Everything You Need To Know About Algebraic Geometry

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Contents

These notes will be a compilation of my knowledge of Algebraic Geometry. It will assume a strong understanding of Commutative Algebra, Differential Geometry, and Algebraic Topology; essentially, anything non-optional material from EYNTKA Undergraduate Algebra, EYNTKA Differential Geometry, and EYNTKA Algebraic Topology will be considered as assumed prerequisites.

The general structure of this book will be as follows:

1. Affine and Projective Varieties: There will be a quick review of the essential facts about $\operatorname{Spec}(R)$ and $\operatorname{Proj}(R)$.
2. *Sheaves*: The essential if Sheave theory
3. *Schemes*: The essential of Scheme theory (and can be thought of as algebro-geometric objects)

All essential category theory will be assumed; it can be reviewed in EYNTKA Undergraduate Algebra.

Throughout this book, we shall use the following terminology:

- Let k be a field, $\overline{k}$ the algebraic closure of k
- Let R an arbitrary ring, $M_n(R)$ the matrix ring over R, R_p the localization of R by $R - p$, R_f the localization of R by $[f, f^2, ...]$, $\text{Frac}(R)$ the field of fractions of R is R is an integral domain
- A is an arbitrary R-algebra.

- $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{N}_{\geq 0} = \{0, 1, 2, ...\}$

The category of rings and commutative rings shall be denoted **Ring** and **CRing** respectively.

Recall Classical Algebraic Geometry

In this chapter, we shall quickly review elementary concepts linking algebra and geometry without going any proofs, that is we are going to explore some classical algebraic geometry. For all the important details, see [3].

0.1 Interpreting Rings as Function Spaces

If A is an R-algebra, then by the universal property of free R-algebras we have if S is a set of generators for A,

$$\begin{tikzcd} R[S] \arrow[r, "\varphi"] & A \\ S \arrow[u, "\iota"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

If S, is finite, then $R[S] \cong R[x_1, x_2, ..., x_n]$ by relabelling S. Since every ring is a $\mathbb{Z}$-algebra, we see that polynomial rings are central to the study of general rings. It is thus fruitful to realize that polynomial rings can naturally be interpreted as functions, that is:

$$f \in R[S],\ R^S \to R[S]^\vee \qquad f \mapsto f(s_i)_{i \in S}$$

In the finite case:

$$f \in R[x_1, x_2, ..., x_n],\ R^n \to R[x_1, x_2, ..., x_n]^\vee \qquad (a_1, a_2, ..., a_n) \mapsto (f \mapsto f(a_1, a_2, ..., a_n))$$

We don't want to consider R^S as having any structure besides being an affine space, hence we shall label it as $\mathbb{A}_R^S$ or in the finite case $\mathbb{A}_R^n$. If the ring is evident in context, we shall write A^n.

For now, we shall stick to the finitely generated case. Note that in $\mathbb{A}_R^n$, $(0, ..., 0)$ has no special property, as we are treating this space as simply a collection of points. To emphasize this geometric interpretation, any $p \in \mathbb{A}_R^n$ will be called a *point*, and for any

$$p = (r_1, r_2, ..., r_n)$$

Each r_i are the *coordinates* of p. Hence, we may more accurately any $f \in R[x_1, x_2, ..., x_n]$ as a function as

$$f : \mathbb{A}_R^n \to R \qquad (r_1, r_2, ..., r_n) \mapsto f(r_1, r_2, ..., r_n)$$

As is common knowledge to the reader, the zeros of polynomials form many interesting algebraic shapes, a prototypical example being $x^2 + y^2 - 1$ over $\mathbb{R}$. Hence, we define:

$$Z(S) = \{p \in \mathbb{A}_R^n \mid f(p) = 0, \ \forall f \in S\}$$

A subset $X \subseteq \mathbb{A}_R^n$ is called an *algebraic set* if there exists $S \subseteq R[x_1, x_2, ..., x_n]$ such that $Z(S) = X$. The collection of algebraic sets are closed under finite union and intersection, and the sets $\mathbb{A}_R^n$ and $\emptyset$ are algebraic, hence the collection of algebraic sets form a topology on $\mathbb{A}_R^n$ known as the *Zariski topology*. commonly, we shall have $R = \mathbb{C}$.

(a lot more words here)

In many ways, the classical Nullstellensatz with finitely generated reduced rings (over algebraically closed fields) shows the symmetries of our "usual" geometry, that is the geometry of zero-sets of polynomials in $\mathbb{R}^n$ or some extension of $\mathbb{Q}^n$. The geometry is naturally trivial in the case of $\mathbb{F} = \mathbb{C}$, however there is a point to only focusing on such rings and studying their properties. Modern algebraic geometry is a departure from this and generalizes the types of rings we use.

0.2 Projective Space

Projective is the natural algebraic setting to solve equations, adding distinct solutions to rational polynomials. (put a summary of basic results from EYNTKA algebra here)

0.3 Noetherian Induction or Well-founded Induction

This is something that will come up in later proofs (for example, see lemma 5.1.14). This is a form of induction that generalizes the usual induction where the well-ordered set is $\mathbb{N}$. Recall that a binary relation R on S is a subset of $S \times S$. For $(a, b) \in R$, we may write $a \preceq b$ or $b \succeq a$. Then the symbol $\preceq$ is sometimes called the *binary relation*. A binary relatoin is a *strict partial order* if it is:

1. Irreflexive: $\forall x \in A, \neg(x < x)$
2. Transitive: $\forall x, y, z \in A, (x < y \wedge y < z) \implies x < z$
3. Asymmetric: $\forall x, y \in A, x < y \implies \neg(y < x)$

Given a set S with a strict partial order $\preceq$, a descending chain is a sequence $a_1, a_2, a_3, ...$ of elements in A such that $a_1 \succeq a_2 \succeq a_3 \succeq ...$

Definition 0.3.1: Well-founded Relation

A strict partial order $\preceq$ on a set A is well-founded if there exist no infinite descending chains in A with respect to $\preceq$.

Definition 0.3.2: Minimal Element

Given a subset S of a well-founded set A, an element $m \in S$ is minimal if there is no element $x \in S$ such that $x \preceq m$.

Theorem 0.3.3: Noetherian Induction

Let $(A, \preceq)$ be a set with a well-founded relation $\preceq$, and let P be a property defined on A. If for all $x \in A$: $[\forall y \in A, (y \preceq x \implies P(y))] \implies P(x)$ then $P(x)$ holds for all $x \in A$.

> ***Proof*** **:**
> see cite:HERE.

The following may be an insightful alternative:

Theorem 0.3.4: Noetherian Induction - Alternative Form

Let $(A, \preceq)$ be a set with a well-founded relation $\preceq$. If $S \subseteq A$ is a subset such that for all minimal elements m of $A \setminus S$, we have $m \in S$, then $S = A$.

> ***Proof*** **:**
> see cite:HERE.

The connection between these forms lies in taking S to be the set of elements where P holds, and showing that if S isn't all of A, then any minimal element of $A \setminus S$ must actually be in S, giving a contradiction.

Part I

Scheme Theory

I believe there will be more than just basic alg geo. This part will go over the core theory, Hartshorne, Vakil, Eisenbud, and so forth. Other parts will surely be specializations.

1

Sheaves

Roughly speaking, Sheaf Theory is the theory of augmenting a topological space[1] that adds information to each open set (adds "local" information) which can be "glued" to get information in larger open sets. The prototypical example of a sheaf is the set of smooth functions on a manifold. Smooth functions on a manifold can be restricted to open subsets of a manifold, and if a smooth function on each part of the manifold is defined in a compatible way (i.e. they agree on intersections), they can be glued back together to form a function on the entire manifold.

A key motivator behind sheaves is that they can hold the "geometric" information of a space. The reader may recall that a vector-space and subspaces can be characterized via their dual spaces, that is all functions from the space to $\mathbb{F}$. A similar result holds for Varieties with the coordinate ring uniquely characterizing a Variety. This duality was found to exist everywhere in mathematics, to name three more examples:

- Riemann manifolds can be characterized by spectral triples
- Radon measure's can be characterized using the duals of $C_c(X)$ where X is a locally compact space.
- Sets with set-function can be thought of as the most "abstract" geometric object (they only have information about containment). There is an appropriate dual called the *complete atomic Boolean algebras.* The reader may be interested in knowing that Boolean algebra's characterize logical operations, which gives an interesting geometric interpretation to logic[2].

There is a long list of such examples (the reader may be curious to checkout this link). This gave rise to the idea that a geometric space can be understood by studying some subset of function on this

[1] more generally, a topoi, which concretely translates in our case to a category whose objects are open sets and morphisms are inclusions

[2] This connection is further refined using Topoi

space with certain. In many cases, there are relatively few functions that are defined globally on a space, but many that are defined on local components (the prototypical example of this would be $\mathbb{C}^3$).[4]

Why do functions capture this geometric information? One can think of functions as tools for representing relations. If our function is of the form $f : X \to \mathbb{R}$, then this is relating X to numbers, i.e. quantification, usually X being a topological space and f being continuous for some notion of closeness. If we have a continuous function of the form $f : X \to \mathbb{R}^n$ (or $\mathbb{C}^n$), with X a topological space, then we may pullback a lot of geometric results from $\mathbb{R}^n$ (or $\mathbb{C}^n$) onto X, depending on how f is defined (ex. it can be shown that f is a diffeomorphism if and only if it pulls back the differential structure of $\mathbb{R}^n$). Very often, it is not possible to define a diffeomorphism $f : X \to \mathbb{R}^n$, however we may define it on components. Functions have the property that we may restrict them or construct them by defining what happening at each point (or in the continuous case, on each open set where they agree on intersections[5]). But you may say that morphisms in a category already capture this intuition. This is indeed the case, what makes functions special is that you can construct larger morphisms from smaller morphisms with an appropriate notion of *gluing* and *restricting*. In particular, the closeness given by the notion of topological spaces and the locality properties inspired by functions shall be the two main properties of sheaves. As is the course of mathematics, it may be asked if these properties of functions can be generalized, and we may ask if we can take special categories which have these features; this is called a *site*. For an introduction to algebraic geometry, limiting to the sites defined on topological spaces is plenty sufficient and shall be the focus of this book.

A particularly important realization will be that commutative rings will in many ways behave like functions on a space, that is if R is a ring, there is a sense in which the elements of R will be the "functions" on a special space (namely, the space $\mathrm{Spec}\,(R)$). This shall be elaborated on in chapter 2 and was commented on in the front-matter. At this stage it is motivating to know that there is a reason why we may care about sheaves other than the sheaf of functions, namely the interpolation between commutative rings and functions.

Motivating Example

Before exploring sheaves in generality, it is useful to have the prototypical example of a sheaf in mind in more detail:

Example 1.1: Sheaf Of Differentiable Functions On Manifold
Let M be a smooth manifold. Then the we shall constuct the sheaf of differentiable function on M. For each open subset $U \subseteq M$, we shall assocaite to it the set of smooth function $C^\infty(U)$. Denote this collection by $\mathcal{O}(U)$ (this will be common notation for sheaves). Then if we have an open subset of U, $V \subseteq U$, we may restrict the function from U to function on V, namely we have a map $\mathrm{res}_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ mapping $f \mapsto f|_V$. Naturally, if we have $W \hookrightarrow V \hookrightarrow U$, then the

[3] entire holomorphic are necessarily power series, while holomorphic on (not necessarily simply connected) open sets are analytic and may even have non-trivial Laurent expansion around holes

[4] see this link for more details

[5] by the gluing lemma

following commutative diagram.

$$\begin{tikzcd} & V \arrow[dr, "\mathrm{res}_{V,W}"] & \\ U \arrow[ur, "\mathrm{res}_{U,V}"] \arrow[rr, "\mathrm{res}_{U,W}"'] & & W \end{tikzcd}$$

This collection of maps satisfy two important properties. In the following let and $U = \bigcup_i U_i$.

1. *Defining Locally*: Let's say $f_1, f_2 \in \mathcal{O}(U)$, Then if f_1 and f_2 agree on all restriction, that is $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_j} f_2$, then $f_1 = f_2$

2. *Gluing*: If there exists a collection $f_i \in \mathcal{O}(U_i)$ such that f_i and f_j agree on intersections, that is $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$, then there is a $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

Then a sheaf will respect these properties. Note that in fact, we may define a manifold M to be $(M, \mathcal{O}_M)$ where M Is a topological space and $\mathcal{O}_M$ is a sheaf of smooth functions on M. If instead $\mathcal{O}(U)$ was the collection of continuous functions, we would get a topological manifold. Even more generally, we may let the functions be set functions, for set functions work well under restrictions and we may define set functions locally and construct set function via gluing.

Note that Diffeology can be thought of as a generalization of this result or of a special case of sheaf theory, and we shall explore it briefly at the end of this chapter.

So if even set functions can form a sheaf, then why did I write "nice" function earlier? As it turns out, what we shall do is abstract the notion of function! In particular, we shall look at systems where we may have the behavior of localizing and gluing "functions" on a topological space. Hence, in the following abstraction, we will consider $\mathcal{O}_M$ to be the data that stores a collection of objects that *behave like functions.*

A key property that distinguishes functions from general morphisms is the notion of evaluation (taking $f(x)$ for some element x and function f). Since we are abstracting the notion of function, we shall need a notion of *evaluation.* To motivate this generalization, let's come back to $(M, \mathcal{O}_M)$ and consider some $\mathcal{O}(U) = C^\infty(U)$ for some open $U \subseteq M$ and take $p \in U$. Recall that $\mathcal{O}(U)$ is a ring and that the *germ* at p is collection of equivalence relations:

$$[f] = \{(f, U) \mid p \in U,\ f \in \mathcal{O}(U)\}$$
$$(f, U) \sim (g, V) \ \Leftrightarrow W \subseteq U, W \subseteq V, p \in W,\ f|_W = g|_W$$

that is, $[f] = [g]$ if and only if there is an open neighborhood $W \subseteq U, V$ containing p ($p \in W$) such that $f|_W = g|_W$ (in the language of sheaf theory, $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). Denote this collection $\mathcal{O}(U)_p$. The key is that the germ of $\mathcal{O}(U)$ at p is a *local ring* with maximal ideal $\mathfrak{m}_p$ of germs such that $[f](p) = 0$ (or simply $f(p) = 0$ if unambiguous). Then:

$$0 \to \mathfrak{m}_p \to \mathcal{O}(U)_p \xrightarrow{f \mapsto f(p)} \mathbb{R} \to 0$$

and all elements $\mathcal{O}(U) \setminus \mathfrak{m}_p$ are invertible. Hence, we shall see that we may thing of "evalutation" of f at some point $p \in U$ as taking first the germ $\mathcal{O}(U)_p$ and quotient it by the maximal ideal $\mathfrak{m}_p$, $\mathcal{O}(U)_p/\mathfrak{m}_p$, and then taking $\overline{f} \in \mathcal{O}(U)_p/\mathfrak{m}_p$ to be the value of f. Note that the above construction required that $\mathcal{O}(U)$ is a *local ring.* This is evidently the case when $\mathcal{O}(U) = C^\infty(U)$, however in general sheaf theory this ring will not in general be local. In fact, $\mathcal{O}(U)$ will not even need to be a ring! When each $\mathcal{O}(U)$ is a ring, then we shall say that X is a *ringed space*, and when all the germs are local rings, then we shall say that X is a *locally ringed space.*

A final word on generalization: we have been working over a topological space, however we may abstract further and work over a general category; in this way, we would define the notion of a "topology" on a category, known as a *Grothendeick topology* (a category with this structure is called a *site*), and develop sheaf theory accordingly. Though this is fruitful, this generlization shall be ommited for now to concentrate on the geoemtric properties at hand when using a topological space.

1.1 Sheaves

We start by formalizing the above intuitions.

Definition 1.1.1: Presheaf Axiomatic Definition

Let X be a topological space. Then a *presheaf* over a category $\mathbf{C}$ on X, denoted $\mathscr{F}_X$ contains the following data:

1. for each open set $U \subseteq X$, there is an object c of $\mathbf{C}$ and denote it by $\mathscr{F}(U)$. If C is a category whose objects are sets with additioanl structure, then the elements of $\mathscr{F}(U)$ are called the *sections* of $\mathscr{F}$ over U. If we say that s is a section of $\mathscr{F}$, then we mean that s is a section of $\mathscr{F}$ over X.

2. For each inclusion map $U \hookrightarrow V$, there is a morphism $\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ called the *restriction* morphism (or *restriction map*) such that

   (a) $\mathrm{res}_{U,U} = \mathrm{id}_U$

   (b) If $W \hookrightarrow V \hookrightarrow U$, then the following diagram commutes

$$\begin{tikzcd} & \mathscr{F}(V) \arrow[dr, "\mathrm{res}_{V,W}"] & \\ F(U) \arrow[ur, "\mathrm{res}_{U,V}"] \arrow[rr, "\mathrm{res}_{U,W}"'] & & F(W) \end{tikzcd}$$

The collection of all presheaves on a topological space X is denoted $\mathbf{PSh}_{\mathbf{C}}(X)$ (or $\mathbf{PSh}(X)$ if unambiguous). If U is clear from context, we shall often write $\mathrm{res}_{U,V} f$ as $f|_V$.

The nomenclature "section" will be made clear in proposition 1.1.9. Notice that it is equivalent to define a presheaf as a *contravariant functor*. Namely

Definition 1.1.2: Presheaf Functorial Definition

Let X be a topological space and let $O(X)$ be the category whose objects are the open sets of X and whose morphism are $\subseteq$ (i.e. the category induced by the partial order given by $\subseteq$). Then a *presheaf* over $\mathbf{C}$ is a contravariant functor from $O(X)$ to $\mathbf{C}$, that is a functor $\mathscr{F} : O(X)^{op} \to \mathbf{C}$.

Usually, our presheaves shall be over $\mathbf{Set}, \mathbf{Mod}_k$ or $\mathbf{Ring}$[6].

[6]Technically, $\mathbf{Ring}$ is $\mathbf{Mod}_{\mathbb{Z}}$, however distinguishing these two is useful as we usually mean k to be a field

An important notion for geometric objects is their behavior locally. The extreme (or limit) of the notation of locality is the behavior around a point p. This idea originated when studying germs of differential functions in Differential Geometry. These are well-defined on presheaves;

Definition 1.1.3: Stalk as Equivalence Relation

let $(X, \mathcal{O}_X)$ be a presheaf. Then the *stalks* at $p \in X$ is the collection of equivalence class:

$$[f] = \{(f, U) \mid p \in U,\ f \in \mathcal{O}(U)\}$$
$$(f, U) \sim (g, V) \qquad \Leftrightarrow \qquad W \subseteq U, W \subseteq V, p \in W,\ \mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$$

The stalks at p is usually denoted $\mathcal{O}_{X,p}$ or $\mathcal{O}_p$ if unambiguous in context.

If $\mathbf{C}$ is a category whose objects are augmented sets[a], then if $p \in U$ and $f \in \mathscr{F}(U)$, $[f] \in \mathscr{F}_p$ is called the *germ* of f at p.

[a] ex. **Set**, **Ring**, **Grp**, etc.

Note that in general, the notion of "evaluation" of a stalk/germ as it is usually defined on differential geometry won't make sense; though the symbol f is written the open sets need not contain functions or objects that act like function. We shall later define nice presheaves in which we may recover the notion of evaluation and treat the elements like functions. There is again a categorical way of interpreting stalks that is useful to us:

Definition 1.1.4: Stalks As Colimits

Let $\mathscr{F} \in \mathbf{PSh}(X)$ be a presheaf. Then a *stalk* at $p \in X$ is:

$$\mathscr{F}_p := \varinjlim \mathscr{F}(U)$$

that is $\mathscr{F}_p$ is the colimit over all $\mathscr{F}(U)$ where $p \in U$.

Hence, this shows us that as long as the colimit exist in our category, we may define the notion of stalks for presheaves over that category without needing to reference the notion of elements. Note that it is the colimit and not the limit since we need $\mathscr{F}_p$ to "represent" the behaviour around p rather than "embed" germs.

One of the most important properties of stalks is their ability to be defined "locally". First, if $U \subseteq X$ is an open set, we can define the sub-presheaf on U by letting $\mathcal{F}|_U(V) = \mathcal{F}(V \cap U)$. The following proposition will be really useful later when we need to work with stalks between larger and smaller (pre)sheaves, and mirrors the property in differential geometry (recall [6, chapter 2.1])

Proposition 1.1.5: Stalk Isomorphism

Let X be a presheaf of sets and $U \subseteq X$ an open set. Then for any $x \in U$:

$$\mathscr{F}_{X,x} \cong_{\mathbf{C}} \mathscr{F}_{U,x}$$

***Proof* :**
We shall prove this in the case where $\mathbf{C} = \mathbf{Set}$. The general case is left as an exercise.

Let us start by defining a map $\varphi : \mathscr{F}_{X,x} \to \mathscr{F}_{U,x}$. Take $[(f, V)] \in \mathcal{F}_{X,x}$ such that $x \in V$. As $x \in U$, $x \in U \cap V$. Hence, notice that:

$$(f, V) \sim (f|_{U\cap V}, U \cap V) \qquad \text{as} \qquad \mathrm{res}_{U,U\cap V} f = \mathrm{res}_{U\cap V,U\cap V} f|_{U\cap V} = f|_{U\cap V}$$

Then define the map between stalk via:

$$\varphi : (f, V) \sim (f|_{U\cap V}, U \cap V) \mapsto (f|_{U\cap V}, U \cap V)$$

This map is well-defined. Take $(f, V_1) \sim (g, V_2)$ where $V_1, V_2 \subseteq X$ so that we have to show that $(f|_{U\cap V_1}, U \cap V_1) \sim (g|_{U\cap V_2}, U \cap V_2)$. As $(f, V_1) \sim (g, V_2)$ there exists $W \subseteq V_1, V_2$ such that

$$f|_W = g|_W$$

Take $U \cap W \subseteq W \subseteq V_1, V_2$. Then by the restriction axioms we have that:

$$f|_{U\cap W} = g|_{U\cap W}$$

So that $(f|_{U\cap W}, U \cap W) \sim (g|_{U\cap W}, U \cap W)$. Then, as

$$\mathrm{res}_{U\cap V_1,U\cap W}(f|_{U\cap V_1}) = \mathrm{res}_{U\cap W,U\cap W}(f|_{U\cap W})$$

(and similarly for V_2), we get:

$$(f|_{U\cap V_1}, U \cap V_1) \sim (f|_{U\cap W}, U \cap W) \sim (g|_{U\cap W}, U \cap W) \sim (g|_{U\cap V_2}, U \cap V_2)$$

and hence the map is well-defined. This map has a natural inverse, namely

$$\varphi^{-1} : \mathscr{F}_{U,x} \to \mathscr{F}_{X,x} \qquad (f, V) \mapsto (f, V)$$

which can be shown to be well-defined and $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = \mathrm{id}$, hence these sets are bijective, which are the isomorphisms in set. To generalize to categories that are augmented sets, take advantage of the restriction maps being morphisms.

Now, presheaves allow for the notion of restricting and localizing, but as we shall show soon not all presheaves are amenable to gluing local information to create global information (think of bounded holomorphic functions on open subsets of $\mathbb{C}$). We must add this condition axiomatically:

Definition 1.1.6: Sheaf using Axioms

Let $\mathscr{F} \in \mathbf{PSh}_C(X)$ be a presheaf over a category $\mathbf{C}$ whose objects are sets possibly with additional structure. Then $\mathscr{F}$ is also a sheaf if it satisfies the following two axioms;

1. *Identity Axiom.* Let $U = \bigcup_i U_i$. If $f_1, f_2 \in \mathscr{F}(U)$, then if $\text{res}_{U,U_i} f_1 = \text{res}_{U,U_i} f_2$ for all i, then $f_1 = f_2$
2. *Gluing Axiom*: Let $U = \bigcup_i U_i$. If there exists a collection $f_i \in \mathscr{F}(U_i)$ such that $\text{res}_{U_i,U_i\cap U_j} f_i = \text{res}_{U_j,U_i\cap U_j} f_j$, then there is a $f \in \mathscr{F}(U)$ such that $\text{res}_{U,U_i} f = f_i$ for all i.
3. $\mathscr{F}(\emptyset)$ is associated with the terminal object (if we're working over set, $\mathscr{F}(\emptyset)$ is a singleton).

If only the identity axiom is satisfied, then our preseaf is called a *separated presheaf.* The collection of all sheaves on a topological space X is denoted $\mathbf{Sh}_C(X)$ (or $\mathbf{Sh}(X)$ if unambiguous)

We may think of the identity axiom telling us there is *at most* one way to glue, while the second axiom is telling us there is *at least* one way to glue. Sometimes, we also specify that $\mathscr{F}(\emptyset)$ is the terminal object in our category (in **Set**, that would be a singleton).

Notice that we are taking elements of each $\mathscr{F}(U)$ because $\mathscr{F}(U)$ is an object in a category whose objects are sets with possibly additional structure. This is a deviation from the generalization presented in the definition of presheaves as $\mathbf{C}$ may be more general. There is in fact a way to define a sheaf for general categories, the solution to this is called the *Grothendeick topology* on a category. This something we shall cover in REF:HERE.

Definition 1.1.7: Sheaf Categorical Interpretation

Let $\mathscr{F} \in \mathbf{PSh}(X)$ be a presheaf, and let $\cdot$ be a terminal object (in **Set**, an arbitrary 1-point set). Let $U = \bigcup_i U_i$, and consider:

$$\cdot \to \mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j)$$

for any i, j Then if the sequence is exact at at $\mathscr{F}(U)$, the identity axiom holds, and if it is exact at $\prod \mathscr{F}(U_i)$, the gluing axiom holds.

Example 1.2: Presheaves And Sheaves

Most of these examples will be on sheaves of sets.

1. Verify that the sheaf of differentiable functions over M, namely $\mathcal{O}_M$, is indeed a sheaf. More generally, if X is any topological space, then the data defined by $\mathscr{F}(U) = \text{Hom}_{\mathbf{C}}(U, \mathbb{R})$ where $\mathbf{C}$ is either **Top** or **Set** is a sheaf.
2. (*Restriction of Sheaves*) Let $\mathscr{F} \in \mathbf{Sh}(X)$ and $U \subseteq X$ an open subset of X. Then the *restriction* of $\mathscr{F}$ to U, denoted $\mathscr{F}|_U$ has as data for each open $V \subseteq U$ $\mathscr{F}|_U(V) = \mathscr{F}(V)$. We shall later see how we may define restrictions of sheaves to arbitrary sets[a]
3. (*Skyscraper Sheaf*) Let X be a topological space, $p \in X$, and S be any set. Let $\iota_p : \{p\} \to X$

be the inclusion map. Then define the sheaf of sets $\iota_{p,*}S$ on X (not on S!) to be:

$$\iota_{p,*}S(U) = \begin{cases} S & p \in U \\ \{e\} & p \notin U \end{cases}$$

This sheaf is called the *skyscraper sheaf* (sometimes denoted $(\iota_p)_*S$). The name comes from the fact that the informal picture for this sheaf looks like a skyscraper at p. For sheaves of Abelian groups, we would $\{e\}$ be the trivial group. More generally, $\iota_{p,*}(U)$ would be the final object if $p \notin U$.

4. (*Constant Presheaf*) Let X be a topological space and S be any set. Define $\underline{S}_{\text{pre}}(U) = S$ for all open $U \subseteq X$ Then it should be verified that $\underline{S}_{\text{pre}}$ forms a presheaf on X, called the *constant presheaf associated to S*. This set is not in general a sheaf if $|S| > 1$: take $X = U_1 \amalg U_2$ to be the disjoint union of two (open) sets, take the constant presheaf so that all restriction maps are constant maps. Chose $s_1 \in S$ and $s_2 \in S$ which are distinct: $s_1 \neq s_2$. Then by the gluing axiom, there would have to exist a $s \in S$ such that $s|_{U_1} = s_1$ and $s|_{U_2} = s_2$ and their intersection is empty and hence tautologically follow the gluing axiom, however as we have the identity map we have $s_1 = s = s_2$ however $s_1 \neq s_2$.

   What makes the sky-scrapper sheaf special is the "distinguishing" of a point making it act like a generalization of an indicator function. In the above example, one of the two sets will not have the point p.

5. A great example of a pre-sheaf on $\mathbb{C}$ that is not a sheaf is the sheaf of bounded holomorphic functions on $\mathbb{C}$. Recall that bounded entire holomorphic functions (defined on all of $\mathbb{C}$) must be constant by Louisville's Theorem, however there do exist bounded functions on open subsets of $\mathbb{C}$. For example , on each disk of radius r, $\mathbb{D}^2_r$, z^2 is bounded, but $\mathbb{C} = \cup_r \mathbb{D}^2_r$ has no holomorphic function for which the restriction to any disk D^2_r is z^2; hence the gluing axiom fails.

   Another example would be the sheaf composed of holomorphic functions admitting holomorphic square roots, as there is no global square-root function on $\mathbb{C}$ due to non-trivial monodromy.

6. (*Constant Sheaf*) Again let X be a topological space and S be any set. For every open $U \subseteq X$, let $\mathscr{F}(U)$ be the collection of all *locally constant* functions from U to S, that is if $f \in \mathscr{F}(U)$, for each $p \in U$ there is an open neighborhood $V \subseteq U$ containing p, $p \in V \subseteq U$, such that $f|_V : V \to S$ is constant (on euclidean space, this is equivalent to having a constant function on each connected component). Then this sheaf is called the *constant sheaf associated to S*, and is denoted $\underline{S}$.

   Another way of describing this sheaf is if we endow S with the discrete topology and let $\mathscr{F}(U)$ be the continuous maps $U \to S$.

7. (*Sheaf of continuous maps*) Let X and Y be topological spaces. Define $\mathscr{F}(U) = \mathrm{Hom}_{\mathbf{Top}}(U, Y)$. This sheaf is a generalization sheaf of differentiable functions and the constnat sheaf, namely

   - The sheaf of differentiable functions has $Y = \mathbb{R}$
   - the constant sheaf has $Y = S$ with the discrete topology.

   If Y is also a topological group, then the continuous maps to Y form a sheaf of groups.

   This to me feels like a "local duality" or "local representation via Y"

8. (*Sheaf of section of a maps*) This is an important special case of the above sheaf. Let X, Y be topological spaces and let $\mu : Y \to X$ be a continuous map. Then the sections of μ (in the sense of the existence of a map $s : X \to Y$ such that $\mu \circ s = \mathrm{id}$) form a sheaf. In particular, for each open $U \subseteq X$, $\mathscr{F}(U)$ is the set of section maps $s : U \to Y$ (continuous maps $s : U \to Y$ such that $\mu \circ s = \mathrm{id}|_U$). Vector bundles and vector fields are a good example of such a sheaf.

[a]see ref:HERE

An important class of sheaves are sheaves over rings ($\mathbf{Sh}_{\mathbf{Ring}}(X)$).

Definition 1.1.8: Ringed Space And Structure Sheaf

Let X be a topological space. Then if we have a sheaf of [commutative] rings over X, we shall denote it $\mathcal{O}_X$, and the tuple $(X, \mathcal{O}_X)$ is called a *ringed space*. The sheaf $\mathcal{O}_X$ is called the *structure sheaf*, and the sections of the structure sheaf $\mathcal{O}_X$ over an open subset U are called *[regular] functions on* U.

1.1.1 Warning Though these are called functions, they are elements of an arbitrary [commutative] ring. We shall see that elements of rings can be thought of as functions, and they are sometimes better characterized as partial functions. The reader should have regular function from [3, chatper 22.3] in mind.

The simplest and prototypical ringed space is again the sheaf of [real/complex] continuous/smooth functions on a manifold/variety with component-wise addition and multiplication.

If $\mathscr{F} \in \mathbf{PSh}(U)$ and $f \in \mathscr{F}(U)$ for some open $U \subseteq X$, then f is called a section. The following proposition motivates this idea

Proposition 1.1.9: Space Of Sections Of (Pre)Sheaf

Let $\mathscr{F}$ be a presheaf of sets. Then there exists a topological space F and a local homeomorphism $\pi : F \to X$ so that $\mathscr{F}$ can be represented as the sheaf of sections from F to X. The space F is called the *étalé space*. In particular, F is the *étalé space* of $\mathscr{F}$.

Furthermore, there is an equivalence between the category $\mathbf{Sh}(X)$ and the category of étalé spaces over X (with bundle morphisms as morphisms).

The idea is that (pre)sheaves can be thought of as continuous sections with the property that around each $x \in F$, there exists an open neighborhood $U \subseteq F$ such that $\pi : U \to \pi(U)$ is a homeomorphism, that is it must remember some information about it's neighborhood. In other words, it is locally homeomorphic to X.

***Proof* :**
Let F be the disjoint union of the stalks on F.

$$F = \coprod_{x \in X} \mathscr{F}_x$$

Then $\pi : F \to X$ is naturally a set map. Define a basis for F as follows: for each $s \in \mathscr{F}(U)$, take the collection

$$\{[s]_x \in \mathscr{F}_x \mid x \in U\}$$

These form a basis for F (note that each stalk has the discrete topology). With the setup, it is now easy to see the equivalence of categories.

The étale space may seem like it ought to be pretty similar to X being locally homeomorphic, however this can be far from the case:

Example 1.3: Non-Hausdorff Étale Space

Take $X = \mathbb{R}$ be the real numbers and take $\mathcal{F}$ to be the smooth functions on $\mathbb{R}$. Take the constant germ around 0 as well as the germ which can be represented by the function:

$$g(x) = \begin{cases} x\sin(1/x) & x\sin(1/x) > 0 \\ 0 & x\sin(1/x) \leq 0 \\ 0 & x < 0 \end{cases}$$

In other words, g is zero when x is negative and when the function $x\sin(1/x)$ would be negative; we want to keep when $x\sin(1/x)$ is positive. Then these two are different points in the étale space, but *cannot* be separated by open sets.

Note that if we were working with $X = \mathbb{C}$ and took analytic or holomorphic functions, then this cannot happen as we have a non-isolated 0 without all of g being zero. This turns out to be the main impingement for the étale space being Hausdorff. In general sheaves of analytic functions shall form Hausdorff étale spaces.

1.1.1 Morphisms of (pre)Sheaves and build-up for Cohomology

Now that we have defined (pre)sheaves, it is natural to define morphism between sheaves to show that $\mathbf{PSh}(X)$ and $\mathbf{Sh}(X)$ form a category. Sheaf morphisms shall be over the same object, and hence can be thought of as the equivalent of looking at section maps between two étale spaces:

Definition 1.1.10: Morphism Of Sheaves

Let $\mathscr{F}, \mathscr{G} \in \mathbf{PSh}(X)$ (note that $\mathbf{Sh}(X) \subseteq \mathbf{PSh}(X)$). Then a *morphism of (pre)sheaves* (over the chosen category) $\varphi : \mathscr{F} \to \mathscr{G}$ contains the data $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ such that if $V \subseteq U$:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}_{U,V}"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"'] & \mathscr{G}(V) \end{tikzcd}$$

In other words, φ is a natural transformation from $\mathscr{F}$ to $\mathscr{G}$. The collection of morphisms between two sheaves is denoted $\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F}, \mathscr{G})$ or $\mathrm{Mor}_{\mathbf{Sh}(X)}(\mathscr{F}, \mathscr{G})$

The image of a presheaf is a presheaf, however the image of a sheaf need not be a sheaf (see example 1.4). Since the morphisms are the same, the category $\mathbf{Sh}(X)$ is a full subcategory of $\mathbf{PSh}(X)$. Given this definition, there is a natural notion of morphism of $\mathcal{O}_X$-module. With morphisms and objects defined, it is easy to show that they form a category. We shall denote the category of sheaves over a category $\mathbf{C}$ on a space X as $\mathbf{C}_X$. Thus:

$$\mathbf{Set}_X \qquad \mathbf{Ab}_X \qquad \mathbf{Ring}_X$$

are the categories of sheaves over sets, abelian groups,and rings. The collection of ringed spaces $(X, \mathcal{O}_X)$ with the above mentioned morphisms will be the category denoted by $\mathbf{Mod}_{\mathcal{O}_X}$. For presheaves, we shall denote the category with an extra superscript:

$$\mathbf{Set}_X^{\text{pre}} \qquad \mathbf{Ab}_X^{\text{pre}} \qquad \mathbf{Ring}_X^{\text{pre}}$$

When working with abelian groups or $\mathcal{O}_X$-modules, both these categories are abelian categories. We shall see that sheaves also form abelian categories, however it requires the notion of sheafification (see section 1.1.3). To be more precise, as the definition of sheaf morphism is that of pre-sheaf morphism, it can be shown that the presheaf kernel of a sheaf morphism is in fact a sheaf, the cokernel need not be a sheaf. As pre-sheaves are Abelian categories, show how to define kernels and cokernels. To see this:

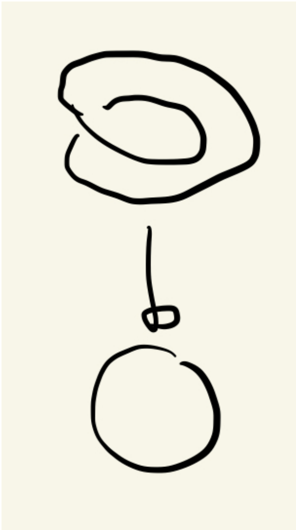
Example 1.4: Difficulty With Cokernels and Images

1. Let $X = \mathbb{C}$ with the usual topology, $\underline{\mathbb{Z}}$ be the constant sheaf on X, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathscr{F}$ the *presheaf* of functions admitting a holomorphic logarithm. Then there is an exact sequence of presheaves on X:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathscr{F} \to 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion, and $\mathcal{O}_X \to \mathscr{F}$ is given by $f \mapsto \exp(2\pi i f)$. Then it should be clear that $\mathscr{F}$ is *not* a sheaf, namely there are functions that don't have a global logarithm but do locally!

2. If you are less comfortable with complex analysis, here is a more rudimentary example, we shall be taking the sheaf of continuous sections of the circle S^1 with the total space for the first sheaf being S^1 and for the second sheaf being the twisted circle, which can be visualized as:



Label this shape S^1_2. Then the first sheaf is all continuous sections on all open subsets of the codomain for the map $\pi : S^1 \to S^1$, while the second is all the continuous sections of all the open subsets of the codomain for the map $\pi : S^1_2 \to S^1$ with a reasonable definition of projection map. Then there is certainly a surjective continuous map φ satisfying the commutative diagram:

$$\begin{tikzcd} S^1_2 \arrow[rr, "\varphi"] \arrow[dr] & & S^1 \arrow[dl] \\ & S^1 & \end{tikzcd}$$

However, there is no surjective maps:

$$\mathcal{F}_2(X) \to \mathcal{F}_1(X)$$

namely since there no continuous section from S^1 to S^1_2 that would satisfy the projection condition.

As presheaves store all local information, we may want that there are some natural functors that preserve some exactness to extract that information. The two natural candidates for functors to would be $-(U)$ taking a (pre)sheaf to its sections over an open set U, and $(-)_p$ the functor taking a (pre)sheaf to it's stalk at p.

Let us start with the stalk map. Let us show that it is indeed a functor:

Proposition 1.1.11: Morphism Of (Pre)Sheaves Induces Morphism On Stalks

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of (pre)sheaves (for simplicity, of sets). Then there is an induced morphism on stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$, that is, it induced a functor $\mathbf{Sh}_{\mathbf{C}}(X) \to \mathbf{C}$.

Proof **:**
This is definition pushing: For each open $O \subseteq X$ we have the morphism $\varphi(O) : \mathscr{F}(O) \to \mathscr{G}(O)$. Define the map:

$$[(f,U)]_p \mapsto [(\varphi(U)(f), U]_p$$

For this map to be well-defined if $(f,U) \sim (f',V)$ then we need:

$$[(f',V)]_p \mapsto [(\varphi(V)(f'), V]_p = [(\varphi(U)(f), U]_p$$

Hence we need that $((\varphi(V)(f'), V) \sim ((\varphi(U)(f), U)$. As $(f,U) \sim (f',V)$ we have there exists $W \subseteq U, V$ such that

$$\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} f' = g$$

Then as φ is a pre-sheaf morphism we have:

$$\begin{tikzcd}
\varphi(U) : f \arrow[r, mapsto] \arrow[d, "\mathrm{res}_{U,W}"] & \varphi(U)(f) \arrow[d, "\mathrm{res}_{U,W}"] & \varphi(V) : f' \arrow[r, mapsto] \arrow[d, "\mathrm{res}_{V,W}"] & \varphi(U)(f') \arrow[d, "\mathrm{res}_{V,W}"] \\
\varphi(W) : \mathrm{res}_{U,W}(f) = g \arrow[r, mapsto] & \varphi(W)(g) & \varphi(W) : \mathrm{res}_{V,W}(f) = g \arrow[r, mapsto] & \varphi(W)(g)
\end{tikzcd}$$

which implies $\mathrm{res} U, W \varphi(U)(f) = \mathrm{res}_{V,W}\varphi(V)(f')$. But that means they are similar, $((\varphi(V)(f'), V) \sim ((\varphi(U)(f), U)$, as we sought to show

From this, it is easy to see that functoriality is given, and hence $(-)_p$ is a functor. It is even an exact functor, but we shall relegate that proof till section 1.3 as it requires more build-up.

It is easier to show that $-(U)$ is an exact functor. By definition, a morphism of schemes $\varphi : \mathcal{F} \to \mathcal{G}$ give a morphisms in $\mathbf{C}$, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$. What is left to show is that monomorphisms/epimorphisms are preserved, or if $\mathbf{C} = \mathbf{Set}$ that injectives and surjective are preserved

Proposition 1.1.12: On Presheaf: Open Set Functor is Exact

The functor $-(U) : \mathbf{PSh}(X) \to \mathbf{C}$ (where $\mathbf{C}$ is abelian) is an exact functor
On $\mathbf{Sh}$, the functor $-(U)$ is left-exact.

Proof **:**
Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a monomorphism so that if $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{H} \to \mathcal{F}$ are morphisms such that $\varphi \circ f = \varphi \circ g$ then $f = g$. Then after applying the functor we get:

$$\varphi(U) \circ f(U) = \varphi(U) \circ g(U) \qquad \text{implies} \qquad f(U) = g(U)$$

Note that any element $f(U), g(U)$ is associated to a presheaf an "indicator sheaf" which is the sheaf where if $V \subseteq U$ then $\mathcal{F}(U) = \{e\}$ and if $V \not\subseteq U$ then $\mathcal{F}(U) = \emptyset$, and so we have that this holds for every possible element (or function $)f(U), g(U)$, giving they are monomorphism, or in the right category injective. A similar argument can be done for epimorphisms/surjective maps using the sky-scrapper sheaf.

The argument for left-exactness works just as well for the sheaf case, and example 1.4 shows why it cannot be extended to be right-exact in general.

Proposition 1.1.13: Exactness of Presheaves On The Level of Open Sets

An exact sequence of presheaves $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \cdots \to \mathscr{F}_n \to 0$ is exact if and only if the sequence $0 \to \mathscr{F}(U)_1 \to \mathscr{F}(U)_2 \to \cdots \to \mathscr{F}(U)_n \to 0$ is exact for all U.

Proof **:**
exercise.

Proposition 1.1.14: Presheaves Over Ab Category Form Ab Category

The category $\mathbf{PSh}_{\mathbf{C}}(X)$ over an abelian category C is an abelian category

Proof **:**
Certainly, if $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ then $\varphi + \psi$ defined via open sets as:

$$\varphi + \psi(U) := \varphi(U) + \psi(U)$$

is a map of presheaf (it respects restriction by the commutativity of both natural transformation), and so the hom-sets form an abelian group. Then as we also have a 0-object, and we have finite products (again, something the reader could verify), we have an additive category.

To show kernels and cokernels exist, by proposition 1.1.13 it suffices to check on the level of open sets. Then we define:

$$(\ker_{\text{pre}} \varphi)(U) = \ker \varphi(U)$$

The key is to consider the following diagram where $U \hookrightarrow V$:

$$\begin{tikzcd}
0 \arrow[r] & \ker_{\text{pre}}(\varphi(V) \arrow[r] \arrow[d, dashed, "\exists!"] & \mathcal{F}(V) \arrow[r] \arrow[d, "\text{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\text{res}_{V,U}"] \\
0 \arrow[r] & {} \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{G}(U)
\end{tikzcd}$$

which shows that this is indeed well-defined. The reader can further check that this does indeed satisfy the universal property of the kernel. By dualizing the argument, we can do the same for the cokernel, and hence the additive category of presheaves has kernels and cokernels, making it abelian.

Hence, the cohomology of presheaves is well-defined can be reduced to the study of it's open sets. By example 1.4, this argument does not work for sheaves as the cokernel is *not* preserved (however kernels are preserved, as shall be proven soon). However, the category shall be shown to be abelian in section 1.1.3. For future reference, we box the following definition:

Definition 1.1.15: Sheaf Hom

Let $\mathscr{F}, \mathscr{G} \in \mathbf{Sh}(X)$. Then the *sheaf hom* is defined to be:

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(U) := \text{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$$

Just like Hom, *Hom* is a (co/contra)variant functor, depending on which position you choose. For future purposes, the dual of $\mathscr{F}$, denoted $\mathscr{F}^\vee$ is $\mathrm{Hom}_{\mathbf{Mod}_{\mathcal{O}_X}}(\mathscr{F}, \mathcal{O}_X)$.

1.1.2 On commuting Hom and stalks You may recall that localizing commutes with many operations. It is tempting to then think that $\mathcal{H}om(\mathscr{F}, \mathscr{G})_p \cong \mathrm{Hom}(\mathscr{F}_p, \mathscr{G}_p)$. However this is *not* the case (think of a counter-example). There is a map from one of these sets to another, what is it?

The concept of the sheaf-Hom has many technicalities, so to practice do the following exercises:

Example 1.5: Sheaf Hom practice

1. Let $\mathscr{F}$ be a sheaf of sets, show that $\mathcal{H}om(\underline{\{p\}}, \mathscr{F}) \cong \mathbb{F}$
2. Let $\mathscr{F}$ be a sheaf of groups, show that $\mathcal{H}om(\underline{\mathbb{Z}}, \mathscr{F}) \cong \mathbb{F}$
3. If $\mathscr{F}$ is an $\mathcal{O}_X$-module, show that $\mathcal{H}om(\mathcal{O}_X, \mathscr{F}) \cong \mathbb{F}$ as $\mathcal{O}_X$-modules.

1.1.2 Stalks and Sheaves

As mentioned earlier, presheaves sheaves induce stalks. It was mentioned that sheaves but *not* presheaves can be characterized on the level of stalks, meaning the stalks of a sheaf shall give us all the information about a sheaf. This in fact almost characterizes sheaves from presheaves, as it in fact characterizes the notion of a *separated presheaf* which is a presheaf along with the identity axiom (but not necessarily Gluability). Though we may carry out the construction on separated presheaves, it shall bring us no strong benefit and so we stick to working with sheaves.

Let us first see that morphisms between sheaves can be determined by their stalks:

Lemma 1.1.16: Sections Determined By Germs

Let $\mathscr{F}$ be a sheaf. Then the map:

$$\mathscr{F}(U) \to \prod_{p \in U} (\mathscr{F}_p)$$

is injective

Proof **:**
Let f be the map and consider $f(s) = (s_x)_{x \in U}$ Say:

$$(s_x)_{x\in U} = (t_x)_{x \in U} \qquad \text{equiv.} \qquad [(s, V_x)]_{x \in U} = [(t, W_x)]_{x \in U}$$

Then by definition there exists a $C_x \subseteq V_x \cap W_x$ such that

$$s|_{C_x} = t|_{C_x}$$

Now as $x \in C_x$ for each $x \in U$ we have $U = \bigcup_{x \in U} C_x$ and furthermore for for any pair C_x, C_y in the open cover we have:

$$\mathrm{res}_{C_x, C_x \cap C_y} s = \mathrm{res}_{C_y, C_x \cap C_y} t$$

hence, by the gluing axiom $s = t$, as we sought to show.

Corollary 1.1.17: Global Section Over Local Sections

Let $\mathscr{F}$ be a sheaf on X and $U_i \subseteq X$ is an open cover. Then:

$$\mathscr{F}(X) \to \prod_i \mathscr{F}(U_i)$$

is injective

Proof **:**
Map $s \mapsto (s|_{U_i})_i$. Consider

$$(s|_{U_i})_i = (t|_{U_i})_i$$

then s and t agree on intersections of U_i, hence there exists a unique element in $\mathcal{F}(X)$ for both these elements, namely $s = t$

Note that this is *not* true for presheaves:

Example 1.6: Injectivity Failure For Presheaves
Let X be any topological space and define:

$$\mathcal{F}(X) = \mathbb{Z} \qquad \mathcal{F}(U) = 0\,(U \subsetneq X)$$

Then this sheaf is gluable, but not separated. If $U \subsetneq X$ is covered, then it is clear that gluing must hold. If $U = X = \cup_i U_i$, then that restrictions holds as any $z \in \mathbb{Z}$ has $z|_{U_i} = 0$ and so they certainly agree on all intersections. Hence, we see that each $z \in \mathbb{Z}$ is glued by a bunch of 0-elements!

Then $\mathcal{F}(X) \to \prod_{x \in X} \mathcal{F}_x$ is *not* injective, as each $\mathcal{F}_x$ consist of a single equivalence class representing the zero element (and hence $\prod_{x \in X} \mathcal{F}_x \equiv \{[0]\}$, and $\mathcal{F}(X)\mathbb{Z}$.

This example shows that we can somehow construct a nonzero object from a bunch of zero-objects. This can be thought of as being the key-blocker, especially in the case of abelian categories where most of the time the pre-image of 0 is only 0. This motivate the following:

Definition 1.1.18: Support of Sections

Let $\mathscr{F}$ be a sheaf of an abelian group on X, and s a global section of $\mathscr{F}$. Then the *support* of S, denoted $\mathrm{Supp}(S)$, is the points $p \in X$ where s has a nonzero germ:

$$\mathrm{Supp}(s) := \{p \in X \mid s_p \neq 0 \text{ in } \mathscr{F}_p\}$$

This notion differs a bit from the usual notion of support in analysis, in particular notice that it is the germ itself that is nonzero and not that the evaluation of the germ s_p at the point p that is nonzero. Using this, show that $\mathrm{Supp}(s)$ is closed. The equivalent in analysis is the definition of the *closed support* of a function. Due to this, the compliment of the support of an element s is the

0-section. The section can also be defined on presheaves, however there are examples of presheaves where there are nonzero sections that "live nowhere" because it is 0 in every stalk[7]

Let us now take a closer look at $\prod_{p\in U}(\mathscr{F}_p)$. This set is much bigger than sections of a sheaf. We thus take the special sub-collection that associates with $\mathscr{F}(U)$ (essentially characterizing the image of $\mathscr{F}(U)$:

Definition 1.1.19: Compatible Germs

Let $\mathscr{F}$ be a sheaf on X and $U \subseteq X$ an open set. Then we say an elements $(s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$ consists of *compatible germs* if there exists some representative open sets U_p for s_p:

$$U_p \subseteq U \text{ open}, \; \tilde{s}_p \in \mathscr{F}(U_p)$$

such that the germ of $\tilde{s}_p$ for all $q \in U_p$ is s_q. In other words, if $\{U_i\}$ is an open cover of U where $p \in U_i$ for all i, then s_p is the germ of f_i at p.

Hence, a compatible germ is the collection of germs that are "compatible" when taking subsets, that is *restrictions*. Using the gluing axiom, it is a good exercise to show that any choice of compatible germs for a sheaf $\mathscr{F}$ over U is the image of a section of $\mathscr{F}$ over U, that is we have successfully identified the right hand side of the equation in lemma 1.1.16. Note how important it is that we use sheaves (or at least separated presheaves) for the proof. The proof also motivate the agricultural terminology of "sheaf", for in agriculture a sheaf is a collection of "Stalks" (which is the main stem of a plant).

1.1.3 Remark When working with sheaves of sets or some structures on sets, the elements of the section can literally be thought of as functions using the above definition, namely function of the form $s : U \to \coprod_{p\in U} \mathscr{F}_p$. The generalization presented above allows for more categories to be considered.

With the right notion of compatible stalks defined, we may show how to determine morphisms of sheaves at the level of stalks:

Proposition 1.1.20: Morphisms Determined By Stalks

Let φ_1, φ_2 be morphisms from a presheaf of sets $\mathscr{F}$ to a sheaf of sets $\mathscr{G}$ that induce the same maps on each stalk. Then $\varphi_1 = \varphi_2$

***Proof* :**
Consider the diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\
\prod_{p\in U} \mathscr{F}_p \arrow[r] & \prod_{p\in U} \mathscr{G}_p
\end{tikzcd}$$

[7] Take $\mathscr{F}$ to be the sheaf of holomorphic functions on $\mathbb{C}$, $\mathscr{G}$ the sheaf of bounded holomorphic functions. Then take the sheaf $\mathcal{H} = \mathscr{F}/\mathscr{G}$. Then show that $\mathcal{H}_x = 0$, however $\mathcal{H}(\mathbb{C}) = \{\text{entire functions}\}/\mathbb{C}$

Proposition 1.1.21: Isomorphisms Determined By Stalks

Let φ be a morphism of sheaves. Then φ is an isomorphism if and only if it induces an isomorphism for all stalks.

Proof **:**
Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. If φ is an isomorphism, then certainly each φ_p is an isomorphism, hence say each φ_p is an isomorphism. Then it must be shown that each $\varphi(U)$ is an isomorphism. For injectivity, say for $s \in \mathscr{F}(U)$ $\varphi(s) = 0 \in \mathscr{G}(U)$. Then for each $p \in U$, $\varphi(s)_p = 0 \in \mathscr{G}_p$. As φ_p is injective at each p, we have $s_p = 0 \in \mathscr{F}_p$. Using the definition of stalk, we have that s is zero on some open set, namely $s|_{W_p} = 0$ for $W_p \subseteq U$. As U is covered by such neighborhoods, by the gluing axiom we have $s = 0$

For surjectivity, let $t \in \mathscr{G}(U)$. Consider $t_p \in \mathscr{G}_p$. As φ_p is surjective, we have $s_p \in \mathscr{F}_F$ such that $\varphi_p(s_p) = t_p$. Now, let $(s(p), V_p)$ be a representative of s_p on the set V_p containing the point p. Then $\varphi(s(p))$ and $t|_{V_p}$ are two elements of $\mathscr{G}(V_p)$ whose germs at p are the same. Thus, replacing V_p by a smaller neighborhood if necessary, we have $\varphi(s(p)) = t|_{V_p} \in \mathscr{G}(V_p)$.

Now, U is covered by open sets V_p where on each V_p we have $s(p) \in \mathscr{F}(V_p)$ and $\varphi(s(p)) = t|_{V_p}$. If p, q are two points, then $s(p)|_{V_p \cap V_q}$ and $s(q)|_{V_p \cap V_q}$ are two sections of $\mathscr{F}(V_p \cap V_q)$ which are both sent to $t|_{V_p \cap V_q}$ via φ. As φ was shown to be injective, they are equal. then by the gluing axiom, there exists an $s \in \mathscr{F}(U)$ such that

$$s|_{V_p} = s(p) \qquad \forall p \in U$$

Finally, to show that $\varphi(s) = t$, notice that $\varphi(s)$ and t are two section of $\mathscr{F}(U)$ where for each p $\varphi(s)|_{V_p} = t|_{V_p}$, hence they glue to create $\varphi(s) = t$, as we sought to show.

This does not imply that if two sheaves have isomorphic stalks, then they are isomorphic! You need to already have a sheaf homomorphism which in turn induces the isomorphic stalks. Conceptually, this is saying that being locally isomorphic does not imply it is globally isomorphic, which there are plenty concrete example form differential geometry to choose from.

Example 1.7: Isomorphic Stalks, Not Isomorphic Sheafs

1. Let X be a Hausdorff space. Consider the constant sheaf $\underline{\mathbb{Z}}$ and the sky-scrapper sheaf $\iota_*(\mathbb{Z})$. Show that the stalks at each point is $\mathbb{Z}$, however these two sheaves are not isomorphic.

2. The following example will rely on Spec for full understanding that is covered in the next chapter, however the reader may rely on their intuitions from [3]. A more geometric example is the following: Take
$$A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$$
which is the 2-sphere in $\mathbb{R}^3$. Take the tangent bundle, i.e. A-module, of all derivations $E = \mathrm{Der}_{\mathbb{R}}(A)$. Then E is a rank 2 locally trivial module since for any open subset or distinguished open subset, $E_x \cong A_x^2$, showing each stalk is isomorphic. However, $E \ncong A^2$.

3. Another famous example would be the trivial line bundle on S^1 vs. the möbius bundle.

It is still useful to know under what conditions do the stalk's of a map induce a sheaf morphism:

Proposition 1.1.22: Characterizing Stalks Inducing Sheaf Morphism

Let X be a topological space with two sheaves $\mathscr{F}, \mathscr{G}$. Let $f_p : \mathscr{F}_p \to \mathscr{G}_p$ be a collection of stalk morphisms for all $p \in X$. Then this gives a map of sheaves $f : \mathscr{F} \to \mathscr{G}$ if and only if for all open subsets $U \subseteq X$, for all sections $s \in \mathscr{F}(U)$, there is an open cover $U = \cup_i U_i$ such that

$$f_p(s_p) = (t_i)_p$$

for all $p \in U_i$ and all i

Proof **:**
write down

1.1.3 Sheafification: Sub and Quotient Sheaves

Just like we can form an abelian group from an abelian semi-group via groupification, we can from a sheaf from a presheaf via sheafification. As is usual, this is done in a universal way that is adjoint to the forgetful functor. Namely:

Theorem 1.1.23: Sheafification

Let $\mathscr{F}$ be a presheaf on X. Then there exists a sheaf $\mathscr{F}^{\text{sh}}$ and a map $\text{sh} : \mathscr{F} \to \mathscr{F}^{\text{sh}}$ such that for any sheaf $\mathscr{G}$ and any presheaf morphism $g : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $f : \mathscr{F}^{\text{sh}} \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathscr{F}^{\text{sh}} \arrow[d, "g"] \\
\mathscr{F} \arrow[ur, "\text{sh}"] \arrow[r, "f"] & \mathscr{G}
\end{tikzcd}$$

The idea will be to remove all section that violate locality, and add sections to respect gluability. Another good take is the following post here.

Proof **:**
Let $\mathscr{F}$ be the presheaf in the question. Define $\mathscr{F}^{\text{sh}}$ as the set of compatible germs of the presheaf $\mathcal{F}$ over U, namely:

$$\mathscr{F}^{\text{sh}}(U) := \{(f_p \in \mathscr{F}_p)_{p \in U} \mid \forall p \in U, \exists p \in V \subseteq U \text{ open and } s \in \mathscr{F}(V) \text{ such that } s_q = f_q, \forall q \in V\}$$

Note this resemble to the étale space. Verify that this is a sheaf, that $\mathscr{F} \to \mathscr{F}^{\text{sh}}$ induces an isomorphism so stalks, and that it satisfies the free-forgetful adjoint.

As practice, show that the constant sheaf is the sheafification of the constant presheaf, that is

$$(\underline{S}_{\text{pre}})^{\text{sh}} = \underline{S}$$

Corollary 1.1.24: Sheafification and Stalks

Let $\mathcal{F} \in \mathbf{PSh}(X)$ be a presheaf. Show that for all $p \in X$

$$\mathcal{F}_p \cong \mathcal{F}_p^{\text{sh}}$$

Proof **:**
Use something similar given in the proof of proposition 1.1.5

In chapter 5, we shall measure the "failure" of a presheaf from being a sheaf using Čech cohomology.

Sub and Quotient Sheaves

We can now return to how cokernels exist in X_{sh}. The main problem with cokernels is that images fail gluing and cokernels fail locality. Sheafification will add enough sections to images to satisfy gluability, and this will give the condition for the cokernel to have locality.

Proposition 1.1.25: Mono- and Epimorhpisms of Sheaves Equivalence

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X (over **Set**, or any abelian category). Then the following are equivalent and characterize kernels:

1. φ is a monomorphism in the category of sheaves
2. φ is injective on the level of stalks: $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ for all $p \in X$
3. φ is injective on the level of pen sets: $\varphi(U) : \mathcal{F}(U) \to \mathcal{F}(U)$ for all $U \subseteq X$

Furthermore, the following are equivalent and characterize cokernels:

1. φ is an epimorphism in the category of sheaves
2. φ is surjective on the level of stalks: $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ for all $p \in X$

Furthermore, if φ is surjective on open sets, it implies the two above conditions (but not the other way around).

Proof **:**
This is combining the results from earlier. In particular, apply the results just proven about stalks to the proof of proposition 1.1.12.

It is important to stress that the last condition given for surjectivity is not implied by the other two:

Example 1.8: Failure of Surjectivity On Open Sets
Let $X = \mathbb{C}$, $\mathcal{O}_X$ be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ the sheaf of invertible holomorphic functions. Let $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$. Then show that there is an open subset $U \subseteq \mathbb{C}$ which fails surjectivity.

Corollary 1.1.26: Exactness of Stalk Functor

Let $\mathbf{Sh}_{\mathbf{C}}(X)$ be the category of sheaves over X on $\mathbf{C}$. Then $(-)_p$ is an exact functor.

Proof **:**
Direct result of proposition 1.1.25

1.2 Sheaf from a base

When defining a differential structure on a manifold, one of the strategies that is taken is to define pieces of the manifold in a compatible way, or said differently define it on some basis of the manifold. This section endeavours to do this for sheaves.

Let X be a topological space with a sheaf $\mathscr{F}$. Take a basis $\{B_i\}$ for X and take the subset of $\mathscr{F}$ consisting of $\mathscr{F}(B_i)$ along with all the appropriate restriction maps. Then there is enough Stalk information to recover all of $\mathscr{F}$ (show this), and hence we my want to start with such a collection to generate a sheaf. This collection alone will generate a presheaf, we will require the basis to satisfy the identity and gluability axioms:

Definition 1.2.1: Sheaf On A Base

Let X be a topological space and $\mathcal{B}$ be the collection defined above. Then this collection is caleld a *sheaf on a base* if it satisfies:

1. **Base identity axiom**: If $B = \cup_i B_i$, then if $f, g \in \mathscr{F}(B)$ are such $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all i, then $f = g$.
2. **Base Gluability**: If $B = \cup_i B_i$, and $f_i \in \mathscr{F}(B_i)$ such that f_i agrees with f_j on any basic open set contained in $B_i \cap B_j$, then there exists $f \in \mathscr{F}(B)$ such that $\mathrm{res}_{B,B_i} f = f_i$ for all i

Theorem 1.2.2: Define Sheaf From Base

Let $\{B_i\}$ be a base for the topological space X and let F be a of sets on the base. Then there is a sheaf $\mathscr{F}$ extending F with isomorphisms $\mathscr{F}(B_i) \cong F(B_i)$ agreeing with restrictions. Furthermore, this sheaf is unique up to unique isomorhpism

Proof **:**
The sheaf $\mathscr{F}$ will be defined as the sheaf of compatible germs of F. To this end, define the stalk of a presheaf F on a base at $p \in X$ by taking:

$$F_p = \varinjlim F(B_i)$$

Define:

$$\mathscr{F}(U) = \\ \{(f_p \in F_p)_{p \in U} \mid \forall p \in U, \exists B \text{ with } p \in B \subseteq U,\ s \in F(B) \text{ that is compatible: } s_q = f_q\ \forall q \in B\}$$

Then the map $F(B) \to \mathscr{F}(B)$ is an isomorphism.

Lots more to cover here that is worth going over

1.3 Sheaves and Abelian Category

What's left is to show that cokernels exist. The main idea is to use stalks. Let us first show that the case for kernels is already dealt with:

Proposition 1.3.1: Sheaf Morphisms Have Kernels

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. Show that it has a kernel

Proof **:**
Go over the proof of proposition 1.1.14 and verify that the kernel argument works sheaves

The above can be proven on the level of open subsets. Let us show that we can bring it to the level of stalks:

Corollary 1.3.2: Commuting Kernel and Stalk

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the stalk of the kernel is he kernel of the stalk:

$$(\ker(\mathcal{F} \to \mathcal{G}))_p \cong \ker(\mathcal{F}_p \to \mathcal{G}_p)$$

Proof **:**
exercise.

Let us now put together the universal properties we've shown so far to get that the category of sheaves also have cokernels:

Proposition 1.3.3: Sheaf Morphisms Have Cokernels

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism. Then it has a cokernel

Proof **:**
Recall that the universal property of the cokernel can be seen as the colimit of the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \\ 0 & \end{tikzcd}$$

As we know, the cokernel forms a presheaf $\mathcal{H}_{\text{pre}}$. By sheafification we have a unique map

$\mathrm{sh} : \mathcal{H}_{\mathrm{pre}} \to \mathcal{H}$. Now let $\mathcal{A}$ be any sheaf. Then consider the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d] \arrow[drr, bend left] & & \\
0 \arrow[r] \arrow[drrr] & \mathcal{H}_{\mathrm{pre}} \arrow[r, "\mathrm{sh}"] & \mathcal{H} \arrow[dr, dashed, "\exists!"] & \\
& & & \mathcal{A}
\end{tikzcd}$$

By the universal property of cokernels for presheaves, we have a unique map $\mathcal{H}_{\mathrm{pre}} \to \mathcal{A}$. By the universal property of sheafification, we will get a unique map composed from the two unique universal maps giving us $\mathcal{H} \to \mathcal{A}$, as we sought to show.

Corollary 1.3.4: Commuting Cokernel and Stalk

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the stalk of the kernel is he kernel of the stalk:

$$(\mathrm{coker}(\mathcal{F} \to \mathcal{G}))_p \cong \mathrm{coker}(\mathcal{F}_p \to \mathcal{G}_p)$$

***Proof* :**
exercise.

Combining this all, we may finally conclude that sheaves on X over abelian category form an abelian category, and we showed that kernels and cokernels can be verified on the level of stalks.

Corollary 1.3.5: Image and Sheafification

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves over an abelian category. Then $\mathrm{im}(\varphi)$ is the sheafification of the image presheaf.

***Proof* :**
exercise

Note how due to the sheafification, the surjectivity of a sheaf is no longer necessarily conditional on the surjectivity of on all open sets, even on $U = X$

Example 1.9: Surjective, But Not On Open Sets

Take $X = \mathbb{C} \setminus \{0\}$, $\mathscr{F}$ the collection of meromorphic functions, and $\mathscr{G}$ the collection of differential 1-forms. Take $\varphi : \mathscr{F} \to \mathscr{G}$ given by $f \mapsto df$. This map is surjective on stalks: indeed locally, each differential 1-form $\omega = df$ for appropriate f (namely as around every point there is a simply-connected open neighborhood). However, on all of X, the form dz/z has nothing mapping to it: indeed the only possible candidate is $\log(z)$ which is not a meromorphic function on $\mathbb{C} \setminus \{0\}$.

Thus, $\varphi(X) : \mathscr{F}(X) \to \mathscr{G}(X)$ is *not* surjective, even though $\varphi : \mathscr{F} \to \mathscr{G}$ *is* surjective!

For a purely algebraic examples, take $X = \mathbb{R}$, $\mathscr{F} = \underline{\mathbb{Z}}$, and $\mathscr{G} = \iota_p(\mathbb{Z}) \oplus \iota_q(\mathbb{Z})$ (i.e. the direct sum of two skyscraper sheaves at two distinct points p, q). Take the natural restriction map φ, show it is surjective on each stalk, but $\text{im}(\mathscr{F}(X)) = \{(n,n) \mid n \in \mathbb{Z}\} \neq \mathbb{Z} \oplus \mathbb{Z}$.

Corollary 1.3.6: Commuting Image and Stalk

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the stalk of the image is he image of the stalk:

$$(\text{im}(\varphi))_p \cong \text{im}(\varphi_p)$$

This last corollary can be interpreted as saying that we can take the stalk of the presheaf image or the stalk of the sheaf image, and get the same result.

Overall, the reason why presheaves are an abelian category is because of the computation "open set by open set", and sheaves are abelian categories because all abelian-categorical notions make sense "stalk by stalk".

Exercise 1.3.1

1. Show that $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if for every $s \in \mathscr{G}(U)$, there exists an open covering $\bigcup_i U_i = U$ and elements $t_i \in \mathscr{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ (hint: recall φ is surjective if and only if it is surjective on stalks).

   This exercise will come back in proposition 4.1.13

1.4 Sheaves between Topological Spaces

The next important question is if we can pushforward and pullback a sheaf given a continuous map $f : X \to Y$. If $U \subseteq V \subseteq Y$ and $f : X \to Y$, then $f^{-1}(U) \subseteq f^{-1}(V)$. Hence, f^{-1} can be thought of as a contravariant functor preserving $\subseteq$. In fact, more is true: f^{-1} also preserves the properties in definition 1.1.7, and hence works on sheaves as well:

Definition 1.4.1: Pushforward Of (Pre)sheaf

Let $f : X \to Y$ be a continuous map and $\mathscr{F} \in \mathbf{PSh}(X)$. Then the *pushforward* or *direct image* of $\mathscr{F}$ through f is defined on every open set $U \subseteq Y$ to be

$$f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$$

and induces a (per)sheaf on Y. The pushforward defines a [contravariant] functor $f^\# : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ sending:

$$f^\# : \mathscr{F} \to f_*(\mathscr{F})$$

If working with a particular category, for example **Set**, we would write $f_* : \mathbf{Set}_Y \to \mathbf{Set}_X$.

It is perhaps unsurprising that f^{-1} preserves pre-sheaves, however it is a little less obvious that it preserves sheaves as well; it is worth thinking this through. To me, this shows that sheaves are a very natural concept, as the usually topological function f^{-1} naturally preserves them. This also motivates the Grothendeick topology that will be introduced much later. We have already seen an example of a pushforward: the skyscraper sheaf is induced that pushing forward along the inclusion map $\iota_p : \{p\} \to X$ the constant sheaf $\underline{S}$.

Using the pushforward, we can define maps between ringed space

Definition 1.4.2: Isomorphism Of Ringed Spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. Then an isomorphism or ringed space is a map $\pi : X \to Y$ where:

1. π is a homeomorphism
2. There is an isomorphism of sheaves Between $\mathcal{O}_X$ and $\mathcal{O}_Y$ where two two sheaves are given via the pushforward π_* (i.e. $\mathcal{O}_Y \cong \pi_*\mathcal{O}_X$ on Y)[a], namely

$$\varphi(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

[a]Once the inverse image sheaf is defined, by adjointness we may also consider $\pi^{-1} : \mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X

There shall be much more to this once it is shown that schemes are ringed spaces. For now, on the level of sheaves, we can see that they behave well with stalks:

Proposition 1.4.3: Pushforward Induces Map On Stalks

Let $f : X \to Y$ be a continuous map and $\mathscr{F} \in \mathbf{Sh}(X)$. Then for each $\pi(p) = q$, there exists a map

$$(f_*\mathscr{F})_q \to \mathscr{F}_p$$

between the stalks of $f_*\mathscr{F}$ and $\mathscr{F}$.

Proof **:**
exercise. This can be verified using the equivalence class definition or the universal property. Doing it using equivalence classes, define the map

$$[s]_q \mapsto [f^{-1}(s)]_p$$

1.4.1 Inverse Image Sheaf

So far, most of the discussion of sheaves where with the assumption of a topological X. In example 1.2, there were examples of sheaves being "pushed-forward" (Definition 1.4.1) via a continuous map $\pi : X \to Y$. Namely there is a sheaf $\pi_*\mathscr{F}$ on Y defined via:

$$\pi_*\mathscr{F}(U) = \mathscr{F}\left(f^{-1}(U)\right)$$

There is an adjoint to it known as the inverse image sheaf[8]. It is not as clear as the case for the push-forward as the image of an open set need not be open, hence the existence is more subtle (similar to the existence of cokernels for groups). The adjoint will be important not only for having more than one perspective on sheaves, but for definition morphisms of schemes.

[8]This is not the pullback, as this will be for a concept with quasi-coherent sheaves

Definition 1.4.4: Inverse Image Sheaf

Let $f : X \to Y$ be a function function. and $\mathscr{F}, \mathscr{G}$ be sheaves on X and Y respectively. Then the *inverse image sheaf* $f^{-1}\mathscr{G}$ on X is the sheaf associated to the presheaf:

$$f^{-1}_{\text{pre}}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$$

or more concisely

$$f^{-1}\mathscr{G} = (f^{-1}_{\text{pre}}\mathscr{G})^{\text{sh}}$$

The construction of the sheaf is done in the usual quotient way: the elements of $f^{-1}_{\text{pre}}\mathscr{F}(U)$ are equivalent classes $(s, U), (t, V)$ where $(s, U) \sim (t, V)$ if there exists a $W \subseteq U \cap V$ where $f(U) \subseteq W$ and the restriction of s, t are equal on W.

The sheafification is crucial:

Example 1.10: Presheaf Inverse Image

Let $f : Y \to X$ where $Y = X \amalg X$ and f is the natural projection map and a sheaf $\mathscr{G}$ defined on X. Then by definition, we would get $f^{-1}_{\text{pre}}\mathscr{G}(U) = \mathscr{G}(U)$, however the sheafification will get:

$$f^{-1}\mathscr{G}(U) = \mathscr{G}(U) \times \mathscr{G}(U)$$

showing the two are in general not equal.

Proposition 1.4.5: push-forward Sheaf Adjoint

Let $f : X \to Y$ be a continuous functions and $\mathscr{F}, \mathscr{G}$ be sheaves on X and Y respectfully. Then there is a bijection:

$$\text{Mor}_X(f^{-1}\mathscr{G}, \mathscr{G}) \leftrightarrow \text{Mor}_Y(\mathscr{G}, f_*\mathscr{F})$$

That is, they are adjoints

Proof **:**
exercise.

One advantage of the inverse image sheaf is the computation of stalk

Proposition 1.4.6: Stalks Via Inverse Image Sheaf

Let $f : X \to Y$ be a continuous function with $\mathscr{G}$ a sheaf on Y. Then:

$$(f^{-1}\mathscr{G})_p = \mathscr{F}_{f(p)}$$

Proof **:**
exercise, follows more naturally via the definition.

In fact, the adjoint of this map is the *skyscraper sheaf*, namely given $f : \{p\} \to X$ where $p \in X$, then $f^{-1}\mathscr{G} = \mathscr{G}_p$. Conversely, if we define the sheaf $\mathscr{F}(\{p\}) = S$, then $f_*\mathscr{F}(U)$ is the skyscraper sheaf.

will get back to this when it is more needed

Exercise 1.4.1

1. Let $U \subseteq Y$ open and $\iota : U \hookrightarrow Y$ be the inclusion. Show that $\iota^{-1}\mathscr{G} \cong \mathscr{G}|_U$.
2. Show that f^{-1} is an exact functor (naturally on abelian categories, however is just as good to show it for any ${}_R\mathbf{Mod}$)
3. Let $U \subseteq X$ be an open subset of a topological space so that $V = X \setminus U$ is closed. Let $\mathscr{F}$ be a sheaf on V, and $\iota : V \hookrightarrow X$ the inclusion.
   a) Show that $(\iota_*\mathscr{F})_p \cong \mathscr{F}_p$ if $p \in V$ and 0 otherwise.
   b) Let $j_!(\mathscr{F})$ be a sheaf on X given by
   $$W \mapsto \mathscr{F}(W) \qquad W \mapsto 0$$
   where the first happens when $W \subseteq V$, and the second in all other cases. Show that $(j_!\mathscr{F})_p \cong \mathscr{F}_p$ if $p \in U$ and 0 otherwise. .
   c) conclude there is a short exact sequence $0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to \iota_*(\mathscr{F}|_V) \to 0$.

1.5 *Diffeology

(would love to add a word here)

2

Schemes

Recall the following correspondence:

$$\left\{\text{Affine Varieties}\right\} \underset{\mathcal{Z}(S)}{\overset{\mathcal{I}(V)}{\rightleftarrows}} \left\{\begin{matrix}\text{Finitely Generated,}\\ \text{Reduced rings}\\ \text{over algebraically closed fields}\end{matrix}\right\} \tag{2.1}$$

This allows for a certain subset of commutative rings to be interpreted as geometric objects, namely zero sets of polynomials, and vice-versa to interpret affine varieties as being determined by the set of functions on them up to functions that vanish on the given variety V. The question naturally arises then if this duality can be extended to general commutative rings. In particular, does there exist some topological space such that:

$$\left\{???\right\} \underset{\mathcal{Z}(S)}{\overset{\mathcal{I}(V)}{\rightleftarrows}} \left\{\text{Commutative Ring}\right\}$$

The necessity of commutativity stems from many of the theorem on prime ideals relying on commutativity[1].

Why do this? This abstraction is not simply for abstractions-sake. It is because the dictionary between algebra and geometry that has been built up to is incredibly useful, and will fix some of the problems of algebraic varieties, for example:

1. There are many algebras over more general rings that have geometric interpretations. For a rather trivial but illustrative example, the coordinate ring of a line whose points are in some

[1]The theory of "non-commutative geometry" using non-commutative rings can be explored in *Spectral Theory* in a class on functional analysis (see [4])

arbitrary ring k (the coordinate ring being $k[x]$) currently does not have a good representation since the Nullstellensatz limits to Algebraically closed fields. Another example that may at first seem "genun-geometric" but turns out to have many strong geometric properties are rings such as $\mathbb{Z}$, $\mathbb{Z}[i]$ (the Gaussian integers), and more generally *algebraic number rings* (see [3, chapter 22]). Mathematicians have found that these corresponds naturally to curves in a plane (see [3, chapter 22]).

2. As we saw in front-matter, localization gives us powerful tools to explore the behavior of a variety at a point. In the current theoretical framework, we cannot consider such rings, for example, if we take the coordinate ring of an affine line, $k[x]$, takes the closure $k(x)$, or study $k[x]_{(x)}$, we loose the fact that we are finitely generated. However, as we saw these certainly still have interpretations as algebraic correspondences to geometric objects.

3. Varieties loose multiplicity information. Saw we have the variety over $\mathbb{C}$ defined by the equations

$$y = x^2 \qquad y = 0$$

then if we take their intersection, we get a point $(0, 0)$. The corresponding affine variety is given by taking the union of quotient of the two coordinate rings, namely:

$$\frac{k[x, y]}{(y, x^2 - y)} = \frac{k[x]}{(x^2)}$$

Usually, we would say omit the x^2 term and just use $k[x]/(x)$ for the coordinate ring, however this eliminate the multiplicity information, namely that the variety $y = x^2$ "doubly intersects" the variety $y = 0$ the variety $y = 0$. The ring $k[x]/(x^2)$ keeps track that we have a "double-point", while $k[x]/(x)$ does not. This is a achieved by the fact that the ring has nilpotent elements.

4. (If you know differential geometry) A geometrical object that has many algebraic parallels is the vector bundle. Most vector bundles are very algebraic, however many are *not* projective (cannot be embedded into $\mathbb{P}^n_k$, or even $\mathbb{P}^n_R$ for some ring R). We thus need need to generalize better our notion from variety. This shall lead to different sheaves on spectra and will be studied in chapter 4.

This build-up will introduce to us the *affine scheme*. Many familiar geometric objects shall look like affine schemes, but not all familiar geometric objects shall look like them. We will need to generalize a bit more and define a *scheme* which is locally an affine scheme. For example, we shall see that projective space is not an affine scheme, but is a scheme. This shall complete liaison that algebraic geometry promises between algebra and geometry.

2.1 Defining Spec

Recall that the maximal ideals of $k[x_1, x_2, \cdots, x_n]/V$ correspond to points in a variety V. It may then be natural to say that the collection of maximal ideals should be considered the "points" of an arbitrary ring R. However, the pre-image of a maximal ideal need not be maximal breaking functoriality. In particular, we would want that the functor translating a ring homomorphism $\varphi : R \to S$ to the associated continuous function mapping the maximal ideal $M \subseteq S$ to a maximal ideal $\varphi^{-1}(M)$ (recall that regular function corresponds contra-variantly to k-algebra homomorphisms), however pre-image of maximal ideals need not be maximal (ex. take $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Then (0) is maximal in $\mathbb{Q}$, but certainly not maximal in $\mathbb{Z}$).

When the theory was originally being developed within functional analysis (see [4]), the spaces were compact Hausdorff spaces, in which case it would have sufficed to work with the maximal ideals as the pre-image of maximal ideals is usually maximal (where the ring was the ring of real continuous functions). However, as presented in the example above this doesn't hold for general rings.

Fortunately, if we loosen our requirement a bit and consider prime ideals, then it is indeed the case that $\varphi^{-1}(P)$ is prime making it the more natural choice of "points" for functorial purposes, even though it shall produce some more "complicated" points (see example 2.1). Working with primes also brings the added benefit of incurring a lot more structure which shall turn out to be very insightful and make the addition of primes not only necessary for the definition to work, but to explain a lot of phenomena[2].

Definition 2.1.1: Spectrum

Let R be an [arbitrary] commutative ring. Then $\operatorname{Spec} R$ is the collection of prime ideals of R. Let $\operatorname{mSpec}(R)$ represent the collection of maximal ideals of R.

We call the collection of elements of R *regular functions*

In the special case where R is a coordinate ring, $\operatorname{Spec} R$ corresponds to points of an affine variety along "extra" points associated to irreducible primes that are not maximal (ex. $(y - x^2) \subseteq k[x, y]$) as well as the zero ideal which is prime in coordinate rings. More Spectrum's of rings shall be described shortly, where the intuition behind having these extra points shall be elucidated.

To avoid confusion between points of $\operatorname{Spec} R$ and ideals of R, we shall write points in square brackets, for example $[\mathfrak{p}] \in \operatorname{Spec} R$ for $\mathfrak{p} \subseteq R$. There is some merit in writing it in the same notation as the equivalence relation notation, as points on in the topology are defined up to the radical of an ideal, for example for $(x^2) \subseteq \mathbb{C}[x]$, $V(\{(x^2)\}) = [(x)]$ (the $V(-)$ function will be defined in section 2.1.2, or see [3, chapter 21]). As we shall see, once a structure sheaf is introduced, the sets $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ and $\operatorname{Spec} \mathbb{C}[x]/(x)$ which each have one element shall not be have the same scheme structure added to them.

Another advantage of working with spectra is that there is no need to reference an ambient space for the algebraic subsets, the information is "within" the space! In particular, every quasi-projective variety in [3, chapter 21] was a subset of $\mathbb{P}^n_k$, while a spectra need not be a subset of $\mathbb{P}^n_k$! We shall later see under which condition we can embed a scheme into $\mathbb{P}^n_R$ for a ring R, see ref:HERE.

When working with coordinate rings, we may consider R to be the set of functions on V, since we may have a notion of "evaluation" for each element of R. We can do the same for a general commutative ring R, motivating the notation f for a general element of a commutative ring R ($f \in R$) and for calling such elements *regular functions*. Notice that if $p(x) \in k[x]$, then we may find $p(a)$ for some $a \in k$ by doing

$$\frac{p(x)}{(x-a)} = p(a) \bmod (x-a)$$

To represent the above more formally every element $f \in k[x]$ (for some algebraically closed field k) is

[2]For example, recall that if k was not algebraically closed, there exists regular functions that are not polynomials. Another is the distinct characterization of divisors in number theory which unifies as describing the same cohomological property of schemes

associated to an element of $\mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Spec}\,(k[x]), \coprod_{(x-a)\in\mathrm{Spec}\,(k[x])} k[x]/(x-a)\right)$, in particular:

$$k[x] \to \mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Spec}\,(k[x]), \coprod_{(x-a)\in\mathrm{Spec}\,(k[x])} \frac{k[x]}{(x-a)}\right)$$

$$f \mapsto ((x-a) \mapsto f \bmod (x-a))$$

Stare at the above until it makes sense. Notably, we can think of a function as holding local information (indeed, the "most local" as it gives point-wise information rather than open-set local, component local, germ local, and so forth) at each point of the spectrum[3]. We extract this point-wise information by evaluating, or in the case of spectra by modding. Notice that the codomain of the hom-set can be thought of as the disjoint union of field of fractions of the germs $k[x]_{(x-a)}$ (as $k[x]_{(x-a)}/\mathfrak{m}_{(x-a)} \cong \mathrm{Frac}(k[x]/(x-a)) = k[x]/(x-a) \cong k$), which motivates the idea that sheaves are the proper object keeping track of evaluation. Generalizing the above, we get the following:

Definition 2.1.2: Value Of Element [on a Spectrum]

Let $f \in R$. Then the "value" of f at the "point" $\mathfrak{p} \in \mathrm{Spec}\,(R)$ is the element:

$$f(\mathfrak{p}) := \overline{f} \in R/\mathfrak{p}$$

Let us now give multiple examples to get some grasp on spec and "functions" on Spec

Example 2.1: Spectrums and [regular] functions

1. Take $\mathrm{Spec}\,(\mathbb{C}[x])$ which is all prime (even maximal) ideal $(x-a)$ for $a \in \mathbb{C}$ and the prime ideal (0). This spectrum is labeled $\mathbb{A}^1_{\mathbb{C}}$; more generally the spectrum of $R[x_1, ..., x_n]$ is labeled $\mathbb{A}^n_R$. As $\mathbb{C}[x]$ is a finitely generated reduced ring over an algebraically closed field, the maximal ideals correspond exactly to all the points $\mathbb{A}_{\mathbb{C}}$ by the Nullstellensatz, with the exception of the special point (0) called the *generic point* which we shall return too soon. To visualize this set, imagine a real line (remember that it is a 1-dimensional object over $\mathbb{C}$):

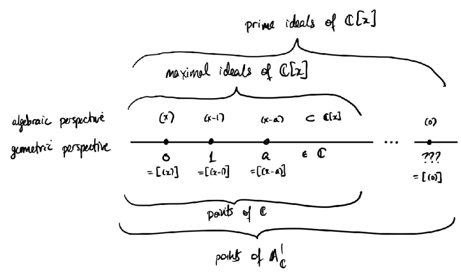


Figure 2.1: Visual Picture of $\mathrm{Spec}\,(\mathbb{C}[x])$

The generic point was located "away" from the line, but that is not in fact accurate. The problem is that it is a single point that is near every other point, while remaining to be a

[3] or more generally, at each point of a geometric space, for example a manifold

single point. What it means for points to be near shall be defined shortly when the Zariski topology on Spectra is introduced, but for now you can imagine that this causes issues when trying to draw out such point on a euclidean piece of paper. There are many ways to visualize it, some write it as a cloud of points in the corner, others at some edge of the visual, and eventually some omit the point when it is clear.

Let us evaluate some regular functions $f \in \mathbb{C}[x]$. Let's say $f = x^2 + 2x + 1$ and input input $[(x-3)]$. Then the answer is

$$16 \bmod (x-3) = f(2) \bmod (x-3)$$

which can be thought of as our usual answer $f(2)$! Notice that $\mathbb{C}[x]/(x-a) \cong \mathbb{C}$, and hence the codomain when evaluating at each point is the same. We shall see this is usually not the case soon. We can think of the fact that all the codomain of each evaluation as being a special symmetry for abstract varieties (that the evaluation information locally can all be embedded into a single global function-space).

Let us now plug in the generic point (0) into our function f (i.e. the point that isn't in our usual visualization of the complex line). We get:

$$f \bmod 0 \in \mathbb{C}[x]$$

that it, it gives back the function itself, and is the only point for which the codomain of f is not isomorphic to $\mathbb{C}$. You can think of 0 as a "1" [complex] dimensional point. When we have spectrum's of higher dimension, say n, then the point will be "n" dimensional. We can't yet define dimension as we are still working generically with sets, this shall be rectified in section ref:HERE.

More generally, this entire discussion works just as well over algebraically closed fields $k = \overline{k}$, as the completeness of $\mathbb{C}$ did not factor into the above results.

Often when working with $\overline{k}$-polynomials over several variables, we shall write the maximal ideals as tuples of point, ex $[(x-2, y-3)] \in \mathbb{A}^2_k$ will be represented as $(2,3) \in \mathbb{A}^2_k$. We shall often then denote the generic point as η to distinguish it from the point 0 which is associated to the ideal (x).

2. Take $y^2 - x^3 + 3x - 1 \in \mathbb{C}[x,y]$ and consider Spec $\frac{\mathbb{C}[x,y]}{(y^2-x^3+3x-1)}$. This this spectrum corresponds to the elliptic curve $y^2 = y^3 - 3x + 1$. It's points will contain all the maximal ideals of the ring $\frac{\mathbb{C}[x,y]}{(y^2-x^3+3x-1)}$ as well as a generic point η corresponding to $[(0)]$ representing the entire space. More generally, all affine algebraic sets correspond to a spectrum, and so a subset of the geometry of spectra's includes the classical geometry of curves, surfaces, and so forth!

3. Here is a Spectra that has no corresponding algebraic set: $\operatorname{Spec}(\mathbb{Z}) = \{(2), (5), (7), \cdots, (0)\}$. In other words, $\operatorname{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ is prime}\}$ and $\operatorname{mSpec}(\mathbb{Z}) = \operatorname{Spec}(\mathbb{Z}) \setminus \{(0)\}$.

   Let us evaluate some of the regular functions $n \in \mathbb{Z}$ at the points $\operatorname{Spec}(\mathbb{Z})$. The regular function 15 at the point $[(7)] \in \operatorname{Spec}(\mathbb{Z})$ is $15 \bmod (7) = 1 \bmod (7)$. The regular function 9 evaluated at $[(3)]$ is $0 \bmod (3)$. We may even say it has a double-zero at 3.

   Hence, functions on $\operatorname{Spec}\mathbb{Z}$ return some information about divisibility by primes.

4. For any field k, $\operatorname{Spec} k = \operatorname{mSpec} k = \{(0)\}$ (or $\operatorname{Spec} k = \{\eta\}$. For example, $\operatorname{Spec}\mathbb{Z}/3\mathbb{Z} = \{0\}$. Thus, fields have "singular" geometries within algebraic algebraic are not so interesting in scheme theory. The term "singular" instead of "trivial" was used as we shall see that

as schemes $\operatorname{Spec} K \ncong \operatorname{Spec} L$ as their sheaves will differ. Though the geometric picture as Spectra is trivial, they shall still hold valuable non-trivial information.

The evaluation at of each function at the singular point shall just return the function, namely it shall be a "constant"

$$3 \bmod 0 = 3 \in k$$

5. Take $\operatorname{Spec} \mathbb{R}[x]$. This consists of $[(0)]$, all prime (even maximal) ideals of the form $[(x-a)]$, and all prime (even maximal) ideals that can be represented by an irreducible 2nd degree polynomial $[(x^2+bx+c)]$, for example $[(x^2+1)]$. Evaluating at any of these points (besides $[(0)]$) will give a field: if evaluating points of the form $(x-a)$ the field will be isomorphic to $\mathbb{R}$, and it evaluating points of the form (x^2+ax+b), the field will be isomorphic to $\mathbb{C}$. As every irreducible quadratic splits into conjugate pairs of linear terms over $\mathbb{C}$, we can think of $\operatorname{Spec}(\mathbb{R}[x])$ as being a copy of $\mathbb{C}$ folded over itself attaching the conjugate pairs (don't continue reading until this is clear)

   Let us take the function x^3-1 and evaluate it as some point. At the point $(x-2)$ we get $7 \bmod (x-2)$, or in our classical interpretation, $f(2)$. At the point (x^2+1), we get:

$$x^3 - 1 \equiv -x - 1 \bmod (x^2+1)$$

   which we can think of as being $-i-1$. Hence, the Spectra of polynomials over fields that are not algebraically closed will behave more like algebraically closed fields as the extra non-maximal primes shall play the role of the required "additional points"! Hence, one piece of the puzzle for the "additional points" in a spectra is that they allow us to think of the spectrum as being "algebraically closed"!

6. Generalizing the above result, show that the points of $\operatorname{Spec} \mathbb{Q}[x]$ (minus the generic point 0) are in bijective correspondence with the points of $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, where the points of $\overline{\mathbb{Q}}$ are glue to their galois-conjugates: $\overline{\mathbb{Q}}/\mathrm{Gal}_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

7. Consider $\operatorname{Spec} \mathbb{Z}[x]$. Unlike $\operatorname{Spec} \mathbb{Q}[x]$, $\mathbb{Z}[x]$ is not a PID, and has prime ideals of the form $(p, f(x))$ where $f(x)$ and $f(x) \bmod p$ is irreducible. This is the first example where the irreducible functions of a polynomial ring are *not* in one-to-one correspondence with the points! The maximal ideals are of the form

$$(p, f(x))$$

   while the prime ideals are of the form:

$$(p) \qquad (f(x)) \qquad (p, f(x))$$

   There are now non-maximal prime ideals, namely the principal ideals (p) for prime p and principal ideals $(f(x))$ where $f(x)$ is irreducible over $\mathbb{Z}$ (or equivalently by Gauss's lemma over $\mathbb{Q}$). To understand this spectrum better, the following is a part of a visual of $\operatorname{Spec} \mathbb{Z}[x]$:

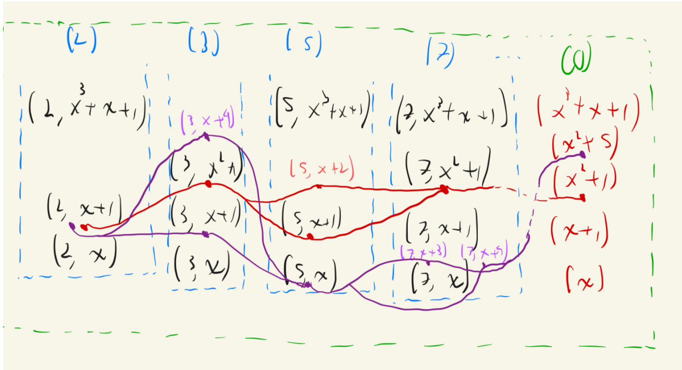


Figure 2.2: Spectrum of $\mathbb{Z}[x]$

Let us parse this image: the maximal points are all of the form $(p, f(x))$ and they are the black points. The prime (p) are contained in each maximal ideal $(p, f(x))$. They are like a more specialized generic point, where they are not near *all* the points, but near every point for which the ideal representing the point contains (p). Due to this, each column is surround by dotted blue lines that indicating the generic location of each prime (p). The entire image is surround by a green dotted line which represents the generic point $[(0)]$ and how it is close to every point. Notice that there are gaps in the columns exactly where the irreducible polynomial $f(x)$ is reducible mod$p$. For example $x^2 + 1 \bmod 2$ is reducible with solution $x = 1$, and so $(2, x^2 + 1)$ is not prime, namely given $\mathbb{Z}[x]/(2, x^2 + 1) = (\mathbb{Z}/2\mathbb{Z})[\omega]$ we have

$$(\omega + 1)(\omega + 1) = \omega^2 + 2\omega + 1 = 1 + 1 = 0$$

showing that $(2, x^2 + 1)$ is not prime, and hence not a point.

Next, each point $(f(x))$ is also a generic point, however their "shape" is not a simple as the points (p). The simplest case is when a point $(p, g(x))$ contains $f(x)$, namely $f(x) = g(x)$. In the image, we see that $[(3, x^2 + 1)]$ is a point and hence $[(x^2 + 1)]$ is near this point. On the other hand, both $(5, x + 2), (5, x + 1)$ are contained in $(x^2 + 1)$, precisely because $x^2 + 1$ factors into a product of those two linear terms mod 5. Hence $[(x^2 + 1)]$ is near $[(5, x + 2)]$ and $[(5, x + 1)]$. This closeness is shown via a red-line passing near these points and represents the fact that $[(x^2+1)]$ is near each of these points. Essentially, this represents the quadratic residues of $x^2 + 1$!

The same thing has been done for the point $[(x^2 + 5)]$. Some extra points have been added to the image to demonstrate more clearly the branching behavior. Notice that $[(x^2 + 5)]$ and $[(x^2 + 1)]$ "intersect", for example at $[(2, x + 1)]$ and at $[(13, x + 5)]$. These give us some number-theoretic information about these equations!

The way I like to interpret these "fuzzy" points corresponding to the irreducible polynomials from some intuition given by complex analysis. This is not unreasonable as the spectrum in many ways mirror the behavior of algebraically closed fields and complex analysis is the analysis of functions that are locally infinite polynomials, and indeed these two fields are very closed linked, something we shall cover in ref:HERE.

For now, consider the following: if we have a "germ" around a single point of a holomorphic function (i.e. the information given by its power-series representation around an arbitrarily small neighborhood around the point), we in fact have information about the entire holomorphic function. From this perspective, it doesn't matter at which point we choose to take the power-series representation, it will return equivalent information. Thus, the information does not live at any point point in particular but "generically" at all these points. If we choose to "specialize" and specify which data we want, we shall evaluate further to a single maximal ideal.

Notice that $\mathbb{Z}[x]$ is seen on a two-dimensional grid. We shall later see that $\operatorname{Spec}\mathbb{Z}[x]$ is naturally seen as a 2-dimensional scheme, while $\operatorname{Spec}\mathbb{Q}[x]$ shall be 1-dimensional. Furthermore, the red-lines and blue dotted lines will be seen as 1-dimensional points, the generic point will be seen as a 2-dimensional point, while all the maximal ideals will be 0-dimensional points.

Finally, let us consider evaluation of functions $f \in \mathbb{Z}[x]$ at points. At maximal points, we shall get values in the field $\mathbb{F}_p(\omega)$[a] where ω is a root of the irreducible polynomial. For example, take $[(7, x^3 + x + 1)] \in \operatorname{Spec}\mathbb{Z}[x]$ and $x^2 - 4 \in \mathbb{Z}[x]$. Then the value will be in $(\mathbb{Z}/7\mathbb{Z})(\omega)$ where ω is an element such that $\omega^3 + \omega + 1 = 0$. Then we would get that the value of $x^2 - 4$ is $\omega^2 + 3 \in \mathbb{F}_7(\omega)$. Another example would be evaluating at the point $[(3, x^2 + 1)]$ the function $x^4 - 1$. In this case, we get

$$x^4 + 2 \bmod x^2 + 1 = x^2 + 2 \bmod x^2 + 1 = 1 \bmod x^2 + 1$$

In other words, $f([(3, x^2 + 1)]) = 1 \in \mathbb{F}_3(i)$. As we are evaluating at maximal ideals, we shall always get a "number" as the output, in particular since we can always embed fields into an algebraic closure, the output of function at points corresponding to maximal ideals is away within an algebraically closed field.

Let us now evaluate at point that corresponds to a non-maximal prime ideal, say at 3. The output of each function $f \in \mathbb{Z}[x]$ will then be:

$$\overline{f} \in (\mathbb{Z}/3\mathbb{Z})[x] = \mathbb{F}_3[x]$$

The way I like to think about these output is that we are asking for *some* of the information stored in f, namely it shall return enough information about f to determine the generic behavior of that function around $[(3)]$. If we wanted the generic behavior around a different generic point, for example $[(x^2 + 1)]$, then we would get:

$$f(i) \in \mathbb{Z}[i]$$

Namely, if we choose any irreducible polynomial, we are choosing which new element of an integral extension we are adding that we then evaluate the function f at. We may then extract further information by modding by different primes. It shall often be the case that evaluating at a generic point corresponds to reducing the information of a function down to a smaller geometric object or to modding information.

8. Take $\operatorname{Spec}(k[x]/(x^2))$, which the spectrum of is a ring with nilpotent elements. This ring can be thought of as an extension of k by an element $\epsilon = x \bmod x^2$ that is thought to be "infinitely small". It has the interesting property that $\epsilon^2 = 0$, even though ϵ itself is nonzero. This can also be thought of as there being "two" point at the origin, 0 and ϵ, and if square ϵ you get 0, though be careful with this intuition as this Spectrum contains only 1 point: $[\epsilon]$ (as (0) is no longer prime!). The ring $k[x]/(x^2)$ is called the *dual numbers* and shall be a recurring ring to study the behaviour of nilpotent elements. This ring has the interesting property of having the function $x \in k(\epsilon)$ which always evaluates to 0 even though it is *not* zero! This gives us our first example of a function[b] that is not defined by what it does at every point, namely because the functions 0 and x have the same outputs.

9. Take $\operatorname{Spec} k[x]_{(x)}$, the localization of $k[x]$ at (x). This ring only has the prime ideals (0) and (x). It shall be shown later that this is a way to look very closely at a point on a particular smooth curve, where most of the information about this point shall be within the sheaf over the spectrum rather than the topology. The idea is that if we restrict to being arbitrarily close found 0, i.e. around $[(x)]$, all non-zero function function are invertible. Hence they all become units in the ring that represents all function where we only care to evaluate near 0. This ring is exactly the ring $k[x]_{(x)}$.

10. Take $\operatorname{Spec} \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}$. As it is a euclidean domain, the points are (0) and $(f(x))$ for all irreducible polynomials over $\mathbb{F}_p$. One of the interesting facets of $\mathbb{A}^1_{\mathbb{F}_p}$ is that there are infinitely many points that correspond to each point in $\mathbb{F}_p$ (one for each possible extension $\mathbb{F}_q/\mathbb{F}_p$). This richness in points will allow us to distinguish functions on $\mathbb{F}_p$ by the evaluation, namely recall that a polynomial is not distinguished by evaluation at all points in $\mathbb{F}_p$, however a polynomial *will* be distinguished by the evaluation of all points in $\operatorname{Spec}(\mathbb{F}_p[x])$, as we shall show using proposition 2.1.4.

11. Take $\operatorname{Spec}(\overline{k}[x,y]) = \mathbb{A}^2_{\overline{k}}$ (ex. $\overline{k} = \mathbb{C}$). Then it should be proven that the points in the set are

$$(0) \qquad (x-a, x-b) \qquad (f(x,y))$$

where $f(x,y)$ is an irreducible polynomial in $\overline{k}[x,y]$. Visually, all the primes of the form $(x-a, x-b)$ correspond to the usual points in $\mathbb{C}^2$. The points $(f(x,y))$ can be thought of as tracing out their curve in $\mathbb{C}^2$, for example $(y-x^2)$ is the point that is the curve $y = x^2$. Notice how The maximal ideal correspond to "0" dimensional points, while the ideal $(y-x^2)$ correspond to "1" dimensional points. Similarly, (0) is a "2" dimensional point. We shall come back to that through krull dimensions in section ref:HERE.

More generally, by the Nullstellensatz the maximal points of $\overline{k}[x_1, ..., x_n]$ are the ideals of the form:

$$(x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$$

The prime ideals of this ring don't lend themselves to a nice classification. Each prime ideal corresponds to an irreducible variety of a certain dimension, and hence the classification of the prime ideals is equivalent to the classification of irreducible varieties, which is a rather difficult task that is carried out in special cases, and more generally leads to the theory of moduli spaces. Some context on this classification in a classical setting was covered in [3, chapter 21.4].

12. Find the points of $\operatorname{Spec}(k[x_1, ..., x_n])$ where k is not algebraically closed, ex. $\mathbb{Q}$; generalize example 5^c.

13. Take $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at $\mathbb{Z} \setminus (p)$. This spectrum contains two points, $[(0)]$ and $[(p)]$. In particular, $\operatorname{Spec} \mathbb{Z}_{(p)}$ is not a single point like $\operatorname{Spec} \mathbb{F}_p$, it somehow has some more "fuzz" around it's single point $[(p)]$. This can be thought of as $\operatorname{Spec} \mathbb{Z}_{(p)}$ "remembering" that there is some arbitrarily small neighborhood around the point $[(p)]$ that it must keep track of. Once we introduce a topology on spectra's, we'll see that the closure of $[(p)]$ is itself (it is a closed point), while the closure of $[(0)]$ is $\{[(0)], [(p)]\}$, motivating the idea that $[(0)]$ "remembers" that there is some more information outside of just the single point! This shall be even more clear once we introduce the notion of dimension: the point $[(p)]$ will be zero-dimensional, while $[(0)]$ will be one-dimensional!

[a]Recall that $k[\omega] = k(\omega)$, or re-prove this
[b]granted, a generalization of the notion of function
[c]If you solved it, note that $(\sqrt{2}, \sqrt{2})$ is glued to $(-\sqrt{2}, -\sqrt{2})$ but *not* $(\sqrt{2}, -\sqrt{2})$

2.1.1 Maps Between Spectra

As spectra are given by rings, define the following natural map induced by rings:

Definition 2.1.3: Induced Spectra Map

let $Y = \mathrm{Spec}\,(S)$ and $X = \mathrm{Spec}\,(R)$ be two spectrum. Then given the ring homomorphism: $\varphi : R \to S$, the induced *spectrum map* $\varphi^a : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is given by:

$$\varphi^a(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$$

The reader ought to verify that if $\varphi : R \to S$ is a ring homomorphism, then $\varphi^a : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ sending $Q \mapsto \varphi^{-1}(Q)$ is functorial (recall or reprove that the pre-image of a prime ideal is prime). We now have two functions happening: φ^a given by ring homomorphisms, and regular functions given by ring elements.

Example 2.2: Maps between Spectra: φ^a

1. Show that the surjective map $R \to R/I$ and the map $R \to S^{-1}R$ induce an injective maps $\mathrm{Spec}\,(R/I) \to \mathrm{Spec}\,(R)$ and $\mathrm{Spec}\,(S^{-1}R) \to \mathrm{Spec}\,(R)$.
2. Here is a concrete example: Take $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[x]$ which by Nullstellensatz we can think of as the traditional points $\mathbb{C}$ along with the generic point (0). Consider the ring homomorphism between these rings defined by:
$$\varphi : \mathbb{C}[y] \to \mathbb{C}[x] \qquad y \mapsto x^2$$
Note this is not the squaring map, as that is not a ring morphism, but rather the map that maps the variable y to x^2 and then extends linearly. Then the map on the sets Spec 's is given by:
$$\pi : \mathrm{Spec}\,(\mathbb{C}[x]) \to \mathrm{Spec}\,(\mathbb{C}[y])$$
$$\pi\left((p)\right) = \varphi^{-1}\left((x-a)\right) = \left\{ f(y) \in \mathbb{C}[y] \;\middle|\; f(x^2) \in (x-a) \right\} = (y - a^2)$$
In one line:
$$\varphi^a\left([(x-a)]\right) = [(y-a^2)]$$
Verify that
$$\pi^{-1}\left([(x-a)]\right) = \{[(x-\sqrt{a})], [(x+\sqrt{a})]\}$$
Perhaps more intuitively, this is the Spec-equivalent of the more usual map $\mathbb{C} \to \mathbb{C}$ given by $x \to y = x^2$ which visually can be thought of as the map that projects the parabola $x = y^2$ onto the line y (namely, interpret the domain as the x-axis, the codomain as the y-axis).
3. More generally, any collection of regular functions can produce a spectra map (a "regular map")[a]: let k be any field and take polynomial $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$. Then define:
$$\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_m] \qquad y_i \to f_i$$

Let A represent the domain and B the codomain. Then show that for any ideal $I \subseteq A$, $J \subseteq B$ such that $\varphi(I) \subseteq J$, φ induces a map of sets:

$$\operatorname{Spec}(k[x_1, ..., x_m]/J) \to \operatorname{Spec}(k[y_1, ..., y_n]/I)$$

More particularly, show that this maps sends the point $[(x_1 - a_1, ..., x_m - a_m)]$ (traditionally, $(a_1, ..., a_m) \in k^m$ to the point traditionally represented as:

$$(f_1(a_1, ..., a_m), ..., f_n(a_1, ..., a_m)) \in k^n$$

4. This example gives a way of thinking of $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[x_1, ..., x_n])$ as an "$\mathbb{A}^n$-bundle" on $\operatorname{Spec}(\mathbb{Z})$. Define the map $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ given by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[x_1, ..., x_n]$. Then show there is a bijection between $\pi^{-1}([(p)])$ where $(p) \neq (0)$ and $\mathbb{A}^n_{\mathbb{F}_p}$! If $(p) = (0)$ then the pre-image is $\mathbb{A}^n_k$. The picture of this bundle can be seen as follows:

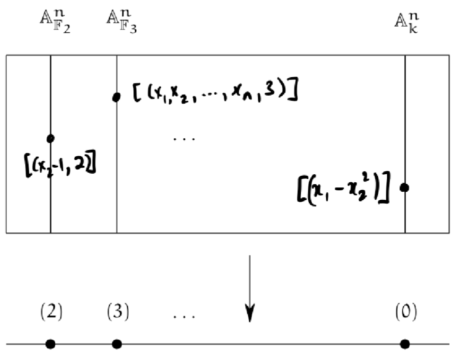


Figure 2.3: Bundle on $\operatorname{Spec} \mathbb{Z}$

5. Consider $\varphi : \mathbb{Z} \hookrightarrow k[x]$. Then $\varphi^a : \operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{Z}$ is the zero map, namely $\varphi^a(\operatorname{Spec} k[x]) = \{[(0)]\}$. From this perspective, we can see that classical field theory has trivial number theory as it has no interesting fibers over $\operatorname{Spec} \mathbb{Z}$.

6. Let $\varphi : R \to S$ be an integral extension. Show that $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is surjective (hint: Lying Over). This results also motivates the terminology "lying over" geometrically. Recall that is R or S is a field, the other must be too. Show that S being a field and the map being surjective can be used to conclude R is a field. What is the geometric argument if R was a field?

[a] In particular, this mirror's regular maps between affine varieties, in particular that for an affine variety being locally regular implies being globally regular

Having defined φ^a, we may introduce our first interesting theoretical result. Recall that there exists functions on a Spectrum of a ring that are not determined by their value at every-point (recall $\operatorname{Spec}(k[\epsilon]/(\epsilon^2)$ with the regular function ϵ). As it turns out, this behaviour can completely be put at the feet of nilpotents.

Proposition 2.1.4: Spectrum And Nilpotent Elements

Let R be a ring and I an ideal of nilpotent elements. Then

$$\operatorname{Spec} R/I \to \operatorname{Spec} R$$

is a bijection. As a consequence, two regular functions (elements of R) have the same value if they differ by a nilpotent element.

Thus, as sets (and even as topologies once introduced), nilpotents have no effect. Their presence is felt within the structure sheaf that is defined in section 2.2

***Proof* :**
Hint: recall that the nilradical is equal to the intersection of all prime elements (see [3, chapter 22]).

Hence if the nilradical is zero, $\sqrt{(0)} = (0)$, functions are determined by their points! In the new language we developed, this say that a function vanishes at every point if and only if it is nilpotent! Note how this contrasts to the case where we required k to be algebraically closed for the same result, showing the greater generality achieved by having the extra prime points!

2.1.2 Zariski Topology

Having defined the set that represents the space, the next thing to do is give a topology to this set. This topology would have to be a generalization of the Zariski topology defined on $k[x_1, ..., x_n]$, and with the definition of regular function this is readily available:

Definition 2.1.5: Zariski Topology

Let R be a commutative ring and $\operatorname{Spec}(R)$ the collection of prime ideals of R. Then define the *closed sets* for the Zariski topology on $\operatorname{Spec}(R)$ to be:

$$V(S) = \{x \in \operatorname{Spec}(R) \mid f(x) = 0, \ \forall f \in S\} = \{[\mathfrak{p}] \in \operatorname{Spec}(R) \mid S \subseteq \mathfrak{p}\}$$

Take a moment to see how this parallels the condition that all the functions in the coordinate ring must equal zero when evaluated at $\mathfrak{p}$. It is an exercise in commutative algebra to check that this is indeed a topology, namely

1. it contain the empty set and $\operatorname{Spec}(R)$ (these are both the vanishing set the right choice of single elements)
2. It is closed under arbitrary unions and finite intersections. Recall (or reprove) that

   $$\cap_i V(I_i) = V\left(\sum_i I_i\right)$$

   for arbitrary indexing set, and that

   $$V(I_1) \cup V(I_2) = V(I_1 I_2)$$

   see [3, chapter 21.1].

Note that the vanishing set $V(-)$ defined for the spectra of rings usually have more points than their counterpart with coordinate rings as they will contains primes that are not maximal ideals (particularly for rings of "dimension" ≥ 2 as we have briefly discused), and these primes will be within $V(S)$ for appropriate S.

Lemma 2.1.6: Induced Spectra Map Is Continuous

Let $\varphi : R \to S$ be a ring homomorphism. Then φ^a is a continuous map

Proof **:**
It suffices to show the pre-image of a closed set is closed, so take $V(S) \subseteq \operatorname{Spec}(R)$ and consider $(\varphi^a)^{-1}(V(S))$. Show that this is equal to $V(\varphi(S))$, showing it is closed.

The map φ^a given by the natural inverting function is a naturally continuous map, that is φ^a is a continuous function between the Spec of two rings with the Zariski topology. Hence Spec is a [contravariant] functor from **CRing** to **Top**.

Note that there will be continuous maps between the spectrum of rings that are not φ^a,

Example 2.3: Continuous Not Induced By Ring Homomorphism
The map from $\mathbb{A}^1_{\mathbb{C}}$ to itself that is the identity on all points except $[(x)]$ and $[(x-1)]$ (which will be represented with 0 and 1) where it swaps these two values is *not* a polynomial and hence cannot have been induced by a ring homomorphism, however it *is* continuous. If $0, 1 \notin V(S)$, then $\varphi^{-1}(V(S)) = V(S)$. Similarly if both values are inside. If 0 or 1 is in $V(S)$, say $0 \in V(S)$, then $\varphi^{-1}(V(S)) = V(S) \setminus \{0\} \cup \{1\}$ which is closed as all finite subsets of $\mathbb{A}^1_{\mathbb{C}}$ are closed.

Can you now find two ring homomorphisms that induce the same Spectra map? There are many example, the idea is to consider nilpotence.

The correspondence is terse as Spec does not carry enough information to fully and faithfully correspond to ring homomorphism. In chapter 3 it shall be shown that scheme morphisms are the right object to carry this information (namely, they shall insure that around every point, that is each stalk, the spectra map behaves like a polynomial)

Just like for varieties, we may define the inverse function $I(-)$. Namely for [nonempty] $S \subseteq \operatorname{Spec}(R)$,

$$I(S) \subseteq R \qquad I(S) = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p}$$

that is, $I(S)$ is a radical ideal (recall the intersection of prime ideals is a radical ideal, think $6\mathbb{Z}$). As usual:

Proposition 2.1.7: Properties of $I(-)$ and $V(-)$

1. $I(S)$ is an ideal of R, and furthermore it is always *radical.*
2. $I(-)$ is inclusion-reversing
3. $I(A \cup B) = I(A) \cap I(B)$, and $I(A \cap B) = \sqrt{I(A) + I(B)}$.
4. $I(\overline{S}) = I(S)$
5. $V(I(S)) = \overline{S}$ and $I(V(J)) = \sqrt{J}$.
6. If S is closed, $V(I(S)) = S$ is S. If I is radical then $I(V(J)) = J$.
7. $I(-)$ and $V(-)$ are an inclusion-reversing bijection between closed subsets of $\text{Spec}\,(R)$ and radical ideals of R.

Hence:

$$\text{closed subset} \subseteq \text{Spec}\,(R) \qquad \Leftrightarrow \qquad \text{radical ideal of } R$$

Proof **:**
exercise.

Thus, we get the bijective correspondence:

$$\big\{\text{Radical ideals of } R\big\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \big\{\text{closed sets of Spec}\,(R)\big\} \tag{2.2}$$

Notice that prime ideals are radical and hence are part of this association. This correspondence does not extend to all ideals until a sheaf, the structure sheaf, is introduced (see section 2.2.1).

As we see, the Zariski topology on $\text{Spec}\,(R)$ is a generalization of the Zariski topology on $k[x]$. The closed sets of $\text{Spec}\,(R)$ will contain the "traditional" points as well as the new points, for example $V(xy, yz)$ will contain all the "traditional‘ points where $y = 0$ or $x = z = 0$, as well as the generic point of the xz-plane (i.e. $[(y)]$ and the generic point of the y axis (i.e. $[(x, z)]$). Just like for affine varieties, $\text{Spec}\,(R)$ is almost never Hausdroff. This is even clearer in $\text{Spec}\,(R)$, for if there are primes that are not maximal, these are *not* closed points (in the Hausdroff topology every point is closed)

Example 2.4: Non-closed Points
Take $\mathbb{A}^2_{\mathbb{C}}$ with $[(y^2 - x)]$ and $[(x-2, y-4)]$. The point $[(y^2 - x)]$ is not closed. Notice that:

$$[(y-4, x-2)] \in V(I([(y^2 - x)])) = V(y^2 - x) \overset{\text{pr. } 2.1.7}{=} \overline{\{[(y^2 - x)]\}}$$

Showing that the point is *not* closed, namely points associated to non-maximal prime ideals are not closed. However, notice the set $V(I([(y^2 - x)]))$ is closed by definition. In a moment, the point $[(y^2 - x)]$ will be seen as the generic point of the above closed set.

In fact, due to Hilbert's Nullstellensatz, we know that in $\mathbb{A}^n_k$ the maximal ideals correspond to the "traditional points", that is the closed points. From here on, we shall call the "traditional points" (more generally points associated to maximal ideals) the *closed points*, and the "extra" or "bonus" points that were being referred to shall be called the *non-closed points*.

With a topology, we can now link points like $[(x-2, y-4)]$ to points like $[(y-x^2)]$. As these are point, it doesn't make sense to say one point contains another. However we we be able to link these points using closure:

Definition 2.1.8: Specialization And Generization

Let $x, y \in X$ be two points in a topological space. Then we say that x is a *specialization* of y and y a *generization* of x if

$$x \in \overline{\{y\}}$$

Algebraically, $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$, hence $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$.

We can now define generic point more precisely:

Definition 2.1.9: Generic Point

Let X be a topological space. Then $p \in X$ is a *generic point* for a closed subset K iff

$$\overline{\{p\}} = K$$

With generic points and the Zariski topology defined, the idea that generic points of a closed set (including all of $\operatorname{Spec}(R)$) are close to all the points in the closed subset (it is contained in all the open subsets (or all the open subsets of $V(S) \cap U$).

Let us end with upgrading proposition proposition 2.1.4

Proposition 2.1.10: Nilpotent Specturm Homeomorphism

Let R be a ring and I an ideal of nilpotent elements. Then

$$\operatorname{Spec} R/I \to \operatorname{Spec} R$$

is a homeomorphism.

***Proof* :**
The map is a bijective continuous map. Show the inverse is continuous.

2.1.3 Distinguished Open Sets

As the topology of $|\operatorname{Spec}(R)|$ is generate by the closed sets $V(S)$, and that these sets are the intersection of:

$$V(S) = \bigcap_{f \in S} V(f)$$

the sets $V(f)$ generate the topology of $\operatorname{Spec}(R)$. We may want to call these sets "distinguished closed sets", however we shall favour their open counter-part due to their more simple behaviour when introducing sheaf theory to spectra[4].

Definition 2.1.11: Distinguished Open Sets

The sets $D(f)$ are called the *distinguished open sets* or *principal open sets* and are defined to be

$$D(f) = X_f = |\mathrm{Spec}\,(R)| \setminus V(f) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid f \notin \mathfrak{p}\}$$

that is, it is the collection of prime ideals of R that do not contain f. Any open set U can be written as:

$$U = \mathrm{Spec}\,(R) \setminus V(S) = \mathrm{Spec}\,(R) \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} D(f)$$

Writing it as $D(f)$ may be helpful, Prof. Ravi Vakil calls these "doesn't-vanish sets" which may serve as a helpful mnemonic. These sets are usually not closed (especially if $\mathrm{Spec}\,(R)$ is connected, in which case unless $D(f) = 0$ it cannot be), and hence they are not the vanishing set for any functions in R. However, they are still isomorphic to a spectrum:

Proposition 2.1.12: Correspondence: Distinguished Open Sets and Localization

Let $D(f) \subseteq \mathrm{Spec}\,(R)$ be a distinguished open set. Then:

$$D(f) \cong_{\mathbf{Top}} \mathrm{Spec}\,(R_f)$$

Justifying the X_f notation. Furthermore, if $g = f_1 f_2 \cdots f_n$, then

$$\bigcap_{i=1,\cdots,n} (\mathrm{Spec}\, R)_{f_i} = (\mathrm{Spec}\, R)_g$$

Proof **:**

Recall that $R_f = R[f^{-1}]$, and that $\mathfrak{p} \subseteq R[f^{-1}]$ iff "$f \notin \mathfrak{p}$ and $\mathfrak{p} \in R$". This is exactly the points of $D(f)$, it is left to check that the map is bi-continuous. The second part is left as an exercise.

A mistake the reader may do is believe that each $D(f)$ can be thought of as the spectrum minus some points, but this need not be the case:

Example 2.5: Spectrum And Distinguished Open Sets

Take $X = \mathrm{Spec}\,(k[x, y])$ and take $U = X \setminus \{[(x, y)]\}$, i.e. X minus the origin (which is a closed set as it is a maximal ideal). Show that $U \ncong D(f)$ for any f (hint: notice that (x, y) is not principle). This shows an important distinction between points and functions!

Show further that U *can* be represented as the union of two distinguished open sets.

Another result we may want to conclude is that each open set $U \subseteq \mathrm{Spec}\,(R)$ corresponds to $\mathrm{Spec}\,(S)$ for an appropriate S, i.e. $U \cong_{\mathbf{Top}} \mathrm{Spec}\,(S)$. This shall be shown to be *false*, the curious reader may jump to example 2.11 for a counter-example.

[4]There shall be natural open subschemes, but infinitely many closed subschemes, one for each possible way of producing nilpotent elements

Furthermore, continuity can be shown on distinguished open sets, namely as they form a basis we can show that:

$$(\varphi^a)^{-1}(D([\mathfrak{p}]) = D(\varphi([\mathfrak{p}]))$$

2.1.4 Subsets of Spectra via quotient and Localization

Two common constructions that came up in a few example is the process of taking the quotient and the localization of a ring R. These form important subsets of $\operatorname{Spec}(R)$, namely by the 4th isomorphism theorem and the prime-correspondence theorem for prime ideals in localization (see [3, chapter 9.3] and [3, chapter 20.5]). By these theorems, $\operatorname{Spec}(R/I), \operatorname{Spec}(S^{-1}R) \subseteq \operatorname{Spec}(R)$. If $I = \mathfrak{p}$ and $S = R \setminus \mathfrak{p}$,the first corresponds to all prime ideals containing $\mathfrak{p}$, while the second corresponds to all prime ideals that are contained in the prime ideal $\mathfrak{p}$[5] These two sets need not union to the entire $\operatorname{Spec}(R)$ (take $\mathbb{Z}$ and 5), however they do correspond to two "opposite" ways of carving out a part of $\operatorname{Spec}(R)$. In the following visual is a very common illustration:

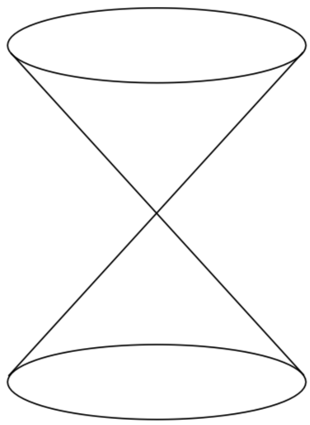


Figure 2.4: partial decomposition of $\operatorname{Spec}(R)$

Geometrically, $\operatorname{Spec}(R/I)$ can be thought of as "carving out" a part of $\operatorname{Spec}(R)$. For example, take $R = \mathbb{C}[x, y, z]$ and $\mathfrak{p} = (x^2 + y^2 - z)$. If we were working with affine complex varieties, then we would say this is the 3D-parabola in $\mathbb{C}^3$, however as we are on $\operatorname{Spec}(R/\mathfrak{p})$ this would come with some extra points (which ones?). For $\operatorname{Spec}(S^{-1}R)$, we may think it as the opposite (we shall cover the special case of $R_{\mathfrak{p}}$ afterwards). For example let's say $S = \{1, f, f^2, ...\}$ and take $S^{-1}R$. Then the primes of $\operatorname{Spec}(S^{-1}R)$ would be all primes that do not containing f, that is all points where f doesn't vanish! As an illustration, the set $\operatorname{Spec}(\mathbb{C}[x, y]_{y-x^2})$ are the points in the affine plane, minus all the points on the parabola $y - x^2$, namely all the singletons (a, a^2) for $a \in \mathbb{C}$ (so $[(x - a, y - a^2)]$), as well as the point $[(y - x^2)]$.

[5]More generally it is all prime ideals not having trivial intersection with the multiplicative set.

Proposition 2.1.13: Closed And Open Subsets

Let R be a commutative ring, $I \subseteq R$ an ideal, $f \in R$, $\mathfrak{p} \subseteq R$ prime, and S a multiplicative set. Then:

1. $\operatorname{Spec}(R/I) \subseteq \operatorname{Spec}(R)$ is a closed subset
2. $\operatorname{Spec}(R_f)$ is an open subset
3. $\operatorname{Spec}(S^{-1}R) \subseteq \operatorname{Spec}(R)$ need not be open or closed in general. In particular $\operatorname{Spec}(R_\mathfrak{p}) \subseteq \operatorname{Spec}(R)$ need not be open or closed. However, it is an embedding, namely $\iota : \operatorname{Spec} S^{-1}R \hookrightarrow \operatorname{Spec} R$ is bijective, continuous, and closed.

Proof **:**

1. Think of $V(I)$ (induce the topology on $\operatorname{Spec}(R/I)$ via the inclusion map)
2. Think $\operatorname{Spec}(R) \setminus V(f)$
3. think of $\operatorname{Spec}(\mathbb{Q}) \subseteq \operatorname{Spec}(\mathbb{Z})$ and $\operatorname{Spec}(\mathbb{Z}_\mathfrak{p}) \to \operatorname{Spec}(\mathbb{Z})$ induced by $\mathbb{Z} \to \mathbb{Z}_\mathfrak{p}$ (this shows that $\operatorname{Spec}(R_\mathfrak{p})$ need not be open.

For the case of $S = R - \mathfrak{p}$, we can think of $\operatorname{Spec}(R_\mathfrak{p})$ as keeping near the $\mathfrak{p}$, namely since the prime ideals away from $\mathfrak{p}$ are turned into units, the primes intersecting $R \setminus \mathfrak{p}$ will be eliminated. For example, if $R = \mathbb{C}[x, y]$ and $\mathfrak{p} = (x, y)$, then we keep all the points corresponding to being "near" the origin, that is the point $[(x, y)]$, i.e. $(0, 0)$, and the 1-dimensional points $[(f(x, y))]$ where $f(0, 0) = 0$ (i.e. where $f \equiv 0 \bmod (x, y)$) that is irreducible curves that pass through the origin. In the special case where $R = k[x]$, then the localization will always only have two points, usually called the classical point and the generic point. These shall eventually be seen as geometric interpretations of DVR.

Using these geometric results, they can lend intuition towards algebraic manipulation:

Example 2.6: Illustration Of Quotient And Localization

Using the above intuitions, we can quickly come up with some isomorphisms. Take $R = \mathbb{C}[x, y]/(xy)$. Visually, this is the union of the y and x-axis. If we localize this at x, then by our above intuition this should eliminated all points where the "function" x vanishes. The function x vanishes at the y-axis, and so we must be left with the x-axis minus the origin. This happens to be $\operatorname{Spec} \mathbb{C}[x]_x$. In fact, these two rings are isomorphic!

$$\left(\frac{\mathbb{C}[x, y]}{(xy)}\right)_x \cong \mathbb{C}[x]_x$$

which the reader should check.

2.1.5 Spectra: Quasicompactness, Connectedness, Irreducibility

Another important consequence is that for any open cover of $\operatorname{Spec}(R)$ given by distinguished open sets, there exists a finite open sub-cover. To see this, notice that $\cup_{i \in I} D(f_i) = \operatorname{Spec}(R)$ For some indexing set I if and only if $(\{f_i\}_{i\in I}) = R$ that is the ideal generated by the set $\{f_i\}_{i\in I}$. Then 1 must be inside that ideal, and must be a finite linear combination. Using the correspondence should then make it follow. This shows that $\operatorname{Spec}(R)$ is always compact in the topological sense. We may wish that this implies that $\operatorname{Spec}(R)$ has the usual nice properties of compactness, this turns out not to be the case (notice this implies $\mathbb{A}^n_{\mathbb{C}}$ is compact). This turns out to be a consequence lacking Hausdroffness. For this reason, a different name is given to compactness:

Definition 2.1.14: Quasicompact

Let X be a topological space. Then if X is compact and non-Hausdorff, X is *quasicompact*

As an exercise, show that if R Is Noetherian, every subset of $\operatorname{Spec}(R)$ is quasicompact. In section 3.3.3, we shall show how the notion of *proper maps* from analysis is the right way of defining the notion of compactness which recovers many of the nice results expected from compactness.

The companion property of [quasi]compactness is connectedness. Recall a set is connected if it cannot be written as a disjoint of two nonempty open sets. Disconnectedness of Spec is equivalent to a ring representable as the product of two rings, $R = R_1 \times R_2$, or if the reader remembers the brief discussion in Representation theory(see [3, chapter 23.2]), if R has idempotent elements

Proposition 2.1.15: Spectrum Connectedness And Idempotent Elements

IF $R = R_1 \times \cdots \times R_n = \prod_i^n R_i$ is a ring, then show that

$$\operatorname{Spec}\left(\prod_i^n R_i\right) = \coprod_i \operatorname{Spec}(R_i)$$

More generally, show that $\operatorname{Spec}(R)$ is not connected if R is isomorphic to the product of two zero-rings, that is there exists $a_1, a_2 \in R$ st $a_1^2 = a_2, a_2^2 = a_2, a_1 + a_2 = 1$.

Proof **:**
good exercise.

Note that for the homeomorphism, the product must be finite, if the product is infinite then $\operatorname{Spec}(\prod_i R_i)$ is larger than the right hand side. In fact, an infinite disjoint union of $\operatorname{Spec}(R_i)$ cannot be represented as $\operatorname{Spec}(R)$ for an appropriate R; it can be seen as the "simplest" example of a scheme that is not an affine scheme (see example 2.9(6)).

A notion similar to connectedness is *irreducibility*. Recall that a topological space is said to be *irreducible* if it is nonempty and it is not the union of two proper closed subsets. Written differently, X is irreducible if whenever $X = Y \cup Z$ with Y, Z closed in X, then $Y = X$ or $Z = X$[6]. Note that Y, Z may have nonempty intersection. In the euclidean topology, this notion is much less important (think $\mathbb{C}$), however in the Zariski topology we shall see that many interesting sets are irreducible. The intuition stems from traditional varieties where irreducibility corresponds to irreducible polynomials, and indeed this intuition when properly generalized translates over to irreducible [closed] sets in the Zariski topology of Spectra.

[6]The finiteness is important, for there may be irreducible sets that contain within them reducible sets of lower dimension

Let $X = Y \cup Z$ for closed sets Y, Z. Then there are radical ideals J_X, J_Y, J_Z such that $V(J_X) = V(J_Y) \cup V(J_Z) = V(J_Y J_Z) = V(J_Y \cap J_Z)$. As these are all radical ideals, $J_X = J_Y \cap J_Z$. Then if J_X is prime, it is an exercise in commutative algebra to show that either $J_Y \subseteq J_Z$ or $J_Z \subseteq J_Y$. Hence, irreducibility is in correspondence with prime ideals:

$$\{\text{prime ideals of } R\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \{\text{closed irreducible sets of Spec}\,(R)\} \tag{2.3}$$

If we look at irreducible closed sets, then $V(-)$ and $I(-)$ gives a bijection between the irreducible closed subsets of Spec (R) and the prime ideals of R, in other words there is a bijection between the *points* of Spec (R) and the irreducible closed subsets of Spec (R). Since it is a bijection, every irreducible closed subset of Spec (R) has *exactly* one generic point associated to it.

As usual, we define the Noetherian property on Spec (R) to mean that the descending chain condition for closed sets is satisfied (to mirror the ascending chain of ideals in R). In the Noetherian case, all closed sets can be decomposed into irreducibles:

Proposition 2.1.16: Noetherian Decomposition

Let X be a Noetherian topological space. Then every nonempty closed subset Z can be expressed uniquely as a finite union

$$Z = Z_1 \cup \cdots \cup Z_n$$

of irreducible closed subsets, none contained in another.That is , any closed subset Z has a finite number of pieces.

Proof **:**
see [3].

The reader should check the equivalent result for prime ideals

Exercise 2.1.1

1. Show that $D(f) \cap D(g) = D(fg)$.
2. $D(f) \subseteq D(g)$ iff $f^n \in (g)$ iff g is invertible in R_f. How would you prove this via "geometric explanation"? For an algebraic explanation, change to $V(f) \supseteq V(g)$ and take $I(-)$ to inverse and conclude.
3. $D(f) = \emptyset$ iff f is in the Nilradical.
4. Show that any connected component is the union of irreducible components (think of the + shape, i.e. Spec $(\mathbb{C}[x, y]/(xy))$).
5. Show that $\mathbb{A}^1_{\mathbb{C}}$ is irreducible.
6. Show that in an irreducible topological space, any nonempty open set is dense.
7. If X is a topological and Z an irreducible subspace, then $\overline{Z}$ is irreducible as well

8. Show that an irreducible space is connected.
9. If A is a integral domain, $\operatorname{Spec}(A)$ is irreducible.
10. Show that $\operatorname{Spec}(\mathbb{C}[x,y]/(xy))$ is connected but reducible.
11. these next two exercises show us how the theory of classical varieties persists with Schemes
    a) Let A be a finitely generated k-algebra. Then show that the closed points of $\operatorname{Spec}(A)$ are dense; show that if $f \in A$ and $D(f)$ is nonempty, then $D(f)$ contains a closed point of $\operatorname{Spec}(A)$[7]
    b) Let $A = \overline{k}[x_1, ..., x_n/I$ where the Nilradical is trivial, and let $X = \operatorname{Spec}(A) \subseteq \mathbb{A}^n_{\overline{k}}$. Show that [regular] functions on X are determined by their valued on the closed points. Hint: if f, g are different [regular] functions on X, then $f - g$ is nowhere zero on an open subset of X. Use the above result.
12. Show that there is a natural bijection irreducible components (maximal irreducible subsets) of $\operatorname{Spec}(R)$ and the *minimal* prime ideals of R.

2.2 Affine Scheme: the Structure Sheaf

Complex manifold have a sheaf of meromorphic functions. It is natural to want to extend these concepts to $\operatorname{Spec}(R)$, namely since holomorphic functions are locally infinite polynomials (in a sense, the "completion" of the case for Spectra). To motivate the following sheaf, recall that on $\mathbb{C}^n$, if we take open subsets there are many more holomorphic functions that could be defined on that open subset, namely that rational functions are well-defined when the roots of the denominator (which are not cancel by the numerator) are outside the open set. For example, on $\mathbb{C}^2 \setminus \{x\text{-axis} \cup y\text{-axis}\}$, the function

$$\frac{x^2 + 4x + y + 1}{x^5 y^4}$$

is well-defined. Moving this to Spec, in $\mathbb{A}^2_{\mathbb{C}}$, we ought to have the above function be well-defined on $D(xy)$. Furthermore, we want the collection of global sections, $\mathscr{O}_X$, to be R, mirroring the "global section" on $\mathbb{C}^2$ being the (non-rational) holomorphic functions in two variables.

As we saw in theorem 1.2.2, it suffice to define a sheaf on a base of $\operatorname{Spec}(R)$, namely on the distinguished open sets. Thus, we may define on $D(f) \subseteq \operatorname{Spec}(R)$ the ring that is the localization of R at the multiplicative subset of all function that do not vanish on $V(f)$ (note how this is subtly different than functions that do not vanish on f, what if f is nilpotent?). This can be represented as:

$$\mathscr{O}_X(D(f)) = \mathscr{O}_X(X_f) \cong R_f$$

First off, this is a presheaf: if $D(f) \supseteq D(g)$, by exercise 2.1.1 $g^n \in (f)$ so $rf = g^n$. Then we may define the restriction map:

$$\operatorname{res}_{X_f, X_g} : R_f \to R_{fg} = R_g \qquad \frac{s}{f^m} \mapsto \frac{r^m s}{r^m f^m} = \frac{r^m s}{g^{nm}}$$

which shows we have defined a presheaf on a base. We next must show that it satisfies the sheaf on base axioms.

[7]Hint: A_f is also a finitely generated k-algebra use generalized Nullstellensatz. Show the closed points of the Spec of a finitely generated k-algebra are those for which the residue field is a finite extension of k

Proposition 2.2.1: Checking Sheaf Axioms For Spec R

Let $X = \operatorname{Spec} R$, and suppose that X_f is covered by distinguished open sets $X_{f_a} \subseteq X_f$. Then:

1. If $g, h \in R_f$ are equal in F_{f_a} when passing through the restriction map, they are equal
2. If for all $a \in R$, there exists a $g_a \in R_{f_a}$ such that for each pair $a, b \in R$, the images of g_a and g_b in $R_{f_a f_b}$ are equal, then there is an element $g \in R_f$ whose image in R_{f_a} is g_a.

In other words, we define a $\mathcal{B}$-sheaf, and by theorem 1.2.2, $\mathcal{O}_X$ extends to a sheaf on X.

***Proof* :**
Let $\operatorname{Spec}(R) = \cup_{i \in I} D(f_i)$ where i indexes over the elements of R or I is the ideal generated by the f_i that creates the entire ring R. What we shall be showing is the identity and Gluability Axiom, or in categorical terms the exactness of:

$$0 \longrightarrow R \longrightarrow \prod_{i \in I} R_{f_i} \longrightarrow \prod_{i \neq j \in I} R_{f_i f_j}$$

Starting with the identity. As $s_1 = s_2$ is equivalent to $s_1 - s_2 = 0$, it suffices to show the result in the case where $s = 0$. By quasicompactness we know there is some finite subset of I where

$$\operatorname{Spec}(R) = \bigcup_{i=1}^{n} D(f_i) \qquad (f_1, ..., f_n) = A$$

Let's first suppose that for some $s \in R$, $\operatorname{res}_{\operatorname{Spec}(R), D(f_i)} s = 0$. It has to be shown that $s = 0$. By definition of R_{f_i} and by the finite number of f_i, there must exist some $m \geq 0$ such that

$$f_i^m s = 0 \qquad \forall i \in \{1, ..., n\}$$

Now, as $D(f_i) = D(f_i^m)$, $(f_1^m, ..., f_n^m) = R$. Thus, for appropriates choices $r_i \in R$ where $\sum_i^n r_i f_i^m = 1$, we get:

$$s = \left(\sum_i r_i f_i^m \right) s = \sum_i r_i (f_i^m s) = 0$$

For general $D(f)$, do the above argument with minor modifications changing R to R_f (why can the sum still be finite? How do you adjust when working with s/f?)

We next show Gluability[a]. Starting with the case of $\operatorname{Spec}(R) = \cup_i D(f_i)$, let's say that there are elements in each R_{f_i} that agree on overlaps $R_{f_i f_j}$ (note that intersections of distinguished open sets are distinguished open sets). As there many be infinitely many sets on which there are overlaps, it cannot be assumed that I is finite. However, the case where I is finite is a good start, say $I = \{1, ..., n\}$. Then we have element $a_i/_{f_i^{l_i}} \in R_{f_i}$ agreeing on the overlaps $R_{f_i f_j}$, meaning:

$$(f_i^{l_i} f_j^{l_j})^{m_{ij}} (f_j^{l_j} a_i - f_i^{l_i} a_j) = 0$$

simplifying notation by writing $f_i^{l_i} = g_i$:

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

As there are finitely many m_{ij}, take $m = \max_{i,j}(m_{ij})$:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

Next, we'll convert to a different set of distinguished open bases: take $b_i = a_i g_i^m$ and $h_i = g_i^{m+1}$ so that $D(h_i) = D(g_i)$. Then on each $D(h_i)$, we have the function $b_i/h - i$, and the overlap condition is:

$$h_j b_i = h_i b_j$$

Noting that $\cup_i D(h_i) = \mathrm{Spec}\,(R)$ giving $1 = \sum_i^n r_i h_i$ for appropriate i, replace h_i with b_i to get :

$$r = \sum_i r_i b_i$$

Then r is the element of R that restricts to each b_j/h_j:

$$rh_j = \sum_i r_i b_i h_j = \sum_i r_i h_i b_j = b_j$$

showing Gluability in the finite case. For the infinite case, we will use the previously shown base-identity axiom. By quasi-compactness, to choose appropriate f_i so that $(f_1, ..., f_n) = R$. Construct r as was just shown. Now, it shall be shown that for any $z \in I \setminus \{1, ..., n\}$, r restricts to the desired element a_z of A_{f_z}. To do this, repeat the argument above on $\{1, ..., n, z, \}$ to get $r' \in R$ which restricts to a_i for $i \in \{1, ..., n, z\}$. But then by the base identity, $r' = r$. Hence, r restricts to $a_z/f_z^{l_z}$.

To finish of the proof, we must do the same for an arbitrary $D(f)$, which shall then be what we wanted to show.

[a] Vakil pointed out that Serre called this as "partition of unity argument", which is the local-to-global principle physically manifested in differential geometry, which is a rather good perspective on what is about to happen

It may be tempting to say that $\mathcal{O}_{\mathrm{Spec}\,(R)}(U)$ for any open subset U is the localization of R at the multiplicative set of all functions that do not vanish at any point of U, however this is not true:

Example 2.7: Subtle Local Section Of Affine Scheme

Let $\mathrm{Spec}\,(R)$ be two copies of $\mathbb{A}_k^2$ glues together at their origins (it shall later be shown this is the spectrum of the ring $R = k[w, x, y, z]/(wy, wz, xy, xz)$). Let U be the compliment of the origin. Then there is a function which is 1 on the first copy and 0 on the second copy (minus the origin). This function will not be of the aforementioned described form.

This is because each ring of section's is given by:

$$\mathcal{O}(U) = \varinjlim_{D(f) \subseteq U} \mathcal{O}(D(f))$$

which does not preserve the localization structure in general.

More explicitly, given an arbitrary open subset U, we can cover it by affine opens $\{U_i\}_{i \in I}$ and cover the intersections $U_i \cap U_j$ by affine opens $\{U_{ijk}\}_{k \in K_{ij}}$. Then the sheaf condition gives that $\mathcal{O}_X(U)$ is the equalizer of:

$$\prod_{i \in I} \mathcal{O}_X(U_i) \rightrightarrows \prod_{i,j \in I, k \in K_{ij}} \mathcal{O}_X(U_{ijk})$$

The general ideal is that $\mathcal{O}_X(U)$ will contain rational functions f/g where g is nonzero on U. In practice, we shall often only require the local sections on distinguished open sets or on affine open sets, and many computations reduce to these two cases.

Adding this information to $\text{Spec}\,(R)$ gives us our first definition we need:

Definition 2.2.2: Affine Scheme

Let $X = \text{Spec}\,(R)$ be the spectrum of the ring R with the Zariski topology, and let $\mathcal{O}_{\text{Spec}\,(R)}$ be the structure sheaf. Then $(\text{Spec}\,(R), \mathcal{O}_{\text{Spec}\,(R)})$ is called an *Affine Scheme*.

With the sheaf defined, we have now "justified" thinking of elements on an arbitrary ring R as functions, as

$$\mathcal{O}_{\text{Spec}\,(R)}(\text{Spec}\,(R)) = \mathcal{O}_{\text{Spec}\,(R)}(\text{Spec}\,(1)) = R_1 = R$$

Even more is true, due to the section's being constructed through localization of R, they may be ratio's of functions, and we give elements of section a particular name:

Definition 2.2.3: Rational Functions

Let $X = \text{Spec}\,(R)$ be an affine scheme and $U \subseteq X$ an open subset. Then set $\mathcal{O}_X(U)$ is called the set of *rational functions*

Note that like complex manifolds (and unlike general smooth manifolds), there are in general very few functions on an affine scheme (in the global section of an affine scheme, all rational functions are regular, that is they are R). On the other hand, $\mathcal{O}_X(U)$ may have many interesting functions. In fact, in many cases all open subsets of X are dense meaning the local behavior can give a lot of fruitful information on the entire scheme (as shall be the case for abstract varieties, see ref:HERE)

Example 2.8: Affine Scheme

All spectra can be equipped with a structure sheaf by definition and so they are all affine schemes. For less obvious examples:

1. If $k \ncong k'$ are distinct fields fields, then as schemes $\text{Spec}\,(k) \ncong \text{Spec}\,(k')$ as their sheaves are different. Since there is only one point $[(0)]$ an all such Spectra, the functions (elements of the sheaf) will all return themselves, which the reader can think of as there only being constant functions, where each sheaf have different constant functions.

2. Let $R = k[x]_{(x)}$. Then $\text{Spec}\,R$ has 3 open sets, namely

$$\emptyset \qquad \{[(0)]\} \qquad \{[(0)], [(x)]\} = \text{Spec}\,R$$

   Note that U and $\emptyset$ are distinguished open sets (since $\{(0)\} = X_x$). The corresponding structure sheaf is:

$$\mathcal{O}_X(X) = R = k[x]_{(x)}$$
$$\mathcal{O}_X(U) = k(x)$$

   The restriction map is the natural inclusion.

3. Let $R = k[[x]]$, the completion of $k[x]$ at x. The only points of $\text{Spec}\,R$ are again $[(0)]$ and

$[(x)]$, and so it has the same open sets. The corresponding structure sheaf is:

$$\mathcal{O}_X(X) = R = k[[x]]$$
$$\mathcal{O}_X(U) = \mathrm{Frac}(k[[x]])$$

This a simple example showing why in the hierarchy of getting close to a neighborhood of a point, completion is considered to be more "refined".

4. If X, Y are affine scheme then the product of X and Y are:

$$\mathrm{Spec}\,(R) \times \mathrm{Spec}\,(S) = \mathrm{Spec}\,(R \otimes_{\mathbb{Z}} S) = \mathrm{Spec}\,(R \otimes S)$$

where the tensor product with no base ring by default assume it is over the initial ring. The generalization of this concept shall bee shown in section 3.2.

2.2.1 Fuzzy Points: Ideal Correspondence in Affine Schemes

So far, the dictionary between certain types of ideals of R and the geometric equivalent in $\mathrm{Spec}\,(R)$ has been establish, namely through $V(-)$ and $I(-)$:

$$\begin{aligned} \text{maximal ideal of } R &\quad\Leftrightarrow\quad \text{closed points of } \mathrm{Spec}\,(R) \\ \text{minimal prime ideal of } R &\quad\Leftrightarrow\quad \text{irreducible components of } \mathrm{Spec}\,(R) \\ \text{prime ideal of } R &\quad\Leftrightarrow\quad \text{irreducible closed subset of } \mathrm{Spec}\,(R) \\ \text{radical ideals of } R &\quad\Leftrightarrow\quad \text{closed subsets of } \mathrm{Spec}\,(R) \end{aligned}$$

What of a general ideal $J \subseteq R$? In section 2.1.2, this information was "lost" as $I(V(J)) = \sqrt{J}$. However, a sheaf would capture this information!

Let's start with the simplest case of $R = \mathbb{C}[x]/(x^2)$ with $X = \mathrm{Spec}\,(R)$ that was mentioned in example 2.1(6). If we consider the set $\mathrm{Spec}\,(R)$, then this is just the origin, i.e. $[(x)]$. Namely, by proposition 2.1.10,

$$\mathrm{Spec}\,(\mathbb{C}[x]/(x^2)) \cong \mathrm{Spec}\,(\mathbb{C}[x]/(x)) \cong \{\overline{(x)}\} \equiv \{0\}$$

Showing that topologically, these two spaces are the same (i.e. they are a single point). To show that the affine scheme $(\mathrm{Spec}\,(\mathbb{C}[x]/(x^2)), \mathcal{O}_{\mathrm{Spec}\,(\mathbb{C}[x]/(x^2))})$ is different from $(\mathrm{Spec}\,(\mathbb{C}[x]/(x)), \mathcal{O}_{\mathrm{Spec}\,(\mathbb{C}[x]/(x))})$, let us first label:

$$(X, \mathcal{O}_X) = \left(\mathrm{Spec}\,\frac{\mathbb{C}[x]}{(x^2)}, \mathcal{O}_{\mathrm{Spec}\,\mathbb{C}[x]/(x^2)}\right)$$

Consider the image of a polynomial $f(x)$ (i.e. an element in the global section) in $\mathbb{C}[x]/(x^2)$. Then we shall see that its evaluation is no longer an element in the field! For example if $\overline{f(x)} = \overline{(x-a)^2} \in \mathbb{C}[x]/(x^2)$, then:

$$\overline{f} = \overline{(x-a)^2} = \overline{x^2 - 2ax + a^2} = \overline{-2ax + a^2} \in \mathbb{C}[x]/(x^2) = \mathcal{O}_X(X)$$

and furthermore evaluating at $\epsilon \in \mathbb{C}(\epsilon)$ we get:

$$\overline{f}(\epsilon) = a^2$$

Notice that a function will have the evaluation at the point 0 information as well as the derivative information (namely, the coefficient of the x term)! How to then visualize the affine scheme $\operatorname{Spec}(R)$ along with its structure sheaf? It is usually done by picturing a "cloud" or "fuzz" around the point:

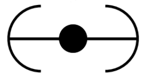
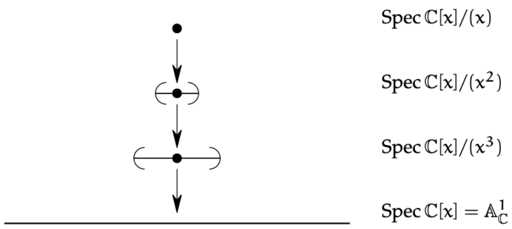


Figure 2.5: Picture of $(X, \mathcal{O}_X)$, not just $X = \operatorname{Spec}(R)$

The sequence of restrictions $\mathbb{C}[x] \to \mathbb{C}[x]/(x^2) \to \mathbb{C}[x]/(x)$ should be thought of as inching closer to $f(0)$:

$$\begin{tikzcd}
\mathbb{C}[x] \arrow[r, two heads] & \mathbb{C}[x]/(x^2) \arrow[r, two heads] & \mathbb{C}[x]/(x) \\
f(x) \arrow[rr, mapsto] & & f(0),
\end{tikzcd}$$

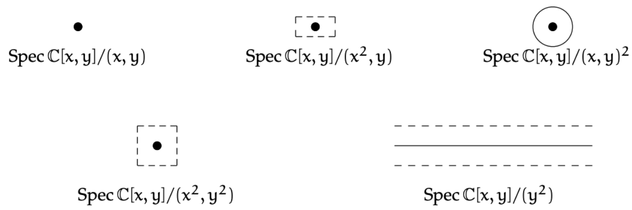
If we take, $\mathbb{C}[x]/(x^3)$, then then the spectrum will again be a single point, and when finding an element from the global section, the value, 1st derivative, and 2nd derivative will be remembered. This can be thought of a larger fuzz. We can have the following chain of inclusions:



Going to higher dimensions and considering:

$$\operatorname{Spec}\left(\frac{\mathbb{C}[x, y]}{(x, y)^2}\right)$$

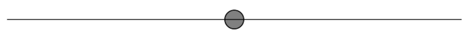
this can be thought of as remembering the derivative in all directions, and can be represented by a circle. The reader should now stare at this image to make sure it makes sense:



Next consider,

$$\operatorname{Spec}\left(\frac{\mathbb{C}[x, y]}{(y^2, xy)}\right)$$

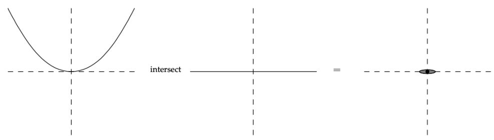
Then this can be thought of as the intersection of a thickened x-axis and the "+" created by xy, that is this affine scheme can be thought of as the x-axis as well as the first-order differential information around the origin.



Once we introduce stalks in section 2.3.2, these points will be more detectible. Finally, consider the following:

$$\begin{aligned}\operatorname{Spec}\left(\mathbb{C}[x/y]/(y - x^2)\right) \cap \operatorname{Spec}\left(\mathbb{C}[x, y]/(y)\right) &= \operatorname{Spec}\left(\mathbb{C}[x, y]/(y - x^2, y\right) \\ &= \operatorname{Spec}\left(\mathbb{C}[x, y]/(y, x^2\right)\end{aligned}$$

That is, intersecting the parabola $y = x^2$ with the y axis should produce produce a fuzzy point to remember the multiplicity:



Exercise 2.2.1

1. Let R be a ring that is not a field. Show that the restriction map on the structure sheaf is almost never surjective. Hence, there are more local functions than global function.
2. Let R be a reduced ring (contains no nilpotent elements). Show that the restriction maps are *injective*, that is each function on a larger open set can only restrict to one function. Give a counter-example for a ring with nilpotent elements.

2.3 Schemes

The last step to define schemes in general is by noticing that many geometric objects are not the spectra of a ring, but are *locally* the spectrum of a ring, as shall be demonstrated after the definition:

Definition 2.3.1: Scheme

Let X be a topological space. Then X is a *scheme* if it comes with a sheaf of rings $\mathcal{O}_X$ such that the ringed space $(X, \mathcal{O}_X)$ is locally affine, that is, X is covered in open sets U_i

$$X = \bigcup_i U_i$$

such that there exists a ring R_i and a homeomorphism $U_i \cong |\mathrm{Spec}\,(R_i)|$ such that

$$\mathcal{O}_X|_{U_i} \cong_{\mathbf{Sh}} \mathcal{O}_{\mathrm{Spec}\,R_i}$$

Given a scheme X, we can always find an open covering by taking the collection of distinguished open subsets of the corresponding $\mathrm{Spec}\,R_i$, which are still open in X as $\mathcal{O}_X|_{U_i}$ is homeomorphic topologically to $\mathrm{Spec}\,R_i$.

The following will define a scheme isomorphism, but *not* a scheme morphism. This is because not all stalks of ringed-space morphisms will preserve the necessary properties (see example 3.2 and theorem 3.1.4).

Definition 2.3.2: Isomorphism of Schemes

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two schemes. Then an *isomorphism of schemes* is an isomorphism of ringed space $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, namely it is a homeomorphism $f : X \to Y$ and a sheaf isomorphism $f^\sharp : f_*\mathcal{O}_Y \to \mathcal{O}_X$ (or equiv. $f^\sharp : \mathcal{O}_Y \to f^{-1}\mathcal{O}_X$)

Example 2.9: Scheme

1. Naturally, all affine schemes are schemes covered by one open set
2. The Riemann sphere is an affine scheme covered by two open sets, the usual stereographic projection maps. Why is it a scheme instead "simply" a complex manifold? What is the structure sheaf?
3. The following is the smallest non-affine scheme. Take

$$X = \{p, q_1, q_2\}$$

Define the topology by making

$$X_1 = \{p, q_1\},\ X_2 = \{p, q_2\}$$

be a sub-base. Define the presheaf $\mathcal{O}$ of rings on X by lettings

$$\mathcal{O}(X) = \mathcal{O}(X_1) = \mathcal{O}(X_2) = K[x]_{(x)} \qquad \mathcal{O}(\{p\}) = K(x)$$

(this is the *completion* at (x), not the localization at x) with the restriction map $\mathcal{O}(X) \to \mathcal{O}(X_i)$ the identify and $\mathcal{O}(X_i) \to \mathcal{O}(\{p\})$ the natural inclusion. Then this presheaf is also a sheaf, and in fact $(X, \mathcal{O})$ is a scheme, but *not* an affine scheme (see exercise ref:HERE). In the geometric context, this is thought of as the "germ of doubled points".

4. By using proposition 2.1.15, the finite disjoint union of schemes is a scheme. Note that an arbitrary disjoint union of affine schemes is not an affine scheme (notice it is not quasicompact).

5. Projective space $\mathbb{P}^n_k$ is a scheme, however we shall delay covering it till later as there is a lot of useful theory that should all be covered at once.

6. If the reader has already encountered blow-ups of a point, then the schemes associated to blow up are usually not affine.

In general, it is not immediate that a scheme candidate $(X, \mathcal{O}_X)$ is or is not covered by a single affine scheme, since there are so many types of rings to choose from. For example, the space $\mathbb{A}^1_{\mathbb{C}} \setminus \{([x])\}$ may seem like it cannot be a non-affine scheme as it is not an affine variety in the classical sense, however space is an affine scheme, namely it is isomorphic to $\mathbb{C}[x, x^{-1}]$. However, in example 2.11 we shall see that this $\mathbb{A}^2_{\mathbb{C}} \setminus \{[(x, y)]\}$ is not affine. We shall soon establish some way of detecting whether a ringed space is affine or not.

With the extra structure, we shall now denote $|X|$ to represent the topological space of X rather than the underlying set. The elements of $\mathcal{O}_X(U)$ are naturally called sections, and sections of $\mathcal{O}_X(X)$ are called *global sections*. Section can be variously denoted as:

$$\mathcal{O}_X(U) \qquad \Gamma(U, \mathcal{O}_X) \qquad H^0(U, \mathcal{O}_X)$$

depending on the use-case[8]. Since stalks of schemes are isomorphic to the stalk in the affine scheme, we call stalks at a point $x \in X$ a *local ring*, and quotient field $\mathcal{O}_{X,x}$ is called the residue field, which is denoted by $\kappa(x)$. We may ask what ringed spaces are isomorphic to affine schemes. The following addresses this question:

Proposition 2.3.3: Locally Ringed Spaces And Affine Schemes

Let $(X, \mathcal{O}_X)$ be a ringed spaced such that $R = \mathcal{O}(X)$. Then if:

1. $\mathcal{O}(U_f) = R[f^{-1}]$
2. The stalks of $\mathcal{O}_x$ of $\mathcal{O}$ are local rings
3. The map $X \to |\mathrm{Spec}\ (\mathcal{O}(X))\,|$ is a homeomorphism

Then $(X, \mathcal{O}) \cong (|\mathrm{Spec}\, R|, \mathcal{O}_{\mathrm{Spec}\, R})$, and we call X a *affine space.* A ring satisfying the first two conditions is called a *locally ringed space.*

***Proof* :**
exercise.

[8]the first is useful if working with only one sheaf, the second is useful once we will introduce multiple different sheaves that we can put on a spectra, and the last is useful for cohomological purposes

The following is put down here for the reader to see that schemes are indeed the geometric notion of rings, however the proof shall be relegated as the corollary of a more general result (or the reader can try to prove it themselves, as it is doable)

Theorem 2.3.4: Correspondence Of Ring Homomorphisms And Affine Schemes

Let R, S be two rings. Then if the ringed-space isomorphism $\pi : \operatorname{Spec}(R) \to \operatorname{Spec}(S)$ induces an isomorphism $\pi^\sharp : S \to R$, which induces a ringed-space isomorphism $\rho : \operatorname{Spec}(R) \to \operatorname{Spec}(S)$,

$$\rho = \pi$$

Proof **:**
corollary of theorem 3.1.4.

The following are useful results for later proofs:

Proposition 2.3.5: Distinguished Open Sets And Schemes

Let $f \in R$. Then:

$$(D(f), \mathcal{O}_{\operatorname{Spec}(R)|_{D(f)}}) \cong (\operatorname{Spec}(R_f), \mathcal{O}_{\operatorname{Spec}(R_f)})$$

If $f = 1$, then:

$$(D(1), \mathcal{O}_{\operatorname{Spec}(R)|_{D(1)}}) \cong (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$$

More generally, the natural set-inclusion map $\operatorname{Spec}(R_f) \hookrightarrow \operatorname{Spec}(R)$ gives a restriction from the Zariski topology on $\operatorname{Spec}(R)$ to the Zariski topology on $\operatorname{Spec}(R_f)$ and the structure sheaf restricts to the structure sheaf on $\operatorname{Spec}(R_f)$

Proof **:**
As X is an affine scheme, it is generated by the distinguished open sets for some (not yet known but existing) ring R. Then as $1 \in R$, $X = D(1)$. Then:

$$\mathcal{O}_X(X) = \mathcal{O}_X(\operatorname{Spec}(R)) = \mathcal{O}_X(D(1)) \stackrel{!}{=} R_1 \cong R$$

Where the $\stackrel{!}{=}$ inequality came from the definition of the structure sheaf, namely localizing R at 1, but 1 is already a unit and so we get:

$$\mathcal{O}_X(X) \cong R$$

From this, show that we have an isomorphism π

$$(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) = (\operatorname{Spec}(\mathcal{O}_X(X)), \mathcal{O}_{\operatorname{Spec}(\mathcal{O}_X(X))}) \xrightarrow{\sim} (X, \mathcal{O}_X)$$

Lemma 2.3.6: Global Section Of Affine Scheme

Let X be an *affine* scheme, meaning $(X, \mathcal{O}_X) \cong (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ for some R not necessarily known. Then we can recover R through the global sections of X. Furthermore, given this ring R, we have $(X, \mathcal{O}_X) \cong (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ as schemes.

The motivation behind this lemma can come if the reader recalls that it wasn't immediate what the global sections of projective space should be, while it was in a sense intrinsic in the definition of an affine variety.

In algebra, we saw that every ring is a $\mathbb{Z}$-algebra, and we can put other actions on a ring, which naturally gives the collection R-algebras. The scheme-equivalent is the following:

Definition 2.3.7: R-scheme and Finite Type Schemes

Let X be a scheme. Then it is an R*-scheme* (or a *scheme over* R) if all the ring of sections are R-algebras, and all restriction maps are R-algebra homomorphisms.

Furthermore, if an R-scheme X can be covered by $\text{Spec}\,(S_i)$ where S_i is a finitely generated R-algebra, we say that X is *locally finite type over* R. If X is quasicompact, then X is said to be *finite type over* R.

Example 2.10: Finite type R-scheme

1. Show that $\mathbb{A}^n_k$ is a k-scheme of finite type.
2. Show that $\text{Spec}\, R[x_1, ..., x_n]/(f)$ for some polynomial $f \in R[x_1, ..., x_n]$ is a finite type R-scheme. More generally, for a finite collection of polynomials $f_1, ..., f_m \in R[x_1, ..., x_m]$, $\text{Spec}\, \mathbb{R}[x_1, ..., x_n]/(f_1, ..., f_n)$ is of finite type. Hence, most intuitions from algebraic sets are about reduced affine $\overline{k}$-schemes of finite type[a].

[a] We shall add one more condition to this, namely being *separated* which prevents having "double points" to recover the notion of a variety

2.3.1 Open and Closed Subschemes

An easy way to get new schemes from known ones is to take open and closed subsets. It is easy to see that it is a topological subspace, The important thing to see is that the structure-sheaf well-defined. Let us start with open subsets as they are a bit easier:

Proposition 2.3.8: Open Subscheme

Let X be a scheme and $U \subseteq X$ an open subset. Then $(U, \mathcal{O}_X|_U)$ is a scheme.

Proof **:**
U is certainly a subspace and a subsheaf with local rings as stalks; it must be shown that U is locally affine.

Let $U \subseteq X$ be open and let $X = \bigcup_i \text{Spec}\,(R^{(i)})$ an affine open cover. As each $\text{Spec}\,(R^{(i)})$ is open, consider the open set $U^{(i)} = U \cap \text{Spec}\,(R^{(i)}) \subseteq \text{Spec}\,(R^{(i)})$. Now consider:

$$\text{Spec}\,(R^{(i)}) = \bigcup_j D(f_j)$$

The same is done to U, that is make it a union of distinguished open sets, however U may be the union of distinguished open sets from many different $\operatorname{Spec}(R^{(\ell)})$, however these shall be "eliminated" since if $D(g_k) \nsubseteq \operatorname{Spec}(R^{(i)})$, by definition $D(g_k) \cap \operatorname{Spec}(R^{(\ell)}) = \emptyset$) for $i \neq \ell$. Then we get:

$$U^{(i)} = \bigcup_{j,k} D(f_j) \cap D(g_k) = \bigcup_{j,k} D(f_j g_k) = \bigcup_{j,k} \operatorname{Spec}(R_{f_j g_k})$$

showing that each $U^{(i)}$ is locally affine, hence U is locally affine.

Definition 2.3.9: Open Subscheme

Let X be a scheme. Then $(U, \mathcal{O}_X|_U)$ is called an *open subscheme.* If U is also an affine scheme, then U is often said to be an *affine open subscheme* or *affine open subset.*

If a scheme is isomorphic to an open subscheme, we shall say that the isomorphism map (onto the image) is an *open embedding*:

Definition 2.3.10: Open Embedding

A map of schemes $\pi : X \to Y$ is an open embedding if:

$$(X, \mathcal{O}_X) \xrightarrow{\sim} (U, \mathcal{O}_Y|_U) \hookrightarrow (Y, \mathcal{O}_Y)$$

2.3.1 Remark about Hartshorne's Definition of Open Subscheme

Note that in Hartshorne, an open subscheme was technically not defined as $(X, \mathcal{O}_X|_U)$ but as $(X, [\mathcal{O}_X|_U])$ where $[\mathcal{O}_X|_U]$ is the equivalence class of all isomorphic sheaves. This is in fact not correct: isomorphic sheaves shall give technically different open subschemes (which are of course isomorphic). Hence, Hartshorne claims that an open subscheme has a *unique* subscheme structure, not *unique up to isomorphism.*

Corollary 2.3.11: Subspace and Zariski Topology On Open Subscheme

Let $U = \operatorname{Spec}(S) \subseteq X$ be affine an open subset and take $(U, \mathcal{O}_X|_U)$. Then the Zariski topology on U is the subspace topology on U with respect to X

***Proof* :**
Let $X = \bigcup_i \operatorname{Spec} R_i$. It suffices to show that the subspace and Zariski topology generate each other. First, if $V \subseteq U$ is open in the subspace topology, $V = \bigcup_j U \cap D(g_j)$ for some collection of g_j from the R_i's. As U is an open subset of X, $U = \bigcup_k D(f_k)$ so that:

$$V = \bigcup_j \left(\bigcup_k D(f_k) \right) \cap D(g_j) = \bigcup_j \left(\bigcup_k D(f_k) \cap D(g_j) \right) = \bigcup_{j,k} D(f_k g_j)$$

Then as $D(f_k g_j) \subseteq D(f_k) \subseteq U$ are open, V is the union of Zariski open subsets.

Conversely, we must show that any Zariski open basis element $D(g) \subseteq U$ is the intersection of U

with an open subset of X. But As U is open, open subsets of U are open subsets of X, and so $U \cap D(g) = D(g)$ gives the desired result.

Note that an open sub-scheme of an affine sub-scheme need not be affine:

Example 2.11: Open Sub-scheme From An Affine Scheme

Let $R = k[x, y]$, $X = \mathbb{A}_k^2 = \mathrm{Spec}\,(k[x, y])$, and consider $U = \mathbb{A}_k^2 \setminus \{(x, y)\}$ which may be considered the plane without the origin. First, Show that U is indeed an open subset (it is the compliment of intersection of certain closed subsets, equivalently the union of certain open sets, there is more than one way of creating it).

Now, there does not exist an R such that $\mathrm{Spec}\,(R) \cong U$. It may be tempting to say that there ought to be a distinguished open set of $\mathbb{A}_k^2$ that produces U, but the reader should think through why this is not the case (hint: (x, y) is not principle). However, U can certainly be described as the union of two distinguished open sets, namely:

$$U = D(x) \cup D(u)$$

What is the global section on the open sub-scheme U? This would be all functions in $D(x) \cup D(y)$ that agree on the intersection, $D(x) \cap D(y) = R_{x,y} = R(x, y)$. As $D(x)$ is all points in $\mathbb{A}_k^2$ that do not vanish in the function x, and $D(y)$ is all the points that do not vanish in the function y, we get:

$$\mathcal{O}_U(D(x)) = R_x = k\left[x, y, \frac{1}{x}\right] \qquad \mathcal{O}_U(D(y)) = R_y = k\left[x, y, \frac{1}{y}\right]$$

Now consider $R_x \cap R_y \subseteq R_{xy}$. These are all rationals functions with only powers of x in the denominator, and only powers of y in the denominator (as it must be in the intersection). But this leaves only the elements of R: $R_x \cap R_y = R$ (where R is identified with it's canonical embedding in R_{xy}). Thus:

$$\mathcal{O}_{\mathbb{A}_k^2}(U) = k[x, y]$$

in other words, the local section of U is identical to the global section of $\mathbb{A}_k^2$. Now, if U was affine, then by lemma 2.3.6, we would have that the affine scheme is determined by the global ring, that is:

$$U \cong \mathbb{A}_k^2$$

that is, this is a scheme isomorphism, and hence also a homeomorphic map between prime ideals that must respect $V(-)$ and $I(-)$. But in U we have:

$$V(x) \cap V(y) = \emptyset$$

while this is certainly not the case in $\mathbb{A}_k^2$ (as it gives (x, y)). Hence U is *not* an affine scheme! Some more interesting things are happening here that is very reminicent of complex geometry, namly this is an algebraic version of *Hartog's Lemma* (see [2]).

When working with closed subschemes, we have to be a little more careful as there can be multiple scheme structure, namely since we can have varying multiplicity of the zero sets, for example $\mathrm{Spec}\,k[x]/(x)$, $\mathrm{Spec}\,k[x]/(x^2)$, $\mathrm{Spec}\,k[x]/(x^3)$, and so forth. This comes down to the very elementary fact that $0^n = 0$ and $n \cdot 0 = 0$. For any non-zero number a, it is not the case that both $a^n \neq a$ and $n \cdot a = a$ $(n \in \mathbb{Z})$. Hence, up to isomorphism, open subschemes have a unique compatible sheaf-structure with the parent scheme

Definition 2.3.12: Closed Embedding and Closed Schemes

Let $f : Y \to X$ be a spectrum map along with $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$. Then if f induces a homeomorphism from Y onto it's image which is *closed* in X, and $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then f is said to be a *closed embedding*.

A *closed subscheme* $Y \subseteq X$ is a closed subset where the structure sheaf on Y makes $\mathrm{id} : Y \hookrightarrow X$ is a closed embedding.

The main difference between a closed subscheme and closed embedding is that a closed embedding comes with an isomorphism, while a closed subscheme would just be Y along with an appropriate structure sheaf. Note that the compliment of a closed subscheme is a unique open subscheme, but the compliment of an open subscheme has multiple closed subscheme, often infinitely many!

Example 2.12: Closed Subscheme

1. Take $\{0\} \subseteq \mathbb{A}^1_k$. This point corresponds to the affine scheme $\mathrm{Spec}\, k[x]/(x) \cong \mathrm{Spec}\, k$. Then $f : \mathrm{Spec}\, k \to \mathrm{Spec}\, k[x]$ mapping $\eta \mapsto 0$ is a closed embedding: its image being closed, the map $f^{\#} : \mathcal{O}_{\mathrm{Spec}\, k[x]}(\mathrm{Spec}\, k[x]) \to \mathcal{O}_{\mathrm{Spec}\, k(x)}(\mathrm{Spec}\, k(x))$ is the map
$$k[x] \to k$$
is naturally surjective, hence all its localizations are surjective, hence we have a closed subscheme.

2. Now consider $\mathrm{Spec}\, k[x]/(x^2) \to \mathrm{Spec}\, k[x]$. The image of f is the same as the above and so is still closed, and the sheaf morphism:
$$k[x] \to k[x]/(x^2)$$
is surjective, hence all its localizations are surjective. Hence, this is also a closed subscheme. More generally, $\mathrm{Spec}\, k[x]/(x^n) \to \mathrm{Spec}\, k[x]$ is a closed subscheme.

   This shows that the single point $\{0\}$ has countably many closed subscheme associated to it, depending how much "infinitesimal data" is preserved!

3. Let $I \subseteq R$ be an ideal. Then $\mathrm{Spec}\, R/I \to \mathrm{Spec}\, R$ is a closed embedding, it's image being closed as it is equal to $V(I)$, and the maps $R \to R/I$ are all surjective.

Lemma 2.3.13: Properties of Closed Embedding

1. f is a closed embedding if and only if for every affine fine open subset $\mathrm{Spec}\,(S) \subseteq Y$ where $f^{-1}(\mathrm{Spec}\,(S)) = \mathrm{Spec}\,(R)$, $S \to R$ is surjective.
2. Every surjective ring homomorphism $S \to R$ induces a closed embedding $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(S)$
3. The composition two closed embeddings is a closed embedding.

***Proof* :**

1. Let f be a closed embedding. Then $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_X$ is surjective (which implies it is surjective on stalks), which implies it is surjective on affine local subsets, which gives the result one way. For the other way, a map is surjective if it is surjective on open subsets that cover the space, which we get by taking the open subsets $D(\varphi)$ and taking their pre-image.

2. exercise

3. Use the two above parts to conclude the result by using the fact that the composition of surjective maps is surjective.

As closed subschemes are given by ideals for each open subsets, it would be good to describe all possible closed subschemes of a scheme by looking at an "ideal sub-sheaf" of the structure sheaf. This requires some more elaboration on the different types of sheaves that can be put on a scheme, which will be covered in section 4.1.2.

2.3.2 Stalks and Evaluation on Schemes

In differential geometry, the stalks of a manifold can be used define the evaluation of a function. This same trick is endeavored for scheme:

Proposition 2.3.14: Stalks of Schemes

Let $\operatorname{Spec}(R)$ be a scheme. Then the stalk at $\mathcal{O}_{\operatorname{Spec}(R)}$ at the point $[\mathfrak{p}]$ is isomorphic to the local ring $R_\mathfrak{p}$.

$$\mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} \cong R_\mathfrak{p}$$

Furthermore, for any $[\mathfrak{p}] \in X$, as $[\mathfrak{p}] \in \operatorname{Spec}(R_i)$ for appropriate R_i, we have

$$\mathcal{O}_{X,[\mathfrak{p}]} \cong \mathcal{O}_{\operatorname{Spec}(R_i),[\mathfrak{p}]}$$

***Proof* :**
For the first part, prove that $R_\mathfrak{p}$ is the colimit of R_f where $f \notin \mathfrak{p}$. The second part comes from proposition 1.1.5.

Thus, the local-to-global results in [3, chapter 22] will eventually be very useful for many arguments. Note that as there are "points within points", namely as there are non-closed points in schemes, then there is also stalks that are subsets of stalks, in particular if $\mathfrak{p}, \mathfrak{m}$ are a prime and maximal ideal respectively where $\mathfrak{p} \subsetneq \mathfrak{m} \in \operatorname{Spec} R$ so that $[\mathfrak{m}] \in \overline{[\mathfrak{p}]}$, then $\mathcal{O}_{X,\mathfrak{p}} \cong R_\mathfrak{p}$ is a localization of $\mathcal{O}_{X,\mathfrak{m}} \cong R_\mathfrak{m}$.

The fact that germs are local rings is not exclusive to sheaves, the same holds for smooth manifolds. Hence, the following definition aims to capture this more general behavior:

Definition 2.3.15: Locally Ringed Space

Let X be a ringed space. Then X is a *locally ringed space* if its stalks are local rings.

The main idea is that the single maximal ideal represents that there is one point at which we can evaluate for each germ. As all local rings have a maximal ideal, this motivates remembering the following notion form commutative algebra:

Definition 2.3.16: Residue Field

Let R be a commutative ring and $\mathfrak{p} \subseteq R$ a prime ideal. Then the ring of fractions of the integral domain domain $R/\mathfrak{p}$ is called the *residue field*, i.e. $\mathrm{Frac}(R/\mathfrak{p})$ is the residue field of $\mathfrak{p}$ (with respect to R). This field is often denoted $\kappa(\mathfrak{p})$.

Using residue fields, we can generalize the evaluation of a function. Recall that:

$$R_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \cong \mathrm{Frac}(R/\mathfrak{p})$$

as localization and taking the quotient commutes (see [3, section 20.5]). This makes it much easier to evaluate a function if we can determine R. Then as the valuation of a function is usually in $R/\mathfrak{p}$, as rational functions are now permitted on (proper) open subsets of X, the evaluation will be in $\mathrm{Frac}(R/\mathfrak{p})$. For a general point $x \in X$ in a scheme, we may diminish to the affine scheme to define evaluation:

Definition 2.3.17: Value Of Element [on a Scheme]

Let X be a scheme, $U \subseteq X$ an open subset, and $f \in \mathcal{O}_X(U)$ a section. Then the value of f at the point $[\mathfrak{p}] \in U$ is the element:

$$f(\mathfrak{p}) := \overline{f} \in \mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \cong \mathrm{Frac}(R/\mathfrak{p}) = \kappa(\mathfrak{p})$$

where $\mathrm{Spec}\,(R)$ is some affine local neighborhood of $\mathfrak{p}$.

As the quotient of any ring by a prime ideal is an integral domain, by taking its field of fraction we are forcing the codomain of each function to be a disjoint union of fields:

$$f : U \to \coprod_{\mathfrak{p}} \mathcal{O}_{U,\mathfrak{p}} \qquad \text{(and in affine case)} \qquad f : U \to \coprod_{\mathfrak{p} \in U} \mathrm{Frac}(R/\mathfrak{p})$$

If $\mathfrak{p}$ is maximal, then $\mathrm{Frac}(R/\mathfrak{p}) = R/\mathfrak{m}$. For certain rings, it suffices sections only over maximal points (namely *Jacobson Schemes*). In the case of affine Jacobson schemes, we can consider:

$$f : U \to \coprod_{\mathfrak{p} \in U} R/\mathfrak{m}$$

Varieties will be an example of Jacobson schemes. If $R/\mathfrak{m}_i \cong R/\mathfrak{m}_j$ for all maximal ideals (which shall be shown to be the case for varieties), then we can simply write:

$$f : U \to R/\mathfrak{m}$$

which recovers the notion of evaluation from classical algebraic geometry!

Example 2.13: Valuation On Affine Scheme's

The key idea is that the stalk $R_\mathfrak{p}$ gives all function's that will not vanish at $\mathfrak{p}$, and hence can divide by the function at that point, and then taking the quotient by the maximal ideal evaluate the function at that point, equivalently considering the value of the rational function in $\mathrm{Frac}(R/\mathfrak{p})$:

1. Take $X = \mathbb{C}[x]$. Then each $f/g \in \mathcal{O}_X(U)$ is a rational polynomial where $g(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. For example if $U = D(x)$, then $h = \frac{x+1}{x} \in \mathcal{O}_X(D(x))$ and:

$$h(3) = \frac{3+1}{3} = \frac{4}{3} \in \mathbb{C} = \frac{\mathbb{C}[x]}{(x-3)} \cong \mathcal{O}_{X,3}/(x-3)$$
$$h(-1) = \frac{-1+1}{-1} = 0 \in \mathbb{C} = \frac{\mathbb{C}[x]}{(x+1)} \cong \mathcal{O}_{X,-1}(x+1)$$

   As the points of $\mathbb{C}[x]$ correspond to the variety, we see that the evaluation will always be in $\mathbb{C}$, and there is no need to take the field of fractions as $\mathrm{Frac}(\mathbb{C}) = \mathbb{C}$.

   Another important example, take (want the vanishing at 0, but something feels off)

2. Take $\mathbb{A}^2_k$ for a field k where char $\neq 2$. Then in the open set U away from the y-axis and the curve $y^2 - x^5$ has in it's local section $\mathcal{O}_{\mathbb{A}^2_k}(U)$ the rational function $(x^2+y^2)/x(y^2-x^5) \in \mathcal{O}_{\mathbb{A}^2_k}(U)$. It's value at the point (2,4) (i.e. $[(x-2, y-4)]$) give a values in $\mathrm{Frac}(R/(x-2, y-4)) = \mathrm{Frac}(k[x,y]/(x-2,y-4)) \cong R_{(x-2,y-4)}/\overline{(x-2,y-4)}$. Then:

$$\frac{x^2+y^2}{x(y^2-x^5)} \bmod (x-2, y-4) = \frac{2^2+4^2}{2(4^2-2^5)} \in \mathbb{C} = \frac{\mathbb{C}[x,y]}{(x-2,y-4)} \cong \mathcal{O}_{\mathbb{A}^2_k,(2,4)}$$

   The value at the point $[(y)]$, which is the generic point of the x axis, is

$$\frac{x^2+y^2}{x(y^2-x^5)} \bmod (y) = \frac{x^2}{-x^6} = -1\frac{1}{x^4} \in \mathbb{C}(x) \cong \mathrm{Frac}(\mathbb{C}[x,y]/(y)) \cong \mathcal{O}_{\mathbb{A}^2_k,[(y)]}$$

3. Take $X = \mathrm{Spec}\,(\mathbb{Z})$ and take the open set $U = \mathrm{Spec}\,(\mathbb{Z}) \setminus \{(2),(7)\}$. This open set has many rational function, but let's take 27/4. Then evaluating this function at different points of U we get:

$$(5):\ 2/(-1) \equiv -2 \equiv 3 \bmod 5$$
$$(0):\ 27/4$$
$$(3):\ 0/1 = 0 \bmod 3$$

Notice that at 3, the function became 0. A (rational) function that whose output is 0 is said to *vanish* at that point. If $\mathfrak{m}_\mathfrak{p}$ represents the maximal ideal of $\mathcal{O}_{X,\mathfrak{p}}$, then functions on an open subset U (i.e. section in $\mathscr{F}(U)$) have values at each point of U whose value lies in $\kappa(\mathfrak{p})$. A function *vanishes* at $\mathfrak{p}$ if its value at $\mathfrak{p}$ is 0. This idea is not well-defined on ringed spaces in general! There is furthermore the following properties mirroring manifold theory:

Proposition 2.3.18: Properties of Functions On Locally Ringed Spaces

Let f be function on a locally ringed space X. Then:

1. The subset of X where f vanishes is closed
2. if f vanishes nowhere, then f is invertible

Note that the property "if f vanishes everywhere, then $f = 0$" is *not* one of the properties. This will be addressed in proposition 2.4.10.

Proof **:**

1. the subset of X where the germ of f is invertible is open
2. If $f \not\equiv 0 \bmod \mathfrak{p}$, then in particular $f \not\equiv 0 \bmod \mathfrak{m}$ for all maximal ideal, meaning f is contained in no maximal ideal. But then f must be a unit. If f where not a unit, then there would exist an $\mathfrak{m}$ that m belongs to, namely the maximal ideal containing f. But only units are not contained in any maximal ideal.

Note how this is is *not* true on a variety defined in the classical sense:

Example 2.14: On Variety: Vanishes Nowhere, Not Invertible

Take

$$\overline{x-100} \in \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}$$

The reader should check that this function vanishes nowhere on $V(x^2+y^2-1)$ as a variety, but it is *not* in invertible (i.e it is not a unit). However, $\overline{x-100}$ *does* vanish on Spec $\mathbb{R}[x,y]/(x^2+y^2-1)$; can the reader find what (non-maximal) prime ideal it vanishes on?

This ties in with the intuition that the spectrum of $\mathbb{R}[x,y]/(x^2+y^2-1)$ would be some "gluing" of $\mathbb{C}[x,y]/(x^2+y^2-1)$ via galois conjugate.

2.3.3 Gluing Schemes

In Differential Geometry, one of the most common way of constructing a manifold, and indeed what is at the heart of geometry, is the process of gluing pieces together with some amount of consistency (see [6, chapter 1]). More generally, gluing manifolds, and gluing schemes, is a special case of a *pushout* or a *fibered coproduct*. The fibered product can be thought of as taking a product of spaces and doing some gluing, while the fibered coproduct can be thought of as taking the disjoint union of spaces and doing some gluing. Just like for manifolds, we can't glue schemes any way we want and get another scheme, we must be a bit more careful. Example 3.10 we shall see how we can "badly" glue schemes[9]. The main added condition we need is a consistency in the maps on local sections:

[9] see this link for more details

Proposition 2.3.19: Constructing Schemes Via Gluing

Let $\{X_i\}_{i\in I}$ be a collection of schemes. Then if there exists:

1. A collection of open subscheme, not necessarily an open cover, $X_{ij} \subseteq X_i$ (where $X_{ii} = X_i$)
2. a collection of isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ (where f_{ii} would be the identity)

such that the isomorphisms agree on triple intersection[a]:

$$f_{ik}|_{X_{ij}\cap X_{ik}} = f_{jk}|_{X_{ji}\cap X_{jk}} \circ f_{ij}|_{X_{ij}\cap X_{ik}}$$

then there is a unique scheme X (up to unique isomorphism) with open subsets isomorphic to X_i respecting the above gluing conditions.

[a]this is called the *cocycle condition*

This is a translation of the techniques from differential geometry (this condition may look more familiar if written as $\mathbf{y} \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap V) \to \mathbf{y}(U \cap V)$). The cocycle condition can be visualized like so:

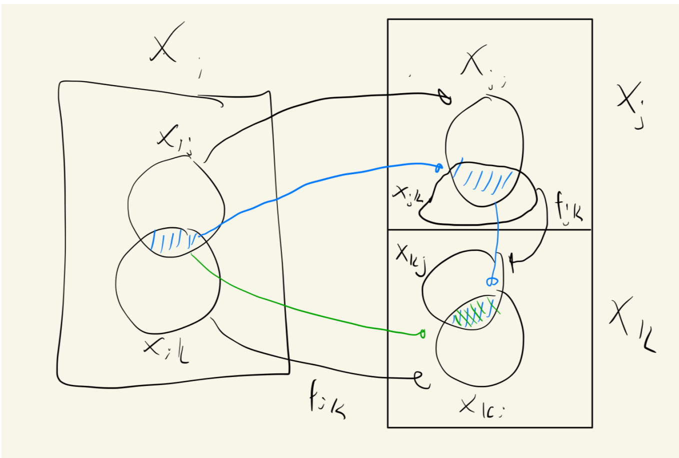


Figure 2.6: Green map must match blue map

Proof **:**
We shall present the high-level idea. Take the following topological space with the quotient topology and quotient-sheaf:

$$X = \frac{\coprod X_i}{(X_{ij} \ni x \sim f_{ij}(x) \in X_{ji})}$$

In particular, we are gluing parts of the scheme together. This gluing is well-defined precisely because of the cocycle condition, it guarantee's transitivity.

This will produce a new topological space; what needs to be shown is that this space is a Scheme.

First, let $X_i = \text{Spec}\,(R_i)$ be affine schemes so that

$$X = \frac{\coprod \text{Spec}\,(R_i)}{(X_{ij} \ni x \sim f_{ij}(x) \in X_{ji})}$$

Then notice that $\pi_i : X \to \text{Spec}\,(R_i)$, $[x_i] \mapsto x_i$ is a well-defined surjective continuous map; since if $x_i \in U_{ij}$ for any j, as f_{ij} (resp. f_{ji}) is an isomorphism, there is only one point in $\text{Spec}\,(R_i)$ representing it, making the map well-defined. It is certainly surjective. For continuity, by the cocycle condition $\pi^{-1}(D(f_i))$ will be open if it intersects any X_{ij}. Then the sets U_i cover X, and I claim that $U_i \cong_{\mathbf{Sch}} \text{Spec}\,(R_i)$, proving X is a scheme in the affine case. The map is bijective as the fibers of the map π_i over each point $x_i \in \text{Spec}\,(R_i)$ have size 1. The inverse map $\iota : \text{Spec}\,(R_i) \to U_i$ mapping $x_i \mapsto [x_i]$ is also continuous since if $V_i \subseteq U_i$ is open, it is representable by points in a set $\cup_{ij} D(f_{ij})$, and so we get a homeomorphism. Furthermore, by it's construction it is naturally also a ringed-space morphism through careful application of sheaf quotients and the cocycle condition. This will give our covering.

Finally, in the general case, we take projection $\pi_i : X \to X_i$ where as $X_i = \bigcup_j \text{Spec}\,(R_{ij})$, we take the covering $\pi_i^{-1}(\text{Spec}\,(R_{ij})$, and proceed in a similar manner, which shall give the desired result.

What would the resulting global section's of a glued scheme look like? Let us do so for the case of two schemes X_1, X_2 being glued on X_{12} and X_{21} with inclusion maps $\iota_i : X_i \to X$. Then we would get:

$$\mathcal{O}_X(U) =$$
$$\left\{ (s_1, s_2) \;\middle|\; s_1 \in \mathcal{O}_{X_1}(\iota_1^{-1}(U)),\ s_1 \in \mathcal{O}_{X_1}(\iota_1^{-1}(U)) \text{ such that } \varphi(s_1|_{\iota_1^{-1}(V)\cap U_1}) = s_1|_{\iota_1^{-1}(V)\cap U_2} \right\}$$

In this case, we have that the global section could be represented via an appropriate tensor product (note the tensor product is the coproduct in R-algebras, where in this case R would be the ring of the intersection). In general, the global sections will be the limit of the diagram of all the inclusion maps along with all the intersection. As it is highly dependent on how we do the gluing, as shall be demonstrated shortly.

Example 2.15: Gluing [affine] Schemes

Let $X = \text{Spec}\,(k[x])$ and $Y = \text{Spec}\,(k[y])$ Take $D(x) \subseteq X$ and $D(y) \subseteq Y$, which is the collection of all points that do not vanish at x and y respectively. As in example 2.5, these subsets are isomorphic to:

$$D(x) \cong \text{Spec}\left(k\left[x, \frac{1}{x}\right]\right) \qquad D(y) \cong \text{Spec}\left(k\left[y, \frac{1}{y}\right]\right)$$

As a diagram, we have:

$$\begin{tikzcd} \text{Spec}\,(k[x]) & & \text{Spec}\,(k[x]) \\ & \text{Spec}\,(k[x,1/x]) \arrow[ul] \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k[x] \arrow[dr] & & k[x] \arrow[dl] \\ & k[x,1/x] & \end{tikzcd}$$

These two open subsets will be glued in two different ways to produce two non-affine schemes that will be familiar to the reader:

1. (Spaces with double-origins). Define the isomorphism $D(x) \cong D(y)$ via the isomorphism $k[x, 1/x] \cong k[y, 1/y]$ with defined by mapping $x \to y$. This is a scheme by proposition 2.3.19.

Label this scheme S_1. This is *not* an affine scheme (what is the global section of S_1?) The same can done in higher dimensions (ex. plane with double-origin, more generally $\mathbb{A}^n_k$ with double origin)

In differential geometry, such spaces are example of non-Hausdorff spaces, however all [non-trivial] schemes are not Hausdorff. This behaviour is not due to a lack of Hausdroffness, but due to the lack of a different condition called *separatedness*. Spaces with double-origins are not separated, while $\mathbb{A}^n_k$ will be an example of a separated space (as a hint: show that in the spaces with double origin, there exists two affine spaces whose intersection is not affine, it is a space that was already covered)

For a more classical proof (if you're comfortable with Hartog's lemma, or a continuity argument), is that any collection of polynomials that vanish every except maybe at $(x, y) = 0$ must in fact also vanish at 0.

2. (projective space) Define the isomorphism $D(x) \cong D(y)$ via the isomorphism $k[x, 1/x] \cong k[y, 1/y]$ with defined by mapping $x \to 1/y$. The reader will recognize this as $\mathbb{P}^1_k$. When $k \in \{\mathbb{R}, \mathbb{C}\}$, this shape is isomorphic to a circle and a sphere, while in higher-dimension, say n it is an example of a space that cannot be embedded in $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively. For schemes, we must be more prudent given the generic zero point (0) and the generic points given by non-linear irreducible polynomials (recall that the linear terms that are galois conjugates are glued together).

   Let us check that $\mathbb{P}^1_k$ is not affine. Conifer the global section $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$. By construction, these are the global sections over X and global sections over Y that agree on the overlap. The sections on X and Y are the polynomials $f(x)$ and $g(y)$ respectively. For them to agree, we would need that, after passing through the isomorphism, $f(x) = g(1/x)$. But such polynomials are only the constant polynomial. Hence:

$$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$$

   Now by lemma 2.3.6 we have

$$\operatorname{Spec} \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \operatorname{Spec}(k) = \{0\}$$

   However, the projective line certainly has more than one point, completing the proof[a].

[a]Those familiar with complex analysis may recall that the projective space ha only constant holomorphic functions defined on it

Vakil gives one more example of interest on p.145

Exercise 2.3.1

1. Let X be the line with two origins, $Y = \mathbb{A}^1_k$ and $f : X \to Y$ the natural projection. Compute $f_*(\mathcal{O}_X)_0$, the stalk of the pushforward at 0.

2.3.4 Projective Schemes

Recall projective space from [6]. In [3] we saw how to construct a projective variety by gluing affine varieties, and hence it may be no surprise that we may construct a projective scheme $\mathbb{P}^n_R$ for a graded ring R. With projective varieties, we had Bézout's theorem generalize the fundamental theorem of algebra. With projective schemes, we shall have an equivalent result in chapter ref:HERE with line bundles on projective schemes giving us "enough functions" (though functions are not quite the right concept, hence why line bundles shall be brought into the discussion). In fact, this can be thought of as the key property of projective schemes; they have "enough" functions[10].

Classically, n-projective space $\mathbb{RP}^n$ or $\mathbb{CP}^n$ is defined on $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ by taking an equivalence class:

$$(x_1, ..., x_{n+1}) \sim (y_1, ..., y_{n+1}) \qquad \Leftrightarrow \qquad (x_1, ..., x_{n+1}) = (\lambda x_1, ..., \lambda, y_{n+1}) \qquad \lambda \neq 0$$

This produces an n-dimensional smooth manifold with the usual patches defined by:

$$U_i = D(x_i) = \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}$$

where the gluing was defined as

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}(x^1, x^2, ..., x^n) &= \varphi_i(x^1 : \cdots x^{j-1} : 1 : x^{j+1} : \cdots : x^n) \\ &= \left(\frac{x^0}{x^i}, ..., \frac{x^{j-1}}{x^i}, \frac{1}{x^i}, \frac{x^{j+1}}{x^i}, ..., \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, ..., \frac{x^n}{x^i}\right) \end{aligned}$$

In [3], we saw we can do the same for any field k, namely since inverses exist. Let us do the same but with a general graded $\mathbb{Z}$-ring R and $S = R[x_0, x_1, ..., x_n]$ and $S_+ \subseteq S$ being the positive-degree elements. Note that we are taking a $\mathbb{Z}$-grading instead of an $\mathbb{N}_{\geq 0}$ grading like in [3] as we shall often be working with rational functions that have negative degrees[11]. As a set, define:

$$\operatorname{Proj} S = \{[\mathfrak{p}] \mid \mathfrak{p} \subseteq S \text{ is a homogeneous prime not containing an irrelevant ideal}\}$$

Recall an irrelevant ideal is an ideal $I_s = (x_0^s, ..., x_n^s)$ for $s > 0$, as it is the zero set of $[0 : \cdots : 0]$ which doesn't exist in projective space. Then $\operatorname{Proj} S$ has the Zariski topology induced by homogeneous functions of positive degree:

$$V_+(T) = \{[\mathfrak{p}] \in \operatorname{Proj} S \mid T \subseteq \mathfrak{p}\}$$

The basis for the topology can be given by the positive degree elements:

$$D_+(f) = \{[\mathfrak{p}] \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$$

These sets are not quite the same as $\operatorname{Spec} R_f$, namely

Lemma 2.3.20: Characterizing Basic Open Sets For Projective Space

$$D_+(f) \cong \operatorname{Spec}(S_{(f)})_0$$

[10] In particular, they will have "ample line-bundles"

[11] More general gradings are sometimes useful, however we shall not need this either in this book or for awhile, I'll get back to this later

Proof :
Let us establish a bijection between these two sets, and then show they are homeomorphic. First, there is a bijective correspondence between the relevant homogeneous $\mathfrak{p} \in S$ where $x_i \notin \mathfrak{p}$, and $\mathfrak{p} \subseteq S_{(x_i)}$ (see [3, chapter 20.3]). Let us now show there is a bijective correspondence between the primes $S_{(x_i)} = A$ and $(S_{(x_i)})_0 = A_0$. First, the map $A_0 \to A$ injectively corresponds the prime ideals of A_0 with A via $\mathfrak{p} \mapsto \mathfrak{p}^e$ (the extension of primes). To show the map is surjective, for each $\mathfrak{p}_0 \subseteq A_0$ define $\mathfrak{p} \subseteq A$ as:

$$\mathfrak{p} = \bigoplus_i \mathfrak{q}_i \qquad \text{where} \qquad \mathfrak{q}_i \subseteq A_i,\ a \in \mathfrak{q}_i \text{ if and only if } \frac{a^{\deg f}}{f^i} \in \mathfrak{p}_0$$

In the case where $\deg f = 1$, we simply get $\frac{a}{f^i} \in \mathfrak{p}_0$. Note that $\mathfrak{q}_0 = \mathfrak{p}_0$, and hence if we show that $\mathfrak{p}$ is a homogeneous ideal in A, we have our desired bijective correspondence. But this almost immediately comes by construction. If $a \in \mathfrak{q}_i$, then $a^2 \in \mathfrak{q}_{2i}$, and if $a, b \in \mathfrak{q}_i$, then $(a+b)^2 = a^2 + 2ab + b^2 \in \mathfrak{q}_{2i}$, and so $a + b \in \mathfrak{q}_i$, showing it is closed under addition. A similar reasoning shows it is closed under scalar multiplication, giving us an ideal. If $ab \in \mathfrak{p}$, then $ab \in \mathfrak{q}_i$ for some i, which implies

$$\frac{(ab)^{\deg f}}{f^i} \in \mathfrak{p}_0 \qquad \Rightarrow \qquad \frac{ab}{f^i} \in \mathfrak{p}_0$$

and using that $\mathfrak{p}_0$, it is not difficult to conclude that either $a, b \in \mathfrak{p}$. We thus have our bijective correspondence, as we sought to show.

Let us apply this to $x_i = f$:

$$\begin{aligned}
D_+(x_i) &= \{[\mathfrak{p}] \in \operatorname{Proj} S \mid x_i \notin \mathfrak{p}\} \\
&= \{[\mathfrak{p}] \in \operatorname{Proj} S \mid x_i \notin \mathfrak{p} \cap S_1\} \\
&\cong \operatorname{Spec} (S_{(x_i)})_0 \\
&\cong \operatorname{Spec} \text{degree 0 elements of } R[x_0, ..., x_n, 1/x_i] \\
&\overset{!}{\cong} \operatorname{Spec} R\left[\frac{x_0}{x_i}, ..., \frac{x_i}{x_i}, ..., \frac{x_n}{x_i}\right] \\
&\cong \operatorname{Spec} \frac{R[x_{0,i}, ..., x_{n,i}]}{(x_{i,i} - 1)} && x_{j,i} = \frac{x_j}{x_i} \\
&\cong \mathbb{A}^n_R
\end{aligned}$$

where $\overset{!}{=}$ comes from the fact that $1/x_i$ is now a unit and a polynomial of degree -1, hence x_j/x_j is of degree 0, and these elements generate $(S_{(x_i)})_0$ over R. Relabeling x_j/x_i as $x_{j,i}$ and "omitting" $x_{i,i}$ as this is equal to one (equivalently, we "set" it to one by taking the quotient, which is similar to de-homogenizing), we see that it is isomorphic to the usual $\mathbb{A}^n_R$, and hence we recover the usual affine cover!

Let us now construct the structure sheaf on $\operatorname{Proj} S$. As we just saw, each $D_+(f)$ is affine and hence we certainly have an affine cover. Let us ensure that the gluing satisfies proposition 2.3.19, namely that all intersections glue appropriately.

Lemma 2.3.21: Structure Sheaf On Proj

Take the topology $\operatorname{Proj} S$. Then this is a scheme.

Proof **:**
Take $D_+(f), D_+(g)$ with non-trivial overlap. It is not difficult to show that $D_+(f) \cap D_+(g) = D_+(fg)$. We shall show that the cocycle condition given by proposition 2.3.19 is satisfied. By theorem 2.3.4, it suffices to show show

$$((S_\bullet)_g)_0[1/(f^{\deg g}/g^{\deg f})] \cong ((S_\bullet)_{fg})_0 \cong ((S_\bullet)_f)_0[1/(g^{\deg f}/f^{\deg g})]$$

as rings. This will induce the desired isomorphism on spectra. We shall show the later isomorphism, the former being almost identical. Define:

$$\varphi : ((S_\bullet)_{fg})_0 \to ((S_\bullet)_f)_0[1/(g^{\deg f}/f^{\deg g})]$$
$$\frac{a}{(fg)^n} \mapsto \frac{a}{f^{n+k}g^n}$$

where k is chosen such that the image has degree 0. This is possible since any element of $((S_\bullet)_{fg})_0$ must have numerator of degree $n(\deg f + \deg g)$.

For the inverse direction, consider the natural localization map at g:

$$\psi : ((S_\bullet)_f)_0[1/(g^{\deg f}/f^{\deg g})] \to ((S_\bullet)_{fg})_0$$

To verify these give an isomorphism, first note that φ is well-defined: if $\frac{a}{(fg)^n} = \frac{b}{(fg)^m}$ in $((S_\bullet)_{fg})_0$, then their images under φ must be equal in the localization at $g^{\deg f}/f^{\deg g}$. Similarly, ψ is well-defined by the universal property of localization. Finally, these maps are inverse to each other by construction, as localizing at g after localizing at f gives the same result as localizing at fg directly.

Thus, we have established the desired ring isomorphism, which induces the required isomorphism of schemes, completing the proof.

Definition 2.3.22: Projective Scheme

Let $S_\bullet = R[x_0, ..., x_n]$. Then the above construction defines the *classical projective n-scheme* or *prototypical projective scheme* over R, and is denoted $\mathbb{P}^n_R$ or $\mathbb{P}^n$ if unambiguous.

If X is a scheme st $X \cong \operatorname{Proj} S_\bullet$, then X is called a *projective R-scheme*. A *quasiprojective R-scheme* is open subscheme of a projective R-scheme.

Notice that it the scalars can be defined for any ring, which generalizes the result from classical algebraic geometry. If $R = k$ is an algebraically closed field, the closed points of $\mathbb{P}^n_k$ will behave late the traditional points of projective space defined in classical algebraic geometry. Just like for $\mathbb{P}^1$, the global section for each $\mathbb{P}^n$ consists only of the constants R, namely $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R) = R$ (check this), and hence $\mathbb{P}^n_R$ is *not* an affine scheme (the case for $n = 1$ was given in example 2.15).

Recall from [3] that homogeneous polynomials cut out sets from $\mathbb{P}^n$. This is because if $a \in R^{n+1}$ is the root of a homogeneous polynomial, so are all multiples of a. In modern algebraic geometry, we consider the global sections to be the "functions" that are defined on all of $\mathbb{P}^n_R$. However, $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R) = R$. So how do we interpret S? We shall see in proposition 4.2.7 that S will be a *line bundle* on $\mathbb{P}^n_R$.

Example 2.16: Projective Scheme Homogeneous Polynomial Cut

There are many example in [3], and hence we shall just put a few as a refresher.

1. Take $C = V(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2_k$ for an algebraically closed field k. . Then if $z = 0$, we would get $[(0,0,0)] \in C$, however this point doesn't exist in $\mathbb{P}^n_k$. Hence consider $z \neq 0$. Then for each solution $[(a,b,c)] = [(\lambda a, \lambda b, \lambda c)] \in C$. In k^3, we have that we get a cone of points that are solutions. Loosely visualizing $\mathbb{P}^2_k$ as a sphere (we should glue the dichotomous points) we see that this resemble the image:

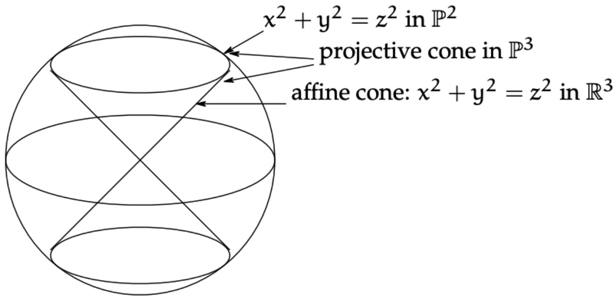


Figure 2.7: Visualization of $V(x^2 + y^2 - z^2)$

   Such vanishing sets of projective space are called *vanishing cones*

2. $\mathbb{P}^0_R = \operatorname{Proj} R[x_0] \cong \operatorname{Spec} R$, hence all affine scheme are 0-dimensional projective schemes.

3. In an affine scheme, two polynomials over a field cut out the same shape if they they differ by a scalar. This is not the case for homogeneous polynomials cutting out parts of $\mathbb{P}^n$, consider $x^2y = 0$ and $xy^2 = 0$. the problem is that these two are homogenizations of different polynomials that produce the same set. However, as we know from section 2.2.1 schemes remember multiplicity. This behaviour shall later be properly taken into account by saying that two polynomials shall cut out the different *closed subscheme* (rather than the same *closed subset*). With this addendum, it shall once again be the case that two polynomials cut out the same closed subscheme if and only if they differ by a scalar multiple, as closed schemes shall essentially correspond to polynomial cut outs (as sets) that have sheaves that remember multiplicity information.

Proposition 2.3.23: Irreducibility of classical projective scheme

The scheme $\mathbb{P}^n_R$ is irreducible and is of finite type.

As irreducible spaces are connected, $\mathbb{P}^n_k$ is connected.

Proof **:**
Recall that the affine scheme is irreducible. Notice the argument is invariant under gluing.

Lemma 2.3.24: Projective Scheme Quasicompact And Quasiseperated

Assume Let X be a projective R-scheme. Then X is quasicompact and quasiseparated

Proof **:**
Recall that the graded ring in the definition is finitely generated.

To conclude this section, let us formally define the affine and projective cone:

Definition 2.3.25: Affine Cone

Let $S_\bullet$ be a graded ring. Then the *affine cone* of $\operatorname{Proj} S_\bullet$ is $\operatorname{Spec}(S_\bullet)$.

Note that $\operatorname{Spec}(S_\bullet)$ is dependent on $S_\bullet$, not just $\operatorname{Proj}(S_\bullet)$. Think of the affine cone as eliminating the points at infinite and adding the origin. In fact, given a field k, there is a natural map $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$ (more generally if $S_\bullet$ is a finitely generated graded ring over a base ring R, there is a natural morphism).

Definition 2.3.26: Projective Cone

Take $\operatorname{Proj} S_\bullet$. Then *projective cone* or *projective completion* is $\operatorname{Proj}(S_\bullet[t])$ for some indeterminate t of degree 1.

Take for example $\operatorname{Proj}(k[x,y,z]/(z^2 - x^2 - y^2)$. Then the projective cone is $\operatorname{Proj}(k[x,y,z,t]/(z^2 - x^2 - y^2)$. Taking $t = 0$ returns the original projective scheme, and taking the compliment (the distinguished set $D(t)$) gives $\operatorname{Spec}(S_\bullet)$. This construction can be thought of as the opposite of the affine cone, where we attach the points at infinity of a conic curve.

2.4 Types of Schemes

In this section, we shall cover various different properties a scheme may have. In the process, we shall see if these properties can be checked affine-locally or stalk-locally, that is whether it suffices to check on an open cover, or if it suffices to check on stalks.

Many of the topological properties inquired about affine schemes can be asked about for general schemes. For example, general schemes need not be quasicompact, while all affine schemes are quasicompact. The reader should think of examples of schemes that are:

(dis)connected, (ir)reducible, quasicompact, Neotherian

2.4.1 Locality

Return to this discussion, because being stalk-local and affine-local must be made more precise

If X is a scheme, it can be covered by affine schemes. Like manifolds, there can be multiple different coverings of X. Translating between these different covering shall be given by the *Affine Communication Lemma*. Given this lemma, there shall be properties for schemes that need only be checked on some affine cover, giving rise to affine-local properties.

Lemma 2.4.1: Build-up Affine Communication

Let $\text{Spec}\,(R), \text{Spec}\,(S)$ be two affine sub-schemes of a scheme X. Then $(\text{Spec}\,(R)) \cap (\text{Spec}\,(S))$ is the union of open sets that are distinguished open sets for both $\text{Spec}\,(R)$ and $\text{Spec}\,(S)$.

Proof **:**
Let $[\mathfrak{p}] \in \text{Spec}\,(R) \cap \text{Spec}\,(S)$. As these are affine scheme, their intersection is the finite union of affine schemes, namely they are given through the appropriate distinguished open sets (recall the proof of proposition 2.3.8). Then there exits an $f \in R$ and $g \in S$ such that $[\mathfrak{p}] \in \text{Spec}\,(R_f), \text{Spec}\,(S_g) \subseteq \text{Spec}\,(R) \cap \text{Spec}\,(S)$. Then $g \in \mathcal{O}_X(\text{Spec}\,(S))$ restricts to an element $g' \in \mathcal{O}_X(\text{Spec}\,(R_f))$.

Now, the points of $\text{Spec}\,(R_f)$ where g vanishes are precisely the points of $\text{Spec}\,(R_f)$ where g' vanishes, hence:

$$\begin{aligned}\text{Spec}\,(S_g) &= \text{Spec}\,(R_f) \setminus \{[\mathfrak{p}] \mid g' \in \mathfrak{p}\} \\ &= \text{Spec}\,((R_f)_{g'})\end{aligned}$$

Defining g'' where $g' = g'/f^n$ (note $g'' \in R$), then

$$\text{Spec}\,((R_f)_{g'}) = \text{Spec}\,(R_{fg''})$$

which gives the desired set.

Theorem 2.4.2: Affine Communication Lemma

Let X be a scheme, and let P be a property that is satisfied by some affine open subsets of X set, say $\text{Spec}\,(R^{(i)})$ for some index i, where $X = \cup_i \text{Spec}\,(R^{(i)})$. Then if

1. for each $f \in R^{(i)}$, $\text{Spec}\,(R^{(i)}_f \hookrightarrow X$ satisfies the property too
2. If $R^{(i)} = (r_1, ..., r_n)$ and $\text{Spec}\,(R^{(i)}_{r_i}) \to X$ satisfies the property, then so does $\text{Spec}\,(R^{(i)}) \to X$

Then X satisfies this property.

Proof **:**
Let $\text{Spec}\,(R) \subseteq X$ be an affine subscheme. As $\text{Spec}\,(R)$ is quasi-compact, by lemma 2.4.1 cover it with a finite number of distinguished open sets $\text{Spec}\,(R_{f_i})$, each of which is distinguished in some $\text{Spec}\,(R_i)$. By (1), each $\text{Spec}\,(R_{f_i})$ has property P, and by (2) $\text{Spec}\,(R)$ has property P, as we sought to show.

Definition 2.4.3: Afine Local and Stalk Local

Let X be a scheme. Then if a collection of open sets satisfy theorem theorem 2.4.2, the property is said to be *affine local*.
If a property P can be checked for X by checking it on each stalk, then the property is said to be *stalk-local*. In particular, a property P is stalk local when X satisfies P if and only if all the stalks of X satisfies P.

Affine local does not imply stalk local, which we shall show is the case with an example that satisfies the following property

Definition 2.4.4: Noetherian Scheme

Let X be a scheme. If X can be covered by affine open sets where each ring R_i of $\operatorname{Spec}(R_i)$ is Noetherian, then X is said to be a *locally Noetherian Scheme*. If X is also quasicompact, then X is said to be a *Noetherian Scheme*.

Recall that If R is a Noetherian ring, then it's localization and extension (via exact sequence) is Noetherian. Conversely, if $R = (f_1, ..., f_n)$ and each R_{f_i} is Noetherian, so is R.

Example 2.17: Affine Local But Not Stalk Local
Take $X = \operatorname{Spec}\left(\prod_i^\infty \mathbb{Z}/2\mathbb{Z}\right)$ (that is, infinite copies of a single point). This scheme is stalk-locally Noetherian, as the only prime ideals are those for which when taking the quotient the result is an integral domain (in this case, the quotient will always be isomorphic to a field, and hence every prime must be maximal). However, the ring is *not* Noetherian as can readily be identified, and hence it is not locally-affine Noetherian.

The converse is true

Proposition 2.4.5: Stalk Local, Then Affine Local

Let X be a scheme for which property P stalk local. Then property P is affine-local

Proof **:**
return to this

2.4.2 Important Types of Schemes

The following establishes some finiteness conditions on the intersection of important open subsets:

Definition 2.4.6: Quasi-separated Scheme

Let X be a scheme. Then if the intersection of any two quasicompact open subsets is quasicompact then X is said to be *quasi-separated*.

Proposition 2.4.7: Characterizing Quasi-separated Schemes

Let X be scheme. Then X is quasi-separated if and only if the intersection of any two affine open subsets is a finite union of affine open subsets

Proof **:**
This is very similar to a compactness argument and is left as an exercise.

Proposition 2.4.8: Affine Scheme and Quasiseparatedness

Let X be an Afine scheme. Then X is quasi-separated

Proof **:**
The intersection of affine open subsets can be covered by finitely many distinguished open subsets. Use proposition 2.4.7.

A large family of schemes that are quasi-separated and quasicompact are projective R-schemes, and as we shall see many schemes we shall be working with can be embedded into a projective R-scheme, and hence these two finiteness properties will be very common. There do exist non quasi-separated schemes:

Example 2.18: Non-quasiseparated Scheme
Let $X = \mathrm{Spec}\,([x_1, x_2, ...]$ and let $U = X \setminus [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, ...)$. Take two copies of X and glue along U to create a double-origin scheme, namely the infinite version of the double-origin scheme seen in example 2.15.Show this is not quasiseparated.

One property that is a little unconventional is that there exists schemes that have no closed points[12]. Fortunately, all quasicompact scheme have closed points, or even better

Proposition 2.4.9: Quasicompact Schemes and Closed Points

Let X be a quasicompact scheme. Then X contains closed points. Furthermore, the closure of any non-closed point contains closed points.

Proof **:**
First, if X is affine, it certainly contains closed points as $\mathcal{O}_X(X) = R$ contain at least one maximal ideal. Furthermore, the closure of any non-closed point would contain maximal ideal as any prime ideal is contained in a maximal ideal.

Now, let X be some quasicompact scheme. Let $X = \bigcup_i^n U_i$ be a finite covering of affine open subsets. Take any closed point $\mathfrak{p}_1 \in U_1$. If it is closed in X we're done. If not, take its closure $\overline{\mathfrak{p}_1}$ in X. As $\mathfrak{p}_1$ is not closed in X, its closure contains more than $\mathfrak{p}_1$, call this point $\mathfrak{p}_2$, and without loss of generality say it is in U_2. If $\mathfrak{p}_2$ is closed in X, we are done. If not, take $\overline{\mathfrak{p}_2}$ in X. Its closure must contain a point $\mathfrak{p}_3$ that is not in U_2 or U_1 (as $\mathfrak{p}_3$ would be in the closure of $\mathfrak{p}_1$ or $\mathfrak{p}_2$ in U_1 and

[12]see `https://math.stanford.edu/~vakil/files/schwede03.pdf`

U_2 respectively). Continue this process until a point is closed, or if we get to $\mathfrak{p}_n \in U_n$, then the closure of this point can't contain any other point in $U_1, ..., U_n$, making $\mathfrak{p}_n$ closed and completing the proof.

Note that $\mathfrak{p}_n$ is in the closure of $\mathfrak{p}_1$ in X: by a topological argument it is easy to show that:

$$\overline{\mathfrak{p}_1} \supseteq \overline{\mathfrak{p}_2} \supseteq \cdots \supseteq \overline{\mathfrak{p}_n} = \mathfrak{p}_n$$

and hence the closure of any non-closed point contains a closed point.

Note that this does not mean that the closed points are *dense*, that is it does not imply that the closure of the collection of closed point is the entire space. Such schemes are called *Jacobson schemes* and are related to *Jacobson rings* and *Jacobson radicals* (see [3, chapter 20.3]).

Recall in proposition 2.3.18 it was mentioned that invertibility can be determined if a function is non-zero at every point, however a function being zero at all points requires more care to study. Part of this is because this is true for quasicompact schemes

Proposition 2.4.10: Quasicompact Schemes and Vanishing Function

Let X be a quasicompact scheme. Then if $f \in \mathcal{O}_X(X)$ vanishes at all points in X, then there exists an n such that $f^n = 0$.

***Proof* :**
Say $f([\mathfrak{p}]) = 0$ so that $f \in \mathfrak{p}$ for all $\mathfrak{p}$. Then on $U_i \cong \operatorname{Spec} R_i \subseteq X$, $f \in \sqrt{R_i}$, implying $f^{n_i} = 0$ for some n. Cover X with these open sets, and by Quasicompactness we only require finitely many such sets, where in each set we have an f_i and n_i such that $f_i^{n_i} = 0$. As X is a scheme, on intersection $U_i \cap U_j$ we have $g_{i,j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ and as ring homomorphisms must map 0 to 0, $g_{i,j}^{\max n_i, n_j} = 0$. Take $n = \max_i n_i$ so that for each intersection $U_i \cap U_j$:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} = g_{i,j} \qquad g_{i,j}^n = 0$$

Now, as these agree on intersections and cover X, by definition there must exist a function $f \in \mathcal{O}_X(X)$ such that $f|_{U_i} = f_i$. Then as $\operatorname{res}_{X,U_i}(f^n) = \operatorname{res}_{X,U_i}(f)^n = f_i^n f_i^{n_i n'} = 0 f^{n'} = 0$, we have:

$$(f^n)|_{U_i} = (f|_{U_i})^n = 0$$

which implies $(f^n)|_{U_i} = 0 \in R_i$. Now, by corollary 1.1.17, the map:

$$\varphi : \mathcal{O}_X(X) \to \prod_i \mathcal{O}_X(U_i) \cong \prod_i R_i$$

is injective. As $\varphi(f^n) = (0, ..., 0)$, and we know that $\varphi(0) = (0, ..., 0)$, it must be that $f^n = 0$, completing the proof.

This is *not* necessarily true for a general scheme, namely since we can have nilpotence of every degree:

Example 2.19: Non-vanishing Function On Scheme

Take

$$X = \coprod_{n\geq 1} \operatorname{Spec}\left(k[x]/(x^n)\right)$$

Then the global function ϵ is zero every-where, namely there is only one point to evaluate to, but essentially by definition there cannot exist an n such that $\epsilon^n = 0$.

A ring with no nilpotent elements is reduced. A scheme that has this property for every section is given a name:

Definition 2.4.11: Reduced Scheme

Let X be a scheme. Then X is a *reduced scheme* if $\mathcal{O}_X(U)$ is reduced for every open set U of X

Proposition 2.4.12: Reducedness Is Stalk-local

Let X be a scheme. Then X is reduced if and only if non of the stalks have nonzero nilpotents. As a consequence, if $f, g \in \mathcal{O}_X(X)$ are two (global) functions and agree on all points, then $f = g$

Proof **:**
Recall the injective embedding $\mathcal{O}_X(U) \hookrightarrow \prod_{p\in U} \mathcal{O}_{X,p}$ and an important characterization of the nilradical.

Lemma 2.4.13: Reducedness For Affine And Projective Schemes

Let R be a reduced ring. Then $\operatorname{Spec}(R)$ is reduced. Hence, $\mathbb{A}^n$ and $\mathbb{P}^n$ are reduced.

Proof **:**
exercise.

Proposition 2.4.14: Reducedness On Quasicompact Schemes

Let X be a quasicompact scheme. Then X is reduced if and only if it is reduced at closed points

Proof **:**
Assume it is reduced at all closed points. As X is quasi-compact, the closure of each non-closed point contains a closed point, $y \in X$ is non-closed and $x \in X$ is a closed point contained in its closure. Then $\mathcal{O}_{X,y}$ is the localization of $\mathcal{O}_{X,x}$. As localization preserves reducedness, $\mathcal{O}_{X,y}$ must be reduced, completing the proof.

The above showed that reducedness for quasicompact schemes is a “closed-point” local condition. It is not an open set local condition. Furthermore, it is not enough to check the global sections:

Example 2.20: Reducedness Edge Cases

- The scheme given by:

$$X = \operatorname{Spec} \frac{\mathbb{C}[x, y_1, y_2, ...]}{(y_1^2, y_2^2, ..., (x-1)y-1, (x-2)y_2, ...)}$$

  is nonreduced at the closed points corresponding to the positive integers. The complement of this set is *not* Zariski open.

  The key is that if X is also locally Noetherian, then reducedness is a locally open condition.

- Another mistake that is easy to make: let X be scheme where the global sections is reduced. This doesn't imply X is reduced (take the scheme cut out by $x^2 = 0$ in $\mathbb{P}^2_k$; note that the global section will be k, but X will be non-reduced).

One of the most important type of reduced ring is an integral domain. If a scheme is an integral domain for each section, it is given a name:

Definition 2.4.15: Integral Scheme

Let X be a scheme. Then X is *integral* if it is nonempty and $\mathcal{O}_X(U)$ is an integral domain for every nonempty open subset $U \subseteq X$.

Proposition 2.4.16: Characterizing Integral Schemes

Let X be a scheme. Then X is an integral scheme if and only if it is irreducible and reduced

Thus, integral schemes can be thought of as schemes that are in "one piece" and have "no fuzz".

Proof **:**
Recall that an affine scheme is irreducible if and only if the nilradical is prime.

Proposition 2.4.17: Characterizing Integrality in Noetherian Schemes

Let X be a Noetherian Scheme. Then X is integral if and only if X is nonempty, connected, and all stalks $\mathcal{O}_{X,\mathfrak{p}}$ are integral domains.

Proof **:**
exercise.

Recall that the product of integral domain is not an integral domain. Similarly, the disjoint union of integral scheme need not be integral. This global behavior is cannot be captured through stalks, as the reader should verify, and hence integrality is not stalk-local.

As integral schemes are irreducible, it has a generic point, say γ. Then if $\operatorname{Spec}(R) \subseteq X$ is a nonempty affine open subset, the stalk at γ (i.e. $\mathcal{O}_{X,\gamma}$) is isomorphic to $\operatorname{Frac}(R)$. This set is given a name:

Definition 2.4.18: Rational Field

Let X be an integral scheme. Then the *function field* is the collection $\text{Frac}(R)$ for any R given by $\text{Spec}\,(R) \subseteq X$, and it is often denoted $k(X)$.

Notice this can be computed on any nonempty open subset of X, and hence on irreducible schemes there is a straightforward way of defining a notion of rational functions on X!

Proposition 2.4.19: Integral Scheme: Restriction Maps An Function Field

Let X be an integral scheme. If $U \subseteq V$ where $V \neq \emptyset$, then

$$\text{res}_{U,V} : \mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V)$$

is an inclusion. Furthermore, given any choice of $\text{Spec}\,(R) \subseteq X$, there is always an inclusion:

$$\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,\eta} \cong k(R)$$

where η is the generic point of X.

Proof **:**
exercise.

Thus, integral schemes have a similar property to that of classical affine and projective varieties, namely that their sections over different open sets can be considered as subsets of a single ring! The restriction maps are always inclusion (on non-empty sets) and gluing is easy as we can see that two elements are glued if and only if they are the same element of $k(X)$.

Exercise 2.4.1

1. Let X be a scheme. Show that X is a locally Noetherian scheme if and only if every open affine subset corresponds to a Noetherian ring.

Normality and Factoriality

Recall that a normal ring R has the property that it an integral domain that is is integrally closed in it's field of fractions, $\overline{R}^{\text{Frac}(R)} = R$. We shall take the time to develop the theory of Schemes for which all the rings of sections are normal. These shall correspond to schemes that have some nicer behaviour for their singularities. It can be thought of as a condition weaker than smoothness (smoothness shall be called regular in section REF:HERE) as normality shall still allow for singularities, however these singularities shall be more tame. In particular, the reader can intuitively have the following geometric picture in mind for subschemes of Noetherian Schemes:

1. In codimension 1, all singularities are resolved, meaning it is "smooth" in codim 1
2. In codimension ≥ 2, being normal means being nice enough so that rational functions defined on the compliment of the singularities can be extended to the singularity (i.e. Algebraic Hartogs's lemma applies)

Definition 2.4.20: Normal Scheme

Let X be a scheme. Then X is said to be *normal* if every stalk $\mathcal{O}_{X,\mathfrak{p}}$ is normal domains.

Proposition 2.4.21: Normal Scheme Stalk Local

Let X be a scheme. Then X is a normal scheme if and only if $\mathcal{O}_{X,\mathfrak{p}}$ is normal for each $\mathfrak{p} \in X$

***Proof* :**
Recall (or reprove) that normality is preserved when localizing (hint: recall the proof that a UFD is normal)

Note that reducedness was stalk-local, and hence normal schemes are reduced. If each stalk is a UFD, then it is immediately a Normal scheme. Let's give a name to such a scheme

Definition 2.4.22: Factorial Scheme

Let X be a scheme. Then X is said to be *normal* if every stalk $\mathcal{O}_{X,\mathfrak{p}}$ is a UFD.

If R is a UFD, then $\operatorname{Spec} R$ is factorial (by the above reasoning), however the converse need be true! Namely, we may have factorial schemes where the stalks are all UFD's but the global sections do not form a UFD. A quintessential example will be the spectra of elliptic curves.

Example 2.21: Normal Schemes

1. The schemes $\mathbb{A}^n_k, \mathbb{P}^n_k$ and $\operatorname{Spec} \mathbb{Z}$ are normal. If R is normal, then $\mathbb{A}^1_R$ is naturally normal.
2. Go over some number theory examples, or Vakil p.165

2.4.3 Associated Points

To motivate the concept, first do the followign exercise:

Example 2.22: Support Of Non-reduced Ring

Let $X = \operatorname{Spec}(k[x,y]/(y^2, xy))$. Show that the suppport of any function X is either empty, the origin, or all of X (hint: for an example, consider the function y). Note that the function y evaluated to 0 everywhere on X, but the support is only the origin!

This will be linked to the irreducible components of the above X, namely the x-axis and the origin. This gives the following definition tha shall be expanded upon:

Definition 2.4.23: Associated Points

Let X be a scheme. Then the associated points are precisely the generic points of irreducible components of the support of some element of X

Proposition 2.4.24: Affine Integral Schemes And Associated Points

Let R be an integral domain. Then the generic point is the only associated point of $\operatorname{Spec}(R)$.

Proof **:**
exercise

Finish this here, p.167

Exercise 2.4.2

1. Find the affine scheme structure for the following rings:
   a) $\operatorname{Spec}\mathbb{C}[x]/(x^3 - x^2)$
   b) $\operatorname{Spec}\mathbb{C}[x]/(x^2 - d)$
   c) $\operatorname{Spec}\mathbb{C}[x]/(x^2 + 1)$
2. Let X be a quasicompact scheme. Show that the closure of each point contains a closed point (this is not true for non quasicompact schemes in general)
3. Let X be a quasicompact scheme. Suppose f vanishes at all point of X. Then there exists some N st $f^n = 0$. Show that this fails when X is not quasicompact (hint: take infinite disjoint unions of $\operatorname{Spec}(R_n)$ where $R_n := k[x]/(x^n)$)
4. Let X be a quasicompact scheme. Then it suffices to cheek reducedness at closed points (show that any nonreduced point has a nonreduced closed point in its closure)
5. Show that reducedness is not in general an open condition by considering:
$$X = \operatorname{Spec}\left(\frac{\mathbb{C}[x, y_1, y_2, ...,}{(y_1^2, y_2^2, y_3^3, ..., (x-1)y_1, (x-2)y_2, (x-3)y_3), ...}\right)$$
and $Y = \operatorname{Spec}[x]$. Show the nonreduced points of X are the closed points corresponding to the positive integers. The compliment of this set is *not* Zariski open.
6. Let X be an integral scheme. Show that if $\emptyset \neq V \subseteq U$, then the map $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is an inclusion. very important, see p.156 under exercise
7. Let X be an integral scheme, γ the generic point, and $\operatorname{Spec}(R) \cong U \subseteq X$ is any nonempty affine open subset of X. Then $\mathcal{O}_X(U) \to \mathcal{O}_{X,\gamma} = K(A)$ is an inclusion. Conclude that gluing is easy: every f_i in U_i for an open cover for U glue together if they are same element in $\operatorname{Frac}(X)$ (recall that $\operatorname{Frac}(X)$ is defined to be $\operatorname{Frac}(R)$ where R is given by $\mathcal{O}_X(\operatorname{Spec}(R))$
8. Show that integrality is not stalk local via the example $\operatorname{Spec}(A) \amalg \operatorname{Spec}(B) = \operatorname{Spec}(A \times B)$.
9. Show that reducedness is an affine local property.
10. Let X be a locally Noetherian Scheme. Then X is quasiseparated.
11. Let X be a Noetherian Scheme. Then it has a finite number of connected components, each a finite union of irreducible components.
12. Show that the ring of sections of a Noetherian Scheme need not be Noetherian.

13. Let X be a scheme of locally finite type over $\mathbb{Z}$ or k. Then X is locally Noetherian.
14. A scheme that is of finite type over any Noetherian ring is Neotherian.
15. Let X be a quasiprojective R-scheme. Then if R is Noetherian, X is a Noetherian scheme, and hence has a finite number of irreducible components.
16. Let X be a locally Noetherian scheme. Then the reduced locus is open. (Vakil p.156)
17. Let $U \subseteq X$ be an open subscheme of a projective R-scheme X. Show that U is locally of finite type R
18. Find a counter-example showing that normal schemes are not in general integral schemes (hint: think of the disjoint union of two normal scheme. For more hint, Vakil p. 162).
19. Let X be a Noetherian scheme. Then X is normal if and only if if is a finite disjoint union of integral normal schemes.

2.5 Dimension

The reader may want to review [3, chapter 20.5] for an in depth look at the algebraic counter-part. Recall that the Krull dimension of a [commutative] ring R is the supremum of the proper chain of prime ideas with indexing starting at 0. Geometrically, for $\operatorname{Spec} R$, and topological spaces more generally (including schemes), its dimension is the supremum of proper chains of closed irreducible subsets with indexing starting at 0. Naturally, $\dim \operatorname{Spec} R = \operatorname{kdim} R$. Also naturally,

$$\dim \operatorname{Spec} R = \dim \operatorname{Spec} R/\sqrt{(0)}$$

If the space is is reducible, then the dimension can vary (for example $\mathbb{A}^1 \cup \mathbb{A}^2 \subseteq \mathbb{A}^3$ interpreted as the x-axis union the yz-plane), and hence we usually work with irreducible sets. If it is pessary, we shall say that a space has *pure dimension* if all of it's irreducible components have the same dimension. Dimension respects subsets: if $X \subseteq Y$ then $\dim X \subseteq \dim Y$. If a space is 1 dimensional, it is called a curve. If it is 2-dimensional, a surface, and ≥ 3 is sometimes generally referred to as an n-fold.

Lemma 2.5.1: Dimension of Open Subset

Let $U \subseteq X$ be an open subset. Then there is a bijection between irreducible closed subsets of U and the irreducible closed subsets of X meeting U.

Proof **:**
exercise

Proposition 2.5.2: Dimension of Scheme

Let X be a scheme. Then X is of dimension n if and only if it admits an open cover of affine open subsets of dimension at most n, where equality must be achieved by at least one affine open subset.

Proof **:**
If X is n-dimensional, certainly each open subset in an affine cover is dimension $\leq n$. For the converse, we may take the contra-positive by supposing $\dim X \neq n$. It certainly can't be less, for at least one $\dim U_i = n$, which then implies $n = \dim U_i = \dim X \cap U_i \geq \dim X < n$, but $n < n$ is a contradiction, so suppose $\dim X > n$. Then there exists a chain of irreducible closed subsets:

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subsetneq X_{n+1}$$

Then given an affine cover, it must be that $X_0 \cap U_i \neq \emptyset$ for at least one open set in the cover. Then by lemma 2.5.1, there is a bijective correspondence between the irreducible closed subsets of U_i and the irreducible closed subsets of X meeting U_i. As $X_0, ..., X_{n+1}$ meet U_i and the correspondence is bijective:

$$X_0 \cap U_i \subsetneq X_1 \cap U_i \subsetneq \cdots \subsetneq X_n \cap U_i \subsetneq X_{n+1} \cap U_i$$

However this is a contradiction as $\dim U_i \leq n$, meaning the last $\subsetneq$ can't be proper. Then this establishes the if and only if, completing the proof.

Lemma 2.5.3: Integral Extensions and Dimension

Let $f : \operatorname{Spec} R \to \operatorname{Spec} S$ correspond to an integral extension. Then

$$\dim \operatorname{Spec} R = \dim \operatorname{Spec} S$$

Proof **:**
use going-up theorem and going-down theorem.

2.5.1 A word on Noetherian Scheme

It is *not* the case that a Noetherian scheme needs to be finite-dimensional: Nagata came up with a counter-example (it is outlined in Vakil in chapter 11 in the first section).

Noetherian Local rings will have to be finite dimensional, and so points of locally Noetherian schemes must have finite codimension.

We are often concerned with spaces that are a few dimensions lower than the ambient space X. As schemes can be reducible, we can have a codimension object that have lower dimensional components. To avoid this, we shall limit codimensions to irreducible subschemes:

Definition 2.5.4: Codimension

Let X be a scheme and $Y \subseteq X$ an irreducible subscheme. Then:

$$\operatorname{codim}_X Y = \sup\{\text{proper chains of irreducible closed subsets starting at } \overline{Y}$$

where indexing starts at 0.

Naturally, for $\operatorname{Spec} R/\mathfrak{p} \subseteq \operatorname{Spec} R$, then the codimension of $\operatorname{Spec} R/\mathfrak{p}$ corresponds to the supremum of lengths of decreasing proper chains of prime ideals starting at $\mathfrak{p}$ with indexing starting at 0: this is called the *height* of $\mathfrak{p}$ (note that $\text{codim}_{\text{Spec}\,R}\text{Spec}\,R/\mathfrak{p} = \text{kdim}R_{\mathfrak{p}}$). For example, the generic point η as a subset of $\text{Spec}\,\mathbb{Z}$ has codimension 0, while any other point has codimension 1. If Y is codimension 0 in X, it must be an irreducible component.

Definition 2.5.5: Hypersurface

Let $Y \subseteq X$ be an irreducible closed subscheme. Then Y is called a *hypersurface* if $\text{codim}_X Y = 1$.

Lemma 2.5.6: Sum of Dimension and Codimension - Scheme

Let X be a scheme and $Y \subseteq X$ an irreducible subscheme. Then:

$$\text{codim}_X Y + \dim Y \leq \dim X$$

Proof **:**
this is a simple topological argument

The equality holds for varieties (see ref:HERE), but does not hold in general; consider the following scheme (image from Vakil)

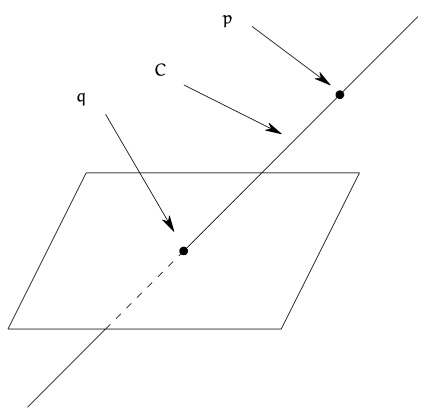


Figure 2.8: Codimension Pathology

The scheme is dimension 2, the point p is dimension 0, but codimension 1. The point q is codimension 2 with no codimension 1 subset in-between.

Working the Krull dimension can be difficult. It is thus useful to remember the following equating the transcendental dimension and krull dimension. First, recall the following definition:

Definition 2.5.7: Transcendental Dimension

Let A be a k-algebra. Then the *transcendental dimension* of A is the transcendental degree of $\text{Frac}(A)$ over k, where $A \cong k[u_1, ..., u_n]$ for appropriate (not necessarily transcendental) generators.

Theorem 2.5.8: Transcendental and Krull Dimension Equality

Let $\operatorname{Spec} A$ be an affine k-scheme where A is a finitely generated integral domain over k. Then:

$$\dim \operatorname{Spec} A = \operatorname{trdeg} \operatorname{Frac}(A)/k$$

Thus, if X is an irreducible k-variety, then

$$\dim X = \operatorname{trdeg} k(X)/k$$

Proof **:**
For the first part, see [3, chapter 21.5]. The second part follows when interpreting variety in the classical sense, and follows for the general sense by theorem ref:HERE (the correspondence between **Var** and a subcategory **Sch**)).

Using this result, we can prove the equality of lemma 2.5.6 using transcendental degree. We shall also requires a few more concepts, one important one establishes which rings "strongly" connect the dimension using prime ideals and transcendental elements:

Definition 2.5.9: Catenary Rings

Let R be a commutative ring. Then R is said to be *catenary* if for every nested chain $\mathfrak{q} \subseteq \mathfrak{p} \subseteq R$ of prime ideals, the maximal chains prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$ all have the same length.

Many rings are catenary rings: localization of finitely generated $\mathbb{Z}$-algebras, complete Noetherian local rings, Dedekind domains, and so forth (see exercise 2.5.1). Importantly, localization of finitely generated k-algebras are catenary (which you need to prove for the following)

Theorem 2.5.10: Sum of Dimension and Codimension - locally finite k-Scheme

Let X be a locally finite k-scheme, $Y \subseteq X$ an irreducible subset, and η the generic point of Y. Then:

$$\operatorname{codim}_X Y + \dim Y = \dim X$$

equivalently:

$$\dim Y + \dim \mathcal{O}_{X,\eta} = \dim X$$

Proof **:**
exercise (you may want to reference [3, chapter 21.5] for some useful algebraic results, Or see Vakil p.311 for a guided exercise).

2.5.1 Krull's Height Theorem and Hartog's lemma

In this section, we shall show how on some nice schemes, the number of equations gives an upper bound on the codimension. This shall give us some nice algebraic computational tools to determine, or at least give information, on what type of objects equations cut out.

Hartog's Lemma

(Cool for bringing in complex analysis into algebraic geometry)

move this to where it's appropriate

Lemma 2.5.11: UfD, Codimension 1 then Principal

Let R be a UFD. Then if $\mathfrak{p} \subseteq R$ has codimension 1, it is principal

Proof **:**
good ring theory reminder exercise.

Exercise 2.5.1

1. some of these questions should eventually be moved to their appropriate section
2. Show that a Noetherian scheme of dimension 0 has finitely many points
3. Let $f : X \to Y$ be an integral morphism. Then the dimension of fibers are 0.
4. If $\nu : \tilde{X} \to X$ is the normalization of a scheme, then $\dim \tilde{X} = \dim X$
5. If K/k is an algebraic field extension and X a k-scheme, $X_k = X \times_k K$ the base change, then $\dim X = \dim X_K$, in particular, $\dim X = n$ if and only if $\dim X_K = n$.
6. Generalize the above result for arbitrary field extensions K/k.
7. Show the following rings are catenary: localization of finitely generated $\mathbb{Z}$-algebras, complete Noetherian local rings, Dedekind domains.

2.5.2 Regular and Smooth schemes

(I'm considering Vakil material optional over Hartshorne rn, so I'll cover this in detail later)

Definition 2.5.12: Tangent Space

Let X be a scheme and $x \in X$ a point Then the *tangent space* at x is given by

$$\mathfrak{m}_x/\mathfrak{m}_x^2$$

where $\mathfrak{m}_x \subseteq R_x$ R is given by an affine local neighborhood of x, $x \in \operatorname{Spec}(R) \subseteq X$.

Definition 2.5.13: Regular Local Ring

Let $(R, \mathfrak{m})$ be a local ring. Then R is regular if:

$$\dim R = \dim_{\mathrm{Frac}(R)} \mathfrak{m}/\mathfrak{m}^2$$

Definition 2.5.14: Regular Scheme

Let X be a scheme. Then X is *regular* if $\mathcal{O}_{X,\mathfrak{p}}$ is a regular local ring for all points $[\mathfrak{p}] \in X$.

3

Morphisms of Schemes

In this chapter, we shall properly define morphisms of schemes, classify many types of morphisms, and interpret many important types of morphisms.

3.1 Definition and Examples

The next goal is to define morphism's of schemes in such a way that ring homomorphisms and scheme morphism are dual to each other. Recall (Definition 1.4.2) that a morphism of ringed spaces $\pi : X \to Y$ is a continuous map of topological spaces along with the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$, i.e. the pushforward map, or equivalently by adjointness the map $\pi^{-1} : \pi^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. As we saw in example ref:HERE, not all spectra maps are induced by ring homomorphisms, and there is no natural way to induce a ring homomorphism from a spectra map. If we add a ringed space structure and get an affine scheme, then the global sections give a natural ring homomorphism. This also allows us to distinguish between affine schemes with different sheaf, or similarly have the same underlying affine scheme but different sheaf morphisms:

Example 3.1: Multiple ringed space maps for Spectra Map
Let $X = Y = \operatorname{Spec} k$ with $\operatorname{char} k = 0$ and $\operatorname{Gal}_{\mathbb{Q}}(k) \neq 0$. Let $f : \operatorname{Spec} k \to \operatorname{Spec} k$ be the identity. Then for each galois automorphism $\varphi \in \operatorname{Gal}_{\mathbb{Q}}(k)$, $f^\#_\varphi : k \to k$ is a distinct sheaf morphism.

As we want our definition to work well for geometric spaces like smooth manifolds, we may ask if a ringed space homomorphism between two affine schemes (or even schemes) induces a unique ring map. However, this is not quite enough: not all ringed space maps between spectra are induced by ring homomorphisms, and there can still be multiple ringed space maps associated to a ring homomorphism.

The key realization is that ring homomorphism induce some more local structure that needs to be preserved! By proposition 1.1.22, the stalks of a sheaf can determine a sheaf morphism (with suitable compatibility on open subsets). By proposition 2.3.14, the stalk of schemes are all local rings. The geometrical interpretation is that $\mathcal{O}_{X,x}$ remembers the information of functions defined around x and whether they evaluate to 0 or not. The unique maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ contains all function defined around x where $f(x) = 0$, and we may only evaluate at the point x[1] (hence why there can only be one maximal ideal, to evaluate at that point).

If $f : R \to S$ is a ring homomorphism and $\mathfrak{p} \subseteq S$, then we get a natural map $f_\mathfrak{p} : R_{f^{-1}(\mathfrak{p})} \to S_\mathfrak{p}$ given by $f_\mathfrak{p}(r/t) = f(r)/s$ where $s = f(f^{-1}(s))$ with $s \notin \mathfrak{p}$. Let $\mathfrak{m}_R$ and $\mathfrak{m}_S$ be the unique maximal ideals of both rings, and notice that $f_\mathfrak{p}(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$. Equivalently, we have $\mathfrak{m}_R \subseteq f_\mathfrak{p}^{-1}(\mathfrak{m}_S)$, and as $\mathfrak{m}_R$ is maximal it must be that $\mathfrak{m}_R = f_\mathfrak{p}^{-1}(\mathfrak{m}_S)$[2]. This need not be true for general homomorphisms between local rings (consider $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$), and so we give a name to such ring homomorphisms between local rings:

Definition 3.1.1: Local Ring Homomorphism

let $\varphi : R \to S$ be a ring homomorphism between local rings with maximal ideals $\mathfrak{m}_R$ and $\mathfrak{m}_S$ respectively Then φ is a *local ring homomorphism* if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

In particular, elements in R that are zero in the quotient must remain being 0 in the image. Now, given a ringed space morphism $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$, does it in general preserve the local-ring structure on stalks? The answer is no:

Example 3.2: Ringed Space Morphism Not Locally Ringed

Take $R = \mathbb{Z}_{(p)}$ and $S = \mathbb{Q}$ so that $\operatorname{Spec}(R) = \{[(p)], [(0)]\}$ and $\operatorname{Spec}(S) = \{[(0)]\}$. The open sets of $\operatorname{Spec}(R)$ are

$$\emptyset \qquad \{[(0)]\} \qquad \operatorname{Spec}(R)$$

and the stalks of $\operatorname{Spec}(R)$ are $\mathbb{Z}_{(p)}$ at (p) and $\mathbb{Q}$ at (0) respectively. The reader can verify that there is only one possible ring homomorphism $\iota : R \to S$; it must be the embedding.

Now, ι induces a morphism $\iota^a : \{(0)\} \hookrightarrow \{(0), (p)\}$:

$$\iota^a : \operatorname{Spec}(S) \to \operatorname{Spec}(R) \qquad\qquad \iota^a(0) = (0)$$

and $\iota^\# : \mathcal{O}_{\operatorname{Spec}(R)} \to (\iota^a)_*\mathcal{O}_{\operatorname{Spec}(S)}$. Now, to show the sheaf morphism is well-defined, we shall show that the map on stalks is well-defined, which suffices by proposition 1.1.22. Indeed, this map is determined by its stalk at 0 and p:

$$\iota_0^\# : \mathbb{Z}_{(p)} \to \mathbb{Q}$$
$$\iota_p^\# : \mathbb{Q} \to \mathbb{Q}$$

Now, we can also define a map φ that maps $(0) \mapsto (p)$. The pre-image of every open set (notably

[1] of course, we can evaluate at the generic point to get back the function as well, but we currently interested in the non-trivial behavior

[2] Note this does not imply $\varphi(\mathfrak{m}_R) = \mathfrak{m}_S$, can you think of a counter-example

the pre-image of zero is now the empty-set). Then the we need the stalk maps:

$$\varphi_0^{\#} : \mathbb{Z}_{(p)} \to \mathbb{Q}$$
$$\varphi_p^{\#} : \mathbb{Q} \to \mathbb{Q}$$

which is well-defined by being an inclusion map. Hence we have another distinct ringed-space morphism. However, we know there is only one ring homomorphism, and hence we managed to produce an extra map.

In general, a ring map $R \to S$ will induce a ringed space map $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ which induces a ring map $R \to S$, however the first mapping is in general injective but *not* surjective, while the second mapping is surjective but *not* injective (example 3.2 being one of the canonical examples). We require that the maps between stalks be a local ring homomorphism, and as we want this definition to work well with other geometric objects like manifolds, we define them for ringed spaces:

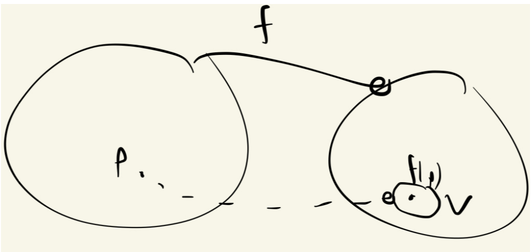
Definition 3.1.2: Locally Ringed Space Morphism

Let X, Y be locally ringed spaces. Then a *morphism of locally ringed spaces* $(\pi, \pi^{\#}) : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks $\pi^{\sharp} : \mathcal{O}_{Y,q} \to \pi_* \mathcal{O}_{X,p}$ (where $\pi(p) = q$) are local ring homomorphisms.

The geometry behind this is better motivated when considering a continuous function between topological spaces $f : X \to Y$. Say $p \mapsto q = f(p)$ and that there is a function g on an open subset $V \subseteq Y$ that vanishes on q: $g(q) = 0$. Then we want that the pullback of g via f to vanish at p, namely:

$$g(f(p)) = 0 \qquad \text{if and only if} \qquad (g \circ f)(p) = 0$$

Visually, we can see this as:



This is "obvious" when g, f are our usual functions since $g(f(p)) = (g \circ f)(p)$, however the above example shows how this is not necessarily true for ringed space morphism, precisely because the domain we are evaluating at differs for g and $g \circ f$. To give a concrete demonstrate, consider again:

$$f : \mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}_{(p)} \qquad f([0]) = [p]$$

with $f^{\#} : \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \to f_* \mathcal{O}_{\mathrm{Spec}\,\mathbb{Q}}$, in particular for each open subset:

$$f^{\#}(V) : \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}_{(p)}}(V) \to \mathcal{O}_{\mathrm{Spec}\,\mathbb{Q}}(f^{-1}(V))$$

As $\operatorname{Spec} \mathbb{Z}_{(p)}$ has two non-empty open subsets

$$f^{\#}(\{[0],[p]\}) : \mathbb{Z}_{(p)} \to \mathbb{Q}$$
$$f^{\#}(\{[0]\}) : \mathbb{Q} \to T$$

where T is the trivial ring where $0 = 1$, since $f^{-1}([0]) = \emptyset$, and $\mathcal{O}_{\operatorname{Spec} \mathbb{Q}}(\emptyset)$ is the terminal object, which for **Ring** is the ring with a single element where $0 = 1$. The 2nd map is the obvious surjection, and the first map must be the identity map.

Now, take the function $p \in \mathcal{O}_{\operatorname{Spec} \mathbb{Z}_{(p)}}(\operatorname{Spec} \mathbb{Z}_{(p)})$. Then:

$$p([p]) = 0 \in \operatorname{Frac}(\mathbb{Z}_{(p)}/(p)) = \mathbb{Q}$$

Then we may "pullback" this function via the map $f^{\#}$:

$$f^{\#}(\operatorname{Spec} \mathbb{Z}_{(p)})(p) = p \in \mathbb{Q} = \mathcal{O}_{\operatorname{Spec} \mathbb{Q}}(f^{-1}(\operatorname{Spec} \mathbb{Z}_{(p)})$$

Then the function p, at the point $f^{-1}([p]) = [0]$ evaluates to:

$$p([0]) = p \in \mathbb{Q} \in \mathbb{Q} = \operatorname{Frac}(\mathbb{Q}/(0))$$

But notice that it no longer evaluates to 0!! Hence, if we consider general ringed space morphisms between spectra, evaluation and composition do not behave well together, we *require* the extra condition on stalks to make it hold!

Now, we know that ring homomorphisms induce local homomorphisms on stalks. Let us use this information to construct ring homomorphism between the local sections defined on distinguished open sets (from which we can readily define ring homomorphisms on local section over any open subset). Let $f : R \to S$ be a ring homomorphism and $f^a : \operatorname{Spec} S \to \operatorname{Spec} R$ be the induced spectrum map. We shall show that we get a map:

$$\mathcal{O}_{\operatorname{Spec}(S)}(D(g)) \to \mathcal{O}_{\operatorname{Spec}(R)}\left((f^a)^{-1}(D(g))\right) \stackrel{!}{=} \mathcal{O}_{\operatorname{Spec}(R)}\left(D(f(g))\right)$$

where $\stackrel{!}{=}$ is the desired equality to make this map well-defined and independent of choice of g. Indeed:

$$\begin{aligned}
(f^a)^{-1}(D(g)) &= \{[\mathfrak{p}] \in \operatorname{Spec}(S) \mid [f^a([\mathfrak{p}])] \in D(g)\} \\
&= \{[\mathfrak{p}] \in \operatorname{Spec}(S) \mid g\left([f^a([\mathfrak{p}])]\right) \neq 0\} \\
&= \left\{[\mathfrak{p}] \in \operatorname{Spec}(S) \;\middle|\; g \bmod f^{-1}(\mathfrak{p}) \neq 0 \in \kappa\left(f^{-1}(\mathfrak{p})\right)\right\} \\
&\stackrel{!}{=} \{[\mathfrak{p}] \in \operatorname{Spec}(S) \mid f(g) \bmod \mathfrak{p} \neq 0 \in \kappa(\mathfrak{p})\} \\
&= \{[\mathfrak{p}] \in \operatorname{Spec}(S) \mid f(g)([\mathfrak{p}]) \neq 0\} \\
&= D(f(g))
\end{aligned}$$

To explain the $\stackrel{!}{=}$ equality, let us focus on $\kappa(f^{-1}(\mathfrak{p}))$ and $\kappa(\mathfrak{p})$. First, notice that:

$$\kappa(f^{-1}(\mathfrak{p})) = \operatorname{Frac}(\mathcal{O}_{\operatorname{Spec} R, f^{-1}(\mathfrak{p})}) = R_{f^{-1}(\mathfrak{p})}/\mathfrak{m}_{f^{-1}(\mathfrak{p})}$$
$$\kappa(\mathfrak{p}) = \operatorname{Frac}(\mathcal{O}_{\operatorname{Spec} S, \mathfrak{p}}) = S_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$$

Next, the ring homomorphism f naturally induces the local ring homomorphism $f_{\mathfrak{p}} : R_{f^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$ given by $\frac{r}{t} \mapsto \frac{f(r)}{f(t)} = \frac{f(r)}{s}$ is well-defined where $s = f(f^{-1}(s))$ for appropriate $s \notin \mathfrak{p}$. Then if $\mathfrak{m}_{\mathfrak{p}} \subseteq S_{\mathfrak{p}}$ is the unique maximal ideal,

$$f^{-1}(\mathfrak{m}_{\mathfrak{p}}) = \left\{ \frac{r}{t} \in R_{f^{-1}(\mathfrak{p})} \;\middle|\; \frac{f(r)}{s} \in \mathfrak{m}_{\mathfrak{p}},\ s \notin \mathfrak{p} \right\}$$

Notice that the elements $r/1$ map into $\mathfrak{m}_s$, and so $\mathfrak{m}_{f^{-1}(\mathfrak{p})} \subseteq f^{-1}(\mathfrak{m}_{\mathfrak{p}})$. As the ideal is maximal, we have $\mathfrak{m}_{f^{-1}(\mathfrak{p})} = f^{-1}(\mathfrak{m}_{\mathfrak{p}})$. Hence, an element is zero in $R_{f^{-1}(\mathfrak{p})}/\mathfrak{m}_{f^{-1}(\mathfrak{p})}$ if and only if it is zero in $S_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. As this must be the case for each point (as $f_{\mathfrak{p}}$ is a local homomorphism for each stalk), this is exactly the condition we need so that $\stackrel{!}{=}$ holds! If starting with a map $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$, so that $\varphi^{\#} : \mathcal{O}_{\operatorname{Spec} S} \to \varphi_* \mathcal{O}_{\operatorname{Spec} R}$, then the codomain of the sheaf map on each distinguished open set has the following interesting interpretation that shows the map is natural

$$\varphi^{\#}(D(f)) : S_f = \mathcal{O}_{\operatorname{Spec} S}(D(f)) \to \mathcal{O}_{\operatorname{Spec} R}(\varphi^{\#}(f)) = R_{\varphi^{\#}(f)} \cong R \otimes_S S_f$$
$$\varphi^{\#}(D(f))\left(\frac{s}{t}\right) = 1 \otimes_S \frac{s}{t}$$

Thus, over general schemes, we want the same structure to be preserved:

Definition 3.1.3: Morphism Of Schemes

Let X, Y be schemes. Then a morphism of locally ringed spaces $\pi : X \to Y$ is called a *morphism of schemes*. Schemes along with scheme morphisms form a category **Sch**.

We have shown how to induce the map between rings and locally ringed space on affine schemes. Let us make sure that these maps are in bijective correspondence:

Theorem 3.1.4: Ring Homomorphism and Locally Ringed Space

Let $\varphi : R \to S$ be a ring homomorphism. Then

1. φ induces a natural morphism of locally ringed space[a]:
$$(\varphi, \varphi^{\sharp}) : (\operatorname{Spec}(S), \mathcal{O}_{\operatorname{Spec}(S)}) \to (\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$$
2. Any morphism of locally ringed space from $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ induces a ring homomorphism $\varphi : R \to S$
3. If the ring homomorphism φ induces a locally ringed space morphism φ^* which induces a ring homomorphism ψ, then $\varphi = \psi$.

[a] ringed space morphism that preserves local rings

Proof **:**

1. Let $\varphi : R \to S$ be a morphism which induces the continuous map $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ (where the notation f is used to not overcrowd notation later in the proof). For each $\mathfrak{p} \in \operatorname{Spec}(S)$, localize S at $\mathfrak{p}$ (i.e. $S \setminus \mathfrak{p}$) to obtain the local homomorphism of the local rings

$\varphi_{\mathfrak{p}} : R_{\varphi^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$. Now, as these stalks follow the condition of proposition 1.1.22, the map $f^\sharp : \mathcal{O}_{\mathrm{Spec}\,(R)}(U) \to \mathcal{O}_{\mathrm{Spec}\,(R)}(f^{-1}(U))$ is a ring homomorphism for all $U \subseteq \mathrm{Spec}\,(U)$. Hence $f^\sharp$ is a morphism of sheaves which induces local rings

2. Let $(f, f^\sharp) : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ be a morphism of locally ringed spaces with the usual structure sheaf. By proposition 2.3.5, , the global section induce a ring homomorphism:

$$\varphi : \mathcal{O}_{\mathrm{Spec}\,(R)}(\mathrm{Spec}\,(R)) \to \mathcal{O}_{\mathrm{Spec}\,(S)}(\mathrm{Spec}\,(S)) \qquad \text{i.e.} \qquad \varphi : R \to S$$

Then by the above discussion for any prime $\mathfrak{p} \in \mathrm{Spec}\,(S)$, the induce map on the stalks $R_{f^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$ is compatible with φ, namely $f(\mathfrak{p}) = (f^\#)^{-1}(\mathfrak{p})$ (i.e. the commutative diagram between $\varphi : R \to S$ and $\varphi_{\mathfrak{p}} : R_{f^\#_{f^{-1}(\mathfrak{p})}} \to S_{\mathfrak{p}}$) forces this equality). Namely, this implies that the map f mappings prime ideals uniquely induces a map φ between rings.:

$$\begin{tikzcd}
X & \mathrm{Spec}\, R \arrow[d, dashed, no head] & \mathrm{Spec}\, S \arrow[l, "f"'] \arrow[d, dashed, no head] & Y \\
\mathcal{O}_X(X) & R \arrow[r, "f^\#(X)=\varphi"] \arrow[d] & S \arrow[d] & f_*\mathcal{O}_Y(Y) \\
\mathcal{O}_{X,f^{-1}(\mathfrak{p})} & R_{f^{-1}(\mathfrak{p})} \arrow[r, "f^\#_{f^{-1}(\mathfrak{p})}"] & S_{\mathfrak{p}} & \mathcal{O}_{Y,\mathfrak{p}}
\end{tikzcd}$$

3. As $f^\sharp$ is a local homomorphism, $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, hence f concise with the map $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ induced by φ. Furthermore, $f^\sharp$ is also induced by φ, since the morphism $(f, f^\sharp)$ of locally ringed spaces comes from the homomorphism φ, as we sought to show.

Affine schemes are the correct category to be the opposite category of [commutative] rings:

Corollary 3.1.5: Equivalence of Rings And Affine Schemes

The category of rings and the opposite category of affine schemes are equivalent.

Furthermore, the functor Spec and Γ (the global section functor) are adjoint.

This shows that schemes are the proper object to capture the information of rings.

***Proof* :**
exercise.

This equivalence will allow us to completely control the morphisms of schemes by looking at ring homomorphisms, namely by reducing a morphism of schemes to a morphism of affine schemes which is then in correspondence with the morphism of affine schemes. First, we must insure that gluing morphisms is well-defined. The result can work in the slightly more general context of ringed spaces:

Lemma 3.1.6: Morphisms Defined on Open Sets

Let X, Y be two ringed spaces where $X = \cup_i U_i$ is an open cover for X. Then if there exists morphism of ringed space morphisms $\pi_i : U_i \to Y$ that agree on overlaps

$$f_i|_{U_I \cap U_j} = f_j|_{U_i \cap U_j}$$

then there is a unique morphism of ringed space $f : X \to Y$ such that $f|_{U_i} = f_i$

***Proof* :**
First, by the gluing lemma in topology the map $f : X \to Y$ is certainly a continuous map, we must show that we have a sheaf morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For each $f_i : U_i \to Y$ we have a sheaf morphism $f_i^{\#} : \mathcal{O}_Y \to (f_i)_*\mathcal{O}_{U_i}$. If $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then:

$$f_i^{\#}(U_i \cap U_j) = f_j^{\#}(U_i \cap U_j)$$

Now for each open $V \subseteq Y$, We can define for each $V \cap f^{-1}(U_i)$

$$f^{\#}(V \cap f^{-1}(U_i)) : \mathcal{O}_Y(V \cap f^{-1}(U_i)) \to \mathcal{O}_X(f^{-1}(V \cap f^{-1}(U_i)))$$
$$f^{\#}(V \cap f^{-1}(U_i)) = f_i^{\#}(V \cap f^{-1}(U_i))$$

where $f_i^{\#}$ and $f_j^{\#}$ agree on $V \cap f^{-1}(U_i) \cap f^{-1}(U_j)$. As these maps cover V and agree on intersections, then for each $s \in \mathcal{O}_Y(V)$, there is a unique section $t \in \mathcal{O}_X(f^{-1}(V))$ where $f^{\#}(V)(s) = t$, namely this section exists by the sheaf axioms and respects restrictions. As this is true for each open $V \subseteq Y$, then this is exactly the criterion for the sheaf morphism $f^{\#}$, completing the proof.

Proposition 3.1.7: Scheme Morphism Locally Affine

Let $\pi : X \to Y$ be a morphism of schemes. Then it is a morphism of locally ringed spaces that locally looks like a morphism of affine schemes, that is if $\operatorname{Spec} R \subseteq X$ and $\operatorname{Spec} S \subseteq Y$ where $\pi(\operatorname{Spec} R) \subseteq \operatorname{Spec}(S)$, then the induced morphism of ringed spaces is a morphism of affine schemes. Conversely, a morphisms of ringed spaces that locally looks like a morphism of affine scheme is a morphism of schemes.

***Proof* :**
Let $\pi : X \to Y$ be a morphism of schemes, and let $Y = \bigcup_j \operatorname{Spec} S_j$ be an affine open cover. Let $y \in \pi(X)$, and consider $x \in \pi^{-1}(y)$. As $y \in \operatorname{Spec} S_j$ for some j, $x \in \pi^{-1}(\operatorname{Spec} S_j)$ and the pre-image is open, and hence can be covered by affine open subschemes, on of which must contain the point x, say this affine scheme is $\operatorname{Spec} R_i$. As the open subset of $\pi^{-1}(\operatorname{Spec} S_j)$ is open in X, we have that each point $x \in X$ is contain in an affine open subscheme $\operatorname{Spec} R_i$ such that $\pi(\operatorname{Spec} R_i) \subseteq \operatorname{Spec} S_j$ and the map is certainly a morphism of affine schemes.

Conversely, if a map $\pi : X \to Y$ between schemes is locally a morphism of schemes on affine schemes, then by lemma 3.1.6 we get a ringed space morphism, which is certainly a locally ringed space morphism and hence $\pi : X \to Y$ is a scheme morphism, completing the proof.

Example 3.3: Morphisms Of Schemes

1. Recall the map $\varphi : \mathbb{C}[y] \to \mathbb{C}[x]$ given by $y \mapsto x^2$, which on closed points can be interpreted as $a \mapsto a^2$. Let's look at the pullback of functions. Consider $3/(y-4)$ on $D([(y-4)]) = \mathbb{A}^1 \setminus \{4\}$. The pull-back is $3/(x^2-4)$ on $\mathbb{A}^1 \setminus \{-2, 2\}$.

2. Let $U \subseteq Y$ be an open subsets of a scheme Y. Then there is a natural morphism of ringed space
$$(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$$
More is true, the above map is an *open embedding*, namely the image of U under the above map is an isomorphism, where the image is also open in Y. Later, this will be shown to be an immersion in the geometrical sense, that is the stalks maps are injective (mirroring the tangent-space maps being injective).

3. Let us look at a non-affine scheme morphisms. Consider:
$$\iota : \mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k \qquad (x_0, x_1, ..., x_n) \mapsto [x_0, x_1, ..., x_n]$$
It certainly continuous, so let's us define a sheaf morphism compatible with it. By lemma 3.1.6, it suffices to show it on an open cover and showing gluing works. Take the usual affine cover of $\mathbb{P}^n_k$, $H_0, ..., H_n$, and define:
$$\iota^{\#}(H_i) : \mathcal{O}_{\mathbb{P}^n_k}(H_i) \to \mathcal{O}_{\mathbb{A}^{n+1}_k}(f^{-1}(H_i))$$
where
$$\mathcal{O}_{\mathbb{P}^n_k}(H_i) \cong k[x_0, ..., \hat{x}_i, ..., x_n]$$
where $(\hat{-})$ represents omitting the element, and $f^{-1}(H_i) = \left\{x \in \mathbb{A}^{n+1}_k \;\middle|\; x_i \neq 0\right\} \cong D(x_i) \subseteq \mathbb{A}^{n+1}_k$. Thus, we define a map:
$$\iota^{\#}(H_i) : k[x_0, ..., \hat{x}_i, ..., x_n] \to k[x_0, ..., x_n]_{x_i} \qquad f \mapsto \frac{f}{1}$$
On $H_i \cap H_j$, we have that the homogeneous coordinates $x_i, x_j \neq 0$ and:
$$\iota^{\#}(H_i \cap H_j) : \mathcal{O}_{\mathbb{P}^n_k}(H_i \cap H_j) \to \mathcal{O}_{\mathbb{A}^{n+1}_k}(f^{-1}(H_i \cap H_j))$$
where the domain is:
$$\mathcal{O}_{\mathbb{P}^n_k}(H_i \cap H_j) = k[x_0, ..., \hat{x}_i, ..., x_n]_{x_j} \cong k[x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n]$$
and the codomain is:
$$\mathcal{O}_{\mathbb{A}^{n+1}_k}(f^{-1}(H_i \cap H_j)) = \mathcal{O}_{\mathbb{A}^{n+1}_k}(D(x_i) \cap D(x_j)) \cong k[x_0, ..., x_n]_{x_i, x_j}$$
And hence we have
$$k[x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n] \to k[x_0, ..., x_n]_{x_i, x_j} \qquad f \mapsto \frac{f}{1}$$
Certainly all the restriction maps commute, and the H_i cover $\mathbb{P}^n_k$, hence we have a well-defined sheaf morphism. Intuitively, we would expect that on every function on a hyper-surface H_i be identically a function on $D(x_i) \subseteq \mathbb{A}^n_k$, and this scheme morphisms shows exactly that intuition!

4. Let $S_\bullet$ be a finitely generated graded R-algebra. Describe the morphism

$$\operatorname{Proj} S_\bullet \to \operatorname{Spec}(R)$$

5. Take the map $\pi : \operatorname{Spec}(\mathbb{C}[x,y]) \to \operatorname{Spec}(\mathbb{C}[x])$ induced by the inclusion $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y]$. This map is the projection map, any closed point $[(x-3, y-a)]$ for all $a \in \mathbb{C}$ map to $[(x-3)]$. What about non-closed points such as $[(y-x^2)]$? Check that this point maps to the generic point $[(0)]$!

If the codomain of a scheme morphism is an affine scheme, then we can completely determine the behaviour of the scheme morphisms by looking at the ring morphism between the global sections:

Proposition 3.1.8: Morphisms to Affine Schemes

Let $\operatorname{Hom}_{\mathbf{Sch}}(X, \operatorname{Spec}(R))$ be the collection of morphism of scheme from a scheme X to an affine scheme $\operatorname{Spec}(R)$. Then it is in natural bijection to the ring homomorhpisms $\operatorname{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_X(X))$:

$$\operatorname{Hom}_{\mathbf{Sch}}(X, \operatorname{Spec}(R)) \cong \operatorname{Hom}_{\mathbf{Ring}}(R, \mathcal{O}_X(X))$$

Furthermore, this is an isomorphism if and only if X is an affine scheme.

Note that the ring $\mathcal{O}_X(X)$ can be very complex, even in nice cases where X is a finite-type k-scheme, for example it need not be finitely generated.

***Proof* :**
If $X = \operatorname{Spec}(S)$, then theorem 3.1.4 gives the bijection. Next, map $\pi : X \to \operatorname{Spec}(R)$ certainly induces a ring homomorphism $\varphi : R \to \mathcal{O}_X(X)$, we shall show that a ring homomorphism $\varphi : R \to \mathcal{O}_X(X)$ induces a unique scheme map $\pi : X \to \operatorname{Spec}(R)$ and that this map is the inverse of the above map. By definition of an affine scheme:

$$\pi : \bigcup_i \operatorname{Spec}(S_i) \to \operatorname{Spec}(R)$$

where we get a restriction map $\pi_i : \operatorname{Spec}(S_i) \to \operatorname{Spec}(R)$ along with sheaf morphisms, for which the global section map is $\varphi_i : R \to \mathcal{O}_X(\operatorname{Spec}(S_i)$. As $\operatorname{Spec} S_i \subseteq X$ is an open subset, there is a natural restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(\operatorname{Spec} S_i)$. We thus get a composition:

$$R \to \mathcal{O}_X(X) \to \mathcal{O}_X(\operatorname{Spec} S_i)$$

The map $R \to \mathcal{O}_X(\operatorname{Spec} S_i)$ is uniquely determined by the ring homomorphism $S_i \to R$, and as X is covered by the $\operatorname{Spec} S_i$, by lemma 3.1.6 there is a unique way to glue the maps of $\mathcal{O}_X(\operatorname{Spec} S_i)$ to $\mathcal{O}_X(X)$, completing the proof.

The converse is not true:

Example 3.4: Maps From Affine To Projective

Take $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\,\mathbb{Q}[x], \mathbb{P}^1_{\mathbb{Q}})$ and $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Q}, \mathbb{Q}[x])$ (recall that $\mathcal{O}_X(\mathbb{P}^1_{\mathbb{Q}}) \cong \mathbb{Q}$). Then $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Q}, \mathbb{Q}[x]) = \{\mathrm{id}\}$, namely since since the units of $\mathbb{Q}$ must map to the units of $\mathbb{Q}[x]$, which are all contains in $\mathbb{Q}$, and as it must be that $1 \mapsto 1$, we are forced to have the identity. However, there are multiple maps from $\mathrm{Spec}\,\mathbb{Q}[x]$ to $\mathbb{P}^1_k$; simply map $\mathrm{Spec}\,\mathbb{Q}[x]$ into the open subset $U_0 \cong k[y] \subseteq \mathbb{P}^1_k$ by taking $x \mapsto p(y)$ for some polynomial $p(y) \in k[y]$. More generally, we can let choose a field k[a].

The intuition is that $\mathcal{O}_X(X)$ in general contains *less* information than X as the global sections on a general scheme need to restrict to local sections on affine open subsets, and hence there may be in general less of them.

Show that the canonical morphism $X \to \mathrm{Spec}\,(\mathcal{O}_X(X))$ is an isomorphism if and only if X is a affine.

[a]Note that an [injective] ring homomorphism $k \to k$ need not be surjective, consider $F(x) \to F(x)$ given by $x \mapsto x^2$, hence the argument requires a bit more work

Corollary 3.1.9: Characterizing Affine Schemes

Let X be a scheme. Then X is an affine scheme if and only if there is a finite subset of elements $f_1, ..., f_n \in \mathcal{O}_X(X)$ such that X_f is affine and $(f_1, ..., f_n) = \mathcal{O}_X(X)$

Proof **:**
use the gluing lemma, the equivalence of categories, and proposition 3.1.8.

Exercise 3.1.1

1. Let $\mathfrak{p} \in X$ be a point in a scheme. Show there is a canonical map
$$\mathrm{Spec}\,(\mathcal{O}_{X,\mathfrak{p}}) \to X$$
2. Using the above, define a canonical morphism $\mathrm{Spec}\,(\kappa(\mathfrak{p})) \to X$.

3.1.1 S-Schemes

Recall that all abelian rings can be seen as $\mathbb{Z}$-algebras, and that $\mathbb{Z}$ is initial in the category of commutative rings. From this perspective, all ring homomorphism $\varphi : R \to S$ are in bijective correspondence with morphisms that make the following diagram commute:

$$\begin{tikzcd} R \arrow[rr] & & S \\ & \mathbb{Z} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then we may generalize the notion of abelian rings to R-algebras by replacing $\mathbb{Z}$ with a ring R. We may further augment the morphisms to R-algebra homomorphisms as we are working with commutative rings. This is the motivation to the following generalization; we can generalize the notion of R-schemes given in definition 2.3.7:

Definition 3.1.10: S-Scheme

The category consisting of all schemes X that have scheme morphisms $X \to S$ and whose morphisms are:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is denoted $\mathbf{Sch}_S$, and it's object are called *S-scheme.* The morphism of schemes $X \to S$ is called the *structure morphism.*

To see what structure is added, let us start with the case of $S = \operatorname{Spec}(R)$, namely if $S = \operatorname{Spec}(R)$, each open subset of X has the structure of an R-algebra, and these generalize definition 2.3.7. If $S = \operatorname{Spec}(R)$, then we would often say that the $(\operatorname{Spec} S)$-scheme is an R-scheme

Working over S-schemes shall restrict to much "simpler" cases than over general schemes. For example, if studying complex analysis and complex varieties, we would expect $\operatorname{Spec}(\mathbb{C})$ to have trivial automorhpisms, and that $\operatorname{Spec}(\mathbb{C}[x]/(x^2+1))$

Example 3.5: S-scheme

1. When working with polynomials, we are often interested in k-homomorphisms $k[x] \to k[x]$ where k is fixed (or else our morphisms would include field endomorphisms, which may be undesirable). We can consider this condition as the following diagram commuting:

$$\begin{tikzcd} k[x] \arrow[rr] & & k[x] \\ & k \arrow[ul, "\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

   This would correspond to k-scheme maps:

$$\begin{tikzcd} \operatorname{Spec}(k[x]) \arrow[dr] & & \operatorname{Spec}(k[x]) \arrow[ll] \arrow[dl] \\ & \operatorname{Spec}(k) & \end{tikzcd}$$

   Note the importance of these maps commuting as scheme morphisms and not just topological maps, namely we need the sheaf maps.

2. Projective schemes over a field k are k-projective schemes.

3. Let L/K be a field extension. Then $\operatorname{Spec} K, \operatorname{Spec} L$ are K-schemes, Notice that Spec $\to$ $\operatorname{Spec} K$ has as a sheaf morphism the injective map $K \to L$, that is it is the field extension L/K. We shall later see that we can find an appropriate scheme that will act as a *covering space* for K, and we shall generalize the Galois group to the *fundamental group scheme* which shall remember multiplicity information of roots of polynomials.

Lemma 3.1.11: Stalks of k-schemes

Let X be a k-scheme and $x \in X$ so that $x = [\mathfrak{p}]$ for some prime $\mathfrak{p} \subseteq R_i = k[u_i]_i$. Then

$$\mathcal{O}_{X,\mathfrak{p}} \cong (k[u_i]_i)_{\mathfrak{p}}$$

In particular, we have that the quotient by $\mathfrak{m}_x$ is a k-algebra:

$$\kappa(x) = k(x)$$

Proof **:**
As X is a k-scheme, each $\mathcal{O}_X(U)$ is a k-algebra, and the colimit of k-algebra's is a k-algebra (they are closed under products and equalizers, see [3]). The rest comes from proposition 2.3.14.

Having a R-algebra structure, we may ask for finiteness over the base ring R:

Definition 3.1.12: Locally Finite Type And Finite Type

Let X, Y be S-schemes. Then a morphism $f : X \to Y$ is *locally of finite type* if there exists an affine open cover $V_j = \operatorname{Spec}(S_j)$ of Y where $f^{-1}(V_i)$ can be covered by affine open subsets $U_{ij} = \operatorname{Spec}(A_{ij})$ where A_{ij} is a finitely generated S_i-algebra, that is

$$f^{\#}(V_i) : S_j = \mathcal{O}_Y(V_j) \to \mathcal{O}_X(f^{-1}(V_i)) \xrightarrow{\text{res}} \mathcal{O}_Y(U_{ij}) = R_{ij}$$

induces a finitely generated S_j-algebra on each R_{ij}. If furthermore each $f^{-1}(V_j)$ can be covered by finite number of affine subsets, f is said to be of *finite type*.

An S-scheme X is said to be of locally finite type (resp. finite type) over S if $f : X \to S$ is of locally finite type (resp. finite type).

Proposition 3.1.13: Final Object Of Scheme

In **Sch**, the scheme $\operatorname{Spec}\mathbb{Z}$ is the final object. This makes the category of schemes isomorphic to the category of $\mathbb{Z}$-schemes.

If k is a field, $\operatorname{Spec} k$ is the final object in the category of k-schemes.

Proof **:**
Recall that $\mathbb{Z}$ is initial in **Ring**, and thus it is easy to show it is initial in **Comm-Ring**, then as Schemes (resp. k-schemes) are the co-categories to rings (resp. k-algebras), the result is immediate.

Proposition 3.1.14: Varieties And Schemes

Let $\overline{k}$ be an algebraically closed field. Then there exists a fully faithful functor $\mathbf{Sch}_{\overline{k}} \to \mathbf{Var}$ where **Var** is the usual category of varieties as defined in [3].

***Proof* :**
See Hartshorne, p.77

3.1.2 S-valued Points

Let's say we have the scheme Spec $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ whose closed points is the real circle[3]. Then if we are looking for the rational-valued points on the scheme, we are looking for the function:

$$\mathrm{ev}_{a,b} : \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)} \to \mathbb{Q}$$

mapping $x \to a$ and $y \to b$ with the property (given by the quotient) that $a^2 + b^2 = 1$. Another way of looking at this situation is that we have a collection of functions $\mathrm{ev} : \mathbb{R}[x, y] \to \mathbb{Q}$ and we are taking the functions where $a^2 + b^2 - 1 = 0$, which is exactly the functions that factor through $\mathrm{ev}_{a,b} : \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)} \to \mathbb{Q}$. Examples of such functions are $\mathrm{ev}_{1,0}$, $\mathrm{ev}_{\frac{3}{5},\frac{4}{5}}$ for $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 - 1 = 0$. This ring homomorphism is part of the global section morphism, namely given the scheme morphism $\mathrm{ev} : \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)} = C$, we have the sheaf morphism on global sections:

$$\mathrm{ev}^{\#}(C) : \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)} \to \mathbb{Q}$$

Now, if we change the ring to $\mathbb{Z}$, or more generally some (local or global) number ring, we shall get different sets of "legal" evaluation functions. For example, if $R = \mathbb{Z}$, then the only evaluation maps are the 4 maps $\mathrm{ev}_{\pm 1,0}$ and $\mathrm{ev}_{0,\pm 1}$. Hence, we may think of the collection of ring homomorphisms:

$$\mathrm{ev}_{a,b} : \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)} \to R$$

as representing the solution over R of an equation. Dualizing this to affine schemes, this is equivalent to asking for a scheme morphism:

$$\operatorname{Spec} \mathrm{ev} : \operatorname{Spec} R \to \operatorname{Spec} \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)}$$

More generally, we may imagine we want to have the values come from a more general object (for example, we are evaluating points on a torus with values coming from a curve). Hence, we may more generally want to have any scheme morphism $S \to X$ to be considered the points of X that take on values on S. We make this precise in the following definition:

Definition 3.1.15: S-valued Points

Let X be a scheme. Then the *S-valued points* of X are the morphisms $S \to X$, and is denoted $X(S)$. If $S = \operatorname{Spec}(R)$, then this is often denoted $X(R)$.

The reader should be aware that the morphism $S \to X$ is called the "S-valued point", and not any particular point in S or the image of S; this terminology can be thought of from the following idea: a morphisms of schemes $X \to Y$ is not determined by the underlying map of points (the continuous map).

[3]Note that it non-closed points make this affine scheme bigger in a meaningful way, recall example 2.14

Proposition 3.1.16: S-valued Points and Scheme Morphisms

Let $X \to Y$ be a scheme morphism. Show that it is determined by all the maps of S-valued points, namely given all S-valued point maps to X and Y, we get a unique scheme morphism map

***Proof* :**
Here is the high-level: first show that any scheme morphism $X \to Y$ induced a map $X(S) \to Y(S)$. Then, take $S = X$. Then we have a version of Yoneda's lemma.

This proof may seem trivial, but the idea behind it will become important when looking at cohomological ideas.

Example 3.6: S-valued Points

1. One of the most common example's is $\overline{k}$-valued points on given on schemes (not yet necessarily over any field). The $\overline{k}$-valued points are the collection of maps $\varphi : \mathrm{Spec}\,(\overline{k}) \to X$. Consider $\varphi([0]) = y \in X$. Taking the stalk-map, where $\varphi_y : \overline{k}O_{X,y} \to \overline{k}$. Taking the residue field, we get: the naturally extended map:
$$\varphi_y : \overline{k}(y) \to \overline{k}$$
This map forces y to be a closed point: If y is not closed, $\overline{k}(y)$ contains transcendental elements (see ref:HERE), say $x \in \overline{k}(y)$, but then there is no morphism from $\overline{k}(y)$ to $\overline{k}$ as for $\varphi(x) \in \overline{k}$ we have $p(\varphi_y(x)) = 0$, but then $\varphi_y(p(x)) = 0$ for $p(x) \in \overline{k}(y)$ which is impossible as field homomorphisms are injective (naturally $\varphi(0) = 0$ as well).

   Hence it is always the case that there exists a natural map $\overline{k}(y) \to \overline{k}$ from $\mathrm{Spec}\,(\overline{k})$ to the closed points of a scheme X. If X is locally finite type over $\overline{k}$, each stalk is a finitely generated $\overline{k}$-algebra and, hence by Zariski's Lemma $\overline{k}(y) = \overline{k}[y]$, and as $\overline{k}$ is algebraically closed, $\overline{k}[y] = \overline{k}$. In fact, we have a one-to-one correspondence:
$$\{\mathrm{Spec}\,(\overline{k}) \to X\} \xrightarrow{\sim} X_{\max}$$
where $X_{\max}$ is the set of closed points of a scheme X of finite type over $\overline{k}$. Show this is even a functor.

   Note that this is another counter-example of the "co" of proposition 3.1.8.

2. Consider $\mathbb{C}(t)$-valued points over $\mathbb{C}$-schemes. Any such point would fit into the following commutative diagram:
$$\begin{tikzcd} \mathrm{Spec}\,\mathbb{C}(t) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Spec}\,\mathbb{C} & \end{tikzcd}$$

3. Next, take $D = \mathbb{C}[\epsilon]/(\epsilon^2)$, the dual numbers. Let us classify all scheme morphisms $\mathrm{Spec}\,(D) \to X$ of finite type over k (recall a scheme morphism is of finite type if every affine open subset $\mathrm{Spec}\,(S) \subseteq X$ has an affine pre-image, $\mathrm{Spec}\,(R) = \varphi^{-1}(\mathrm{Spec}\,(S)$ whose ring R is a finitely generated S-algebra). As D has a unique maximal ideal (ϵ), $\mathrm{Spec}\,(D)$ has a unique maximal

closed point, let's label it $\overline{o}$. Then the homomorphism $D \to k$ with kernel (ϵ) is associated to the canonical embedding $\iota : \operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(D)$ with $\overline{o} = \iota(o)$ (hence the bar notation). Then any morphism $\varphi : \operatorname{Spec}(D) \to X$ is in correspondence to the composite-morphism $\varphi \circ \iota : \operatorname{Spec}(k) \to X$, which itself is in correspondence to the closed points of X. If $x \in X$ is a closed points, then for appropriate φ we have $x = \varphi(\overline{o})$.

Now, for a choice of $x \in X$, pick an affine open neighborhood $x \in \operatorname{Spec}(R) \subseteq X$, and take $\operatorname{Hom}_x(\operatorname{Spec}(D), X) \cong \operatorname{Hom}_x(\operatorname{Spec}(D), \operatorname{Spec}(R))$ where Hom_x is the collection of scheme morphisms st $\varphi(\overline{o}) = x$. Then all these morphisms are determined by a local k-homomorphism $f : R \to D$ where $f(\mathfrak{m}_x) \subseteq (\epsilon)$ by locality. Then as R, as a vector-space, is equal to

$$R = k + \mathfrak{m}_x$$

we have that the homomorphism is determined by what it does to $\mathfrak{m}_x$, namely it defines some linear transformation $\mathfrak{m}_x \to (\epsilon) \cong k$. As $\epsilon^2 = 0$, $f(\mathfrak{m}_x^2) = 0$, and so f is a *linear function* on $\mathfrak{m}_x/\mathfrak{m}_x^2$. Convesely, any function f to extended to be 0 on $\mathfrak{m}_x^2$ determines a homomorphism $R \to D$ taking $\mathfrak{m}_x$ into (ϵ).

Thus, if X is a k-scheme and $x \in X$, the set

$$\operatorname{Hom}_x(\operatorname{Spec}(D), X) \cong \mathfrak{m}_x/\mathfrak{m}_x^2 = \Theta_{X,x}$$

Furthermore, if $f : X \to X'$ is another morphism, this induces a differential

$$\mathrm{d}_x : \Theta_{X,x} \to \Theta_{X',f(x)}$$

A lot one the when it is nicer, on locally closed embeddings, etc. Vakil p.238-240

3.1.3 Rational Maps

When working with [quasiprojective] varieties, we had the weaker notion of a rational function that was defined almost everywhere. A rational function was more flexible, allowing for some important classification results, the definition of Kodaira dimension, study singularities, more invariances by looking at the birational automorphisms, and some "local to global" principles. The following brings these concepts over to schemes

Definition 3.1.17: Rational Map

Let X, Y be schemes. Then a *rational map* $\pi : X \dashrightarrow Y$, is a nequivalence relation:

$$(\alpha U \to Y) \sim (\beta : V \to Y) \qquad \text{if and only if} \qquad Z \subseteq U \cap V \text{ s.t. } \alpha|_Z = \beta|_Z$$

where U, V, Z are all dense open subsets[a].

If X, Y are S-schemes, then rational maps of S-schemes are defined similarly.

If the image of one of the representatives is dense, then the rational map is said to be *dominant* or *dominating*.

[a]When Y is separated, we will let $Z = U \cap V$

Note that a rational map is is an equivalence relation, not necessarily defined on a single domain.

Definition 3.1.18: Birational Map

A rational map $\pi : X \dashrightarrow Y$ is *birational* if it is dominant, and it has a dominant rational inverse. Two schemes X, Y are *birationally equivalent* if there is a birational map $\pi : X \dashrightarrow Y$.

Example 3.7: Rational Map

1. A scheme morphisms always gives a rational map, similarily to how all regular functions are special rational functions
2. The projection $\mathbb{P}^n_R \dashrightarrow \mathbb{P}^{n-1}_R$ projecting down a dimension is a rational map (recall the same reasoning for rational functions in [3, chapter 22.3]

Lemma 3.1.19: Dominating Rational Map on Irreducible Schemes

Let $\pi : X \dashrightarrow Y$ be a rational map between irreducible schemes. Then π is dominating iff it sends the generic point of X to the generic point of Y.

***Proof* :**
recall the properties of generic points in irreducible schemes.

We may naturally want to have a result that is similar to the correspondence of rational functions and field extensions as given in [3]. The full correspondence requires irreducible varieties (integral finite type k-schemes), however we may at least always induce a map of function fields:

Proposition 3.1.20: Rational Maps of Integral Schemes induces Function Field Map

The collection of integrals schemes with dominant maps induces morphisms of function fields in the opposite direction.

***Proof* :**
First show that the form a category (importantly, composition works). Then think of how to create this correspondence.

Corollary 3.1.21: Birational Map on Irreducible Schemes

Let $\pi : X \dashrightarrow Y$ be a birational map between irreducible schemes. Then the induced map of function fields is an isomorphism.

For the converse failing

Example 3.8: Function Field Not Inducing Rational Map
Take Spec $(k[x])$ and Spec $(k(X)$ which have the same function field $k(x)$. Then certainly there is no corresponding rational map

$$\operatorname{Spec} k[x] \dashrightarrow \operatorname{Spec} k(x)$$

of k-schemes, as such a morphism would correspond to a morphism from $U \subseteq \operatorname{Spec} k[x]$ (ex. $U = \operatorname{Spec} k[x, 1/f(x)]$) to $\operatorname{Spec} k(x)$, but not map of rings $k(X) \to k[x, 1/f(x)]$ satisfies the right properties for any $f(x)$.

Proposition 3.1.22: Reduced Schemes and Birational Maps

Let X, Y be reduced schemes. Then X and nY are birational if and only if there exists a dense open subscheme $U \subseteq X$ and $V \subseteq Y$ such that

$$U \cong V$$

Proof **:**
Generalize the proof from [3] (If stuck, look at Vakil p. 190)

Vakil looks here at rational maps of irreducible varieties that look like translating the results over from varieties, so I will omit for now

I like his translation of the Pythagorean problem into a scheme problem

non-rational complex curve, cool presentation

3.1.4 Maps on Projective Schemes

As ring homomorphisms correspond to scheme morphisms, we would like that graded ring homomorphism correspond to morphisms on projective schemes.

3.1.5 Group Schemes

I'm putting this here because I looked into algebraic groups and so I want a place to put it down

Exercise 3.1.2

1. If $\varphi : A \to B$ is a ring homomorphism, then the corresponding morphism Spec $(B) \to$ Spec (A) is dominant if and only if $\ker \varphi$ is contained in the nilradical of A.

3.2 Fibered Product and Fibers

With scheme morphisms defined, we may bring in one more tool from commutative algebra to analyze schemes, namely we can translate the tenors product to the fibered product (or pullback). As a high-level idea, the fibered product can be thought of as a product of schemes with gluing, while the fibered coproduct is the disjoint union plus gluing, and will be a scheme if it satisfies the conditions of proposition 2.3.19, but it will not exist in general, see example 3.10.

Theorem 3.2.1: Fibered Product Of Schemes

Let X, Y be schemes over S. Then then the fibered product of X and Y over S exists, that is the limit of the diagram exists:

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The fibered product is denoted $X \times_S Y$

More commonly, we would say that the fibered product satisfies the universal property that if $Z \to X$ and $Z \to Y$ commutes in the diagram, then there exists a unique morphism $Z \to X \times_S Y$ such that

$$\begin{tikzcd} & Z \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & X \times_S Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes. If S is the final object, or more concisely it is empty or "doesn't matter", then we have the usual product. If S is not specified, the product shall be defined as $X \times Y := X \times_{\mathrm{Spec}\,(\mathbb{Z})} Y$

***Proof* :**
We shall construct the products of affine schemes, and then glue them together.

Let $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$, and $S = \mathrm{Spec}\,(R)$ be affine. Then A, B are R-algebras, and the product of X and Y over S is:

$$\mathrm{Spec}\,(A \otimes_R B)$$

This comes form the universal property of tensor products along with the property of adjoints. Indeed, if $Z \to \mathrm{Spec}\,(A \otimes_R B)$ is a scheme morphism, then this gives a ring homomorphism $A \otimes_R B$ into $\mathcal{O}_Z(Z)$. By the universal property of tensor products, a morphism of $A \otimes_R B$ into is the same as a morphism from A and B to that ring. Then by proposition 3.1.8, this gives a morphism of Z into $\mathrm{Spec}\,(A \otimes_R B)$, and this morphism will satisfy the universal property of fibered products given by the universal property of tensor products.

Let us now construct the more general case via gluing. Let X, Y be schemes. To do this, we shall reduce to a covering of $X \times_S Y$. For any open affine $U \subseteq X$, if the product existed then $\pi_1^{-1}(U) \cong U \times_S Y$ (check this[a]). Now if $\{X_i\}$ is an open covering for x, then I claim that if $X_i \times_S X$ exists, so does $X \times_S Y$. Indeed, for each $U_{ij} = \pi_1^{-1}(X_i) \subseteq X_i \times_S Y$, by the uniqueness of the product there are unique isomorphisms: $\varphi_{ij} : U_{ij} \to U_{ji}$ for each i, j for compatible projections. Furthermore, these isomorphisms are compatible with the gluing condition (as you should check if you are not convinced!). As a remind, the condition was:

$$\varphi_{ik}|_{X_{ij}\cap X_{ik}} = \varphi_{jk} \circ \varphi_{ij}|_{X_{ij}\cap X_{ik}}$$

Thus, we get that we may glue them together by lemma 3.1.6 we have that this glued scheme is isomorphic to $X \times_S Y$, and through this isomorphism we can check that the universal property is satisfied.

By interchanging X and Y, we can now conclude from the proof on affine schemes that if X and Y are any schemes, and S is an affine, then $X \times_S Y$ exists. To show that S can be an arbitrary scheme, let $\{S_i\}$ be an open affine cover of S. Let $\alpha : X \to S$ and $\beta : Y \to S$ be two scheme morphisms, and let $X_i = \alpha^{-1}(S_i)$, $Y_i = \beta^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ is well-defined.

Now, what's key is that this scheme is the same as the scheme $X_i \times_S Y$. To see this, we must invoke the uniqueness of the universal property, and indeed if $f : Z \to X_i$ and $g : Z \to Y$ are scheme morphisms over S, by the properties of the commutative diagram we must have that the image of v must land inside Y_i. But by universality they must be equal. Thus $X_i \times_S Y$ is well-defined for each i, and so by the above argument it glues to create $X \times_S Y$, as we sought to show.

[a] If $f : Z \to U$, compose with the inclusion map and invoke the appropriate universal properties

Example 3.9: Fibered Products

1. Show that $\mathbb{A}^n_k \times_k \mathbb{A}^m_k \cong \mathbb{A}^{n+m}_k$ (as these space are affine, this should be "immediate" from the algebra)

2. Show that the fibered product of $\operatorname{Spec}\mathbb{C}$ and $\operatorname{Spec}\mathbb{C}$ over $\operatorname{Spec}\mathbb{R}$ is in fact the disjoint union $\operatorname{Spec}(\mathbb{C}) \amalg \operatorname{Spec}(\mathbb{C})$. More generally show that if L/K is a finite galois extension, then $L \otimes_K L$ is isomorphic to $L^{|G|}$ ($|G|$ copies of L), hint: this is the linear independence of characters proof in geometric form. Hence, over fields fibered products capture some idea properties of galois groups.

3. Recall that the intersection of rings is a ring. This is a special case of the fibered product in the category of commutative rings, namely that if $S \to R$ and $S' \to R$ exists, then:

$$\begin{tikzcd} & S \cap S' \arrow[dl, dashed] \arrow[dr, dashed] & \\ S \arrow[dr] & & S' \arrow[dl] \\ & R & \end{tikzcd}$$

Let us see how this translates to schemes and affine schemes. Recall that the product and coproduct of affine schemes is an affine scheme:

$$\operatorname{Spec} S \times \operatorname{Spec} S' = \operatorname{Spec} S \otimes_{\mathbb{Z}} S'$$
$$\operatorname{Spec} S \amalg \operatorname{Spec} S' = \operatorname{Spec} S \times S'$$

If $U, U' \subseteq \operatorname{Spec} R$ are open affine, $U = \operatorname{Spec} S$ and $U' = \operatorname{Spec} S'$, Then it is *not* the case that:

$$\operatorname{Spec} S \cap \operatorname{Spec} S' = \operatorname{Spec} S \cap S'$$

namely because adjoint relation between **CRing** and **Af-Sch** is contravariant. Consider for example $R = \mathbb{C}[x]$, $D(t), D(t-1) \subseteq \operatorname{Spec} R$. Then $D(t) \cap D(t-1) = D(t(t-1))$ and

$$\operatorname{Spec} D(t(t-1)) = \operatorname{Spec} \mathbb{C}\left[t, \frac{1}{t(t-1)}\right]$$

while $\mathbb{C}[t, \frac{1}{t}] \cap \mathbb{C}[t, \frac{1}{t-1}] = \mathbb{C}[t]$, hence:

$$\operatorname{Spec} \mathbb{C}[t]_{t(t-1)} = D(t) \cap D(t-1) = \operatorname{Spec} \mathbb{C}[t]_t \cap \operatorname{Spec} \mathbb{C}[t]_{t-1} \neq \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}$$

We may *not* take $S \cup S'$, as this is not a ring, and as these contain units $\langle S, S' \rangle = R$. Instead, we get:

$$\operatorname{Spec} S \cap \operatorname{Spec} S' = \operatorname{Spec} S \otimes_R S'$$

As the tensor product is a very versatile operation, the fibered product is as well. Let us demonstrate by translating some important results:

Proposition 3.2.2: Changing Basis Strategies

The following are common simple fibered products:

1. Let $\iota : U \to Z$ be an open embedding and $\varphi : Y \to Z$ be any scheme morphism. Then $U \times_Z Y$ is
$$(\varphi^{-1}(U), \mathcal{O}_Y|_{\varphi^{-1}(U)})$$
2. Let X be an R scheme, then $S \times_R X$ for $\varphi : R \to S$ is an S-scheme
3. Let $Y \subseteq X$ be a closed scheme. Then the fibered product with a closed subscheme is a closed subscheme of the fibered product

***Proof* :**
In each case 2, 3 and 4, you can reduce to the affine case (just like for the proof of the fibered product)

1. trace the argument to show this is the only option. A special consequence of this is that if $U, V \to Z$ are open embeddings, this is just their intersection.
2. They key result is that $S[t] \cong S \otimes_R R[t]$.
3. the key result is that if $\varphi : S \to R$ is a ring morphism with $I \subseteq S$ and $I^e \subseteq R$ the extension, then:
$$R/I^e \cong R \otimes_S S/I$$
Which can be shown using the right-exactness of $- \otimes_S R$ on $I \to S \to S/I$.
4. The key result is that if $\varphi : S \to R$ is a morphism $D \subseteq S$ a multiplicative subset, Then:
$$R[\varphi(D)^{-1}] \cong R \otimes_S (S[D^{-1}])$$

This motivates an important definition generalizing affine and projective schemes (idk if this should be here)

Definition 3.2.3: Affine Scheme Over A Scheme

Let X be a scheme. Then:
$$\mathbb{A}^n_X = X \times_{\mathbb{Z}} \mathbb{Z}[x_1, ..., x_n]$$

Example 3.10: Pushout Need Not Exist

Consider two copies of $\mathbb{A}^1_{\mathbb{C}}$. Consider:

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[dl] \arrow[dr] & \\ \mathbb{A}^1_{\mathbb{C}} & & \mathbb{A}^1_{\mathbb{C}} \end{tikzcd}$$

where $[(0)] \mapsto [(0)]$ in both cases. This is like the double-origin example, except every point is a double origin. This shall be shown to not be a scheme as it is "too non-separated".

We cannot fix this by limiting to the category of Affine schemes. Consider for example:

$$\begin{tikzcd} & \operatorname{Spec} k[t]_t \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} k[t] & & \operatorname{Spec} k[t] \end{tikzcd}$$

where $\operatorname{Spec} k[t] = \mathbb{A}^1_k \setminus \{0\}$, and the maps are the maps given by $t \mapsto t$ and $t \mapsto 1/t$. Then we saw in example 2.15 that the colimit of this diagram is projective space, $\mathbb{P}^1_k$. Hence affine schemes are not in general closed under pushout. Note that the corresponding fibered product does exist in **CRing**, namely it will have to be:

$$\begin{tikzcd} & k \arrow[dl, dashed] \arrow[dr, dashed] & \\ k[t] \arrow[dr] & & k[t] \arrow[dl] \\ & k[t]_t & \end{tikzcd}$$

However, applying Spec to this will not return the pushout, namely the pair of adjoint functors $(\operatorname{Spec}, \Gamma)$ does not preserve pushouts.

Note however that colimits do exist in the category of locally ringed spaces. See this link.

Fibers and family of Schemes

This gives a huge class of new object's that we may define. Let us start with defining fibers:

Definition 3.2.4: Fibers

Let $f : X \to Y$ be a morphism of scheme. Let $y \in Y$, and $\kappa(y)$ be the residue field of y. Let $\operatorname{Spec}(\kappa(y)) \to Y$ be the natural morphism[a]. Then the *fiber* of the morphism f over the point y is the scheme

$$X_y = X \times_Y \operatorname{Spec}(\kappa(y))$$

In the case where $X = \operatorname{Spec}(R), Y = \operatorname{Spec}(S)$ are affine, then for a point $\mathfrak{p} \in Y$:

$$f^{-1}(\mathfrak{p}) = \operatorname{Spec} R \times_S \operatorname{Spec} \kappa(\mathfrak{p}) = \operatorname{Spec}(R \otimes_S S_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}})$$

[a] exercise ref:HERE

Proposition 3.2.5: Fiber Property

Let $f : X \to Y$ be a morphism of schemes, $y \in Y$, and X_y the fiber of f over y. Then X_y is homeomorphic to $f^{-1}(y) \subseteq X$.

If this is unclear, we should expect the fibers of the projection $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ to be lines, that is $\mathbb{A}^1_{\mathbb{C}}$, and indeed for some $a \in \mathbb{A}^1_{\mathbb{C}}$:

$$\begin{aligned}
\mathbb{A}^2_{\mathbb{C}} \times_{\mathbb{A}^1_{\mathbb{C}}} \operatorname{Spec}(\kappa(a)) &= \operatorname{Spec}\left[(\mathbb{C}[x,y]) \otimes_{\mathbb{C}[x]} \left(\frac{\mathbb{C}[x]_{(x-a)}}{\mathfrak{m}_{(x-a)}}\right)\right] \\
&\cong \operatorname{Spec}\left[\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \operatorname{Frac}\left(\frac{\mathbb{C}[x]}{(x-a)}\right)\right] \\
&= \operatorname{Spec}\left[\frac{\mathbb{C}[x,y]}{(x-a)} \otimes_{\mathbb{C}[x]} \operatorname{Frac}\left(\frac{\mathbb{C}[x]}{(x-a)}\right)\right] \\
&\cong \operatorname{Spec}\left[\mathbb{C}[y] \otimes_{\mathbb{C}[x]} \mathbb{C}\right] \\
&\cong \operatorname{Spec} \mathbb{C}[y] \\
&= \mathbb{A}^1_{\mathbb{C}}
\end{aligned}$$

Proof **:**

Let $X'_y = f^{-1}(y) = \{x \in X \mid f(x) = y\}$ and consider the following commutative diagram:

$$\begin{tikzcd}
& Z \arrow[d, dashed] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "h"] & \\
& f^{-1}(y) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & & \{y\} \arrow[dl, "\mathrm{id}"] \\
& Y &
\end{tikzcd}$$

As $f(g(Z)) = h(Z) = y$, $f(g(Z)) = y$, so $f^{-1}(y) = g(Z)$. Hence, there is a natural map from Z to $f^{-1}(y)$ that makes the diagram commute. Now letting $\{y\} = \operatorname{Spec} \kappa(y)$, $f^{-1}(y)$ is homeomorphic to X_y by uniqueness of the universal object. As the subspace topology of $f^{-1}(y)$ and the Zariski topology match, $f^{-1}(y)$ is scheme-isomorphic to X_y, completing the proof.

In some ways, fibers of schemes are more "regular" than fibers of sets, as the extra information of schemes will provide some more possible pre-images:

Example 3.11: Fibers

1. Take the familiar projection of the parabola onto the x axis induced by the ring map $\mathbb{Q}[x] \to \mathbb{Q}[y]$ mapping $x \mapsto y^2$. We can think of this as the map $\mathbb{Q}[x] \to \mathbb{Q}[y^2]$, where the codomain has a nice isomorphism:

$$\mathbb{Q}[y^2] \cong \frac{\mathbb{Q}[x,y]}{(y^2-x)}$$

This makes the computations of fibers easier. The pre-image of 1 is:

$$\begin{aligned}
\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x, x-1) \\
&\cong \operatorname{Spec}\left(\mathbb{Q}[y]/(y^2-1)\right) \\
&\cong \operatorname{Spec}\left(\mathbb{Q}[y]/(y-1)(y+1)\right) \\
&\overset{\text{C.R.T}}{\cong} \operatorname{Spec}\left(\mathbb{Q}[y]/(y-1) \times \mathbb{Q}[y]/(y+1)\right) \\
&\cong \operatorname{Spec}\left(\mathbb{Q}[y]/(y-1)\right) \amalg \operatorname{Spec}\left(\mathbb{Q}[y]/(y+1)\right)
\end{aligned}$$

That is, it is the points ± 1. The pre-image of 0 is:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x, x) \cong \operatorname{Spec}\mathbb{Q}[y]/(y^2)$$

Notice that the pre-image remembers multiplicity information!

As the spectrum of a polynomial ring over rationals is the quotient by the galois action on the polynomial ring over $\overline{k}$, the pre-image of -1 exists! It is:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x, x+1) \cong \operatorname{Spec}\mathbb{Q}[y]/(y^2+1) \cong \operatorname{Spec}\mathbb{Q}[i] = \operatorname{Spec}\mathbb{Q}(i)$$

This can be thought of as "size 2" point over the base field. The pre-image of the generic point is a reduced point, and can be thought of as "size 2" over the residue field:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}(x) \cong \operatorname{Spec}\mathbb{Q}[y] \otimes_{\mathbb{Q}[y]} \mathbb{Q}(y^2)$$

Try working through the pre-image of other points that are not commonly associated to $\mathbb{Q}$. Notice that the pre-image is the two points, or you get nonreduced behaviour that is "of degree 2" or you get a field extension of degree 2.

2. Find the fibers of $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ over different points.

3. As $X \to \operatorname{Spec}(\mathbb{Z})$ is always well-defined and unique, there is always fibers over $\operatorname{Spec}(\mathbb{Z})$ and hence they are usually given a name: The fiber over (0) is often labeled $X_{\mathbb{Q}}$ (over $\mathbb{Q}$), and the fibers for all the primes are X_p (over $\mathbb{F}_p$). A common terminology is that X_p *arises by reduction* mod p of the scheme X.

On p. 260 of Vakil he gives more on general fibers, generic fibers, and generically finite morphisms

The reader may recall that we used tensor products to extend R-modules to S-modules and called it a *base change*. We can now motivate this terminology. If we have collection S-scheme, i.e. scheme morphisms $X \to S$, then we may want to change the "base" S to another scheme S'. This would be done by a *base change map* $S \to S'$. The resulting new S'-scheme would be $X' = X \times_S S'$. More generally

Definition 3.2.6: Scheme-theoretic Pullback

Let $Y \to Z$ and $X \to Z$ be scheme morphisms. Then the map $X \times_Z Y \to X$ is called the *scheme-theoretic pullback* of Y (or just pullback if it is clear we are working with schemes).

Example 3.12: Parameterization of Curves using Schemes

Take

$$X = \operatorname{Spec} \frac{k[x,y,t]}{(ty - x^2)}$$
$$Y = \operatorname{Spec} k[t]$$

Let $f : X \to Y$ be given by the natural ring homomorphism $k[t] \to \frac{k[x,y,t]}{(ty-x^2)}$. Note both these schemes are integral schemes of finite type over k. Now, identify the closed points of Y with k. For $a \neq 0 \in k$, we get the fibers correspond to the parabola's $ay = x^2$ or $y = x^2/a$. All of these are irreducible, reduced curves. At $a = 0$, we get $x^2 = 0$, which is a reduced scheme. Hence, the scheme $\operatorname{Spec} k[t]$ *paramaterized* the curves!

More generally, S-schemes can be geometrically interpreted as schemes paramaterized by the points of S whose maps are fiber-bundle morphisms.

A quick note must be done about general products and why it may be preferable to take the fibered product over a field:

Example 3.13: Strange Fibered Products

1. Show that $\operatorname{Spec}(\mathbb{Z}/m\mathbb{Z}) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(\mathbb{Z}/n\mathbb{Z}) = \emptyset$.
2. Another product of taking the "absolute" fibered product is that if $X = \operatorname{Spec} \mathbb{Z}[x]$ and $Y = \operatorname{Spec} \mathbb{Z}[y]$ then:

$$\dim X \times Y = \dim(X) + \dim(Y) - \dim(\operatorname{Spec} Z) = \dim X + \dim Y - 1$$

If we restrict to k-schemes for a field k, these two pathologies do not occur.

As these examples show, the fibered product of two schemes need not be the schemes of these two fibered products. In a concrete sense, this is because we must be more specific with what types of points we are dealing with. If we specify the points, we in fact recover the usual "sets"[4].

[4] The topological product can also be thought of as being over the right valued points

Proposition 3.2.7: Fibered Product With Z-valued Points

Let X, Y be schemes and take Z-valued points on each scheme (i.e. morphisms in $\text{Hom}(Z, X)$ and $\text{Hom}(Z, Y)$. Then show that the fibered product will be:

$$\text{Hom}(W, X \otimes_W Y) = \text{Hom}(Z, X) \times_{\text{Hom}(Z,W)} \text{Hom}(Z, Y)$$

Proof **:**
exercise.

Put an appropriate definition here for the following

Hence, given a surjective map $Y \to X$, Y can be regarded as a collection of schemes paramaterized by Y. Conversely, we may want to study a scheme X_0, and see if there are some deformations we may want to do. Then a common way of defining a *family of algebraic deformation* is by a scheme morphisms $f : X \to Y$ where one of the fibers of f is isomorphic to X_0.

Some more here on representable Functors

3.2.1 Product of Projective Scheme: Serge Embedding

I will elaborate on this later, these are good exercises

Exercise 3.2.1

1. Let $\varphi : X \to Y$ be a morphism of S-scheme of finite type. Show that if $S \to S'$ is any base extension, $\varphi' : X' \to Y'$ is an S'-scheme morphism preserves finite type.
2. Let $\varphi : X \to Y$ be a scheme morphism of integral schemes. Show the fibers needn't be irreducible or reduced.
3. Perhaps put here some base-change exercises relating the tensor product exercises you had in [3] and now give them their geometric counter-part.
4. Let $\varphi : B \to A$ be a ring morphism, $S \subseteq B$ a multiplicative subset, so $\varphi(S)$ is a multiplicative subset of A. Show that $\varphi^(S)^{-1}A \cong A \otimes_B (S^{-1}B)$. What does this imply for the relation between localization and the fibered product?
5. Show that if $U, V \subseteq X$ are open subschemes, then $U \times_Z V \cong U \cap V$
6. Let X be a k-scheme. Show that if L/k is a field extension, $X \otimes_{\text{Spec}\,(k)} \text{Spec}\,(L)$ is an L-scheme. Show that this in fact gives a functor (define the appropriate maps, and verify).

3.3 Types of Morphism

Many of these concepts are the generalizations of the types of schemes covered so far. All of these concepts can be expressed in terms of morphisms.

Go over this entire section and do the exercises

As schemes are rather general notions, so to scheme morphisms can be very general. The following notions are guiding principles presented by Vakil on what a good type of morphism should have. This terminology will be very useful in labeling many of the following lemmas an propositions. Let's say that C is a collection of morphisms between schemes. Then:

1. it is *local on target* if
   (a) $\pi : X \to Y$ is in the class, then for any open subset $V \subseteq Y$, the restricted homomorphism $\pi^{-1}(V) \to V$ is in the class
   (b) $\pi : X \to Y$ is a morphism, $\cup_i V_i = Y$ is an open cover where each $\pi^{-1}(V_i) \to V_i$ is in this class, then π is in this class

   Usually we consider affine opens, but it need not be the case.

2. is *local on source* if there is
   (a) $\pi : X \to Y$ is in the class, then for any open subset $U \subseteq X$, the restricted homomorphism $U \to \pi(U)$ is in the class
   (b) $\pi : X \to Y$ is a morphism, $\cup_i U_i = X$ is an open cover where each $U_i \to \pi(U_i)$ is in this class, then π is in this class

3. it is closed under composition

4. it is closed under fibered products (Go over exercise 9.4.B in Vakil, p.262. In general, that section can be put here)
   (a) As a special case, is closed under change of basis from an S-scheme to a T-scheme

3.3.1 Finiteness Conditions

Definition 3.3.1: Quasicompact Morphism

Let $\pi : X \to Y$ be a morphism of scheme. Then π is *quasicompact* if every open affine subset $U \subseteq Y$ has quasicompact pre-image, that is $\pi^{-1}(U)$ is quasicompact.

Example 3.14: Quasicompact Morphism

1. Let $f : \operatorname{Spec} R \to \operatorname{Spec} S$ be a scheme morphism. Then any $f^{-1}(U)$ for open affine $U \subseteq \operatorname{Spec} S$ is quasi-compact,
2. (give a non-affine example)
3. (give a non-quasicompact example, or at least outline it)

Definition 3.3.2: Quasiseparated Morphism

Let $\pi : X \to Y$ be a morphism of scheme. Then π is *quasiseparated* if every open affine subset $U \subseteq Y$ has quasicompact pre-image

Example 3.15: Quasiseparated Morphism

1. here

Definition 3.3.3: Affine Morphism

Let $\pi : X \to Y$ be a morphism of schemes. Then π is *affine* if for every affine open subset $U \subseteq Y$, $\pi^{-1}(U)$ is an affine scheme (interpreted as an open subscheme of X)

This condition would be nice to be affine-local on target. It takes some build-up:

Lemma 3.3.4: Affine Morphisms and Qcqs

Let $\pi : X \to Y$ be an affine morphism. Then it is quasicompact and quasiseparated.

***Proof* :**
recall that affine schemes are quasiseparated.

Lemma 3.3.5: Qcqs Lemma

Let X be a quasicompact and quasiseparated scheme and $s \in \mathcal{O}_X(X)$. Then

$$\mathcal{O}_X(X)_s \cong \mathcal{O}_X(X_s)$$

This theorem can be thought of as the generalization of the result where $X = \text{Spec}\,(R)$, we we already know that:

$$\mathcal{O}_{\text{Spec}\,(R)}\left(\text{Spec}\,(R)\right)_s \cong, \mathcal{O}_{\text{Spec}\,(R)}(\text{Spec}\,(R)_s) = \mathcal{O}_{\text{Spec}\,(R)}(D(s))$$

***Proof* :**
If X is quasicompact, cover it with finitely many affine open sets $U_i = \text{Spec}\, R_i$. take $U_{ij} = U_i \cap U_j$. Then the following is exact

$$0 \to \mathcal{O}_X(X) \to \prod_i R_i \to \prod_{i,j} \mathcal{O}_X(U_{ij})$$

As X is quasiseparated, each U_{ij} is covered by finitely many affine open sets $U_{ijk} = \text{Spec}\,(R_{ij})$ so that the following is exact:

$$0 \to \mathcal{O}_X(X) \to \prod_i R_i \to \prod_{(i,j),k} R_{ijk}$$

Now, localization (being an exact functor) gives:

$$0 \to \mathcal{O}_X(X)_s \to \left(\prod_i R_i\right)_s \to \left(\prod_{(i,j),k} R_{ijk}\right)_s$$

which is still exact. As localization commutes with finite products (importance on finiteness), we get for appropriates values:

$$0 \to \mathcal{O}_X(X)_s \to \prod_i (R_i)_{s_i} \to \prod_{i,j,k} (R_{ijk})_{s_{ijk}}$$

which again is still exact, where the s_i and s_{ijk} is is the restriction to the rings R_i and R_{ijk}. Now, the scheme X_s can be covered by the affine open sets $\text{Spec}\,(R_i)_{s_i}$ and $\text{Spec}\,(R_i)_{s_i} \cap \text{Spec}\,(R_{ijk})_{s_{ijk}}$ giving almost the same exact sequence:

$$0 \to \mathcal{O}_X(X_s) \to \prod_i (R_i)_{s_i} \to \prod_{i,j,k} (R_{ijk})_{s_{ijk}}$$

Hence, the kernel of the map

$$\mathcal{O}_X(X)_s \to \mathcal{O}_X(X_s)$$

is the same, and we get an isomorphism.

Proposition 3.3.6: Affine Morphisms Are Affine-local On Target

Let $\pi : X \to Y$ be a scheme morphism. Then π is an affine morphism if and only if there is a affine open cover $\cup_i U_i = Y$ such that $\pi^{-1}(U_i)$ is affine.

Proof **:**
We shall want to apply the affine communication lemma. For that, we must show the conditions are satisfied. First, suppose $\pi : X \to Y$ is affine over $\text{Spec}\,(S)$ so that $\pi^{-1}(\text{Spec}\,(S)) = \text{Spec}\,(R)$. Then notice that for any $s \in S$:

$$\pi^{-1}(\text{Spec}\,(S_s)) = \text{Spec}\,(R)_{\pi^\# s}$$

showing the first condition. For the next, suppose that $\pi : X \to \text{Spec}\,(S)$ and $(s_1, ..., s_n)$ with $X_{\pi^\# s_i} = \text{Spec}\,(R_i)$. We'll show that X is affine as well. To that end, let $R = \mathcal{O}_X(X)$. Then $X \to \text{Spec}\,(S)$ factors through $\alpha : X \to \text{Spec}\,(R)$ (given by the isomorphism $R \to \mathcal{O}_X(X)$), giving us:

$$\begin{tikzcd} \bigcup_i X_{\pi^\# s_i} \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & \text{Spec}\,(R) \arrow[dl, "\beta"] \\ & \bigcup_i D(s_i) = \text{Spec}\,(S) & \end{tikzcd}$$

Then, as we are working with scheme morphisms, the pre-image of a distinguished open set is a distinguished open set and so

$$\beta^{-1}(D(s_i)) = D(\beta^\#(s_i)) \cong \text{Spec}\, R_{\beta^\# s_i}$$

and $\pi^{-1}(D(s_i)) = \text{Spec}\,(R_i)$. Now, as X is quasicompact and quasiseparated, by the Qcqs lemma we have the isomorphism:

$$A_{\beta^\# s_i} = \mathcal{O}_X(X)_{\beta^\# s_i} = \mathcal{O}_X(X_{\beta^\# s_i}) = R_i$$

Hence, α induces an isomorphism $\text{Spec}\,(R_i) \cong \text{Spec}\,(R_{\beta^\# s_i})$, and so α is an isomorphism over each $\text{Spec}\,(R_{\beta^\# s_i})$, which covers $\text{Spec}\,(R)$, and hence α is an isomorphism, and so $X \cong \text{Spec}\,(R)$, completing the proof.

Corollary 3.3.7: Hypersurfaces and Affine Morphissm

Let X be a closed subset of $\text{Spec}\,(R)$ which is locally cut out by one equation ($\text{Spec}\,(R)$ is covered by open subsets Z which are each given by the quotient of one equation). Then the complement of Z, Y, is affine

(note: in the global case, $Y = \text{Spec}\,(R_f)$, but Y need not be of this form for the local case)

Proof **:**
exercise.

Example 3.16: Affine Morphisms

1. Let L/K be a field extension. Then $\text{Spec}\,(K) \to \text{Spec}\,(L)$ is an affine morphism.

The next type of morphisms are finite morphisms.

Definition 3.3.8: Finite Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *finite* if for every affine open set $\text{Spec}\,(S) \subseteq Y$, $\pi^{-1}(\text{Spec}\,(S))$ is the spectrum of a S-algebra that is finitely generated as a S-module.

Naturally, all finite morphisms are affine morphisms. Note how being a finitely generated S-module is a rather stronger condition than being a finitely generated S-algebra, consider $k[x]$ over k.

Proposition 3.3.9: Finite Morphism Affine Local On Target

Let $\pi : X \to Y$ be a scheme morphism. Then it is finite if there is an affine cover $\bigcup_i \text{Spec}\,(S_i) = Y$ such that $\pi^{-1}(\text{Spec}\,(S_i)$ is the spectrum of a finite S_i-algebra.

Proof **:**
exercise.

Proposition 3.3.10: Finite Morphism Composition Closed

The composition of two finite morphisms is finite

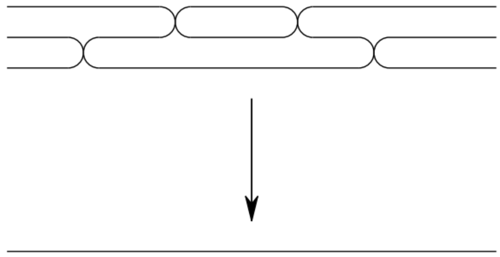
Proof **:**
exercise.

Example 3.17: Finite Morphism

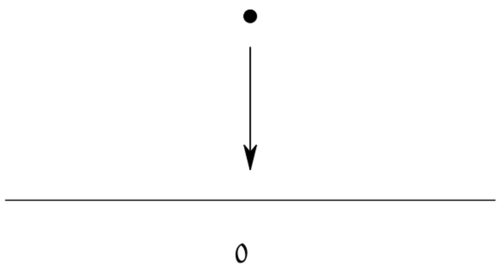
1. If L/K is a finite extension, $\text{Spec}\,(K) \to \text{Spec}\,(L)$ is a finite morphism.

2. All closed embeddings are finite morphisms.

3. Take the map $\text{Spec}\, k[t] \to \text{Spec}\, k[u]$ given by $u \mapsto p(t)$ for some degree n polynomial $p(t) \in k[t]$. Then this map is finite: notice that $k[t]$ is generated as a $k[u]$-module by $1, t, ..., t^{n-1}$. This can be visualized as:



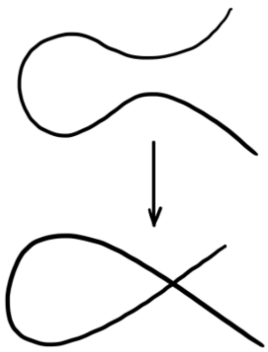
4. Let $I \subseteq R$ be an ideal and take the natural schemes and map $\text{Spec}\,(R/I) \to \text{Spec}\,(R)$. Then this is a finite morphism, as R/I is generated as an R-module by $1 \in R/I$. This can be visualized as:



5. Take $\text{Spec}\, k[t] \to \text{Spec}\, \frac{k[x,y]}{(y^2 - x^2 - x^3)}$ corresponding to the ring map given by:

$$x \mapsto t^2 - 1 \qquad y \to t^3 - t$$

Namely, this shows this curve is rational. This is a finite morphism, since $k[t]$ is generated as a $\frac{k[x,y]}{(y^2-x^2-x^3)}$-module by 1 and t. This can be visualized as:



As the reader may have noticed, there is only one point that is 2-to-1. The reader should check that this will induce an isomorphism between $D(t^2-1)$ and $D(x)$, i.e. if the node is removed there is an isomorphism. This shall come up many times as an example of normalizing a curve.

6. Let $X \to \operatorname{Spec} k$ be a scheme morphism. Show that if the morphism if finite, X is the finite union of points with the discrete topology, each points having as residue field a finite extension of k. This can be visualized as:

Notice that all the fibers were finite. This is no happenstance of the chosen examples:

Proposition 3.3.11: Finite Morphism is Finite-to-one

Let $\pi : X \to Y$ be a finite morphism. Then it's fibers are finite

Proof **:**
exercise. (Vakil p.214 for hint if you're stuck)

The converse is not the case

Example 3.18: Finite Fibers But Not Finite Morphisms

The open embedding $\mathbb{A}^2_k \setminus \{(0,0)\} \hookrightarrow \mathbb{A}^2_k$ has finite fibered (they are one-to-one), but it is not affine (as $\mathbb{A}^2_k \setminus \{(0,0)\}$ is not affine).

It also possible to have finite fibers and affine, but not finite: Take $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ (intuitively, localizing is "changing perspectives" to the infinitesimal).

It looks important to add 7.3.I to get a grasp on finite morphisms to affine spectrums relation to projective scheme

Definition 3.3.12: Integral Morphism

Let $\pi : X \to Y$ be a scheme morphism. The nit is an *integral morphism* if π is affine, and for every affine open subset $\operatorname{Spec}(S) \subseteq Y$, where $\pi^{-1}(\operatorname{Spec}(S)) = \operatorname{Spec}(R)$, the induced map $S \to R$ is an integral extension.

As integral extensions must be finitely generated over the source ring, an finite morphism is a integral morphism, however an integral morphism need not be finite (think $\operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Q}$). It is further easy to verify that the composition of integral maps is integral.

Proposition 3.3.13: Integral Morphisms Affine-local On Target

Let $\pi : X \to Y$ be an scheme morphism. Then π is an integral morphism if and only if there is a affine open cover $\cup_i U_i = Y$ such that we get integral extension for each pre-image.

***Proof* :**
exercise.

Proposition 3.3.14: Integral Morphism Are Closed Maps

Let $\pi : X \to Y$ be a Integral morphism. Then it is a closed map: it maps closed subsets to closed subsets

As a consequence, finite morphisms are closed

***Proof* :**
idea: reduce first to ring map., and consider the induced map. Now take $I \subseteq R$ and $J = (\pi^{\#})^{-1}(I)$ and consider the integral extension $S/J \subseteq A/I$. Apply the Lying Over theorem.

Definition 3.3.15: Locally Finite Type Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *locally of finite type* if for every affine open subset $\operatorname{Spec}(S) \subseteq Y$, and every affine open subset $\operatorname{Spec}(R) \subseteq \pi^{-1}(\operatorname{Spec}(R))$, the induced morphism $S \to R$ makes R a finitely generated S-algebra. Equivalently, $\pi^{-1}(\operatorname{Spec}(S))$ can be covered by affine open subsets $\operatorname{Spec}(R_i)$ where each R_i is a finitely generated S-algebra.

Definition 3.3.16: Locally Finite Type Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *of finite type* if it is locally of finite type and quasicompact.

Proposition 3.3.17: Locally Finite Type Morphism Affine-local On Target

Let $\pi : X \to Y$ be an scheme morphism. Then π is locally of finite type (resp. of finite type) if there is a cover of Y by affine open sets $\text{Spec}\,(S_i)$ such that $\pi^{-1}(\text{Spec}\,(S_i)$ is locally finite type over S_i.

Proof **:**
exercise.

Example 3.19: Finite Type Morphisms

1. All Schemes that are finite type over k are example of finite type scheme morphisms $X \to \text{Spec}\, k$
2. The map $\mathbb{P}^n_R \to \text{Spec}\, R$ is of finite type ($\mathbb{P}^n_R$ is covered by $n+1$ open sets)

Proposition 3.3.18: Integral and Finite Type iff Finite

1. finite morphisms are of finite type
2. A morphism is finite if it is integral and of finite type

Proof **:**
exercise.

Definition 3.3.19: Quasifinite Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *quasifinite* if it is of finite type, and for each $\mathfrak{q} \in Y$, $\pi^{-1}(\mathfrak{q})$ is a finite set.

By proposition proposition 3.3.11 and proposition 3.3.18(1), finite morphisms are quasifinite. The converse is not true (think of $\mathbb{A}^2_k \setminus \{(0,0)\} \to \mathbb{A}^2_k$. The finite type condition is necessary, there exists morphisms with finite fibers that are not quasifinite

Example 3.20: Finite Fibers, But Not Quasifinite

The morphism $\text{Spec}\,(\mathbb{C}(t)) \to \text{Spec}\,(\mathbb{C})$ has finite fibers, but is not quasifinite. Similarly to $\text{Spec}\,(\overline{\mathbb{Q}}) \to \text{Spec}\,(\mathbb{Q})$.

Quasifinite morphisms shall return when discussing Zariski Main Theorem (see ref:HERE).

Definition 3.3.20: Locally Finite Presentation Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *locally of finite presentation* if for every affine open subset $\mathrm{Spec}\,(S) \subseteq Y$, and every affine open subset $\mathrm{Spec}\,(R) \subseteq \pi^{-1}(\mathrm{Spec}\,(R))$, the induced morphism $S \to R$ makes R a finitely presented S-algebra. Equivalently, $\pi^{-1}(\mathrm{Spec}\,(S))$ can be covered by affine open subsets $\mathrm{Spec}\,(R_i)$ where each R_i is a finitely presented S-algebra.

There is one exception for the definition of finitely presented schemes that must be underlined in the following definition:

Definition 3.3.21: Finite Type Morphism

Let $\pi : X \to Y$ be a scheme morphism. Then π is said to be *finitely presented* if it is locally of finite type and quasicompact *and quasiseparated.*

Vakil promises exercise 9.3.H shows why this would naturally arise

Exercise 3.3.1

1. Let $\pi : X \to Y$ be an open embedding. Then if Y is locally Noetherian, X is locally Noetherian. If Y is Noetherian, X is Noetherian.
2. Let $\pi : X \to Y$ be an open embedding. Show that if Y is quasicompact, X need not be (there is an affine counter-example)
3. Show that being an open embedding is not local on the source.
4. Let B/A be an integral extension. Then show that any ring homomorphism $B \to C$ induces an integral extensions $C \otimes_B A/C$. Conclude that integrality is preserved by base change (see scalar extensions in [3] if you are unsure what it means to change base).
5. Show that every open embedding is locally of finite type. Hence, every quasicompact open embedding is of finite type.
6. Every open embedding into a locally Noetherian scheme is of finite type
7. The composition of two morphisms locally of finite type is locally of finite type.
8. Let $\pi : X \to Y$ be locally of finite type and Y locally Noetherian. Then X is locally Noetherian. IF $\pi : X \to Y$ is of finite type and Y is Noetherian, then X is Noetherian,
9. Show that quasifinite morphisms to $\mathrm{Spec}\,(k)$ are finite)
10. Let $k = \overline{\mathbb{F}_p}$. Show that the morphism $F : \mathbb{A}^n_k \to \mathbb{A}^n_k$ of k-schemes corresponding to the bijection $x_i \mapsto x_i^p$ is a bijection, and no power of F is the identity (as sets)
11. (Generalizing the above) Let X be an $\mathbb{F}_p$-scheme. Define an endomorphism $F : X \to X$ such that for each affine open subset $\mathrm{Spec}\,(R) \subseteq R$¡ F corresponds to the map $R \to R$ given by $f \mapsto f^p$. The morphism F is called the *absolute Frobenius morphism.*

3.3.2 Separated Morphisms

We now fix the notion of Hausdroffness through the notion of *separatedness*. The idea is that we want to avoid cases where there can be "double points" or even more points that are intuitively arbitrarily close to each other. In the topological setting, we saw X is Hausdorff if and only if $\Delta \subseteq X \times X$ is closed.

Definition 3.3.22: Separated Scheme

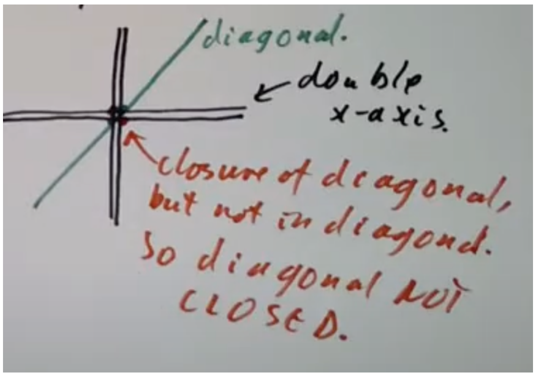
Let X be a scheme. Then X is *separated* if the map $X \to X \times X$ is a closed embedding.

More generally, a map $X \to S$ between schemes is *separated* if the map $X \to X \times_S X$ is a closed embedding.

Note that the first condition is equivalent to saying that the morphism $X \to \operatorname{Spec}(\mathbb{Z})$ is separated

Example 3.21: Separated Morphisms

1. Take the line with two origins over k, label it X. Then when considering $X \times_{\operatorname{Spec} k} X$, we can think to it as in image bellow. The double origin will correspond to 4 points, but the diagonal will only have 2 points. Taking the closure of the diagonal will include all 4 points, showing that it is *not* separated.



   Note that if we take $X \to X$ to be the identity map, then this *is* a separated morphism, since:
$$X \cong X \times_X X$$

2. Let $X = \operatorname{Spec}(A) \to \operatorname{Spec}(B) = Y$ be any morphism of schemes between affine schemes. Then this is *always* a separated morphism. Unwind the definitions: when converting into rings we get the map:
$$A \otimes_B A \to A$$
   which is surjective, and hence we get:
$$A \cong A \otimes_B A/\mathfrak{a}$$
   that is, we are taking the quotient by an ideal, which we know will give us a closed subscheme of $X \times_Y X$ isomorphic to X.

As a historical footnote, the term scheme used to mean separated scheme back before the 1960s, and the term pre-scheme was used for the modern term scheme.

Proposition 3.3.23: Characterizing Separated Morphisms

Let $f : X \to Y$ be a scheme morphism. Then f is separated if and only if the image of the diagonal morphism if closed in $X \times_Y X$

Proof **:**
exercise (if stuck, Hartshorne, p.96)

The next concept will not be too useful to us for awhile, and shall come back up when we see algebraic stacks and algebraic spaces, but is useful to define:

Definition 3.3.24: Quasi-separated

Let X be a scheme. Then X is *quasi-separated* if the image of $X \to X \times X$ is quasi-compact.

More generally, a map $X \to S$ between schemes is *quasi-separated* if the image map of $X \to X \times_S X$ is quasi-compact.

Proposition 3.3.25: Separated Implies Quasi-separated

Let X be a separated scheme. Then then X is quasi-separatd

Proof **:**
Show that any closed embedding is quasi-compact.

We shall next characterize separated schemes and morphisms in using valuation rings. This will be analogous to the fact that in a Hausdorff space, the limit of a sequence has at most 1 point. From this result, think of a continuous map $(0, 1) \to X$. If we want to extend this to a map $[0, 1] \to X$, if X is Hausdroff, there is only one possible extension. If X wasn't, say it had two origins, then there can be more than one map that can extend (the new 0 point will have two possible places to map to).

Let us now extend this idea to a map $X \to Y$, and we have an embedding $(0, 1) \to X$ an identity map $(0, 1) \to [0, 1]$ and an embedding $(0, 1] \to Y$ where we are only adding one point. Then there should only be one lift. Note that this lift is not dependent on X and Y. Let's say Y was the double-origin scheme and X was the double y-axis scheme. Then the embedding $(0, 1)$ can be extended to two way into Y, but that will correspond to a unique lift.

With this intuition in mind, let us now look at what is the equivalent of the open interval for schemes. This would be a Discrete valuation ring (DVR) R. It has only two points. We can think of as $\mathrm{Spec}\,(\mathrm{Frac}(R))$ as the open interval and $\mathrm{Spec}\,(R)$ as adding the closed 0 (notice that the new point in $\mathrm{Spec}\,(R)$ is also a closed point, while the other point is "open" and is the generic point that can be thought of was closed to everywhere else. Then we have the following situation

$$\begin{tikzcd} \operatorname{Spec}(\operatorname{Frac}(R)) \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

With the hope that having at most one lift corresponding to the map being separable. This is *sometimes* the case

Go over Harshorne's proofs

Theorem 3.3.26: Valuation Criterion for Separatedness

Let $f : X \to Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if for any field K, and for any valuation ring R with quotient field K, $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, and the morphism $i : U \to T$ is the morphism induced by the inclusion $R \subseteq K$, given a morphism of T to Y, and given a morphism of U to X, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

and there is at most one morphism of T to X making the whole diagram commutative.

Proof **:**
see Hartshorne theorem 4.3

This allows us to prove some important properties of proper morphism:

Proposition 3.3.27: Properties of Proper Morphism

Consider only Noetherian schemes. Then:

1. A closed embedding to X is proper
2. A composition of proper maps is proper
3. Proper morphism are stable under base-extension
4. Product of proper morphisms are proper
5. If $f : X \to Y$ and $g : Y \to Z$ are morphisms, and g is separated, and $g \circ f$ is proper, then f is proper.
6. Properness is local on the base

Proof **:**
exercise

Proposition 3.3.28: Intersection of Separated Affine Schemes

Let $X \to S$ be a separated scheme morphism where S is affine. Then if $U, V \subseteq X$ are open affine
$$U \cap V$$
is open affine.

Proof **:**
Here is the high-level

1. First show that $U \times_S V \subseteq X \times_S X$ is open and affine (it is the tensor-product of the corresponding algebras)
2. $U \cap V$ is the inverse image of $U \times_S V$ of the diagonal map.
3. For any closed embedding, the inverse image of an open affine is open-affine. Hence as f is separated, $U \cap V$ is affine, completing the proof.

Rational maps to Separated Schemes

here

3.3.3 Proper Morphism

Proper morphisms capture the idea of compactness. Compactness allows us to introduce some finiteness condition to our problem. In our case, this shall translate to the map from a proper scheme to another being closed over *any* basis. We shall see that almost all schemes we have encountered so far that are proper shall be able to be embedded into projective space, and projective space is in some way the prototypical proper scheme, and maps to projective space are the prototypical proper morphisms.

Recall that scheme morphisms need not be closed: their image need not be the vanishing set. It was shown in [3] that when working with projective varieties, their image is always closed with the intuition being that there are no "gaps" that can be created (the canonical example being $V(xy-1)$ projected onto the x-axis). For schemes, we shall require something a bit stronger, as we can have closed maps where when we change basis it is no longer closed (hence, it was somehow intrinsic to the choice of basis of the scheme). To fix this, we essentially *force* the maps to always be closed under any basis:

Definition 3.3.29: Universally Closed

Let $f : X \to Y$ be a continuous map. Then f is *universally closed* if for every map $Z \to Y$, the induced map $Z \times_Y X \to Y$ is closed.

The idea is that we want the image to always be closed, even if we extend the map by a "non-proper" space. For example $\mathbb{R} \to 0$ is a closed map, but $\mathbb{R} \times \mathbb{R} \to 0 \times \mathbb{R}$ is no longer a closed map, as it take $V(xy-1)$ to a non-closed set. To link this to the usual notion of proper:

A continuous map $f : X \to Y$ on locally compact Hausdroff spaces with a countable bases is universally closed if and only if it is proper (i.e. the pre-image of compact subsets are compact)

Another useful equivalent is $f : X \to Y$ is proper if f is closed and the fibers are compact.

On schemes, we shall also want the maps to be locally finite and be separate so that proper maps behave very similarly to embedding in projective space[5]

Definition 3.3.30: Proper Morphism

A morphism $f : X \to Y$ is called *proper* if it is separated, finite type, and universally closed. A scheme X is said to be proper if the morphism $X \to \text{Spec}\,(\mathbb{Z})$ is proper, and a k-scheme is proper if the maps $X \to \text{Spec}\,(k)$ is proper. An R-scheme is said to be *proper over* R if the map $X \to \text{Spec}\,(R)$ is proper.

Example 3.22: Proper Morphisms

1. The map $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ is separated, finite type, and closed. It is not proper, as we can take another $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$, take the fibered product::
$$\mathbb{A}^2_{\mathbb{C}} \cong \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$$
and we see that $V(xy-1) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is not closed in the image, namely it is $D(x) \subseteq \mathbb{A}^1_{\mathbb{C}}$.
2. All closed embeddings are proper

Proposition 3.3.31: Finite Morphisms Are Proper

Let $f : X \to Y$ be a finite morphism. Then f is proper

***Proof* :**
Here is the high-level: A finite morphism is certainly separated and finite type, so let us show it is universally closed. First, let's say $X \to Y$ is finite and we have a morphism $Z \to Y$. Then the fibered product $X \times_Y Z$ along with the map $X \times_Z Y \to Z$ is finite. Hence, it suffices to show that finite morphism maps are closed. By covering Y in affine open subsets, we reduce to the affine case. Hence, the problem comes down to having $\varphi : R \to S$ be a ring homomorphism where S is a finite R-module, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is closed. Next, if $I = \ker\varphi$, then $R \to R/I \subseteq S$ is a composition of closed maps, and so we can assume $R \subseteq S$.

Now, a closed set of S is given by an ideal $J \subseteq S$. Hence we get:
$$\frac{R}{R \cap \varphi^{-1}(J)} \subseteq \frac{S}{J}$$
As $\operatorname{Spec} \frac{R}{R\cap\varphi^{-1}(J)} \subseteq \operatorname{Spec} R$, we have now reduced to showing the map $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective. But this is given by the Going-up Theorem (see [3, chapter 20.5]), which completes the proof.

[5] We shall define *projective* morphisms later. The two concepts are similar, but a proper scheme need not have an ample bundle

Note that quasi-finite morphisms need not be proper: $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$ is one of the popular counter-examples.

Another very important family of proper morphism are projective morphism. Let us first define general projective space:

Definition 3.3.32: Projective Space Over Scheme

Let Y be a scheme. Then the *projective space* over Y is the scheme:

$$\mathbb{P}^n_Y := Y \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$$

Definition 3.3.33: Projective Morphism

Let $f : X \to Y$ be a scheme morphism. Then f is said to be *projective* if it factors into the following:

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "f"] & \mathbb{P}^n_Y \arrow[r] & Y \end{tikzcd}$$

A morphism is said to be *quasiprojective* if it factors into an open immersion $X \to X'$ followed by a projective morphism $X' \to Y$.

We shall show that projective morphisms are proper, and along with finite morphisms shall form a large class of proper morphisms (though not *all* proper morphisms fall under these two categories, see example ref:HERE). To show that projective morphisms are proper, we require a local description of properness similar to the one given for separatedness. To motivate it, let us consider the case in euclidean topology. Consider a function $f : \mathbb{R}_{>0} \to \mathbb{P}^1$. Then such a function may or may not have a limit as $x \to 0$, namely if it isolates arbitrarily fast. As the codomain is $\mathbb{P}^1$ we shall say it does have a limit of f shoots off to infinity. Now, for rational functions, it is impossible to oscillate arbitrarily quickly. Hence, it always has a limit as we tend to 0.

The fact that the limit always exists is a consequence of $\mathbb{P}^1$ being compact. Consider now a diagram:

$$\begin{tikzcd} \mathbb{R}_{>0} \arrow[r, "f"] \arrow[d, "\iota"] & X \arrow[d, "\pi"] \\ \mathbb{R}_{\geq 0} \arrow[r, "\text{extension}"'] \arrow[ur, dashed] & \mathbb{R} \end{tikzcd}$$

Then if there is always a lift, represented by the dotted line, we have a proper space. Visually, we can see this as:

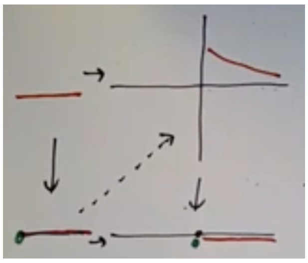


Figure 3.1: Visualizing point-wise condition

To change this into the language of scheme theory, we would replace $\mathbb{R}_{>0}$ with $\operatorname{Spec}\operatorname{Frac}(R)$ and $\mathbb{R}_{\geq 0}$ with $\operatorname{Spec} R$ for a DVR (or just a valuation ring) R. Then the top-right and bottom-right should be X and Y, and we want the map to be proper. Just like for separatedness, this works with a couple of extra relatively weak conditions:

Theorem 3.3.34: Valuation Criterion For Properness

Let $f : X \to Y$ be a scheme morphism of finite type and X is Noetherian. Then f is proper if and only if for every valuation ring R and every morphism $U \to X$ and $T \to Y$ there exists a unique lifting $T \to X$ such that following diagram commutes:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Proof **:**
First assume that f is proper. Then by definition f is separated, so the morphism $T \to X$ is unique once we know it exists. For the existence, we consider the base extension $T \to Y$, and let $X_T = X \times_Y T$. We get a map $U \to X_T$ from the given maps $U \to X$ and $U \to T$.

$$\begin{tikzcd} U \arrow[r] & X_T \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ & T \arrow[r] & Y \end{tikzcd}$$

Let $\xi_1 \in X_T$ be the image of the unique point t_1 of U. Let $Z = \{\xi_1\}^-$. Then Z is a closed subset of X_T. Since f is proper, it is universally closed, so the morphism $f' : X_T \to T$ must be closed, so $f'(Z)$ is a closed subset of T. But $f'(\xi_1) = t_1$, which is the generic point of T, so in fact $f'(Z) = T$. Hence there is a point $\xi_0 \in Z$ with $f'(\xi_0) = t_0$. So we get a local homomorphism of local rings $R \to \mathcal{O}_{\xi_0,Z}$ corresponding to the morphism f'.

Now the function field of Z is $k(\xi_1)$, which is contained in K, by construction of ξ_1, and R is maximal for the relation of domination between local subrings of K. Hence R is isomorphic to $\mathcal{O}_{\xi_0,Z}$, and in particular R dominates it. Hence by lemma ref:HERE (4.4 in Hartshorne) we obtain a morphism of T to X_T sending t_0, t_1 to ξ_0, ξ_1. Composing with the map $X_T \to X$ gives the desired morphism of T to X.

Conversely, suppose the condition of the theorem holds. To show f is proper, we have only to show that it is universally closed, since it is of finite type by hypothesis, and it is separated by theorem 3.3.26. So let $Y' \to Y$ be any morphism, and let $f' : X' \to Y'$ be the morphism obtained from f by base extension. Let Z be a closed subset of X', and give it the reduced induced structure.

$$\begin{array}{ccc} Z \subseteq X' & \longrightarrow & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

We need to show that $f'(Z)$ is closed in Y'. Since f is of finite type, so is f' and so is the restriction of f' to Z (Ex. 3.13). In particular, the morphism $f' : Z \to Y'$ is quasi-compact, so by (4.5) we have only to show that $f(Z)$ is stable under specialization. So let $z_1 \in Z$ be a point, let $y_1 = f'(z_1)$, and let $y_1 \rightsquigarrow y_0$ be a specialization. Let $\mathcal{O}$ be the local ring of y_0 on $\{y_1\}^-$ with its reduced induced structure. Then the quotient field of $\mathcal{O}$ is $k(y_1)$, which is a subfield of $k(z_1)$. Let $K = k(z_1)$, and let R be a valuation ring of K which dominates $\mathcal{O}$.

From this data, by lemma ref:HERE (4.4 in Hartshorne) we obtain morphisms $U \to Z$ and $T \to Y'$ forming a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow \\ T & \longrightarrow & Y' \end{array}$$

Composing with the morphisms $Z \to X' \to X$ and $Y' \to Y$, we get morphisms $U \to X$ and $T \to Y$ to which we can apply the condition of the theorem. So there is a morphism of $T \to X$ making the diagram commute. Since X' is a fibred product, it lifts to give a morphism $T \to X'$. And since Z is closed, and the generic point of T goes to $z_1 \in Z$, this morphism factors to give a morphism $T \to Z$. Now let z_0 be the image of t_0. Then $f'(z_0) = y_0$, so $y_0 \in f'(Z)$. This completes the proof.

Proposition 3.3.35: Projective Morphism Are Proper

1. Let $f : X \to Y$ be a projective morphism between Noetherian Schemes. Then f is proper.
2. Let $f : X \to Y$ be a quasi-projective morphism between Noetherian Schemes. Then f is of finite type and separated.

***Proof* :**
It suffices to show that $X = \mathbb{P}^n_{\mathbb{Z}}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Recall that X is a union of open affine subsets $V_i = D(x_i)$, and that V_i is isomorphic to $\operatorname{Spec}\mathbb{Z}[x_0/x_i, \ldots, x_n/x_i]$. Thus X is of finite type. To show that X is proper, we will use the valuation criterion for proneness and imitate the proof of that a regular map on a projective variety minus a point can uniquely be extended to that point. So suppose given a valuation ring R and morphisms $U \to X$, $T \to \operatorname{Spec}\mathbb{Z}$ as shown:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}. \end{tikzcd}$$

Let $\xi_1 \in X$ be the image of the unique point of U. Using induction on n, we may assume that ξ_1 is not contained in any of the hyperplanes $X - V_i$, which are each isomorphic to $\mathbb{P}^{n-1}$. In other words, we may assume that $\xi_1 \in \bigcap V_i$, and hence all of the functions x_i/x_j are invertible elements of the local ring $\mathcal{O}_{\xi_1}$.

We have an inclusion $k(\xi_1) \subseteq K$ given by the morphism $U \to X$. Let $f_{ij} \in K$ be the image of x_i/x_j. Then the f_{ij} are nonzero elements of K, and $f_{ik} = f_{ij} \cdot f_{jk}$ for all i, j, k. Let $v : K \to G$ be the valuation associated to the valuation ring R. Let $g_i = v(f_{i0})$ for $i = 0, \ldots, n$. Choose k such that g_k is minimal among the set $\{g_0, \ldots, g_n\}$, for the ordering of G. Then for each i we have

$$v(f_{ik}) = g_i - g_k \geq 0$$

hence $f_{ik} \in R$ for $i = 0, \ldots, n$. Then we can define a homomorphism

$$\varphi : \mathbb{Z}[x_0/x_k, \ldots, x_n/x_k] \to R$$

by sending x_i/x_k to f_{ik}. It is compatible with the given field inclusion $k(\xi_1) \subseteq K$. This homomorphism φ gives a morphism $T \to V_k$, and hence a morphism of T to X which is the one required. The uniqueness of this morphism follows from the construction and the way the V_i patch together.

Exercise 3.3.2

1. Show that two copies of Spec $(k[x_1, x_2, ...])$ minus the origin, glued at the origin, is not quasi-separated.

3.3.4 Étale Morphisms: Smoothness

smooth vs regular

3.4 Application: Variety

Using the definitions above, we may recover the notion of a variety in the language of schemes:

Definition 3.4.1: k-Variety

Let X be a k-scheme. Then if X is reduced, irreducible, separated, and of finite type over k, then X is said to be a *variety over* k or a k*-variety*. If X is affine/(quasi)projective, then it is an affine/(quasi)projective variety over k respectively.

It is important to note that in [3], a variety was an affine algebraic set that was irreducible, which motivates adding irreducible in the definition. However some authors drop the condition and "variety" would be "algebraic set". Furthermore, some authors require the field $k = \overline{k}$ be algebraically closed as part of the definition to more closely reflect the geometric results of the Nullstellensatz. Note too that we have define k-varieties and not varieties in general.

Definition 3.4.2: Complete Variety

Let X be a k-variety. Then it is complete if it is proper

The name complete for a proper variety comes from the fact that we are in some way thinking of a complete variety as an extension into projective Space. It can be shown (see ref:HERE[6]) that if C is complete, then there exists a morphism $C \to \mathbb{P}^n$ to a projective scheme that is isomorphic to an open dense subset. There do exist complete non-projective varieties, but they are hard to come by, partly due to how badly the singularity must be[7].

Proposition 3.4.3: Affine and Projective Varieties

1. $\mathbb{A}^n_k/I$ is an affine k-variety if and only if $I \subseteq k[x_1, ..., x_n]$ is a radical ideal
2. Let $I \subseteq k[x_1, ..., x_n]$ be a radical graded ideal. Then $\operatorname{Proj} k[x_0, ..., x-n]/I$ is a projective k-variety [a]

[a] I need not be radical, ex. $I = (x_0^2, x_0x_1, ..., x_0x_n)$

Proof **:**
exercise.

In proposition 3.1.14, we shall show that a certain subset of schemes will fully and faithfully map to the category of $\overline{k}$-varieties.

(Put somewhere that the fibered product of finite type k-schemes over a finite type k-scheme is a finite type k-scheme, and so varieties behave well under fibered product)

[6] see Borcherds scheme video *abstract and projective varieties*

[7] again see the same Borcherds video for an overview. The example he gave also allows for producing *algebraic spaces* that are not schemes

4

Sheaves of Modules

So far, we have only worked with the structure sheaf $\mathcal{O}_X$ over a scheme X; we can put other sheaves that shall give use many important insights, similarity to how different [vector] bundles on manifolds provide a lot of useful information. All of these sheaves shall have an $\mathcal{O}_X$-action on them, making them the generalization of modules to ringed spaces. The most important of these sheaves shall be ones that are locally isomorphic to sheaves corresponding to R_i-modules. Such sheaves shall be called *quasi-coherent sheaves*, and if they are finitely generated they shall be called *coherent shaves*. The term "coherent" comes from Serre who saw these sheaves as "cohering" (i.e. sticking together). The additional "quasi-" for non-finitely generated modules comes from loosing some results when omitting finite generation (for example, they are not in general preserved under pushforward).

By working with quasi-coherent sheaves, we shall be able to define *ideal sheaves*, *image schemes*, *twisted sheaves* on projective space (which leads to recovering $S_\bullet$ from $\operatorname{Proj} S_\bullet$), and *differential*, among many more new interesting sheaves on schemes that will give lots more information on schemes.

4.1 Definition and Basic Properties

Let us start by defining the module-equivalent for ringed spaces:

Definition 4.1.1: $\mathcal{O}_X$-Module

Let $\mathcal{O}_X \in \mathbf{Sh}_{\mathbf{Ring}}(X)$. Then an *$\mathcal{O}_X$-module* on X is a sheaf $\mathscr{F} \in \mathbf{Sh}_{\mathbf{Ab}}(X)$ that has the following data:

1. For each open $U \subseteq X$ $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module
2. if $U \subseteq V$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathscr{F}(V) \arrow[r, "\curvearrowright"] \arrow[d, "\mathrm{res}_{V,U} \times \mathrm{res}_{V,U}"'] & \mathscr{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\
\mathcal{O}_X(U) \times \mathscr{F}(U) \arrow[r, "\curvearrowright"] & \mathscr{F}(U)
\end{tikzcd}$$

If $\mathscr{F}, \mathscr{G}$ are $\mathcal{O}_X$-modules, then a *morphism of $\mathcal{O}_X$-modules* $\varphi : \mathscr{F} \to \mathscr{G}$ is a sheaf-morphism such that for each open $U \subseteq X$, $\mathscr{F}(U) \to \mathscr{F}(X)$ is a $\mathcal{O}_X(U)$-module homomorphism. Their collection is often denoted $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ or sometimes just $\mathrm{Hom}_X(\mathscr{F}, \mathscr{G})$ if it is unambiguous.

These should really be thought of as *fibre bundles* or even *vector bundles* on X. The latter will be more appropriate for quasi-coherent sheaves, which can be thought of as the "closure" of the category of vector-bundles on a scheme X to make the category abelian. Nevertheless, keeping bundles in mind when working with $\mathcal{O}_X$-modules will help build up intuition.

We define the pushforward and pullback of $\mathcal{O}_X$-modules on ringed spaces for completeness, even though this won't come back with the exception of a couple of exercises:

Definition 4.1.2: $\mathcal{O}_X$-module Pushforward and Pullback

let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a ringed space morphism, and $\mathscr{F}$ is a $\mathcal{O}_X$-module on X. Then $f_*\mathscr{F}$ is a $f_*\mathcal{O}_X$-module on Y. Via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we see that this has a natural $\mathcal{O}_Y$-module structure, and it is called the *direct image* of $\mathscr{F}$ via f. We can do a similar chain of reasoning for $f^{-1}\mathcal{O}_Y$ creating the *inverse image* of $\mathscr{F}$ which is adjoint to the direct image.

Example 4.1: $\mathcal{O}_X$-Modules

1. Take $\mathscr{F} = \mathcal{O}_X$ with action on each section being defined by multiplication. Then then this $\mathcal{O}_X$-module as any ringed space $\mathcal{O}_X$ is a $\mathcal{O}_X$-module over itself. As each restriction map is a ring-homomorphism, it is naturally $\mathcal{O}_X(U)$-module homomorphism, and hence the action is preserved.
2. Just like how every abelian group is a $\mathbb{Z}$-module, every sheaf of abelian groups on some space X is a $\underline{\mathbb{Z}}$-module (where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$), showing how this is a natural generalization.
3. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a $\mathcal{O}_X$-module homomorphism, then the kernel, image, and cokernel are all $\mathcal{O}_X$-modules (generalizing the result form the case of sheaf morphisms). It is also closed under products and coproducts, making $\mathcal{O}_X$-modules an abelian category.

We can define the *sheaf of ideals* on X, $\mathscr{I}$, which has for it's sheaf structure $\mathscr{I}(U) \subseteq \mathcal{O}_X(U)$ given by the kernel of an $\mathcal{O}_X$-module map with $\mathcal{O}_X$ a structure sheaf.

4. For cohomological purposes, we can always define the sheaf-hom $\mathcal{O}_X$ module given by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

where $\mathcal{O}_X|_U$ is given the natural restriction to a $\mathcal{O}_X|_U$-module on U.

5. The tensor product $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is the sheafification of the presheaf given by $U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$. The sheafification is necessary and shall in fact cause very non-trivial behaviour, namely it shall not be the case that:

$$\mathscr{F} \otimes \mathscr{G}(U) \cong \mathscr{F}(U) \otimes \mathscr{G}(U)$$

and in fact it can go spectacularly wrong, see example 4.12

An important class of examples (and historically the first studied class of examples) were inspird by vector bundles. Recall that a vector-bundle can be thought of as a local-trivialization, that is for every point $x \in M$ there is a open neighborhood $x \in U \subseteq M$ such that there is a map $\pi^{-1}(U) \cong_{\mathbf{Top}} U \times \mathbb{R}^k$ respecting projecting and isomorphic on fibers. In other words, it is locally free. If there is a global trivialization, than it is simply free. These shall be some of the basic examples we shall keep in mind:

Definition 4.1.3: Free $\mathcal{O}_X$-module

Let X be a ringed space and $\mathscr{F}$ a $\mathcal{O}_X$-module. Then $\mathscr{F}$ is *free* if

$$\mathscr{F} \cong_{\mathcal{O}_X} (\mathcal{O}_X)^n$$

$\mathscr{F}$ is *locally free* if X can be covered in open sets U such that $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module. The rank of each $U \subseteq \mathscr{F}$ is the number of copies of $\mathcal{O}_X$ (the "dimension", whether finite or infinite).

If X is connected then the rank must be globally the same.

Example 4.2: Locally Free Sheaves and Vector Bundles

For a more detailed review of the differential geometry portion of this example, see [6, chapter 4.1]. Let $(M, \mathcal{O}_M)$ be a smooth n-manifold with the sheaf of smooth functions and let $\pi : V \to M$ be a vector bundle over M. Then the sheaf of sections $\sigma : M \to V$ is an $\mathcal{O}_M$-module; label this sheaf $\mathscr{F}$ so that the sections over U are denoted $\Gamma(U, \mathscr{F})$. It is indeed a $\mathcal{O}_M$-module: given a smooth function $f \in \mathcal{O}_M(U) = C^\infty(U)$ where $U \subseteq M$ and $s \in \Gamma(U, \mathscr{F})$ then $f \cdot s$ is defined as

$$(f \cdot s)_p = f(p) \cdot s_p$$

where s_p is the value of the section at p. Then given the trivialization $\rho : \pi^{-1}(U) \to U \times \mathbb{R}^n$, the section s can be seen as a section over the trivialization:

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "s"'] \end{tikzcd}$$

in Hence, every section s can be seen as a map:

$$s : U \to U \times \mathbb{R}^n \qquad p \mapsto (p, f_1(p), ..., f_n(p))$$

where each $f_i : U \to R$ is smooth. From this, we can naturally see that:

$$\Gamma(U, \mathscr{F}) \cong (\mathcal{O}_U)^n$$

Hence, it is locally free of rank n. You should check that the transition condition and the cocycle condition match, showing that every vector bundle induces a locally free sheaf. Conversely, every locally free sheaf can induce "vector bundle" (see ref:HERE for the important part of the construction). We get the following equivalence:

$$\left\{\begin{matrix}\text{Vector Bundles } V \\ \text{on } X\end{matrix}\right\} \underset{\mathscr{F} \mapsto \operatorname{Spec} \operatorname{Sym}(\mathscr{F}^\vee)}{\overset{V \mapsto \mathscr{F}}{\rightleftarrows}} \left\{\begin{matrix}\text{locally free} \\ \text{sheaves on } X\end{matrix}\right\}$$

A subset of locally free sheaves to point out are those that are rank 1. A locally free $\mathcal{O}_X$-module of rank 1 (over a *locally* ringed space) is called an *invertible sheaf*. If the reader knows some K-theory (see [1]) to quickly draw the connection recall that we can take the monoid of $\mathcal{O}_X$-module along with the tensor product. The prototypical example in algebra is the monoid of k-modules. Then if $\mathscr{F} \otimes \mathscr{F}^\vee \cong \mathcal{O}_X$ (or for R-modules $M \otimes_R M^\vee \cong R$), then these would be the *invertible* elements in this monoid. Then every locally free sheaf of rank 1 is invertible. If X is a locally ringed space (*not* just a ringed space) then this becomes and iff. Hence, it is common to call them a line bundle as well. Due to the fact that they make a group, they shall appear a lot, and shall play a central role in section 4.3.

Definition 4.1.4: Invertible Sheaf or Line Bundle

Let X be a ringed space and $\mathscr{F}$ a $\mathcal{O}_X$-module. Then $\mathscr{F}$ is called an *invertible sheaf* or *line bundle* if it is locally free of rank 1.

4.1.1 Quasi-Coherent and Coherent Sheaves

A natural question then is whether the category of R-modules R-**Mod** and the category of $\mathcal{O}_X$-modules $\mathcal{O}_X$-**Mod** are equivalent when $X = \operatorname{Spec} R$. As it turns out, there are in fact too many $\mathcal{O}_X$-modules:

Example 4.3: Too Many $\mathcal{O}_X$-modules Over Affine Scheme

Let $X = \operatorname{Spec} \mathbb{Z}_{(2)}$ which consists of two open sets X and $\{\eta\}$. For an $\mathcal{O}_X$-module, we shall need to define a pair of modules, an $\mathbb{Z}_{(2)}$-module M and a $\mathbb{Q}$-module N, along with a restriction module homomorphism:

$$M \to N$$

But now, we would need that $N = M\left[\frac{1}{2}\right]$. With this condition, notice that we can set:

$$M = \mathbb{Z}_{(2)} \qquad N = 0$$

and this would *not* be a sheaf of modules coming form over R!

To fix this, we require that our $\mathcal{O}_X$-module must locally look like sheaf modules. Let us start with defining the "affine scheme", namely the sheaf modules over $X = \operatorname{Spec} R$. Then given any R-module M, we shall want an $\mathcal{O}_X$-modules $\widetilde{M}$ that locally looks like modules with a structure sheaf. We saw that every ring R has is naturally a scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. In the same vein, every R-module is naturally a $\mathcal{O}_{\operatorname{Spec} R}$-module

Definition 4.1.5: Sheaf Associated to M [on affine scheme]

Let R be a ring and M and R-module. Then the *sheaf associated to* M on $\operatorname{Spec} R$ is a sheaf given by the basis:

$$\widetilde{M}(D(f)) = M_f \qquad \text{(an } R_f\text{-module)}$$

that is, the localization of module M at the ring element f, and define the natural restriction maps. Denote this $\mathcal{O}_{\operatorname{Spec} R}$-module $\widetilde{M}$.

The judicious reader may check that this is indeed s sheaf on $\operatorname{Spec} R$. As we have now introduced another sheaf structure on X, we will start being conscious of our representation of the local and global sections we may write $\Gamma(X, \widetilde{M})$. The usual $\mathcal{O}_X(X)$ notation for a structure sheaf can be written as $\Gamma(X, \mathcal{O}_X)$.

Definition 4.1.6: Quasi-coherent Sheaf

Let $(X, \mathcal{O}_X)$ be a scheme. Then a $\mathcal{O}_X$-module $\mathscr{F}$ is called *quasi-coherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} R_i$ such that for each i there exists an R_i-module M_i such that:

$$\mathscr{F}|_{U_i} \cong \widetilde{M_i}$$

We shall also have the finitely generated equivalent of the above, however we must be a bit careful as the kernel of between two finitely generated R-modules need not be finitely generated, for example take the ring homomorphism between these modules over themselves

$$k[x_1, x_2, ...] \to k[x_1, x_2, ...] \qquad x_i \to 0$$

for all $i \in \mathbb{N}$. Then the kernel is $(x_1, x_2, ...)$ which is not finitely generated. The reader could check that the image and cokernel are always finitely generated, hence this is specifically a kernel problem. We simply force the required condition

Definition 4.1.7: Coherent Module

Let R be an R-module. Then if R is finitely generated and all kernel of maps $\operatorname{Hom}_R(R^n, M)$ are finitely generated for all $n \in \mathbb{N}$, then M is called *coherent*.
A ring is called coherent if it is a coherent R-module over itself.

Then if a sheaf if covered by coherent modules, it is coherent

Definition 4.1.8: Coherent Sheaf

Let $(X, \mathcal{O}_X)$ be a scheme. Then a $\mathcal{O}_X$-module $\mathscr{F}$ is called *quasi-coherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} R_i$ such that for each i there exists a coherent R_i-module M_i such that:

$$\mathscr{F}|_{U_i} \cong \widetilde{M_i}$$

For the reader interested in complex geometry, coherent and quasi-coherent sheaves can be defined on ringed spaces, and this is done in particular by complex geometers to include their setting.

4.1.1 Distinction from Hartshorne n Hartshorne, coherent sheaves are defined as sheaves over finitely generated modules, but then says they they are a bit pathological and so we always impose the Noetherian condition, which for us is a sufficient condition for a module to be coherent.

We will immediately show that quasi-coherent sheaves are independent of choice of covering:

Proposition 4.1.9: Quasi-coherence is Affine-local

Let X be a scheme. Then a $\mathcal{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} R \subseteq X$, there is an R-module M such that

$$\mathscr{F}|_U \cong_R \widetilde{M}$$

In other words, we can apply the affine communication lemma (theorem 2.4.2). If X is Noetherian, then we can replace quasi-coherent with coherent and M is a finitely generated R-module

Proof **:**
As $X = \bigcup_i U_i$ can be covered in affine open sets and each of these are covered by distinguished open sets, we may reduce to proving this in the case of $X = \operatorname{Spec} R$. To that end, let $M = \Gamma(X, \mathscr{F})$. We need to show there is an isomorphism $\varphi : \widetilde{M} \to \mathscr{F}$. As $\mathscr{F}$ is quasi-coherent, for each $g_i \in X$ we have:

$$\mathscr{F}|_{D(g_i)} \cong \widetilde{M_i}$$

for some R_{g_i}-module M_i. But then

$$\mathscr{F}(D(g_i)) \cong M_{g_i}$$

and so $M_i = M_{g_i}$, and so raising these maps we get that φ is an isomorphism.

By simply addition the conditions of X being Noetherian and M finitely generated we get the corresponding strengthening.

Hence, we have the equivalence of the correspondence between rings and Affine schemes for [quasi] coherent sheaves:

Corollary 4.1.10: Equivalence of Categories: Module Version

If $X = \operatorname{Spec} R$, then the functor $M \mapsto \widetilde{M}$ is an equivalence of categories between R-modules and quasi-coherent $\mathcal{O}_X$-modules with inverse function $\mathscr{F} \mapsto \Gamma(\operatorname{Spec} R, \mathscr{F})$. If R is Noetherian, then the equivalence is between finitely generated R-modules and coherent $\mathcal{O}_X$-modules.

***Proof* :**
Follows immediately

Example 4.4: Quasi-Coherent Sheaves

1. Certainly, the usual structure sheaf $\mathcal{O}_X$ on X is quasi-coherent (even coherent). At each point $[\mathfrak{p}]$, we have the $R_\mathfrak{p}$-module $R_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}$, that is every point has the germ as it's stalk. We can think of all of these stalks as being one-dimension vector-spaces (as they are all $R_\mathfrak{p}$-modules over themselves).

2. Let $X = \operatorname{Spec} k[x, y]$ and Take the $k[x, y]$-module $k[x, y]/(f)$ for some irreducible f. Now take a point $[\mathfrak{p}] \in X$. If $f \notin \mathfrak{p}$, then
$$\left(\frac{k[x, y]}{(f)}\right)_\mathfrak{p} \cong 0$$
On the other hand, if $f \in \mathfrak{p}$, we essentially get an 1-dimensional module. Visually, there is a line-bundle on $\operatorname{Spec} k[x, y]/(f) \subseteq \operatorname{Spec} k[x, y]$ which is zero everywhere else.

   For example, if $f = (x)$, and $x \in \mathfrak{p}$ then $(k[y])_\mathfrak{p}$ is just the germ at that point.

   More generally, let $Y \subseteq X$ be a closed subscheme defined by an ideal $\mathfrak{a} \subseteq R$. Then given the inclusion map $\iota : Y \to X$, $\iota_* \mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module (even coherent); show it is isomorphic to $\widetilde{R/\mathfrak{a}}$.

3. Take $X = \operatorname{Spec} \mathbb{Z}$, let $M = \mathbb{Z}/2\mathbb{Z}$ be the usual $\mathbb{Z}$-module, and take $\widetilde{M}$. Then this sheaf has a single non-trivial stalk at 2 for:
$$M_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$$
For any other prime p (and 0):
$$M_{(p)} \cong 0$$
Hence, we can think of this module just being a fibre bundle with only one non-trivial fiber. For another example, take the $\mathbb{Z}$-module $M = \mathbb{Z}/12\mathbb{Z}$. Then we have:
$$M_{(2)} \cong \mathbb{Z}/4\mathbb{Z} \qquad M_{(3)} \cong \mathbb{Z}/3\mathbb{Z}$$
and zero everywhere else. We can picture this as a stalk at 2 with "twice" the size of the vector space (note that $\mathbb{Z}/2\mathbb{Z}$ is a vector-space over itself), and the stalk at 3 has just a vector-space.

4. Let $U \subseteq X$ be an open subscheme with inclusion map $\iota : U \to X$. Inspired by exercise 1.4.1, define the sheaf $\iota_!(\mathcal{O}_U)$ by extending $\mathcal{O}_U$ by zero outside of 0. Then this is an $\mathcal{O}_X$-module, but *not* in general quasi-coherent! To see this, let X be any integral scheme so that each

$\mathcal{O}_X(W)$ is an integral domain. Take $V = \operatorname{Spec} R$ to be an open affine subset of X not contained in U but $U \cap V \neq \emptyset$. Then $\iota_!(\mathcal{O}_U)|_V$ contains no global sections of V, but is *not* the zero sheaf. But then it cannot be of the form $\widetilde{M}$ for any R-module M.

5. It is important the we took the pushforward in the above example. Let Y be a closed subscheme of X (not necessarily from an ideal). Then $\mathcal{O}_X|_Y$ is not in general quasi-coherent on Y, and usually not even a $\mathcal{O}_Y$-module. Let $X = \operatorname{Spec} k[x, y]$ and Y be the closed subscheme given by (y^2), so that $Y = \operatorname{Spec} k[x, y]/(y^2)$. Then in $\mathcal{O}_Y(U)$ where $U \subseteq Y$, we have that the function y is nilpotent: $y^2 = 0$. However, on $\mathcal{O}_X|_Y(U)$ there is no nilpotence! In particular, when restricting, we are restricting to Y as a *set*! Hence, we see that $\mathcal{O}_X|_Y$ is not even a $\mathcal{O}_Y$-module

6. Some of the simplest quasi-coherent sheaves are *free sheaves* where $\mathscr{F} \cong \mathcal{O}_X^{\oplus I}$, and locally free sheaves, where there is an open cover such that $\mathscr{F}|_{U_i} \cong \mathcal{O}_X^{\oplus I}$. In fact, we can think of quasi-coherent sheaves as the smallest sub-category of $\mathcal{O}_X$-**Mod** that is an abelian category and includes locally-free sheaves (show that the category of locally-free sheaves along with $\mathcal{O}_X$-homomorphisms is not abelian, take the trivial line bundle on $\mathbb{A}^1_{\mathbb{C}}$ with coordinate t and multiply it by t). The way we take the closure is simply by adding the kernels and cokernels of of locally-free sheaves, showing that they can be thought of as the "building blocks" of quasi-coherent sheaves.

7. Let us give the scheme equivalent of the Hedgehog (or Hairy Sphere[a]) Theorem. Take $X = \operatorname{Spec} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$. All the closed points of this spectrum correspond to the sphere, and there are many non-closed points. Labeling this ring as R, take the R-module M to be the submodule of $R \oplus R \oplus R$ of elements (X, Y, Z) where $xX + yY + zZ = 0$ (note that x, y, z here represent the indeterminates of the ring). This module represent all vectors that are orthogonal to the points of the sphere. Then this sheaf will on each open set look like $\mathcal{O}_X \oplus \mathcal{O}_X$ (similar to the sphere example), but it cannot look like it globally (or else its change of basis to $\mathbb{C}$ and then analytification would contradict the hairy ball theorem).

8. As mentioned, quasi-coherent sheaves are the (proper) closure of locally free sheaves, which means there must be non-locally free sheaves. We shall see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$ (more generally, for elliptic curve), every nonempty set has a nontrivial invertible sheaves, see section 6.1.1.

9. (If you know some number theory) Let K be a number field and $\mathcal{O}_K$ the ring of integers. Then the fractional ideals form an invertible sheaf. Show that two invertible sheaves that differ by a principal ideal yield the same invertible sheaf.

   Let us do a concrete example: Take $R = \mathbb{Z}[\sqrt{-5}]$ which the reader should recall is not a UFD as
   $$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$
   It was shown in number theory that $J = (2, 1 + \sqrt{-5})$ is not principal. We shall use this to find a non-trivial line bundle. First, if $I \subseteq R$ is principal, it is easy to see that $I \cong_R R$, and this is not the case if I is not principal[b]. Next, $D(2), D(3)$ over R. Over both of these, J becomes principal:
   $$D(2) : (2, 1 + \sqrt{-5}) = (2) \qquad \text{as} \qquad 1 + \sqrt{-5} = \frac{2 \cdot (1 + \sqrt{-5})}{2}$$

$$D(3) : (2, 1 + \sqrt{-5}) = (1 + \sqrt{-5}) \qquad \text{as} \qquad 2 = \frac{(1 + \sqrt{-5})(1 - \sqrt{-5})}{3}$$

hence, taking J as an R-module, we see that it is locally free, but is is *not* globally free.

10. Take $\mathbb{A}^1_k$ and let $\mathscr{F}$ be the sky-scrapper sheaf supported at the generic point with (abelian) group $k(t)$. Show this is a quasi-coherent sheaf. Show that this sheaf is not locally free. More generally, let X be an integral Noetherian scheme. Put the constant sheaf $\underline{\kappa(X)}$ on it. Then this is a quasi-coherent $\mathcal{O}_X$-module, but not in general finitely generated on open subsets (and hence not coherent, unless X is restricted to a point).

11. Let $X = \mathbb{P}^n_k$. Then there is a natural line bundle on X where each point $p \in X$ is associated to the 1-dimensional sub-space in $\mathbb{A}^{n+1}_k$ of the points that make up the equivalence class. We shall later see that all line bundles on $\mathbb{P}^n_k$ will be isomorphic to this line bundle, an n-times tensor product of this line bundle or a n-times tensor product of the dual of the bundle (which can be thought of as the inverse).

12. On affine space $\mathbb{A}^n_k$ locally free $\mathcal{O}_X$-modules must be trivial. It is highly nontrivial to prove (the Quillen–Suslin Theorem, formally known as Serre's Theorem[c]), but it is enlighetning: Every projective module over a polynomial ring is free. This translates to every vector bundle over affine space is globally trivial. We shall see that we want non-trivial quasi-coherent sheaves on $\mathbb{A}^n_k$.

13. Let $X = \operatorname{Spec} R$ be an affine scheme and let M be a projective R-module. Recall projective modules over local rings are free. Then the germ $\widetilde{M}_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module (see [3, chapter 20.3]). From this perspective, projective modules are one of the closest modules to how we usually view non-trivial vector-bundles

14. let $X = \operatorname{Spec} R$ again and let M be a flat R-module. Then taking $\widetilde{M}$, the localization at each point remains flat (see [3, chapter 20.3]). This can geometrically be interpreted as the fibers of M over X varying smoothly!

[a] also known by the more humorous name *Hairy Ball Theorem*

[b] Notice that this means we can take $\widetilde{I}$, something we can't do with schemes as ideals are not rings. We shall return to this in ref:HERE

[c] The title got moved as there are enough important things named Serre's Theorem

Let us establish some basic properties of quasi-coherent sheaves. The following proposition is a direct translation of the results from structure sheaves and affine schemes and is boxed here for reference:

Proposition 4.1.11: Properties of Sheaf Module

Let M be an R-module, $X = \operatorname{Spec} R$, $\widetilde{M}$ the associated sheaf of M, and $f : R \to S$ a ring homomorphism with $f^a : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ Then:

1. The stalks of $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to $M_{\mathfrak{p}}$
2. $\Gamma(\operatorname{Spec} R, \widetilde{M}) \cong M$
3. $M \to \widetilde{M}$ is an exact fully faithful functor from R-**Mod** to $\mathcal{O}_{\operatorname{Spec} R}$-**Mod**.
4. If N is another R-module
$$(\widetilde{M \otimes_R N}) \cong \widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec} R}} \widetilde{N}$$
5. If $\{M_i\}_{i \in I}$ is a collection of R modules:
$$\widetilde{\left(\bigoplus_{i \in I} M_i\right)} \cong \bigoplus_{i \in I} \widetilde{M_i}$$
6. If N is an S-module,
$$f_*(\widetilde{N}) = \widetilde{{}_R N}$$
where ${}_R N$ is where we consider N with the natural R-module structure given by f. Furthermore, $f_*(-)$ is an *exact* functor (note: X *must* be affine)
7. If $\widetilde{N}$ is the associated sheaf of N and $\widetilde{M}$ is some sheaf module, then
$$f^*(\widetilde{M}) = \widetilde{M \otimes_R S}$$

It is important for exactness that $f_*(-)$ is pushing a quasi-coherent sheaf over an affine scheme. This fails for non-affine schemes, as we shall see with some of the simplest exact with projective schemes. In chapter 5, we shall find right-derived functors for this map.

Proof **:**

1. Direct generalization of proposition 2.3.14
2. Direct generalization of corollary 3.1.5
3. The fully faithful part can be deduced by generalizing corollary 3.1.5. For exactness, recall from [3] that localization is exact.
4. Recall from [3] that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$.
5. exercise
6. exercise
7. exercise

Let us next show that being 0 on an open subset is almost globally zero, and that it is also relatively easy to find global sections form local section on (distinguished) open subsets:

Lemma 4.1.12: Quasi-coherence and Extending From $D(f)$

Let $X = \operatorname{Spec} R$ be an affine scheme, $f \in R$, $D(f) \subseteq X$, and $\mathscr{F}$ a quasi-coherent sheaf on X. Then:

1. If $s \in \Gamma(X, \mathscr{F})$ is a global section of $\mathscr{F}$ whose restriction to $D(f)$ is zero, then there exists some $n > 0$ such that $f^n s = 0$
2. If $t \in \mathscr{F}(D(f))$, then or some $n > 0$, $f^n t \in \mathscr{F}(X)$.

The idea behind (2) is that we can eliminate the denominator to have a globally defined function; think $1/x \in \mathcal{O}_X(\mathbb{A}^1_k \setminus \{0\})$. For part (1), think of this as reflecting the same property as complex holomorphic functions, but having to take into consideration nilpotent elements. This idea works for "vector bundles" on affine schemes.

Proof **:**

1. Suppose $s \in \Gamma(X, \mathscr{F})$ such that $s|_{D(f)} = 0$. Cover X in finitely many $D(g_i)$ (as X is quasicompact), and consider $s|_{g_i} = s_i \in M_i = \mathscr{F}(D(g_i))$. Then as $D(f) \cap D(g_i) = D(fg_i)$,
$$\mathscr{F}|_{D(fg_i)} = \widetilde{M_i}_f$$
Then the homomorphism restriction map must imply $s_i = 0 \in (M_i)_f$, which by definition of localization implies $f^{n_i} s = 0$. Doing this for each M_i and taking the $n = \max_i n_i$, by gluing we get that $f^n s = 0$.
2. Let $t \in \mathscr{F}(D(g))$, and like above consider $t_i \in \mathscr{F}(D(fg_i)) = (M_i)_f$. Then by definition, for some $n > 0$ there is some element $t_i \in M_i = \mathscr{F}(D(g_i))$ that restricts to $f^{n_i} t$ on $D(fg_i)$. Again, take $n = \max_i n_i$ to make it independent of choice.

   With n chosen to be large enough, on $D(g_i) \cap D(g_j) = D(fg_ig_j)$, the two sections t_i, t_j restrict to $f^n t$. Hence, by part a there exists an $m > 0$ such that
$$f^{m_{ij}}(t_i - t_j) = 0 \qquad \text{on} \qquad D(g_ig_j)$$
   Naturally take $m = \max_{i,j} m_{ij}$. Now the local sections $f^m t_i \in \mathscr{F}(D(g_i))$ glue to give a global section $s \in \mathscr{F}(X)$ where $s|_{D(f)} = f^{n+m} t$.

Next, recall that the global section functor need not be exact. In the case where $X = \operatorname{Spec} R$, and we are working with quasi-coherent sheaves, it is exact:

Proposition 4.1.13: Exactness of Global Section with Sheaves of Modules

Let $X = \operatorname{Spec} R$ be an affine scheme and $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ an exact sequence of $\mathcal{O}_X$-modules, where at least $\mathscr{F}'$ is quasi-coherent. Then:

$$0 \to \Gamma(\operatorname{Spec} R, \mathscr{F}') \to \Gamma(\operatorname{Spec} R, \mathscr{F}) \to \Gamma(\operatorname{Spec} R, \mathscr{F}'') \to 0$$

is exact

***Proof* :**
By proposition 1.1.12, Γ is left-exact, hence it suffices to show it is right-exact when working with $\mathcal{O}_X$-modules and at least $\mathscr{F}'$ is quasi-coherent, namely we must show the map is surjective.

Let $s \in \Gamma(X, \mathscr{F}'')$. As $\mathscr{F} \to \mathscr{F}''$ is surjective, by exercise 1.3.1[1] there is an open neighborhood $D(f)$ of x such that $s|_{D(f)} \in \mathscr{F}''(D(f))$ lifts to a section $t_x \in \mathscr{F}(D(f))$. I claim now that there exists an appropriate $n > 0$ such that $f^n s \in \Gamma(\mathscr{F}'', X)$ lifts to a global section $t \in \mathscr{F}(X)$.

As X is an affine scheme, it is quasicompact, and so cover it by a finite principal open sets $D(g_i)$, and for g_i $s|_{D(g_i)}$ lifts to a section $t_i \in \mathscr{F}(D(g_i))$. Then, on $D(f) \cap D(g_i) = D(fg_i)$, there is $t_x, t_i \in \mathscr{F}(D(fg_i))$ which both lift $s_{D(fg_i)}$.

$$t_x - t_i \in \mathscr{F}'(D(fg_i))$$

As $\mathscr{F}'$ is quasi-coherent, by lemma 4.1.12[1], there exists some $n_i > 0$ such that $f^{n_i}(t_x - t_i)$ extends to a section:

$$u_i \in \mathscr{F}'(D(g_i))$$

Pick $n = \max_i n_i$. Now, taking the image of u_i (as the function is injective), consider $t_i' = f^n t_i + u_i \in \mathscr{F}(D(g_i))$. Then t_i' is a lifting of $f^n s|_{D(g_i)}$, and

$$t_i'|_{D(g_i)} = f^n t_x|_{D(g_i)}$$

Then on $D(g_i g_i)$, t_i', t_j' both lift $f^n s$, and so $t_i' - t_j' \in \mathscr{F}'(D(g_i g_j))$. As $t_i'|_{D(fg_i g_j)} = t_j'|_{D(fg_i g_j)}$, by lemma 4.1.12[2] there exists an m_{ij} such that $f^{m_{ij}}(t_i' - t_j') = 0$ on $D(g_i g_j)$. As usual, take $m = \max_{i,j} m_{ij}$. Then the section $f^m t_i' \in \mathscr{F}(D(g_i))$ glue to a global section $t'' \in \mathscr{F}(X)$, which is a lift of $f^{n+m}s$.

Now, to show each s has a nonempty pre-image. As X is quasicompact, cover X by $D(f_1), ..., D(f_r)$. Then we know that each $s|_{D(f_i)} \in \mathscr{F}''(D(f_i))$ lifts to a section in $\mathscr{F}(D(f_i))$. Then by what was just shown there exists a global section $t_i \in \mathscr{F}(X)$ such that t_i is a lift of $f_i^n s$ where n is the maximum value of all lifts.

Now, As $D(f_i)$ cover X, $(f_1^n, ..., f_r^n) = R$, so let

$$1 = \sum_i a_i f_i^n$$

Take $t = \sum_i a_i t_i \in \mathscr{F}(X)$. Then the image t is

$$\sum_i a_i f_i^n s = s$$

showing each s has a nonempty preimage, as we sought to show.

The reader could think of the above result as something special about affine schemes. In chapter 5, we shall see that it is exactly the non-affine schemes that will have a non-exact global-section function (over quasi-coherent sheaves), see theorem 5.1.17.

Proposition 4.1.14: All Objects Are Quasi-Coherent

Let X be a scheme. Then

1. the kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent.
2. Any extension are quasi-coherent.

If X is Noetherian, the same is true for coherent. If $f : X \to Y$ is a morphism of schemes then:

1. If $\mathscr{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f^*\mathscr{G}$ is a quasi-coherent $\mathcal{O}_X$-module
2. If X, Y and Noetherian, $\mathscr{G}$ is coherent,then $f^*\mathscr{G}$ is coherent
3. If either X is Noetherian of f is quasicompact and separated, then if $\mathscr{F}$ is quasi-coherent, $f_*\mathscr{F}$ is quasi-coherent sheaves.

Proof **:**
As shown in chapter 1, these properties can be shown locally, so it suffices to assume X is affine. Then as $M \mapsto \widetilde{M}$ is exact and fully faithful, kernels, cokernels, and images of quasi-coherent sheaves are preserved. We must still show that the extension of two quasi-coherent sheaves is quasi-coherent, so consider an exact sequence of $\mathcal{O}_X$-modules where $\mathscr{F}', \mathscr{F}''$ is quasi-coherent:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

Then by proposition 4.1.13, the global sections are exact as $\Gamma(X, -)$ is an exact functor:

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}') \to 0$$

relabel for simplicity to:

$$0 \to M' \to M \to M'' \to 0$$

Now apply the $\widetilde{(-)}$ exact functor. Then there is a natural commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}'' & \longrightarrow & 0 \end{array}$$

As $\mathscr{F}', \mathscr{F}''$ are quasi-coherent, Γ and $\widetilde{(-)}$ returns itself by proposition 4.1.11, we have that $\mathscr{F}' \to M'$ and $\mathscr{F}'' \to M''$ are isomorphism. By the 5-lemma, $M \to \mathscr{F}$ is an isomorphism. Naturally, if X is Noetherian, all sheaves are coherent.

Now let $f : X \to Y$ be a scheme morphism. As these properties are local we may assume X, Y are affine. As Γ and $\widetilde{(-)}$ are equivalence of categories and inverses of each other, and

$f^*(\widetilde{M}) \cong (\widetilde{M \otimes_R S})$, a quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$ get's pushed out to a quasi-coherent $\mathcal{O}_X$-module $f^*\mathscr{G}$. If Y is Noetherian, everything becomes coherent.

the push-forward requires more conditions (see example 4.5). If X is Noetherian of f is quasi-compact, X can be covered by finitely many affine open subsets, and as X is Noetherian of f is separated, the intersection of two affine open subsets is either affine or the union of affine open subsets. Label the affine open U_i and the affine local subsets whose union is the intersections $U_i \cap U_j$ as U_{ijk}. Now for any $s \in \mathscr{F}(f^{-1}(V))$, $V \subseteq Y$ open, there is a unique collection of $s_i \in \mathscr{F}(f^{-1}(V) \cap U_i)$ whose restriction to $f^{-1}(V) \cap U_{ijk}$ must all be equal. Thus, we get the following exact sequence:

$$0 \to f_*\mathscr{F} \to \bigoplus_i f_*(\mathscr{F}|_{U_i}) \to \bigoplus_i f_*(\mathscr{F}|_{U_{ijk}})$$

where f_* is a bit overloaded for the second and third position, As it should be the function induced by $f : U_i \to Y$ and $f : U_{ijk} \to Y$. Now, as U_i are affine, by proposition 4.1.11[4], $f_*(\mathscr{F}|_{U_i})$ and $f_*(\mathscr{F}|_{U_{ijk}})$ are quasi-coherent. As the kernel of a $\mathcal{O}_X$-module morphism of a quasi-coherent sheaf is quasi-coherent, $f_*(\mathscr{F})$ is quasi-coherent, completing the proof.

If X were not Noetherian or f not quasi-compact and separated, the pushforward need not be quasi-coherent:

Example 4.5: Pushforward Not Quasi-coherent

Let $X = \coprod_{i\in\mathbb{N}} \operatorname{Spec}\mathbb{Z}$, $Y = \operatorname{Spec}\mathbb{Z}$, and $f : X \to Y$ be the identity for each disjoint element of $\operatorname{Spec}\mathbb{Z}$. Take $\mathscr{F} = \mathcal{O}_X$.

Now, for any quasi-coherent sheaf $\mathscr{G}$ on Y, as we have a change of basis for each open set we must have:

$$\mathscr{G}(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(U) \xrightarrow{\sim} \mathscr{G}(U)$$

We'll show this is not the case when $\mathscr{G} = f_*\mathcal{O}_X$ and $U = D(2) \subseteq \operatorname{Spec}\mathbb{Z}$. In this case we get:

$$\left(\prod_{i\in\mathbb{N}} \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \to \prod_{i\in\mathbb{N}} \mathbb{Z}\left[\frac{1}{2}\right]$$

Next, for $t \in \left(\prod_{i\in\mathbb{N}} \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right]$, we have hat:

$$t = \left(\frac{a_i}{2^n}\right)_{i\in\mathbb{N}}$$

for some fixed n. The image ought to be the identity. However, the codomain contains elements that are not of this form, namely there are:

$$s = \frac{b_i}{2^{n_i}} \in \prod_{i\in\mathbb{N}} \mathbb{Z}\left[\frac{1}{2}\right]$$

where n_i grows arbitrarily large and cannot be an image the image for any t!

If X, Y are Noetherian, it is *not necessarily* the case that f_* of a coherent is a coherent sheaf!

Example 4.6: Pushforward of Noetherian Not Coherent
Take $f : \operatorname{Spec} k[x] \to \operatorname{Spec} k$ given by $k \to k[x]$. Then $f_*(\mathcal{O}_{\operatorname{Spec} k[x]}) = k[x]$ is not a finitely generated k-module.

There are however many sufficient properties that can be added to f to make the image coherent, for example finite, projective, or proper. (see ref:HERE)

4.1.2 Closed subschemes and Ideal Sheaves

Let us return to closed subschemes, which were defined to be closed subsets $X \subseteq Y$ where $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjective map. As the map is surjective, the kernel at each open set must give an ideal. Given a closed embedding $\pi : X \to Y$, there is an *ideal sheaf* that we can put on Y that is given by the kernel of the map:

$$\mathcal{O}_Y \to \pi_*\mathcal{O}_X$$

Definition 4.1.15: Ideal Sheaf

Let $\mathcal{O}_X$ be a structure sheaf for a scheme X. Then an *ideal sheaf* is a $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

Lemma 4.1.16: Kernel and Ideal Sheaves

Let Y be a closed subscheme of X and $\iota : Y \to X$ the inclusion map. Then the kernel of $\iota^\# : \mathcal{O}_X \to \iota_*\mathcal{O}_Y$ is an ideal sheaf, and is usually denoted $\mathcal{I}_Y$.

***Proof* :**
exercise

The reader should find no difficulty in seeing this is a coherent sheaf over $\mathcal{O}_Y$. We get the following short exact sequence of $\mathcal{O}_Y$-modules:

$$0 \to \mathcal{K} \to \mathcal{O}_Y \to \pi_*\mathcal{O}(X) \to 0$$

For each affine open subset with a closed embedding $\operatorname{Spec}(R) \to Y$, let $I(R)$ be the corresponding ideal (they define a base for the sheaf). We may ask if we can study the closed embedding condition locally using $\mathcal{K}$ and the ideals $I(R)$. The first thing we may want is an equivalent of the fact that every ideal sheaf should correspond to a closed subscheme (i.e. make lemma 4.1.16 and iff). This is not the case, though mainly since we have not limited ourselves to quasi-coherent sheaves. The following necessary property for ideal sheaves given by kernels shall be shown to not apply to all ideal sheaves:

Proposition 4.1.17: Sheaf Ideal For Closed Embedding

Let $X \to Y$ be a scheme morphism. Then $X \to Y$ is a closed embedding if and only if for each affine open subset $\operatorname{Spec} S \subseteq Y$ and corresponding inclusion $\operatorname{Spec} S \to Y$ giving the ideals $I(S) \subseteq S$, the restriction morphism $S \to S_f$ induces an isomorphism:

$$I(S)_f \cong I(S_f)$$

for all $f \in S$. In other words the closed embeddings $\operatorname{Spec} S/I \to \operatorname{Spec} S$ glue to get a closed embedding $X \to Y$.

Proof **:**
The $(\Rightarrow)$ is the easier direction. For the $(\Leftarrow)$ direction, take an ideal $I(S) \subseteq S$ corresponding to an affine open subset $\operatorname{Spec}(S) \subseteq X$. Suppose that for each closed embedding $\operatorname{Spec}(S) \to Y$ and $f \in S$, the restriction $S \to S_f$ induces an isomorphism $I(S)_f \cong (S_f)$. Using this, show you can glue together closed embeddings $\operatorname{Spec}(S/I) \to \operatorname{Spec}(S)$ in a well-defined manner to get a closed embedding $X \to Y$.

Example 4.7: Not All Ideal Sheaves Correspond To Closed Embeddings
Take $X = \operatorname{Spec} k[x]_{(x)}$. This scheme has two points: the closed point and the generic point, say η. Then take:

$$\mathcal{K}(X) = 0 \subseteq \mathcal{O}_X(X) = k[x]_{(x)}$$
$$\mathcal{K}(\eta) = k(x) = \mathcal{O}_X(\eta)$$

Then using the above proposition, we see that this sheaf cannot define a closed embedding.

We recover a correspondence if we limit to quasi-coherent sheaves:

Proposition 4.1.18: Quasi-coherent Ideal Sheaves

Let X be a scheme. Then for any closed subscheme $Y \subseteq X$, the corresponding ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X. If X is Noetherian, $\mathcal{I}_Y$ is coherent.

Conversely, any quasi-coherent sheaf of ideals on X corresponds uniquely to a closed subscheme of X.

Proof **:**
This follows by the Qcqs lemma (lemma 3.3.5) to get the required condition for proposition 4.1.17.

Corollary 4.1.19: Ideal Sheaves On Affine Schemes

Let $\operatorname{Spec} R$ be an affine scheme. Then there is a one-to-one correspondence between the ideals $\mathfrak{a} \subseteq R$ and the closed subscheme $Y \subseteq X$ given by

$$\mathfrak{a} \mapsto \operatorname{Spec} R/\mathfrak{a}$$

With further control of closed subschemes, we can define the following interesting analogy to vanishing sets:

Definition 4.1.20: Vanishing Scheme

Let X be a scheme and $s \in \mathcal{O}_X(X)$. Then a *vanishing scheme* corresponding to s, $V(s)$, is the scheme corresponding to gluing $\operatorname{Spec}(R/(s_R))$ where $s_R = s|_{\operatorname{Spec}(R)}$. More generally, we can take the ideal $(S) \subseteq \mathcal{O}_X(X)$ generated by the elements S.

The reader should use proposition 4.1.17 to insure this is well-defined. The advantage of working with vanishing schemes over vanishing sets is that it remembers extra information, such as the multiplicity of intersection.

Proposition 4.1.21: Properties of Vanishing Schemes

Let X be a scheme. Then:

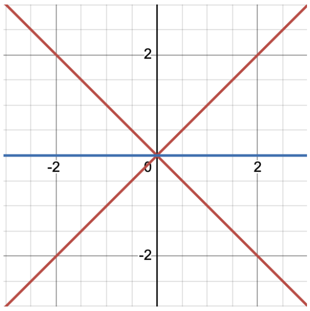
1. $\cup_i^n V(S_i) = V(\cap_i^n S_i)$
2. $\cap_{i \in \mathcal{I}} V(S_i) = V(\langle S_i \mid i \in \mathcal{I} \rangle)$

Proof **:**
Note that the underlying sets of the schemes will just be finite unions and intersections.

The harder point to verify is that the union becomes intersection. In the case of vanishing sets, the proof takes advantage of the fact that $I \cap J \subseteq IJ \subseteq (I \cap J)^2$. Note that it must be the intersection, as we would need that that $V(S) \cap V(S) = V(S)$ and not $V(S^2)$.

Example 4.8: Vanishing Scheme Intersection

1. Take $V(y - x^2), V(y) \subseteq \mathbb{A}_k^2$. Then with vanishing sets, this would just be the point $(0, 0)$. However, this point now has multiplicity 2.
2. Take $V(y^2 - x^2), V(y) \subseteq \mathbb{A}_k^2$. The visual of these two are:



Use this to show that if X, Y, Z are closed subscheme of a scheme W, then

$$(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z$$

Hence, not all properties of unions and intersection can carry over!

4.1.3 Locally Closed Embedding

Recall that the line without the origin is not a closed subscheme of $\mathbb{A}^2_k$. This is This is however still a pretty reasonable subset, and almost a closed subscheme: it is a closed subscheme of an open subscheme of $\mathbb{A}^2_k$ (namely the subscheme given by $k[x, 0]$). This motivates the following:

Definition 4.1.22: Locally Closed Embedding

Let $\pi : X \to Y$ be a scheme morphism. Then it is a *locally closed embedding* if π can be factored into:

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$$

where α is a closed embedding, and b is an open embedding.

The above is such an example: more formally the morphism $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ is a locally closed embedding. Note that we could have also taken an open subset of a closed subset (in exercise ref:HERE you make sure these are equivalent)

Proposition 4.1.23: Locally Closed embedding is Locally Finite

Let $\pi : X \to Y$ be a locally closed embedding. Then it is locally finite.

Proof **:**
exercise.

Give a characterization of intersection of locally closed embeddings as you've moved the fibred product to earlier

4.1.4 Image Scheme

Definition 4.1.24: Image Scheme

Let $\iota : Z \to Y$ be a closed subscheme, which gives the short exact sequence:

$$0 \to \mathcal{K}_{Z/Y} \to \mathcal{O}_Y \to \iota_* \mathcal{O}_Z \to 0$$

Then the *image* of $\pi : X \to Y$ *lies in* Z if the composition

$$\mathcal{K}_{Z/Y} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X$$

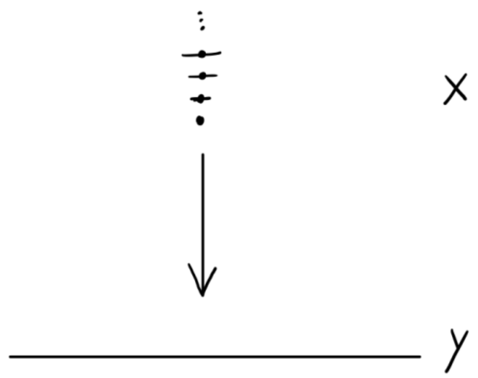
is zero. Then the *scheme-theoretic image* of π (a closed subscheme of Y) is the smallest closed subscheme containing the image

Intuitively, locally functions vanishing on Z pullback to zero functions on X. If the image of π line sin some subschemes Z_j, it lies in their intersection, and hence it is the scheme-theoretic image. Note that the scheme theoretic image may not necessarily be an open subscheme:

Example 4.9: Scheme-theoretic Image

1. Take
$$\pi : \text{Spec}\,(k[\epsilon]/(\epsilon^2)) \to \text{Spec}\,(k)$$
given by $x \mapsto \epsilon$. Then the scheme-theoretic image of π ks
$$\text{Spec}\,(k[x]/(x^2))$$
as the polynomials pulling back to 0 are precisely the multiples of x^2.
2. Now take
$$\pi : \text{Spec}\,(k[\epsilon]/(\epsilon^2)) \to \mathbb{A}^1_k$$
given by $x \mapsto 0$. Then the scheme-theoretic image is
$$\text{Spec}\,(k[x]/(x))$$
which is reduced.
3. Take
$$\text{Spec}\, k[t, t^{-1}] = \mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k = \text{Spec}\,(k[u])$$
given by $u \mapsto t$. Then any function $g(u)$ which pulls back to 0 as a function of t must be the zero function. Hence,the scheme-theoretic image is
$$\text{Spec}\,(k[u])$$
4. Consider the following more pathological example:
$$\pi : \coprod \text{Spec}\left(\frac{k[\epsilon_n]}{(\epsilon_n)^n}\right) \to \text{Spec}\, k[x]$$

which well denote $\pi : X \to Y$, and it is given by $x \mapsto \epsilon_n$ on the nth component. Now, if $g(x)$ on Y pulls back to 0 on X, then its Taylor expansion must be 0. But then the scheme-theoretic image is $V(0)$ on Y, that is it is Y, even though the scheme-theoretic image is the origin (the closed point 0). Visually:



4.1.5 Closed subscheme of Projective Schemes

Let us now take a moment to introduce some practical applications to projective schemes.

Definition 4.1.25: Projective Hypersuface

A *hypersurface* of $\mathbb{P}_k^n$ is a closed subscheme given by a single homogeneous polynomial f. The *degree* of the hypersurface is the degree of the homogeneous polynomial.

A hypersurface of degree 1 is called a *hyperplane*. A hypersurface of degree $2, 3, 4, 5...$ is called a *quartic*, *cubic*, *quartic*, *quintic*, and so froth.

If $n = 2$ ($\mathbb{P}_k^2$) a hypersurface is called a *curve*. A hypersurface of degree $1, 2, 3$ is called a *line*, *conic curve* (or *conic*), and *surface* respectively.

Recall that $\mathbb{P}_k^n$ has only constant functions defined globally, and hence any such cut-out be done by elements of the defining graded ring.

Example 4.10: Closed Projective Subschemes

1. *Twisted Cubic*: Take $V(xy - zw, x^2 - wy, y^2 - xz) \subseteq \mathbb{P}_k^3$. This shape is called the *twisted cubic*. This is a *curve*, and it is isomorphic to $\mathbb{P}_k^1$ by the induced map:

$$k[w, x, y, z] \to k[s, t] \qquad (w, x, y, z) \mapsto (s^3, s^2t, st^2, t^3)$$

To see it is an isomorphism, take the standard open sets and show it gives closed embeddings, for example if $w \neq 0$, $s \neq 0$ then:

$$k[x/w, y/w, z/w] \to k[t/s] \qquad x/w \mapsto t/s,\ y/w \mapsto t^2/s^2,\ z/w \mapsto t^3/s^3$$

2. *Surjective graded ring map*: Show that $\mathbb{R}_\bullet \twoheadrightarrow S_\bullet$ is a surjective graded ring homomorphism, then it induces a closed embedding $\operatorname{Proj} S_\bullet \to \operatorname{Proj} R_\bullet$.

3. Let $X \to \operatorname{Proj} S_\bullet$ be a closed embedding in a projective R-scheme (so that $S_\bullet$ is a finitely generated graded R-algebra). Show that X is projective as $\operatorname{Proj}(S_\bullet/I)$ (mirror the proof for the affine case, see Shafarevich).

4. Any injective linear transformation $V \to W$ induces closed embedding $\mathbb{P}V \to \mathbb{P}W$. The closed subscheme defined here is called a *linear space.*

5. Take $S_\bullet$ to be a finitely generated graded ring generated by elements of degree 1. Note that S_1 is a finitely generated S_0-module, and the irrelevant ideal S_+ is generated by degree 1 elements. Then $\operatorname{Proj} S_\bullet$ is isomorphic to a closed subscheme of $\mathbb{P}^n_R$. To see this, take the free module R^{n+1} generated by $t_0, ..., t_n$. Then take:
$$\operatorname{Sym}^\bullet(R^{n+1}) = R[t_0, .., t_n] \twoheadrightarrow S_\bullet \qquad t_i \mapsto x_i$$
Then this implies:
$$S_\bullet = R[t_0, t_1, ..., t_n]/I$$
Thus, we always get that such rings can be interpreted as closed projective subschemes! Conceptually, this is the generalization of the affine case where if S is a finitely generated R-algebra, w may choose n generators of S and describe $\operatorname{Spec}(S)$ as a closed subscheme of $\mathbb{A}^n_R$; we are choosing coordinates (now in the projective case).

6. The following shows a way to parameterize certain projective varieties. Take $V(wz-xy) \subseteq \mathbb{P}^3_k$ (with projective coordinates w, x, y, z) Then for any two elements $\alpha, \beta \in k$ that are not both zero, notice that as $[c, d] \in \mathbb{P}^1_k$ varies, $[\alpha c, \beta c, \alpha d, \beta d]$ is a line on the quadratic surface. Show there is one other parameterization of this surface via lines, an show it must be the case that these are the only 2 (hint: one parameterization contains $V(w, x)$ and the other $V(w, y)$). The following gives a visualization of the parameterization that contains $V(w, x)$:

A common important closed projective subscheme is those given by the different gradings $S_{d\bullet}$:

Definition 4.1.26: Veronese Embedding

Let $S_\bullet = k[a_0, ..., a_n]$. Let $N = \dim_k S_{d\bullet} = \binom{n+d}{d}$. Then $\operatorname{Proj} S_{d\bullet} \subseteq \mathbb{P}^{N-1}$ is called the *d-uple Veronese Embedding.*

Most of the projective schemes are associated to graded rings generated by elements of degree 1. Having different degrees, or weighting, can

Definition 4.1.27: Weighted Projective Space

Let $S_\bullet = k[a_1, ..., a_n]$ where each a_i has degree d_i. Then $\operatorname{Proj} k[a_1, ..., a_n]$ is called the weighted projective space and is denoted $\mathbb{P}(d_1, ..., d_n)$

Example 4.11: Weighted Projective Space

1. Show that $\mathbb{P}(m,n) \cong \mathbb{P}$
2. Show that $\mathbb{P}(1,1,2) \cong \operatorname{Proj} k[u,v,w,z]/(uw - v^2)$

Proposition 4.1.28: Maps To Projective Space - Special Case

Let R be a ring, X a R-scheme, and $f_0, ..., f_n$ functions on X with no common zeros. Then there exists a scheme morphism $[f_0 : \cdots f_n] : X \to \mathbb{P}^n_R$.

Proof **:**
exercise

Special case: $X = \mathbb{A}^{n+1}_k \setminus \{0\}$ and the functions are $x_0, ..., x_n$. This will give the usual injection $\mathbb{A}^{n+1}_k \setminus \{0\} \hookrightarrow \mathbb{P}^n_R$. This does not classically *all* embeddings, ex. $\mathbb{P}^1_R \xrightarrow{\text{id}} \mathbb{P}^1_R$ does not fall under this method, but we shall show now how to find it.

Exercise 4.1.1

1. Put Exercise 8.1.M from Vakil
2. Show that the composition of two locally closed embeddings is locally closed

4.2 Quasi-Coherent Sheaves on Projective Schemes

Let us now explore quasi-coherent shaves on projective schemes. One of the main results we are aiming for is to show that most schemes can be embedded into projective space. We shall also see how to recover the graded ring that constructed the projective scheme.

Let us start by defining the natural way we can have a sheaf over a projective scheme:

Definition 4.2.1: Sheaf Associated to M [on projective scheme]

Let S be a graded ring and M a graded S-module. Then the *sheaf associated to M* on $\operatorname{Proj} S$, denoted $\widetilde{M}$ is given by:

$$\widetilde{M}|_{D_+(f)} = (\widetilde{M[f^{-1}]_0})$$

for homogeneous $f \neq 0$.

Like in projective schemes, if S is the usual polynomial ring with grading by degree, then open sets $D(x_i) \subseteq \operatorname{Proj} S$ are isomorphic to $\operatorname{Spec}(S_{x_i})_0$, and so the module $\widetilde{M}(D(x_i))$ becomes a module over an affine scheme.

Proposition 4.2.2: Sheaf on Projective Scheme is Quasi-Coherent

Let S be a graded ring, M a graded S-module, and $X = \operatorname{Proj} S$. Then:

1. $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$
2. $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module, and if S is Noetherian and M finitely generated, it is coherent.

Proof **:**
a repeat of similar result earlier

Let us start by showing how we can define some natural sheaves on $\operatorname{Proj} S$ whose direct sum of global sections will recover S:

Definition 4.2.3: Twisting Sheaf

Let $S_\bullet = \bigoplus_n S_n$ be a graded ring and $X = \operatorname{Proj} S$. Then for every $n \in \mathbb{Z}$, define the sheaf:

$$\mathcal{O}_X(n) := \widetilde{S(n)}$$

where $S(n)$ is the graded ring $S(n)_i = S_{n+i}$. The sheaf $\mathcal{O}_X(1)$ will be called the *twisting sheaf.*

Note that $S(0) = S$, so $\mathcal{O}_X(0) = \mathcal{O}_X$ which implies $\Gamma(\operatorname{Proj} S, S(0)) = \Gamma(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ represents the constant global sections. Let us next consider $\widetilde{S(n)}$. Let us work on our usual $S = R[x_0, ..., x_n]$ (proposition 4.2.5 shall generalize this discussion). On $D(x_i)$ we will then have our usual localization, except we will not have the degree 0 elements but degree n elements:

$$\widetilde{S(m)}(D_+(x_i)) = \text{degree } n \text{ of } R[x_0, ..., x_n x_i^{-1}]$$

Note that this is no longer a ring, but a $(R[x_{0,i}, ..., x_{n,i}, x_i^{-1}])_0$-module where x_0/x_i is degree 0 (hence closed under multiplication. As an abelian group, it is generated by:

$$x_0^m, x_0 x_1^{m-1}, x_0 x_2^{m-1}, x_3^{m-6} x_2^3 x_8^3, ..., x_n^m$$

that is, all homogeneous elements. Note that multiplying by any element in the ring $(R[x_{0,i}, ..., x_{n,i}, x_i^{-1}])_0$ keeps the degree the same.

Is this module any different from $\widetilde{S}(D_+(x_i))$ as modules? In fact they are not! Note that on $D_+(x_i)$, multiplying by x_i is an isomorphism from the degree m component of $R[x_0, ..., x_n x_i^{-1}]$ to its degree $m+1$ component! Thus, $\widetilde{S(n)}$ is locally isomorphic to $\widetilde{S}$, and hence it is locally free of rank 1, that is a *line bundle* (or invertible sheaf).

These sheaves are not be isomorphic, because the *gluing* of these sheaves is subtly different and shall motivate why it is called the *twisted* sheaves. Let us first distinguish the $\widetilde{S(n)}$ by showing they have different global sections. Going about it a bit informally, let's consider $\widetilde{S(n)}$ and cover it with $D(x_i)$ so that $\Gamma(\widetilde{S(n)}, D(x_i)) = (R[x_0, ..., x_n x_i^{-1}])_n$. Then whenever gluing two $D(x_i), D(x_i)$, their intersection shall have polynomials in degree n that have no x_i, x_j term. The gluing is very similar to what we've done before: we can think of the transition from $D(x_i)$ to $D(x_j)$ on $D(x_i) \cap D(x_j)$ would map $x_k \mapsto x_k$ and

$$\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j}$$

Then that means that the global section is:

$$\Gamma(\widetilde{S(n)}, \operatorname{Proj} S) = S_n$$

and so we can use dimension counting to show they are distinct for each classical projective space!

$\dim \Gamma(\widetilde{S(n)}, \operatorname{Proj} S)$	$\mathbb{P}^0_R$	$\mathbb{P}^1_R$	$\mathbb{P}^2_R$	$\mathbb{P}^3_R$	$\mathbb{P}^4_R$
$S(0)$	1	1	1	1	1
$S(1)$	1	2	3	4	5
$S(2)$	1	3	6	10	15

After introducing some properties of twisted sheaves, we shall show the $\widetilde{S(-n)}$ (given by the multiplicity of poles, or similarly by the dual space of $S(n)$) are not isomorphic as well

We should take a moment to justify the term "twisting". First off, only $\widetilde{S(0)}$ is globally free, hence there is something going on for each $\widetilde{S(n)}$. To see that it is a twisting, let us look more closely at the transition for $\widetilde{S(1)}$ on $\mathbb{P}^1_R$. On $D(x_0) \cap D(x_1)$, we only have the degree 1 terms. Then we see from the map that

$$\frac{x_0}{x_1} \mapsto \frac{x_0 x_0}{x_1 x_0} = \frac{x_1}{x_0}$$

as we see, we would have to "flip twice", to get back to the original. In higher twists, the power would be greater, showing the number of twists increases.

Let us now introduce the grading to any sheaf defined on projective space:

Definition 4.2.4: n-twisted sheaf of $\mathscr{F}$

For any $\mathcal{O}_X$-module $\mathscr{F}$, let:

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

we call this sheaf the n-twisted sheaf of $\mathscr{F}$.

Proposition 4.2.5: Properties of Twisted Sheaves

Let S be a graded ring and $X = \operatorname{Proj} S$. Let S be generated by S_1 as an S_0-algebra. Then

1. The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf and $\mathcal{O}_X(n) \not\cong_{\mathcal{O}_X} \mathcal{O}_X(m)$ for $n \neq m$
2. For any graded S-module M, $\widetilde{M}(n) = \widetilde{M(n)}$
3. $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$
4. Let T be a graded ring generated by T_1 as a T_0-algebra. Let $\varphi : S \to T$ be a homogeneous homomorphism, and take $U \subseteq \operatorname{Proj} T$. Given $f : U \to X$:

$$f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U \qquad \text{and} \qquad f_*(\mathcal{O}_Y(n)|_U) \cong (f_* \mathcal{O}_U)(n)$$

Proof **:**

1. We must show that $\mathcal{O}_X(n)$ is an invertible sheaf, i.e. locally free of rank 1. We shall use the fact that S is generated by S_1 as an S_0-algebra, meaning it is finitely generated and hence will be "locally S_n". Take $\mathcal{O}_X(n)|_{D(f)} \cong \widetilde{(S_n)_f}$ on Spec $(S_f)_0$. Now, notice that $(S_n)_f$ is isomorphic to $(S_0)_f$ via the map $s \mapsto f^n s$. As S is generated by S_1 as an S_0-algebra, we see that each open subsets are isomorphic to $\widetilde{S_n}$, and hence it is an invertible sheaf (i.e. a line bundle).

   For the second claim, copy formalize the global section calculations from earlier.

2. immediate

3. Show that for any two graded modules $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ (not too difficult generalization of an earlier result), and that $((M \otimes_S N)_f)_0 = (M_f)_0 \otimes_{(S_f)_0} (N_f)_0$.

4. Generalize form the affine case.

Using the above properties, show that the negative degree elements are not isomorphic and fill out the table:

$\dim \Gamma(\widetilde{S(n)}, \operatorname{Proj} S)$	$\mathbb{P}^0_R$	$\mathbb{P}^1_R$	$\mathbb{P}^2_R$	$\mathbb{P}^3_R$	$\mathbb{P}^4_R$
$S(-2)$	1	0	0	0	0
$S(-1)$	1	0	0	0	0
$S(0)$	1	1	1	1	1
$S(1)$	1	2	3	4	5
$S(2)$	1	3	6	10	15

We shall show that $S(-d) = 0$ in theorem theorem 5.2.7. Let us now show there is a natural graded ring associated to each projective scheme:

Definition 4.2.6: Graded s-module Associated to $\mathscr{F}$

Let S be a graded ring, $X = \operatorname{Proj} S$, and $\mathscr{F}$ a sheaf of $\mathcal{O}_X$-modules. Then the *graded S-module associated to $\mathscr{F}$* as:

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

where the product is given by the properties of proposition 4.2.5.

We can now naturally recover S in the case of it being a polynomial ring:

Proposition 4.2.7: Recovering Graded Ring

Let R be a ring, $S = R[x_0, ..., x_n]$, and $X = \operatorname{Proj} S$. Then:

$$\Gamma_*(\mathcal{O}_X) \cong S$$

***Proof* :**
We cover X with the open sets $D_+(x_i)$. Then to give a section $t \in \Gamma(X, \mathcal{O}_X(n))$ is the same as giving sections $t_i \in \mathcal{O}_X(n)(D_+(x_i))$ for each i, which agree on the intersections $D_+(x_i x_j)$. Now t_i is just a homogeneous element of degree n in the localization S_{x_i}, and its restriction to $D_+(x_i x_j)$ is just the image of that element in S_{x_i, x_j}. Summing over all n, we see that $\Gamma_*(\mathcal{O}_X)$ can be identified with the set of $(r+1)$-tuples $(t_0, \dots, t_r)$ where for each i, $t_i \in S_{x_i}$, and for each i, j, the images of t_i and t_j in S_{x_i, x_j} are the same.

Now the x_i are not zero divisors in S, so the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i, x_j}$ are all injective, and these rings are all subrings of $S' = S_{x_0 \cdots x_r}$. Hence $\Gamma_*(\mathcal{O}_X)$ is the intersection $\bigcap S_{x_i}$ taken inside S'. Now any homogeneous element of S' can be written uniquely as a product $x_0^{i_0} \cdots x_r^{i_r} f(x_0, \dots, x_r)$, where the $i_j \in \mathbb{Z}$, and f is a homogeneous polynomial not divisible by any x_i. This element will be in S_{x_i} if and only if $i_j \geq 0$ for $j \neq i$. It follows that the intersection of all the S_{x_i} (in fact the intersection of any two of them) is exactly S.

Note that if S is not a polynomial ring, this result need not hold, though it is be related (see Hartshorne exercise II.5.14). A case where it fails is if we have that $M_0 = k$ (or some finite dimensional vector space) and $M_n = 0$ for all other integers; then $\tilde{M} = 0$. Hence, this shows that can't start with an M, do $\widetilde{M}$ and recover M by Γ_*. Unfortunately, the map $M \mapsto \Gamma_*(\widetilde{M})$ is neither injective nor surjective. However, if you *start* with a coherent sheaf $\mathscr{F}$, apply Γ_* to get $\Gamma_*(\mathscr{F})$, and then do $\widetilde{\Gamma_*(\mathscr{F})}$, then:

$$\widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$$

To prove this, we first require a technical lemma:

Lemma 4.2.8: Extending Sections for Invertible Sheaves

Let X be a scheme, $\mathcal{L}$ an invertible sheaf on X, $f \in \Gamma(X, \mathcal{L})$, X_f be the open set of points $x \in X$ where $f_x \notin \mathfrak{m}_x \mathcal{L}_x$, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Now:

1. Suppose that X is quasi-compact, and $s \in \Gamma(X, \mathscr{F})$ is a global section of $\mathscr{F}$ whose restriction to X_f is 0. Then for some $n > 0$, we have $f^n s = 0$, where $f^n s$ is considered as a global section of $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$.

2. Suppose furthermore that X has a finite covering by open affine subsets U_i, such that $\mathcal{L}|_{U_i}$ is free for each i, and such that $U_i \cap U_j$ is quasi-compact for each i, j (it is quasi-separated). Given a section $t \in \Gamma(X_f, \mathscr{F})$, then for some $n > 0$, the section $f^n t \in \Gamma(X_f, \mathscr{F} \otimes \mathcal{L}^{\otimes n})$ extends to a global section of $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$.

***Proof* :**
This lemma is a direct generalization of lemma 4.1.12, with an extra twist due to the presence of the invertible sheaf $\mathcal{L}$.

1. First, cover X with a finite number (possible since X is quasi-compact) of open affines $U = \operatorname{Spec} A$ such that $\mathcal{L}|_U$ is free. Let $\psi : \mathcal{L}|_U \cong \mathcal{O}_U$ be an isomorphism expressing the freeness of $\mathcal{L}|_U$. Since $\mathscr{F}$ is quasi-coherent, by proposition 4.1.13 there is an A-module M with $\mathscr{F}|_U \cong \widetilde{M}$. Our section $s \in \Gamma(X, \mathscr{F})$ restricts to give an element $s \in M$. On the other hand, our section $f \in \Gamma(X, \mathcal{L})$ restricts to give a section of $\mathcal{L}|_U$, which in turn gives rise to

an element $g = \psi(f) \in A$. Clearly $X_f \cap U = D(g)$. Now $s|_{X_f}$ is zero, so $g^n s = 0$ in M for some $n > 0$, just as in the proof of lemma 4.1.12. Using the isomorphism

$$\mathrm{id} \times \psi^{\otimes n} : \mathscr{F} \otimes \mathcal{L}^n|_U \cong \mathscr{F}|_U,$$

we conclude that $f^n s \in \Gamma(U, \mathscr{F} \otimes \mathcal{L}^n)$ is zero. This statement is intrinsic (i.e., independent of ψ). So now we do this for each open set of the covering, pick one n large enough to work for all the sets of the covering, and we find $f^n s = 0$ on X.

2. Proceed as in the proof of lemma 4.1.12, keeping track of the twist due to $\mathcal{L}$ as above. The hypothesis $U_i \cap U_j$ quasi-compact is used to be able to apply part (1) there.

Proposition 4.2.9: Partial Equivalence For Graded Rings

Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$ and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

In particular, all quasi-coherent sheaves on projective schemes come from a graded module.

Proof **:**

First, define the morphism β for any $\mathcal{O}_X$-module $\mathscr{F}$. Let $f \in S_1$. Since $\widetilde{\Gamma_*(\mathscr{F})}$ is quasi-coherent in any case, to define β, it is enough to give the image of a section of $\widetilde{\Gamma_*(\mathscr{F})}$ over $D_+(f)$. Such a section is represented by a fraction m/f^d, where $m \in \Gamma(X, \mathscr{F}(d))$, for some $d \geq 0$. We can think of f^{-d} as a section of $\mathcal{O}_X(-d)$, defined over $D_+(f)$. Taking their tensor product, we obtain $m \otimes f^{-d}$ as a section of $\mathscr{F}$ over $D_+(f)$. This defines β.

Now let $\mathscr{F}$ be quasi-coherent. To show that β is an isomorphism we have to identify the module $\Gamma_*(\mathscr{F})_{(f)}$ with the sections of $\mathscr{F}$ over $D_+(f)$. We apply lemma 4.2.8, considering f as a global section of the invertible sheaf $\mathcal{L} = \mathcal{O}(1)$. Since we have assumed that S is finitely generated by S_1 as an S_0-algebra, we can find finitely many elements $f_0, \ldots, f_r \in S_1$ such that X is covered by the open affine subsets $D_+(f_i)$. The intersections $D_+(f_i) \cap D_+(f_j)$ are also affine, and $\mathcal{L}|_{D_+(f_i)}$ is free for each i, so the hypotheses of lemma 4.2.8 are satisfied. The conclusion of lemma 4.2.8 tells us that $\mathscr{F}(D_+(f)) \cong \Gamma_*(\mathscr{F})_{(f)}$, which is just what we wanted.

We shall later show this that all resolution of quasi-coherent sheaves by locally free sheaves must have finite resolutions, see ref:HERE. For now, this shows we can translate our characterization of closed subschemes with ideal sheaves to projective schemes, and that

Corollary 4.2.10: Characterizing Projective Schemes

Let R be a ring. Then:

1. If Y is a closed subscheme of $\mathbb{P}^r_R$, then there is a homogeneous ideal $I \subseteq S = R[x_0, \ldots, x_r]$ such that Y is the closed subscheme determined by I.

2. A scheme Y over $\operatorname{Spec} R$ is projective if and only if it is isomorphic to $\operatorname{Proj} S$ for some graded ring S, where $S_0 = R$, and S is finitely generated by S_1 as an S_0-algebra.

***Proof* :**

1. Let $\mathcal{I}_Y$ be the ideal sheaf of Y on $X = \mathbb{P}^r_R$. Now $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_X$; the twisting functor is exact; the global section functor Γ is left exact; hence $\Gamma_*(\mathcal{I}_Y)$ is a submodule of $\Gamma_*(\mathcal{O}_X)$. But by proposition 4.2.9, $\Gamma_*(\mathcal{O}_X) = S$. Hence $\Gamma_*(\mathcal{I}_Y)$ is a homogeneous ideal of S, which we will call I. Now I determines a closed subscheme of X , whose sheaf of ideals will be $\widetilde{I}$. Since $\mathcal{I}_Y$ is quasi-coherent, we have $\mathcal{I}_Y \cong \widetilde{I}$ by proposition 4.2.9, and hence Y is the subscheme determined by I. In fact, $\Gamma_*(\mathcal{I}_Y)$ is the largest ideal in S defining Y .

2. Recall that by definition Y is projective over $\operatorname{Spec} R$ if it is isomorphic to a closed subscheme of $\mathbb{P}^r_R$ for some r. By part (1), any such Y is isomorphic to $\operatorname{Proj} S/I$, and we can take I to be contained in $S_+ = \bigoplus_{d>0} S_d$, so that $(S/I)_0 = R$. Conversely, any such graded ring S is a quotient of a polynomial ring, so $\operatorname{Proj} S$ is projective.

We conclude this section by finally showing why tensoring quasi-coherent sheaves is not closed:

Example 4.12: Lack of Closure For Quasi-coherent Sheaves
Take $\mathscr{F} = \mathcal{O}(-1)$ and $\mathscr{G} = \mathcal{O}(1)$. Then let us compare:

$$(\mathscr{F} \otimes \mathscr{G})(U) \qquad \text{and} \qquad (\mathscr{F}(U) \otimes \mathscr{G}(U))$$

The left hand side is $\tilde{R}$ and hence is 1 dimensional, while the right hand side is 0 dimensional (tensoring a 0 dimensional space with a 2 dimensional space).

Another example I came across is very interesting:

Example 4.13: Free Sheaves Need Not Be Projective Objects
Really cool! here!

4.3 Divisors

Divisors are a way of measuring how far away the geometry of your space is constructed by "principal" objects (think of principal ideals of a ring). There are two fundamental approaches to this: by seeing how far away the zero set of functions are from being constructed by 1 element, and by looking at the irreducible codimension 1 subschemes. The first approach shall yield *Cartier Divisors* while the second approach shall yield *Weil Divisors*. In most nice cases (think Integral Noetherian Schemes), these two shall give the same notion of divisor. If you have read [3, chapter 22.7], then in this section we shall generalize the notion of divisors to general schemes and show hey they relate to a *Picard Scheme*.

The intuition for divisors works well when starting with examples from complex analysis (in particular, see [2, chapter 3 and 6]). In this case, the two notions of divisors shall coincide and so we shall be finding the divisors for Riemann surface X. Starting with $X = \mathbb{C}$, consider the free-abelian group generated by the points of X, so that each element of this group is of the form:

$$\sum_i n_i p_i \qquad n_i \in \mathbb{Z},\ p_i \in X$$

We may ask if there is a meromorphic function associated to this function. As we are interested in algebraic constructions, we limit ourselves to function with a finite number of poles and zero, that is all *rational functions*. Then We get that for each $\sum_i n_i p_i$, we can associated to it the meromorphic function;

$$(z - p_i)^{n_i}$$

Let us call the set of meromorphic functions the *principle divisors*, and the finite formal $\mathbb{Z}$-linear combinations the *divisors*. Then the quotient of these two groups is zero:

$$\frac{\text{Divisors}}{\text{Principal Divisors}} = 0$$

Let us now consider X to be the Riemann surface. We shall once again take the divisor group to be the free group. This time, the principal divisor group (the group of meromorphic functions) has a fundamental limitations placed on it: recall that the sum of the zero's and pole of a meromorphic function is zero[1]. In this case, the sum of each divisor is some integer $n \in \mathbb{Z}$. Using this, you can show that:

$$\frac{\text{Divisors}}{\text{Principal Divisors}} \cong \mathbb{Z}$$

Finally, consider an elliptic curve given by $\mathbb{C}/L$ where L is a lattice. Richard Borcherds, finish off!

4.4 Maps to Projective Space

Definition 4.4.1: Twisting Sheaf On $\mathbb{P}^n_X$

Let X be a scheme. Then the *twisting sheaf* on $\mathcal{O}(1)$ on $\mathbb{P}^n_X$ is $g^*(\mathcal{O}(1))$ where:

$$g : \mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{Z}}$$

where $\mathbb{P}^n_X = X \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ by definition 3.3.32.

[1] Recall that by adding the point at infinite, we shall add the "compliment" of each zero/pole; convince yourself that z^2 has two zeros and two poles

Definition 4.4.2: Very Ample Sheaf

Let X be a Y-scheme, and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is said to be *very ample* relative to Y if there is a locally closed embedding $\iota : X \to \mathbb{P}^n_Y$ for some n such that

$$\iota^*(\mathcal{O}(1)) \cong \mathcal{L}$$

If Y is Noetherian, then X is projective if and only if X is both proper and has a very ample sheaf relative to Y. Indeed, if X is projective over Y,

Recall proposition 4.1.28 that showed that if if X a R-scheme and $f_0, ..., f_n$ are functions on X (global sections) with no common zeros. Then there exists a scheme morphism $[f_0 : \cdots f_n] : X \to \mathbb{P}^n_R$. It was commented that not all maps to projective space arise in this form, for example $\mathrm{id} : \mathbb{P}^n_k \to \mathbb{P}^n_k$. In this section, we remedy that. To start, we shall look for sheaves that are generated by their global section. The most important example will be when $S = k[x_0, ..., x_n]$ and $X = \operatorname{Proj} S$; then the global section of a coherent sheaf *must* be finitely generated.

Definition 4.4.3: Generated By Global Sections

Let X be a scheme and $\mathscr{F}$ an $\mathcal{O}_X$-module. Then $\mathscr{F}$ is *generated by global sections* if there is a family of global sections $\{s_i\}_{i\in I} \subseteq \Gamma(X, \mathscr{F})$ such that for each $x \in X$, the images of s_i in the stalk $\mathscr{F}_x$ generate $\mathscr{F}_x$ as an $\mathcal{O}_{X,x}$-module.

Naturally, $\mathscr{F}$ is generated by global sections iff $\mathscr{F}$ can be written as the quotient of a free-sheaf.

Example 4.14: Generated By Global Sections

1. Let $X = \operatorname{Spec} R$. Then any $\mathscr{F} = \widetilde{M}$ is generated by global sections, simply pick generators for M as an R-module
2. Let $X = \operatorname{Proj} S$ for a graded ring S genrated by S_1 as an S_0-algebra. Then the elements of S_1 can give global sections of $\mathcal{O}_X(1)$ that generate it.
3. The scheme $\mathcal{O}_X(-1)$ is *not* generated by global sections, as it has an empty global section.

Theorem 4.4.4: Global Section Generation of twisted Sheaf over Proj Noetherian Scheme

Let X be a projective scheme over a Noetherian ring R, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X, and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathscr{F}(n)$ can be generated by a finite number of global sections.

***Proof* :**
Let $i : X \to \mathbb{P}^r_R$ be a closed embedding of X into a projective space over R, such that $i^*(\mathcal{O}(1)) = \mathcal{O}_X(1)$. Then $i_*\mathscr{F}$ is coherent on $\mathbb{P}^r_R$ as closed embedding are finite morphisms and projective schemes are Noetherian, and $i_*(\mathscr{F}(n)) = (i_*\mathscr{F})(n)$ by proposition 4.2.5, and $\mathscr{F}(n)$ is generated by global sections if and only if $i_*(\mathscr{F}(n))$ is (in fact, their global sections are the same), so this can be reduced to the case $X = \mathbb{P}^r_R = \text{Proj } R[x_0, \ldots, x_r]$.

Now cover X with the open sets $D_+(x_i)$, $i = 0, \dots, r$. Since $\mathscr{F}$ is coherent, for each i there is a finitely generated module M_i over $B_i = R[x_0/x_i, \dots, x_n/x_i]$ such that $\mathscr{F}|_{D_+(x_i)} \cong \tilde{M}_i$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate this module. By lemma 4.1.12 there is an integer n such that $x_i^n s_{ij}$ extends to a global section t_{ij} of $\mathscr{F}(n)$. As usual, we take one n to work for all i, j. Now $\mathscr{F}(n)$ corresponds to a B_i-module M_i' on $D_+(x_i)$, and the map $x_i^n : \mathscr{F} \to \mathscr{F}(n)$ induces an isomorphism of M_i to M_i'. So the sections $x_i^n s_{ij}$ generate M_i', and hence the global sections $t_{ij} \in \Gamma(X, \mathscr{F}(n))$ generate the sheaf $\mathscr{F}(n)$ everywhere, as we sought to show.

Corollary 4.4.5: Coherent Sheaf is quotient of "free twisted sheaves"

Let X be projective R-scheme. Then any coherent sheaf $\mathscr{F}$ on X can be written as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of twisted structure sheaves $\mathcal{O}(n_i)$ for various integers n_i.

Proof **:**
Let $\mathscr{F}(n)$ be generated by a finite number of global sections. Then we have a surjection $\bigoplus_{i=1}^r \mathcal{O}_X \to \mathscr{F}(n) \to 0$. Tensoring with $\mathcal{O}_X(-n)$ we obtain a surjection $\bigoplus_{i=1}^r \mathcal{O}_X(-n) \to \mathscr{F} \to 0$ as required.

From this, we can now classify a large number of coherent sheaves of on projective space:

Theorem 4.4.6: Finite Generation of Coherent Sheaf Over Projective Scheme

Let k be a field, R a finitely generated k-algebra, $X = \mathbb{P}_R^n$, and $\mathscr{F}$ a *coherent* $\mathcal{O}_X$-module. Then $\Gamma(X, \mathscr{F})$ is a finitely generated R-module. If $R = k$, then $\Gamma(X, \mathscr{F})$ is a finite dimensional k-vector space.

This proof can be thought of the generalization of the proof that $\mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}_k^n) = k$. Not too that this is *not* true if the space is not projective, for example over $\mathbb{A}_k^n$ for any n the resulting space is *not* a finite dimensional k-vector space, as $\Gamma(\mathbb{A}_k^n, \mathcal{O}_{\mathbb{A}_k^n}) = k[x]$ which is infinite dimensional!

Proof **:**
Let $X = \operatorname{Proj} S$, where S is a graded ring with $S_0 = R$ which is finitely generated by S_1 as an S_0-algebra, corollary 4.2.10. Let M be the graded S-module $\Gamma_*(\mathscr{F})$. Then by proposition 4.2.9, $\tilde{M} \cong \mathscr{F}$. On the other hand, by theorem 4.4.4, for n sufficiently large, $\mathscr{F}(n)$ is generated by a finite number of global sections in $\Gamma(X, \mathscr{F}(n))$. Let M' be the submodule of M generated by these sections. Then M' is a finitely generated S-module. Furthermore, the inclusion $M' \hookrightarrow M$ induces an inclusion of sheaves $\tilde{M}' \hookrightarrow \tilde{M} = \mathscr{F}$. Twisting by n we have an inclusion $\tilde{M}'(n) \hookrightarrow \mathscr{F}(n)$ which is actually an isomorphism, because $\mathscr{F}(n)$ is generated by global sections in M'. Twisting by $-n$ we find that $\tilde{M}' \cong \mathscr{F}$. Thus $\mathscr{F}$ is the sheaf associated to a finitely generated S-module, and so we have reduced to showing that if M is a finitely generated S-module, then $\Gamma(X, \tilde{M})$ is a finitely generated R-module.

Now, recall from [3] that there is a finite filtration

$$0 = M^0 \subseteq M^1 \subseteq \dots \subseteq M^r = M$$

of M by graded submodules, where for each i, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogeneous prime ideal $\mathfrak{p}_i \subseteq S$, and some integer n_i. This filtration gives a filtration of $\tilde{M}$, and the short exact sequences

$$0 \to \widetilde{M^{i-1}} \to \widetilde{M^i} \to \widetilde{M^i/M^{i-1}} \to 0$$

give rise to left-exact sequences

$$0 \to \Gamma(X, \widetilde{M^{i-1}}) \to \Gamma(X, \widetilde{M^i}) \to \Gamma(X, \widetilde{M^i/M^{i-1}}).$$

Thus to show that $\Gamma(X, \tilde{M})$ is finitely generated over R, it will be sufficient to show that $\Gamma(X, (S/\mathfrak{p})^{\sim}(n))$ is finitely generated, for each $\mathfrak{p}$ and n. Thus we have reduced to the following special case: Let S be a graded integral domain, finitely generated by S_1 as an S_0-algebra, where $S_0 = R$ is a finitely generated integral domain over k. Then $\Gamma(X, \mathcal{O}_X(n))$ is a finitely generated R-module, for any $n \in \mathbf{Z}$.

Let $x_0, \ldots, x_r \in S_1$ be a set of generators of S_1 as an R-module. Since S is an integral domain, multiplication by x_0 gives an injection $S(n) \to S(n+1)$ for any n. Hence there is an injection $\Gamma(X, \mathcal{O}_X(n)) \to \Gamma(X, \mathcal{O}_X(n+1))$ for any n. Thus it is sufficient to prove $\Gamma(X, \mathcal{O}_X(n))$ finitely generated for all sufficiently large n, say $n \geq 0$.

Let $S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$. Then S' is a ring, containing S, and contained in the intersection $\bigcap S_{x_i}$ of the localizations of S at the elements $x_0, \ldots, x_r$.

Next, let's show that S' is integral over S. Let $s' \in S'$ be homogeneous of degree $d \geq 0$. Since $s' \in S_{x_i}$ for each i, we can find an integer n such that $x_i^n s' \in S$. Choose one n that works for all i. Since the x_i generate S_1, the monomials in the x_i of degree m generate S_m for any m. So by taking a larger n, we may assume that $ys' \in S$ for all $y \in S_n$. In fact, since s' has positive degree, we can say that for any $y \in S_{\geq n} = \bigoplus_{e \geq n} S_e$, $ys' \in S_{\geq n}$. Now it follows inductively, for any $q \geq 1$ that $y \cdot (s')^q \in S_{\geq n}$ for any $y \in S_{\geq n}$. Take for example $y = x_0^n$. Then for every $q \geq 1$ we have $(s')^q \in (1/x_0^n)S$. This is a finitely generated sub-S-module of the quotient field of S'. It follows by a well-known criterion for integral dependence (see [3]), that s' is integral over S. Thus S' is contained in the integral closure of S in its quotient field.

To complete the proof, since S is a finitely generated k-algebra, by commutative algebra we know that S' will be a finitely generated S-module. It follows that for every n, S_n is a finitely generated S_0-module, as we sought to show.

In fact, this proof shows that $S_n = S'_n$ for all sufficiently large n, see exercises ref:HERE

Note that it was not technically needed that R is a finitely generated k-algebra, but just a "Nagata ring". See ref:HERE for a generalization. As almost all (Noetherian) rings are Nagata, this isn't much of a generalization.

Corollary 4.4.7: Pushforward of Coherent Sheaf by Projective Morphism

Let $f : X \to Y$ be a projective morphism of schemes of finite type over a field k. Let $\mathscr{F}$ be a coherent sheaf on X. Then $f_*\mathscr{F}$ is coherent on Y.

Recall this is *not* true if the map is not projective, see example 4.6.

***Proof* :**

The question is local on Y, so we may assume $Y = \operatorname{Spec} R$, where R is a finitely generated k-algebra. Then in any case, $f_*\mathscr{F}$ is quasi-coherent by proposition 4.1.14, so $f_*\mathscr{F} = \Gamma(Y, f_*\mathscr{F})^{\sim} = \Gamma(X, \mathscr{F})^{\sim}$.

But $\Gamma(X, \mathscr{F})$ is a finitely generated R-module by the theorem, so $f_*\mathscr{F}$ is coherent.

This result shall have a cohomological proof which will allow us to generalize to proper morphisms, see ref:HERE.

4.5 Differentials

Definition 4.5.1: A-derivation

Let A be a commutative ring, B an A-algebra, and M a B-module. Then an A-*derivation* of B into M is a map $d : B \to M$ such that

1. $d(b + b') = d(b) + d(b')$
2. $d(bb') = d(b)b' + bd(b')$
3. $d(a) = 0$ for all $a \in A$

Definition 4.5.2: Module of Relative Differential Forms

Let A be a commutative ring, B an A-algebra. Then a *module of relative differential form* is a B-module $\Omega_{B/A}$ with an A-derivation $d : B \to \Omega_{B/A}$ which satisfies the universal property of relative differential forms: if M is another B-module and $d' : B \to M$ an A-derivation, then there is a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M \end{tikzcd}$$

Certainly $\Omega_{B/A}$ exists via the usual construction: take the free B-module F generated by the primitives $d(b)$ for $b \in B$ and quotient by the submodule generated by the relationships:

$$d(b + b') - d(b) - d(b') \qquad \text{and} \qquad d(bb') - d(b)b' - bd(b') \qquad \text{and} \qquad d(a)$$

Then the map $d : B \to \Omega_{B/A}$ is given by $b \mapsto d(b)$. Then it is an exercise in algebra to see that this satisfies the universal property and $(\Omega_{B/A}, d)$ is unique up to unique isomorphism.

5

Cohomology of Schemes

Let X be a scheme. There are a few natural cohomological questions we may ask about X:

1. Can we quantify the gluing of the affine open sets that make of X so that the cohomology of affine scheme's is trivial?
2. If we have open sets $U \subseteq V \subseteq X$, to what degree $\text{res} : V \to U$ fails to be surjective?

These shall be answered in this chapter. There shall be many cohomological theories presented, which shall all turn out to be describing the same phenomena from different angles.

The reader may want to review [3, Part VII] to get a refresher on (co)homology theory.

5.1 Definition and Basic Properties

Recall that if R is a ring, then every R-module is isomorphic to a submodule of an injective module (see [3, chatper 16.3]). From this, we shall be able to conclude that ringed space have enough injectives

Proposition 5.1.1: Ringed Space Has Enough Injectives

Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of sheaves of $\mathcal{O}_X$-modules, ${}_{\mathcal{O}_X}\mathbf{Mod}$ has enough injectives.

Proof **:**
Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then for each $x \in X$, the stalk $\mathscr{F}_x$ is an $\mathcal{O}_{x,X}$-module. Then there is an injection

$$\mathscr{F}_x \to I_x$$

where I_x is an injective $\mathcal{O}_{x,X}$-module. We can do this for each point. Now, Let $j : \{x\} \to X$ denote the inclusion of the one-point space for each point, let $\{x\}$ have I_x as a sheaf, and define the sheaf:

$$\mathscr{I} = \prod_{x \in X} j_*(I_x)$$

where j_* is the usual pushforward. This shall be our injective object.

For any $\mathcal{O}_X$-module $\mathscr{G}$, consider

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G}, \mathscr{I}) = \prod \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G}, j_*(I_x))$$

Then for each point $x \in X$, notice that:

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{G}, j_*(I_x)) \cong \operatorname{Hom}_{\mathcal{O}_{x,X}}(\mathscr{G}_x, I_x)$$

From this, we see that if we pick $\mathscr{G} = \mathscr{G}$, we get a natural injective $\mathcal{O}_X$-module map $\mathscr{F} \to \mathscr{I}$ given by $\mathscr{F}_x \to I_x$. Furthermore, $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathscr{I})$ is exact as it is the product $\operatorname{Hom}_{\mathcal{O}_{x,X}}(-, I_x)$ which is exact as I_x is injective, and the stalk functor $\mathscr{G} \mapsto \mathscr{G}_x$ which is exact by 1.1.26. Hence, $\operatorname{Hom}(-, \mathscr{I})$ is exact, making $\mathscr{I}$ an injective $\mathcal{O}_X$-module, completing the proof.

Corollary 5.1.2: Sheaves of Abelian's Has Enough Injectives

Let X be a topological space. Then the category of sheaves of abelian groups on X has enough injectives

Proof **:**
Take $\mathcal{O}_X = \mathbb{Z}$ and apply proposition 5.1.1

Definition 5.1.3: Global Section Functor

Let X be a topological space. Then the *global section functor*

$$\Gamma(X, -) : \mathrm{Ab}(X) \to \mathrm{Ab}$$

which is left- but not right-exact (by proposition 1.1.12). Then the cohomology functors $H^i(X, -)$, the right derived functors of $\Gamma(X, -)$ are the *cohomology groups* (with respect to a sheaf).

To be able to create resolutions, recall that it is enough to find acyclic objects. We shall show that the following types of objects shall be acyclic:

Definition 5.1.4: Flasque Sheaf

Let X be a topological space. Then a sheaf $\mathscr{F}$ on X is called *flasque* if for every inclusion $U \subseteq V$, $\mathrm{res} : \mathscr{F}(V) \to \mathscr{F}(U)$ is surjective.

Proposition 5.1.5: Injective Sheaf Modules Are Flasque

Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque

Proof **:**
For any open subset $V \subseteq X$, let $\mathcal{O}_V$ denote the sheaf $j_!(\mathcal{O}_X|_V)$, which is the restriction of $\mathcal{O}_X$ to V, extended by zero outside V. Now let $\mathscr{I}$ be an injective $\mathcal{O}_X$-module, and let $V \subseteq U$ be open sets. Then we have an inclusion $0 \to \mathcal{O}_U \to \mathcal{O}_V$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathscr{I}$ is injective, we get a surjection $\mathrm{Hom}(\mathcal{O}_V, \mathscr{I}) \to \mathrm{Hom}(\mathcal{O}_U, \mathscr{I}) \to 0$. But $\mathrm{Hom}(\mathcal{O}_U, \mathscr{I}) = \mathscr{I}(U)$ and $\mathrm{Hom}(\mathcal{O}_V, \mathscr{I}) = \mathscr{I}(V)$, so $\mathscr{I}$ is flasque.

Corollary 5.1.6: Trivial Homology For Flask Sheaf

Let $\mathscr{F}$ be a flasque sheaf on X. Then

$$H^i(X, \mathscr{F}) = 0 \qquad i > 0$$

Proof **:**
As there are enough injective, embed $\mathscr{F}$ into $\mathscr{I}$, and take it's quotient $\mathscr{G}$. Most importantly, check that the quotient sheaf of two flasque sheaves is flasque, and so we have a short exact sequence of flasque sheaves:

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$$

As $\mathscr{F}$ is flasque, we have the $\Gamma(X, -)$ will preserve this exact sequence:

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{G}) \to 0$$

As $\mathscr{I}$ is injective, $H^i(X, \mathscr{I}) = 0$ for $i > 0$, and so by taking the long exact sequence of cohomology:

$$H^1(X, \mathscr{F}) = 0, \qquad H^i(X, \mathscr{F}) \cong H^{i-1}(X, \mathscr{G}),\ i \geq 2$$

As $\mathscr{G}$ is also flasque, we get by induction that $H^i(X, \mathscr{F}) = 0$ for $i > 0$, completing the proof.

Hence, flasque sheaves are acyclic for $\Gamma(X, -)$, and so we may compute cohomology by taking Flasque resolutions! Now, let us link this to schemes. First, if $(X, \mathcal{O}_X)$ is a ringed space, by taking $R = \Gamma(X, \mathcal{O}_X)$, we always have a natural R-module structure on any $\mathcal{O}_X$-modules $\mathscr{F}$. Thus, the cohomology groups of $\mathscr{F}$ have a natural R-module structure on them, and the associated exact sequence are naturally R-modules. Then for R-schemes, the cohomology groups of any $\mathcal{O}_X$-module will have natural R-modules structure.

Let us now prove our first sheaf cohomology result: let us try to bring in our results on dimension and Noetherian space. To prove this, let us first build-up the appropriate results for the cohomology of Noetherian space

Lemma 5.1.7: Noetherian Space and Flasque Sheaves

Let X be a Noetherian topological space. Then the colimit of flasque sheaves is flasque.

Proof **:**
Let $(\mathscr{F}_\alpha)$ be a directed system of flasque sheaves. Then for any inclusion $V \subseteq U$, for each α we have $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$ is surjective. As $\varinjlim$ is an exact functor, we get that the map:

$$\varinjlim_\alpha \mathscr{F}_\alpha(U) \to \varinjlim_\alpha \mathscr{F}_\alpha(V)$$

is surjective. Then on a Noetherian topological space:

$$\varinjlim \mathscr{F}_\alpha(U) = (\varinjlim \mathscr{F}_\alpha)(U)$$

Hence we have:

$$(\varinjlim_\alpha \mathscr{F}_\alpha)(U) \to (\varinjlim_\alpha \mathscr{F}_\alpha()V)$$

is surjective, hence $\varinjlim_\alpha \mathscr{F}_\alpha$ is flasque, completing the proof.

Lemma 5.1.8: Colimit and Cohomology Commuting

Let X be a Noetherian topological space and $(\mathscr{F}_\alpha)$ a direct system of abelian sheaves. Then there is a natural isomorphism for each $i \geq 0$:

$$\varinjlim H^i(X, \mathscr{F}_\alpha) \cong H^i(X, \varinjlim \mathscr{F}_\alpha)$$

Proof **:**
For each α we have a natural map $\mathscr{F}_\alpha \to \varinjlim \mathscr{F}_\alpha$. This induces a map on cohomology, and then we take the direct limit of these maps. For $i = 0$, this becomes

$$\Gamma(X, \varinjlim \mathscr{F}_i) \cong \varinjlim \Gamma(X, \mathscr{F}_i)$$

Which is readily true for Noetherian topological spaces. For the general case, consider the category $\text{ind}_A(\mathbf{Ab}(X))$ consisting of all directed systems of objects of $\mathbf{Ab}(X)$, indexed by A. This is an abelian category. Furthermore, since $\varinjlim$ is an exact functor, we have a natural transformation of δ-functors

$$\varinjlim H^i(X, -) \to H^i(X, \varinjlim -)$$

from $\text{ind}_A(\mathbf{Ab}(X))$ to $\mathbf{Ab}$. They agree for $i = 0$, so to prove they are the same, it will be sufficient to show they are both effaceable for $i > 0$. For in that case, they are both universal, and hence by uniqueness of universal object must be isomorphic.

So let $(\mathscr{F}_\alpha) \in \text{ind}_A(\mathbf{Ab}(X))$. For each α, let $\mathscr{G}_\alpha$ be the sheaf of discontinuous sections of $\mathscr{F}_\alpha$. Then $\mathscr{G}_\alpha$ is flasque, and there is a natural inclusion $\mathscr{F}_\alpha \to \mathscr{G}_\alpha$. Furthermore, the construction of $\mathscr{G}_\alpha$ is functorial, so the $\mathscr{G}_\alpha$ also form a direct system, and we obtain a monomorphism $u : (\mathscr{F}_\alpha) \to (\mathscr{G}_\alpha)$ in the category $\text{ind}_A(\mathbf{Ab}(X))$. Now the $\mathscr{G}_\alpha$ are all flasque, so $H^i(X, \mathscr{G}_\alpha) = 0$ for $i > 0$. Thus $\varinjlim H^i(X, \mathscr{G}_\alpha) = 0$, and the functor on the left-hand side is effaceable for $i > 0$. On the other hand, $\varinjlim \mathscr{G}_\alpha$ is also flasque. So $H^i(X, \varinjlim \mathscr{G}_\alpha) = 0$ for $i > 0$, and we see that the functor on the right-hand side is also effaceable, as we sought to show.

Corollary 5.1.9: Direct sum and cohomology commuting

Let X be a Noetherian topological space. Then there is a natural isomorphism for each $i \geq 0$:

$$\bigoplus_\alpha H^i(X, \mathscr{F}_\alpha) \cong H^i(X, \bigoplus_\alpha \mathscr{F}_\alpha)$$

Proof **:**
The direct is a special case of a colimit.

Lemma 5.1.10: Cohomology of Closed Subsets

Let $Y \subseteq X$ be a closed subset of X, $\mathscr{F}$ a sheaf of abelian groups, and $j : Y \to X$ the natural inclusion. Then:

$$H^i(Y, \mathscr{F}) = H^i(X, j_*\mathscr{F})$$

where $j_*\mathscr{F}$ is the extension of $\mathscr{F}$ by zero on Y (see exercise 1.3.1)

Proof **:**
If $\mathscr{I}*$ is a flasque resolution of $\mathscr{F}$, $j_*\mathscr{F}^*$ is a flasque resolution of $j_*\mathscr{F}$, and so for each i:

$$\Gamma(Y, \mathscr{I}^i) = \Gamma(X, j_*\mathscr{I}^*)$$

But then

$$H^i(Y, \mathscr{F}) = H^i(X, j_*\mathscr{F})$$

as we sought to show.

Due to this lemma, we shall often write $\mathscr{F}$ instead of $j_*\mathscr{F}$ without worrying about ambiguity. With these lemmas, we can prove our first main result:

Theorem 5.1.11: Noetherian Space and Dimension

Let X be a Noetherian topological space of dimension n. Then for all $i > n$, all sheaves of abelian groups $\mathscr{F}$ on X vanish:

$$H^i(X, \mathscr{F}) = 0 \qquad i > n$$

Proof **:**
Let us first set some notation: let $Y \subseteq X$ be a closed subset, $\mathscr{F}$ be any sheaf on X, and $\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$ where $j : Y \to X$ is the inclusion. Similarly for open subset $U \subseteq X$ where $\mathscr{F}_U = i_*(\mathscr{F}|_U)$ with the inclusion $i : U \to X$. If $U = X \setminus Y$, we get the short exact sequence:

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0$$

With this setup, we shall prove the result by induction on the dimension of X.

Step 1. Reduction to the case X irreducible. If X is reducible, let Y be one of its irreducible components, and let $U = X - Y$. Then for any $\mathscr{F}$ we have an exact sequence

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X, \mathscr{F}_Y) = 0$ and $H^i(X, \mathscr{F}_U) = 0$ for $i > n$. But Y is closed and irreducible, and $\mathscr{F}_U$ can be regarded as a sheaf on the closed subset $\bar{U}$, which has one fewer irreducible components than X. Thus using lemma 5.1.10 and induction on the number of irreducible components, we reduce to the case X irreducible.

Step 2. Suppose X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. For otherwise, X would have a proper irreducible closed subset, and $\dim X$ would be ≥ 1. Thus $\Gamma(X, \cdot)$ induces an equivalence of categories $\mathbf{Ab}(X) \to \mathbf{Ab}$. In particular, $\Gamma(X, \cdot)$ is an exact functor, so $H^i(X, \mathscr{F}) = 0$ for $i > 0$, and for all $\mathscr{F}$.

Step 3. Now let X be irreducible of dimension n, and let $\mathscr{F} \in \mathbf{Ab}(X)$. Let $B = \bigcup_{U \subseteq X} \mathscr{F}(U)$, and let A be the set of all finite subsets of B. For each $\alpha \in A$, let $\mathscr{F}_\alpha$ be the subsheaf of $\mathscr{F}$ generated by the sections in α (over various open sets). Then A is a directed set, and $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$. So by (2.9), it will be sufficient to prove vanishing of cohomology for each $\mathscr{F}_\alpha$. If α' is a subset of α, then we have an exact sequence

$$0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a sheaf generated by $\#(\alpha - \alpha')$ sections over suitable open sets. Thus, using the long exact sequence of cohomology, and induction on $\#(\alpha)$, we reduce to the case that $\mathscr{F}$ is generated by a single section over some open set U. In that case $\mathscr{F}$ is a quotient of the sheaf $\underline{\mathbb{Z}}_U$ (where $\underline{\mathbb{Z}}$ denotes the constant sheaf $\mathbb{Z}$ on X). Letting $\mathcal{K}$ be the kernel, we have an exact sequence

$$0 \to \mathcal{K} \to \underline{\mathbb{Z}}_U \to \mathscr{F} \to 0.$$

Again using the long exact sequence of cohomology, it will be sufficient to prove vanishing for $\mathcal{K}$ and for $\underline{\mathbb{Z}}_U$.

Step 4. Let U be an open subset of X and let $\mathcal{K}$ be a subsheaf of $\underline{\mathbb{Z}}_U$. For each $x \in U$, the stalk $\mathcal{K}_x$ is a subgroup of $\mathbb{Z}$. If $\mathcal{K} = 0$, skip to Step 5. If not, let d be the least positive integer which occurs in any of the groups $\mathcal{K}_x$. Then there is a nonempty open subset $V \subseteq U$ such that $\mathcal{K}|_V \cong d \cdot \underline{\mathbb{Z}}|_V$ as a subsheaf of $\underline{\mathbb{Z}}|_V$. Thus $\mathcal{K}_V \cong \underline{\mathbb{Z}}_V$ and we have an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \mathcal{K} \to \mathcal{K}/\underline{\mathbb{Z}}_V \to 0.$$

Now the sheaf $\mathcal{K}/\underline{\mathbb{Z}}_V$ is supported on the closed subset $(U - V)^-$ of X, which has dimension $< n$, since X is irreducible. So using lemma 5.1.10 and the induction hypothesis, we know $H^i(X, \mathcal{K}/\underline{\mathbb{Z}}_V) = 0$ for $i \geq n$. So by the long exact sequence of cohomology, we need only show vanishing for $\underline{\mathbb{Z}}_V$.

Step 5. To complete the proof, we need only show that for any open subset $U \subseteq X$, we have $H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_Y \to 0.$$

Now $\dim Y < \dim X$ since X is irreducible, so using lemma 5.1.10 and the induction hypothesis, we have $H^i(X, \underline{\mathbb{Z}}_Y) = 0$ for $i \geq n$. On the other hand, $\underline{\mathbb{Z}}$ is flasque, since it is a constant sheaf on an irreducible space (II, Ex. 1.16a). Hence $H^i(X, \underline{\mathbb{Z}}) = 0$ for $i > 0$ by (2.5). So from the long exact sequence of cohomology we have $H^i(X, \underline{\mathbb{Z}}_U) = 0$ for $i > n$.

5.1.1 Application: Cohomology of Noetherian Affine Scheme

In this section, we shall make rigorous the intuition that the cohomology of schemes measure some form of gluing by showing that if $X = \operatorname{Spec} R$, is a Noetherian affine scheme, then for any quasi-coherent sheaf $\mathscr{F}$ over X,

$$H^i(X, \mathscr{F}) = 0 \qquad \forall i > 0$$

The key insight is that if I is an injective R-module, $\tilde{I}$ on $\operatorname{Spec} R$ is flasque. This will motivate the idea that cohomology of schemes should be thought of how the gluing of schemes can be measured, which shall motivate introducing Čech cohomology in the next section and showing it is equivalent to the cohomology given by flasque resolutions, and shall be computationally much more manageable.

First, let M be an R-module and $\mathfrak{a} \subseteq R$ be an ideal. Define the $\mathfrak{a}$-torsion submodule

$$\operatorname{Tor}_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0, \text{ for some } n > 0\}$$

5.1.1 Note

This result is in fact true without the Noetherian condition, however it is harder to prove. The extra effort for building it up is not necessary for the main theory, for the curious see ref:HERE.

Lemma 5.1.12: $\mathfrak{a}$-Torsion Module Injective

Let R be a Noetherian ring, $\mathfrak{a} \subseteq R$ an ideal, and I an injective R-module. Then the submodule

$$J = \operatorname{Tor}_{\mathfrak{a}}(I)$$

is injective

***Proof* :**

To show that J is injective, it will be sufficient to prove Baer's Criterion, namely to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi : \mathfrak{b} \to J$, there exists a homomorphism $\psi : A \to J$ extending φ. Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some power of $\mathfrak{a}$, so there exists an $n > 0$ such that $\mathfrak{a}^n \varphi(\mathfrak{b}) = 0$, or equivalently, $\varphi(\mathfrak{a}^n \mathfrak{b}) = 0$. Now applying Artin-Rees lemma (see [3, chapter 20.3]) to the inclusion $\mathfrak{b} \subseteq A$, we find that there is an $n' \geq n$ such that $\mathfrak{a}^n \mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^{n'}$. Hence $\varphi(\mathfrak{b} \cap \mathfrak{a}^{n'}) = 0$, and so the map $\varphi : \mathfrak{b} \to J$ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$. Now we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] & A/\mathfrak{a}^{n'} \arrow[dr, dashed, "\psi'"] & & \\
\mathfrak{b} \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "\varphi"'] & \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'}) \arrow[u, hook] \arrow[r] & J \arrow[r] & I
\end{tikzcd}$$

Since I is injective, the composed map of $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$ to I extends to a map $\psi' : A/\mathfrak{a}^{n'} \to I$. But the image of ψ' is annihilated by $\mathfrak{a}^{n'}$, so it is contained in J. Composing with the natural map $A \to A/\mathfrak{a}^{n'}$, we obtain the required map $\psi : A \to J$ extending φ, as we sought to show.

Lemma 5.1.13: Flasque on Affine Scheme I

Let I be an injective R-module over a Noetherian ring R. Then for any $f \in R$, the natural map $I \to I_f$ is surjective

Proof **:**
Let $f \in R$ and $\varphi : I \to I_f$ be the natural map. I claim this map is surjective. Pick $x \in I_f$. By definition of localization and of φ:

$$\frac{\varphi(y)}{f^n} = x$$

We shall find an element that maps to the left hand side. To find this map, for each $i > 0$, let $\mathfrak{b}_i$ be the annihilator of f^i. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$, and as R is Noetherian there exists an m such that $\mathfrak{b}_m = \mathfrak{b}_{m+1} = \cdots$. Now, define from the ideal to the injective R-module $\psi : (f^{n+m}) \to I$ where $f^{n+m} \mapsto f^m y$. As I is injective, by Baer's criterion this map extends to $\Psi : A \to I$. Label $\psi(1)$ as z. Then we have $f^{n+m}z = f^m z$. But then we see that Ψ and φ match, in particular we have:

$$\varphi(z) = \frac{\varphi(y)}{f^n} = x$$

showing surjectivity, and completing the proof.

Lemma 5.1.14: Flasque on Affine Scheme II

Let I be an injective R-module over a Noetherian ring R. Then $\tilde{I}$ on $X = \operatorname{Spec} R$ is flasque

Note that $\widetilde{I}$ is flasque even if R is not Noetherian, however this proof would be adds technicality without adding insight.

Proof **:**
We will use Noetherian induction on $Y = (\operatorname{Supp} \tilde{I})$ (see section (0.3) for an overview of Noetherian induction). If Y consists of a single closed point of X, then $\tilde{I}$ is a skyscraper sheaf which is certainly flasque.

In the general case, it suffices to show that for any open set $U \subseteq X$, $\Gamma(X, \tilde{I}) \to \Gamma(U, \tilde{I})$ is surjective. If $Y \cap U = \emptyset$, we are done, so say $Y \cap U \neq \emptyset$. Then there is an $f \in A$ such that the open set $X_f = D(f)$ is contained in U and $X_f \cap Y \neq \emptyset$. Let $Z = X - X_f$, and consider the following diagram:

$$\begin{tikzcd}
\Gamma(X, \tilde{I}) \arrow[r] & \Gamma(U, \tilde{I}) \arrow[r] & \Gamma(X_f, \tilde{I}) \\
\Gamma_Z(X, \tilde{I}) \arrow[r] \arrow[u, hook] & \Gamma_Z(U, \tilde{I}), \arrow[u, hook] &
\end{tikzcd}$$

where Γ_Z denotes sections with support in Z. Now given a section $s \in \Gamma(U, \tilde{I})$, consider its image s' in $\Gamma(X_f, \tilde{I})$. By proposition 4.1.11, $\Gamma(X_f, \tilde{I}) = I_f$, so by lemma 5.1.13 there is a $t \in I = \Gamma(X, \tilde{I})$ restricting to s'. Let t' be the restriction of t to $\Gamma(U, \tilde{I})$. Then $s - t'$ goes to 0 in $\Gamma(X_f, \tilde{I})$, so it has support in Z. Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X, \tilde{I}) \to \Gamma_Z(U, \tilde{I})$

is surjective.

Now, let $J = \Gamma_Z(X, \tilde{I})$. If $\mathfrak{a}$ is the ideal generated by f, then $J = \Gamma_{\mathfrak{a}}(I)$ so by lemma 5.1.12, J is also an injective A-module. Furthermore, the support of $\tilde{J}$ is contained in $Y \cap Z$, which is strictly smaller than Y. Hence by our induction hypothesis, $\tilde{J}$ is flasque. Since $\Gamma(U, \tilde{J}) = \Gamma_Z(U, \tilde{I})$, we conclude that

$$\Gamma_Z(X, \tilde{I}) \to \Gamma_Z(U, \tilde{I})$$

is surjective, as we sought to show.

Theorem 5.1.15: Affine Noetherian Scheme: Trivial Cohomology

Let $X = \operatorname{Spec} R$ be the spectrum of a Noetherian ring R. Then for all quasi-coherent sheaves $\mathscr{F}$ on X,

$$H^i(X, \mathscr{F}) = 0 \qquad \forall i > 0$$

Proof **:**
Let $\mathscr{F}$ be a quasi-coherent sheaf on the affine scheme $X = \operatorname{Spec} R$ so that we have a module $M = \Gamma(X, \mathscr{F})$ and by corollary 4.1.10 $\mathscr{F} = \widetilde{M}$. Then from [3, chapter 16.3], we know we can take the injective resolution $0 \to M \to I^*$. Then as $\widetilde{(-)}$ is exact, we have that $0 \to \widetilde{M} \to \widetilde{I}^*$ is exact. By lemma 5.1.14, each $\widetilde{I}^i$ is flasque, hence by the principles of homology we can use this sequence to compute the cohomology groups. But then, applying the (in this case exact) Γ we have the sequence of R-modules $0 \to M \to I^*$ and we immediately get:

$$H^0(X, \mathscr{F}) = M \qquad H^i(X, \mathscr{F}) = 0 \qquad i > 0$$

as we sought to show.

Corollary 5.1.16: Injective Embedding of Sheaves On Neotherian Schemes

Let X be a Noetherian scheme and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $\mathscr{F}$ can embed in a quasi-coherent flasque sheaf.

This corollary shall in fact use the Noetherian condition in a meaningful way, as shall be pointed out in the proof

Proof **:**
As X is Noetherian, let $X = \bigcup_i^n \operatorname{Spec} R_i = \bigcup_i^n U_i$ be a finite open affine cover, and let $\mathscr{F}|_U = \widetilde{M_i}$ for each i. Embed each M_i into an injective R_i-module I_i, and define:

$$\mathscr{G} = \bigoplus_i^n \iota_*(\widetilde{I_i})$$

where ι is the inclusion of $U_i \to X$ for each U_i (where we are overloading notation letting ι represent each inclusion). As for each i we have the injective map $\mathscr{F}|_{U_i} \to \widetilde{I}_i$, we get a map the injective maps $\mathscr{F} \to f_*(\widetilde{I}_i)$. Then taking the direct sum, we get the injective map $\mathscr{F} \to \mathscr{G}$. Now, each $\widetilde{I}_i$ is flasque by lemma 5.1.14 and by definition quasi-coherent on U_i. Then $f_*(\widetilde{I}_i)$ is also flasque (exercise) and quasi-coherent (proposition 4.1.14, this uses the Noetherian assumption!). Then certainly the direct sum is flasque, completing the proof.

The final theorem shall show that having trivial cohomology > 0 characterizes affine schemes (with the necessary Noetherian condition to be consistent with what we've proven):

Theorem 5.1.17: Characterizing Affine Schemes using Cohomology

Let X be a Noetherian scheme. Then the following are equivalent:

1. X is affine
2. $H^i(X, \mathscr{F}) = 0$ for $i > 0$ and all for all quasi-coherent sheaves $\mathscr{F}$
3. $H^1(X, \mathscr{I}) = 0$ for $i > 0$ and all for all coherent sheaves of ideals $\mathscr{I}$

Proof **:**

(1) $\Rightarrow$ (2) Theorem 5.1.15

(2) $\Rightarrow$ (3) A-fortiori

(3) $\Rightarrow$ (1) We shall prove X is affine using corollary 3.1.9. First we show that X can be covered by open affine subsets of the form X_f, with $f \in R = \Gamma(X, \mathcal{O}_X)$. Let $\mathfrak{p}$ be a closed point of X, let U be an open affine neighborhood of $\mathfrak{p}$, and let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathscr{I}_{Y \cup \{\mathfrak{p}\}} \to \mathscr{I}_Y \to k(\mathfrak{p}) \to 0,$$

where $\mathscr{I}_Y$ and $\mathscr{I}_{Y \cup \{\mathfrak{p}\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{\mathfrak{p}\}$, respectively. The quotient is the skyscraper sheaf $k(\mathfrak{p}) = \mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ at $\mathfrak{p}$. Now from the exact sequence of cohomology, by (3) we get an exact sequence

$$\Gamma(X, \mathscr{I}_Y) \to \Gamma(X, k(\mathfrak{p})) \to H^1(X, \mathscr{I}_{Y \cup \{\mathfrak{p}\}}) = 0.$$

So there is an element $f \in \Gamma(X, \mathscr{I}_Y)$ which goes to 1 in $k(\mathfrak{p})$, i.e., $f_\mathfrak{p} \equiv 1 \pmod{\mathfrak{m}_\mathfrak{p}}$. Since $\mathscr{I}_Y \subseteq \mathcal{O}_X$, we can consider f as an element of R. Then by construction, $\mathfrak{p} \in X_f \subseteq U$. Furthermore, $X_f = U_{\bar{f}}$, where $\bar{f}$ is the image of f in $\Gamma(U, \mathcal{O}_U)$, so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f. By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \dots, f_r \in R$.

Let us verify that $(f_1, \dots, f_r) = R$. We use $f_1, \dots, f_r$ to define a map $\alpha : \mathcal{O}_X^r \to \mathcal{O}_X$ by sending $\langle a_1, \dots, a_r \rangle$ to $\sum f_i a_i$. Since the X_{f_i} cover X, this is a surjective map of sheaves. Let $\mathscr{F}$ be the kernel:

$$0 \to \mathscr{F} \to \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \to 0.$$

Filter $\mathscr{F}$ as follows:

$$\mathscr{F} = \mathscr{F} \cap \mathcal{O}_X^r \supseteq \mathscr{F} \cap \mathcal{O}_X^{r-1} \supseteq \cdots \supseteq \mathscr{F} \cap \mathcal{O}_X$$

for a suitable ordering of the factors of $\mathcal{O}_X^r$. Each of the quotients of this filtration is a coherent sheaf of ideals in $\mathcal{O}_X$. Thus by (3) and the long exact sequence of cohomology, we climb up the filtration and deduce that $H^1(X, \mathscr{F}) = 0$. But then $\Gamma(X, \mathcal{O}_X^r) \to \Gamma(X, \mathcal{O}_X)$ is surjective, which tells us that $f_1, \ldots, f_r$ generates R, as we sought to show.

5.2 Computation: Čech Cohomology

The Čech Cohomology computes to what degree a topological space X with a sheaf of abelian groups is covered. In the case where X is a Noetherian separated scheme with a quasi-coherent sheaf and the covering is an affine open covering, then the Čech cohomology groups shall be coincide with the cohomology groups defined by the Global section functor. Čech cohomology shall provide practical way to compute the cohomology. The following construction will be familiar to those who have seen the computation of group cohomology or have covered algebraic topology (see [5, chapter 4])

Let X be a topological space and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering. Give I a well-ordering[1]. Then for any finite set of indices $i_0, ..., i_p \in I$, let

$$U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$$

Let $\mathscr{F}$ be a sheaf of abelian groups on X. Define $C^*(\mathfrak{U}, \mathscr{F})$ to be:

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0, \ldots, i_p})$$

where each element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is given choosing an element $\alpha_{i_0, \ldots, i_p} \in \mathscr{F}(U_{i_0, \ldots, i_p})$. For each $(p+1)$-tuple $i_0 < \cdots < i_p$. Define the coboundary map $d : C^p \to C^{p+1}$ to be:

$$(d\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \widehat{i_k}, \ldots, i_{p+1}} |_{U_{i_0, \ldots, i_{p+1}}}$$

As usual, $d^2 = 0$, meaning we have a complex and so

Definition 5.2.1: Čech Cohomology

Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathscr{F}$ on X, define the *Čech cohomology group* of $\mathscr{F}$ with respect to $\mathfrak{U}$ to be:

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) = h^p(C^*(\mathfrak{U}, \mathscr{F})) = \frac{\ker\, d^{+1}}{\operatorname{im} d}$$

Note that $\check{\mathrm{H}}^p(\mathfrak{U}, -)$ is *not* in general a δ-functor, that is if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves of abelian groups on X, $\check{\mathrm{H}}^p(\mathfrak{U}, -)$ is not a δ-functor:

[1] Alternatively, make sure that if indices repeat then the element is 0, and rearranging the indices gives multiplies the function by$(-1)^\sigma$

Example 5.1: Čech cohomology not δ-functor

Say $\mathfrak{U}$ consists of only X. Then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > 0$ because all intersection are trivial. Furthermore, $\check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{F}) = \check{\mathrm{H}}^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$. Hence, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, then applying $\check{\mathrm{H}}^*(\mathfrak{U}, -)$ we get the "long exact sequence":

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

But $\Gamma(X, -)$ need not be right-exact, and hence this is *not* a long exact sequence!

This is one of the few Cohomologies theories which *does not* work with derived functors, even though they are central in most of homological algebra. Čech cohomology is very useful for computing.

Example 5.2: Čech Cohomology

1. Let $X = \mathbb{P}^1_k$, $\mathscr{F}$ be sheaf of differentials Ω, and $\mathfrak{U}$ the usual open covering of two $\mathbb{A}^1_k$ with coordinates x and $1/x$; label these sets U, V. Then we get two terms:

$$\begin{aligned} C^0 &= \Gamma(U, \Omega) \times \Gamma(V, \Omega) \\ C^1 &= \Gamma(U \cap V, \Omega) \end{aligned}$$

   Then:

$$\begin{aligned} \Gamma(U, \Omega) &= k[x]dx \\ \Gamma(V, \Omega) &= k[y]dy \\ \Gamma(U \cap V, \Omega) &= k\left[x, \frac{1}{x}\right] dx \end{aligned}$$

   The differential map $d : C^0 \to C^1$ is given by:

$$\begin{aligned} x &\mapsto x \\ y &\mapsto \frac{1}{z} \\ dy &\mapsto \frac{1}{x^2} dx \end{aligned}$$

   Hence, $\ker d$ is all elements $(f(x)dx, g(y)dy) \in C^0$ such that

$$f(x) = \frac{-1}{x^2} g\left(\frac{1}{x}\right)$$

   But this is true if and only if $f = g = 0$. Hence:

$$\check{\mathrm{H}}^0(\mathfrak{U}, \Omega) = 0$$

   Next, for $\check{\mathrm{H}}^1$, the image of d is all expression of the form:

$$\left(f(x) = \frac{1}{x^2} g\left(\frac{1}{x}\right)\right) dx$$

   This is a sub-space of $k[x, 1/x]dx$ generated by all $x^n dx$ with $-1 \neq n \in \mathbb{Z}$. Hence

$$\check{\mathrm{H}}^1(\mathfrak{U}, \Omega) = k$$

   which is generated by the image of $x^{-1}dx$.

2. Let us compute the Čech cohomology in a more classical setting: take S^1 and put the constant $\mathbb{Z}$ sheaf on it, $\underline{\mathbb{Z}}$. Let U, V be two open semi-circle where $U \cap V$ is the disjoint small open intervals.

Let us now build-up towards showing the cohomology given by the derived functor on Γ matches the Čech cohomology

Lemma 5.2.2: 0th Cohomology Matching

Let X be a topological space, $\mathfrak{U}$ an open covering, and $\mathscr{F}$ a sheaf of abelian groups on X. THen:

$$\check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$$

Proof **:**

By definition, $\check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{F}) = \ker(d : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}))$. If $\alpha \in C^0$ is given by $\{\alpha_i \in \mathcal{F}(U_i)\}$, then for each $i < j$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. So $d\alpha = 0$ says the sections α_i and α_j agree on $U_i \cap U_j$. But then by the sheaf axioms $\ker d = \Gamma(X, \mathcal{F})$.

We next work towards fixing the lack of a δ-functor. Let $X, \mathfrak{U}, \mathscr{F}$ be as usual, $V \subseteq X$ an open subset, and $\iota : V \to X$ the natural inclusion map. Define the complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ of sheaves of on X for each $p \geq 0$ as:

$$\mathscr{C}(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}})$$

and define the differential map $d : \mathscr{C}^p \to \mathscr{C}^{p+1}$ in the same way. Importantly, this was defined so that:

$$\Gamma(X, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})) = C^p(\mathfrak{U}, \mathscr{F})$$

Lemma 5.2.3: Čhech Cohomology Resolution

Let X be a topological space, $\mathfrak{U}$ an open covering, and $\mathscr{F}$ a sheaf of abelian groups. Then the complex $\mathscr{C} * (\mathfrak{U}, \mathscr{F})$ is a resolution of $\mathscr{F}$, namely we have a long exact sequence:

$$0 \to \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \to \cdots$$

for appropriate natural map $\epsilon : \mathscr{F} \to \mathscr{C}^0$

Proof **:**

Define $\epsilon : \mathscr{F} \to \mathfrak{U}^0$ by taking the product of the natural maps $\mathscr{F} \to f_*(\mathscr{F}|_{U_i})$ for $i \in I$. Then the exactness at the first step follows from the sheaf axioms for $\mathscr{F}$.

For the exactness of the complex $\mathscr{C}$ for $p \geq 1$, it is enough to check exactness on the stalks. Let $x \in X$, and suppose $x \in U_j$. For each $p \geq 1$, define the map $k : \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x \to \mathcal{C}^{p-1}(\mathfrak{U}, \mathscr{F})_x$ as follows: for $\alpha_x \in \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x$, it is represented by a section $\alpha \in \Gamma(V, \mathscr{C}^p(\mathfrak{U}, \mathscr{F}))$ over a neighborhood V of x, which we may choose so small that $V \subseteq U_j$. Now for any p-tuple $i_0 < \ldots < i_{p-1}$, set

$$(k\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}}$$

This makes sense because $V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{j,i_0,\ldots,i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the required map k. Now one checks that for any $p \geq 1$, $\alpha \in \mathscr{C}_x^p$,

$$(dk + kd)(\alpha) = \alpha$$

Thus k is a homotopy operator for the complex $\mathscr{C}_x$, showing that the identity map is homotopic to the zero map! It follows that the cohomology groups $h^p(\mathscr{C}_x)$ of this complex are 0 for $p \geq 1$.

Lemma 5.2.4: Čhech and Flasque Sheaves

Let X be a topological space, $\mathfrak{U}$ be an open covering, and let $\mathscr{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) = 0$.

Proof **:**
Consider the resolution $0 \to \mathscr{F} \to \mathcal{C}^{\cdot}(\mathfrak{U}, \mathscr{F})$ given by lemma 5.2.3. As $\mathscr{F}$ is flasque, the sheaves $\mathcal{C}^p(\mathfrak{U}, \mathscr{F})$ are flasque for each $p \geq 0$. Indeed, for any $i_0, \ldots, i_p$, $\mathscr{F}|_{U_{i_0,\ldots,i_p}}$ is a flasque sheaf on $U_{i_0,\ldots,i_p}$, f_* preserves flasque sheaves, and a product of flasque sheaves is flasque. Thus, we can use this resolution to compute the usual cohomology groups of $\mathscr{F}$. But $\mathscr{F}$ is flasque, so $H^p(X, \mathscr{F}) = 0$ for $p > 0$. On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X, \mathcal{C}^{\cdot}(\mathfrak{U}, \mathscr{F}))) = \check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F})$$

But then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > 0$, as we sought to show.

Lemma 5.2.5: Chech Cohomology Functorial Map

Let X be a topological space and $\mathfrak{U}$ an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathscr{F}$,

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

Proof **:**
This follows from comparing two acyclic resolution (see [3, chapter 27.4]), namely:

$$0 \to \mathscr{F} \to \mathscr{I}^*$$
$$0 \to \mathscr{F} \to \mathscr{C}^*(\mathfrak{U}, \mathscr{F})$$

which gives a morphism of complexes $\mathscr{C}^*(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^*$ which induces the identity map on $\mathscr{F}$ and is unique up to homotopy. By applying the functors $\Gamma(X, -)$ and h^p, we extract the desired maps.

Theorem 5.2.6: Čech Cohomology Equivalence

Let X be a Noetherian separated scheme, $\mathfrak{U}$ an open affine cover of X, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, the natural maps

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

are isomorphisms.

***Proof* :**
For $p = 0$ we have an isomorphism by lemma 5.2.2. For $p > 0$, by corollary 5.1.16 embed $\mathscr{F}$ in a flasque, quasi-coherent sheaf $\mathcal{G}$, and let $\mathcal{Q}$ be the quotient:

$$0 \to \mathscr{F} \to \mathcal{G} \to \mathcal{Q} \to 0$$

For each $i_0 < \ldots < i_p$, the open set $U_{i_0,\ldots,i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme. Since $\mathscr{F}$ is quasi-coherent, we therefore have an exact sequence

$$0 \to \mathscr{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{Q}(U_{i_0,\ldots,i_p}) \to 0$$

of abelian groups by proposition 4.1.13. Taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^{\cdot}(\mathfrak{U}, \mathscr{F}) \to C^{\cdot}(\mathfrak{U}, \mathcal{G}) \to C^{\cdot}(\mathfrak{U}, \mathcal{Q}) \to 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. Since $\mathcal{G}$ is flasque, its Čech cohomology vanishes for $p > 0$, so we have an exact sequence

$$0 \to \check{\mathrm{H}}^0(\mathfrak{U}, \mathscr{F}) \to \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{G}) \to \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{Q}) \to \check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{F}) \to 0$$

and isomorphisms

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{Q}) \to \check{\mathrm{H}}^{p+1}(\mathfrak{U}, \mathscr{F})$$

for each $p \geq 1$. Now comparing with the long exact sequence of usual cohomology for the above short exact sequence, using the case $p = 0$, the fact it is flasque , we conclude that the natural map

$$\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism. But $\mathcal{Q}$ is also quasi-coherent (proposition 4.1.14), so we obtain the result for all p by induction, completing the proof.

5.2.1 Application: Cohomology of Projective Spaces

Let us now compute the cohomology for the twisted sheaves on projective space. Let R be a Noetherian ring, $S_\bullet = R[x_0, ..., x_n]$, and $X = \operatorname{Proj} S = \mathbb{P}^n_R$. Let $\mathcal{O}_X(1)$ be the twisted sheaf of serre. If $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-module, let $\Gamma_*(\mathscr{F})$ be the graded S-module

Theorem 5.2.7: Twisted Sheaf Cohomology

Let R be a Noetherian ring, and let $X = \mathbb{P}^n_R$, with $n \geq 1$. Then:

1. $H^0(X, \mathcal{O}_X(-d)) \cong 0$ for $d \in \mathbb{N}$
2. $H^n(X, \mathcal{O}_X(-n-1)) \cong R$.
3. The natural map
$$H^0(X, \mathcal{O}_X(n)) \times H^n(X, \mathcal{O}_X(-d-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \cong R$$
is a perfect pairing (isomorphic bilinear map) of finitely generated free R-modules, for each $d \in \mathbb{Z}$.
4. $H^i(X, \mathcal{O}_X(d)) = 0$ for $0 < i < n$ and all $d \in \mathbb{Z}$

Overall:
$$H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) = \begin{cases} (R[x_0, ..., x_n])_d & i = 0,\ d \geq 0 \\ \left(\frac{1}{x_0 x_1 \cdots x_n} R[1/x_0, ..., 1/x_n]\right)_d & i = n,\ d < 0 \\ 0 & \text{otherwise} \end{cases}$$

***Proof* :**

Let us first setup the Čech cohomology. Let $\mathscr{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d)$. As cohomology commutes with direct sums on Noetherian space, the cohomology on $\mathscr{F}$ will be the direct sum of the cohomology on $\mathcal{O}_X(d)$ for $d \in \mathbb{Z}$. Hence, we can compute $\mathscr{F}$ to find the cohomologies.

Let $D(x_i)$ cover $\operatorname{Proj} S$, $0 \leq i \leq n$: label this open cover $\mathcal{U}$. Then for any direct subset of indices, we have:
$$\mathscr{F}(U_{i_0,\ldots,i_p}) = S_{x_{i_0},\ldots,x_{i_p}})$$
Thus, the grading of $\mathscr{F}$ is given by the grading of S, and we get:
$$C^*(\mathcal{U}, \mathscr{F}) : \prod S_{x_{i_0}} \to \prod S_{x_{i_0}, x_{i_1}} \to \cdots \to S_{x_{i_0},\ldots,x_{i_n}}$$

1. $H^0(X, \mathscr{F})$ is the kernel of d^0 (the first map), which is just S by proposition 4.2.7.
2. $H^0(X, \mathcal{O}_X(-d))$ must be 0 as we have the isomorphism above.
3. $H^n(X, \mathscr{F})$ is the cokernel of of d^{n-1} (the last map). To see what this is, notice that $S_{x_{i_0},\ldots,x_{i_n}}$ is a free R-module generated by monomial $\left\{x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n} \;\middle|\; l_i \in \mathbb{Z}\right\}$. The image of d^{n-1} is generated by the sub-basis where at least one $l_i \geq 0$. Hence, $H^n(\mathscr{F}, X)$ is a free R-module generated by the negative monomials:
$$\left\{x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n} \;\middle|\; l_i \in -\mathbb{N}\right\}$$
Note that this is graded by $\sum_i l_i$. Now there is only one monomial of degree $-n-1$, namely:
$$x_0^{-1} \cdots x_n^{-1}$$

hence:

$$H^n(X, \mathcal{O}_X(-n-1)) \cong R$$

4. Note that $H^n(X, \mathcal{O}_X(-d-n-1)) \cong 0$ since there are no negative monomials of degree bigger than 0 for $m < -n-1$. For $d \geq 0$, $H^0(X, \mathcal{O}_X(d))$ has a basis consisting of the usual monomials of degree d, i.e., $\{x_0^{m_0} \cdots x_n^{m_n} | m_i \geq 0 \text{ and } \sum m_i = d\}$. The natural pairing with $H^n(X, \mathcal{O}_X(-n-d-1))$ into $H^n(X, \mathcal{O}_X(-n-1))$ is determined by

$$(x_0^{m_0} \cdots x_n^{m_n}) \cdot (x_0^{l_0} \cdots x_n^{l_n}) = x_0^{m_0+l_0} \cdots x_n^{m_n+l_n},$$

where $\sum l_i = -d-n-1$, and the object on the right becomes 0 if any $m_i + l_i \geq 0$. So it is clear that there is a perfect pairing under which $x_0^{-m_0-1} \cdots x_n^{-m_n-1}$ is the dual basis element of $x_0^{m_0} \cdots x_n^{m_n}$.

5. Let us do induction on n. If $n = 1$ there is nothing to prove, so let $n > 1$. If we localize the complex $C^*(\mathfrak{U}, \mathscr{F})$ with respect to x_n, as graded S-modules, we get the Čech complex for the sheaf $\mathscr{F}|_{U_n}$ on the space U_n, with respect to the open affine covering $\{U_i \cap U_n | i = 0, \ldots, n\}$. By (4.5), this complex gives the cohomology of $\mathscr{F}|_{U_n}$ on U_n, which is 0 for $i > 0$ by (3.5). Since localization is an exact functor, we conclude that $H^i(X, \mathscr{F})_{x_n} = 0$ for $i > 0$. In other words, every element of $H^i(X, \mathscr{F})$, for $i > 0$, is annihilated by some power of x_n.

   Next let's show that for $0 < i < n$, multiplication by x_n induces a bijective map of $H^i(X, \mathscr{F})$ into itself. Then it will follow that this module is 0. Consider the exact sequence of graded S-modules

   $$0 \to S(-1) \xrightarrow{x_n} S \to S/(x_n) \to 0.$$

   This gives the exact sequence of sheaves

   $$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

   on X, where H is the hyperplane $x_n = 0$. Twisting by all $d \in \mathbb{Z}$ and taking the direct sum, we have

   $$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0,$$

   where $\mathscr{F}_H = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_H(d)$. Taking cohomology, we get a long exact sequence

   $$\cdots \to H^i(X, \mathscr{F}(-1)) \to H^i(X, \mathscr{F}) \to H^i(X, \mathscr{F}_H) \to \cdots.$$

   Considered as graded S-modules, $H^i(X, \mathscr{F}(-1))$ is just $H^i(X, \mathscr{F})$ shifted one place, and the map $H^i(X, \mathscr{F}(-1)) \to H^i(X, \mathscr{F})$ of the exact sequence is multiplication by x_n.

   Now H is isomorphic to $\mathbb{P}_R^{n-1}$, and $H^i(X, \mathscr{F}_H) = H^i(H, \bigoplus \mathcal{O}_H(d))$ by lemma 5.1.10. So we can apply our induction hypothesis to $\mathscr{F}_H$, and find that $H^i(X, \mathscr{F}_H) = 0$ for $0 < i < n-1$. Furthermore, for $i = 0$ we have an exact sequence

   $$0 \to H^0(X, \mathscr{F}(-1)) \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}_H) \to 0$$

   by (1), since $H^0(X, \mathscr{F}_H)$ is just $S/(x_n)$. At the other end of the exact sequence we have

   $$0 \to H^{n-1}(X, \mathscr{F}_H) \xrightarrow{\partial} H^n(X, \mathscr{F}(-1)) \xrightarrow{x_n} H^n(X, \mathscr{F}) \to 0.$$

   Indeed, we have described $H^n(X, \mathscr{F})$ above as the free R-module with basis formed by the negative monomials in $x_0, \ldots, x_n$. So it is clear that x_n is surjective. On the other hand, the kernel of x_n is the free submodule generated by those negative monomials $x_0^{l_0} \cdots x_n^{l_n}$ with $l_n = -1$. Since $H^{n-1}(X, \mathscr{F}_H)$ is the free R-module with basis consisting of the negative monomials in $x_0, \ldots, x_{n-1}$, and ∂ is division by x_n, the sequence is exact. In particular, ∂ is injective.

Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $x_n : H^i(X, \mathscr{F}(-1)) \to H^i(X, \mathscr{F})$ is bijective for $0 < i < n$, as we sought to show.

6

Application: Curves and Surfaces

Placeholder my notes from Milne, Hartshorne, and Vakil. (first want to write out more fully the previous chapters)

6.1 Curves

Here

6.1.1 Elliptic Curves

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6.2 Surfaces

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Bibliography

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