we sought to show. □

Proposition 2.1.1: Convergence In Inner Product

Let $x_n \to x$ and $y_n \to y$. Then $\langle x_n, y_n\rangle \to \langle x, y\rangle$

***Proof* :**

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle| \\
&= \|x_n - x\|\|y_n\| + \|x\|\|y_n\|
\end{aligned}$$

which tends to zero since $x_n, y_n \to 0$.

When it comes to the right notion of morphism for Hilbert spaces, this is actually not as straight forward as it may seem. To start with a special case, it is straightforward to imagine an isomorphism:

Definition 2.1.3: Unitary Map

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, and let $U : \mathcal{H}_1 \to \mathcal{H}_2$. Then U is called a *unitary map* if if it is an invertible linear map that perserves inner products:

$$\langle Ux, Uy \rangle_2 = \langle x, y \rangle_1$$

The appropriate notion of a morphism that matches this is not so easy. We may result to the fact that all Hilbert spaces are banach spaces and use continuous linear operators, however such operators do not capture the inner product structure (it cannot be recovered using this information). There is a way to recover this information if we instead upgrade our category with an involution, this is known as a dagger category:

$$(-)^\dagger : \mathrm{Hom}(a, b) \to \mathrm{Hom}(b, a)$$

this is a morphism between homsets that satisfy certain propertie given by adjoint (which we'll see in the next section). Then we recover an inner product and unitary maps. We won't go down the road of looking at dagger categories, but it is insightful to think about what information is useful in these categories.

Note that by taking $x = y$, we get that $\|x\|_1 = \|Ux\|_2$, i.e. it is an isometry. In fact, the converse is true too: every surjective isometry is unitary. We can define the operator norm on $\|T\|$ differently using the inner product:

$$\|T\| = \sup_{\|x\|=1,\ \|y\|=1} |\langle T(x), y \rangle|$$

the two are shown to be equivalent whith not too much work:

Proof. For $\geq$, using Cauchy-Swartz we get:

$$\sup_{\|x\|=1,\ \|y\|=1} |\langle T(x), y \rangle| \leq \|T(x)\| \leq \|T\|$$

For the other direction

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|T(x)\| \\ &= \sup_{\|x\|=1,\ T(x)\neq 0} \frac{\langle T(x), T(x) \rangle}{\|T(x)\|} \\ &\leq \sup_{\|x\|=\|y\|=1} |\langle T(x), y \rangle| \end{aligned}$$

□

Subspaces

Just like for Banach spaces, close spaces of hilbert spaces are hilbert spaces. As we saw in section 1.3, we often want to separate an operator by how it *annihilates* a subspace. For Hilbert spaces, such subspaces are really easy to find:

Definition 2.1.4: Orthogonal

We say that x is *orthogonal* to y if $\langle x, y \rangle = 0$. We will write $x \perp y$

If two vectors are orthogonal, the triangle inequality becomes instant:

Theorem 2.1.4: Pythagoras

$$x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

***Proof* :**

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2$$

We might wonder if we take $E \subseteq H$, can we find all vectors in H that are somehow "dual" to E? Define it's *orthogonal compliment* to be:

$$E^\perp = \{x \in H \mid \langle x, y\rangle = 0 \ \forall y \in E\}$$

Notice that E is *always* a closed subspace:

Proof. The fact it's a subspace follows from linearity of the second component.t

for closed, suppose there is a sequecne $\{x_n\} \subseteq E^\perp$ where $x_n \to x \in H$. Then:

$$|\langle x, y\rangle| \leq |\langle x - x_n, y\rangle| \leq \|x - x_n\|\|y\| \to 0 \quad \text{as } n \to \infty$$

□

We can also find closed (and not closed) subspace of hilbert spaces:

Example 2.3: Subspaces Examples

1. If $\mathcal{H}$ is a finite-dimensional Hilbert space, then all subspace are closed
2. If $\mathcal{H}$ is infinite-dimensional, consider $\mathcal{H} = L^1([0,1])$ with the L^2 inner product. and take $P([0,1]) \subseteq L^1([0,1])$ to be the subspace of all polynomial functions. This is is certainly not equal to $L^1([0,1])$, however polynomial functions are dense in $L^1([0,1])$ (they can arbitraily approximate continuous bump characteristic functions, and hence simple functions). However, their closure is $L^1([0,1])$, showing that $P([0,1])$ cannot be closed.
3. Assume $E \subseteq \mathcal{H}$ is an open subspace. Then since $0 \in E$, there exists a $r > 0$ such that

   $$B_r(0) \subseteq E$$

   However, for any nonzero element $x \in \mathcal{H}$,

   $$\frac{r}{2\|x\|}x \in B_r(0) \subseteq E$$

   Since E is closed under scalar multiplication, this shows that $x \in E$, so $\mathcal{H} \subseteq E$, so $\mathcal{H} = E$, showing that the only open subspace of $\mathcal{H}$ is $\mathcal{H}$ itself.

Thus, we are interested in closed subspace since are closed under limits, meaning we can use our tools from analysis to get the following decomposition result:

Theorem 2.1.5: Hilbert Space Decomposition

Let $E \subseteq \mathcal{H}$ be a closed subspace. Then:

$$\mathcal{H} = E \oplus E^{\perp}$$

In particular, for any $x \in H$, there is a unique $y \in E$ and $z \in E^{\perp}$ such that $x = y + z$.

Proof **:**
Let $x \in \mathcal{H}$, and let $\delta = \inf\{\|x - y\| = \rho(x, y) \mid y \in E\}$ (which exists by completeness of $\mathbb{R}^n$ and $\mathcal{H}$). let $\{y_n\} \subseteq E$ be a sequence such that $\|x - y_n\| \to \delta$. We will show that $\{y_n\}$ is a cauchy sequence, and then find a limit point for it. By the parralelogram law:

$$2(\|x - y_n\| + \|x - y_m\|) = \|y_n - y_m\|^2 + \|(y_n + y_n) - 2x\|^2$$

Since $y_n + y_m \in E$, for sufficinetly large N so that $n, m > N$:

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\|\frac{1}{2}(y_n + y_n) - x\|^2 \\ &\leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\delta^2\end{aligned}$$

and as $n, m \to \infty$, this sequence goes to $4\delta^2 - 4\delta^2 = 0$, and so $\|y_n - y_m\|\rho(y_n, y_m) \to 0$, meaning $\{y_i\}_i$ is a cauchy sequence. Take $y \lim y_n$. Then since E is closed, $y \in E$. By construction $\|x - y\| = \delta$. Now let $z = x - y$.

I claim that $z \in E^{\perp}$, that is $\langle z, u\rangle = 0$ for all $u \in E$. So, consider $\langle u, z\rangle$ for any $u \in E$. If $\langle u, z\rangle$ is complex, multiply it by a [complex] unit-length scalar to make it real ($\langle u, z\rangle = 0$ if and only if $a\langle u, z\rangle = 0$). With the fact that it's real, we will do a sneaky trick where we take advantage of optimization of differentiable functions. In particular, let

$$f : \mathbb{R} \to \mathbb{R} \quad f(t) = \|z + tu\|^2 = \|z\|^2 + 2t\langle z, u\rangle + \|u\|^2$$

f is certainly a differentiable function, being a polynomial with a square-root on $\mathbb{R}$. Furthermore, $z = x - y$ is the minimal value of $\|x - y\|$, and so the minimal value of $\|x - y\|^2$. It follows that $z + ut$ is not minimal, and so this equation is has a minimum when $t = 0$. Thus, we get $0 = f'(0) = 2\langle u, z\rangle$, implying that $\langle u, z\rangle = 0$. Since u was arbitrary, we get that $z \in E^{\perp}$.

To show that z is the minimal distance from x, let's say there was another $z' \in E^{\perp}$ that satisfied the equation. Then by the Pythagorean theorem:

$$\begin{aligned}\|x - z'\| &= \|(x - z) + (z - z')\|^2 \\ &= \|x - z\|^2 + \|z - z'\|^2 && x - y \in E, z - z' \in E^{\perp} \\ &\geq \|x - z\|^2\end{aligned}$$

So equality holds if and only if $z = z'$, meaning z is of minimal distance to x. A similar reasoning shows the same for y.

To show uniqueness, let's say $y + z = x = y' + z'$. Then $y - y' = z - z' \in E \cap E^{\perp} = \{0\}$, so $y - y' = z - z' = 0$, which can be used to show that $y = y'$ and $z = z'$, completing the proof.

Corollary 2.1.1: Double Dual Property

For each subset $Y \subseteq H$, the smallest closed subspace of H containing Y is $(Y^{\perp})^{\perp}$. In particular, if Y is a subspace of H, then $\overline{X} = (X^{\perp})^{\perp}$

Proof **:**
exercise

We said that orthogonal vector are somehow dual to each other. However, there is the more classical notion of "dual" vector spaces. Let $\mathcal{H}^*$ be set of all linear functions from $\mathcal{H}$ to the underlying field (usually $\mathbb{C}$, but sometimes $\mathbb{R}$); such linear functions are called functionals. It turns out that all linear functionals can be written in the form $f_y(x) = \langle x, y\rangle$, and we can define a norm on $\mathbb{H}^*$ such that $\|f_y\| = \|y\|$. Thus, $y \mapsto f_y$ will define a conjugate-linear isometry from $\mathcal{H}$ to $\mathcal{H}^*$. The following theorem shows tha this map is surjective:

Theorem 2.1.6: Natural Isomorphism To $\mathcal{H}^*$ (Riesz Representation)

Let $f \in \mathcal{H}^*$. Then there exists a $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$ for all $x \in X$.

Proof **:**
It easy easy to check uniqueness: if $\langle x, y\rangle = \langle x, y'\rangle$ for all x, then let $x = y - y'$. Manipulating the equation, we get that $\|y - y'\| = 0$, so $y = y'$.

If f is the zero functional, then take $y = 0$. If otherwise, then let $\mathcal{M} = \{x \mid f(x) = 0\}$ be the kernel of f. It is clearly a subspace, and any squence will also necesarily converge within $\mathcal{M}$, and so it is also closed. Furthermore, $\mathcal{M}$ is a proper subspace, since $f \neq 0$. Thus, $\mathcal{M}^{\perp} \neq \{0\}$ by the previous proof.

Now, pick any $z \in \mathcal{M}^{\perp}$ with $\|z\| = 1$. Then if we set $u = f(x)z - f(z)x$, we have that $u \in \mathcal{M}$. Hence:

$$0 = \langle u, z\rangle = f(x)\|z\|^2 - f(z)\langle x, z\rangle = f(x) - \langle x, \overline{f(z)}z\rangle$$

re-arrangine and setting $y = \overline{f(z)}z$, we get

$$f(x) = \langle y, z\rangle$$

as we sought to show.

The uniqueness makes this map to be injective, and by linearity of f and the first component of the inner product makes it is an isomorphism. Furthermore, since $\|f_y\| = \|y\|$, limits are also preserved, and so this in fact defines an isomorphism between $\mathcal{H} \cong \mathcal{H}^*$. Notice how this differs from vector-spaces where there is no natural isomorphism between V and V^* (there is one between V and V^{**}).

As we've done with banach spaces, we should define the topology on the dual of H. By the Riesz representation theorem, we see that any $f \in H^*$ is simply $\langle \cdot, y\rangle$ for appropriate $y \in H$. Hence, the weak topology on H is the initial topology corresponding to the functionals $x \mapsto \langle x, y\rangle$.

Note that it is important that the space is *complete*. The following shows what can happen otherwise:

Example 2.4: Functional Without Riesz Representation

Let $V = \mathbb{C}[x]$ where we treat any $f \in \mathbb{C}[x]$ as a function $f : [0,1] \to \mathbb{C}$. Then:

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g}(t)dt$$

Note that V is not complete: Letting $V \subseteq C([0,1])$ we see there is a Cauchy sequence in V that converges to $|x - 0.5|$ in $C([0,1])$. Let's say that Riesz Representation theorem still holds, so that for $L_z \in V^*$ where $L(f) = f(z)$, there exists a $g \in V$ such that

$$L(f) = \langle f, g \rangle = \int_0^1 f(t)\overline{g}(t)dt$$

Then

$$f(z) = \int_0^1 f(t)\overline{g}(t)dt$$

For any $f \in V$. Define $h(z) = x - z$ so that $hf \in V$ and $(hf)(z) = h(z)f(z) = 0$. Hence:

$$0 = \int_0^1 h(t)f(t)\overline{g(t)}dt$$

If $f = \overline{h}g \in V$ then

$$0 = \int_0^1 |h(t)|^2 ||g(t)|^2 dt$$

since $h \neq 0$, it must be that $g = 0$ a.e. But then $\langle f, g \rangle = 0$ a.e., so $L_z(f) = 0$ almost everywhere. But that's a contradiction, as certianly L_z is not 0 a.e.

However, notice that if we take the completion, we will have enough elements so that L_z does indeed have a Riesz Representation.

Orthogonal Basis

In this section, we will show how Hilbert spaces have much nicer dense sets than Banach spaces. In particular, we shall see that we will be able to upgrade the notion of a basis more readily to Hilbert spaces. Since we are studying infinite dimensional vector-spaces, we often care more about "countable linear combination" rather than "finite linear combination" as we have for vector-spaces. To distinguish the two, we will call *Hamel basis* a set of linearly independent elements for which all *finite linear combinations* can represent every element of the vector space. We will call a *Schauder basis* a set of linearly independent elements for which all *countable inear combinations* can represent every element of the vector space. If every element in a Schauder basis has norm 1 and is orthogonal, we call it an *orthonormal basis*. Since we have an inner-product that is complete, it would be nicest if this vector-space has an orthonormal basis. We strive to this end in the following build-up:

Theorem 2.1.7: Bessel's Inequality

If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$:

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

As a consequence, $\{\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$ is countable.

***Proof* :**
It suffices to show that $\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$ for any finite subset $F \subseteq A$, since if it fails for some countable amount, it will fail for some finite amount. Then we see that:

$$\begin{aligned}
0 &\leq \left\| x - \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\| \\
&= \|x\|^2 - 2 \operatorname{Re} \left\langle x, \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\rangle + \left\| \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\| \\
&= \|x\|^2 - 2 \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 + \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 && \text{Pythagorean Theorem} \\
&= \|x\|^2 - \sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2
\end{aligned}$$

and manipulating the final equation we get:

$$\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

as we sought to show

This in particular is important for uncountable sets. In the following we establish when equality holds, and thus when an orthonormal set is an orthonormal basis

Theorem 2.1.8: Orthonormal Basis Condition

Let $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal set in $\mathcal{H}$. Then the following are equivlent:

1. (Completenss) If $\langle x, u_\alpha \rangle = 0$ for all α, then $x = 0$
2. (Parseval's Identity) $\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2$ for al l$x \in \mathcal{H}$
3. For each $x \in \mathcal{H}$, $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$ where the right hand side sum has only countably many nonzero terms and converges in the norm topology no matter the order of the terms (so converges absolutely?)

Proof **:**
p.175 Folland

Definition 2.1.5: Orthonormal Basis

Let $\{u_\alpha\}_{\alpha\in A}$ be an orthonormal set. Then if the set satisfies any of the 3 conditions in theorem 2.1.8, it is called an *orthonormal basis*.

Theorem 2.1.9: Hilbert Space Basis

Every Hilbert space has a basis

Proof **:**
Application of Zorn's lemma: The collection of ordered orthonormal sets ordered by inclusion has a maximal element, and maximality is equivalent to the first property of theorem 2.1.8.

Now that we have established a basis for Hilbert spaces, we will seperate them into 3 categories: Those with finite, countable, and uncountable basis. In most cases, we will be working with countable basis due to the following property:

Proposition 2.1.2: Seperable Hilbert Spaces

A Hilbert space $\mathcal{H}$ is seperable if and only if it has a countable orthonormal basis, in which case every orthonormal basis for $\mathcal{H}$ is countable.

Here, we are using the word seperable to mean countable dense subset.

Proof **:**
p. 176

With the definition of an isomorphism, we can see that all Hilbert spaces are isomorphic to $\ell^2(A)$ given basis A:

Theorem 2.1.10: Classifying Hilbert Spaces

Let $\{e_\alpha\}_{\alpha\in A}$ be an orthonormal basis for $\mathcal{H}$. Then the correspondence

$$x \mapsto \hat{x} \qquad \hat{x}(\alpha) = \langle x, u_\alpha \rangle$$

is a unitary map from $\mathcal{H}$ to $\ell^2(A)$.

Proof **:**
We'll show it's a surjective isometry, which is equivalent to it being unitary.

First off, the map $x \to \hat{x}$ is linear since the inner product is linear in each component, and it is an isometry by Parseval identity: $\|x\|^2 = \sum |\hat{x}(x)|^2$.

For surjectivity, let $f \in \ell^2(A)$, so $\sum_{\alpha \in A} |f(\alpha)^2| < \infty$, so the Pythagorean shows that all partial sums of the series $\sum f(\alpha)u_\alpha$ are Cauchy [a], so $x = \sum_\alpha f(\alpha)u_\alpha$ exists in $\mathcal{H}$ and $\hat{x} = f$.

[a] Note that only countably many terms are nonzero

2.2 Operators on Hilbert Spaces

In this section, we try to generalize many notion's we've seen before, including:

1. adjoints, self-adjoint, and normal operators
2. sign and absolute value and when can we decompose a function into this form
3. diagonalizable
4. polar decomposition (think of how for every $z \in \mathbb{C}$, $z = \arg(z)|z|$)

Adjoints

Hilbert spaces have very simple dual structure, since $\mathbb{H} \cong \mathbb{H}^*$ in a very natural way with the inner product of $\mathbb{H}$. Thus, the problem of finding the structure of dual spaces is fully finished for hilbert spaces. Due to this very simple dual structure, we will be able to properly define an "opposite" operator given any operator $T : \mathbb{H}_1 \to \mathbb{H}_2$. In essense, there is an interesting symmetry in the structure of any map: Given a map $T : \mathbb{H}_1 \to \mathbb{H}_2$ between two hilbert spaces, there is a natural dual map $T^* : \mathbb{H}_2^* \to \mathbb{H}_1^*$. Hence, we have

$$(-)^* : \mathrm{Hom}(\mathbb{H}_1, \mathbb{H}_2) \to \mathrm{Hom}(\mathbb{H}_2^*, \mathbb{H}_1^*)$$

and since in Hilbert spaces $\mathbb{H}_i \cong \mathbb{H}_i^*$, we may envision this map as "flipping" the direction of arrows in a natural way:

$$(-)^* : \mathrm{Hom}(\mathbb{H}_1, \mathbb{H}_2) \to \mathrm{Hom}(\mathbb{H}_2, \mathbb{H}_1)$$

Not only does this map exist, but it is an isometric isomrophism (in **Ban**)! We will start with the special case of $\mathrm{Hom}(\mathbb{H}, k) = \mathbb{H}^*$. We will find that $(-)^*$ is a map

$$(-)^* : \mathrm{Hom}(\mathbb{H}, \mathbb{H}) \to \mathrm{Hom}(\mathbb{H}, \mathbb{H})$$

The following shows how we can define such a dual and some consequences:

Lemma 2.2.1: Endomorphism and And Inner Products

Let $\mathbb{H}$ be a complex hilbert space. Then there is a bijective isometric correspondence between $\mathrm{End}_k(\mathbb{H})$ and bounded sesquilinear forms on $\mathbb{H}$ given by tne map $T \to B_T$ where

$$B_T(x, y) = \langle x, T(y) \rangle$$

***Proof* :**
Let $T \in \text{End}_k(\mathbb{H})$. Then certainly B_T is a sequilinear form on $\mathbb{H}$ bounded by $\|T\|$ since

$$\begin{aligned}\|B_T\| &= \sup\{|B_T(x,y)| \mid \|x\| \leq 1,\ \|y\| \leq 1\} \\ &= \sup\{|\langle x, T(y)\rangle| \mid \|x\| \leq 1, \|y\| \leq 1\} \\ &\leq \|T\|\end{aligned}$$

On the other hand

$$\begin{aligned}\|T(x)\|^2 &= \langle T(x), T(x)\rangle \\ &= B_T(T(x), x) \\ &\leq \|B_T\|\|T(x)\|\|x\| \\ &\leq \|B_T\|\|T\|\|x\|^2\end{aligned}$$

showing that $\|T\| \leq \|B_T\|$, and so $\|T\| = \|B_T\|$.

Theorem 2.2.1: Existence Of Adjoint

Let $T \in \text{End}_k(\mathbb{H})$. Then there exists a unique T^* in $\text{End}_k(\mathbb{H})$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for all $x, y \in \mathbb{H}$. The map $T \mapsto T^*$ is conjugate linear, contravariant with respect to multiplication, an isometry of $\text{End}_k(\mathbb{H})$ onto itself, an involution, and satisfies

$$\|T^*T\| = \|T\|^2$$

***Proof* :**
Fix $T \in \text{End}_k(\mathbb{H})$, and let $B_T(x,y)$ be a continuous sesquilinear form defined as $(x,y) \mapsto \langle T(x), y\rangle$. Then by lemma 2.2.1, there exists a $T^* \in \text{End}_k(\mathbb{H})$ such that $T^* \to B_T$. In particular

$$B_T(x,y) = \langle x, T^*(y)\rangle$$

Thus,

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

giving us existence of such an operator. We now establish all the properties named in the theorem. By conjugate symmetry

$$\begin{aligned}\langle x, T^{**}(y)\rangle &= \langle T^*(x), y\rangle \\ &= \overline{\langle y, T^*(x)\rangle} \\ &= \overline{\langle T(y), x\rangle} \\ &= \langle x, T(y)\rangle\end{aligned}$$

Thus, T^{**} and T correspond to the same sesquilienar form, thus $T^{**} = T$. This also shows that $T \mapsto T^*$ is an involution. Furthermore:

$$\langle x(ST)^*y\rangle = \langle S(T(x)), y\rangle = \langle T(x), S^*(y)\rangle = \langle x, T^*(S^*(y))\rangle$$

thus $(S \circ T)^* = T^* \circ S^*$ showing $(-)^*$ is a contravariant endofunctor. Since the operator norm is submiltiplicative, we have $\|T^*T\| \leq \|T\|\|T^*\|$. On the other hand, applying $\angle T(x), y\rangle = \langle x, T^*(y)\rangle$ to T^* we get

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*(T(x)), x\rangle = \|T^*T\|\|x\|^2$$

showing that $\|T\|^2 \leq \|T^*T$
. Combining these we get $\|T\| \leq \|T^*\|$, hence $\|T\| = \|T^*\|$ (since $T^{**} = T$). This finishes the proof.

Note that if $\mathbb{H}$ is a finite dimensional complex Hilbert space, then each operator T has a matrix representation, say $(\alpha_{ij)})$. Then $T^* = (\overline{\alpha_{ji}})$, i.e. $T^* = \overline{T^t}$ where T^t is the transpose given the matrix representation.

Definition 2.2.1: Adjoint And Self-Adjoint

Let $T \in B(\mathbb{H})$ be a continuous linear operator between Hilbert spaces. Then T^* is called the *adjoint* of T. If $T = T^*$, then T is said to be *self adjoint* (or *Hermitian*).

Proposition 2.2.1: Adjoint Properties I

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then

$$\ker(T^*) = (T(\mathbb{H}))^\perp$$

Proof **:**
Since $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ we have $y \in \ker(T^*)$ when $y \in (T(\mathbb{H}))^\perp$. Conversely, if $y \in (T(\mathbb{H}))^\perp$, then $T^*(y) \in \mathbb{H}^\perp = \{0\}$.

For ease of reference, we shall list some of the results we have proven in the theorem:

Proposition 2.2.2: Adjoint Properties II

Let $T, S \in B(\mathbb{H})$. Then:

1. $(T + S)^* = T^* + S^*$
2. $(cT)^* = \overline{c}T^*$
3. $(TS)^* = S^*T^*$
4. $(T^*)^* = T$
5. $I^* = I$

Proof **:**
If you're not convinced of any of these, double check the theorem or try proving it again.

Proposition 2.2.3: Adjoint Properties III

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then the following are equivalent:

1. T is invertible (so $T^{-1} \in \mathrm{End}_k(\mathbb{H})$)
2. T^* is invertible
3. T and T^* are bounded away from zero.
4. T and T^* are injective, and $T(\mathbb{H})$ is closed
5. T is injective and $T(\mathbb{H}) = \mathbb{H}$

Proof **:**
Since T is invertible, $T^{-1}T = TT^{-1} = I$, and so $(T^{-1})^*$ is the inverse of T^*, giving (1) $\Leftrightarrow$ (2).

For (1) $\Rightarrow$ (3), we say that an operator F is bounded away from zero if

$$\|x\| \leq C\|F(x)\|$$

for some constant C. In our case:

$$\|x\| = \|T^{-1} \circ T(x)\| \leq \|T^{-1}\|\|T(x)\|$$

Since (1) $\Leftrightarrow$ (2), the same holds for T^*.

For (3) $\Rightarrow$ (4), We have $\|T(x)\| \geq \epsilon\|x\|$ and $\|T^*(x)\| \geq \epsilon\|x\|$ for some $\epsilon > 0$ and all $x \in \mathbb{H}$. Thus certainly both T and T^* are injective: if $T(x) = T(y)$ then

$$0 = \|T(x) - T(y)\| = \|T(x-y)\| \geq \epsilon\|x-y\|$$

hence $\|x-y\| = 0$, so $x = y$. Furthermore, $\|T(x) - T(y)\| \geq \|x-y\|$ imples that $T(\mathbb{H})$ is complete, hence a closed subspace of $\mathbb{H}$.

For (4) $\Rightarrow$ (5), by corollary 2.1.1 ($\overline{Y} = (Y^\perp)^\perp$) and proposition 2.2.1 we get

$$\overline{T(\mathbb{H})} = (T(\mathbb{H})^\perp)^\perp = (\ker(T^*))^\perp = \{0\}^\perp = \mathbb{H}$$

where the last equality come from T^* bein injective.

Finally, for (5) $\rightarrow$ (1), we use the open mapping theorem.

This allows us to generlize this result to the followin

Proposition 2.2.4: Inversing Maps

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be hilbert spaces and consider an operator $T \in \mathrm{Hom}_k(\mathbb{H}_1, \mathbb{H}_2)$. Then there exists a unique operator $T^* \in B(\mathbb{H}_2, \mathbb{H}_1)$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

show that the map $T \to T^*$ is conjugate linear isometry of $\mathrm{Hom}_k(\mathbb{H}_1, \mathbb{H}_2)$ to $\mathrm{Hom}_k(\mathbb{H}_2, \mathbb{H}_1)$ and satisfies

$$\|T^*T\| = \|T^2 = \|TT^*\|$$

Proof **:**
Let

$$\hat{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in \mathrm{End}_k(\mathbb{H}_1 \oplus \mathbb{H}_2)$$

By theorem 2.2.1, there is a unique function $\hat{T}^*$ such that

$$\langle \widehat{T}(x_1, x_2), (y_1, y_2)\rangle = \langle (x_1, x_2), \widehat{T}^*(y_1, y_2)\rangle$$

By definition of $\widehat{T}$, we see

$$\widehat{T}^* = \begin{pmatrix} 0 & T^* \\ 0 & 0 \end{pmatrix}$$

Hence we have that T^* is a function with domain and codomain $T^* : \mathbb{H}_2 \to \mathbb{H}_1$. It follows that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

and by theorem 2.2.1, T and T^* respect all the desired properties, completing the proof.

For the rest of this seciton, we will focus on $T \in B(\mathbb{H})$ insead of $T \in B(\mathbb{H}_1, \mathbb{H}_2)$. The reason for this is to find suitable decompositino theorems of T generlizing the notion of eigenvalues.

Recall that over finite dimensional vector spaces, $TT^* = T^*T$ implied T was diagonalizable, with the usual special case of $T = T^*$ (self-adjoint) being looked into. We would like to know if this notion of diagonalizability for normal operators remains true for infinite dimensional Hilbert spaces. We shall see that this is not the case, some more conditions is required (for ex. a special case of digaonalizable operators will be comapct normal operators). Before we get into the technicalities, I want to put a word on how I think about $TT^* = T^*T$. In my mind, when two functions commute, it means that one function is somehow "invariant" with respect to the other. Think of PDE's with equations that commute with the differential operator that makes it invariant to the Lorenzian metric. In this case, I like to think of T being somehow "symmetric". Since $TT^*(x) = T^*T(x)$ for all $x \in \mathbb{H}$, and T^* is in some ways the dual (think of ref:HERE, the generalization), we see that T is "invariant" to the effect of the dual T^*. We shall see in a moment that the invariant to the dual is translated to T and T^* being metrically identical. In the simpleset case of $T = T^*$, it is evident why this is the case. If $T = T^*$, we can deepen our intuition with complex numbers: if we consider $T : \mathbb{C} \to \mathbb{C}$ where $T(z) = z$, then $T^*(z) = \overline{z}$. Hence, $T = T^*$ if and only if $T(\mathbb{C}) \subseteq \mathbb{R}$, since only the real number are invariant under conjugation.

Definition 2.2.2: Normal Operator

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then T is said to be *normal* if it commutes with its adjoint, i.e. if $T^*T = TT^*$

As an immedaite consequence, we have

$$\|T(x)\| = \langle T^* \circ T(x), x\rangle^{1/2} = \langle T \circ T^*(x), x\rangle^{1/2} = \|T^* x\|$$

showing that T and T^* are metrically identical. In fact, by the palalalogram law (also known as polarization identity) if $T \in \mathrm{End}_k(\mathbb{H})$ such that T and T^* ar emetrically identical, we can show $\langle T^* \circ T(x), y\rangle = \langle T \circ T^*(x), y\rangle$ for all x, y and so T is normal by lemma 2.2.1. By proposition 2.2.3, an operator is normal if and only if it is bounded away from zero.

Positive Operators

Definition 2.2.3: Positive Operator

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then T is said to be *positive* if $T = T^*$ and $\langle T(x), x\rangle \geq 0$ for all $x \in \mathbb{H}$. If $k = \mathbb{C}$, the positivey condition implies self-adjointness by the polarization identity.

Certainly, the sum of two positive operators is positive (geometrically, positive operatros form a cone in $\mathrm{End}_k(\mathbb{H})$). However the product is not positive, and need not be self-adjoint.

Example 2.5: Positive And Non-Positive Operators

here

However, if the opeartors commute (if the product is self-adjoint), we see that there is a product that is positive since ($ST = S^{1/2}TS^{1/2} \geq 0$ by what we'll show in the next proposition)

Let $B(\mathbb{H})_{sa} \subseteq \mathrm{End}_{\mathbb{R}}(\mathbb{H})$ be the collection of self-adjoint operators. Then this is a closed subspace, this subspace intersected with the conte of positive operators defines an ordder $S \leq T$ if $T - S \geq 0$. By taking the case of 2 dimensional real subspaces and $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$, this order is not total, not even a lattice order.

We require the following lemma to gain some important results (which will be an immedate consequence of the spectral theorem in ref:HERE)

Lemma 2.2.2: Square Root Lemma

If $S \leq T$ in $B(\mathbb{H})_{sa}$, then $A^*SA \leq A^*TA$ for every $A \in \mathrm{End}_k(\mathbb{H})$. If morever) $\leq S$, then $\|S\| \leq \|T\|$

Proof **:**
Since $S \leq T$, we know that $\langle (T - S)(y), y\rangle \geq 0$ for evey $y \in \mathbb{H}$. Replacing y with $A(x)$ for appropraite $x \in \mathbb{H}$, it follows that $A^*(T - S)A \geq 0$, so $A^*SA \leq A^*TA$.

If $0 \leq S \leq T$, by the Cauchy-Schwarz inequality on the positive sesquilinear form $\langle S(\cdot), \cdot\rangle$, we get that for every pair of unit vectors $x, y \in \mathbb{H}$

$$|\langle S(x), y\rangle|^2 \leq \langle S(x), x\rangle\langle S(y), y\rangle \leq \langle T(x), x\rangle\langle T(y), y\rangle \leq \|T\|^2$$

Hence, $\|S\|^2 \leq \|T\|^2$ by lemma 2.2.1.

Lemma 2.2.3: Approximation Lemma

There is a sequence $(p_n) \subseteq \mathbb{R}[x]$ of polynomials with positive coefficients such that $\sum p_n$ converges uniformally on $[0,1]$ to

$$t \mapsto 1 - (1-t)^{1/2}$$

Proof **:**
This is immediate from real analysis, but we can also construct it explicitly.If $q_0 = 0$ and $q_n(t) = \frac{1}{2}(t + q_{n-1}(t)^2)$, then we can see that each coefficent of q_n is positive and that $q_n(t) \leq 1$ for every $t \in [0,1]$. Morever, since

$$2(q_{n+1} - q_n) = q_n^2 - q_{n-1}^2 = (q_n - q_{n-1})(q_n + q_{n-1})$$

it follows by induction that each polynomial $p_n = q_n - q_{n-1}]$ has positive coefficients. Taking the q_n as elements of $C([0,1])$, we see they converge pointwise to $[0,1]$ to a function q such that $2q(t) = t + q(t)^2$. Thus, $q(t) = 1 - (1-t)^{1/2}$. Sicne $q \in C([0,1])$ and $[0,1]$ is compact, by the monotone ocnvergence $q_n \nearrow q$ si in fact uniform by Dini's lemma, hence

$$\sum p_n = \lim q_n = q$$

completing the proof.

Proposition 2.2.5: Square-Rooting Positive Operator

Let $T \in \mathrm{End}_k(\mathbb{H})$ be a positive operator. Then there is a unique positive operator $T^{1/2}$ such that $(T^{1/2})^2 = T$. Moreover, $T^{1/2}$ commutes with every operator commuting with T.

Proof **:**
Analysis Now p.92

Proposition 2.2.6: Invertible Positive Operator

A positive operator T is invertible in $\mathrm{End}_k(\mathbb{H})$ if and only if $T \geq \epsilon I$ for some $\epsilon > 0$. I that case, $T^-1 \geq$ and $T^{1/2}$ is invertib,e and $(T^{-1})^{1/2} = (T^{1/2})^{-1}$. If $T \leq S$, then $S^{-1} \leq T^{-1}$

Proof **:**
Analayis now p.94

Projection

Let $W \subseteq \mathbb{H}$ be a closed subspace of a hilbert space H. Then we know that $\mathbb{H} = W + W^{\perp}$. In particular, for eah $y \in \mathbb{H}$, we have $y = w + w^{\perp}$, $w \in W$ and $w^{\perp} \in W^{\perp}$. Since

$$\alpha y_1 + \beta y_2 = (\alpha x_1 + \beta x_2) + (\alpha x_1^{\perp} + \beta x_2^{\perp})$$

is the decomposition of a linear combination, it follows that $P : \mathbb{H} \to \mathbb{H}$ given by $P(y) = x$ is an operator in $\mathrm{End}_k(\mathbb{H})$ with $\|P\| \leq 1$. Note P is idempotent ($P^2 = P$), self-adjoint, and positive since

$$\langle P(y_1), y_2 \rangle = \langle x_1 < x_2 + x_2^{\perp} \rangle = \langle x_1, x_2 \rangle = \langle x_1 + x_1^{\perp}, x_2 \rangle = \langle y_1, P(y_2) \rangle$$

and

$$\langle P(y), y \rangle = \langle x, x + x^{\perp} \rangle = \|x\|^2 \geq 0$$

Definition 2.2.4: (Orthogonal) Projection

The map P in the above is called the *(orthogonal) projection.*

If P is a self-adjoint and idempotent in $\mathrm{End}_k(\mathbb{H})$, then

$$W = \{x \in \mathbb{H} \mid P(x) = x\} = P(\mathbb{H})$$

is a closed subsapce of $\mathbb{H}$. If $w^{\perp} \in W^{\perp}$, we have (since $P = P^*$) that

$$\|P(w^{\perp})\|^2 = \langle w^{\perp}, p^2(w^{\perp}) \rangle = 0$$

Thus, P is an orthogonal projection of $\mathbb{H}$ onto W. Note that $I - P$ is the projection on $W^{\perp}$.

Projections are some of the east operators we may construct, and can be thought as the "characteristic function" from real analysis. The analogue of simple function is then obtained by decomposing $\mathbb{H}$ into a finite orthogonal sum

$$\mathbb{H} = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

corresponding to the projections $P_1, ..., P_n$ which are pairwise orthogonal ($P_i P_j = 0$ for $i \neq j$ satisfying $\sum P_i = I$). We may then consider

$$T = \sum \lambda_n P_n$$

for scalars $\lambda_1, \lambda_2, ..., \lambda_n \in k$. This may look familiar to the notion of *diagonalization*. Indeed, we may generalize the concept to infintie dimensions like so:

Definition 2.2.5: Diagonalizable

Let $T \in \mathrm{End}_k(\mathbb{H})$ be an operator. Then T is said to be *diagonalizable* if there is an orthonormal basis $\{e_i \mid i \in I\}$ for $\mathbb{H}$ and a bounded set $\{\lambda_i \mid i \in I\} \subseteq k$ such that

$$T(x) = \sum \lambda_i \langle x, e_i \rangle e_i$$

The number $\langle x, e_i \rangle$ are the coordiantes for x in the basis $\{e_i\}$ and each λ_i is an *eigenvalue* for T corresponding to the eignevector $e_i >$ Denotign P_i to be the projecti nofo $\mathbb{H}$ onto ke_i, we have $P_i(x) = \langle x, e_i \rangle e_i$, so we may write the operator as:

$$T = \sum \lambda_i P_i$$

Note that the above sum is not necessarily convergent in the norm topology on $\mathrm{End}_k(\mathbb{H})$, but only pointwise convergent (i.e. in convergent in the strong operator topology). If T is diagonalizable, then T^* is diagonalizable with

$$T^*(x) = \sum \overline{\lambda_i}\langle x, e_i\rangle e_i$$

with eigenvalues $\{\overline{\lambda_i}\}$. Furthermore, if T is diagonalizable, $T^*T = TT^*$ (with eigenvalues $|\lambda_i|^2$ with basis $\{e_i\}$), and hence T is normal. T is self-adjoint if and only if all λ_i are real and T is positive if and only if $\lambda_i \geq 0$ for all i. If $\mathbb{H}$ is finite-dimensional, every normal operator is diagonalizable. This is not generally true for infinite dimensions. In general, most "interesting" (normal) operators on infinite-dimensional hilbert sapces are nondiagonalizable, and must be handled with a continuous analogue of the concept of a basis (this will be the spectrum, see section 2.3 and chapter 3).

Unitary Operators

Recall $T : \mathbb{H} \to \mathbb{H}$ is unitary if T is an isometric isomorphism from $\mathbb{H}$ onto itself (if $k = \mathbb{R}$ we say it is an *orthogonal operator*). By the polarization identity, we see that $U^*U = UU^* = I$, so that U is normal and invertible with $U^{-1} = U^*$. Conversely, if $U \in \mathrm{End}_k(\mathbb{H})$ such that $U^{-1} = u^*$ then U is unitary since

$$\|U(x)\|^2 = \langle U^*U(x), x\rangle = \|x\|^2$$

So U is an isometry and is surjective since U is invertible. If $\{e_i \mid i \in I\}$ is an orthonormal basis for $\mathbb{H}$ and U is unitary, then $\{Ue_i \mid i \in I\}$ is a basis for $\mathbb{H}$. Conversely, we saw that every transition between orthonormal bases is given by a unitary operator. For future reference the denote the group of unitary operators $U(\mathbb{H}) \subseteq \mathrm{End}_k(\mathbb{H})$ (it should be evident it forms a group).

Definition 2.2.6: Unitary Equivalent

Let $S, T \in \mathrm{End}_k(\mathbb{H})$. Then we say that S and T are *unitarily equivalnet* of $S = UTU^*$ for some unitary operator U.

It should be clear that unitary equivalent preserves norm, self-adjointness, normality, diagonalizability, and unitarity.

Definition 2.2.7: Partial Isometry

Let $U \in \mathrm{End}_k(\mathbb{H})$. Then U is said to be a *partial isometry* if there is a closed subpspace $X \subseteq \mathbb{H}$ such that $U|_X$ is isometric and $U|_{X^\perp} = 0$

This immediately implies that $X^\perp = \ker(U)$, hence $X = \bar{(}U^*(\mathbb{H}))$. Setting $P = U^*U$, then

$$\langle P(x), x\rangle = \|U(x)^2 = \|x\|^2$$

for every $x \in X$, hence $P(x) = x$ by the Cauchy-Schwarz inequality. Furthermore, $P(x^\perp) = 0$ for every $x^\perp \in X^\perp$, so P is a projetion of $\mathbb{H}$ onto X. Conversely, if $U \in \mathrm{End}_k(\mathbb{H})$ such that $U^*U = P$ for some projeciotn P, taking $X = P(\mathbb{H})$ and see from the equation $\|U(x)$
$^2 = \langle p(x), x\rangle$ that U is isometric on X and 0 on $X^\perp$, and so U is a partial isometry. Using this, we can also show that U^* is a partial isometry. Since $U(I - P) = 0$,

$$(UU^*)^2 = UU^*UU^* = UPU^* = UU^*$$

so UU^* is self-adjoint idempodent, i.e. a projection. Note that U^* is the projectio nof $U(\mathbb{H})$ back onto $U^*(\mathbb{H})$ so that U^* is a *partial inverse* of U.

Be weary that there exist isometries that are not unitary (since they need not be surjective). For example, if $\mathbb{H}$ is a separable and has countable basis $\{e_n \mid n \in \mathbb{N}\}$, we have

$$S\left(\sum \alpha_n e_n\right) = \sum \alpha_n e_{n+1}$$

Then S (the unilateral shift operator) is an isometry of $\mathbb{H}$ ont othe subspace $\{e_1\}^{|}perp$ and consequently S^* is a patial isometry of $\{e_1\}^\perp$ onto $\mathbb{H}$. In particular, $S^*S = I$, wheas SS^* is the projection onto $\{e_1\}^\perp$.

Theorem 2.2.2: Polar Decomposition of T

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then there is a unique positive operator $|T| \in \mathrm{End}_k(\mathbb{H})$ such that

$$\|T(x)\| = \||T|(x)\|$$

and $|T| = (T^*T)^{1/2}$. Moreover, there is a unique partial isometry U with kernel $\ker(U) = \ker(T)$ and $U \circ |T| = T$. In particular

$$U^*U|T| = |T| \qquad U^*T = |T| \qquad UU^*T = T$$

Proof **:**
Analysis Now p.96

We may think of U as the generalized "sign" of T and $|T|$ as the generalized "absolute value" of T. By the formula

$$T^* = |T|U^* = U^*(U|T|U^*)$$

and the unique of polar decomposition, we see that U^* is the sign of T^*, whereas $U|T|U^*$ is the absolute value fo T^*. In general, the absolute value of a sum or product of noncommuting operators doens't have much relation to the sum or product of the absolute values. Similarly, the sign (in the finite dimensional case), can't always be chosen to be unitary (the unilateral shift is an immedaite counter example). For T to have the form $U|T|$ for some unitary operator U, it is necessary and sufficient that the cloesd subspaces $\ker(T)$ and $\ker(T^*)$ have the same dimension (finite or infinite). Easy cases of this general theorem are estalblisehd in the next result

Proposition 2.2.7: Polar Decomposition For Isomorphisms

Let $T \in \mathrm{End}_k(\mathbb{H})$ be invertible. Then the partial isometry in its polar decomposition is unitary

Proof **:**
Analysis Now p.97

Proposition 2.2.8: Polar Decomposition For Normal Operators

Let $T \in \text{End}_k(\mathbb{H})$ be normal. Then there is a unitary operator W commuting with T, T^*, and $|T|$ such that $T = W|T|$

Proof **:**
Analysis Now p.97

I want to add one more thing, I just don't konw where: for any $T \in B(\mathbb{H})$, $T = T_1 + iT_2$ for self-adjoint operators T_1, T_2. This lets us generalize the spectral theorem for noraml operators, since we will konw it for self-adjoint operator.

Lemma 2.2.4: lemma I

If $T = T^*$ and $\|T\| \leq 1$, the operator $U = T + i(I - T^2)^{1/2}$ is unitary and $T = \frac{1}{2}(U + U^*)$

Proof **:**
Analysis Now p.97

Lemma 2.2.5: Lemma II

Let $S \in \text{End}_k(\mathbb{H})$ with $\|S\| < 1$. Then for every unitary U, there are unitaries U_1, V_1 such that

$$S + U = U_1 + V_1$$

Proof **:**
Analysis Now p.98

Theorem 2.2.3: Russo-Dye-Gardner Theorem

Let $T \in \text{End}_k(\mathbb{H})$ with

$$\|T\| < 1 - \frac{2}{n} \qquad n > 2$$

Then there are unitary operators $U_1, U_2, ..., U_n$ such that

$$T = \frac{1}{n}(U_1 + U_2 + \cdots + U_n)$$

Proof **:**
Analysis Now p.98

Since the self-adjoint elements span $\text{End}_k(\mathbb{H})$, we see from lemma 2.2.4 that every $T \in \text{End}_k(\mathbb{H})$ is the linear combination of (at least 4) unitary operators The estimate in the Russo-Dye-Gardner

Theorem is much more precise with respect to the norm, and shows that every element in the open unit ball of $\text{End}_k(\mathbb{H})$ is a convex combination (indeed, a mean) of unitary operators. On the surface of the ball this need not be the case, in fact the unilateral shift cannot be expressed as a convex combination of unitaries (exercise 3.2.8 in Gert K. Pederson Analysis Now).

Proposition 2.2.9: Numerical Radius Of Operator

Let $\mathbb{H}$ be a complex hilbert space and $T \in \text{End}_{\mathbb{C}}(\mathbb{H})$. Then we deifn eth *numerical radius* of T as

$$|||T||| = \sup \{\langle T(x), x\rangle \mid x \in \mathbb{H}, \|x\| \leq 1\}$$

Then $\|\cdot\|$ is a norm on $\text{End}_{\mathbb{C}}(\mathbb{H})$ satisfying $\frac{1}{2}\|\cdot\| \leq |||\cdot||| \leq \|\cdot\|$. More over, $|||T^2||| \leq |||T|||^2$ for every T and $|||T||| = \|T$
if T is normal

Proof **:**
Analysis Now p.99

2.3 Compact Operators

Throughout this section, let $\mathbb{H}$ be a hilbert space over $k \in \{\mathbb{R}, \mathbb{C}\}$ unless otherwise specified.

Definition 2.3.1: Finite Rank Operator

Let $T \in \text{End}_k(\mathbb{H})$. Then T is said to have *finite rank* if $T(\mathbb{H})$ is finite-dimensional subspace of $\mathbb{H}$. Denote by $B_f(\mathbb{H}) \subseteq \text{End}_k(\mathbb{H})$ the set of all finite rank opeartors.

Clearly, $B_f(\mathbb{H})$ is a subspace, even a subalgebra, even an ideal of $\text{End}_k(\mathbb{H})$. If $T \in B_f(\mathbb{H})$, by proposition 2.2.1, we get an orthogonal decomposition $\mathbb{H} = T(\mathbb{H}) \oplus \ker(T^*)$, which shows that $T^*(\mathbb{H}) - T^*T(\mathbb{H})$, so T^* has finite rank. Thus, $B_f(\mathbb{H})$ is *self-adjoint* ideal in $\text{End}_k(\mathbb{H})$ $((B_f(\mathbb{H}))^* = B_f(\mathbb{H})$ as a set). The subset $B_f(\mathbb{H}) \subseteq \text{End}_k(\mathbb{H})$ is similar in relation to $C_c(X) \subseteq C_b(X)$ when X is a locally compact Hausdorff space. In particular, these classes describe the local phenomena on $\mathbb{H}$ and on X. Passing to a limit in norm may destroy the "locality". but enogh structure is preservd to make these "quasilocal" operators and functions very useful and interesting. We shall stud the closure of $B_f(\mathbb{H})$ in this section as a noncommutative analogue of $C_0(X)$ in function theory.

Lemma 2.3.1: Net And Compact Operators

There is a net $(P_\lambda)_{\lambda \in \Lambda}$ of projection in $B_f(\mathbb{H})$ such that $\|P_\lambda(x) - x\| \to 0$ for each $x \in \mathbb{H}$.

that is, the projection are "dense" in $B_f(\mathbb{H})$.

Proof **:**
We want to find a next P_n such that $\|P_n(x) - x\| \to 0$. Let $\{e_i \mid i \in I\}$ be an orthonormal basis for $\mathbb{H}$ and let N be a net of finite subsets of I ordered by inclusion. For each $n \in N$, let P_n denote the projection of $\mathbb{H}$ onto span $\{e_i \mid i \in n\}$ so that $(P_n)_{n \in N}$ is a net in $B_f(\mathbb{H})$.

Now, if $x \in \mathbb{H}$, then $x = \sum a_i e_i$, thus

$$\|P_n(x) - x\|^2 = \sum_{i \notin n} |a_i|^2$$

By Parseval's identity, this tends to zero, completing the proof.

Theorem 2.3.1: Closed Unit Balls In Hilbert Spaces

Let $B \subseteq \mathbb{H}$ be a closed unit ball in a hilbert space $\mathbb{H}$. Then the following conditions on an operator $T \in \mathrm{End}_k(\mathbb{H})$ are equivalent

1. $T \in \overline{B_f(\mathbb{H})}$
2. $T|_B$ is a weak-norm continuous function from B into $\mathbb{H}$
3. $T(B)$ is compact in $\mathbb{H}$
4. $\overline{T(B)}$ is compact in $\mathbb{H}$
5. each net in B has a subnet whose iamge under T converges in $\mathbb{H}$

Proof **:**
Analysis Now p.105

Definition 2.3.2: Compact Operators

Let $T \in B(\mathbb{H})$. Then T is compact if $T(B)$ is compact. The set of compact operators is denoted by $B_0(\mathbb{H})$ to show that these operators "vanish at infinity".

Lemma 2.3.2: Diagonalizable Operator Is Compact

Let $T \in \mathrm{End}_k(\mathbb{H})$. Then T is comapct if and only if its eigenvalues $\{\lambda_i \mid i \in I\}$ belongs to $c_0(I)$

Proof **:**
Let $T(x) = \sum \lambda_i \langle x, e_i \rangle e_i$. If $T \in B_0(\mathbb{H})$, then for any $\epsilon > 0$, define

$$I_\epsilon = \{i \in I \mid |\lambda_i| \geq \epsilon\}$$

If I_ϵ is infinite, the net $(e_i)_{i \in I_\epsilon}$ converges weakly to zero for any well-ordering of I_ϵ since $\langle e_i, x \rangle \to 0$ by Parsaval's identity. Sicne $\|T(e_i)\| = |\lambda_i| \geq \epsilon$ for $i \in I_\epsilon <$ this contradicts theorem 2.3.1(3). Thus, I_ϵ must be finite for each $\epsilon > 0$, hence λ_i must vanish at infinity.

Conversely, if I_ϵ is finite for all $\epsilon > 0$. et

$$T_\epsilon = \sum_{i \in I_\epsilon} \lambda_i \langle \cdot, e_i \rangle e_i$$

Then T_ϵ has finite rank, and

$$\begin{aligned}\|(T-T_\epsilon)(x)\|^2 &= \left\|\sum_{i\notin I_\epsilon}\lambda_i\langle x,e_i\rangle e_i\right\|^2\\ &= \sum_{i\notin I_\epsilon}|\lambda_i|^2|\langle x,e_i\rangle|^2\\ &\leq \epsilon^2\|x\|^2\end{aligned}$$

hence, $\|T-T_\epsilon\|\leq\epsilon$, thus $T\in B_0(\mathbb{H})$ by theorem 2.3.1(1), completing the proof.

Lemma 2.3.3: Eigen Vectors Between Adjoint

Let x be an eigenvector for a normal operator $T\in\text{End}_k(\mathbb{H})$ corresponding to an eigenvalue λ. Then x is an eigenvector for T^* corresponding to the eigenvalue $\overline{\lambda}$. Eigenvectors for T correspondign to different eigenvalues are orthogonal.

Proof **:**
Note that the operator $T-\lambda I$ is normal and it's adjoitn is $T^*-\overline{\lambda}I$. COnversely,

$$\|(T^*-\overline{\lambda}I)(x)\| = \|(T-\lambda I)(x)\| = 0$$

If $T(y)=\gamma y$ where $\gamma\neq\lambda$, then we may certainly assume $\lambda\neq 0$ so that

$$\begin{aligned}\langle x,y\rangle &= \lambda^{-1}\langle T(x),y\rangle\\ &= \lambda^{-1}\langle x,T^*(y)\rangle\\ &= \lambda^{-1}\langle x,\overline{\gamma}y\rangle\\ &= \lambda^{-1}\gamma\langle x,y\rangle\end{aligned}$$

Hence, $\langle x,y\rangle=0$, showing they are different eigenvectors, completing the proof.

Lemma 2.3.4: Normal Compact Operator, then Eivenvalue Exists

Every normal, compact operator T on a complex Hilbert space $\mathbb{H}$ has an eigenvalue λ with $|\lambda|=\|T\|$

Proof **:**
Analysis now p. 107

Theorem 2.3.2: Normal Compact, Then Diagonalizable

Every normal compact operator $T\in\text{End}_k(\mathbb{H})$ is diagonalizable and its eigenvalues (counted-with multiplicity) vanish at infinite. Conversely, each such operator is normal and compact.

Proof :
Analysis now p.108

It will be convenient for future sections to introduce a new notation: Let $x \odot y$ represent the rank one operator in $\text{End}_k(\mathbb{H})$ determined by the vectors $x, y \in \mathbb{H}$ and the formula

$$(x \odot y)(z) = \langle z, y \rangle x$$

The map $(x, y) \mapsto x \odot y$ is sesquilinear map of $\mathbb{H} \times \mathbb{H}$ into $B_f(\mathbb{H})$. If $\|e\| = 1$ then $e \odot e$ is the one-dimensional projection of $\mathbb{H}$ on $\mathbb{C}e$. Every normal compact operator $\mathbb{H}$ can now be written by the above theorem in the form

$$T = \sum \lambda_i e_i \odot e_j$$

For suitable orthonormal basis $\{e_i \mid i \in I\}$. The sum will converge in norm, since either the set $I_0 = \{i \in I \mid \lambda_i \neq 0\}$ is finite (so that $T \in B_f(\mathbb{H})$) or else countably infinite in which case the sequence $\{\lambda_i \mid i \in I_0\}$ converges to zero.

Definition 2.3.3: Spectrum

We say that the compact set

$$\text{sp}(T) = \{\lambda_i \mid i \in I_0\} \cup \{0\}$$

is the *spectrum* of T.

For every continuous function f on $\text{sp}(T)$, define

$$f(T) = \sum f(\lambda_i) e_i \odot e_i$$

Then $f(T)$ is compact if and only if $f(0) = 0$ and the map $f \mapsto f(T)$ Is an isometric $*$-preserving homomorhpism of $C(\text{sp}(T))$ into $\text{End}_k(\mathbb{H})$. If $f(z) = \sum \alpha_{nm} z^n \overline{z}^m$ is a polynomial of two commuting variables z and $\overline{z}$, then $f(T) = \sum \alpha_{nm} T^n (T^*)^m$. This result is the *spectrial (mapping) theorem* for normal compact operators. In the next chapter, we shall generalize this to any normal operator.

Calkin Algebra and Index

Since $B_0(\mathbb{H}) \subseteq B(\mathbb{H})$ is a closed ideal, we may define $B(\mathbb{H})/B_0(\mathbb{H})$ with the quotient norm[1]. This quotient is given a name

Definition 2.3.4: Calkin Algebra

Let $B_0(\mathbb{H}) \subseteq B(\mathbb{H})$. Then the algebra

$$B(\mathbb{H})/B_0(\mathbb{H})$$

is called the *Calkin Algebra* of $\mathbb{H}$

[1]This is even a C^*-algebra

We shall see that many properties of $B(\mathbb{H})$ are conveniently expressed in terms of the Calkin algebra.

Definition 2.3.5: Compact Pertubation

If S and T are elements in $B(\mathbb{H})$ and $S - T \in B_0(\mathbb{H})$, we say that S is a *compact pertubation* of T.

This is antoher way of saying that S and T hav the same image in the Calkin algebra. Such "local" pertubation happen frequently in application and properties of an operator that are invariatn under compact pertubations are highly used. The best known of these we shall study know is known as the *index*

Theorem 2.3.3: Atkinson's Theorem

Let $T \in B(\mathbb{H})$. Then the following are equivalent:

1. There is a unique operator $S \in B(\mathbb{H})$ such that ST and TS are the projections on $(\ker(T))^\perp$ and $\ker(T^*)^\perp$, respectively, and both projections have finite co-rank
2. For some operator $S \in B(\mathbb{H})$ both operators $ST - I$ and $TS - I$ are compact
3. the image of T is invertible in the Calkin algebra $B(\mathbb{H})/B_0(\mathbb{H})$
4. Both $\ker(T)$ and $\ker(T^*)$ are finite-dimensional subsapces and $T(\mathbb{H})$ is closed

***Proof* :**
Analysis now p.109

In essence, Fredholm operators are operators that are are almost invertible up to some finite-dimension beign ignored, in particular they are invertible modulo compact operators.

Definition 2.3.6: Fredholm Operator

The operator satisfying the condition's in Atkinson's theorem are called *Fredholm operators*, and the class of them is dneoted $F(\mathbb{H})$.

Definition 2.3.7: Index

For each $T \in F(\mathbb{H})$, define the *index* of T to be

$$\text{index}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$$

If we shooe S for T as in (1) of Atkinson's theorem and write $suchthat = I - P$ and $TS = I - Q$ with P and Q projection of rank, then another way of writing this would be

$$\text{index}(T) = \text{rank}(P) - \text{rank}(Q)$$

By (3) of Atkinson's Theorem, the product of Fredholm operators is again a Fredholm operator. In particular, every product $RT \in F(\mathbb{H})$ if $T \in F(\mathbb{H})$ and R is invertible in $B(\mathbb{H})$, and in this case

$$\text{index}(RT) = \text{index}(TR) = \text{index}(T)$$

since R is bijective. It should also be clear that $T^* \in F(\mathbb{H})$ if $T \in F(\mathbb{H})$ with index $\text{index}(T^*) = -\text{index}(T)$. Finally, the partial inverse S to T in (1) of Atkinson's theorem is a Fredholm operator with index $\text{index}(S) = -\text{index}(T)$. F

Definition 2.3.8: Indexed Fredholm Operators

For each $n \in \mathbb{Z}$ define
$$F_n(\mathbb{H}) = \{T \in F(\mathbb{H}) \mid \text{index}(T) = n\}$$

All of these subsets are nonemtpy, since if S is the unilateral shift, then $S^n \in F_{-n}(\mathbb{H})$ and $(S^*)^n \in F_n(\mathbb{H})$ for every $n > 0$.

Lemma 2.3.5: Finite Rank And 0 Index

If $A \in B_f(\mathbb{H})$, then $I + A \in F_0(\mathbb{H})$

Proof **:**
Analysis now p.110

Corollary 2.3.1: Fredholm Alternative

If $A \in B_0(\mathbb{H})$ and $\lambda \in \mathbb{C} \setminus \{9\}$, then either $\lambda I - A$ is invertible in $B(\mathbb{H})$ or λ is an eigenvalue of A with finite multiplicity, in which case $\overline{\lambda}$ is an eigenvalue for A^* with the same multiplicity

Proof **:**
Let $T + I - \lambda^{-1}A$. Since $T \in F_0(\mathbb{H})$, by lemma 2.3.5 we know that $T(\mathbb{H})$ is closed and that
$$\dim(\ker(T)) = \dim(\ker(T^*)) < \infty$$
If these dimension are 0, then $T(\mathbb{H}) = (\ker(T^*))^{\perp} = \mathbb{H}$ and $\ker(T) = \{0\}$, hence T is invertible by proposition 2.2.3, completing the proof.

This lemma implies that the spectrum of a compact operator (i.e. those λ such that $\lambda I - T$ is not invertible, as we'll see in the next chapter) consists of $\{0\}$ and the eigenvalues for T. IN particualr, it is a countable subset of $\mathbb{C}$ with 0 as the only possible accumulatpion point.

Theorem 2.3.4: Index Invariant Under Compact Operator

Let $T \in F(\mathbb{H})$ and A be a compact operator. Then
$$\text{index}(T + A) = \text{index}(T)$$

Proof **:**
Analysis now p.111

Proposition 2.3.1: Fredholm Class Is Open

Each Fredholm class $F_n(\mathbb{H})$ for each $n \in \mathbb{H}$ is open in $B(\mathbb{H})$

Proof **:**
analysis now p.112

Proposition 2.3.2: Grading Structure Of Freidholm Operators

If $T_1 \in F_n(\mathbb{H})$ and $T_2 \in F_m(\mathbb{H})$ then $T_1T_2 \in F_{n+m}(\mathbb{H})$

Proof **:**
Analysis now p. 112

We close with an interesting topological fact that brings some light to the index .Let G denote the group of invertible elements of the Calkin algebra $B(\mathbb{H})/B_0(\mathbb{H})$. and G_0 is the connected component of the identity (which is the group generated by the elements of the form $\exp(A)$ for some $A \in B(\mathbb{H})/B_0(\mathbb{H})$). Then G_0 is an open and closed subgroup of G and G/G_0 is a discrete group that labels the connected components in G. In this case, G/G_0 is isomorphic to $\mathbb{Z}$, and the index of a Fredholm operator T is the image of T under the compsoed quotinet maps from $F(\mathbb{H})$ to G and from G to G/G_0 $(= \mathbb{Z})$

2.4 Trace

3

Spectral Theory

In short, Spectral theory generlizes the notion of eigeen vectors and values to infinite dimensional operators. In the previous chapter, we saw how there is a notion of diagonalizability for compact normal operators. In this chapter, we extend this result to normal operators. We also along the way learn some important dualities such as the Gelfand duality which has some useful applications, and is an intersting motivator for classical algebraic geometry.

3.1 Banach Algebras

Recall the definition of a banach algebra

Definition 3.1.1: Banach Algebra Revisited

A *Banach Alagebra* is an algebra A with a submultiplicative norm (so that it is continuous) st the unerlying vector space is a banach space.

All the same applies for Banach Algebras as we expect. Note that for A/I to be a Banach algebra for an ideal I, I has to be closed, just like for subspaces. If A is unital and $A \neq \{0\}$, then it is certainly unique and $\|I\| \geq 1$ (why not =?? Analysis now p.128) If A has no units, we may try to isometrically embed it into a larger Banach Algebra $\widehat{A}$ such that $A \subseteq \widehat{A}$ becomes an ideal such that every nonzero ideal of $\widehat{A}$ has a nonzero intersection with A (such an ideal is called an *essential ideal*). This can be thought as the algebraic counterpart of the notion of compactification of a topological space. For classical Banach algebra, there is usually a natural way of *adjoining a unit*, but there is always a general method of doing so that is the counter-part of one-point compactification. Take the direct sum $A \oplus k = \widehat{A}$ along with the product

$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$$

and the submultiplicative 1-norm. Then $A \oplus k$ is a unital Banach algebra with $I(0, 1)$ and contains A as an ideal of co-dimension 1.

Definition 3.1.2: Regular Representation

Let A be a Banch Algebra. Then the (left) *regular representation* $\rho : A \to B(A)$ by $\rho(x)(y) = xy$ for $x, y \in A$.

It should be verified that ρ is a norm decreasing algebra homomorphism, and if A is unital, the inequalities

$$\|A\| \leq \|\rho(A)\|\|I\| \leq \|A\|\|I\|$$

show that ρ is a homeomrophism. Renorming A by using the equivalent norm from $B(A)$, we may assume that $\|I\| = 1$ (if $A \neq \{0\}$), which we do.

Definition 3.1.3: Approximate Unit

A net $(E_\lambda)_{\lambda \in \Lambda}$ in the unit ball of a Banah algebra A is an *approximate unit* if

$$\lim E_\lambda x = \lim x E_\lambda = x$$

for all $x \in A$.

Example 3.1: Approximate Units

Can you think of approximate units for the non-unital Banach algebra $C_0(X)$, $B_0(\mathbb{H})$ and $L^1(\mathbb{R}^n)$? Show that the existence of an approximate unit guarantees that the regular representation ρ is an isometry

We shall denote $A^\times = \mathrm{GL}(A)$ the set of all invertible elements of A (in algebra, we'd usually call these elements unital, but the term unit in functional analysis is more commonly reserved for the identity)

Lemma 3.1.1: Almost Invertible Elements

Let $x \in A$ where A is a unital banach algebra with $\|A\| < 1$. Then $I - A \in \mathrm{GL}(A)$ and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$$

***Proof* :**

Since the norm is submultiplicative $\|x^n\| \leq \|x\|^n$, thus the series $\sum x^n$ converges in x to an element y. Sicne $xy = yx = y - I$ by manipulating the above power series, it follwos that $y = (I - x)^{-1}$, completing the proof.

Proposition 3.1.1: Properties Of $\mathrm{GL}(A)$

Let A be a unital algebra and $\mathrm{GL}(A)$ be it's multiplicative group. Then $\mathrm{GL}(A)$ is an open subset of A, an the map $x \mapsto x^{-1}$ is a homeomorphism of $\mathrm{GL}(A)$

***Proof* :**
Let $x \in \mathrm{GL}(A)$ and $y \in A$, Then by lemma 3.1.1

$$y = x - (x - y) = x(I - x^{-1}(x - y)) \in \mathrm{GL}(A)$$

provided that $\|A^{-1}(A - B)\| < 1$. In particular, the ball $B_\epsilon(x)$ is contained in $\mathrm{GL}(A)$ for $\epsilon < \|x^{-1}\|^{-1}$, which proves that $\mathrm{GL}(A)$ is open. If $y \to a$, we see from the above equation and lemma 3.1.1 that $y^{-1} \mapsto a^{-1}$, thus inverting is a continuous process and therefore a homeomorphism since $(A^{-1})^{-1} = A$, completing the proof.

From ehre on out, all Banach algebra will be over $\mathbb{C}$, i.e. all Banach algebras will be *complex*. See exercise ref:HERE on the complexicication of general real Banach algebras (or E.2.1.13 in Analysis Now).

Definition 3.1.4: Spectrum

Let A be a complex unital Banach algebra. Then for every $x \in A$, define the *spectrum* of x t obe the set

$$\mathrm{sp}(x) = \{\lambda \in \mathbb{C} \mid \lambda I - x \notin \mathrm{GL}(A)\}$$

The comliment of $\mathrm{sp}(x)$ is the *resolvent set*.

On the resolvent set, we can define the resolvent function

$$R(x, \lambda) = (\lambda I - A)^{-1}$$

Definition 3.1.5: Spectral Radius

Let A be a complex unital Banach algera, $x \in A$, and $\mathrm{sp}(x)$ be its spectrum. Then the smalleest $r \geq 0$ such that $\mathrm{sp}(x) \subseteq B_r(0)$ is called the *spectral radius* of x, and is denoted $r(x)$. THus

$$r(x) = \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(x)\}$$

Lemma 3.1.2: Build-Up Lemma I

Let $x \in A$ and $f(z) = \sum \alpha_n z^n$ be a holomorphic function in a region contined in the closed disk $B_r(0)$ with $\|x\| \leq r$. then we can define $f(x) = \sum \alpha_n x^n$ in A. Morever, if $\lambda \in \mathrm{sp}(x)$ and $|\lambda| \leq r$, then $f(\lambda) \in \mathrm{sp}(f(x))$

***Proof* :**
The series $f(x)$ is absolutely convergence ($\sum |\alpha_n| \|x\|^n < \infty$), so certainly $f(x) \in A$. Moerover

$$\begin{aligned} f(\lambda)I - f(x) &= \sum \alpha_n(\lambda^n I - x^n) \\ &= (\lambda I - x) \sum \alpha P_{n-1}(\lambda, x) \\ &= (\lambda I - x) y \end{aligned}$$

where $P_{n-1}(\lambda, x) = \sum_{k=0}^{n-1} \lambda^k A^{n-k-1}$ so that $\|P_{n-1}(\lambda, x)\| \leq n r^{n-1}$, which implies the series of polynomials converges in A to an element y commuting with x. We see form this computation that if $f(\lambda)I - f(x) \in \mathrm{GL}(A)$ with inverse v, then yv will be the inverse of $\lambda I - x$, so tha $\lambda I - x \in \mathrm{GL}(A)$

Lemma 3.1.3: Build-Up Lemma II

For every $x\ inA$, we have

$$r(x) \leq \inf \|A^n\|^{1/n}$$

***Proof* :**
If $|\lambda| > \|A\|$, thn by lemma 3.1.1

$$(\lambda I - x)^{-1} = \lambda^{-1}(I - \lambda^{-1}x)^{-1} = \sum \lambda^{-n-1} x^n$$

which shows that $r(x) \leq \|x\|$. Now if $\lambda \in \mathrm{sp}(x)$, then $\lambda^n \in \mathrm{sp}(x^n)$ by lemma 3.1.2. Thus $|\lambda^n| \leq \|x^n\|$ from what we've just shown. It follows that $r(x) \leq \|x^n\|^{1/n}$ for every n, completing the proof.

Theorem 3.1.1: Spectrum Is Compact

Let $x \in A$ be an element of a complex unital Banach algebra A. Then the spectrum of x, $\mathrm{sp}(x)$ is a compact nonempty subset of $\mathbb{C}$, and the spectral radius of x is the limit of the convergence sequence $\|A^n\|^{1/n}$

***Proof* :**
Analysis now p.131, really cool proof that I should take the time to write down!

A cool consequence of this is that we get a classical result of Representation theory known as Artin-Wedderburn for algebraically closed algebras:

Corollary 3.1.1: Algebraically Closed Division Algebra $\mathbb{C}$

Let A be a unital Banach algebra over $\mathbb{C}$. Then if A is a division ring, $A = \mathbb{C}$

Proof **:**
For each $x \in A$, there is some $\lambda \in sp(x)$ by the above theorem. Thus, $\lambda I - x \notin \mathrm{GL}(A)$, hence $x = \lambda I$.

3.2 Continuous Functional Calculus

Let $T \in B(\mathbb{H})$ and consider $\sigma(T)$. Consider $B(\sigma(T))$, which is a $*$-algebra where $*$ is complex onjugation. We may consider this algebra with the supremum norm. In this section, we'll be looking itno various sub $*$-algebra of $B(\sigma(T))$, for example the polynomials functions on $\sigma(T)$ or the continuous functions on $\sigma(T)$

Definition 3.2.1: Functional Calculus

Let $T \in B(\mathbb{H})$. Then a *functinoal calculus* of T is a continuous $*$-homomorphism from $*$-algebra of functions $(\sigma(T) \to \mathbb{C})$ to $B(\mathbb{H})$ which sents the constant function 1 to the idneitty operator,a and sends the identity function $x \mapsto x$ to T

Since it's a $*$-homomorphism, we have

$$\varphi(\overline{f}) = \varphi(f)^*$$

since we have the supremum norm and the $*$-homomorphism is continuous, $f_n \to f$ in the supremum norm then $f_n(T) \to f(T)$ in the norm topology of $B(\mathbb{H})$.

(more here, this is really cool!)

3.3 Gelfand Duality