Everything You Need To Know About MAT354

Nathanael Chwojko-Srawley

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Contents

For now, I copied over what was in the course description

Complex numbers, the complex plane and Riemann sphere, Möbius transformations, elementary functions and their mapping properties, conformal mapping, holomorphic functions, Cauchy's theorem and integral formula. Taylor and Laurent series, maximum modulus principle, Schwarz's lemma, residue theorem and residue calculus.

This textbook will assume some basic familiarity with complex numbers and construction of complex number. Might do a history section at some point (the number i was added not to solve $x^2 + 1 = 0$, but polynomials of degree 3 that so happened to need an "imaginary" number to get to the real value).

(A quick word on the real origins of complex numbers)

(A word of warning by Tao: many theorems that we'll cover fail spectacularly for non-holomorphic functions. This is not so much the case for real-differentiable and continuous functinos due to distribution theory)

1

Complex Numbers and Functions

In this chapter, we go over how to upgrade the real numbers into the complex numbers. The complex numbers are a wonderful mathematical object to work with: they form the smallest complete algebraically closed characteristic 0 field – about as good with properties in analysis (completeness), algebra (algebraic closure, commutative, characteristic 0), and geometry (i can be interpreted as rotation, inner product space giving us angles, norms, and metric), and with a very simple construction from $\mathbb{R}$ (it's extension is only of dimension 2!). These many nice properties provide us a variety of flexibility in the manipulation of complex numbers: we can represent them in polar or euler form, we can find many useful identities, and we will gain an important tool on dealing with infinities.

Complex functions will also have many important properties. Complex linear transformations will be those that, interpreted as $\mathbb{R}^2$ transformations, will preserve angle and orientation. Complex polynomials will always have all their roots, ratios of complex polynomials can be analyzed using our new technics dealing with $1/0$, and they can be decompsoed into simpler sums of of rational functions of the form $1/(z - a_i)$.

1.1 Arithmetics and Conjugate

The real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$, making it a very natural field to work with in analysis. However, $\mathbb{R}$ is not algebraically closed, meaning there are polynomials in $\mathbb{R}[x]$ that do not have solutions. The complex numbers recitfy this by being the *algebraic closure* of $\mathbb{R}$. Due to this, I will take an algebraic approach to defining the complex numbers. To an algebraist, the complex numbers are often defiend as the ring $\mathbb{R}[x]$ quotient by the maximal ideal $(x^2 + 1)$

$$\mathbb{C} := \mathbb{R}[x]/(x^2 + 1)$$

with $\overline{x}$ being identified with $\sqrt{-1}$. We may also identify $\overline{x}$ with $-i$, this field will be labled $\overline{\mathbb{C}}$; more on this field soon. The field $\mathbb{C}$ is a degree 2 extension, and so all the element of $\mathbb{C}$ can be written of the from $a + b\sqrt{-1}$ for the root $\sqrt{-1}$ of $x^2 + 1$. By convention (established by Euler), we will write $i := \sqrt{-1}$, and we will also call the elements of $\mathbb{C}$ *numbers* (or *complex numbers* to be more precise when the contexts necessitates it).

Addition and multiplication is defined as a natural consequence of the extension, that is:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where 1 and i are the basis elements, and hence all elements of $\mathbb{C}$ can be written as a linear combination of $a+bi$ for $a, b \in \mathbb{R}$. We usually denote the real part of an element $z \in \mathbb{C}$ by $\mathrm{Re}(z)$ and the imaginary part by $\mathrm{Im}(z)$. Since $\mathbb{C}$ is a field, division is defined. As a mneumonic for the computation, if we want to find:

$$\frac{1}{a + bi}$$

then we then multiply this by a fancy 1;

$$\frac{1}{a + bi}\frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right) i$$

And so, we have that $\mathbb{C}$ is a complete algebraically closed field. In chapter ref:HERE, we will show $\mathbb{C}$ being algebraically closed means any polynomial $p(x) \in \mathbb{C}[x]$ has a root. We are taking this property for granted for now[1].

As mentioned earlier, $\mathbb{R}[x]/(x^2 + 1)$ has two possible isomorphisms since $(\overline{x}^2 + 1) = (\overline{x} - 1)(\overline{x} + 1)$. We usually pick $\overline{x} = \sqrt{-1} =: i$ and label the resulting field $\mathbb{C}$, however we could have chosen $-i$. We know from algebra that there is a field isomorphism between $\mathbb{C}$ and $\overline{\mathbb{C}}$, showing the two are indistinguishable as fields. In particular, the isomrophism is:

$$a + bi \mapsto a - bi$$

Since the base field is $\mathbb{R}$, $\mathbb{C}$ and $\overline{\mathbb{C}}$ are even $\mathbb{R}$-algebra isomorphic. This function is usually denoted $\overline{z}$. Since it's an $\mathbb{R}$-algebra isomorphism:

$$\overline{zw} = \overline{z}\overline{w} \qquad \overline{z + w} = \overline{z} + \overline{w} \qquad \overline{rz} = r\overline{z}$$

Importantly, $\overline{z}$ is *not* $\mathbb{C}$-linear, in particular for any $c \in \mathbb{C}$, $\overline{c \cdot z} = \overline{c} \cdot \overline{z}$. This is perhaps obvious since this is another formulation of the field isomorphism property, however it is in fact very important to remember this in the context of complex differentiation: complex-differentiable functions will be locally $\mathbb{C}$-linear. The function $\overline{z}$ is called the *conjugate* of z. Geometrically, conjugation can be thought as the complex equivalent of reflection function $z \mapsto -z$, however algebraically it has many different properties. Conjugation let's us define many useful functions, for example

$$z\overline{z} = a^2 + b^2 \in \mathbb{R} \qquad z + \overline{z} = 2a$$

that is, the maps $z \mapsto z\overline{z}$ and $z \mapsto z + \overline{z}$, which both maps $\mathbb{C} \to \mathbb{R}$. On the other hand,

$$z - \overline{z} = 2bi$$

letting us also isolate the complex part. Since $\overline{\overline{z}} = z$, we get that $\overline{(\_)}$ is it's own inverse, which by definition makes it a type of function called an *involution*, which show up a lot in mathematics and

[1]This can also be proven using only the Intermediate value theorem if we used some group and field theory; see my EYTNKA undergraduate algebra for an example

usually serve a useful function (ex. if $x^2 = e$ in a group, then the group can be split into a product of groups, or if $A^2 = I$ for a matrix, the matrix is diagonalizable, or if $f^2 = 1$ for some function, then f is bijecive, or $\neg\neg A = A$ for some logical statement A). For complex numbers, complex conjugation will give us a useful arithmetic tool that will sipmlify many identities (some examples of which we'll list bellow). If we have any polynomial $p(x)$ over $\mathbb{C}$, or even any rational function[2] $p(x)/q(x)$, conjugation distributes over it:

$$\overline{\left(\frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}\right)} = \frac{\overline{a_0} + \overline{a_1x} + \overline{a_2x^2} + \cdots + \overline{a_nx^n}}{\overline{b_0} + \overline{b_1x} + \overline{b_2x^2} + \cdots + \overline{b_mx^m}}$$

Four important uses of conjugation are:

1. we can represent the real or imaginary part of a complex number through the following equations:
$$\operatorname{Re} a = \frac{a + \overline{a}}{2} \qquad \operatorname{Im} a = \frac{a - \overline{a}}{2i}$$

2. Another example that is useful is the ability to represent the inverse of a complex number:
$$z^{-1} = \frac{\overline{z}}{z\overline{z}} \tag{1.1}$$
Naturally, in equation (1.1), the $\overline{z}$ cancel out, but computationally it is easier to understand it in this form since we often treat $\mathbb{C}$ as a vector-space over $\mathbb{R}$ and we want all its elements to bewritten in the form $a + bi$.

3. We will soon define the norm on $\mathbb{C}$, but it is already useful to point out that
$$|z|^2 = z\overline{z}$$
showing a connectin between the notion of size and how it relates to the agebra of complex number multiplication.

4. If r is a root of the $p(x)$ ($p(r) = 0$), then $\overline{r}$ is a root of $\overline{p(x)}$ (prove this). If $p(x) \in \mathbb{R}[x]$, then r and $\overline{r}$ are rotos of $p(x)$.

1.2 Geometry and Analysis

In the following sections, we will explore the geometric properties of complex numbers

1.2.1 Norms and Inequalities

When extending from the real to the complex numbers, they loose their total ordering (there is not "natural" ordering on them in which any two points have a finite distance and if $a < b$ and $b < a$ then $a = b$). However, $\mathbb{C}$ does a notion of size to every vector given by

$$N(z) = z\overline{z}$$

[2]the numerator and denominator are polynomials

notice that $N(z) = \overline{N(z)}$ and so its output is real. This is called the *norm form* and is defined for any field extension. It has the two properties of:

$$N(zw) = N(z)N(w) \qquad N(z) = 0 \Leftrightarrow z = 0$$

This particular norm has an important property: $N(z) > 0$ if $z \neq 0$, that is it is *positive definite* (this is not the case for all norms, consider $\mathbb{Q}(\sqrt{2})$). Since it is positive definite, we can define an inner product via:

$$\langle v, w \rangle = \mathrm{Re}(z\overline{w})$$

This makes $\mathbb{C}$ is an inner vector space over $\mathbb{R}$ [3]. We would write the induced norm as:

$$|z| := \langle z, z \rangle = N(z)^{1/2}$$

where the $|\cdot|$ notation is reminiscent of the absolute value notation from $\mathbb{R}$ (notice it in fact it continuously extends $|\cdot|$). One can verify that the axioms of a norm are satisfied by $\|\cdot\|$, that is, for any $x, y \in \mathbb{C}$ and $s \in \mathbb{R}$:

Definition 1.2.1: Modulus (Norm)

Let $z \in \mathbb{C}$. Then define the function:

$$\|z\| = |z| = \sqrt{a^2 + b^2}$$

which is called the *modulus* of z and can be shown to satisfy the following properties:

1. **Subadditivity or Triangle Inequality**: $\|x + y\| \leq \|x\| + \|y\|$
2. **Absolute Homogeneity**: $\|sx\| = |s| \, \|x\|$
3. **Positive definiteness**: If $\|x\| = 0$ then $x = 0$. Note that in tandem with (2) this means that $\|x\| = 0$ if and only if $x = 0$

making it a norm

Since we'll be treating $\mathbb{C}$ as "numbers" in it of itself, then we will often abuse notation and write $|z|$ for the norm of z, and call it the *absolute value* or *modulus* of z. This norm naturally defines the metric $|z - w|$, and so the same topology as $\mathbb{R}^2$, and so any continuous function on $\mathbb{R}^2$ is continuous on $\mathbb{C}$. We get some important results about convergent sequences and series from real analysis that carry over to complex analysis that will simplify proofs are:

1. $z_n \to z$ if and only if $\sum_k^n |z_k| < \infty$ (i.e. (z_n) is convergent if and only if the series of $(|z_n|)$ is absolutely convergent)
2. $\sum_k z_k$ is convergent if and only if $z_k \to 0$
3. Notice that this also makes the norm *continuous*. Using the limit definition, this means that:

$$|\lim_{n\to\infty} z_n| = \lim_{n\to\infty} |z_n|$$

[3] in particular, a complete normed vector space, making it a Banach space

4. $|z| = |\overline{z}|$ and $|\operatorname{Re}(z)|, |\operatorname{Im}(z)| \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$. Use these inequalities to show that $z_n \to z$ if and only if the real and imaginary part converge and that Re and Im passes through the limit.

Using conjugation, we can come up with the simpler formula of

$$|z|^2 = z\overline{z}$$

This can also make some calculations much simpler. For example, using the definition of an absolute value directly, we get:

$$\begin{aligned}
|xy|^2 &= |(ac-bd)+(ad+cd)i|^2 \\
&= (ac-bd)^2 + (ad+cd)^2 \\
&= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2(adbc) + (cd)^2 \\
&= (ac)^2 + (bd)^2 + (ad)^2 + (cd)^2 \\
&= (a^2+b^2)(c^2+d^2) \\
&= |x|^2|y|^2
\end{aligned}$$

This extends to finite products and quotients. If we use the fact that $|z| = z\overline{z}$, we get

$$|ab|^2 = ab\overline{ab} = ab\overline{a}\overline{b} = a\overline{a}b\overline{b} = |a|^2|b|^2$$

Using the conjugation definition also allows us to find many more identities in relatoin to the norm, for example:

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + (z\overline{w} + \overline{z}w) + w\overline{w}$$

which simplifies to

$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(a\overline{b})$$

and similarly, if we take the difference of the two numbers, we get:

$$|z-w|^2 = |z|^2 + |w|^2 - 2\operatorname{Re}(a\overline{b})$$

Combining these two identities, we get the following interesting result:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

which is the *parallelogram law*, showing that this norm induces an *inner product*:

$$\langle z, w \rangle = \frac{\|z+y\|^2 - \|x-y\|^2}{4}$$

Another inequality we get using the absolute the following:

$$-|a| \leq \operatorname{Re} a \leq |a|$$

where equality holds if $b = 0$ and $a \geq 0$. Using this fact, we can combine it with equation $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(a\overline{b})$ to get

$$|z+w|^2 \leq |z|^2 + |b|^2$$

where equality will only hold if $a\bar{b} \geq 0$. If $b \neq 0$, we can re-write it as $|b|^2(a/b) \geq 0$, i.e. $a/b \geq 0$. Repeating this for an arbitrary finite length triangle inequlaity:

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

we see that $a_i/a_j \geq 0$ for all i, j. In this way, we see how very similar complex numbers are to real numbers (this time, more than vector's in $\mathbb{R}^2$) since this is the property of real numbers!

(there are a couple more inequalities, but I will for now conclude with this one)

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

moving $|b|$ to the other side, we get

$$|a| - |b| \leq |a - b|$$

Similarly if we do the same for $|b|$: $|b| - |a| \leq |a - b|$. This meaning that either the positive or negative value will both be less than $|a - b|$ and so

$$||a| - |b|| \leq |a - b|$$

Finally, we state the famous Cauchy inequality:

$$\left|\sum a_i b_i\right|^2 \leq \sum |a_i|^2 \sum |b_i|^2$$

We can show that $+, \cdot$, and $-$ are all continuous functions, and so our usual limit laws hold. Since conjugation is continuous, we also have:

$$\lim_{n\to\infty} \overline{z_n} = \overline{\lim_{n\to\infty} z_n}$$

Exercise 1.2.1

1. Show that on S^1, $\frac{1}{z} = \overline{z}$.
2. show that $\mathbb{C}^\times \cong (0, \infty) \times S^1$ as groups.
3. Let $\omega \in S^1$. show that $z \mapsto \omega z$ is an isometry of $\mathbb{C}$, that is

$$|\omega z - \omega w| = |z - w|$$

1.2.2 Change of Variables: Polar and Euler Form

Visually, we can picture a complex number as an arrow (or vector) in the complex plane. Note that $\mathbb{C}$ over $\mathbb{R}$ can be visualized as a plane. Usually, the x-axis will be called the real axis, and the y axis will be called the imaginary axis:

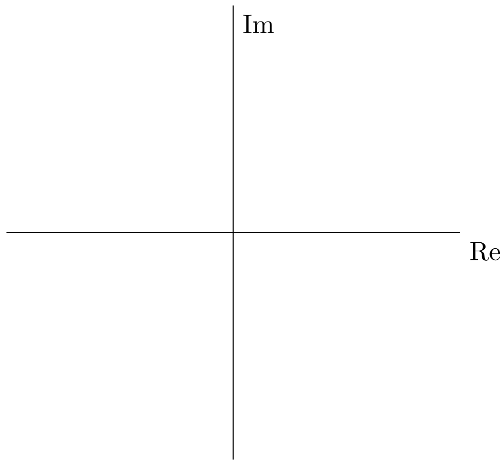


Figure 1.1: Complex Plane

When adding two vectors, it is like moving one vector to the tip of the others, the tip of the combination being the new complex number. This can also be thought as a line through the following parallelogram:

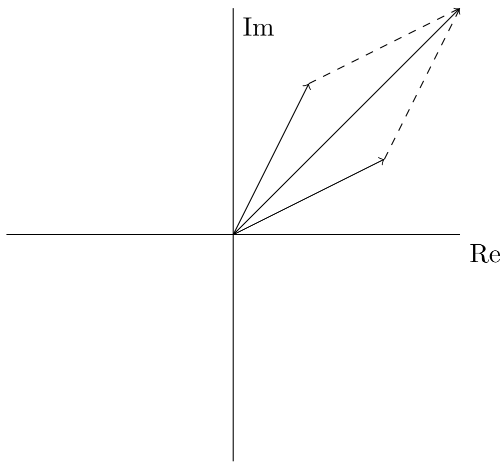


Figure 1.2: Adding Complex Numbers

When it comes to multilication, we require a little more intuition of what i does in the complex plane. In particular, Notice that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, so i returns to itelf in after multiplying by itself 4 times. Thus, multiplying by i can be thought of as rotating 90 deg. or $\pi/2$ times. This intuition can be captured more rigorously: take $\mathbb{C}^\times$ to be set of nonzero complex numbers. Then notice we can represent any element in here by two numbers: $|z|$ and $z/|z|$. The first represents the magnitude, while the second captures the angle. In particular, $|z/|z|| = |z|/|z| = 1$, and so they all lie on the unit circle S^1. As long as $z \neq 0$, $|z| \neq 0$. We thus can imagine there being a group isomorphism:

$$\mathbb{C}^\times \cong_{\mathbf{Grp}} (0, \infty) \times S^1$$

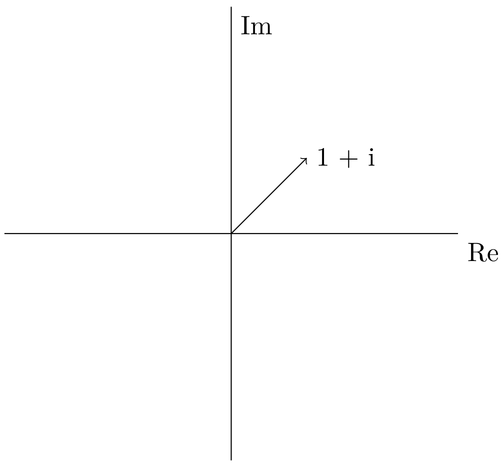
and indeed there is. First, for any $z/|z|$, using pythagoras and some highschool trigonometry we can represent this as $\cos(\theta) + i\sin(\theta)$ for appropriate θ. Then we get $x + iy \mapsto (|x + iy|, \cos(\theta) + i\sin(\theta))$. The key is that multiplication is indeed preserved since:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$$

Thus, we can represent complex numbers in terms of rotation! For example:

1. $$i = \cos(\pi/2) + i\sin(\pi/2)$$
   $$1 = \cos(0) = i\sin(0)$$

2. What about representing $1 + i$? Notice that in the complex plane, this would form a triange:



by Pythagoras, the hypotenuse of the triangle will have length $\sqrt{2}$, which the norm $1 + i$! Furthermore, since we know the adjacent and the opposite length, we can find the angle by $\tan(b/a)$. Therefore, we get

$$1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)$$

This can be properly formalized as the *change of variables* from cartesian coordiante functions π_x, π_y to the polar coordiante functions, which for any $z \in \mathbb{C}$:

$$z = |z|(\pm\cos(\theta) \pm i\sin(\theta))$$

Sometimes, the notation is collapsed to $z = |z|\operatorname{cis}(\theta)$ where cis the $\cos + i\sin$. The positive or negative value being added depends in what quadrant the complex number resides in. This also means that numbers in polar representation are defined up to $2k\pi$ rotations, since $2k\pi + \theta = \theta$ in terms of rotation. The value of $|z|$ is called the *magnitude* and the angle θ is called the *argument* (measured by starting on the positive real axis and moving anti-clockwise). The magnitude is usually represented with the variable r, so $a + bi = r(\cos(\theta) + i\sin(\theta))$. we usually write the argument of a complex number as $\arg(z)$. Thus, $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$. By doing some more manipulation, it is not hard to show that

$$z_1/z_2 = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

In other words, $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$. This should be ringing a bell: log has these properties. We will return to this in section 2.4.3. In combination with trigonometric identities, these formulas are quite easy to manipulate, for example:

1. $z^{-1} = \frac{1}{(\cos(\theta)+i\sin(\theta))} = r^{-1}(\cos(\theta) - i\sin(\theta)) = r^{-1}(\cos(-\theta) + i\sin(-\theta))$
2. $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ (known as *Moivre's Formula*)
3. $\sqrt[n]{z} = \sqrt[n]{r}(\cos(\theta/n) + i\sin(\theta/n))$. There are in fact multiple solutions: we need that $n\psi = \theta$. Naturally $\psi = \theta/n$ is a solution, but so is $\psi = \theta/n + 2k\pi/$ for $0 \leq k \leq n-1$ since
$$n\psi = \theta + 2k\pi = \theta$$

Unfortunately, adding and subtracting does not have a nice represetentation in polar form, however we can comfortably add and subtract vector in cartesian form, so this is of no real loss to us.This gives an easy geometric interpretation of adding, subtracting, multiplying, and dividing complex numbers.

For a proper change of variables from cartesian to polar, some care must me made so that $\arg(z)$ is in fact continuous and bijective. As we've seen, it is not surjective and is not continuous (namely, it jumps when crossing over from $2\pi - \epsilon < 2\pi < \pi$). To make it continuous, we have to make a choice of how we measure angle and how what interval we pick. The most common is called the *fundamental domain* or the *principal branch* (for reasons that shall be made clear once we study log) with it's elements being called the *standard argumetns* and is $(-\pi, \pi)$ with:

$$\arg(1) = 0 \qquad \arg(i) = \pi/2 \qquad \arg(-1) = \pi, \arg(-i) = -\pi/2$$

Notice we have eliminated $(-\infty, 0]$ In order to make the function continuous; there is no way around this due to non-trivial *monodramy.* The angle function defined on $\mathbb{C} \setminus (-\infty, 0]$ for which $\arg(1) = 0$ will be denoted Arg.

Another coordiante represetnation of complex numbesr is *euler coordinates.* The functions cos and sin are defined through an infinite power series that converges everywhere:

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \tag{1.2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{1.3}$$

Given appropriate manipulation of these power series, we can make them resembles the power series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{1.4}$$

which after the appropraite manipulation, we get:

$$|r|e^{\theta i} = |r|(\cos(\theta) + i\sin(\theta))$$

Thus, we the Euler representation of a complex number as well. We have similar niceties manipulating complex numbers as Euler representations:

1. $zw = |r||s|e^{(\theta+\psi)i}$
2. $z^{-1} = |r|^{-1}e^{-\theta i}$

and so on. Sometimes, they allow for simpler expressions for solutions of polynomials. For example, take the equation $x^n - 1 = 0$ or equivalently $x^n = 1$. By the fundamental theorem of Algebra, $x^n - 1$ has n roots. Since $x^n = 1$, these roots are called the *roots of unity*, and are studied extensively. Given Euler's representation of complex numbers, we know that $1 = e^0$, and so the roots of unity of $x^n - 1$ can be represented as

$$z_k = e^{\frac{2k\pi}{n}} \qquad k = 1, 2, ..., n$$

Notice that $(z_k)^n = 1$ for each k since $2k\pi = 0$. For example, here is the 5th root of unity: (some reaosn not rendering)

Square roots and Monodromy

The function $\theta(z)$ which gives the angle of z is *not* a continuous function: it is discontinuous at the line of non-negative real numbers. This creates the issue that the function $f : \mathbb{C} \to \mathbb{C}$ $f(z) = \sqrt{z}$ is discontinuous. As we mentioned earlier, f has *two* possible solutions. We can define f to have one of these, but then it will always have discontinuities! For example

$$f(z) = \sqrt{|z|}\left(\cos\left(\frac{\theta(z)}{2}\right) + i\sin\left(\frac{\theta(z)}{2}\right)\right)$$

has discontinuities since $\theta(z)$ is discontinuous. It is even impossible to define a function f where $f^2 - z = 0$ even on the domain of $|z| = 1$ (you can use a limit argument to show this). There is in fact a nice visual argument that can show us this. Let's say we want to make it continuous on S^1. Let's pick $\sqrt{1} = 1$, and now try to make it continuous on all of S^1 If we start moving counter-clockwise, we see that the $\sqrt{z}$ must always be half the angle, so we will keep halfing the angle but keeping the magnitude. Doing so, we will eventually end up at $\sqrt{-1}$, which by the way we've been choosing our outputs forces us to pick $\sqrt{-1} = i$. if we keep going, we keep halfing the angle until we reach back 1, at which point we have $\sqrt{1} = -1$... however we already established that $\sqrt{1} = 1$. This problem cannot be fixed even if we try a different approach. Another insightful way of seeing it that is worth mentioning is if we always take the smallest angle between the positive real axis and our complex number on the unit circle. The argument will remain the same when going counter clockwise towards -1, but now do this argument but going clock-wise. We again the angle in half again everytime, but notice that once we reach $\sqrt{-1}$, we are forced to pick $-i$! A visual might be helpful to see this:

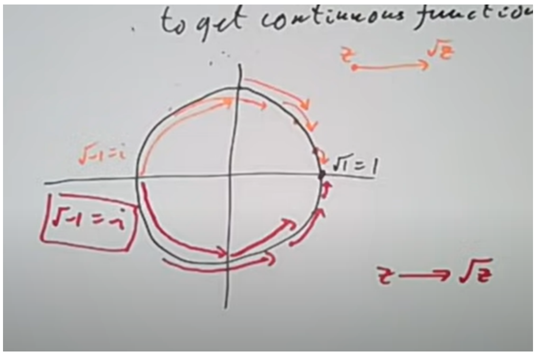


Figure 1.3: Continuity breaks for square root

Hence, $\sqrt{-}$ cannot be continuous on S^1, and hence more generally on $\mathbb{C}$. We would have to eliminate some points in order to make it continuous, usually the choice of poince is $(-\infty, 0)$. The fact that going around in different ways doesn't give us the same result is called *non trivial monodromy*.

Another approach some authors pick is to keep the entire domain $\mathbb{C}$ and to say that $\sqrt{-}$ is in fact a *mult-valued function*, meaning the output has both possibilities. This is not very desirable; so many of the tools we have worked with (topology, inner product and smoothness to name a notable few) all are based on the idea of function being single valued, we'd have to re-think all of these notions to incorporate multi-valuedness and the disconinuity that comes up becaues of the multi-valuedness!! To make the notion of "multi-valuedness", we will come up with a new domain for which f is in fact single-valued. To make this rigorous is too much at the moment, but the motivating idea is quite interesting. We will do so for $\sqrt{-}$. Take two copies of $\mathbb{C} \setminus (-\infty, 1]$. Label the bottom odge of the first two copies 1 and 3 respectively and the top edges 2 and 4. Then the resulting shape can no longer be embedded into $\mathbb{C}^2$, requiring $\mathbb{C}^3$ to embbedd it without intersection, but we may still visualize it like so:

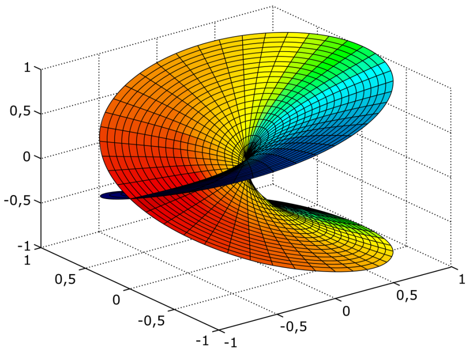


Figure 1.4: Branch Covering for $\sqrt{-}$

Let R represent this surface, known as a *Riemann surface*. We will now lift the function $\sqrt{-}$ to R so that it is again a function. In particular, notice that R is a covering space for $\mathbb{C}$, usually known as a *branch covering*. Then we define map f from R to $\mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\sqrt{-}"'] & \end{tikzcd}$$

Then f is now a *single valued* function. This same process will be done for many other function to "recover" a well-defined function. Famously, there is no proper complex log, there are infinitely many outputs. The corresponding branch covering of $\mathbb{C}$ for complex log will be:

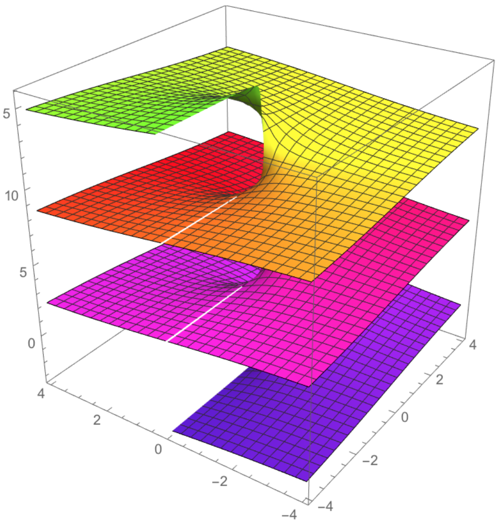


Figure 1.5: Branch Covering for complex log

Exercise 1.2.2

1. If $a, c \in \mathbb{C}$ and $\operatorname{Im} a\bar{c} = 0$, then $c = r \cdot a$ where $r \in \mathbb{R}$
2. Let $z = a + bi \in \mathbb{C}$ and define $\dot{z} = b + ai$. Show that $z \cdot \dot{z}$ is strictly imaginary unless $\dot{z}z = 0$ case it's (naturally) both real and imaginary. Is there a geometric interpretation of this result?

1.2.3 Riemann Sphere

There is one more important geometric interpretation of the complex numbers that must be mentioned that allow for better manipulation of poles at infinity: the extended complex numbers. In $\mathbb{R}$, we can "compactify" $\mathbb{R}$ by adding two points, $\pm\infty$ and defining appropriate operations. In fact, we can define $\mathbb{R}$ by simply defining one point at infinity, $\mathbb{R} \cup \{\infty\}$ (this is called the *one-point compactification*, see my EYNTKA Topology). We will label this as $\mathbb{R}_\infty$. Notice that $\mathbb{R}_\infty$ is homeomorphic to S^1 via the following homeomorphism: if we take any $z \in S^1$ as a tuple (x, y), then

$$(x, y) \mapsto \frac{x}{1 - y}$$

and $(0,1) \mapsto \{\infty\}$. This shows that $\mathbb{R}_\infty$ is indeed compact, and furthermore we can try to work to incorporate infinity into our analysis. However, we have the problem that $\mathbb{R}_\infty$ looses the well-ordering of $\mathbb{R}$, namely since ∞ is both smaller and larger than any real number, in particular $\infty < 0 < \infty$. However, if we instead take $x \mapsto |x|$, then ∞ is the natural value tha appears as either $\lim_{x\to-\infty} |x|$ or $\lim_{x\to\infty} |x|$. Thus, it is natural to think of $\mathbb{R}_\infty$ as the codomain of for $|\cdot|$, and more generally some norm $\|\cdot\|$ on a real vector-space rather than an element that we can add to the ordering of $\mathbb{R}$, and is "closed" under convergent and divergent sequences which are eventually strictly increasing or decreasing[4].

Though order is lost, some arithmetic properties are still well-defined (these are shown to be well-defined through the use of arbitrrary sequences):

1. $a + \infty = \infty + a = \infty$ for all finite a
2. $b \cdot \infty = \infty \cdot b = \infty$ for $b \neq 0$, including ∞

It is impossible (as shown in 1st year calculus) to define $\infty + \infty$ or $0 \cdot \infty$ since there can be sequence that approach these value's differently depending on how "fast" each term goes to ∞ or 0 (recall the l'hopital rules). It is however possible to define the convention that $a/0 = \infty$ or $b/\infty = 0$ for $b \neq \infty$. Though there is some arithmetics well-defined in $\mathbb{R}_\infty$, if we transfer over to S^1 via the stereogrpahic homeomoprhism, it is much harder to perform arithmetics, and so we would usually take the S^1 interpretation of $\mathbb{R}_\infty$ for it's geometric benifits.

Let's now work with $\mathbb{C} \cup \{\infty\}$. Like with the real case, let's label it $\mathbb{C}_\infty$. Some other common labels are $\widehat{\mathbb{C}}$ and $\overline{\mathbb{C}}$. Just like for $\mathbb{R}$, we are extending $\mathbb{C}$ by a single point instead of a circle of points since we are more so interesting in the "size" of points approaching infinity rather than any sequence converging to some "appropriate" point at infinity (like a $\infty(\cos(\theta) + i\sin(\theta))$). Let's make precise the relation between $\mathbb{C}_\infty$ and S^2. Identify $\mathbb{C}_\infty \setminus \{\infty\}$ with the x-axis and y-axis of $\mathbb{R}^3$, where the x-axis representing the real component and the y-axis representing the imaginary component. Naturally, no point in $\mathbb{R}^3$ is identified with the point ∞ in $\mathbb{C}_\infty$. Take a unit sphere centered at the origin of $\mathbb{C}_\infty$. Now, starting from the north-pole, project a down through the sphere and into the complex plane. This is a one-to-one correspondence. If $S^2 = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ x, y, z \in \mathbb{R}\}$ then

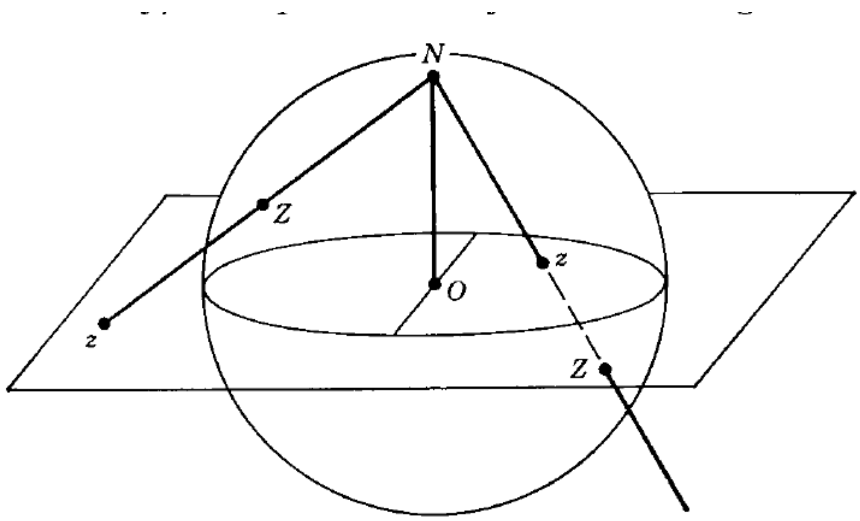
$$z = \frac{x_1 + ix_2}{1 - x_3}$$

By taking the square of the modulus ($|z|^2$), we can re-arrange that equation to get

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2} \quad x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)} \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{1.5}$$

By letting the pole $(0, 0, 1)$ correspond to the point at infinity, we get that the sphere is homeomoprhic to $\mathbb{C}_\infty$ (and hence, we get a nice geometric object representing $\mathbb{C}_\infty$ without needing to "distinguish" a point). This process is called *stereographic projection*, and visually looks like so:

[4] Note that it is not closed for any type of sequence, for example for oscillating sequences



Note that the points where $x_3 < 0$ corresponds to the disk $|z| < 1$, and when $x_3 > 0$ the points correspond to $|z| > 1$ while if $x_3 = 0$ then $|z| = 1$. We may also stereographically project from the south-pole, giving us

$$z' = \frac{x_1 - ix_2}{1 + x_3}$$

Note that z an z' are related by the equation $zz' = 1$ unless z or z' is a pole N or S.

For future purposes, we will quickly say a word on how circles on the Riemann sphere map to $\mathbb{C}_\infty$. Any cirlce *not* passing through the pole maps a circle on the plane. If a point of a circle is at the north pole, then the cirlce maps to a *line* in $\mathbb{C}_\infty$. To see this, notice that a cirlce on the Riemann sphere lies in the plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$ where we can assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \leq \alpha_0 < 1$. From equation (1.5), we can re-write this previous equation as:

$$\alpha_1(z + \overline{z}) + \alpha_2 i(z - \overline{z}) = \alpha_3(|z|^2 - 1) = \alpha_0(|z|^2 + 1)$$

or:

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0$$

When $\alpha_0 \neq \alpha_3$, this is the equation of a circle, and when $\alpha_0 = \alpha_3$, this is the equation of a straight line. Conversely, the equation of a cricle and a line can always be written in this form.

Finally, on S^2, it is easy to calcualte the distance between the two points (i.e. the geodesic) and so have a finite metric on $\mathbb{C}_\infty$: If $z = (x_1, x_2, x_3)$ and $z' = (x_1', x_2', x_3')$, we first get the equation of the geodesic:

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 = (x_3 - x_3')^2 = 2 - 2(x_1 x_1 + x_2 x_2' + x_3 x_3')$$

(Some more computation that I will skip for now, but have to return to get computational intuition, p. 20 Ahlfors)

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

or if $z' = \infty$:

$$d(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}$$

1.3 Complex Functions

In this section, we will go over how to think about complex functions and emphasize geometric intuitions. For convenience, we will use z, w as complex numbers and x, y as real numbers. a *real function* is a function with codomain $\mathbb{R}$, while a *complex function* has a codomain $\mathbb{C}$. Whether the domain is real, complex, or anything else (ex. a manifold, a vector space, a topological space) is usually deduced from context.

1.3.1 Complex Linear Transformations

Since a function is differentiable if it is locally linear, it is worth taking a moment to understand complex linear functions. Let's start with the simplet $\mathbb{C}$-linear transformation: a map $f : \mathbb{C} \to \mathbb{C}$ where

$$f(z) = az \qquad a \in \mathbb{C}$$

representing $z = x + yi$ and $a = a + bi$, we get:

$$(a + bi)(x + yi) = (ax - by) + (bx + ay)i$$
$$\Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}$$

Notice that this linear translation is a rotation and a scaling. In particular, it *preserved the angles*. A function that is angle-preserving is called *homothetic*. Furthermore, we cannot have any mirror symmetry. For example

$$z \mapsto \overline{z}$$

is homothetic, but it is *not* $\mathbb{C}$-linaer. The way I like to think about it is that if you are in one "mirror-space", so to speak, then you will always measure the angle going from a fixed choice of either left or right. If you change mirror spaces, then you change which side you measure your angle from. Thus, when we think of angle preserving we will think of angle preserving with respect to a fixed side. From this persepctive, $z \mapsto \overline{z}$ maps an angle θ to $2\pi - \theta$, which in most cases is not angle perserving.

Lemma 1.3.1: Real Angle Preserving Then Complex

Let $T : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear function that perserves angles. Then there exists an $a \in \mathbb{C}$ such that $T(z) = az$ or $T(z) = a\overline{z}$

Proof **:**
Since T is homothetic, it must be bijective. Let S be a homethetic function such that $S^{-1}T(1, 0) = (1, 0)$. $S^{-1}T$ is alos homothetic, so it must be that Since $S^{-1}T(0, 1) = (0, c)$ for some $c \neq 0$. Since $S^{-1}T$ is linear, $S^{-1}T(1, 1) = (1, c)$. Since $S^{-1}T$ is angle preserving, it must be that $c = \pm 1$. depending on which this is, we get $T(z) = az$ or $T(z) = a\overline{z}$, completing the proof.

1.3.2 Polynomial and Rational Functions

Polynomial functions (with real coefficients) are easy example of real differentiable functions. So too will complex polynomials. It will turn out that a slight-generalization of these polynomials give a huge example of complex differentiable functions, in particular for allowing the *quotient* of polynomials. Functions that are polynomials over polynomials are called *rational polynomials* or *rational functions*. In this section, we give some common names to notions for rational functions. By the fundamental theorem of algebra, we get

$$P(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$$

for n not necessarily distinct roots (we will prove this theorem in section ref:HERE). For now, we will assume differentiation of complex polynomials is identical to that of real polynomials (which will turn out to be the case, see section 2.1). If $P(\alpha_i) = 0$, we call α_i a zero of P. Since the roots need not be distinct, we call the order of a zero the number of roots that are equal. Note that if α has order h for P, then $P^{(h-i)}(\alpha) = 0$ for $1 \leq i \leq h-1$, and $P^{(h)}(\alpha) \neq 0$. Note that if the order of an α is 1, then $P(\alpha) = 0$ but $P'(\alpha) \neq 0$. We call a zero of order 1 a simple zero.

Definition 1.3.1: Rational Function

Let $p(x)$ and $q(x)$ be polynomials over $\mathbb{C}$ where $\gcd(p(x), q(x)) = 1$ (i.e., they don't have any similar roots). Then

$$R(x) : \mathbb{C}_\infty \to \mathbb{C}_\infty,\ R(x) = \frac{p(x)}{q(x)}$$

is called a *rational function* or a *Fracitonal Linear Transformation* over $\mathbb{C}_\infty$.

For more information, see `https://encyclopediaofmath.org/wiki/Fractional-linear_mapping`. We let the codomain be the extended complex numbers in order that the denominator be able to accept 0 values. In particular, we'll define

$$R(x) = \frac{p(x)}{q(x)} = \frac{n}{0} = \infty$$

This is another reason to allow only one point of compactification, to allow this extension to be well-defined. This also motivates the name pole: these points get mapped to the north pole of the Riemann sphere. It also allows us to rigorously capture the idea by doing:

$$R(\infty) = \lim_{z\to\infty} R(z)$$

and so R defined on $\mathbb{C}_\infty$ is indeed well-defined. since it is posible that $R(z) = \infty$, it would be convenient to find a way to find these "poles" of the functions (a pole is when the polynomial function in the denominator is zero). To find these, we will define

$$R_1(z) = R(1/z)$$

And define $R(\infty) = R_1(0)$. From this, we can define the order of zeros (i.e. roots) or poles at ∞ to be the zeros or poles of the origin of R_1. Now, given some R:

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

Then

$$R_1(z) = z^{m-z}\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}$$

From this we, can see the order of zero's or poles at ∞. If $m > n$, then the zeroes at infinity has order $m - n$. Conversely, if $m < n$, then the poles at infinity have order $n - m$. If $n = m$, Then we either get

$$R(\infty) = a_n/b_m \neq 0 \quad \text{or} \quad \infty$$

with all the zeros and poles established, we can count the number of zeros or poles a given rational function has, including the zero's or poles at ∞; namely by the observations in the above paragraph we have that the number of zeroes/poles is always $\max(n, m)$. Since this is a consistent number, we call this number the *order* of the rational function. The order of a rational function is quite stable. The function $R - a$ has the same number of poles and zeros and hence the same order (meaning any $R(z) = a$ of order p has p roots). Furthermore, $R - a$ (called a *parrallel transition*) keeps the point ∞ stable, while the rational function $1/z$ (called the *inversion*) interchanges 0 and ∞.

Treating differentiation of rational functions as a formal operation [5], we also see that the poles are stable under differenitation, which we can see using the quotient rule. As an exercise, check that if a pole of $R(z)$ has order k, then a pole of $R'(z)$ has order $k + 1$.

We may also use differentiation and the inverse function to find the order of a pole: Using $R_1(z)$ to flip the poles and the roots, we can then take the formal derivative until we get a nonzero value.

1.3.3 Möbius Transformation

(see this website: here)

A rational function of particular interest which we will focus on in section ref:HERE is those of of order 1. Due to their importance, they are given a name for reference:

Definition 1.3.2: Möbius Transformatino

Let f be a rational function of degree 1

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \alpha\delta - \beta\gamma \neq 0$$

Then f is called a *Möbius transformation*.

The reason we have $\alpha\delta - \beta\gamma \neq 0$ is so that there is 1 pole and 1 root. Let's say that

$$S(-b/a) = \frac{0}{c\frac{-b}{a} + d} = 0$$

If the denominator was 0 we would get 0/0, that is the root would not be well defined. That would happen if:

$$c\frac{-b}{a} + d = 0 \quad \Leftrightarrow \quad -cb + ad = 0$$

Hence, we will require sthat $ad - bc \neq 0$. If $ad - bc = 0$, then $c = \frac{ad}{b}$, which substituing in the 1st order FLT we get:

$$\frac{az + b}{\frac{ad}{b}z + d} = \frac{az + b}{\frac{d}{b}(az + b)} = \frac{b}{d}$$

[5] think of it as a new algebraic operation on the ring of rational functions over $\mathbb{C}$

and hence has no poles. Manipulating the equation, we get that the inverse is:

$$S^{-1}(w) = \frac{dw - b}{-cw + a}$$

showing the inverse is also a first order rational function. The key that we care about these functions is that the pole is sent to a single point. Once we define complex differentiation, we will see that Möbius transformations are the diffeomorphissm of $\overline{\mathbb{C}}$, making them worthwhile to study. Already we can see that such a function is certainly a homeomorphism, and hence they belong to $\mathrm{Aut}_{\mathbf{Top}}(\mathbb{C})$.

Given Möbius transformations are such nice function, they should be relatively simple to imagine. Indeed, if $\gamma = 0$, then we have $\frac{\alpha z}{\delta} + \frac{\beta}{\delta} = az + b$ for some constants a and b showing that this möbius transformation is just scaling and transformation, while if $\gamma \neq 0$ then

$$\begin{aligned}
\frac{az+b}{cz+d} &= \frac{az}{cz+d} + \frac{b}{cz+d} \\
&= \frac{a}{c} + \frac{\frac{-ad}{c}}{cz+d} + \frac{b}{cz+d} \\
&= \frac{a}{c} + \frac{b-ad}{cz+d} \\
&= \frac{a}{c} + \frac{(1/c)(b-ad)}{z+d/c} \\
&= \alpha + \frac{\beta}{z+\gamma}
\end{aligned}$$

showing that in this case a möbius transformation is a scaled by β, translated by α, recipricated around γ function. This is nice to keep in mind, though when it comes to manipulating the original möbius transformation it might be hard to see what's going on. Another way we can decompsose a Möbius transformation is given if $c \neq 0$:

$$f_1(z) = z + d/c \quad f_2(z) = 1/z \quad f_3(z) = \frac{bc - ad}{c^2}z \quad f_4(z) = z + a/c$$

then

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{az+b}{cz+d}$$

The following proposition may help visualize the images of this type of function:

Proposition 1.3.1: Properties Of Möbius Transformations

Let $f : \mathbb{C} \to \mathbb{C}$ be a möbius transformation. Then:

1. f circle's and lines to circles and lines
2. f is uniquely determined by where it takes three distinct points of $\mathbb{C}$ to any three distinct points in $\mathbb{C}$.
3. f fixes 1, 2, or all points.

Proof :

1. in Alhors
2. Suppose z_1, z_2, z_3 are distinct points in $\mathbb{C}$ and w_1, w_2, w_3 are distinct points in $\mathbb{C}$. We'll start by showing there exists a map such that $z_1 \mapsto 1$, $z_2 \mapsto 0$ and $z_3 \mapsto \infty$. First, invert about any cirlce centered at z_3, whihc takes z_3 to ∞. The points z_1, z_2 get mapped to z_1', z_2', neither can be ∞, Next, translate z_2' to 0, which keeps z_1 mapped to ∞ and maps z_3' to z_3''. Finally rotate and dilate about the origin in such a way that $z_3'' = 1$. This keeps 0 and ∞ fixed and hence works. In order for this to be a Möbius transformation, relfect across the real axis: this keeps $1, 0$, and ∞.

   The last steps of the proof are left as an exercise.

We next analyze the global properties of Möbius transformations. These will be important for us in section ref:HERE, where we prove two important theorems (note that biholomorphic means complex diffeomorphic)

1. *Riemann mapping theorem*: If U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f from U onto the open unit disk
2. *Uniformization theorem*: Let R be a Riemann surface (a complex 1-manifold) and $U \subseteq R$ a simply connected connected open subset of R. Then U is biholomorphic to one of the following:
   (a) Riemann sphere (which we'll eventually show is $\mathbb{C}\mathrm{P}^1$)
   (b) $\mathbb{C}$
   (c) $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$

In other words, we will use Möbius transformation in classifying all one-dimensional complex manifolds, for which there is surprisingly only 3 up to isomorphism! These following theorems give us the appropriate global mapping properties that will be lemma's for the above two theorems.

Proposition 1.3.2: Möbius Transformation On Upper Half Plane

Let f be a Möbius transformtion with domain $\mathbb{H}$. Then:

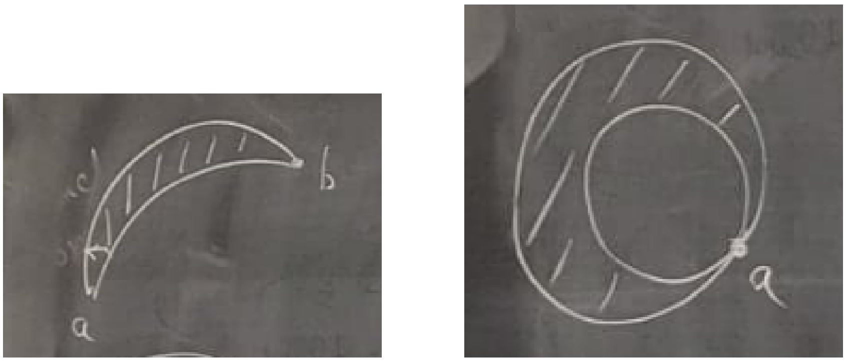
1. if $f(\mathbb{R}) \subseteq \mathbb{R}$, then $a, b, c, d \in \mathbb{R}$
2. if $f : \mathbb{H} \to \mathbb{D}$, then f must be of the form:

$$\eta \frac{z - \alpha}{z - \overline{\alpha}}$$

   with $|\eta| = 1$ and $\operatorname{im} \alpha > 0$.

Proof :
exercise, or look at your notes

To better understand such maps, we will see how they map certain shapes to either $\mathbb{H}$ or $\mathbb{D}$. These are commonly occuring shapes in complex analysis and so knowing where they map is useful, in particular they will be useful when constructing Riemann surfaces via gluing. A circular wedge is a shape of one of these two forms:



(a) distinct endpoints (b) same endpoints

Figure 1.6: cirtulcar wedges

For figure (a), map a to 0 and b to infinity likeso:

$$\frac{z-a}{z-b}$$

This will stretch out (a) so that it looks like so

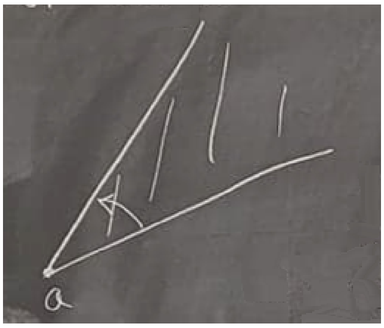


Figure 1.7: apply $\frac{z-a}{z-b}$

Now just rotate so that one of the wedge lie on the x-axis, and scalr by an appropriate z^{α} so that it maps to $\mathbb{H}$. For figure (b), map it by $\frac{1}{z-a}$ so that it looks like so:

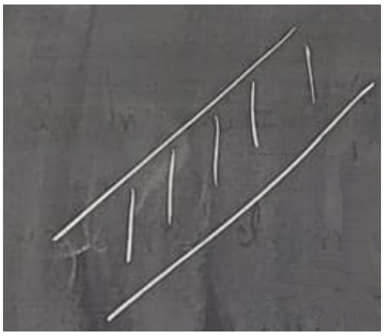
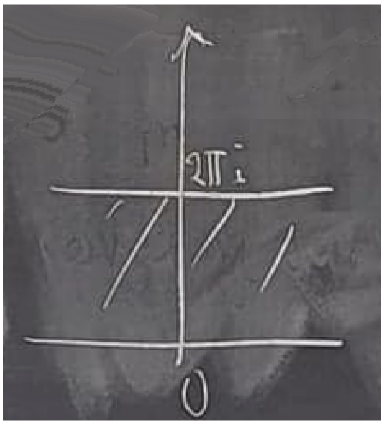


Figure 1.8: apply $\frac{1}{z-a}$

then rotate it and dilate it so that we get:



and finally exponentiate it to get $\mathbb{H}$. The last shape we will care to map if $\mathbb{C} - I$ for some open interval I. Without loss of generailty say $I = (-1, 1)$. First apply the map $\frac{z+1}{z-1}$ to map $-1 \mapsto 0$, $1 \mapsto \infty$ and $0 \mapsto 0$, then apply the square root function $z^{1/2}$ for your choice of square root function, which will create a $\mathbb{H}$ but with the y-axis, then map that to the disk $\mathbb{D}$ so that we may map it to $\mathbb{H}$. this will make us end up with the equation $w = z - \sqrt{z^2 - 1}$.

1.3.4 Partial Fractions

Every rational function can be represented as a *partial fraction* which splits the rational function as a sum of two rational function with the degree of the numerator being smaller than the degree of the denominator. In order for this to be the case, we need that $R(z)$ has a pole at infinity. Such a decomposition is well-defined since polynomial devition is well-defined[6].

[6] If you've taken algebra, recall that $\mathbb{C}[x]$ is a Euclidean Domain

Definition 1.3.3: Partial Fraction Decomposition

Let $R(z)$ be a rational function. Then the resulting partial fraction decomposition will be of the form

$$R(z) = G(z) + H(z)$$

where $G(z)$ is a *polynomial* without a constant term, and $H(z)$ is finite at ∞. The degree of G is the order of the pole at ∞. The polynomial G is called the *singular part of* R *at* ∞.

Example 1.1: Partial Fraction Decomposition

here

We can manipulate this equation further to get some nice results that allow for easier integration of more complex functions. Let $b_1, ..., b_n$ be the poles of $R(z)$. Define a new function $R'_j(\zeta) = R(B_j + 1/\zeta)$. R' has a pole at ∞. By definition:

$$R'_j(\zeta) = R\left(b_j + \frac{1}{\zeta}\right) = G_j(\zeta) + H_j(\zeta)$$

Or, with some simple change of variables:

$$R(z) = G_j\left(\frac{1}{z - b_j}\right) + H_j\left(\frac{1}{z - b_j}\right)$$

Notice how now $R(z)$ is written in a different from: the polynomial G_j has now $\frac{1}{z-b_j}$ as its indeterminate. The polynomial G_j is called the *singular part of* $R(z)$ *at* b_j. By construction, $H_j\left(\frac{1}{z-b_j}\right)$ is finite for $z = b_j$. Now, consider the expression

$$R(z) - G_j\left(\frac{1}{z - b_j}\right) + H_j\left(\frac{1}{z - b_j}\right) = 0$$

(words in Althors boringly explaining here on page 32 why we get)

$$R(z) = G(z) + \sum_{i=1}^{q} G_j\left(\frac{1}{z - b_j}\right)$$

which is the expression used in calculus to simplify integrals.

2

Complex Differentiation

Now that we've studied the arithemtics, geometry, and analytical/topological properties of $\mathbb{C}$, we will move on to studying *differentiable complex functions*. Since $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, we see that continuity of $\mathbb{C}$ works identicaly to that of $\mathbb{R}^2$. We will find that differentiability will produce some very different results, which will in fact greatly restrict which functions are differentiable and give *much* nicer results than the theory of differentiability of real numbers. We will in fact rather quickly answer lot of the "usual" questions that we had to build up a lot of work for in real analysis rather quickly in complex analysis, and be seeking out "pathological" cases to work with since things are so nice.

After defining and expanding and exploring complex-differentiable functions and finding some necessary and sufficient conditions for a functions to be complex-differentiable, we will start building up a repertoire of complex-differentiable function. We will show that complex-linear functions are complex-differentiable (as we hope they should be). Then since the sum, product, and quotient of complex-differentiable functions is complex-differentiable, we will get that all polynomials and rational functions are complex-differentiable. The next step would be to explore infinite polynomials, that is power-series. Naturally, power-series need not converge everywhere and so a power-series will only be defined up to domain of convergence which will be well defined by a radius of convergence (with the exception of the border, for which we have to be more careful). Using power-series, we can show that many of our common functions from real analysis are complex-differentiable, most importantly $e^x, \sin, \cos, \log$. Using the technics we've built up, we can show many properties of these functions, like periodicity or the algebra of the exponential ($e^{x+y} = e^x e^y$).

2.1 Build-up

let f be a real or complex function. We may ask whether some point a in the codomain satisfies:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-Df(a)(h)}{\|h\|}=0$$

The answer will really depend on the domain and codomain. Let's say $f:\mathbb{C}\to\mathbb{R}$ is a differentiable function (i.e. the limit above exists for all $z\in\mathbb{C}$). Since we're working in "one-dimension" over $\mathbb{C}$, let's use the usual definition of the derivative:

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(h)}{h}$$

Remember that the value $f'(z)$ must be independent of how h approaches 0. Let's first approach $h\to 0$ as only real numbers (i.e., immediately make the imaginary component 0 and keep approaching that way). Then $f'(z)$ is the quotient of two real numbers, and hence will be real. On the other hand, if we $h\to 0$ from the complex component (i.e. immediately make the real component 0 and keep approaching that way), then $f'(z)$ is the quotient of a real number by an imaginary number. As we know

$$\frac{1}{i}=-i$$

and so the resulting number must be imaginary. The only number to be both real and imaginary is 0, and hence if f is indeed differentiable, then it's derivative must be 0 (or f is not differentiable). As a consequence of this, by what we know of integration, the resulting function must be *constant*. This makes sense if we recall $\mathbb{C}$-linear functions preserve angles, and projecting to $\mathbb{R}$ would certainly not perserve angles locally.

If we switch the order of the domain and co-domain and consider $f:\mathbb{R}\to\mathbb{C}$, then this case can be thought of as like $f:\mathbb{R}\to\mathbb{R}^2$. Consider that any such function can be represented as $f(t)=x(t)+iy(t)$ for appropriate functions $x:\mathbb{R}\to\mathbb{R}$ and $y:\mathbb{R}\to\mathbb{R}$; it might be more convincing if I write this in vector form:

$$f(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

The element i will act simply as a scalar (or the vector component $(0,1)$) since it is in the numerator, and so

$$f'(t)=x'(t)+iy'(t)$$

and so f is differentiable if x and y are differentiable. Thus, the case of $f:\mathbb{C}\to\mathbb{C}$ is truly different than the real case, so the rest of this section comes to figuring the structure of such a function

2.2 Holomorphic Functions

To find out the conditions for which a function $f:\mathbb{C}\to\mathbb{C}$ must be complex-differentiable, we will define a function to be differentiable, and then find then necessary condition's that this will impose onto f. We will give a special name to functions that are complex-differentiable:

Definition 2.2.1: Holomorphic (Analytic) Function

let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a function with Ω open. Then if for every point $z \in \mathbb{C}$, $f'(z)$ is defined, that is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(h)}{h}$$

then f is said to be *holomorphic* or *analytic*

You might recall from calculus that an analytic function is a function such that the function can be represented as a power-series at every point. Soon, we will show how this is true for all Holomorphic functions, and in fact is both a necessary and sufficient condition for a holomorphic function, and so the words Analytic and Holomorphic are used interchangeably by most Mathematicians.

The key example of holomorphic functions are:

Example 2.1: Holomorphic Functions

1. the function $f(z) = c$ for some $c \in \mathbb{C}$ is a holomorphic function
2. the function $f(z) = z$ is a holomorphic function
3. the function $f(z) = az$ for some $a \in \mathbb{C}$ is a holomorphic function (in other words, $\mathbb{C}$-linear functions ought to be complex-differentiable).

Holomorphic functions work well under our usual binary operations, and have many properties we would expect of them:

Proposition 2.2.1: Properties of Holomorphic Functions

Let $f : \mathbb{C} \to \mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions. Then

1. $f + g$, fg, $g \circ f$, and zf (for some $z \in \mathbb{C}$) are holomorphic functions (hence, with the result from example 2.1, all polynomials are holomorphic) with the same derivative formula as the real equivalent. Similarly, f/g is an holomorphic function, provided that $g(z) \neq 0$ for all $z \in \mathbb{C}$ (i.e. g does not vanish). Hence, all well-defined rational polynomials are holomorphic functions. Sometimes, we abuse notation and say f/g is holomorphic even if $g(z) = 0$ for some points by excluding those points from the domain.
2. f is continuous.
3. f is real-differentiable (interpreting f as $f : \mathbb{R}^2 \to \mathbb{R}^2$)
4. There exists functions u, v such that f can be written as $f(z) = u(z) + iv(z)$. Both u and v are continuous. In particular, if $z = x + iy$, then we have $f(z) = u(x, y) + iv(x, y)$. This is called the *standard representation of* f.

***Proof* :**

1. This is the same breaking down the limit trick as seen in elementary calculus

2. Let
$$f(z+h)-f(z)=\frac{h}{h}(f(z+h)-f(h))=h\cdot\left(\frac{f(z+h)-f(h)}{h}\right)$$
and, evaluating the limit, we get
$$0\cdot f'(z)=0$$
and so
$$\lim_{h\to 0} f(z+h)-f(z)=\Leftrightarrow \lim_{h\to 0} f(z+h)=f(z)$$
and since this is true for all h, f is continuous

3. Since f is holomorphic, then f', exists, that is for any $z\in\mathbb{C}$, the limit
$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$
exists. Take the absolute value of the right hand side:
$$\left|\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\right|$$
since $|\cdot|$ is continuous, it passes through limits:
$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right|=\lim_{h\to 0}\frac{|f(z+h)-f(z)|}{|h|}$$
which, since the topology of $\mathbb{C}$ is identical to that of $\mathbb{R}^2$, this is equivalent to saying that f, if interpreted as $\mathbb{R}^2\to\mathbb{R}^2$, is real-differentiable!

4. Recall that every element of $\mathbb{C}$ can be written of the form $a+bi$. Thus, the output of any function $f(z)$ will be of the form $a+bi$. The real and imaginary part are dependent on both a and b, and $f(a+bi)=u(a,b)+iv(a,b)$ is a well-defined function where we have $u,v:\mathbb{R}^2\to\mathbb{R}$. Furthermore, since f is holomorphic, it is continuous, and so restricting to only going through the real or imaginary part must also be continuous since projections are continuous, and hence u and v are continuous. Conversely, if u and v are continuous, then so so is adding them (mutiplying by i is equivalent to specifying a basis element), and so f is continuous.

   Notice that since the projection map $\pi:\mathbb{R}^2\to\mathbb{R}$ is a differentiable function, both u and v are real-differentiable when interpreted as $u,v:\mathbb{R}^2\to\mathbb{R}$. Note that they *cannot* be interpreted as complex-differentiable, since all functions of the fomr $u,v:\mathbb{C}\to\mathbb{R}$ are constant, and u,v are certainly not always constant.

The collection of holomorphic function $f:\mathbb{C}\to\mathbb{C}$ is sometimes denoted $C^\omega(\mathbb{C})$. By the above proposition, $C^\omega(\mathbb{C})$ is a $\mathbb{C}$-algebra, and is almost a division $\mathbb{C}$-algebra[1]. We have yet to establish if we can put any properties on u,v to make f holomorphic; we will do so in section 2.2.2. The following proposition we will outline a necessary condition that needs to be put on u,v:

[1] In section 2.5.3, we shall show how we may loosen the definition a bit to make it into a field

Proposition 2.2.2: Cauchy-Riemann Condition

Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function with dummy variable h that approaches 0. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be the standard representation of f, so that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are well-defined. Then

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Or equivalently, if we represent f as $f(z) = u(z) + iv(z)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations are called the *Cauchy-Riemann Equations*.

Since we can take the limit any way we want, using the Cauchy-Riemann equation, we have that:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

making it relativly easy to find the complex-derivaive of f by simply using real-differentiation. If $a = \frac{\partial u}{\partial x}$ and $b = \frac{\partial u}{\partial y}$, then in matrix form these equations are:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which should strongly remind the reader of complex-linear function from section 1.3.1.

***Proof* :**
Recall that the limit at z must be the same regardless of how h approaches 0. If we approach strictly from the real part, then we will get the partial derivative

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

On the other hand, if we approach from a purely imaginary number, we get an intriguing result: let $h = ik$ for some $k \in \mathbb{R}$. Then

$$f'(z) = \lim_{ik\to 0} \frac{f(z+ik) - f(ik)}{ik} = \lim_{ik\to 0} -i \cdot \frac{f(z+ik) - f(z)}{k} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since f' must be equal no matter which direction we approach from, it must be that:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

or equivalently, since the real and imaginary part of these equations must agree, we get that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

as we sought to show

Thus, for a function to be holomorphic this partial differential equation *must* be satisfied! Note that the converse is not true: If we have two functions u, v satisfying the Cauchy-Riemann condition, it does not imply that $f = u+iv$ is holomorphic (as we'll show in example 2.2). On the other hand, with some added regularity to f (or to it's partials or u, v), the resulting function is complex-differentiable. We will explore more on what regularity condition's we can pose. For now, the simplest we can give is that f is real-differentiable and satisfies C.R. then it is Holomorphic (note that this means that C.R. does not imply differentiability, as we'll see a counter-example soon)

Proposition 2.2.3: C.R + Real-Differentiable Implies Complex Differentiable

Let $f : \mathbb{C} \to \mathbb{C}$ be a real-differentiable[a] functions satisfying the Cauchy-Riemann equations. Then f is holomorphic.

[a]Sometimes called Fréchet differentiable due to it's generalization to normed vector spaces

***Proof* :**
Since f is real-differentiable, there exists a matrix A such that

$$F((x, y) + h) = F(x, y) + Ah + o(|h|) \tag{2.1}$$

as $h \to 0$ with

$$A = \begin{pmatrix} \frac{\partial u}{\partial x}(x, y) & \frac{\partial u}{\partial y}(x, y) \\ \frac{\partial v}{\partial x}(x, y) & \frac{\partial v}{\partial y}(x, y) \end{pmatrix}$$

By the Cauchy-Riemann equations, the matrix A represents the multiplication by the complex number

$$\lambda = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y)$$

and so we can re-write equation (2.1) in complex terms with $z = x + iy$ as

$$f(z + h) = f(z) + \lambda h + o(|h|)$$

as $h \to 0$. But this implies f is holomorphic, completing the proof.

As we know, a function is differentiable if it is locally linear. Thus, we would expect that a holomorphic function is locally $\mathbb{C}$-linear. As we know from multi-variable calculus, a function f real-differentiable at a produces a tangent map df_a which is $\mathbb{R}$-linear between tangent spaces. We will show that if $f = u + iv$ is holomorphic, then df_a is $\mathbb{C}$-linaer.

Proposition 2.2.4: Holomorphic, Then Locally Complex Linear

Let f be holomorphic, $a \in \mathbb{C}$, and define df_a to be the tangent map of a real-differentiable function as defined above and $f = u + iv$ be the standard represetnation. Then u, v satisfy the Cauchy-Riemann condition if and only if df_a is $\mathbb{C}$-linear, and:

$$\frac{\partial f}{\partial z}(a)(\zeta) = df_a(\zeta) \qquad \forall \zeta \in \mathbb{C}$$

***Proof* :**
Let $d_p f : \mathbb{C} \to \mathbb{C}$ be the tangent map at $p \in \mathbb{C}$. Then as we showed in proposition 2.2.3, the linear approximation satisfies the Cauchy-Riemann equations, which forces our matrix to be of the form representing complex multiplication. But then $d_p f$ is $\mathbb{C}$-linear.

Example 2.2: Holomorphic and non-Holomorphic Functions

1. Let
$$f(x + yi) = \sqrt{|x||y|}$$
Then f is Cauchy-Riemann, but is no Holomorphic (approach from $h = (h_1, h_2) = (t, t)$ and $h = (h_1, h_2) = (t, -t)$). Notice that f is not real-differentiable, showing satisfying the Cauchy-Riemann condition is insufficient for f to be complex-differnetiable (which should make sense, recall we needed C^1 for existence of partials implying existence of total derivative)

2. You should check that $f(z) = \overline{z}$ (i.e. the conjugation function) does not satisfy the Cauchy-Riemann. However, $f(z)$ is real-differentiable. Thus, though it looks very differentiable (it's real-differentiable), it cannot be!

3. The $f(z) = x^2 - y^2 - i2xy = z^2$ will satisfy the Cauchy-Riemann equations and is real-differentiable.

4. More generally, any polynomial
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$
satisfies the Cauchy-Riemann equations and are holomorphic, with derivative
$$f'(z) = n a_n z^{n-1} + (n-1) z_{n-1} z^{n-1} + \cdots + a_1$$

5. $R = f/g$ be a ratioanl function. Let $\mathbb{C} - P$ be the domain of R, where P is the set of poles. Then R is holomorphic. R is almost holomorphic on all of $\mathbb{C}$, only failing on a finite set of points. A function which is holomorphic on all but a discrete set of points is called a *meromorphic* functions (from the greek *mero* meaning "part").

6. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then we see in the proof of the C.R. equations that $f' = \frac{\partial f}{\partial x}$. Thus, the notion of real and complex differentiation are compatible. If we take $f|_{U \cap \mathbb{R}} : U \cap \mathbb{R} \to \mathbb{C}$ to be the restriction of f to the real line, then the real-derivative of $f|_{U \cap \mathbb{R}}$ exists and is equal to the restriction of $f : U \to \mathbb{C}$ to the real line.

7. All of these functions are not Holomorphic since non satisfy the Cauchy Riemann equations:
$$|z|, \qquad \operatorname{Re} z, \qquad \operatorname{Im} z, \qquad \arg(z), \qquad \overline{z}$$
This can also be seen since any real holomorphic function must be constant (by the observations we've made earlier), but none of these are constant.

Before continuing onto further explorations of holomorphic functions, a comment must be made: there is often a game in mathematics about how weak of a condition can you impose on your building blocks to make our original condition hold. In this case, we have that $f = u + iv$, and we may ask how weak of conditions can be impose on u and v (or f) to make f Holomorphic. We saw that if f is real-differentiable and satisfies the Cauchy condition, then f is complex-differentiable. In fact, we can assume even more weakly that f is continuous and u, v each have first partial derivatives. This is the *Looman-Menchoff Theorem*, which we'll prove in section ref:HERE. We will take a moment to prove a stronger claim than proposition 2.2.3 to practice working with partial derivatives:

Theorem 2.2.1: C^1 + C.R. Then Holomorphic

Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be conjugate harmonic functions (or more weakly, u, v are both C^1 and satisfy the Cauchy-Riemann equations). Then:

$$f = u + iv$$

is a holomorphic function.

Note We will later be able to say if and only if to the above theorem by showing that f is Holomorphic, then $f \in C^\infty(\mathbb{C})$.

***Proof* :**
Let $u, v \in C^1(\mathbb{R}^2)$ where u, v satisfy the Cauchy Riemann equations. As we know, the derivative is a linear approximation and so gives rise to the following equation learnt in elementary calculus:

$$u(x+h, y+k) - u(x, y) = \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1$$

$$v(x+h, y+k) - v(x, y) = \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2$$

where ϵ_1 and ϵ_2 tend to zero faster than h and ik, and so in particular more rapidly than $h + ik$, i.e.

$$\frac{\epsilon_1}{h+ik} \to 0 \qquad \frac{\epsilon_2}{h+ik} \to 0$$

Using the Cauchy-Riemann equations, we can re-write $f = u + iv$ as:

$$f(z + (h+ik)) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h+ik) + \epsilon_1 + i\epsilon_2$$

dividing by $h + ik$ and taking the limit as $(h + ik) \to 0$, we get:

$$\lim_{(h+ik)\to 0} \frac{f(z+(h+ik)) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

showing the limit exists. Since u and v also satisfy the Cauchy-Riemann condition, f must be holomorphic, as we sought to show

2.2.1 Relation to Conjugate

Consider a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. We can interpret such a function as a real-differentiable function $f : \mathbb{R}^2 \to \mathbb{C}$ that satisfies the Cauchy-Riemann equations: $f(x, y)$. We will analyze such a map and see that it will give us some useful relations between $f(z)$, $f(\overline{z})$ and $\overline{f}(z)$. Since

$$z = x + iy \qquad \overline{z} = x - iy$$

we can isolate x and y and get:

$$x = \frac{z + \overline{z}}{2} \qquad y = \frac{z - \overline{z}}{2i}$$

Then applying the chain rule imagning we can to real-differentiation, we get the following equations that are important enough to be given a name:

Definition 2.2.2: Wirtinger Derivatives

Let f be a real-differentiable function. Then:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

We may think of these as operator's that we may apply to any real differentiable function with complex codomain:

$$\frac{\partial}{\partial z}(f) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Let's say f is a holomorphic function. Applying the Cauchy-Riemann conditions, we see that:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = f' \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}} = 0$$

This shows that we can think of a holomorphic function f as being a function of *a complex variable*, rather than a function of *two real variables* with a complex output (in other word, you can think that the imaginary part of a holomorphic function is essentially determined, up to a constant, by the real part, and vice-versa). We can also think of holomorphic functions as being independent of thier conjugate, meaning if we interpret a complex function as being $f(z, \overline{z})$ then we would get $f(0, \overline{z}) = c$ for some constant $c \in \mathbb{C}$ (so we cannot have any conjugate terms in a complex function). Taking the above definition, we have that:

$$\frac{\partial}{\partial z} z = 1 \qquad \frac{\partial}{\partial z}\overline{z} = 0$$
$$\frac{\partial}{\partial \overline{z}} z = 0 \qquad \frac{\partial}{\partial \overline{z}}\overline{z} = 1$$

Using this, we see that if

$$f(z) = \sum_{n,m=0}^{k} a_{m,n} z^m \overline{z}^n$$

Then f is holomorphic if and only if $a_{m,n} = 0$ for all $n \neq 0$, hence no complex polynomial which is holomorphic has $\overline{z}$ as a variable. If we have a holomorphic function f, then we can define $\overline{f}$ to be $\overline{f}(z) = \overline{f(z)}$. Applying Wirtinger's derivative we get:

$$\frac{\partial \overline{f}}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial \overline{f}}{\partial \overline{x}} = \frac{\partial f}{\partial x} = f'$$

We'll show $\frac{\partial \overline{f}}{\partial z} = 0$, leaving the other identity as an exercise. Computing:

$$\begin{aligned} 0 &= \overline{0} \\ &= \overline{\frac{\partial f}{\partial \overline{z}}} \\ &= \frac{1}{2}\overline{\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial \overline{z}}\right)} \\ &= \frac{1}{2}\left(\overline{\frac{\partial f}{\partial x}} - i\overline{\frac{\partial f}{\partial y}}\right) \\ &= \frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x} - i\frac{\partial \overline{f}}{\partial y}\right) \\ &= \frac{\partial \overline{f}}{\partial z} \end{aligned}$$

showing that taking the conjugate of a function "flips" the behavior of the Wirtinger derivative.

In terms of differential forms, if we take $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, then $dz = dx + idy$ and $d\overline{z} = dx - idy$. If we re-write, we get

$$dx = \frac{1}{2}(dz + d\overline{z}) \qquad dy = \frac{1}{2i}(dz - d\overline{z})$$

giving us df in terms of dz and $d\overline{z}$ to be

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\overline{z}$$

we can then shorten this by putting in the Wirtinger derivative to get:

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

and thus we get

$$df = \frac{\partial f}{\partial z}dz$$

which recovers our original notion of derivative of differential form.

(I have yet to understand the point of this next part)

By similar formal arguments we can derive a very simple method which allows us to compute, without use of integration, the analytic function $f(z)$ whose real part is a given harmonic function $u(x,y)$. We remark first that the conjugate function $\overline{f(z)}$ has the derivative zero with respect to z and may, therefore, be considered as a function of $\bar{z}$; we denote this function by $\bar{f}(\bar{z})$. With this notation we can write down the identity

$$u(x,y) = \tfrac{1}{2}[f(x+iy) + \bar{f}(x-iy)].$$

It is reasonable to expect that this is a formal identity, and then it holds even when x and y are complex. If we substitute $x = z/2$, $y = z/2i$, we obtain

$$u(z/2, z/2i) = \tfrac{1}{2}[f(z) + \bar{f}(0)].$$

Since $f(z)$ is only determined up to a purely imaginary constant, we may as well assume that $f(0)$ is real, which implies $\bar{f}(0) = u(0,0)$. The function $f(z)$ can thus be computed by means of the formula

$$f(z) = 2u(z/2, z/2i) - u(0,0).$$

A purely imaginary constant can be added at will.

In this form the method is definitely limited to functions $u(x,y)$ which are rational in x and y, for the function must have a meaning for complex values of the argument. Suffice it to say that the method can be extended to the general case and that a complete justification can be given.

2.2.2 Harmonic Functions

Let $f = u + iv$ be a holomorphic function. Due to the Cauchy-Riemann equations, u and v have some interesting analytical properties. The two equalities of the partial derivatives of u and v can be re-written in many forms:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{2.2}$$

which is the Jacobian. This should make sense, as f when interpreted as a real-differentiable function would require the jacobian to be nonzero in order for $f'(z) \neq 0$. There is another powerful result we get from the Cauchy-Riemann equations: the derivative of a holomorphic function will be holomorphic! This contrasts to differentiable functions for which the derivative is not necessarily differentiable (and hence we require the definition of C^n and C^∞). It will be proven that all holomorphic functions are "C^∞"!

Since we will eventually show that f will be C^∞, the following discussion is in fact broader then the restrictions we shall put it in, but for now since we have yet to prove f being differentiable once implies infinitely differentaible, we will explicitely assume that the partials of f are at least twice differentiable, meaning the partials of u and v are at least twice differentaible. Then we have:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial x \partial y} - \frac{\partial v}{\partial x \partial y} = 0$$

and similarly for v: $\triangle v = 0$. Note that we can switch the order of the mixed derivative since we are integrating stirctly in the real direction each time, and so the results of real differentiation applies. Thus, if f is holomorphic, we have that $\Delta u = \Delta v = 0$. For future reference we box this result:

Lemma 2.2.1: Holomorphic Then Harmonic

Let f be an holomorphic functions and represent it as $f(x+iy) = u(x,y) + iv(x,y)$ whose partials are twice real-differentiable. Then u and v must be harmonic:

$$\Delta u = 0 \qquad \Delta v = 0$$

Proof **:**
This was proved in equation (2.2).

Functions u, v satisfying the above criterion are actualy quite important in PDE's, and so we will label them for future reference:

Definition 2.2.3: Laplace Operator And Harmonic Functions

Let f be a function that is at least twice differentiable. Then the *Laplace operator* on f is defined as:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

If $\Delta f = 0$, then f is said to be *harmonic*.

The intuition that I have for harmonic functions is given any point $f(z)$, the "average" of all of it's neighbouring points is zero. Think about this in terms of the 2nd derivative for a function from $\mathbb{R} \to \mathbb{R}$. If the 2nd derivative is zero, then you have a line (as you can check). At any point on the line, the "average" of its neighbouring point is zero. If f'' was not zero, then if you look at point's around it (say for the function x^2 at 0 where $f''(0) > 0$) then the average of the points around it are not zero (in this example, they are positive)! Thus, if f is holomorphic and u, v are obtained via the standard represetnation, we have $\Delta u = \Delta v = 0$. This gives us a stronger necessary condition for a function f to be holomorphic.

The converse is also true: if u, v satisfy the Cauchy-Riemann equations and are each Harmonic, then we can in fact construct a holomorphic function f via $f := u + iv$. Note that being harmonic does not imply the Cauchy-Riemann equations: $z \mapsto \overline{z}$ is harmonic, but not holomorphic. However, since being harmonic implies being at least C^1, by proposition 2.2.3 the function is holomorphic.

Definition 2.2.4: Harmonic Conjugate Function

If u is a harmonic function and v is the harmonic function that makes $f = u + iv$ an holomorphic function, then v is called the *harmonic conjugate of* u. In particular, u, v are harmonic functions satisfying the Cauchy-Riemann equations

Example 2.3: Finding Conjugate harmonic Function

Let $u(x,y) = x^2 - y^2$. This is a harmonic function, and so we can proceed in finding it's harmonic compliment. Notice that

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

Thus

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

We can integrate one or the other equation to solve for v. To make an arbitrary choice, let's pick the first one and integrate it with respect to x to get

$$v = 2xy + \varphi(y)$$

where $\varphi(y)$ is some functions based on y. Deriving with respect to y gives us

$$\frac{\partial v}{\partial y} = 2x + \varphi'(y) = 2x$$

telling us that $\varphi'(y) = 0$ so $\varphi(y) = 0$. Thus we get

$$f(z) = f(x + iy) = x^2 + y^2 + i(2xy) = z^2$$

You can now verify that $f(z) = x^2 + y^2 + i2xy$ is indeed holomorhpic. Check that this function is actually $f(z) = z^2$. What's the derivative? Remember that you can write it in terms of partial derivatives of u and v. Can you write it in terms of z?

For reference, here is a list of harmonic conjugates

u	v	$u + iv$
x	y	z
x	$y + 1$	$z + i$
y	$-x$	$-iz$
$x^2 - y^2$	$2xy$	z^2
$e^x \cos(y)$	$e^x \sin(y)$	e^z
$\frac{x}{x^2+y^2}$	$\frac{-y}{x^2+y^2}$	$\frac{1}{z}$

Note that for the last example we must exclude the origin from the domain. Being harmonic gives us some important regularity on functions maxima's and minima's:

Theorem 2.2.2: Maximum Principle Of Harmonic Functions

Let U be an open subset of $\mathbb{C}$ and let $u : U \to \mathbb{R}$ be a harmonic function. Let K be a compact subest of U, and let ∂K be the boundary of K. Then:

$$\sup_{z \in K} u(z) = \sup_{z \in \partial K} u(z)$$

and

$$\inf_{z \in K} u(z) = \inf_{z \in \partial k} u(z)$$

***Proof* :**
We'll prove it for sup, since inf follows similarly (note that $-u$ is also harmonic). IT is always the

case that $\sup_{z\in K} u(z) \geq \sup_{z\in\partial K} u(z)$, so for the sake of contradiction let's say

$$\sup_{z\in K} u(z) > \sup_{z\in\partial K} u(z)$$

As u is continuous and K is comapct, u achieve's it's maximum at some point, say $z_0 \in K$. By our assumption, z_0 is in the interior. Sicne z_0 is a loca lmaximum of u, and u is twice differntiable, we must have:

$$\frac{\partial^2 u}{\partial x^2}(z_0) \leq 0 \qquad \frac{\partial^2 u}{\partial y^2}(z_0) \leq 0$$

This almost contradicts the harmonicity of u, but it is still possibelthat both of these partial derivatives vanish. TO fix this, we wiggle around an epsilon amount of room to add some convexity. Letting $\epsilon > 0$ be a small number we'll choose later, let $u_\epsilon : U \to \mathbb{R}$ be the modified function

$$u_\epsilon(x+iy) := u(x+iy) + \epsilon(x^2+y^2)$$

Since K is compact, the function x^2+y^2 is boudned on K. If ϵ is small enough, then by our assumption on u we must also have on u_ϵ the relation:

$$\sup_{z\in K} u_\epsilon(z) > \sup_{z\in\partial K} u_\epsilon(z)$$

By the same arguement u_ϵ achieves its maximum at some interior point z_ϵ of K, thus:

$$\frac{\partial^2 u}{\partial x^2}(z_\epsilon) \leq 0 \qquad \frac{\partial^2 u_\epsilon}{\partial y^2}(z_\epsilon) \leq 0$$

Sine u is harmonic, we must have:

$$\frac{\partial^2 u_\epsilon}{\partial x^2} + \frac{\partial^2 u_\epsilon}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + 2\epsilon + \frac{\partial^2 u}{\partial y^2} + 2e = 4\epsilon > 0$$

on U. But that's a contradiction, completing the proof.

Corollary 2.2.1: Maximum Principle For Holomorphic Functions

Let $f : U \to \mathbb{C}$ be a continuously twice differentiable holomoprhic function on an open set U an K a compact subset of U. Then

$$\sup_{z\in K} |f(z)| = \sup_{z\in\partial K} |f(z)|$$

This reuslt is also known as the *maximum modulus principle*

***Proof* :**
Use the fact that $|w| = \sup_{\theta\in\mathbb{R}} \mathrm{Re}(we^{i\theta})$.

Proposition 2.2.5: Consequences Of Harmonic Functinos

Let $U \subseteq \mathbb{C}$ be an open subset.

1. If $f : U \to \mathbb{C}$ is twice continuously differentiable

$$\nabla f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial f}{\partial z}$$

2. If f is a complex polynomial:

$$f(z) = \sum_{\substack{n,m\geq 0 \\ n+m\leq d}} c_{n,m} z^n \overline{z}^m$$

then f is harmonic on $\mathbb{C}$ if and only if $c_{n,m}$ vanishes whenever n and m are both positive (f only contains terms $c_{n,0}z^n$ or $c_{0,m}\overline{z}^m$)

3. if $u : U \to \mathbb{R}$ is a real polynomial

$$u(x+iy) = \sum_{\substack{n,m\geq 0 \\ n+m\leq d}} a_{n,m} x^n y^m$$

then u is harmonic if and only if it is the real part of a compelx polynomial

$$f(z) = \sum_{n=0}^{d} c_n z^n$$

Proof **:**
exercise

We leave with a generalization of the notion of harmonic conjugate. We saw that every harmonic polynomial has at least one harmonic conjugate (up to constant). We may ask whether this holds for more general harmonic functions. If $U = \mathbb{C}$ is the entire plane, this is indeed the case:

Proposition 2.2.6: Entire Harmonic Conjugate On $\mathbb{C}$

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function. Then there exists a harmonic conjugate $v : \mathbb{C} \to \mathbb{R}$ of u> Furthermore, this harmonic conjguate is unique up to constants: if v, v' are two harmontic conjugate sof u, then $v' - v$ is a constant funcion

Proof **:**
proposition 28 here

This proof generlizes for some other domains like rectangles, but is left for when we have built up the notion of contour integrals. In some cases (in particualr, when U is not simply connected), harmonic functions will not have harmonic conjugates!

2.2.3 Properties of Holomorphic Functions

In this seciton, we upgrade some clasical results of real-differentiability to complex differentiability taking advantage of the fact that f must satisfy the C.R. equations. We then show a very important geometrical property of holomorphic functions, namely that they must be *conformal*.

Proposition 2.2.7: Derivative Zero Then Constant

Let f be holomorphic and $f' = 0$. Then f is constant

Proof **:**
Since f is holomorphic and $f' = 0$:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = 0$$

Thus, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Now the standard real-differentiable argument shows that f is constant.

(Ahlfors has a more elaborate proof that doesn't use the real result on p.72)

Proposition 2.2.8: Equivalent to Derivative Zero

Let f be holomorphic. Then:

1. If $|f|$ is constant, then f is constant
2. If $\operatorname{Re} f$ is constant, then f is constant

Note how this contrasts to a real-differentiable function $f : U \subseteq \mathbb{R}^2 \to \mathbb{C}$, if $|f|$ is constant, this does not imply f is constant (think $U = \mathbb{R}^2 \setminus (0,0)$ and $(x,y) \mapsto (x,y)/\sqrt{x^2+y^2} = (x,y)/\|(x,y)\|$), and similarly for $\operatorname{Re} f$ (which we can think of as taking the function $\pi \circ f$ with any nonconstant function f)

Proof **:**

1. $|f|^2 = f\overline{f}$. Then taking the Wirtinger derivative on both sides and noting that $\frac{\partial \overline{f}}{\partial z} = 0$, we get

   $$0 = \frac{\partial f}{\partial z} \cdot \overline{f}$$

   If $\overline{f} = 0$ at some point, say p then $|f(p)| = 0$, but $|f|$ is constant so $|f| = 0$, and so $f = 0$ is constant. Thus, assume it's nonzero everywhere, so $\frac{\partial f}{\partial z} = 0$. But then by proposition 2.2.7 f is constant.

2. We have $\operatorname{Re} f = \frac{1}{2}(f + \overline{f})$ where the left hand side is constant. Then

   $$0 = \frac{1}{2}\frac{\partial f}{\partial dz}dz + \frac{\partial \overline{f}}{\partial z}dz$$

   cancelling terms out we get:

   $$0 = \frac{\partial f}{\partial z}dz$$

completing the proof.

As an exercise show that if $\log|f|$ or $\arg(f)$ is constant, then f is constant.

One important result that is "invisible" in $\mathbb{R}$ but becomes apparent in $\mathbb{C}$ is that (real and complex) differentiable functions preserve angles! Being complex differentiable will also mean the function perserves orientation:

Definition 2.2.5: Conformal Maps

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Then f is said to be *conformal* for every z_0 in the domain, f preserves the angles between two curves passing through z_0 and perserves orientation. If f does not perserve orientation, it is called *homothetic* or *angle-preserving*.

A conformal map may be interpreted as preserving the shape of any sufficiently small figure, while possibly rotating and scaling (but not reflecting) it. If f is holomorphic, we will says it's conformable if $Df(a)$ interpreted as a real 2 by 2 matrix is an angle-preserving map and orientation preserving. If it is simply angle-preserving, then *a priori* we don't know if if f will filp-flop bebetween orientations. If f is additionally at least real C^1, then f *has* to strictly stay within one orientation:

Proposition 2.2.9: C^1 and Angle Preserving Functions

Let $\Omega \subseteq \mathbb{C}$ be a conencted open set, $f : \Omega \to \mathbb{C}$ a real C^1 function with a nonzero Jacobian determinant at every point. Then if f preserves angles at every point of Ω (that is, if it is homothetic), then either:

$$\frac{\partial f}{\partial z} = 0 \qquad \text{or} \qquad \frac{\partial f}{\partial \bar{z}} = 0$$

where both can't be zero (or else the jacobian's determinant is zero). In the second case f is holomorphic, in the first case f is anti-holomorphic (locally $\overline{\mathbb{C}}$-linear)

***Proof* :**
Since f is $\mathbb{R}$-differnetiable, Df exists. Since it is angle preserving, by lemma 1.3.1 Df is either of the form:

$$Df(x)(p) = ap \qquad \text{or} \qquad Df(x)(p) = a\overline{p}$$

that is, it is either a rotation, or a rotation and a reflection. Consider the sets:

$$\left\{ z \in \Omega \;:\; \frac{\partial f}{\partial z}(z) = 0 \right\} \qquad \left\{ z \in \Omega \;:\; \frac{\partial f}{\partial \bar{z}}(z) = 0 \right\}$$

These two sets are disjoint, since if both $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}} = 0$, then f a zero Jacobian, contradiction the assumption of the quesiton. Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, the sets

$$\left\{ \frac{\partial f}{\partial z} = 0 \right\} \qquad \left\{ \frac{\partial f}{\partial \bar{z}} = 0 \right\}$$

are closed sets. By what we've shown they are disjoint and their union is Ω. By connectivity of Ω, one of these sets must be empty.

This compelte the proof. If the second one is empty, then Df is $\mathbb{C}$-linear and hence f is holomorphic. If the first one is empty, then Df is anti $\mathbb{C}$-linear (that is $Df_p(az) = \overline{a}Df_p(z)$) and f is called *anti-holomorphic*.

We can re-state the theorem a bit by saying that the Jacobian determinant is nonzero and f is a C^1 angle-preserving function if and only if f is either (strictly) holomorphic or anti-holomorphic. In the above theorem, we have assumed that being locally $\mathbb{C}$-linear is sufficient to be conformal, and indeed it is. However, we there is a nice proof that goes about this more directly that I think is worth showing to show that it is in fact a very simple result:

Proposition 2.2.10: Holomorphic Then Conformal

Let f be a holomorphic function. Then f is a conformal map.

***Proof* :**
let $\gamma : (-\epsilon, \epsilon) \to U \subseteq \mathbb{C}$ be some differentiable cruve with $\gamma(0) = z_0$ and $f : U \to \mathbb{C}$ a holomorphic function. We can think of $\gamma'(0)$ as being the velocity vector of a particle passing through z_0. Take $f \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{C}$. By the chain rule:

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$$

Representing $f'(z_0)$ in polar coordinates we get $f'(z_0) = re^{i\theta}$. Thus, we get:

$$(f \circ \gamma)'(0) = re^{i\theta}\gamma'(0)$$

that is, f transforms the velocity vector by multiplying the speed by a factor of r and rotating it counte-clockwise by a fixed angle θ. If we know consider two trajectories γ_1, γ_2 passing through z_0 at $t = 0$, then the map f will preserve the angle between the velocity vector $\gamma_1'(0)$ and $\gamma_2'(0)$ as well as their orientation, completing the proof.

This gives us another way of eliminating non-holomorphic functions. For example $f(x + iy) = x + i(x + y)$ perserves orientaiton, but not angle, while $f(z) = \overline{z}$ preserves angle, but not orientiaton. Note too how we in fact have the same phenomena happening for real-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ for functions of the form $f : I \to \mathbb{R}$, but this is much less interesting since the only two possible angles between velocity vectors are 0 and π; this shows the 2-dimensionality of the complex plane makes conformality a much more “rigid” property for complex differentiable functions.

The final result we shall state shall build upon our earlier observation's that the derivative of f resemble the Jacobi identity, making it tempting to say that holomorphic implies coninuously holomorphic. This may be jumping the gun, but for the next theorem we shall assume that this is indeed the case (in section ref:HERE we shall prove it). We shall use this to show the complex version of the inverse function theroem. This theorem naturally applies holomorphic, since a holomorphic function is already real-differentiable, but we would want the inverse to also be complex differentiable:

Theorem 2.2.3: Holomorphic Inverse Function Theorem

Let f be holomorphic in a neighbourd of z_0 and $f'(z_0) \neq 0$. Then there exits a U, v, $z_0, f(z_0) = w_0$ such that $f|_U$ is a homeomoprhism onto V with inverse $g : f(U) \to U$. Furthermore, g is holomorphic and

$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{f'(z)}$$

where $w = f(z)$

Proof **:**
With our assumption that holomorphic implies continuously holomorphic, the real version of the inverse function gives us all but g being complex-differentiable. Sicne f is holomorphic, we have that in the standard basis we may represent $f'(z)$ as:

$$[f'(z)] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then we compute:

$$g'(w) = [f'(z)]^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which shows g satisfies the Cauchy-Riemann equations, and thus is holomorphic, completing the proof.

2.2.4 Polynomial and Rational Functions

As we saw in example 2.1, constant and linear functions are holomorphic. By proposition 2.2.1, the sum and product of holomorphic functions is holomorphic. Therefore, all polynomials are holomorphic

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with derivative

$$P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$$

To not repeat the same restrictions in the following theorems, if we write out a polynomial as we have done for $P(z)$, we will assume $a_n \neq 0$, and 0 is not a polynomial [2].

Found this video (commented to compile) :

A defining feature of polynomials is that they are defined by their roots (up to a constant) along with one more point. Having theorems on how the roots behave under differentiation will give us some good insights on polynomial. The following two theorems give geometric insight on the roots:

Theorem 2.2.4: Roots Of Derivative Within Convex Hull (Gauss-Lucas Theorem)

Let P be a polynomial. Then the roots of P' are within the convex hull of the roots of P

[2] for formal reasons we will soon see, it's degree will have to be $-\infty$, see ref:HERE

Proof :
The proof is broken down into steps:

1. Suppose that $P(z)$ has degree n and zeros $b_1, ..., b_n$ (each zero listed as many times as multiplicity). Show that
$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - b_k}$$

   Proof. This is simply a matter of computation. First, by the fundamental theorem of algebra:
$$P(z) = c(z - b_1) \cdots (z - b_n)$$
   then:
$$P'(z) = c \sum_{i=1}^{n} \prod_{k \neq i} (z - b_k)$$
   Then dividing, we get:
$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - b_k}$$
   as we sought to show □

2. Show that if $P'(z) = 0$, then
$$\left(\sum_{k=1}^{n} \frac{1}{|z - b_k|^2} \right) \overline{z} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$

   Proof. If $P'(z) = 0$, then:
$$0 = \sum_{k=1}^{n} \frac{1}{z - b_k}$$
   Now, for each term in the summand, multiply it by $\frac{\overline{z} - \overline{b_k}}{\overline{z} - \overline{b_k}}$ and the summand in two:
$$0 = \sum_{k=1}^{n} \frac{\overline{z}}{|z - b_k|^2} - \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$
   Moving the values around, we get:
$$\left(\sum_{k=1}^{n} \frac{1}{|z - b_k|^2} \right) \overline{z} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$
   as we sought to show □

3. Deduce that if $P'(z) = 0$, then z lies within the convex hull of the points b_k.

Proof. Let $P'(z_0) = 0$. Then by part (b):

$$\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right) \overline{z_0} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z_0 - b_k|^2}$$

Conjugating both sides and isolating z_0, we get:

$$z_0 = \frac{\sum_{k=1}^{n} \frac{b_k}{|z_0 - b_k|^2}}{\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right)} = \sum_{k=1}^{n} \frac{\frac{b_k}{|z_0 - b_k|^2}}{\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right)} = \left(\sum_{k=1}^{n} \frac{\frac{1}{|z_0 - b_k|^2}}{\left(\sum_{j=1}^{n} \frac{1}{|z_0 - b_j|^2}\right)} b_k\right)$$

where the value in the summand is non-negative. Notice here that the coefficients of the b_k's add up to 1. Since the b_k's form a convex hull, this is a convex combination, and so z_0 must be in the convex hull, as we sought to show. $\square$

A rational function of particular interest is the Möbius transformation

$$f(z) = \frac{az + b}{cz + d} \qquad ad - bc \neq 0$$

Note that since

$$f^{-1}(w) = \frac{dw - b}{-cw + a}$$

then f is in fact *biholomorphic*, meaning it is in fact a symmetry of $\hat{\mathbb{C}}$. In fact, $\mathrm{Aut}(\hat{\mathbb{C}})$ is the collection of all Möbius transformation. If you did not take differential geometry, then the following discussion may be skipped and the statement may be taken for granted for now. Recall that $S^2 \cong \mathbb{C}\mathrm{P}^1$ as smooth manifolds. Thinking of S^2 as the Riemann sphere $\hat{\mathbb{C}}$, we see that finding the symmetries of S^2 is finding the symmetries of $\mathbb{C}\mathrm{P}^1$, i.e. the automorphism group of complex diffeomorphisms. Recall that $\mathrm{GL}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})$ and let $\mathrm{GL}(\mathbb{C}) \curvearrowright \mathbb{C}$ in a natural way. We may descend this action to $\mathrm{PGL}(\mathbb{C}) \curvearrowright \mathbb{C}\mathrm{P}^1$ by taking $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/Z(\mathbb{C})$ where $Z(\mathbb{C})$ is the center of $\mathrm{GL}_2(\mathbb{C})$, i.e. the collection of all diagonal matrices. To see what this descension looks like, take

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(\mathbb{C})$$

Then:

$$[z, 1]\begin{pmatrix} a & c \\ b & d \end{pmatrix} = [az + b, cz + d] = \left[\frac{az + b}{cz + d}, 1\right]$$

which is exactly a Möbius transformation! Thus, we see that the collection of Möbius transformations, which we may label as $\mathrm{PGL}(\mathbb{C})$, are all the automorphisms (i.e. the biholomorphic functions) of $\hat{\mathbb{C}}$.

2.3 Power Series

Since polynomials are holomorphic functions, it is natural to ask whether the completion of polynomials, power series[3], are also holomorphic functions. The answer was already revieled earlier when we

[3] in particular $\varprojlim(k[x]/(x)^n) = k[[x]]$

said that another common name for holomorphic functions are analytic functions (since an analytic function is locally given by a convergent power-series).

Definition 2.3.1: Formal Power Series

Let z_0 be any complex number. Then a *formal power series* with complex coefficeints around the point z_0 is a formal series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some complex numbers $a_0, a_1, ...$, with z an indeterminate

These are called "formal" since we require to establish an atual domain and codomain in order for them to be functions, at which point we usually call them power series. Another way of thinking about power series is that they are the completion of the ring of polynomials $\mathbb{C}[z - z_0]$, and hence they are the "formal" result of this completion. We can naturally try and define an evaluation function by replacing z with some value in $\mathbb{C}$ and see if it converges. The following theorem gives us a systemic way of finding which values can be plugged in:

Theorem 2.3.1: Abel's Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a power series. Then there exists a number R, $0 \leq R \leq \infty$, called the *radius of convergence*, with the following properties:

1. The series converges absolutely for every z with $|z| < R$. If $0 \leq \rho < R$, the convergence is uniform for $|z| \leq \rho$.
2. If $|z| > R$, the terms of the series are unbounded, and the series thus diverges
3. In $|z| < R$, the sum of the series is a holomorphic function. The derivative is obtained by term-wise differentiation, and the derived series has the same radius of convergence.

Note that when $|z| = R$, the result depends on the power-series[4]. This circle is called the *circle of convergence*.

***Proof* :**
Without loss of generailty, let $z_0 = 0$ (this is simply shifting the polynomial over). Recall the important formula by *Hadamards* that relates R to given the coefficeints of a power-series:

$$1/R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

If $|z| < R$, we can find a ρ such that $|z| < \rho < R$. Then $1/\rho > 1/R$, and by the definition of the limit superior, there exists an n_0 such that $|a_n|^{1/n} < 1/\rho$, meaning

$$|a_n| < \frac{1}{\rho^n} \qquad \forall n \geq n_0$$

[4]This problem is related to the same tricky problem of asking when a Fourier series converges

Thus:

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \qquad \forall n \geq n_0$$

and for any fixed z, the right hand side is a geometric series which for $\rho > 1$ converges by Hadamard's formula. To show uniform convergence for $|z| \leq \rho < R$, we take advantage of Weiestrass M-test. Pick a ρ' with $\rho < \rho' < R$. Then we can get for large enough n_0:

$$|a_n z^n| \leq \left(\frac{\rho}{\rho'}\right)^n \qquad n \geq n_0$$

Since the majorant (the power series given by the right hand side) is convergetn and has constant terms, by the Weiestrass's M test the power series is uniformally convergent. If $|z| > R$, then we can do the same manipulation we've done before but instead choose $R < \rho < |z|$ to get

$$|a_n z^n| > \left(\frac{|z|}{\rho}\right)^n \qquad \forall n \geq n_0$$

Thus, fixing an n_0, we get that

$$|a_{n_0} z^{n_0}| + |a_{n_0} z^{n_0}| < |a_{n_0} z^{n_0}| + |a_{n_0+1} z^{n_0+1}| + \cdots$$

showing the right hand side is unbounded, and all $|z| > R$ diverge.

Next, we will show that the derived series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence. First, we will show that $\sqrt[n]{n} \to 1$

Proof. Set $\sqrt[n]{n} = 1 + \delta_n$. Then $\delta_n > 0$, and by the binomial theorem

$$n = (1 + \delta_n)^n > 1 + \frac{1}{2} n(n-1)\delta_n^2$$

Manipulating, we get $\delta_n^2 < 2/n$, and so $\delta_n 0 > 0$. □

We now continue the proof. For $|z| < R$, decompose the power-series two:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = s_n(z) + R_n(z)$$

where

$$s_n(z) = a_1 + a_1 z + \cdots + a_{n-1} z^{n-1}$$
$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k$$

write the "desired" derivative of f as

$$f_1(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \lim_{n \to \infty} s_n'(z)$$

we want to show that $f'(z) = f_1(z)$. To that end, write:

$$= \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \tag{2.3}$$

$$= \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0)\right) + (s'_n(z_0) - f_1(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0}\right) \tag{2.4}$$

where we naturally assume that $z \neq z_0$ and both $|z|, |z_0|$ are $< \rho < R$. Recalling that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$, the last term in the above can be rewritten as

$$\sum_{k=n}^{\infty} a_k(z^{k-1} + z^{k-2}z_) + \cdots + zz_k^{k-1} + z_0^{k-1})$$

since $|z|, |z_0| < \rho$ we have:

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} k|a_k|\rho^{k-1}$$

the right hand side of the above expression is the remainder term of a convergent series, and so we can find a large enough n_0 such that

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| < \frac{\epsilon}{3}$$

The same reasoning can be applied For the second term in equation (2.4) for some $n \geq n_1$. For the 1st term, choose some $n \geq n_0, n_1$. Then by definition of the derivative there exists a $\delta > 0$ such that for $0 < |z - z_0| < \delta$ implies

$$\left|\frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0)\right| < \frac{\epsilon}{3}$$

Combining all these equations, we get that:

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0)\right| < \epsilon$$

when $0 < |z - z_0| < \delta$. But then that is exactly the condition we need to show that $f'(z_0)$ exists and is equal to $f_1(z_0)$, completing the proof.

The last reasoning can be repeated indefinitely, meaning a power series with positive radius of convergence has derivatives of all order with the same radius of convergence. Now that we have a radius of convergence, we can define the *power series* of a formal power series to be the function $F : B_R(z_0) \to \mathbb{C}$ such that

$$F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

We will most the time not make a distinction between formal power series and power series and call them both power series unless the situation calls for nuance. At this point, it is a good idea to recall some tests to check whether a power series converges:

1. **root test**: if $\limsup_{n\to\infty}(a_n)^{1/n} < 1$, then the series converges, if > 1 it diverges, and if $= 1$ it

is inconclusive.

2. **Ratio test** : if $\lim_{N\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series converges. if > 1 it diverges, and $= 1$ it is inconclusive.

3. **Comparison test**: If $\sum_{n=0}^{\infty} a_n z^n$ converges and $b_n < a_n$, then $\sum_{n=0}^{\infty} b_n z^n$ converges

4. **Integral test**: If $f[1,\infty) \to \mathbb{R}_+$ is a non-negative monotonically decresing function such that $f(n) = a_n$, then if
$$\int_1^{\infty} f(x)dx < \infty$$
then the series also converges, and if it diverges so does the series

5. If $\sum_i^{\infty} |a_n||z^n|$ converges for some z, so does the power series. If $a_n \nrightarrow 0$, then the $\sum_n^{\infty} a_n z^n$ for any z does not converge.

We mentioned that what happens at the border is not determined. We in fact get some subtle behavior that is worth going into. Take the power sries $\sum_{n=0}^{\infty} z^n$. Then it has radius of convergence of 1. if $z \in B_1(0)$, then since the series converges uniformally we have:

$$\begin{aligned} z\sum_{n=0}^{\infty} z^n &= \sum_{n=0}^{\infty} z^{n+1} \\ &= \sum_{n=1}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} z^n - 1 \end{aligned}$$

Thus, we some algebraic manipulation we get the closed form of:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

It is evident to see that as long as $z \in B_1(0)$ the function $z \mapsto \frac{1}{1-z}$ does not extend continuously to the boundary point $z = 1$ of the disk. however, it *does* extend continuously, even smoothly, to the rest of the boundary. In fact, even more remarkably (though perhaps not so remarkable if we take a moment to think about it), it can be *holomorphically extended* to all of $\mathbb{C} \setminus \{1\}$. However, $\sum_{n=0}^{\infty} z^n$ diverges at every point in the boundary (when $|z| = 1$, the coefficeints of z^n of the series do not converge to zero), and evidentily it diverges outside the unit ball. Thus, we see that the function that matches the power series can be well-defined much beyond the radius of convergence of the power series. We'll prove in the next proposition (proposition 2.3.1) that the power series matching $1/(1-z)$ is unique and hence we can't find a power series that somehow fixes this.

We can play around with the fact that $\sum_{n=0}^{\infty} z^n$ and $\frac{1}{1-z}$ must be uniquely identified by *formally* identifying the two on all of $\mathbb{C} \setminus \{1\}$. For example, if we plug in $z = 2$, then this identification would lead us to say that:
$$1 + 2 + 2^2 + 2^3 + \cdots = -1$$
which under our current understanding of convergence would certainly be absurd. However, there is a way of interpreting this beyond the classical notion of convergence, leading to concepts like the

generalised summation methods ilke the zeta function regularisation. These will be further discussed in ref:HERE.

(see also Tao's blog here and here (example 11))

Proposition 2.3.1: Taylor Expansion Is Unique

Let F, G be two power series centered at z_0 that agree on some neighbourd U. Then the coefficients of each term are equal. In particular, the Taylor series expantion is unique.

Note that if the point around which the power series is centered is different, we can no longer compare coefficients easily. For example, we can see that both $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(z+1)^n$ both converges to $\frac{1}{1-z}$ on $B_1(0)$ but have diferent coefficients.

Proof **:**
exercise?

(in this article, exercise 18 gives a way to go back and forth betwen two power series centered around different points)

As we have seen, the power series (as a function) can be well behaved as one approaches the boundary of the disk of convergence, while being divergent at the boundary. However the converse of this, where the power series converges at the boundary but does not behave well (as a function) as one approaches the boundary, does not occur:

Theorem 2.3.2: Abel's Limit Theorem

If $\sum_0^{\infty} a_n$ converges, then $f(z) = \sum_0^{\infty} a_n z^n \to f(1)$ as z approaches 1 in such a way that

$$\frac{|1-z|}{1-|z|}$$

remains bounded (sometimes known as the *Stolz angle*) or *non-tangent angle approach.*

Geometrically, this condition can be interpreted to mean that as we approach 1, we do so in such a way that the angle our approaching curve is taking is not tangent to the point 1, see the following link here.

Proof **:**
We amy assume $\sum_0^{\infty} a_n = 0$ by shifting over the sequence and have $\sum_0^{\infty} a_n$ be a_0. Writing the partial sum $s_n = a_0, a_1, ..., a_n$, we get

$$\begin{aligned} s_n(z) = a_0 + a_1 z + \cdots + a_n z^n \\ &= s_) + (s_1 - s_0)z + \cdots + (s_n - s_{n-1})z^n \\ &= s_0(1-z) + s_1(z - z^2) + \cdots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\ &= (1-z)(s_0 + s_1 z + \cdots + s_{n-1} z^{n-1}) + s_n z^n \end{aligned}$$

Since $s_n z^n \to 0$, we get the representation

$$f(z) = (1-z)\sum_0^\infty s_n z^n$$

Next we are assuming that $|1-z| \leq K(1-|z|)$ for some choice $K \in \mathbb{R}$. Since $s_n \to 0$, choose m large enough so that $|s_n| < \epsilon$ for $n \geq m$. The remainder of the series $\sum s_n z^n$ form $n = m$ onwards, is the ndomianted by the geometric sers

$$\epsilon \sum_m^\infty |z|^n = \epsilon \frac{|z|^m}{1-|z|} < \frac{\epsilon}{1-|z|}$$

It follows that

$$|f(z)| \leq |1-z| \left| \sum_0^{m-1} s_k z^k \right| + K\epsilon$$

The term on the right hand side can be made arbitrarily small by choosing z sufficinetly close to 1, and so we can conclude that $f(z) \to 0$ as $z \to 1$ subject to our given Stolz angle restriction.

We continue now to explore the properties of power series. Important opreations we are used to on our functions is adding, scaling, multiplying, inversing, or composing them. We shall see how these operations interact with power series. One important comment must be made: the problem of convegence of the composition of power is one that is still being inverstigated. In particular, sufficient condition's for convergence is an area of research, I found this paper (here) . Hence, to avoid such problems when they arrise, we may treat our power series as formal power series, the derivative as a formal operation, and defien $f(0) := a_0$ (since we are not treating f as a function).

Proposition 2.3.2: Composing Power Series

Let f, g be formal power series. Then $g(f(z))$ is well-defined if $b_0 = 0$.

***Proof* :**
When composing power series, we get:

$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z = b_2 z^2 + \cdots)^2 + \cdots$$

where each coefficient is an infinite sum, and so we must ask for their convergence conditions. One thing we can do to guarantee this is by setting $b_0 = 0$, which will guarantee that all the coefficients are finite sums and hence well-defined.

Theorem 2.3.3: Inverse Function Theorem On Formal Power Series

Let $f(z)$ be a formal power series. Then there is a formal power series $g(z)$ such that $b_0 = 0$ and $f \circ g = \text{id}$ (where id is the power series $\text{id}(z) = z$) if and only if $a_0 = 0$ and $f'(0) = 0$. In this case, g is unique and $g \circ f$ is also the identity. If f has positive radius of convergence, so does g

Proof **:**
We want to a g such that $f(g(z)) = z$. We want:

$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z = b_2 z^2 + \cdots)^2 = z$$

expanding and using the method of undetermined coefficients expanding the first two terms we get:

$$a_0 = 0 \qquad a_1 b_1 = 1$$

This shows us that $a_0 = 0$ and $a_1 = f'(0) \neq 0$ are necessary conditions. To show they are sufficient conditions, we can see that we may deduce all other coefficeints given these conditions.For example, we may deduce the coefficient of z^n as $a_0 + a_1 g(z) + \cdots + a_n g(z)^n$, so

$$a_1 b_n = P_n(a_2, ..., a_n, b_1, ..., b_{n-1})$$

Thus, we would start by doing $b_1 = 1/a_1$, and $b_2, b_3, ...$ are defiend recursively (this resulting polynomials is called *Bells polynomial*). With our construction, we have that g satisfies $b_0 = 0$ and $b_1 \neq 0$. We can do the same thing to find the other-side inverse, that is $g(f_1(w)) = w$. These inverses are equal:

$$f_1 = \mathrm{id} \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1) = f \circ \mathrm{id} = f$$

where we assumed the associativity of composition, something that can be checked. Finally, for the radius of convergence, we shall delay such considerations for now, noting that when we show that holomorphic functions are analytic we can use the inverse function theorem to estimate the values.

Proposition 2.3.3: Convergence On Operations Of Powerseries

Let f, g be convergent power series with radius of convergence $R(f)$ and $R(g)$. Then $g \circ f$ converges too. In particular, if we take some $r > 0$ so that $\sum_{n=0}^{\infty} |a_n| r^n < R(g)$ (i.e. is less then the absolute power series), then

1. $R(g \circ f) \geq r$
2. $|f(z)| < R(f)$ if $|z| < r$
3. $(g \circ f)(z) = g(f(z))$

Proof **:**
This is simply finding good bounds using the real power series. If we have

$$(g \circ f)(z) = \sum_p^{\infty} b_p \left(\sum_n^{\infty} a_n z^n \right)^p = \sum_k^{\infty} c_k z^k$$

then taking the aboslute value and using the natural generalization of the triangle inequality:

$$\left| \sum_p^{\infty} b_p \left(\sum_n^{\infty} a_n z^n \right)^p \right| \leq \sum_p^{\infty} |b_p| \left(\sum_n^{\infty} |a_n| |z|^n \right)^p = \sum_k^{\infty} \gamma_k |z|^k$$

where we evidently see by the use of the triangle inequality that $|c_k| \leq \gamma_k$. Thus, using the limi comparison test, we get that $(g \circ f)$ convreges absolutely if $|z| < r$, telling us that the radius of convergence of $(g \circ f)$ Is at lesat r and that $|f(z)| < R(g)$. The final fact now becomes the fact that we may re-arrange the terms of an absolutely convergent series.

Proposition 2.3.4: Operations On Power Series

Let f, g be power series with radius of convergence R_1 and R_2:

$$\sum_{n=1}^{\infty} a_n z^n \qquad \sum_{n=1}^{\infty} b_n z^n$$

Then:

1. $f + g$ is a power series given by term-wise addition with radius of convergence of at lesat $\min(R_1, R_2)$ (though it may be much larger)
2. fg is a power series with

$$\left(\sum_{k}^{\infty} a_n (z-c)^k\right)\left(\sum_{k}^{\infty} b_k (z-c)^k\right) = \sum_{k}^{\infty}\left(\sum_{i}^{\infty} a_i b_{k-1}\right)(z-c)^k$$

3. If $a_0 \neq 0$, then there is a unique power series such that $f(z)g(z) = 1$. If f has a positive radius of convergence, so does g

***Proof* :**

1. here
2. here
3. Without loss of generailty we may assume $a_0 = 1$ by dividing by a_0. Then we may write $f(z) = 1 - h(z)$ where $h(z)$ is now a power series satisfying $h(0) = 0$. Then it is a classical result that

$$\frac{1}{1-w} = 1 + \sum_{n=1}^{\infty} w^n$$

substituting $w = h(z)$, we get $g(z) = (1 - h(z))^{-1}$. For the radius of convergence, we may use the same trick as used in the inverse function theorem for power series.

2.4 Exponential, Trigonometric, and Logarithmic Function

We know explore 3 imporant examples of power series which will come up again and again. They will all be based on the single power series which will be known as the *exponential function*

Exponential Function

The exponential function is the function that satisfies the solution to the ODE

$$f'(z) = f(z) \qquad f(0) = 1$$

We can solve this by setting

$$\begin{aligned} f(z) &= a_0 + a_1 z + \cdots + a_n z^n + \cdots \\ f'(z) &= a_1 + 2a_2 z \cdots + n a_n z^{n-1} + \cdots \end{aligned}$$

which requires that $a_{n-1} = n a_n$ with $a_0 = 1$. Using induction, we get that

$$a_n = \frac{1}{n!}$$

This solution is usually denoted as either e^z or $\exp(z)$

Definition 2.4.1: Exponential Function

Let f be the function such that $f = f'$. Then the solution to this ODE is called the *exponential function*:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

it is usually denoted as either e^z or $\exp(z)$.

Notice that $\sqrt[n]{n!} \to \infty$, and hence e^z converges on the entire complex plane. the exponential function satisfies some nice properties. By definition it is it's own derivative. It is easy to manipulate algebraically:

$$e^{a+b} = e^a \cdot e^b$$

To see this, notice that

$$D(e^z e^{c-z}) e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$$

and so $e^z \cdot e^{c-z}$ is a constant (found by setting $z = 0$). Thus $e^z \cdot e^{c-z} = e^c$. Letting $z = a$ and $c = a + b$ gives us our desired result. Another way of doing this is by taking the partial sums of e^{a+b} and doing some algebraic manipulation to get the result, and concluding it works as $n \to \infty$. We can thus see that we have a homomorphism $r \mapsto e^{ir}$ mapping $\mathbb{R} \to S^1$. and so e^z is *never zero* with kernel $2\pi\mathbb{Z}$ so $\mathbb{R}/2\pi\mathbb{Z} \cong S^1$. This is in fact a topological homomorphism, and we can consider S^1 having the quotient topology. As an exercise, this topology can be verified to be the same as the subsapce topology of $S^1 \subseteq \mathbb{C}$. By our observation that $\mathbb{C}^\times \cong S^1 \times (0, \infty)$, we in fact get that

$$\exp : \mathbb{C} \to \mathbb{C}^\times$$

is a surjective homomorphism from $\mathbb{C}$ with addition to $\mathbb{C}^\times$ with multiplication, allowing us to link these two operations! As we've pointed out earlier $\ker(\exp) = 2\pi i\mathbb{Z}$.

Since $e^z \cdot e^{-z} = 1$, choosing $z = x \in \mathbb{R}$ to be real, notice that $e^x > 0$ for $x > 0$. Since e^x and e^{-x} are recipricals, this means that $0 < e^x < 1$ for $x < 0$. Since the series ahs real ceofficients, $e^{\overline{z}}$ is the complex conjugate of e^z. Thus,

$$|e^{iy}|^2 = |e^{iy} \cdot e^{-iy}| \qquad |e^{x+iy}| = e^x$$

2.4.1 Trigonometric Functions

Since the addition, composition, multiplication, and dividions of holomorphic functions is holomorphic, we have the following two holomorphic functions:

Definition 2.4.2: Trigonometric Functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{2.5}$$

By doing some substitution and computation, we get:

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$
$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

Using equation (2.5), There is a natural way of linking these formulas to e^z. First, since $e^{x+iy} = e^x e^{iy}$, we only need to expand for e^{iy}:

$$e^{iy} = 1 + iy - i^2\frac{y^2}{2!} + i^3\frac{y^3}{3!} - \cdots$$

Since e^{iy} is absolutely convergent, we can re-arrange the terms, getting us:

$$e^{iy} = (1 - \frac{y^2}{2!} + \cdots) + i(y - \frac{y^3}{3!} + \cdots)$$

where the right term on the right hand uses the fact that we can factor constants form converging series. We thus get:

$$e^{iz} = \cos(z) + i\sin(z)$$

Be mindful that for a complex number z, we usually get

$$z = |z|(\cos(\arg(z)) + i\sin(\arg(z)))$$

while for the expression e^{iz} we put z dirrectly into cos and sin. Using this formula, there is an easy way of remembering the definitions of sin and cos: take $e^{iz} = \cos(z) + i\sin(z)$, and take $e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z)$. Add these two together and divide by 2 to get the definition of cos. Similarly for sin. By doing some algebraic manipulations, we can get:

$$\cos^2(z) + \sin^2(z) = 1$$

Similarly, we get that

$$D(\cos(z)) = -\sin(z) \qquad D(\sin(z)) = \cos(z)$$

and we also get the identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

Any other trigonometric function and their properties can now be quickly derived using these (for example, tan, cot, and so forth). An important and counter-intuitive fact to point out is that cos and sin are defined on all of $\mathbb{C}$ (since the radius of convergence is ∞) and that the solution to the equation $\cos(z) = w$ has a solution for any $w \in \mathbb{C}$, not just $w \in [-1, 1]$!

2.4.2 Periodicity

A function is periodic if there exists a (nonzero) c such that $f(z+c) = f(z)$ for all $z \in \mathbb{C}$. We shall show that e^z has a period, that in particular there exists a nonzero c such that $e^{z+c} = e^z e^c = e^z$. From the properties of e^z, we immeditately get if the period exists then $c = iw$ for some real number w. Since

$$D\sin y = \cos(y) \leq y \qquad \sin(0) = 0$$

we get by the mean value theorem that $\sin(y) < y$ for $y > 0$. Similarly

$$D\cos(y) = -\sin(y) > -y \qquad \cos(0) = 1$$

which gives

$$y > 1 - \frac{y^2}{2}$$

which in turn gives

$$\sin(y) > y - y^3/6$$

and so

$$\cos(y) < 1 - y^2/2 + y^4/24$$

Using this, we get that

$$\cos(\sqrt{3}) < 0$$

Thus, by the Mean value theorem there exists a y_0 between 0 and $\sqrt{3}$ such that $\cos(y_0) = 0$. Since

$$\cos^2(y_0) + \sin^2(y_0) = 1$$

we have that $\sin y_0 = \pm 1$ and that $e^{iy_0} = \pm i$. Hence $e^{4iy_0} = 1$, giving us a period of $4y_0$.

This period is in fact the smallest possible priod. Take $0 < y < y_0>$ Since

$$y > y(1 - y^2/6) > y/2 > 0$$

wwe have that $\cos(y)$ is strictly decreasing. Since $\sin(y)$ is positive and $\cos^2(y) + \sin^2(y) = 1$, it follow that $\sin(y)$ is strictly increasing, and hence $\sin(y) < \sin(y_0) = 1>$ The double inequality $0 < \sin(y) < 1$ guarantees that e^{iy} is niether ± 1 nor $\pm i$. Thus, $e^{4iy} \neq 1$, and so $4y_0$ is indeed the smallest positive period. Let's say $\omega_0 = 4y_0$.

Let's now say ω is an arbitrary period. We'll show that there exists an n st

$$n\omega_0 \leq \omega < (n+1)\omega_0$$

If ω is not equal to $n\omega_0$, then $\omega - n\omega_0$ would be a positive period $< \omega_0$. But we just showed this is impossible, hence every periodmust be an integral multiple of ω_0.

We will denoted the smallest positive period of e^{iz} by 2π. Important consequence's of the proof we've just done is the following identities:

$$e^{\pi i/2} = i \quad e^{\pi i} = -1 \quad e^{2\pi i} = 1$$

2.4.3 Logarithm

The inverse of the exponential function is called the *logarithm*, denoted $\log(z)$. In particular we would want that $e^{\log(z)} = z$. Since e^z is never 0, the number 0 has no logarithm. If we write $z = \log(\omega)$ and choose $\omega \neq 0$, the equation $e^{x+iy} = \omega$ is equivalent (by euler's formula) to

$$e^x = |\omega| \quad e^{iy} = \frac{\omega}{|\omega|}$$

the first equation has a unique solution $x = \log|\omega|$, that is the real logarithm of the positive number $|\omega|$. The second equation is a complex number with absolute value 1, and thus it has only one solution in the interval $0 \leq y < 2\pi$. It is also satisfied by all y that differ form this solution by an integer multiple of 2π. Thus, every complex number has infinitely many logarithms which differ from each other by multiples of $2\pi i$. The imaginary part of $\log(\omega)$ is called the *argument* and is sometimes denoted $\arg(\omega)$. It is usually seen as the angle (measuring from the right side) between the x axis and $\omega/|\omega|$. By definition, the argument has infinitely many answers, for example:

$$\arg(i) = \{\pi/2, 5\pi/2, 9\pi/2, ...\}$$

Overall, the logarithm can be decomposed into the family of equations:

Definition 2.4.3: Logarithm

Given $\log|z|$ and $\arg(z)$, we get the following countable faimly of functions

$$\log(\omega) = \log|\omega| + i\arg(\omega)$$

Definition 2.4.4: Branch

Let $f(z)$ be a continuous function defined on a connected set Ω. then f is said to be a *branch* of log if

$$e^{f(z)} = z$$

For future purposes, we wil box the following result:

Lemma 2.4.1: Difference Between Brancehs

Let $f(z)$ be a branch of $\log(z)$ in a connected open set Ω. Then any other branch will have ht form $g(z) = f(z) + 2k\pi i$ for some $k \in \mathbb{Z}$. Conversely, for all $k \in \mathbb{Z}$, $f(z) = 2k\pi i$ is a branch

***Proof* :**
Let f, g be two branches and let

$$h(z) = \frac{f(z) - g(z)}{2\pi i}$$

Since h Is continuous on a connected set, we know it's image isconnected as well. The image of h is in $\mathbb{Z}$ and thus is only one point in the image, implying h must be constant, completing the proof.

When we choose $f(z)$ for which $\arg(1) = 1$, we shall call $f(z)$ the *principal branch*, and denote the argument function as Arg to emphasize we are now using the continuous function with a fixed choice of angle.

Using log and exp, we may define z^α.

Definition 2.4.5: General Exponent

Let $a, b \in \mathbb{C}$. Then:

$$a^b := \exp(b \log a)$$

When $z \in \mathbb{R}$, then by convention we will take the real logarithm unless stated otherwise. If a is restricted to positive numbers, then $\log(a)$ shall be real and a^b has a single value. Ohterwise, we would consdier $\log(a)$ to be the complex logarithm with a^b having infinitely many values, each different from one antoher by $e^{2\pi i n b}$. There is only a single value if and only if b is an integer n and a^b can be interpreted as a power of a or a^{-1}. If b is a rational numbe with reduced form p/q, then a^b has exactly q values that can be represetned as $\sqrt[q]{a^p}$. We also have the equality of sets (remember these sets are infinite)

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log(z_1 z_2) = \arg z_1 + \arg z_2$$

Using this, we can also finte inverse sin and cos by solving the equation

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) = \omega$$

Manipulating, we get

$$e^{iz} - 2w + e^{iz} = 0 \qquad \Leftrightarrow \qquad (e^{iz})^2 - 2we^{iz} + 1 = 0$$

This a quadratic equation with e^{iz} as the root $e^{iz} = \omega \pm \sqrt{\omega^2 - 1}$, which implies $iz = \log(\omega \pm \sqrt{\omega^2 - 1})$. Thus:

$$z = \arccos(\omega) = -i \log(\omega \pm \sqrt{\omega^2 - 1})$$

Since $\omega + \sqrt{\omega^2 - 1}$ and $\omega - \sqrt{\omega^2 - 1}$ are recipricals, we can also write:

$$z = \arccos(\omega) = \pm i \log(\omega + \sqrt{\omega^2 - 1})$$

The inveres of $\sin(z)$ is now easily seen as:

$$\arcsin(\omega) = \frac{\pi}{2} - \arccos(\omega)$$

Proposition 2.4.1: Derivative Of Logarithm

Let $f(z)$ be a branch of $\log(z)$ on a domain Ω. Then $f(z)$ is holomorphic and

$$f'(z) = \frac{1}{z}$$

Proof **:**
We simply comptute:

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\\
&=\lim_{h\to 0}\frac{f(z+h)-f(z)}{e^{f(z+h)}-e^{f(z)}}\\
&=\lim_{w\to f(z)}\frac{w-f(z)}{e^{w}-e^{f(z)}}\\
&\overset{!}{=}\frac{1}{e^{f(z)}}\\
&=\frac{1}{z}
\end{aligned}$$

where the $\overset{!}{=}$ equality comes from the fact that

$$\lim_{w\to f(z)}\frac{e^{w}-e^{f(z)}}{w-f(z)}$$

exists since e^z is differentiable, and so the reciprical exists too with limit $1/e^z$.

Proposition 2.4.2: Log Power Series

For $|z|<1$, the power series
$$f(z):=\sum_{n}^{\infty}(-1)^{n+1}\frac{z^n}{n}$$
converges and is equal to the principal branch of $\log(1+z)$

Proof **:**
It can directly be checked that $f(z)$ and $g(w)=\sum_n^\infty \frac{w^n}{n!}$ (which is the series expansion of e^w-1) are inverses of each other, and hence $g(f(z))=z$. Thus, $e^{f(z)}=z+1$, which by definition makes $f(z)$ the branch of $\log(z+1)$. Evaluating f at 0 shows that this is a principal branch.

Finally, we take a moment to think about a Riemann surface that will make log single-valued. Consider the covering
$$X=\left\{(z,w)\in\mathbb{C}^2\ :\ z=e^w\right\}$$
Then log would just be a mapping to the second coordinate. By the periodicity of log, we see that for any small neighbourd around $z\in\mathbb{C}$, we have countably many open sets laying above it in X. Furthermore, if we let $\log(1)=0$, then as we go around counter-clockwise on S^1 with the assumption that log is continuous, we will get that we would get back to $\log(1)=2\pi$, and everytime we go around we get another 2π factor: $\log(1)=2k\pi$. This motivates a visual like so:

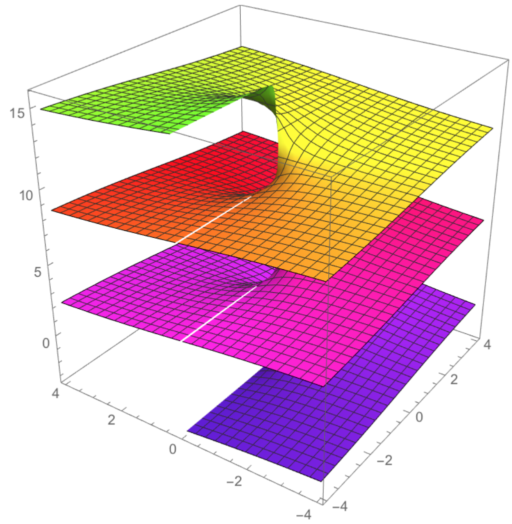


Figure 2.1: Branch Covering for complex log

which we have already briefly mentioned at the end of section 1.2.2.

2.5 Analytic Functions

Definition 2.5.1: Analytic Function

Let A be a function on an open set Ω. Then f is said to be *analytic* if every $f(z)$ can be written as a convergent power series centered at some $z_0 \in \Omega$. In particular, for every $z_0 \in \Omega$ there is a conergent power series centered at z_0 such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where z satisfies $|z - z_0| < r$ and r is less than the radius of convergence of the above power series.

Analytic functions have many of the same nice properties as power series. For example, they are infinitely differentiable, and their integral is easy to find:

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

which has the same radius of convergence by the root test. The easiest example of analytic functions are power series

Proposition 2.5.1: Power Series Are Analytic

Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a convergent power series with radius of convergence R. Then $f(z)$ is analytic in $|z - z_0| < R$

Proof **:**
Without loss of generailty we may assume $z_0 = 0$ since we are simply translating the entire function to be around 0. Pick any $|r_0| < R$. We shall convert the power series $\sum_n^{\infty} a_n z^n$ into one centered at z_0. Computing:

$$\begin{aligned} f(z) &= \sum_n^{\infty} a_n(z_0 - (z - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z - z_0)^k \end{aligned}$$

Since the z in the above is within the radius of convergence at the point 0, we get that the above power series is absolute continuous and hence we may re-arrange the terms:

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k$$

If this re-arrangement looks foreign, think of the layer-cake representation for double integrals. This power series evidentily respects $|z - z_0| < r$ for any $r < R - |z_0|$, and hence is a valid power series representation, completing the proof.

2.5.1 Principle of Analytic Continuation

Given a domain $\Omega \subseteq \mathbb{C}$ on which a polynomial is defined, we may enlarge Ω to a larger open set, and the polynomial will uniquely the enlarged set. Naturally, this generalizes to power series too. What is great is that this principle extends to analytic functions! The key idea is that if all derivatives at a point are zero for a polynomial and a power series, then it is zero for a neighbour (for a polynomial, it is zero everywhere, and for a power series it is zero on the radius of convergence)

Theorem 2.5.1: Principle of Analytic Continuation

Let $f(z)$ be an analytic function in a connected open set Ω and $z_0 \in \Omega$. Then the following are equivalent:

1. $f^{(n)}(z_0) = 0$ for $n \in \mathbb{N}$
2. f is identically 0 in a neighbourhood of z_0
3. f is 0 in Ω

Proof **:**
(3) $\Rightarrow$ (1) is immediate, and (1) $\Rightarrow$ (2) comes from Taylor's Theorem. We only require to show (2) $\Rightarrow$ (3).

Let $\omega' = \{z \in \Omega\ :\ f \equiv 0$ in a neighbourhood of z in $\Omega\}$. The set Ω' is nonempty since z_0 is in Ω'. it is also certainly open by definition. If we show Ω' is closed, then it is clopen and hence must be equal to Ω. Let $z \in \overline{\Omega'}$. Since the derivative is continuous, it's values can be extended uniquely to it's border and hence $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}$. Then, by (1) $\Rightarrow$ (2), f is identically 0 in a neighbourhoodof z, and so in fact $z \in \Omega'$. But then $\Omega' = \overline{\Omega'}$, showing it's closed But then by connectedness $\Omega' = \Omega$.

Note that there are smooth functions witll derivatives zero that are only 0 at a single points, e^{-1/x^2} is an example, and hence this result cannot extend to general smooth functions.

Corollary 2.5.1: Analytic Continuation Of Functions

Let f, g be analytic in a connected open domain Ω and $f = g$ in a neighbourhood of some point. Then $f = g$ in Ω

Proof **:**
Simply take the function $f - g$ and 0.

Corollary 2.5.2: Analytic Continuation Agreeing On Interval

Let f, g be analytic functions on U that agree a non-isolated set of points S. Then $f = g$ on U

Proof **:**
Consider $f - g$ on S. Then since $f - g$ is an analytic function, it must have isolated roots or else it's constant. Since S has non-isolated points, $f - g$ has non-isolated roots, hence $f - g$ must be constant on all of U, but then $f = g$

Corollary 2.5.3: Analytic On $\mathbb{C}$, Then Power Series

Let f be an entire analytic function. Then f is a power series.

Proof **:**
Take $f(0) = \sum a_n z^n$ on some disk $\mathbb{D}$. Represent f on $\mathbb{D}$ as g. Then since $f - g$ is 0 in the neighorhood of $\mathbb{C}$, g extends to all of $\mathbb{C}$, completing the proof.

Hence, the extension of an analytic function must be *unique*, and we only need two analytic functions to agree on a non-isolated set of points! This is in stark constrast with general smooth functions which may have multiple different possible extensions on their entire domain. For example take

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} x \geq 0 \end{cases}$$

then f on $(-2, -1)$ agree's with the zero function, but the two will certainly not extend to the same function! One interesting algebraic fact we get from this is the following

Corollary 2.5.4: Ring Of Analytic Functions

The ring of analytic functions on an open domain Ω, $\mathcal{A}(\Omega)$ forms an integral domain

***Proof* :**
Let $f, g \in \mathcal{A}(\Omega)$ and consider $fg = 0$. If $f \neq 0$, then f must be nonzero in some open domain. Hence g must be zero in some open domain. But then g is identically 0 in Ω, and hence $g = 0$. But then $\mathcal{A}(\Omega)$ is an integral domain, completing the proof.

(move from later, need to incorporate)

The next concept we will tackle is known as *analytic continuation*. Say we have a real differentiable function $f : (0, 1) \to \mathbb{R}$. If I ask you to tell me what is $f(-1)$, then you'll say this is obviously undefined. We may extend f smoothly to have it defined at -1, but there are certainly many ways of doing this. In the complex case, we will get that if f is determined to be complex-differentiable in some connected region, then it automatically determines uniquely on any larger connected open set, that is there is always a unique holomorphic extension to a larger connected open set. The most infamous analytic continuation is that of the *Riemann-zeta function*

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{s^k}$$

As of writing this, there is a million dollars to them that could show that all the zero's of $\zeta(s)$ are either real or have real part 1/2:

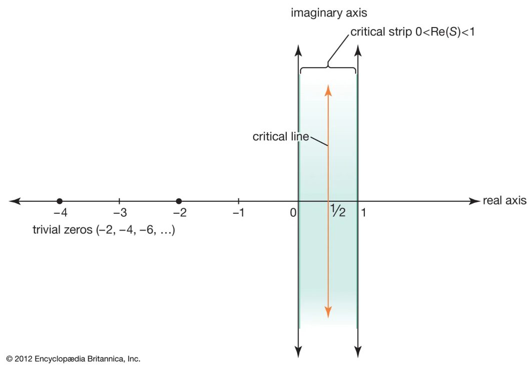


Figure 2.2: Visualizing where the critical points lie

2.5.2 Zeros and Poles of Analytic Functions

We shall explore the poles and roots of analytic functions. Let's say f is analytic in a neighbourhood of z_0. Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ given a z in a close enough neighbourhood of z_0. Let's say $f(z_0) = 0$ but f is not identically 0. Let k be the smalest integer such that $f^{(k)}(z_0) \neq 0$ (which is equivalent to saying $a_k \neq 0$). Then we may write:

$$f(z) = (z - z_0)^k g(z)$$

Then in a small enough neighbourhood, $g(z) \neq 0$ and $(z - z_0)^k \neq 0$ if $z \neq z_0$, and hence $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$ for appropriate $\epsilon > 0$. Hence, the root's of f must be isolated, just like for polynomials. The value k is called the *order* or *multiplicity* of the root z_0 at f. Doing a coordinate change on the above makes f into quite a simple function: if $\zeta := (z - z_0) g(z)^{1/k}$, then

$$f(z(\zeta)) = \zeta^k$$

This also means that any compact subset of the domain can only contain finitely many zeros.

2.5.3 Meromorphic Functions

Let's now generalize to a quotient of analytic function: $\frac{f(z)}{g(z)}$ where g is not identically 0. If $g(z_0) \neq 0$, then $\frac{f(z)}{g(z)}$ is well-defined and analytic in a neighbourhood of z_0 which we see by taking fg^{-1} in proposition 2.3.4. Then similarly to have we've done polynoials, if z_0 is a root of f and g, we me re-write both as $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$ where $f_1(z_0), g_1(z_0) \neq 0$. Then:

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}$$

If $k \geq l$, then $\frac{f}{g}$ extends to be analytic at z_0. If $k < l$, then z_0 is a pole of $\frac{f}{g}$ of order $l - k$, and we get

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$$

hence, it is natural to extend $\frac{f}{g}$ as a function on the Riemann sphere. Due to this non-analytic behavior on only a discrete set, we label them:

Definition 2.5.2: Meromorphic Functions

A *meromorphic function* in an open set Ω is a function that is well-defined and analytic in the compliment of a discrete set and expressible in a neighbourhoodof any point in Ω as a quotinet of analytic function $\frac{f}{g}$ where g is not identically 0.

It is clear that meromorphic funtions from a *field*: they are the localization of the analytic functions which form an integral domain.

3

Complex Integration

When we do real integration (whether it is signed definite, or unsigned definite like Lebesuge, or antiderivative giving a function), there is always a simple choice for orientation. For example, if we take

$$\int_a^b f(x)dx$$

there is only one direction we can go from a to b[1]. As we mentioned earlier, we can already define the Lebesuge integral on $\mathbb{C}$ (given the Borel σ-algebra given by the euclidean topology on $\mathbb{R}^2$). Instead, we would want to integrate along a path in $\mathbb{C}$. In our usual calculus setting, this leads us to defining the notion of differential form an integrating a form along a path:

$$\int_\gamma \omega$$

where $\omega = f_1 dx + f_2 dy$ for some (at least continuous, usually C^1) function f and a (at lest piece-wise continuous, usually piece-wise C^1) path γ. We saw that under the right condition, $\int_\gamma \omega$ is path independent (on a connected set), and sometime there exists an F such that $\omega = dF$. If f has the first property, it is said to be *conservative*, and if it has the second property it is said to have a primitive. If f is holomorphic, then we are *guaranteed* that $f dz$ ($dz = dx + idy$) is conservative, and if the domain is simply connected then it even has a primitive! In a sense, this can be seen as a feature that would be greatly lacking if it wasn't the case: By the FTC this is true of f is real differentiable, and so we would certainly expect it to be true for complex-differentiable. On $\mathbb{R}$, a non simply-connected set is simply not a connected set, which makes the introduction of requiring a simply connected set a novelty in the case of $\mathbb{C}$ which is not "visible" in the case of $\mathbb{R}$.

Complex integration will give us many tools that allow us to integrate integral, including real integrals. Two intersting example would be

[1]Granted, we may change the speed at which we get there, but the change of variables theorem tells us the value does not change

1. evaluating $\int_0^\infty \frac{\sin(x)}{x} dx$ and get $\frac{\pi}{2}$
2. Showing that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

One very important result we'll show is that if a function is once complex differentiable, it is infinitely complex differentiable, and even better it is analytic. This is completly false for real-differentiable function: consider any (legesgue) integrable function f. Then $F(x) = \int_0^x f(t)dt$ is (real) differentiable by the FTC. However, it can be far away from being twice differentiable. The fact that the Cauchy-Riemann equations force this relation is thus at first hand certainly quite a surprising result, and I will later put down here why this is "natural". The closest intuition I have so far was given to me by a PDE professor who said "it's a real-differentiable function that has to satisfy a PDE, there is bound to be incredible symmetry there". I will certainly have to think about it some more TBD.

3.1 Integration Review

(Tao's blog can also be helpful: here)

As mentioned, we need to specify a path on which we are integrating. A path (or curve) will be a continuous function $\gamma : [a, b] \to U \subseteq \mathbb{C}$. It is differentiable if γ is differentiable. The notatation $-\gamma$ will represent reversing the path, that is

$$-\gamma(x) = \gamma\left(\frac{b-x}{a}\right)$$

Now consider $f : U \to \mathbb{C}$. Just like we've done for Riemann integrals, we can take any path γ in a plane and split it up via linear connections, then take their sum:

$$\sum_{i=1}^{n} f(z_c)(z_{i+1} - z_i)$$

where $z_i \in \gamma([a, b])$. Intuitively, we can then take the limit of this value[2] to define $\int_\gamma f(x)dz$. This limit would exist in cases were, say, f is continuous. If you remember your theorem's from calculus, when the integral exists we can also think of this as taking the supremum over all possible path approximations. Overall, this notion gives a good intuition on what is the integral, but not a good way of computing it. We can find a good computational tool if the path γ is C^1. In this case, we can perform a change or variables. By letting $dz = \gamma(x)$, then $dz = \gamma'(x)dx$, which is saying that

$$z_{i+1} - z_i \approx \gamma'(x)(x_{i+1} - x_i)$$

and this approximatin becomes better as the partition gets finer. Then the integral becomes:

$$\int_a^b f(\gamma(x))\gamma'(x)dx$$

But now, we have this integral back down along the real axis, and so like we mentioned earlier we can split it along the real and imaginary parts. This integral is also well-defined since γ' is continuous.

[2] which can be shown to be independent of partition

Sometimes, we will want γ to have some sharp points, which in derivative language means it has critical points. This is fine, since it will be only a zero-measure set amount of points, and so we would split γ into a piece-wise differentiable path and define the integral to be the sum over the pieces.

Another way of looking at this is by coming back to vector-valculus. Given $dz = dx + idy$ and $f = u + iv$, we get

$$\begin{aligned} \int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy) \\ &= \int_\gamma (udx - vdy) + \int_\gamma (vdx + udy) \end{aligned}$$

This will be important when we use Green's theorem in a later proof (TBD).

The integral is independent of the re-paramaterization up to change of direction: if $\gamma' : [a,b] \to U$ defines the same path, the integarl is the same, but if $\gamma' = -\gamma$, then that would change the sign of the integral. Here are some important properties we will want to reference later

Proposition 3.1.1: Properties Of Line Integrals

1. $\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \cup \gamma_2} f(z)dz$
2. $\int_\gamma f + gdz = \int_\gamma fdz + \int_\gamma gdz$
3. $\left|\int_\gamma f(z)dz\right| \leq M|\gamma|$ where M is an upper bound for $|f|$ and $|\gamma|$ is the length of γ.

***Proof* :**
These are all proved in a calculus class. If you're re-proving them, notice that (3) is the generlization of the usual result of $\left|\int_a^b f(x)dx\right| \leq M|b-a|$.

Lemma 3.1.1: Paths In Connected Set

Any two points in a connect set $\Omega \subseteq \mathbb{R}^2$ can be joined by a piecewise C^1 curve

***Proof* :**
This is inherently just a connectedness problem. Let $a \in \Omega$. Let

$$E = \{b \in \Omega \ : \ a, b \text{ can be joined by a piecewise curve}\}$$

. Clearly $a \in E$, and for any point in E, any open neighbourood around a point can be joined by a straightline and hence E is open. Let $b \in \overline{E}$ and any neighbourood around b. This neighbourood certainly intersects E by assumption of b beinga border point, and hence a path in E connects to b by a straight line. But then $b \in E$, so $E = \overline{E}$, and hence E is clopen and so $E = \Omega$ since by definition Ω has no nontrivial clopen subsets.

Many of our results will necessitate the property of being simply connected, which says our space does not have a hole (or more technically that all closed paths are nullhomotopic). We shall give an example of what can go wrong to see why we care:

Example 3.1: Complex Integration not Simply Connected

1. We'll integrate z^n for different n's and different paths where $\gamma(0) = \gamma(1) = 1$. Naturally, if the path is constant the $\int_\gamma z^n = 0$. Let's instead take the path $\gamma(\theta) = e^{2\pi i\theta}$ so that it forms a circle around 0. Then, by also changing $z^n = e^{inx}$ and doing a change of variables we get:

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^{2\pi} e^{inx} i e^{ix} dx \\ &= i\int_0^{2\pi} e^{i(n+1)x} dx \\ &= i\left(\frac{e^{i(n+1)x}}{n+1}\bigg|_0^{2\pi}\right)\end{aligned}$$

   we now get to a scenario where the value we get depends on n. If $n \neq -1$ we get that the integral is equal to 0. If $n = -1$, then we in fact get that the integral is $2\pi i$! Looking bakc at our function, we see that our integral is $\int_\gamma 1/z dz$. This is essentially exactly a definition of the real $\log(x)$. In the complex case, notice that $1/z$ is not defined at $z = 0$, meaning the domain of this function is $\mathbb{C} \setminus \{0\}$, in particular this is *not a simply connected set.* This is a good example of what goes wrong if your set is not simply connected.

3.2 Differential Forms Review

Differential forms are the natural tool to use to hold geometric information when integrating, which in our case would be paths. In this section, a map $\gamma : [a, b] \to \Omega$ where $\Omega \subseteq \mathbb{R}^2$ will be C^1 or piecewise C^1, and $\gamma(t) = (x(t), y(t))$. A 1-form will be written as

$$\omega = Pdx + Qdy$$

with P, Q being (real or complex) continuous functions on Ω. Then define:

$$\int_\gamma \omega = \int_a^b \gamma^*\omega = \int_a^b F(t)dt$$

where $F(t) = P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)$, which we get from the pullback:

$$\gamma^*(Pdx + Qdy) = \gamma^*(P)\gamma^*(dx) + \gamma^*(Q)\gamma^*(dy)$$

where $\gamma^*(P) = P \circ \gamma$ and $\gamma^*(dx) = d(\gamma^* x) = d(x \circ \gamma) = d(x(t)) = x'(t)$. As usual, reparametirization of the curves does not affect the value of the curve up to a sign: if $u : [c, d] \to [a, b]$ is a strictly positive diffeomorphism so that we have a new path $t \mapsto \gamma(u(t))$, then if this new path is γ', then:

$$\int_{\gamma'} \omega = \int_c^d f(\gamma(u(t)))u'(t)dt = \int_a^b f(u)du = \int_\gamma \omega$$

If it is strictly negative then $\int_{\gamma'} \omega = -\int_\gamma \omega$. a diffeomorphism can only be striclty positive or negative since if it's derivative is 0 at any point is is not a diffeomrophism. If we have a piecewise curve, then we define:

$$\int_\gamma \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega$$

If $\gamma(a) = \gamma(b)$ (if the endpoints are the same), we shall say that γ is a closed curve. In this case, we may start by integrating at any point (show this if you're not comfortable with it). One of the nicest 1-forms are those of the form $\omega = dF$. These are particularely nice to compute. By the FTC:

$$\int_\gamma dF = \int_a^b F'(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a))$$

which we can see visually as thinking of the graph F defining the surface for which gravity "pulls down" on, and then dF represents the "potnetial" energy given by walking up or down this surface. Then the overall potential energy is only dependent on your height, giving us the above formula. If we return to the same point, then we get zero net potential energy, hence

$$\oint_\gamma dF = 0$$

Since we are trying to define integration of 1-forms $f(z)dz$, it would be nice if they always have an antiderivative just like the real 1-forms $f(x)dx$. We lead up to this by developing the theory for general complex 1-forms:

Definition 3.2.1: Primitive

given a 1-form ω, a *primitive* of ω is a C^1 function $F : \Omega \to \mathbb{C}$ such that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Outside of complex, such forms are usually called *exact*.

Note that this gives another quick proof that if Ω is connected and dF is zero then F is constant (which in the gravity interpretation can be thought as never gaining any potential energy). Using this, we see that if $\omega = dG$ for some other function, then $d(F - G) = 0$, showing that F and G differ by a constant and hence primitives are unique up to a constant. Not all complex 1-forms have a primitive (as we shall see after the next proposition). The next proposition gives the exact property needed for complex 1-forms to have a primitive:

Proposition 3.2.1: Criterion for Primitive Existing

Let ω be a 1-form on an open set $\Omega \subseteq \mathbb{C}$. Then ω has a primitive if and only if $\oint_\gamma \omega = 0$ for every piecewise C^1 closed curve γ

***Proof* :**
If ω has a primitive so that $\omega = dF$, then certainly

$$\int_\gamma \omega = F(\gamma(b)) - F(\gamma(a)) = 0$$

since $\gamma(a) = \gamma(b)$. Conversely, fix some $p = (x_0, y_0) \in \Omega$ and define

$$F(x, y) = \int_\gamma \omega$$

where γ is a path from (x_0, y_0) to (x, y). F is well-defined since if δ is another path from (x_0, y_0) to (x, y) then going along γ then back down δ forms a closed curve, by assumption has value 0, and hence since we can split integrals over piece-wise paths the integral over both paths are equal. I claim that $dF = \omega$. Letting $\omega = Pdx + Qdy$, then taking a straight path (since it is path independent) we get:

$$F(x + h, y) - F(x, y) = \int_x^{x+h} P(t, y)dt$$

Then

$$\lim_{h\to 0} (F(x + h, y) - F(x, y)) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} P(t, y)dt = P(x, y)$$

where the least equality comes form the FTC. But now we do the same for Q, giving us our final result and completing the proof.

It would be nice if we can check a subset of curves. For example, if we can only check all rectangles paths are zero, we would have a much easier time computing. In general this is not possible: we will require our domain to be *simply connected*. On a simply connected surface, like the open disk, this is a sufficient condition

Proposition 3.2.2: Existence Of Primitive On Disk

Let D be an open disc. If $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle contained in D with sides parallel to the axes, then ω has a primitive in D

the key is that every rectangle must be defined, and hence our space cannot have any holes.

***Proof* :**
Let (x_0, y_0) be the center of the disk and $(x, y) \in D$ be any point of D. Then we have two paths γ_1, γ_2 each starting at (x_0, y_0) and ending at (x, y) which each are two sides of the rectangle (one going horizontal then vertical, the other first goes vertically then horizontally). Then by assumtpion $\int_R \omega = \int_{\gamma_1+\gamma_2} \omega = 0$ and so $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ We may now define

$$F(x, y) = \int_{\gamma_1} \omega$$

where we arbitrarily chose one of the two intervals. We an repeat the same argument to show that

$$\frac{\partial F}{\partial x} = P \qquad \frac{\partial F}{\partial y} = Q$$

where $\omega = Pdx + Qdy$, completing the proof.

Thus, instead of checking on all curves, it suffices to check for all γ that are boundary of rectangles, or in particular sufficiently small rectangles, if Ω is an open disk.

As we mentioned, this works since there are no holes on open disks. On spaces that have holes, the above argument breaks down, and we can find a form ω that does not have a primitive (dz/z on $\mathbb{C} \setminus \{0\}$ is now an example of this by proposition 3.2.1). For 1-forms on $\mathbb{R}^2$, this would usually be the it. However, for forms of the form $f(z)dz$, we will fortunately still have that it has *local primitives* for any simply connected subset. We thus define the following:

Definition 3.2.2: Closed From

Let $\omega = Pdx + Qdy$ be a 1-form on an open set Ω. Then we say that ω is *closed* if for any point $z \in \Omega$, there is an open neighbourood in which ω has a primitive.

Another more common definition in the world of geometry is that a form is a closed form if $d\omega = 0$. We can always assume we can pick a small enough neighbourood around $z \in D$ to make it a disk, hence a form ω is closed if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle contained in a disk D.

In general, a closed differential form on a disk has a primitive, but a closed differential form on Ω needn't have a primitive as we saw in example 3.1. In the example, we had the form

$$\frac{dz}{z} = \frac{dx + idy}{x + iy} = \frac{xdx + ydx}{x^2 + y^2} + i\frac{xdy - ydx}{x^2 + y^2}$$

Let's consider a new form which is the imaginary part of this form:

$$w = \frac{xdy - ydx}{x^2 + y^2}$$

This is a closed form in the plane with the origin excluded, and has no primitive since if γ is the unit circle then

$$\int_\gamma \frac{xdy - ydx}{x^2 + y^2} = 2\pi$$

If we inspected this form more closely, we would see that it's equal ot $d\arctan\left(\frac{y}{x}\right)$, which is a *many-valued function*, showing us how ω fails to have a primitive.

If ω is a closed form, it only has local primitives. We may wonder if instead of finding a global primitive, we find a primitive given a chosen path γ. The following makes precise what this may mean:

Definition 3.2.3: Primitive Along A Path

Let $\gamma : [a, b] \to \Omega$ be a path contained in an open set Ω and let ω be a closed form in Ω. A continuous function $f : [a, b] \to \Omega$ is called a *primitive of ω along γ* if for any $\tau \in [a, b]$, there exists a primitive F of ω in a neighbourood of the point $\gamma(\tau) \in \Omega$ such that

$$F(\gamma(t)) = f(t)$$

for t near enough τ.

To understand the above intuitively, we shall give an example. Recall that $\int_\gamma dz/z$ has is nonzero if γ is a circle, hence it is not a primitive form (however it is closed since it is locally a primitive). Letting $\omega = dz/z$, if we take $f(t) = \int_0^t \gamma^*\omega$, then $f(0) = 0$ and $f(2\pi) = 2\pi i$. If ω was a primitive, we would have $f(0) = f(2\pi) = 0$, i.e. we would be on a single surface, or the graph of a function, thus f would represent the "height" at the point $f(t)$. However, this is not the case, as $\gamma(0) = \gamma(2\pi)$. This is probably some foreshadowing, but we may see f as hinting at the existence of a *Riemann surface* on which dz/z does have a primtive in some correct generalization.

Theorem 3.2.1: Primitive Along A Path

For any path γ, such that $f(t) = \int_0^t \gamma^*\omega$ is continuous, f is a primitive, always exists, and is unique up to addition of a constant.

***Proof* :**
Cartan p. 58, doable proof

Primitive of $\log(z)$ primitiveIntegerValueRmk

We may use this theorem to show things like any closed path which does not pass throug the origin for the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

is an integer, since the primitive along the path will be a branch of $\log(z)$, and so we will get $f(b) - f(a)$, which will be the different between two branches of $\log(z)$ at $\gamma(a) = \gamma(b)$, which we showed in lemma 2.4.1 is $2k\pi i$, and hence we have an integer.

Relation to Homotopies

Since holes are the main problem in defining a primitive, we shall take some time showing that up to being "stuck" or "wound " around a hole, the value of the ingral is path independent. The natural concept to introduce then is that of homotopy. Let $I = [0, 1]$. In this section, we show that integrals of paths in closed forms are homotopy invariant.

Definition 3.2.4: Homotopy

Let $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ be two path with the same initial and final points. Then these paths are *homotopic* (in D) with fixed endpoints if there exists a continuous map $(t, u) \to \delta(t, u)$ of $I \times I$ onto D st

$$\begin{cases} \gamma(t, 0) = \gamma_0(t) & \gamma(t, 1) = \gamma_1(t) \\ \delta(0, u) = \gamma_0(0) = \gamma_1(0) & \delta(1, y) = \gamma_0(1) = \gamma_1(1) \end{cases}$$

Similarly, we have a homotopy between two closed paths γ_0, γ_0 if there is a continuous maps $(t, u) \to \delta(t, u)$ of $I \times I$ into D st

$$\begin{cases} \delta(t, 0) = \gamma_0(t) & \delta(t, 1) = \gamma_1(t) \\ \delta(0, u) = \delta(1, u) & \text{for all } u \end{cases}$$

we say a closed path is null homotopic if it is homotopic to a constant function.

We shall show that on a closed form, two homotopic paths (with fixed points or that are closed) have the same value. We first upgrade theorem 3.2.1 to homotopies

Definition 3.2.5: Primitive Along Homotopies

Let $(t, u) \to \delta(t, u)$ be a continuuous mapping of a rectangle

$$a \leq t \leq b \qquad a' \leq u \leq b'$$

into the open et D and let ω be a clseod form in D. A *primitive of ω following the mapping* δ is a continuous function $f(t, u)$ in the rectangle satisfying the property that for any point (τ, ν) of the rectangle, there exists a primitive F of ω in a neighbourood of $\delta(\tau, \nu)$ such that

$$F(\delta(t, u)) = f(t, u)$$

at any point (t, u) sufficinetly near (τ, ν)

Lemma 3.2.1: Primitive Along Homotopies

There always exists an f as given in the above definition and it is unique up to addition of a constant.

Proof **:**
I will skip writing this proof down for now.

Theorem 3.2.2: Closed Forms And Homotopic Paths

Let γ_0, γ_1 be two homotopic paths of D with fixed end points or two homotopic closed paths. Then if ω is any closed from in D, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

Proof **:**
ibid.

Definition 3.2.6: Simply Connected

Let D be a set. We say that D is *simply connected* if D is connected and any closed path in D Is nullhomotopic.

Simply connected sets are the exact condition needed to guarantee a closed form is a primitive:

Theorem 3.2.3: Closed Form Has Primitive In Simply Connected Set

Let ω be a closed differential form on a simply connected open set D (i.e. ω has local primitives). Then ω has a global primitive in D

Proof **:**
By theorem 3.2.2, we have $\int_\gamma = 0$ for any closed paths contained in D, which by proposition 3.2.1 means that ω has a primitive in D.

This means that the closed form dz/z on any simply connected open set of $\mathbb{C}$ not containing zero has a primitve. This primitive would be a branch of $\log(z)$.

Winding Number of Closed Path

We keep bringing up the 1-form dz/z as our example of a closed non-primitive form. We may slightly expand this to a family of such 1-forms by translating:

$$\frac{dz}{z-a}$$

The reason for this particular closed (but not primitive) form is because they show up in the limit definition:

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

We shall in proceeding sections integrate the above value to great effect, and so understanding the behavior of the denominator will be very important. The value the integral

$$\int_\gamma \frac{dz}{z-a}$$

Is dependnet on the path γ and where how often it "winds"

Definition 3.2.7: Winding Number Of Closed Path

Let γ be a closed path in $\mathbb{C}$ and let $a \in \mathbb{C}$ such that a is not in the image of γ. Then the *winding number* of γ with respect to a (sometimes also called the *index* of γ with respect to a) is defiend to be the value of the integral

$$I(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

By remark primitiveIntegerValueRmk, $I(\gamma, a)$ is an integer.To find the winding number, we must find some continuous complex-valued function $f(t)$ defined for $0 \leq t \leq 1$ st

$$e^{f(t)} = \gamma(t) - a$$

which then gives us

$$I(\gamma, a) = \frac{f(1) - f(0)}{2\pi i} = n$$

where n depends on the number of times γ winds. Some notable properties of the winding number are

1. If a is fixed ,$I(\gamma, a)$ remains constant when the closed path γ is continuously diffformed without passing through a.
2. If we fixed the closed path γ and varry a in the compliment of the image of γ, then $I(\gamma, a)$ is a locally constant function with respect to a. Hence $I(\gamma, a)$ is a functio nof a which is constant in each connected component in the compliment of the image of γ
3. If $\gamma(I)$ is conatined in a simply connected open set Ω where the image doens't contain a, then $I(\gamma, a) = 0$.
4. If γ is a cirlce described in the positiv esense (so that $I(\gamma, 0) = 1$), then $I(\gamma, a) = 0$ for a outside the circle, and $I(\gamma, a) = 1$ for a inside the circle.

The winding number has a few intersting mapping consequences

Proposition 3.2.3: Image Of Cirlce

Let $f : \mathbb{D}_r \to \mathbb{R}^2$ be a coninuous function from $\mathbb{D}_R = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq r^2\right\}$ and let γ be the restriction of f to $x^2 + y^2 = r^2$. If a point a of the plane does not belong to the image of Γ and if $I(\gamma, a) \neq 0$, then f take sthe value of a at at least one poitn in the open disk $x^2 + y^2 < r^2$

Proof **:**
For the sake of contradiction, let's say this was not the case. The restriction of f to the concentric circle of 0 defines a continuous deformation of the closed path γ to a point. Thus, $\int_\gamma \frac{dz}{z-a} = 0$, contradicting our hypothesis, as we sought to show.

(Product of paths, p. 64 Cartan)

(idk if I should add information about oriented boundaries here)

3.3 Cauchy's Theorems

We now bring our discussion to 1-forms of the form $\omega = f(z)dz$ where f is a *holomorphic function.* The first fundamental result is that on simply connected open sets, ω has a primitive, hence generalizing the FTC for complex integration of curves:

Theorem 3.3.1: Cauchy's (Integral) Theorem

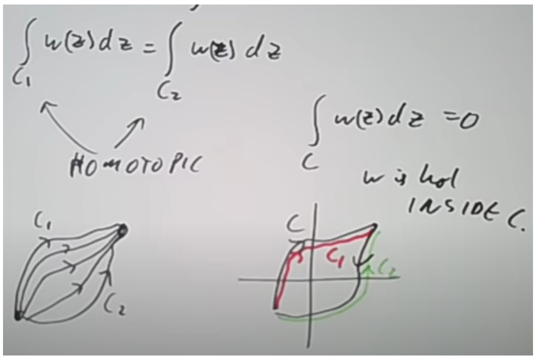
Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $\gamma : [a, b] \to U$ be a smooth closed curve. If γ is homotopic to a constant curve, then

$$\int_\gamma f(z)dz = 0$$

Equivalently, if γ_1 and γ_2 are homotopic, then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Another way of stating this theorem is that if $f(z)$ is holomorphic in an opn set D of the complex plane, then $f(z)dz$ is a closed form in D (where $dz = xdx + iydy$). To see why we may either take a closed form to be zero or two paths with the same fixed point, you may stare at this image:



There are many proofs of this, depending on how strong of assumptions we have. Here is a quick intuitive proof if we also have access to Green's Theorem. Recall that Green's theorem states that for a closed path γ:

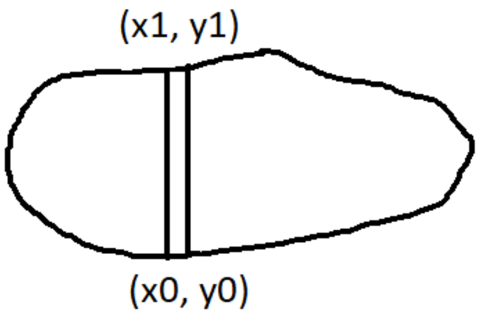
$$\int_\gamma f(x, y)dx + g(x, y)dy = \int\int_{\text{int}(\gamma)} \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dxdy$$

that is, we may integrate over the boundary of γ, or we integrate over the interior region. We'll recall the sketch of the proof of Green's theorem on a convex region for simplicity. We may simpilfy further by just showing

$$\int_\gamma f(x,y)dx = \int\int_{\text{int}(\gamma)} -\frac{\partial f}{\partial y}dxdy$$

since we would do the same proof for the g component and sume the two. Take some convex region:

Take some vertical strip, and select two points on the bottom and top border:



Now, by the fundamental theorem of calculus, we get that:

$$\int_{y=y_0}^{y_1} \frac{\partial f}{\partial y}dy = f(y_1) - f(y_0)$$

Now, take the above equation, and integrate both sides with respect to x:

$$\int\int_{\text{int}(\gamma)} \frac{\partial f}{\partial y}dy = \int_x \int_{y=y_0}^{y_1} \frac{\partial f}{\partial y}dy = -\int_\gamma f(x,y)dx$$

where the integarl is negative because we integrated anti-clockwise, if we integrated clockwise we'd get a positive. But this is the result. Then for most cases of Cauchy's theorem, we would have a finite union of convex regions, which suffices for our purposes.

Proof **:**
Let γ be a closed path and f a holomorphi function. With Green's Theorem, we simply compute:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy) \\ &= \int_\gamma udx - vdy + i\int_\gamma udy + vdx \\ &= \int\int_{\text{int}(\gamma)} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + \int\int_{\text{int}(\gamma)} \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy \\ &\stackrel{!}{=} 0\end{aligned}$$

where the $\stackrel{!}{=}$ inequality comes by using the Cauchy Riemann equations!

Using Wirtinger derivatives, we may compress the entire theorem into the following one-liner:

$$\int_\gamma f(z)dz = \int\int_{\text{int}(\gamma)} \frac{\partial f}{\partial \bar{z}} d\bar{z}dz = 0$$

where the first equality is green's theorem and the rest is the compressed version of what we just did.

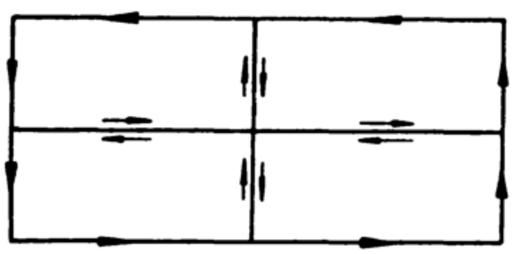
see this video for where the proof comes from: (Richard Brocherd in here)

The theorem can be proven by just assuming f is holomorphic without yet knowing the partials are continuous:

Proof **:**
We want to show $f(z)dz$ is closed, so we must prove that the integral $\int_\gamma f(z)dz$ is zero along the boundary γ of any rectangle R contained (with its interior) in D. Let's say that γ is the boundary of R and

$$\int_\gamma f(z)dz = p(R)$$

We want to show $p(R) = 0$. Subdivide R into 4 equal sided rectangles, labeled R_i and define paths γ_i like so:



It is easy to see that

$$\int_\gamma f(z)dz = \sum_i^4 f(z)dz = \sum_i^4 p(R_i)$$

Thu,s there must be at least one rectangle such that

$$|p(R_i)| \geq \frac{1}{4}|p(R)|$$

Let' say this recantlges is $R^{(1)}$. Then we may subdivide $R^{(1)}$ into four equal rectangles, and again at least one of which, say $R^{(2)}$ will satisfy the condition $|p(R^{(2)})| \geq \frac{1}{4^2}|p(R)|$. We can repeat this indefinitely, and from it define a sequence sequence that converges to $z_0 \in D$ (in particular f is holomorphic at z_0). Thus

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)|z - z_0| \tag{3.1}$$

where the error term approaches 0 as $z \to z_0$, that is $\lim_{z\to z_0} \epsilon(z) = 0$.

Now, if $\gamma(R^{(k)})$ is the orineted boundary of the rectangle $R^{(k)}$, then we get

$$\left| \int_{\gamma(R^{(k)})} f(z)dz \right| \geq \frac{1}{4^k}|p(R)| \tag{3.2}$$

Computing the left hand side using equation (3.1), we get

$$\int_{\gamma(R^k)} f(z)z = f(z_0)\int_{\gamma(R^{(k)})} dz + f'(z_0)\int_{\gamma(R^{(k)})} (z - z_0) + \int_{\gamma(R^{(k)})} \epsilon(z)|z - z_0|dz$$

on the right hand side of the above equation, the first two integrals are zero, and the third becomes negligible with the area of the rectagnle $R^{(k)}$ as k increases indefinitely, and hence is negligible as compared with $\frac{1}{4^k}$. Thus, when comparing with equation (3.2), we get that $p(R) = 0$. Hence, we get that

$$\int_\gamma f(z)dz = 0$$

completing the proof.

Corollary 3.3.1: Local Primitive For Holomorphic Functions

Let f be a holomorphic function in an open neighbourood $\Omega \subseteq \mathbb{C}$. Then f locally has a primitive whihc is holomorphic

Proof **:**

By Cauchy's Theorem, $f(z)dz$ is a closed form, and hence has a local primitive.

For future integration purposes and that of our next theorem, we shall require a slightly more general result to Cauchy's Theorem

Corollary 3.3.2: Cauchy Theorem On All But line of Points

Let $f(z)$ be a continuous function in an open set D which is holomorphic at every point of D except perhaps at the points of a line Δ parallel to the real axis. Then the form $f(z)dz$ is closed. In particular, if f is holomorphi at any point of D except perhaps at some isolated poitns, then the form $f(z)dz$ is closed

Proof **:**
This is simply an epsilon argument bounding away the bad points with a negligible extra area.

We want to show that $\int_\gamma f(z)dz$ is zero for the boundary γ of any rectangle contained in D. This is obvious if the rectangles does not intesect the line Δ, so suppose it has a side contained in Δ. lEt $u, u+a, u+ib, u+a+ib$ be the four corners of the rectangle, and $u, u+a$ being on the line Δ; a and b are real and without loss of generailty we may assume $b > 0$. Let $R(\epsilon)$ be the rectangel with corners

$$u + i\epsilon, u + a + i\epsilon, u + ib, u + a + ib$$

with $\epsilon > 0$. The integral $\int f(z)dz$ is zero on the boundary of $R(\epsilon)$, and as $\epsilon \to 0$, this integral tends to the integral round the boundary γ of the rectangle R. Certainly the limit passes through since we are dealing with compect sets, and hence $\int_\gamma f(z)dz = 0$.

Finally, let's say Δ intersects the rectangle without containing the entire boundary (i.e. it intersects it at two points). Then the line Δ splits R into two rectangles R', R'', and we may repeat the previous argument to show that $\int f(z)dz$ is zero around the boundaries of both R' and R'', and so the sum of these rectangles is equal to the integarl $\int_\gamma f(z)dz$, completing the proof.

This next theorem combines many results of holomorphic functions that at first seems to state a fact that should be generally true, but is in fact quite particular to our current build-up:

Theorem 3.3.2: Cauchy's Integral Formula

Let f be a holomorphic function in an open set Ω. Let $a \in \Omega$ and let γ be a closed path of Ω which does not pass through a and which is homotopic to a point in Ω. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = I(\gamma, a)f(a)$$

where $I(\gamma, a)$ denotes the winding number of the closed path γ with respect to a. In particular, if γ winds around a once, then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a)$$

Proof **:**
Let $g : \Omega \to \mathbb{C}$ be the function in Ω defiend by

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(z) & z = a \end{cases}$$

which is continuous by the definition of the derivative. g is holomorphic at any point of Ω except the point a. By corollary 3.3.2 we thus have for any path γ that is nullhomotopic:

$$\int_\gamma g(z)dz = \int_\gamma \frac{f(z) - f(a)}{z - a} dz = 0$$

which implies:

$$\int_\gamma \frac{f(z)}{z-a} = \int_\gamma \frac{f(a)}{z-a}$$

For the right hand side, by definition:

$$\int_\gamma \frac{f(a)}{z-a} dz = 2\pi i I(\gamma, a) f(a)$$

Thus, substituting back in we get and moving terms around we get:

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = I(\gamma, a) f(a)$$

If γ winds around a once, then:

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = f(a)$$

completing the proof.

Some important observations are in order:

1. There is no analogy for this in $\mathbb{R}$. Any path in $\mathbb{R}$ that is closed will cannot go around the singularity $x-t$. Even if we tried to fix this somehow, we would get $0 = \int_0^0 f(t)/(x-t)dt = f(x)$, which is simply not true, there are many smooth non-constant functions.
2. We cannot generlize this for $\mathbb{R}^n$ differentiation since we cannot divide ($\mathbb{R}^n$ is not a field[3]). In fact, the only complete divison algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, and only $\mathbb{R}$ and $\mathbb{C}$ are fields, hence Cauchy's Integral theorem is a particularity of the structure of $\mathbb{C}$
3. Notice how γ can be *any* path as long as it's nullhomotopic, while $f(a)$ is fixed. This shows the rigiditiy the global nature of holomorhpic functions!

The key result we may conclude with this theorem is that in an open disc, any holomorphic function can be expanded as a power series

Theorem 3.3.3: Taylor Expansion Of Holomorphic Function

Let $f(z)$ be a holomorphic function in the open disc $|z| < \rho$. Then f can be expressed as a power series in this disk. In particular, there exists a power series $p(x) = \sum_i^\infty a_n x^n$ whose radius of convergence is $r \geq \rho$ and whose sum $p(z)$ is equal to $f(z)$ for $|z| < \rho$.

[3] In particular, there is not field extension from $\mathbb{R}$ that has degree greater than 2

***Proof* :**
Let $r < \rho$. We shall find a power series which uniformally convergent to $f(z)$ for $|z| \leq r$. By the uniquness of the power series expansion, this power series is indepednent of our choice of r.

Choose an r_0 such that $r < r_0 < \rho$. Then by theorem 3.3.2, by taking γ to be the radius of the circle r_0 centered at 0 going counter-clockwise (i.e. in the positive direction), then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t - z} dt$$

for $|z| \leq r$. The function $1/(t - z)$ can be expanded to a series for $|z| < |t|$, in particular

$$\frac{1}{t - z} = \frac{1 - t}{1 - z/t} = \frac{1}{t}(1 + \frac{z}{t} + \frac{z^2}{t^2} + \cdots + \frac{z^n}{t^n} + \cdots)$$

Thus, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}} dt$$

The series converges uniformally for $|z| \leq r$ and $|t| = r_0$. Thus, we may integrate term by term an we obtain a uniformally convergent series for $|z| \leq r$:

$$\begin{aligned} f(z) &= \int_\gamma \frac{1}{2\pi i} \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}} dt \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t^{n+1}} dt \right) z^n \\ &= \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

where the coefficient are given by the integral formula

$$a_n = \frac{1}{2\pi i} \int_{|t| = r_0} \frac{f(t)}{t^{n+1}} dt$$

as we sought to show.

Corollary 3.3.3: Holomorphic, Then Analytic

Let f be a holomorphic function. Then f is analytic

***Proof* :**
For any point $a \in \Omega$ of the domain of the holomorhpic function f, by theorem 3.3.3 we may write f as a power series, and hence it is locally a power series, and hence it is analytic.

As a consequence, every holomorphic function is smooth, and every derivative is itself holomorphic.
With this, we can prove the converse of Cauchy's theorem:

Theorem 3.3.4: Morera's Theorem

Let $f(z)$ be a continuous function in an open (connected) set $\Omega \subseteq \mathbb{C}$. If the differential form $f(z)dz$ is closed, then the function $f(z)$ is holomorphic in D

The connected condition is simply to make the proof nicer.

Proof **:**
$f(z)dz$ has a primitive g locally, and by definition Since $g'(z) = f(z)$, hence g is holomorphic. Since $f = g'$ is the derivative of a holomorphic function, so is itself holomorphic. Sicne Ω is connected, we see that $g' = f$ on Ω. Hence, f is holomorphic.

Thus, we have characterized the 1-forms ω on $\mathbb{R}^2$ that are closed, namely they are of the form $\omega = f(z)dz$ for holomorphic $f(z)$. From this perspective, the Cauchy-Riemann equations are a geometric property: they guarantee that $d\omega = 0$.

Corollary 3.3.4: Holomorphic On All But Some Points, Then Holomorphic

Let f be continuous and D and holomorphic at all points of D except perhaps at points of some lien Δ. Then f is holomorphic on all points of D

Proof **:**
Without loss of generailty we may suppose Δ is parallel to the real axis by rotating if necessary. Then we have by corollary 3.3.2 that $f(z)dz$ is closed. Then by Morera's theorem f is holomorphic at all points of D, completing the proof.

From this perspective, corollary 3.3.2 only seemed like a generalization, however it was important to establish for technical reasons.

Corollary 3.3.5: Entire, Then Power Series

Let f be holomorphic on $\mathbb{C}$. Then f is a power series.

Proof **:**
In theorem 3.3.3, we may take $r \in \mathbb{R}_+$ since $\rho = \infty$.

Note that we could have established this as well by using analytic continuity.

Corollary 3.3.6: Uniform Convergence Of Holomorphic Functions

Let $f_n \to f$ be a sequence of holomorphic function converging uniformly to f on an open disk $\mathbb{D}$. Then f is holomorphic

***Proof* :**
Since $\mathbb{D}$ is connected, for any closed path in $\mathbb{D}$ we have

$$\int_\gamma f_n(z)dz = 0$$

Then by uniform convergence:

$$\int_\gamma f(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0$$

Thus, by Morera's Theorem f is holomorphic, completing the proof.

Hence, if $A(\Omega)$ is the collection of all analytic function $f : \Omega \to \mathbb{C}$ (i.e. all Holomorphic functions), thn $A(\Omega)$ is a Banach space with respect to the supremum norm.

Using corollary 3.3.4, we may holomorphically extend a function in the way that we are about to describe, this type of extension is known as *Schwarz' principle of symmetry.* Let D be a non-empty, connected, open set D which is symmetric with respect to the real axis. Let D^+ be the intersection of D with the half-plane $y \geq 0$ and D^- be the intersection of D with the half plane $y \leq 0$. Let's say $f(z)$ is continuous on D^+ and holomorphic on the points of D^+ where $y > 0$. Then we may extend f to all of D like so: consider $g(z)$ defined on D^- like os

$$g(z) = \overline{f(\overline{z})}$$

By our discussion in section 2.2.1, g is holomorphic on all D^- where $y < 0$. Then the function $h(z)$ which is $f(z)$ on D^+ and $g(z)$ D^- is continuous on D and holomorphic on all of D except perhaps the points on the real axis. Thus, by corollary 3.3.4, it is holomorphic on all of D.

3.4 Consequence of Taylor Expansion

If f is holomorphic in an open disc $\mathbb{D}$ centered at the origin, then $f(z) = \sum_{n=0}^\infty a_n z^n$ which converges in $\mathbb{D}$. We know that the coefficients of this power series can be recovered by taking derivatives of f, namely

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

We shall show two very important results:

1. that the growth of these coefficient is bounded in an important way
2. that $f(z)$ is always the average of a the integral of a circle of points (it satisfies the man value property, the integral equivalent of Harmonic)

Both of these results are found if we look at the Taylor expansion of a holomorphic function with $z = re^{i\theta}$ for $0 \leq r < \rho$ where ρ denotes the radius of the disc $\mathbb{D}$. We then get:

$$f(re^{i\theta}) = \sum_{n=0}^\infty a_n r^n e^{in\theta} \tag{3.3}$$

Fixing r and allowing θ to vary, $f(re^{i\theta})$ is a periodic function with respect to θ and the right hand side converges uniformally when θ varies and r is fixed. In fact equation (3.3) is the fourier expansion of f. Since the series converges uniformally, we may integrate term by term the following usual integral from fourier analysis:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})d\theta = \sum_{p\geq 0}\frac{1}{2\pi}\int_0^{2\pi} a_p r^p e^{i(p-n)\theta}d\theta = \sum_{p\geq 0}\frac{1}{2\pi}a_p r^p\int_0^{2\pi} e^{i(p-n)\theta}d\theta$$

One the right hand side, we see that integrating against $p \neq n$ gives 0, while integratin against $p = n$ gives 2π, which cancels with $1/2\pi$, and hence we get a single constant giving us:

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})d\theta \tag{3.4}$$

Note that we could have deduced this from theorem 3.3.3 by using

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}}dt$$

which we leave as an exercise. With equation (3.4), we can upper bound the value of a_n: Let $M(r) = \sup_\theta |f(re^{i\theta})|$ where r is fixed. Then the right hand side of equation (3.4) becomes

$$|a_n r^n| = \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|e^{-in\theta}\right| d\theta M(r) = M(r)$$

Hence, we get:

$$|a_n| \leq \frac{M(r)}{r^n} = \frac{\sup_\theta(f(re^{i\theta}))}{r^n} \qquad n \in \mathbb{N}$$

or written differnetly:

$$|f^{(n)}(0)| \leq \frac{M(r)n!}{r^n}$$

these inequalities are important enough to be given a name:

Definition 3.4.1: Cauchy Inequalities

Let f be a holomorphic function and $f(z) = \sum a_n z^n$. Then for each coefficient we have the bound:

$$|a_n| \leq \frac{M(r)}{r^n} = \frac{\sup_\theta(f(re^{i\theta}))}{r^n} \qquad n \in \mathbb{N}$$

These inequalities are called the *Cauchy inequalities*.

The Cauchy inequalities can be used to great effect as they force the bounds on the growth of f. Note that if f is smooth, it is not guaranteed that these bounds are finite, in fact if the bounds are finite as they are in Cauchy's inequalities then a real smooth function is analytic, showing that the real limitor behind analyticity is rate of growth (see Pugh's Real Analysis). It should be shown that if f is holomorphic in a disk $|z| < R$, and that if $|z| \leq r < R$, then:

$$|f^{(n)}(z)| \leq \frac{M(r)n!}{(R-r)^n}$$

This limitations on the growth of coefficients puts lots of important restraint on holomorphic functions, perhaps the most famous such example is the following theorem:

Theorem 3.4.1: Louville's Theorem

Let f be a holomorphic function on $\mathbb{C}$. Then if f is bounded, f is constant

Notice how this is certainly not true for $\mathbb{R}$, $\sin, \cos, e^{-1/x^2}$ are all smooth on $\mathbb{R}$ and bounded but certainly not constant.

Proof **:**
Since holomorphic functions are analytic, express f in terms a of a power series around 0. By the discussion above, each coefficient a_n are bounded by Cauchy's inequalities.

$$|a_n| \leq \frac{M(r)}{r^n}$$

Since f is bounded, $M(r) \leq M$ for some positive M independent of r, and so we have

$$|a_n| \leq \frac{M}{r^n}$$

for any $r \in \mathbb{R}$. But then that means means that the above inequality holds true for all r, and so it must be that $a_n = 0$. since this is true for all $n \geq 1$, we see that f is in fact constant

One famous consequence of this is that any complex polynomial must have a root

Theorem 3.4.2: D'Alembert's Theorem

Let $p(x)$ be a non-constant complex polynomial. Then $p(x)$ has at least one root.

Proof **:**
For the sake of contradiction, assume $p(x)$ is non-constant and and $p(z) \neq 0$ for any $z \in \mathbb{C}$. Then $1/p(z)$ is holomorphic on the whole plane. It is bounded, since

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \right)$$

for $a_n \neq 0$ tends to infinity as $|z|$ tends to inifnity, so there is comapct disc outside of which $|1/p(z)|$ is bounded. On the other hand $|1/p(z)|$ is bounded inside the compact disc since it is a continuous function. Hence $1/p(z)$ is bounded in the whole plane and hence is constant by Louiville's theorem. But that implies $p(z)$ is constant, contradicting our hypothesis, and completing the proof.

Proposition 3.4.1: Bounding Growth Of Holomorphic Function

Let $f(z)$ is an entire function (i.e. holomorphic in $\mathbb{C}$). Then:

1. If $|f(z)| < 1 + |z|^{1/2}$, f is constant
2. If $|f(z)| < 1 + |z|^n$, f is a polynomial of degree $\deg(f) \leq n$

Proof **:**

1. By the taylor expansion of Holomorphic functions, let

$$f(z) = \sum a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt$$

Since f is entire, f agrees with this power series on all of $\mathbb{C}$ (we may use analytic continuity or directly use the Taylor expansion theorem). Then, bounding the coefficients, we get:

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt \right| \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(t)}{t^{n+1}} \right| dt \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \frac{1 + t^{1/2}}{t^{n+1}} dt \\ &= \frac{2\pi}{2\pi} \frac{r(1 + r^{1/2})}{r^{n+1}} \\ &= \frac{1 + r^{1/2}}{r^n} \end{aligned}$$

since this holds as $r \to \infty$, we see that $|a_n| = 0$, hence f is constant. Since $f(0) = a_0$, we get that $|f(0)| \leq 1$, so f is a constant function in with image in $[-1, 1]$, completing the proof.

2. By the taylor expansion of Holomorphic functions, let

$$f(z) = \sum a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt$$

Then

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt \right| \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(t)}{t^{n+1}} \right| dt \\ &\leq \frac{1}{2\pi} \int_{|z|=r} \frac{1+t^k}{t^{n+1}} dt \\ &= \frac{2\pi}{2\pi} \frac{r(1+r^k)}{r^{n+1}} \\ &= \frac{1+r^k}{r^n} \end{aligned}$$

Hence, if $n > k$, we see that the limit as $r \to \infty$ goes to 0, and so we see that the Taylor expansion of $f(z)$ must be

$$f(z) = \sum_{k=0}^{n} a_n z^n$$

and so it's a polynomial. Since f is entire, by analytic continuity it must be that f is equal to this polynomial everywhere, hence f is a polynomial completing the proof.

We know examine the bahavior of $f(z)$ instead of the growth of its derivative, which corresponds to the case where $n = 0$. We get that

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

which, if we center our series at 0, is equivalently written as:

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

This means that $f(0)$ is the average (the mean value) on the any circle centered at 0 of radius r. Thus, if S is a closed disk contained in an open set D and f is holomorphic in the interior of S, the value of f at the centre of S is its mean value of the boundary of S.

Definition 3.4.2: Mean Value Property

Let f be a complex-valued continuous function. Then f is said to respect the *mean value property* if for all $r \in \mathbb{R}$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

As we discussed in section 2.2.2, all harmonic functions have this property, and and in fact all functions satisfying the mean value property are harmonic, but not all harmonic functions are holomorphic (easy counter-example is $z \mapsto \operatorname{Re} z$).

Theorem 3.4.3: Maximum Modulus Principle via Averaging

Let f be a continuous complex-valued function in an open set D of the plane $\mathbb{C}$. If f has the mean value property and $|f|$ has a relative maximum at a point $a \in D$ (i.e. $|f(z)| \leq |f(a)|$ for any z sufficinetly close to a), then f is constant in some neighbourood of a

Proof **:**
If $f(a) = 0$, then the result is immediate, so suppose that $f(a) \neq 0$. By multiplying f by a compelx constant if necessary, we can reduce the theorem to the case where $f(a)$ is real and > 0. Define:

$$M(r) = \sup_{\theta} |f(a + re^{i\theta})|$$

by maximality hypothesis, we have that for sufficicently small $r \geq 0$, we have $M(r) \leq f(a)$. By the mean value property,

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \leq M(r) \tag{3.5}$$

thus $f(a) \leq M(r)$, and so $f(a) = M(r)$. Thus, the following function is always ≥ 0 for sufficinetly small $|z - a| = r$

$$g(z) = \mathrm{Re}(f(a) - f(z))$$

and $g(z) = 0$ if and only if $f(z) = f(a)$. By equation (3.5), the mean value of $g(z)$ on the circle $|z - a| = r$ is zero. Since g is continuous and ≥ 0, this means that g is identically zero on this circle, and so $f(z) = f(a)$ when $|z - a| = r$ when r is sufficinetly small. Since this is true for all r smaller, we have that f is constnat on a sufficiently small neighbourood, completing the proof.

Corollary 3.4.1: Harmonic and max Attained in Interior, then Constant

Let D be a bounded, connected, open subset of $\mathbb{C}$ and f be a complex-valued continuous function defined on $\overline{D}$ having the mean value property in D. Then if M is the upper bound of f and $|f(a)| = M$ at some point $a \in D$, then f is constant

Proof **:**
Recall by theorem 2.2.2 that $\sup_{z \in \overline{D}} |f(z)| = \sup_{z \in \partial\overline{D}} |f(z)| = M$. Let's say that there is a $a \in D$ such that $|f(a)| = M$. Then by theorem 3.4.3 f is constnat in some neighbourood of a. This set is certainly open, and by doing a limit argument we see that it is also certainly closed, and it contains a meaning it's non-empty, and so it must be all of $\overline{D}$. Hence $f(z) = f(a)$ for all $z \in \overline{D}$

The maximum modulus principle allows us to upgrade the bounds given by the Cauchy inequality to a bound on the growth of $f(z)$. If $n = 0$, then the Cauchy inequality tells us that:

$$|a_0| = |f(z)| \leq \frac{M(r)}{r^0} = M(r) \qquad z \in \partial B_r(z)$$

We may upgrade this result to be true for all z in the ball:

Corollary 3.4.2: Cauchy Inequalities And Maximum Modulus Principle

Let f be holomorphic at z and $M(r) = \sup_\theta |f(z + re^{i\theta})|$. Then

$$|f(z)| \leq M(r) \qquad \forall |z| \leq r$$

***Proof* :**
By the maximum modulus principle, the maximum of $|f(z)|$ in some compact neighbourood is achieved on the boundary. let $f(z_0)$ is the point such that $|f(z_0)|$ is the maximum, then $|f(z_0)| = M(r)$, hence

$$|f(z)| \leq M(r) \qquad |z| \leq r$$

completing the proof.

As we see, the Maximum modulus principle puts strong constraints on the growth of holomorphic functions. The following theorem is one application of this, and shows how it can even force linearity of a function in the right situation

Theorem 3.4.4: Schwarz' Lemma

Let f be a holomorphic function in the disk $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Assume that:

$$f(0) = 0 \qquad |f(z)| < 1 \qquad \forall z \in \mathbb{D}$$

Then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$
2. if for a $z_0 \neq 0$, we have $|f(z_0)| = |z_0|$, then in fact

   $$f(z) = \lambda z$$

   for some λ such that $|\lambda| = 1$

***Proof* :**
By Taylor's expansiont theorem, $f(z) = \sum_{n=0}^\infty a_n z^n$ in some neighbourood of 0, and so the coefficient a_0 is zero since $f(0) = 0$. Thus, $f(z)/z$ is holomorphic in $\mathbb{D}$ as well. Since $|f(z)| < 1$ by assumption, we ahve

$$\left|\frac{f(z)}{z}\right| < \frac{1}{|z|} = \frac{1}{r} \qquad |z| = r$$

By the maximum modulus principle, this holds true for all $|z| \leq r$. Fixing $z \in \mathbb{D}$, we get $|f(z)| < |z|/r$ for any $r \geq |z|$ and $r < 1$.In the limit as $r \to 1$, we get

$$|f(z)| \leq |z|$$

which establishes (1).

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then the holomorphic function $f(z)/z$ attains the upper bound of its modulus at a point in the interio $|z| < 1$, thus by corollary 3.4.1

$$f(z)/z = \lambda$$

for some λ such that $|\lambda| = 1$. But then $f(z) = \lambda z$, completing the proof of (2).

Exercise 3.4.1

1. Show that every power series has a root

3.5 Laurent's Expansions

So far, we've been studying holomorphic function's without much concern for singularities (for example, $f(z) = 1/z$ is holomorphic with a singularity at $z = 0$). When it comes to studying the singularities, Taylor series start getting limited. We thus extend the notion of Taylor series in effect in insure a good tool for studying singularities:

Definition 3.5.1: Laurent Series

A *Laurent series* is a formal infinite series of the form:

$$\sum_{n\in\mathbb{Z}} a_n x^n$$

where n can vary accros all integers.

Any Laurent series is assocaited with two formal power series

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=-1}^{-\infty} a_n x^{-n}$$

If ρ_1 was the radius of convergence of the first and $1/\rho_2$ was the radius of convergence of the second one, then the original laurent series would have an *annulus of convergence* $\rho_2 < z < \rho_1$.

Proposition 3.5.1: Laurent Series Is Holomorphic

Let f be a Laurent series convergent in $\rho_2 < z < \rho_1$. Then f is holomorphic and the laurent series converges uniformally in the anulus $\rho_2 \leq z \leq \rho_1$ and absolutely in $\rho_2 < z < \rho_1$.

***Proof* :**
Let's say that we have convergent series:

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for} \qquad |z| < \rho_1 \tag{3.6}$$

$$f_2(z) = \sum_{n=-1}^{-\infty} a_n z^n \qquad \text{for} \qquad |z| > \rho_2 \tag{3.7}$$

We start by showing that f_2 is holomorphic with respect to z. Putting $z = \frac{1}{u}$, we get

$$g(u) = \sum_{n=0}^{\infty} a_{-n} u^n$$

which is holomorphic for $|u| < 1/\rho_2$ and it's derivative is given by

$$g'(u) = \sum_{n=0}^{\infty} n a_{-n} u^{n-1}$$

Then differentiating and using the chain rule, we get

$$f_2'(z) = \frac{-1}{z^2} g'(1/z) = \sum_{n=-1}^{-\infty} n a_n z^{n-1}$$

Thus, the series f_2 is differentiable term by term for $|z| > \rho_2$, and hence holomorphic. From now on, assume $\rho_2 < \rho_1$. Then the sum of the esries

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

is holomorphic in the annulus $\rho_2 < |z| < \rho_1$ and it's derviative $f'(z)$ is the sum of the series $\sum_n a_n z^{n-1}$ given by differentiating term by term. Note that Laurent series converge uniformally in the annulus $r_2 \leq |z| \leq r_1$ where

$$\rho_2 < r_2 < r_1 < \rho_1$$

Finally, for uniqueness, in the uniformally convergent part of the annulus, if we integrate

$$f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n r^n e^{in\theta}$$

with respect to θ, we get just like before the integral formula

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta$$

and thus given a function f, the coefficients a_n of the laurent expansion (when it exists) is determiend uniquely by the above relation.

Definition 3.5.2: Laurent Expansion

Let f be a function. Then f is said to have a *Laurent expansion* in the annulus $\rho_2 < |z| < \rho_1$ if there is a laurent series $\sum_{n \in \mathbb{Z}} a_n z^n$ which converges in this anulus and that is equal to $f(z)$ at each point in the annulus.

Naturally, by our pervious comment, if f is locally a laurent series, f is holomorphic in the anulus and uniformally converges in $r_2 \leq |z| \leq r_1$ where $\rho_2 < r_2 < r_1 < \rho_1$.

Theorem 3.5.1: Laurent Expansion Of Holomorphic Functions

Let f be a holomorphic function. Then given ρ_2, ρ_1, in an annulus $\rho_2 < |z| < \rho_2$, f an a laurent expansion

Proof **:**
Pick ρ_1, ρ_2 and choose r_1, r_2 such that

$$\rho_2 < r_2 < r_1 < \rho_1$$

By our earlier observations of the uniqueness of the Laurent expansion, the laurent series we will find will not depend on the choice of r_1 and r_2, and so the laurent series will converge on $\rho_2 < |z| < \rho_1$. Choose r_1', r_2' such that

$$\rho_2 < r_2' < r_2 < r_1 < r_1' < \rho_1$$

and consider the compact annulus $r_2' \leq |z| \leq r_1'$ whose oriented boundary is the difference of the circles γ_1 with the center 0 and the raidus r_1' in the positive direction, and the circle γ_2 with the center 0 and radius r_2' in the positive direction. By Cauchy's integral formula, we have for $r_2 \leq |z| \leq r_1$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)}{t-z}dt$$

For the first integral on the right hand side, we have $|t| = r_1'$ and $|z| \leq r_1 < r_1'$, so we can write the series exapnsion

$$\frac{1}{t-z} = \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}$$

which converges uniformally when t descrives the circle of center 0 and radius r_1'. We may replace $\frac{1}{t-z}$ in the first integral by this series as we've done before and by uniform convergence integrate term by term and we again get

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t-z}dt = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t^{n+1}}dt$$

for $n \geq 0$. For the second integral, we have

$$|t| = r_2' \qquad |z| \geq r_2 > r_2'$$

and so we ahve

$$\frac{1}{t-z} = -\frac{1}{z}\frac{1}{1-t/z} = -\sum_{n=-1}^{-\infty} \frac{z^n}{t^{n+1}}$$

replacing $1/(t-z)$ in the second integral by this series and then using uniform convergence to integrate term by term, we get

$$-\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)}{t-z} = \sum_{n=-1}^{-\infty} a_n z^n \qquad a_n = \frac{1}{1\pi i}\int_{\gamma_2} \frac{f(t)}{t^{n+1}}dt$$

for $n < 0$. combining these two, we get

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

for $r_2 \leq |z| \leq r_1$ which converges uniformally, completing the proof.

Proposition 3.5.2: Holomorphic Function Decomposition In Annulus

Let f be a holomorphic function in the annulus $\rho_2 < |z| < \rho_1$. Then there exists holomorphic functions $f_1(z)$ in the disk $|z| < \rho_1$ and a holomorphic function $f_2(z)$ for $|z| > \rho_2$ such that

$$f(z) = f_1(z) + f_2(z)$$

Furthermore, this decomposition is unique if f_2 tends to 0 as $|z|$ tends to infinity.

***Proof* :**
Let $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ be the Laurent expansion of f and let

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \qquad f_2(z) = \sum_{n=-1}^{-\infty} a_n z^n$$

Certainly we have found the desired decomposition and $|f_2(z)|$ tends to zero as $|z|$ tends to infinity. Now, suppose that

$$f(z) = g_1(z) + g_2(z)$$

is another such decompoition where g_2 decays to 0 as $|z| \to \infty$. We'll show that $f_1 = g_1$ and $f_2 = g_2$. Let h be the holomorphic function which is equal to $f_1 - g_1$ for $|z| < \rho_1$ and equal to $g_2 - f_2$ for $|z| > \rho_2$. This function is holomorphic on $\mathbb{C}$ and tends to 0 as $|z| \to \infty$. By the maximum modulus principle or Louville's theorem, the function h must be identically 0, but then we get $f_1 = g_2$ and $f_2 = g_2$, showing uniqueness and completing the proof.

An immediate use of Laurent expansion is showing that e^{-1/z^2} is *not* analytic. Indeed, the Laurent series converges everywhere except $z = 0$. There is this nice visual from wikipedia for this:

`https://en.wikipedia.org/wiki/Laurent_series#/media/File:Expinvsqlau_GIF.gif`

(A word on Cauchy's inequalities for Laurent Series?)

3.6 Study of Isolated Singularities

Let's now use our new local expansion of f in terms of Laurent series to study singularities. Let f be a holomorphic on the punctured disk $\mathbb{D}$ punctered at 0. When can f be holomorphically extended so that f is holomorphic on all of $\mathbb{D}$? In $\mathbb{R}^2$, it is certainly the case that there exists smooth function that cannot be extended, consider $\|x\|$ on the punctured disk. Such an extension is certainly unique (by analytic continuation, or simply because any continuous function is uniquely extended to its boundary), and it certainly has to be bounded in a neighbourood of the puncture. For complex function's being bounded is actually the extact condition needed for the extension to be holomorphic:

Proposition 3.6.1: Holomorphic Extension To Singularities

Let f be a holomorphic function on $\mathbb{D} \setminus \{0\}$. Then f extends holomorphically to 0 if and only if f is bounded in some neighbourood of 0

Proof **:**
It is clearly a necessary condition, hence we show it's sufficient. In the punctured disk $0 < |z| < \rho$, take the Laurent expansion of the holomorphic function f:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

In an appropriately small neighbourood, there exists a number M for which $|f(z)|$ is bounded above for $|z| = r$ for sufficinetly small r. By Cauchy inequalities, we get

$$|a_n| \leq \frac{M}{r^n}$$

for small r and $n < 0$. In particular, for negative n we get $a_n = 0$. Thus, the Laurent expansion of f reduces to a Taylor series. Then the Taylor series defines the natural required extension, and by uniqueness of the extension we have our desired result, completing the proof.

If we cannot extend, we define the following:

Definition 3.6.1: Isolated Singularity

Let $f(z)$ be a holomorphic function in the punctured disk $0 < |z| < \rho$. The origin 0 is said to be an *isolated singularity* of f if the function f cannot be extneded to a holomorphic function on the entire disk $|z| < \rho$.

Certainly, if not all negative coefficient of the laurent expantion of f are zero, then 0 is an isolated singularity. This splits into two cases

1. There are only finitely many $n < 0$ such that $a_n \neq 0$. In this case, there is a positive n such that $z^n f(z) = g(z)$ is a holomorphic function in some neighbourood around the origin. Thus, $f(z) = g(z)/z^n$ is *meromorphic* in some neighbourood of the origin (in a sense, the singularity does not grow too fast, it does not grow faster than polynomial growth)

2. There are infinitely many negative integers such that $a_n \neq 0$. in this case, $f(z)$ is not a meromorphic function in a neighbourood of the origin.

These lead to the following definition

Definition 3.6.2: Pole And Essential Singularities

If f cannot be extended to a holomorphic function at the origin but is meromorphic, then 0 is a *pole* of f. If f is not meromorphic on $\mathbb{D}$, then 0 is an *essential singularity* of f

With the introduction of laurent expansion, we have a few more interesting application of Louville's Theorem. One that can be proven easily if is if $|f| \leq |g|$, then it must be that $f = cg$ for some constnat c (in other words, there is no real "faster growing functions", or no holomorphic functions that "dominate"). The following result is a bit harder:

Theorem 3.6.1: Weiestrass Theorem

If 0 is an isolatd essential singularity of a holomorphic function f on the punctured disk $0 < |z| < \rho$, then for any $\epsilon > 0$, the image of the punctured disk $0 < |z| < \epsilon$ under f is everywhere dense in the plane $\mathbb{C}$

Proof **:**
Suppose for the sake of contradiction that there exists a disc centered at a of radius $r > 0$ which is outside the image of the puncutered disk $0 < |z| < \epsilon$ under f. Then

$$|f(z) - a| \geq r \qquad 0 < |z| < \epsilon$$

The function $g(z) = \frac{1}{f(z)-a}$ is then holomorphic and bounded in the puncutered disc $0 < |z| < \epsilon$. By proposition 3.6.1 this function can be extended to a holomorphic function on $|z| < \epsilon$, which we will re-denote as $g(z)$. Thus $1/f(z)$ will be meromorphic in the disc $|z| < \epsilon$ and $f(z) = a + \frac{1}{g(z)}$ will also be meromorphic, which contradicts the hypothesis that 0 is an essential singularity of $f(z)$.

Note that the case where z_0 is an essential singulariy is naturally reduecd to the case when$z_0 = 0$ by replacing z by $z - z_0$.

The following theorem is a generalization of Weiestrass, but will not be proven:

Theorem 3.6.2: Picard's Theorem

Let 0 be an isolated singularity of the holomorphic function f. Then the image of f of any punctered disk $0 < |z| < \epsilon$ is either the whole plane $\mathbb{C}$ or the plaen C with one missing point

Proof **:**
see ref:HERE

Take $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$ which is holomorphic in the punctered plane $z \neq 0$ and has an isolated essential singularity at the origin since the coefficeint of $1/z^n$ is nonzero for all $n \geq 0$. This function never take the value 0. Any other value *is* taken for any z in $0 < |z| < \epsilon$.

3.7 Residue Theorem

We know generalize Cauchy's Theorem and Cauchy's Integral theorem for the case where we allow poles, i.e. for meromorphic functions. The result can be seen as a special case of the generalized Stokes theorem. Naturally, since poles are involved, this leads us to talk about the Riemann sphere, and define holomorphic functions on the Riemann sphere. The way we define a function f to be holomorphic on S^2 is how we do it for [complex] manifolds: let $f : S^2 \to \mathbb{C}$, and take the coordinate charts eliminating the north and south pole respectivey by:

$$z = \frac{x + iy}{1 - u} \qquad z' = \frac{x - iy}{1 + u}$$

Then if $D \subseteq S^2$ is an open subset, f is holomorphic on D if and only if for any $p \in D$ that is not the north pole, f is holomorphic in coordinates z, and for any $p \in D$ that is not the south pole, f is holomorphic in coordinates z' (i.e., f represented in our two charts is holomorphic). We will say that $f : \mathbb{C} \to \mathbb{C}$ is "holomorphic at infinity" if by doing a change of variables $z' = 1/z$ and a neighbourood $|z| > r$, f is holomorphic at 0 in $|z'| < 1/r$. Similarly f is meromorphic at infinity and has an isolated essential singularity at infinity if the function $f(1/z)$ is meromorphic or has an isolated essential singularity at 0.

We can now generlize Cauchy's theorem in an anulus

Theorem 3.7.1: Cauchy's Theorem On An Anulus

Let $f(z)$ be holomorphic in an anulus $\rho_2 < |z| < \rho_1$ centered at the origin. Then if γ is a closed path contained in the anulus, then

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = I(\gamma, 0)a_{-1}$$

where $I(\gamma, 0)$ is the widning number of the path γ with respect to to origin 0 and a_{-1} is the coefficient of $1/z$ in the Laurent expansion of f

***Proof* :**
f has a Laurent expansion, and so we may write:

$$f(z) = \frac{a_{-1}}{z} + g(z) \qquad g(z) = \sum_{n \neq 1} a_n z^n$$

Note that $g(z)$ is certainly holomorphic in the anulus (it is the difference of two holomorphic function in the naulus), and its primitive is equal to

$$\sum_{n \neq 1} \frac{a^n}{n+1} z^{n+1}$$

We can see it was crucial that $n \neq -1$. We thus have:

$$\int_\gamma f(z)dz = a_{-1} \int_\gamma \frac{dz}{z} + \int_g g(z)dz$$

Now

1. $\int_\gamma g(z)dz = 0$ since g has a primitive
2. $\int_\gamma dz/z = 2\pi i I(\gamma, 0)$ by definition of the winding number

Plugging in these results, we get

$$\int_\gamma f(z)dz = 2\pi i a_{-1} I(\gamma, 0)$$

In particular, if γ winds around once in the positive sense, we get

$$\int_\gamma f(z)dz = 2\pi i a_{-1}$$

completing the proof.

Note that if f is holomorphic in $|z| < \rho_1$, then $a_{-1} = 0$ (since f has a Taylor expansion), and hence this theorem properly generalizes Cauchy's Theorem.

Definition 3.7.1: Residue

Let f be a function defined on $0 < |z| < r$ with an isolated singularity at 0 (which may be either a pole or an essential singularity). When γ is a closed path in some neighbourood of 0 which does not pass throguh 0, then the coefficient a_{-1} of the laurent expansion is called the *residue* of the function f at the singular point 0.

For a residue at the point at infinity, we require a little more technicality. Let $f(z)$ be holomorphic for $|z| > r$ and let $z = \frac{1}{z'}$. Then

$$f(z)dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'$$

Definition 3.7.2: Resisude At The Point At Infinity

Let f be holomorphic in $|z| > r$. Then the residue of the point at infinity is the residue of the function $-\frac{1}{z'^2} f\left(\frac{1}{z'}\right)$ at the point $z' = 0$. If $\sum_{n\in\mathbb{Z}} a_n z^n$ is the Laurent expansion of $f(z)$ in a neighbourood of the point at infinity, the residue of f at infinity is $-a_{-1}$.

Theorem 3.7.2: Residue Theorem

Let $\Omega \subseteq S^2$ be an open subset of the Riemann sphere and let f be a holomorphic function in Ω except perhaps at isoalted points which are singularities of f. Let Γ be the oriented boundary of a compact subset $A \subseteq \Omega$ and suppose that Γ does not pass through any singularities of f or the point at infinity. Then only a finite number of singularities z_k are contained in A, and

$$\int_\gamma f(z)dz = 2\pi i \left(\sum_k \operatorname{Res}(f, z_k) \right)$$

where $\operatorname{Res}(f, z_k)$ denotes the residue of the function f at the point z_k. The summation extends over all singularities $z_k \in A$ including the point at infinity if it qualifies

Proof **:**
We shall split the proof into two cases: When the point at infinity is in A, and when it is not.

If the point at infinity is not in A, then A is a compact set of $\mathbb{C}$. Each singular point z_k is the center of a clsoed disc S_k in the interior of A, and we can choose small enough radii so that these discs are disjoint (since there are finitely many of them). Let γ_k be the boundary of the disk S_k and

described in the positive sense let $\delta_k = S_k \setminus \gamma_k$. Let

$$A' = A - \bigcup_k \delta_k$$

the boundary of A' is the difference between Γ and the circles γ_k, that is $\Gamma \bigcup -\gamma_k$. Now, since f is holomorphic in an open neighbourood of A',

$$\int_\Gamma f(z)dz = \sum_k \int_{\gamma_k} f(z)dz$$

and for each integral:

$$\int_{\gamma_k} f(z)dz = 2\pi i \operatorname{Res}(f, z_k)$$

Thus:

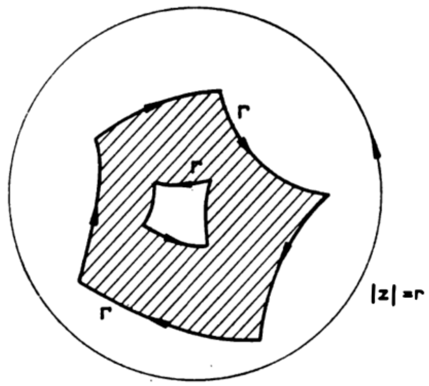
$$\int_\Gamma f(z)dz = \sum_k 2\pi i \operatorname{Res}(f, z_k)$$

completing the proof in the case of A not containing the point at infinity.

Let's now let A contain the point at infinity. Let $r \in \mathbb{R}_{>0}$ such that $|z| \geq r$ is a neighbourood of the point at infinity which does not intersect Γ and such that $f(z)$ is holomorphic in this neighbourood (the point at infinity being excluded). Let

$$A' = A - \{z \in \mathbb{C} \ : \ |z| > r\}$$

This will leave us with a region like so where the shadded region is the compliment of A



The oriented boundary of A' is the sum of the oriented boundary of Γ and of $|z| = r$ described in the positive sense now. Now, f is holomorphic in an open neighbourood of A', and so applying what we've proved in the first case we get:

$$\int_{\Gamma} f(z)dz + \int_{|z|=r} f(z)dz = 2\pi i \sum_k \operatorname{Res}(f, z_k)$$

where the sum on the right hand side is over all singularities z_k in A *except* the point at infinity. Now, by definition of the residue at infinity, we get

$$\int_{|z|=r} f(z)dz = -2\pi i \operatorname{Res}(f, \infty)$$

Thus, substituting and moving around we get

$$\int_{\Gamma} f(z)dz = 2\pi i \left(\operatorname{Res}(f, \infty) + \sum_k \operatorname{Res}(f, z_k) \right)$$

which is exactly the relation we were seaking, completing the proof.

We shall show how to use the residue theorem to compute many integrals without the explicit need to find anti-derivatives. Before we move on to giving many useful tools, we immediately point out on consequence of the Residue theorem when $A = S^2$ is the entire Riemann sphere. Then $\Gamma = \emptyset$ and so $\int_{\Gamma} f(z)dz = 0$, giving us:

$$\sum_k \operatorname{Res}(f, z_k) = 0$$

For example, rational functions are holomorphic at all but a finite number of isolated singularities, and so the sum of residues of a ratioanl function (includign the residue at infinity) is zero.

3.7.1 Residue Calculus

We'll now run through a few standard stricks to solve a couple more integrals. We shall first establish some general facts we may use with the Residue theorem, then find technics for more particular types of solutiosn

Preliminaries

Let f have a simple pole z_0 (pole of multiplicity 1) that is not at infinity. Then we may write

$$f(z) = \frac{1}{z - z_0} g(z)$$

where g is holomorphic in a neighbourood of z_0 and $g(z_0) \neq 0$. Let

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be its Taylor expansion at z_0. Then the Laurent expansion of f at z_0 is (by uniqueness):

$$f(z) = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}$$

Since $a_0 = g(z_0)$, we see that $b_{-1} = g(z_0)$ at $f(z)$, $z \neq z_0$. Thus:

$$\operatorname{Res}(f, z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} (z - z_0) f(z) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} g(z) = g(z_0)$$

Let's now consider $f = P/Q$ where P and Q are holomorphic in a neighbourood of z_0 and z_0 is a simple zero fo Q with $P(z_0) \neq 0$. Then:

$$f(z) = \frac{1}{z - z_0} \frac{P(z)}{Q_0(z)}$$

Now $\frac{P(z)}{Q_0(z)}$ is holomorphic in a neighbourood of z_0, so it has power series:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Hence:

$$f(z) = \frac{a_0}{z - z_0} \sum_{n=0}^{\infty} a_n (z - z_0)^{n-1}$$

Hence

$$\operatorname{Res}(f, z_0) = a_0 = \frac{P(z_0)}{Q_0(z_0)} = \frac{P(z_0)}{Q'(z_0)}$$

Example 3.2: Application I

Let $f(z) = \frac{e^{iz}}{z^2+1}$, which has two simple poles at $z = \pm i$. By our above observations, we see that we must find $P/Q' = \frac{1}{2z} e^{iz}$ so that:

$$\operatorname{Res}(f, i) = -\frac{i}{2e} \qquad \operatorname{Res}(f, -i) = \frac{ie}{2}$$

Let's now examine the case of multiple poles so that

$$f(z) = \frac{1}{(z - z_0)^k} g(z)$$

where $g(z)$ is holomorphic in an open neighbourood of z_0 where $g(z_))\neq 0$. Repeating the above computations for finding the Laurent series of f, we get:

$$f(z) = \frac{a_0}{(z - z_0)^k} + \cdots + \frac{a_{k-1}}{z - z_0} + a_k = \sum_{n=k+1}^{\infty} a_n (z - z_0)^{n-k}$$

Hence, $\operatorname{Res}(f, z_0) = a_{k-1}$ where a_{k-1} is the coefficient of the Taylor expansion of g. Hence, to fidn the residue, we simply have to partly do the Taylor expansion of g at z_0.

Example 3.3: Application II

Let $f(z) = \frac{e^{iz}}{z(z^2+1)^2}$. We'll find $\operatorname{Res}(f, i)$. Then:

$$f(z) = \frac{1}{(z-i)^2} \frac{e^{iz}}{z(z+i)^2} = \frac{1}{(z-i)^2} g(z)$$

For computational simplicity, let $t = z - i$ so that $z = i + t$. We shall find the coeficient of t in the Taylor expansion of:

$$h(t) = \frac{e^{i(i)t}}{(i+t)(2i+t)^2}$$

To so do so, we write down a partial Taylor expansion of the 3 terms in $h(t)$. In our case, it suffices to find the degree 1 term:

$$e^{i(i+t)} = e^{-1}(1 + it + \cdots)$$

$$(i+t)^{-1} = -i(1 - it)^{-1} = -i(1 + it + \cdots)$$

$$(2i+t)^{-2} = -\frac{1}{4}\left(1 - \frac{i}{2}t\right)^{-2} = -\frac{1}{4}(1 + it + \cdots)$$

Hence, we get

$$h(t) = \frac{i}{4^e}(1 + 3it + \cdots)$$

Thus, we get

$$\operatorname{Res}(f, i) = -\frac{3}{4^e}$$

The next thing we shall see is that we may use residues to detect the degree of a meromorphic function at a point z_0:

Proposition 3.7.1: Counting Zeros And Poles

Let $f(z)$ be a meromorphic function which is not constant in a nopen set D and let Γ be the orinted boundary of a compact set K contained in D. Suppose that the function f has no poles on Γ and does not take the value α on Γ. THen:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z - P \tag{3.8}$$

where Z denotes the sum of the orders of multiplicity of the roots contained in K of the equation

$$f(z) - a = 0$$

and P denotes the sum of the orders of multiplicity of the poles of f contained in K.

Proof **:**
Let f be meromorphic in an open neighbourood of z_0. Since it is meromorphic, then any root has finite degree so:

$$f(z) = (z - z_0)^k g(z)$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$. If f is holomorphic, $k \geq 0$, and if z_0 Is a hole then $k < 0$. Taking the logarithmic derivative of f, we get

$$\frac{f'}{f} = \frac{k}{z - z_0} + \frac{g'}{g}$$

Thus, f'/f has z_0 as a simple pole, and hence:

$$\text{Res}(f'/f, z_0) = k$$

Now, by very similar equations, we can find:

$$\text{Res}\left(\frac{f'}{f-\alpha}, z_0\right) = Z - P$$

Thus, by the Residue theorem, for Γ as described in the proposition:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z - P$$

completing the proof.

If f is holomorphic, then the above equation (3.8) becomes:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z$$

Hence, this integral both finds zeros of polynomials and their multiplicity.

Proposition 3.7.2: Stability of Roots

Let z_0 be a root of order k of the equation $f(z) = a$, f beign a non-constnat holomorphic function in some neighbouroodof z_0. For any sufficinetly small enough of U of z_0, and for any b sufficiently near to a and not equal to a ($b \neq a$), the equation $f(z) = b$ has exactly k simple solutions in V.

Proof **:**
Let Γ be a circle centered at z_0 with sufficiently small readius so that z_0 isthe only solution to the equation $f(z) = a$ conatined in the clsoed disc bounded by γ. Furthermore, suppose that the radius of γ is sufficiently small so that $f'(z) \neq 0$ at any point of the disc except the center z_0. Consider the integral:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz$$

Then the above remains constant when b varies in a connected cpoment of the compliment of the image of γ under f. Thus, for any b sufficinetly near to a, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}dz = k$$

Thus, the equation $f(z) = b$ has exaclty k roots in the interio of γ, if each root is counted with its order of multiplicity. But for b sufficinetly near to a but $\neq a$, the root of the equation $f(z) = b$ are all simple because the derivative of $f'(z) \neq 0$ t any point of z sufficinetly near to z_0 and $\neq z_0$, completing the proof.

3.7.2 Method of Residues

We now introduce some intergration tricks. Note that there is no general way of solving integrals, we just gained some more tools.

Consier an integral of the form

$$I = \int_0^{2\pi} R(\sin(t), \cos(t))dt$$

where $R(x, y)$ is a rational function without a pole on the cirlce $x^2 = y^2 = 1$. Let $z = e^{it}$ so that z describes the unit circle as t increase from 0 to 2π. Thus, I wil be equal to $2\pi i$ times the sum of the residues of:

$$\frac{1}{iz}R\left(\frac{1}{2i}\left(z-\frac{1}{z}\right), \frac{1}{2}\left(z+\frac{1}{z}\right)\right)$$

at the poles contained in the unit disc, that is:

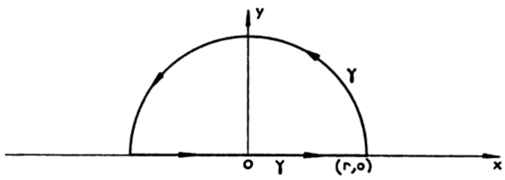
$$I = 2\pi\sum\left\{\frac{1}{iz}R\left(\frac{1}{2i}\left(z-\frac{1}{z}\right), \frac{1}{2}\left(z+\frac{1}{z}\right)\right)\right\}$$

Example 3.4: Residue Method I

Consider
$$\int_0^{2\pi} \frac{1}{a+\sin(t)}dt$$
where $a \in \mathbb{R}_{>1}$. Decomposing, we get
$$I = 2\pi \sum \operatorname{Res} \frac{2i}{z^2+2iaz-1}$$
The only pole contaiend in the unit disc is
$$z_0 = -ia = i\sqrt{a^2-1}$$
it's residue is $\frac{i}{z_0+ia} = \frac{1}{\sqrt{a-1}}$. Thus:
$$I = \frac{2\pi}{\sqrt{a^2-1}}$$

Our next type of integral will be of the form:
$$I = \int_{-\infty}^{\infty} R(x)dx \tag{3.9}$$
where R is a rational function without a real pole. Natuarlly, we need ot assume the integral is convergent, which is equivalent (i.e. it is a necessary and sufficient condition) is that the principal part of $R(x)$ at infinity is of the form $\frac{1}{x^n}$ for integer $n \in \mathbb{N}_{\geq 2}$. Equivalently, we must have:
$$\lim_{|x|\to\infty} xR(x) = 0$$

To calculate integrals of the form in equation (3.9), we will integrate the function $R(z)$ in the complex plane with the boundary of a half-disc centered at 0 wit hradius r and situated in the half-plane $y \geq 0$:



For a sufficiently large r, $R(z)$ is holomorphic on the boundary, and so by the Residue theorem $\int_\gamma R(z)dz$ is equal to the sum of the residues of the poles of R inside γ. We thus have the following breakdown of the integral:
$$\int_{-r}^{r} R(x)dx = \int_{\delta(r)} R(z)dz = 2\pi i \sum \operatorname{Res}(R(z)) \tag{3.10}$$
where $\delta(r)$ is the rest of the unit circle without the $[-r, r]$ interval, i.e. the upper half circle centered at 0 with radius r described in the positive sense. Our goal now is to show that as $r \to \infty$, the second

integral in equation (3.10) tends to 0. This will leave us with

$$\int_{-\infty}^{\infty} R(x)d = 2\pi i \sum \mathrm{Res}(R(z))$$

or, if we were going in the lower half plane, we'd get

$$\int_{-\infty}^{\infty} R(x)d = -2\pi i \sum \mathrm{Res}(R(z))$$

where we sum over all the poles in the lower half plane.

Lemma 3.7.1: Residue Method Lemma I

Let $f(z)$ be a continuous function defined in the sector $\theta_1 \leq \theta \leq \theta_2$ and r, θ denote the modulus and the argument of z. Then if

$$\lim_{|z|\to\infty} zf(z) = 0 \qquad (\theta_1 \leq \arg(z) \leq \theta_2)$$

Then $\int f(z)dz$ extended over the arc of the circle $|z| = r$ contained in the sector tends to 0 as $r \to \infty$.

***Proof* :**
Let $M(r)$ be the maximum of $|f(z)|$ on $|z| = r$. Then

$$\left|\int f(z)dz\right| \leq M(r)r(\theta_2 - \theta_1) \xrightarrow{r\to\infty} 0$$

Example 3.5: Residue Method II
Let

$$I = \int_0^{\infty} \frac{1}{1+x^6}dx$$

The complexification $1/(1+z^6)$ has 6 poles, all on the unit circle, three in the upper half plane and three in the lower. Sticking to the upper-half plane:

$$e^{i\frac{\pi}{6}} \qquad e^{i\frac{3\pi}{6}} \qquad e^{i\frac{5\pi}{6}}$$

Calculating the residues at each poles we get $\frac{1}{6z^5} = \frac{-z}{6}$. Thus:

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}dx \\ &= -\frac{\pi i}{6}\left(e^{i\frac{\pi}{6}} + e^{i\frac{3\pi}{6}} + e^{i\frac{5\pi}{6}}\right) \\ &= \frac{\pi}{6}\left(2\sin(\pi/6) + 1\right) \\ &= \frac{\pi}{3} \end{aligned}$$

We next study integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)e^{ix}dx$$

where f is holomorphic in a neighbouroodof of each point of the closed half plane $y \geq 0$, except perhaps at a finite number of points. We shall split up this integral into the cases where the singularities or on the real axis and when they are not on the real axis.

Let's start wit hteh case where the singularities are *not* on the real axis. Then the integral

$$\int_{-r}^{r} f(x)e^{ix}dx$$

is well-defined, and as $r \to \infty$, we naturally get

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx$$

if the integral is convergent. Then it is relatively easy to find the value, it shall again be the poles in the upper-half plane. This time, it will we will not be able to take the equivalent path in the lower half plane since e^{iz} grows quickly in the lower half plane. To get the result, we need the following lemma

Lemma 3.7.2: Residues Method Lemma II

Let $f(z)$ be a function defined in a sectro of the half-plane $y \geq 0$. If

$$\lim_{|z|\to\infty} f(z) = 0$$

Then $\int_{|z|=r} f(z)e^{iz}dz$ contained in the sector tends to 0 as $r \to \infty$.

***Proof* :**
Let $z = re^{i\theta}$ and $M(r) = \max_{\theta\in S} |f(re^{i\theta})$ where S is the sector. Then:

$$\left|\int f(z)e^{iz}dz\right| M(r) \int_0^{\pi} e^{-r\sin(\theta)}rd\theta$$

I claim that the integral on the right hand side is bounded by π. Indeed, first notice that

$$\int \int_0^{\pi} e^{-r\sin(\theta)}rd\theta = 2\int_0^{\pi/2} e^{-r\sin(\theta)}rd\theta$$

Next, for $0 \leq \theta \leq \pi/2$, we have $\frac{2}{\pi} \leq \frac{\sin(\theta)}{\theta} \leq 1$. Thus:

$$\int_0^{\pi/2} e^{-r\sin(\theta)}rd\theta \leq \int_0^{\pi/2} e^{-\frac{2}{\pi}r\theta}rd\theta \leq \int_0^{\infty} e^{-\frac{2}{r}r\theta}rd\theta = \frac{\pi}{2}$$

Sinec $f(z) \xrightarrow{|z|\to\infty} 0$, $M(r) \to 0$, completing the proof.

Proposition 3.7.3: Evaluating Type III Integrals

If $\lim_{|z|\to\infty} f(z) = 0$ for $\mathrm{Im}(z) \geq 0$, then

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum \mathrm{Res}(f(z), e^{iz}) \tag{3.11}$$

where the summation is over the singularities of $f(z)$ contained in the upper half-plane $y > 0$.

Proof **:**
This is now an immediate consequence of the above lemma and our previous observations.

Example 3.6: Residue Method III
Consider

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+1}dx = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx\right)$$

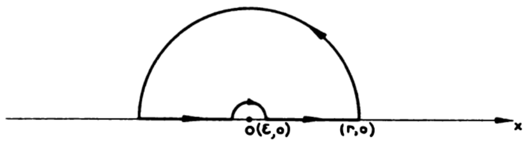
Then it is equal to

$$\pi i \sum \mathrm{Res}\left(\frac{e^{iz}}{z^2+1}\right)$$

where the summatino govers over the poles in the upper half-plane. Since there is only the pole $z = i$, it is simple, and hence it's residue is $\frac{1}{2ei}$. Thus:

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+1}dx = \frac{\pi}{2e}$$

Let's now say that $f(z)$ has singularities on the real axis. For simplicity, let's say it is at the origin (the proof for other points is a translation of this special case). What we shall do is modify our path of integration to bypass the singualrity:



Let's say that the smaller semi-circle has path $\gamma(\epsilon)$. Then:

Lemma 3.7.3: Residues Method Lemma III

Let $z = 0$ be a simpel pole of $g(z)$. Then

$$\int_{t\to 0}\int_{\gamma(\epsilon)} g(z)dz = \pi i\,\mathrm{Res}(g, 0)$$

***Proof* :**
First, we have $g(z) = \frac{a}{z} + h(z)$ where h is holomorphic at the origin. The integral $\int_{\gamma(\epsilon)} h(z)$ tends to 0 as $\epsilon \to 0$, and the integral $\int_{\gamma(\epsilon)} \frac{a}{z} dz$ tends to $\pi i a$, but that complete the proof.

We may now apply this lemma to the function $g(z) = f(z)e^{iz}$

Example 3.7: Residue Method IV
Let

$$I = \int_0^\infty \frac{\sin(x)}{x} dx$$

Then

$$I = \int_0^\infty \frac{\sin(x)}{x} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin(x)}{x} dx = \frac{1}{2i} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_\epsilon^\infty \frac{e^{ix}}{x} dx \right)$$

Then:

$$\frac{1}{2i} \lim_{\epsilon \to 0} \int_{\gamma(\epsilon)} \frac{e^{iz}}{z} dz = \frac{\pi}{2} \operatorname{Res}\left(\frac{e^{iz}}{z}, 0 \right) = \frac{\pi}{2}$$

Using our current results, we may find out how to integrate integrals of the form $\int_{-\infty}^\infty f(x)e^{-ix} dx$. Note that we would have to integrate in the *lower* half-lane instead of the upper, for $|e^{-iz}|$ is bounded in the lower half-plane. More generally, an integral of the form

$$\int_{-\infty}^\infty f(x)e^{ax} dx$$

where $a \in \mathbb{C}$ can be evaluated in the half-plane where $|e^{az}| \leq 1$. Now, what about:

$$\int_{-\infty}^\infty f(x)\sin^n(x) dx \qquad \int_{-\infty}^\infty f(x)\cos^n(x) dx$$

We have that $\sin(z)$ and $\cos(z)$ are always unbounded in any half plane, so in order to evaluate such an expression, we would have to express the trigonometric functions in terms of complex exponentials so that our preceeding methods can be applied.

The forth type of integral we'll consdier is:

$$I = \int_0^\infty \frac{R(x)}{x^\alpha} dx$$

where $\alpha \in (0, 1)$, and $R(x)$ is a rational fucntion with no poles in the positive real axis $x \geq 0$.