Everything You Need To Know About Complex Analysis

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Contents

Summary

On a high-level, Complex analysis is the beginning of the study of Riemann surfaces, which are 1-dimensional complex manifolds. The first four chapters focus on studying the properties of the base-space, namely $\mathbb{C}$ and [simply-connected] open subsets of $\mathbb{C}$.

This book will start by covering the properties of complex numbers, including their arithmetics, algebraic, geometric, and analytic properties. It then covers important examples complex function, including fractional linear transformations and complete functions defined on the Riemann sphere (on the 1-point compactification of $\mathbb{C}$). After having defined the elementary properties of $\mathbb{C}$ and complex functions, we move on to introducing the main concept of the book: complex differentiabile function. We will explore the consequence of a function being complex-differentiable including harmonic and conformal properties, and give a list of important examples of complex-differentiable functions. After proving all power series are complex-differentiable, we explore the cannonical power series: $e^x, \cos(x), \sin(x), \log(x)$ and their related functions. We finish off by generalizing power series to functions that are locally power series, analytic functions, and gain some very important properties that make analytic functions much more powerful than most smooth functions.

Having built-up the notion of complex-differentiability, we shall explore integrating complex functions, more particularily 1-forms $f(z)dz$ where f is complex-differentiable. This will bring us many fruitful results in algebra, geometry, and PdEs, and will give us reulsts suchas Cauchy's Theorem, Cauchy's Integral Formula, Maximum Modulus Principle, and Residue Theory.

(After this, it will be Complex II material, so I will add it next semester)

This textbook will assume some basic familiarity with complex numbers and construction of complex number. Might do a history section at some point (the number i was added not to solve $x^2 + 1 = 0$, but polynomials of degree 3 that so happened to need an "imaginary" number to get to the real value).

(A quick word on the real origins of complex numbers)

(A word of warning by Tao: many theorems that we'll cover fail spectacularly for non-holomorphic functions. This is not so much the case for real-differentiable and continuous functions due to

distribution theory)

1

Complex Numbers and Functions

In this chapter, we go over how to upgrade the real numbers into the complex numbers. The complex numbers are a wonderful mathematical object to work with: they form the smallest complete algebraically closed characteristic 0 field – about as good with properties in analysis (completeness), algebra (algebraic closure, commutative), and geometry (i can be interpreted as rotation, inner product space giving us angles, norms, and metric), and with a very simple construction from $\mathbb{R}$ (it's extension is only of [real] dimension 2!). These many nice properties provide us a variety of flexibility in the manipulation of complex numbers: we can represent them in polar or Euler form, we can find many useful identities, and we will gain an important tool on dealing with infinities.

What makes complex numbers interesting is the power of that complex functions bring. In complex analysis, a [single-valued] function with domain and codomain $\mathbb{C}$. In this way, the input will be thought of having the properties of $\mathbb{C}$, while the properties of the funciton shall also be interpreted through $\mathbb{C}$ (since $\mathbb{C}$ shall be it's codomain). It shall become very apparant in the rest of these chapters that complex-differentiable functions will have incredible properties that shall seem very foreign to anyone with intuition from real analysis. In this chapter, we shall start by getting comfortable with interpreting complex linear functions and complex polynomials. We shall then see that it is rather practical to consider continuous functions f which may take ∞ as an input and output ∞. We shall do this by compactifying $\mathbb{C}$ and analyzing the result. This shall lead us to a “new” linear function, the fractional linear function.

1.1 Arithmetic's and Conjugate

The rational numbers are the smallest set of numbers in which the 4 arithmetic operations $(+, -, \times, \div)$ is well-defined. These make them the ideal starting point in the study of arithmetic's, however, the rationals lack two keys properties:

1. polynomials are central to algebra, however not all solutions to polynomials have solutions over $\mathbb{C}$. We solve this by constructing a larger set of numbers called the *algebraic closure* of $\mathbb{Q}$ and usually denote it $\overline{Q}$

2. Some sequences of rational numbers do not have a limit even when they seem to approach a limit (i.e. a Cauchy sequence). We rectify this by taking the *completion*, and usually denote it $\mathbb{R}$

These are two important properties, the first in the study of algebra and the second in the study of analysis. The complex numbers are the combination

The real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$ with the euclidean metric, making it a very natural field to work with in analysis. However, $\mathbb{R}$ is not algebraically closed, meaning there are polynomials in $\mathbb{R}[x]$ that do not have solutions. The complex numbers recitify this by being the *algebraic closure* of $\mathbb{R}$. Due to this, I will take an algebraic approach to defining the complex numbers. To an algebraist, the complex numbers are often defined as the ring $\mathbb{R}[x]$ quotient by the maximal ideal $(x^2 + 1)$

$$\mathbb{C} := \mathbb{R}[x]/(x^2 + 1)$$

with $\overline{x}$ being identified with $\sqrt{-1}$. We may also identify $\overline{x}$ with $-\sqrt{-1}$, this field is sometimes labled $\overline{\mathbb{C}}$ (the "conjugate" complex field), and we shall temporarily use this notation in this section (the notation shall be used for a more common structure later). The field $\mathbb{C}$ is a degree 2 extension, and so all the element of $\mathbb{C}$ can be written of the from $a + b\sqrt{-1}$ for the root $\sqrt{-1}$ of $x^2 + 1$. By convention (established by Euler), we will write $i := \sqrt{-1}$, and we will also call the elements of $\mathbb{C}$ *numbers* (or *complex numbers* to be more precise when the contexts necessitates it). Addition and multiplication is defined as a natural consequence of the extension, that is:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where 1 and i are the basis elements, and hence all elements of $\mathbb{C}$ can be written as a linear combination of $a+bi$ for $a, b \in \mathbb{R}$. We usually denote the real part of an element $z \in \mathbb{C}$ by $\mathrm{Re}(z)$ and the imaginary part by $\mathrm{Im}(z)$. Since $\mathbb{C}$ is a field, division is defined. As a mneumonic for the computation, if we want to find:

$$\frac{1}{a + bi}$$

then we then multiply this by a fancy 1;

$$\frac{1}{a + bi}\frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \left(\frac{a}{a^2 + b^2}\right) - \left(\frac{b}{a^2 + b^2}\right) i$$

It is not difficult, to show that $\mathbb{C}$ is also complete, and so we have that $\mathbb{C}$ is a complete algebraically closed field. In theorem 3.2.6, we will show $\mathbb{C}$ being algebraically closed means any polynomial $p(x) \in \mathbb{C}[x]$ has a root. We are taking this property for granted for now[1].

As mentioned earlier, $\mathbb{R}[x]/(x^2 + 1)$ has two possible isomorphisms since $(\overline{x}^2 + 1) = (\overline{x} - 1)(\overline{x} + 1)$. We usually pick $\overline{x} = \sqrt{-1} =: i$ and label the resulting field $\mathbb{C}$, however we could have chosen $-i$. We know from algebra that there is a field isomorphism between $\mathbb{C}$ and $\overline{\mathbb{C}}$, showing the two are indistinguishable as fields. In particular, the isomrophism is:

$$a + bi \mapsto a - bi$$

[1]This can also be proven using only the Intermediate value theorem if we used some group and field theory; see my EYTNKA undergraduate algebra for an example; or by using the homotopy group; see EYNTKA Algebraic Topology

Since the base field is $\mathbb{R}$, $\mathbb{C}$ and $\overline{\mathbb{C}}$ are even $\mathbb{R}$-algebra isomorphic. This function is usually denoted $\overline{z}$. Since it's an $\mathbb{R}$-algebra isomorphism:

$$\overline{zw} = \overline{z}\overline{w} \qquad \overline{z+w} = \overline{z} + \overline{w} \qquad \overline{rz} = r\overline{z}$$

Importantly, $\overline{z}$ is *not* $\mathbb{C}$-linear, in particular for any $c \in \mathbb{C}$, $\overline{c \cdot z} = \overline{c} \cdot \overline{z}$. This is perhaps obvious since this is another formulation of the field isomorphism property, however it is in fact very important to remember this in the context of complex differentiation: complex-differentiable functions will be locally $\mathbb{C}$-linear, and hence conjugation as a function $\mathbb{C} \to \mathbb{C}$ shall not be complex-differentaible. The function $\overline{z}$ is called the *conjugate* of z. Notice that this is *not* vertical equivalent to the function $z \mapsto -z$, since $-z$ doesn't reflect, but *rotate* $\mathbb{C}$ by π degree (think $1 + i \mapsto -1 - i$). conjugation has many important algebraic properties, for example, conjugation let's us define many useful functions, such as[2]:

$$z\overline{z} = a^2 + b^2 \in \mathbb{R} \qquad z + \overline{z} = 2a \in \mathbb{R}$$

that is, the maps $z \mapsto z\overline{z}$ and $z \mapsto z + \overline{z}$, which both maps $\mathbb{C} \to \mathbb{R}$. On the other hand,

$$z - \overline{z} = 2bi$$

letting us also isolate the complex part. Since $\overline{\overline{z}} = z$, we get that $\overline{(-)}$ is it's own inverse, which by definition makes it a type of function called an *involution*, which show up a lot in mathematics and usually serve a useful function (ex. if $x^2 = e$ in a group, then the group can be split into a product of groups, or if $A^2 = I$ for a matrix, the matrix is diagonalizable, or if $f^2 = 1$ for some function, then f is bijecive, or $\neg\neg A = A$ for some logical statement A). For complex numbers, complex conjugation will give us a useful arithmetic tool that will simplify many identities (some examples of which will be listed bellow). If we have any polynomial $p(x)$ over $\mathbb{C}$, or even any rational function[3] $p(x)/q(x)$, conjugation distributes over it:

$$\overline{\left(\frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}\right)} = \frac{\overline{a_0} + \overline{a_1x} + \overline{a_2x^2} + \cdots + \overline{a_nx^n}}{\overline{b_0} + \overline{b_1x} + \overline{b_2x^2} + \cdots + \overline{b_mx^m}}$$

Three important uses of conjugation are:

1. we can represent the real or imaginary part of a complex number through the following equations:
$$\operatorname{Re} a = \frac{a + \overline{a}}{2} \qquad \operatorname{Im} a = \frac{a - \overline{a}}{2i}$$

2. Another example that is useful is the ability to represent the inverse of a complex number:
$$z^{-1} = \frac{\overline{z}}{z\overline{z}} \tag{1.1}$$
Naturally, in equation (1.1), the $\overline{z}$ cancel out, but computationally it is easier to understand it in this form since we often treat $\mathbb{C}$ as a vector-space over $\mathbb{R}$ and we want all its elements to bewritten in the form $a + bi$.

3. If r is a root of the $p(x)$ ($p(r) = 0$), then $\overline{r}$ is a root of $\overline{p(x)}$ (prove this). If $p(x) \in \mathbb{R}[x]$, then r and $\overline{r}$ are rotos of $p(x)$.

[2]Notice that these are the *trace* and *determinant* of the appropriate complex matrix
[3]the numerator and denominator are polynomials

1.2 Geometry and Analysis

In the following sections, we will explore the geometric properties of complex numbers.

1.2.1 Norms and Inequalities

When extending from the real to the complex numbers, they loose their total ordering (there is no "natural" ordering on them in which any two points have a finite distance and if $a < b$ and $b < a$ then $a = b$). However, $\mathbb{C}$ does a notion of size to every vector given by

$$N(z) = z\overline{z}$$

notice that $N(z) = \overline{N(z)}$ and so its output is real. This is called the *norm form* and is defined for any field extension. It has the two properties of:

$$N(zw) = N(z)N(w) \qquad N(z) = 0 \Leftrightarrow z = 0$$

This particular norm has an important property: $N(z) > 0$ if $z \neq 0$, that is it is *positive definite*[4] Since it is positive definite and has nice linearity properties, we can define an inner product via:

$$\langle v, w\rangle = \mathrm{Re}(z\overline{w})$$

This makes $\mathbb{C}$ is an inner vector space over $\mathbb{R}$ [5]. We would write the induced norm as:

$$|z| := \langle z, z\rangle = N(z)^{1/2}$$

where the $|\cdot|$ notation is reminiscent of the absolute value notation from $\mathbb{R}$ (notice it in fact it continuously extends $|\cdot|$). One can verify that the axioms of a norm are satisfied by $\|\cdot\|$. Due to historical precedents, the norm is called the *modulus*:

Definition 1.2.1: Modulus (Norm)

Let $z \in \mathbb{C}$. Then define the function:

$$\|z\| = |z| = \sqrt{a^2 + b^2}$$

which is called the *modulus* of z and can be shown to satisfy the following properties:

1. **Subadditivity or Triangle Inequality**: $\|x + y\| \leq \|x\| + \|y\|$
2. **Absolute Homogeneity**: $\|sx\| = |s|\,\|x\|$
3. **Positive definiteness**: If $\|x\| = 0$ then $x = 0$. Note that in tandem with (2) this means that $\|x\| = 0$ if and only if $x = 0$

making it a norm

Since we'll be treating $\mathbb{C}$ as "numbers" in it of itself, then we will often abuse notation and write $|z|$ for the norm of z, and call it the *absolute value* or *modulus* of z. This norm naturally defines the

[4]this is not the case for all norms, consider $\mathbb{Q}(\sqrt{2})$ where $N(a + \sqrt{2}) = a^2 - 2b^2$
[5]in particular, a complete normed vector space, making it a Banach space

metric $|z-w|$, and so the same topology as $\mathbb{R}^2$, and so any continuous function on $\mathbb{R}^2$ is continuous on $\mathbb{C}$. We get some important results about convergent sequences and series from real analysis that carry over to complex analysis that will simplify proofs are:

1. $z_n \to z$ if and only if $\sum_k^n |z_k| < \infty$ (i.e. (z_n) is convergent if and only if the series of $(|z_n|)$ is absolutely convergent)
2. $\sum_k z_k$ is convergent if and only if $z_k \to 0$
3. Notice that this also makes the norm *continuous*. Using the limit definition, this means that:
$$|\lim_{n\to\infty} z_n| = \lim_{n\to\infty} |z_n|$$
4. $|z| = |\overline{z}|$ and
$$\max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$$
Use these inequalities to show that $z_n \to z$ if and only if the real and imaginary part converge and that Re and Im passes through the limit (i.e. they are continuous)

By consequence of the definition we have:

$$|z|^2 = z\overline{z}$$

which ties the analytic property of the norm to the algebraic property of conjugation. This can make some calculations much simpler. For example, using the definition of an absolute value directly, we get:

$$\begin{aligned}
|xy|^2 &= |(ac-bd)+(ad+cd)i|^2 \\
&= (ac-bd)^2 + (ad+cd)^2 \\
&= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2(adbc) + (cd)^2 \\
&= (ac)^2 + (bd)^2 + (ad)^2 + (cd)^2 \\
&= (a^2+b^2)(c^2+d^2) \\
&= |x|^2|y|^2
\end{aligned}$$

This extends to finite products and quotients. If we use the fact that $|z| = z\overline{z}$, we get

$$|ab|^2 = ab\overline{ab} = ab\overline{a}\overline{b} = a\overline{a}b\overline{b} = |a|^2|b|^2$$

Using the conjugation definition also allows us to find many more identities in relatoin to the norm, for example:

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + (z\overline{w} + \overline{z}w) + w\overline{w}$$

which simplifies to

$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(a\overline{b})$$

and similarly, if we take the difference of the two numbers, we get:

$$|z-w|^2 = |z|^2 + |w|^2 - 2\operatorname{Re}(a\overline{b})$$

Combining these two identities, we get the following interesting result:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

which is the *parallelogram law*, showing that this norm induces an *inner product*:

$$\langle z, w \rangle = \frac{\|z+y\|^2 - \|x-y\|^2}{4}$$

Another inequality we get using the absolute the following:

$$-|a| \leq \operatorname{Re} a \leq |a|$$

where equality holds if $b = 0$ and $a \geq 0$. Using this fact, we can combine it with equation $|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(a\bar{b})$ to get

$$|z+w|^2 \leq |z|^2 + |b|^2$$

where equality will only hold if $a\bar{b} \geq 0$. If $b \neq 0$, we can re-write it as $|b|^2(a/b) \geq 0$, i.e. $a/b \geq 0$. Repeating this for an arbitrary finite length triangle inequality:

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

we see that $a_i/a_j \geq 0$ for all i, j. In this way, we see how very similar complex numbers are to real numbers (this time, more than vector's in $\mathbb{R}^2$) since this is the property of real numbers!

(there are a couple more inequalities, but I will for now conclude with this one)

$$|a| = |(a-b) + b| \leq |a-b| + |b|$$

moving $|b|$ to the other side, we get

$$|a| - |b| \leq |a-b|$$

Similarly if we do the same for $|b|$: $|b| - |a| \leq |a-b|$. This meaning that either the positive or negative value will both be less than $|a-b|$ and so

$$||a| - |b|| \leq |a-b|$$

Finally, we state the famous Cauchy inequality:

$$\left|\sum a_i b_i\right|^2 \leq \sum |a_i|^2 \sum |b_i|^2$$

We can show that $+, \cdot$, and $-$ are all continuous functions, and so our usual limit laws hold. Since conjugation is continuous, we also have:

$$\lim_{n\to\infty} \overline{z_n} = \overline{\lim_{n\to\infty} z_n}$$

Exercise 1.2.1

1. Show that $\mathbb{C}$ has no total ordering, meaning that no total ordering $\leq$ exists such that if $x \leq y$, then for all $z \in \mathbb{C}$, $zx \leq zy$.

2. Show that on S^1, $\frac{1}{z} = \overline{z}$.
3. show that $\mathbb{C}^\times \cong (0, \infty) \times S^1$ as topological groups.
4. Let $\omega \in S^1$. show that $z \mapsto \omega z$ is an isometry of $\mathbb{C}$, that is

$$|\omega z - \omega w| = |z - w|$$

5. Let $\left|\frac{z-1}{z}\right| = 1$. Show that $\operatorname{Re} z = \frac{1}{2}$.

1.2.2 Change of Variables: Polar and Euler Form

Visually, we can picture a complex number as an arrow (or vector) in the complex plane. Note that $\mathbb{C}$ over $\mathbb{R}$ can be visualized as a plane. Usually, the x-axis will be called the real axis, and the y axis will be called the imaginary axis:

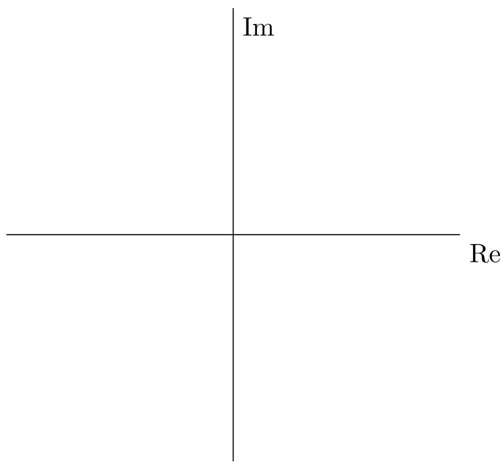


Figure 1.1: Complex Plane

When adding two vectors, it is like moving one vector to the tip of the others, the tip of the combination being the new complex number (equivalently, this comes from the definition of addition in $\mathbb{R}[x]/(x^2+1)$). This can also be thought as a line through the following parallelogram:

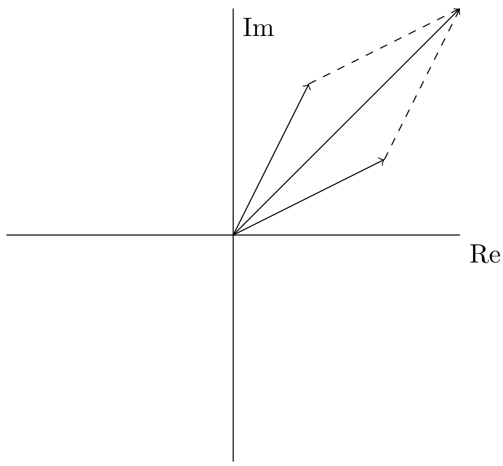


Figure 1.2: Adding Complex Numbers

When it comes to multilication, we require a little more intuition of what i does in the complex plane. In particular, Notice that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, so i returns to itelf in after multiplying by itself 4 times. Thus, multiplying by i can be thought of as rotating $90\,\text{deg}$. or $\pi/2$ times. This intuition can be captured more rigorously: take $\mathbb{C}^\times$ to be set of nonzero complex numbers. Then notice we can represent any element in here by two numbers: $|z|$ and $z/|z|$. The first represents the magnitude, while the second captures the angle. In particular, $|z/|z|| = |z|/|z| = 1$, and so they all lie on the unit circle S^1. As long as $z \neq 0$, $|z| \neq 0$. We thus can imagine there being a group isomorphism:

$$\mathbb{C}^\times \cong_{\mathbf{Grp}} (0, \infty) \times S^1$$

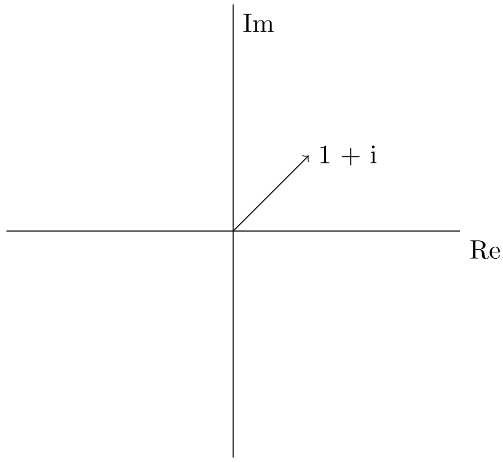
and indeed there is. First, for any $z/|z|$, using pythagoras and some highschool trigonometry we can represent this as $\cos(\theta) + i\sin(\theta)$ for appropriate θ. Then we get $x + iy \mapsto (|x + iy|, \cos(\theta) + i\sin(\theta))$. The key is that multiplication preserves angles since:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$$

Thus, we can represent complex numbers in terms of rotation! For example:

1. $$i = \cos(\pi/2) + i\sin(\pi/2)$$
$$1 = \cos(0) = i\sin(0)$$

2. What about representing $1 + i$? Notice that in the complex plane, this would form a triange:



by Pythagoras, the hypotenuse of the triangle will have length $\sqrt{2}$, which the norm $1 + i$! Furthermore, since we know the adjacent and the opposite length, we can find the angle by $\tan(b/a)$. Therefore, we get

$$1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)))$$

This can be properly formalized as the *change of variables* from cartesian coordiante functions π_x, π_y to the polar coordiante functions, which for any $z \in \mathbb{C}$:

$$z = |z|(\pm\cos(\theta) \pm i\sin(\theta))$$

Sometimes, the notation is collapsed to $z = |z| \operatorname{cis}(\theta)$ where cis the $\cos + i\sin$. The positive or negative value being added depends in what quadrant the complex number resides in. This also means that numbers in polar representation are defined up to $2k\pi$ rotations, since $2k\pi + \theta = \theta$ in terms of rotation. The value of $|z|$ is called the *magnitude* and the angle θ is called the *argument* (measured by starting on the positive real axis and moving anti-clockwise). The magnitude is usually represented with the variable r, so $a + bi = r(\cos(\theta) + i\sin(\theta))$. we usually write the argument of a complex number as $\arg(z)$. Thus, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. By doing some more manipulation, it is not hard to show that

$$z_1/z_2 = (r_1/r_2)(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

In other words, $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$. This should be ringing a bell: log has these properties. We will return to this in section 2.4.3. In combination with trigonometric identities, these formulas are quite easy to manipulate, for example:

1. $z^{-1} = \frac{1}{(\cos(\theta)+i\sin(\theta))} = r^{-1}(\cos(\theta) - i\sin(\theta)) = r^{-1}(\cos(-\theta) + i\sin(-\theta))$
2. $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ (known as *Moivre's Formula*)
3. $\sqrt[n]{z} = \sqrt[n]{r}(\cos(\theta/n) + i\sin(\theta/n))$. There are in fact multiple solutions: we need that $n\psi = \theta$. Naturally $\psi = \theta/n$ is a solution, but so is $\psi = \theta/n + 2k\pi/$ for $0 \leq k \leq n-1$ since

$$n\psi = \theta + 2k\pi = \theta$$

Unfortunately, adding and subtracting does not have a nice representation in polar form, however we can comfortably add and subtract vector in Cartesian form, so this is of no real loss to us. This gives an easy geometric interpretation of adding, subtracting, multiplying, and dividing complex numbers.

Another coordinate representation of complex numbers is *Euler coordinates*. The functions cos and sin are defined through an infinite power series that converges everywhere:

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \tag{1.2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{1.3}$$

Given appropriate manipulation of these power series, we can make them resembles the power series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{1.4}$$

which after the appropriate manipulation, we get:

$$|r|e^{\theta i} = |r|(\cos(\theta) + i\sin(\theta))$$

Thus, we the Euler representation of a complex number as well. We have similar niceties manipulating complex numbers as Euler representations:

1. $zw = |r||s|e^{(\theta+\psi)i}$
2. $z^{-1} = |r|^{-1}e^{-\theta i}$

and so on. Sometimes, they allow for simpler expressions for solutions of polynomials. For example, take the equation $x^n - 1 = 0$ or equivalently $x^n = 1$. By the fundamental theorem of Algebra, $x^n - 1$ has n roots. Since $x^n = 1$, these roots are called the *roots of unity*, and are studied extensively. Given Euler's representation of complex numbers, we know that $1 = e^0$, and so the roots of unity of $x^n - 1$ can be represented as

$$z_k = e^{\frac{2k\pi}{n}} \qquad k = 1, 2, ..., n$$

Notice that $(z_k)^n = 1$ for each k since $2k\pi = 0$. For example, here is the 5th root of unity.

1.2.3 Monodromy

For a proper change of variables from Cartesian to polar, some care must me made so that $\arg(z)$ is in fact continuous and bijective. We have so far defined it as a function $\arg(z) : \mathbb{C}^* \to \mathbb{R}$ that gives back the angle of z. Note that $\arg(f(0) = 0$. To see function is not continuous take the parameterization $f : [0, 2\pi] \to S^1 \subseteq \mathbb{C}^*$ taking $f(\theta) = \cos(\theta) + i\sin(\theta)$. In particular, $\arg(f(\theta)) = \theta$. If $\arg(z)$ were continuous, $\arg(f(\theta))$ would be continuous. Now, notice as $\theta \to 2\pi$, the output of $f(t)$ approaches 1, which we already know has argument 0, however the angle is still *increasing*. If the function were continuous, the limit would be preserved and would be 2π, however we know that $\arg(f(2\pi)) = \arg(1) = 0$; we must suddenly drop.

More generally as θ ranges over every real number it is not continuous (namely, it jumps when crossing over in the interval $(2\pi - \epsilon, 2\pi + \epsilon)$), and if we make it continuous by limiting the domain it won't be surjective. To make it continuous, we have to make a choice of how we measure angle and how what interval we pick. The most common is called the *fundamental domain* or the *principal branch* (for reasons that shall be made clear once we study log) with it's elements being called the *standard arguments* and is $(-\pi, \pi)$ with:

$$\arg(1) = 0 \qquad \arg(i) = \pi/2 \qquad \arg(-1) = \pi, \arg(-i) = -\pi/2$$

As mentioned earlier, shrinking the dmaoin will make this function loose surjectivity from $\mathbb{C}^*$. To re-gain surjectivity, we shall eliminate $(-\infty, 0]$ from the domain. There is no way around this due to non-trivial *monodromy*, if we define a different angle function on a different domain, we shall eliminate a different ray from $\mathbb{C}^*$. This phenomena is given a name: we say that the angle function has non-trivial *monodromy*. The angle function defined on $\mathbb{C} \setminus (-\infty, 0]$ for which $\arg(1) = 0$ will be denoted Arg, i.e.

$$\text{Arg} : \mathbb{C}^* \setminus (-\infty, 0] \to (-\pi, \pi)$$

Though this is a fix, the fact that the function $\theta(z)$ not a continuous function shall give problems when trying to show certain standard functions are continuous. For example, this creates the issue that the function $f : \mathbb{C} \to \mathbb{C}$ $f(z) = \sqrt{z}$, which is discontinuous. To see this, take:

$$f(z) = \sqrt{|z|}\left(\cos\left(\frac{\theta(z)}{2}\right) + i \sin\left(\frac{\theta(z)}{2}\right)\right)$$

Note we could have defined a second square-root function if we measured the angle by turning clock-wise (which represents the second solution to the equation $f^2 - z = 0$), however the exact same problems shall arise. There is in fact a nice visual argument that can show us this. Let's say we want to make it continuous on the sub-domain S^1. Let's pick $\sqrt{1} = 1$, and now try to make it continuous on all of S^1 If we start moving counter-clockwise, we see that the $\sqrt{z}$ must always be half the angle, so we will keep halving the angle but keeping the magnitude. Doing so, we will eventually end up at $\sqrt{-1}$, which by the way we've been choosing our outputs forces us to pick $\sqrt{-1} = i$. If we keep going, we keep halving the angle until we reach back 1, at which point we have $\sqrt{1} = -1$... However we already established that $\sqrt{1} = 1$. This problem cannot be fixed even if we try a different approach. Another insightful way of seeing it that is worth mentioning is if we always take the smallest angle between the positive real axis and our complex number on the unit circle. The argument will remain the same when going counter clockwise towards -1, but now do this argument but going clock-wise. We again the half the angle every time, but notice that once we reach $\sqrt{-1}$, we are forced to pick $-i$! A visual might be helpful to see this:

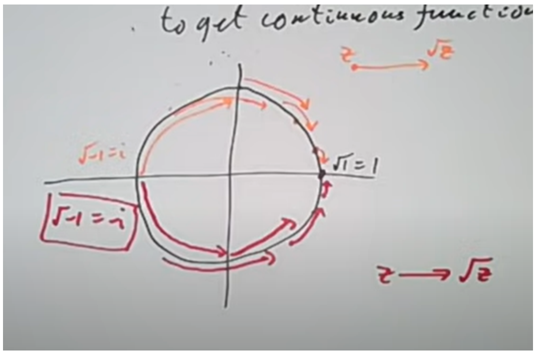


Figure 1.3: Continuity breaks for square root

Hence, $\sqrt{-}$ cannot be continuous on S^1, and hence more generally on $\mathbb{C}$. We would have to eliminate some points in order to make it continuous, usually the choice of points is $(-\infty, 0)$. The fact that going around in different ways doesn't give us the same result is, again, called *non trivial monodromy*.

Another approach some authors pick is to keep the entire domain $\mathbb{C}$ and to say that $\sqrt{-}$ is in fact a *multi-valued function*, meaning the output has both possibilities. This is not very desirable; so many of the tools we have worked with (topology, inner product and smoothness to name a notable few) all are based on the idea of function being single valued, we'd have to re-think all of these notions to incorporate multi-valuedness and the discontinuity that comes up because of the multi-valuedness!! To make not resort notion of "multi-valuedness", we will come up with a new domain for which f is in fact single-valued. To make this rigorous is too much at the moment, but the motivating idea is quite interesting. We will do so for $\sqrt{-}$. Take two copies of $\mathbb{C} \setminus (-\infty, 1]$. Label the bottom edge of the first two copies 1 and 3 respectively and the top edges 2 and 4 Now glue $1, 2$ and $3, 4$ (more formally, take the quotient where these points are in the same equivalent class). Then the resulting shape requires $\mathbb{C}^3$ to embed it without intersection, but we may still visualize it in"in $\mathbb{R}^3$+$\mathbb{R}$ for colours", or $\mathbb{C}^2$, like so:

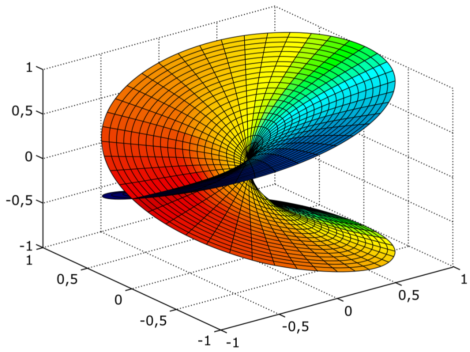


Figure 1.4: Branch Covering for $\sqrt{-}$

Let R represent this surface, known as a *Riemann surface*. We will now lift the function $\sqrt{-}$ to R so that it is a well-defined single-valued function without the need to cut out part of it's domain. In particular, notice that R is a covering space for $\mathbb{C}$, usually known as a *branch covering*. Then we define map f from R to $\mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\sqrt{-}"'] & \end{tikzcd}$$

Then f is now a *single valued* function. This same process will be done for many other function to "recover" a well-defined function. Famously, there is no proper complex log, there are infinitely many outputs. The corresponding branch covering of $\mathbb{C}$ for complex log will be:

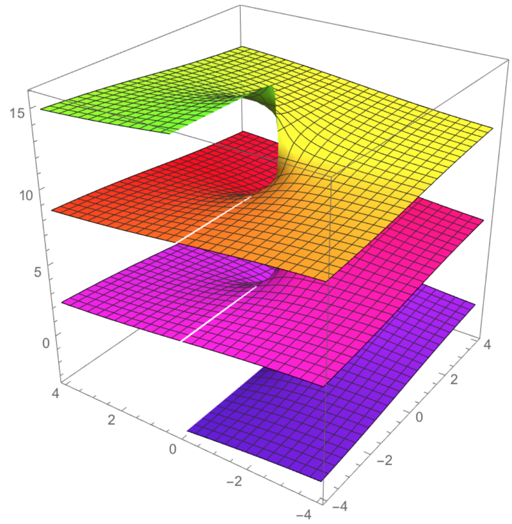


Figure 1.5: Branch Covering for complex log

Exercise 1.2.2

1. If $a, c \in \mathbb{C}$ and $\operatorname{Im} a\bar{c} = 0$, then $c = r \cdot a$ where $r \in \mathbb{R}$
2. Let $z = a + bi \in \mathbb{C}$ and define $\dot{z} = b + ai$. Show that $z \cdot \dot{z}$ is strictly imaginary unless $\dot{z}z = 0$ case it's (naturally) both real and imaginary. Is there a geometric interpretation of this result?

1.2.4 Riemann Sphere: adding a point at infinity

When working with $1/x$, we have a discontinuity at 0, namely since the graph "jumps" at 0. This can be remedied with the 1-point compactification and some differential geometry by taking $f : S^1 \to S^1$ and taking $z \mapsto \overline{z}$ which extends $1/z$ to all of S^1, and with the stereographic projection this can be seen as a way of "introducing" $1/x$ in a well-defined manner (we shall make this rigorous in a few paragraphs). However this does not remedy the fact that in coordinates, a connected interval around 0 becomes two disconnected open rays (going to $\pm\infty$), namely when looking restricting back to $\mathbb{R}$, the functions are no longer continuous. Naturally, on S^1, the map is continuous (even real-differentiable), however we have now lost some of the tools are accustomed to analyzing $\mathbb{R}$.

We require at least one more dimension to be able to image the image connected[6]. In this way, we shall soon consider the complex function $1/z : \mathbb{C} \to \mathbb{C}$ to be a continuous function, and even a complex-differentiable function.

In order to consider $1/z$ well-defined function, we need to extended complex numbers. Let us first start wit what happens when we do this to $\mathbb{R}$. As mentioned earlier, in $\mathbb{R}$ we can "compactify" $\mathbb{R}$ by adding one point at infinity, $\mathbb{R} \cup \{\infty\}$ (this is called the *one-point compactification*, see my EYNTKA Topology). We will label this as $\mathbb{R}_\infty$. Notice that $\mathbb{R}_\infty$ is homeomorphic to S^1 via the following

[6] in essence, because we require at least two dimensions to embed a circle

homeomorphism: if we take any $z \in S^1$ as a tuple (x, y), then

$$(x, y) \mapsto \frac{x}{1-y}$$

and $(0,1) \mapsto \{\infty\}$. This shows that $\mathbb{R}_\infty$ is indeed compact, and furthermore we can try to work to incorporate infinity into our analysis. The first thing we may notice is that we have the problem that $\mathbb{R}_\infty$ looses the well-ordering of $\mathbb{R}$, namely since ∞ is both smaller and larger than any real number, in particular $\infty < 0 < \infty$. However, if we instead take $x \mapsto |x|$, then ∞ is the natural value that appears as either $\lim_{x\to-\infty}|x|$ or $\lim_{x\to\infty}|x|$. Thus, it is natural to think of $\mathbb{R}_\infty$ as the codomain of $|\cdot|$, and more generally some norm $\|\cdot\|$ on a real vector-space rather than an element that we can add to the ordering of $\mathbb{R}$, and is "closed" under convergent and divergent sequences which are eventually strictly increasing or decreasing[7].

Though order is lost, some arithmetic properties are still well-defined (these are shown to be well-defined through the use of arbitrary sequences):

1. $a + \infty = \infty + a = \infty$ for all finite a
2. $b \cdot \infty = \infty \cdot b = \infty$ for $b \neq 0$, including ∞

It is impossible (as shown in 1st year calculus) to define $\infty + \infty$ or $0 \cdot \infty$ since there can be sequence that approach these value's differently depending on how "fast" each term goes to ∞ or 0 (recall the l'hopital rules). It is however possible to define the convention that $a/0 = \infty$ or $b/\infty = 0$ for $b \neq \infty$. Though there is some arithmetics well-defined in $\mathbb{R}_\infty$, if we transfer over to S^1 via the stereogrpahic homeomoprhism, it is much harder to perform arithmetics, and so we would usually take the S^1 interpretation of $\mathbb{R}_\infty$ for it's geometric benifits.

Let's now work with $\mathbb{C} \cup \{\infty\}$. Like with the real case, let's label it $\mathbb{C}_\infty$. Some other common labels are $\widehat{\mathbb{C}}$ and $\overline{\mathbb{C}}$. Just like for $\mathbb{R}$, we are extending $\mathbb{C}$ by a single point instead of a circle of points since we are more so interesting in the "size" of points approaching infinity rather than any sequence converging to some "appropriate" point at infinity (like a $\infty(\cos(\theta) + i\sin(\theta))$). Let's make precise the relation between $\mathbb{C}_\infty$ and S^2. Identify $\mathbb{C}_\infty \setminus \{\infty\}$ with the xy-axis in $\mathbb{R}^3$, where the x-axis representing the real component and the y-axis representing the imaginary component. Naturally, no point in $\mathbb{R}^3$ is identified with the point ∞ in $\mathbb{C}_\infty$. Take a unit sphere centered at the origin of $\mathbb{C}_\infty$. Now, starting from the north-pole, project a down through the sphere and into the complex plane. This is a one-to-one correspondence with the xy-axis, omitting the north-pole . If $S^2 = \big\{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1, \ x, y, z \in \mathbb{R}\big\}$ then

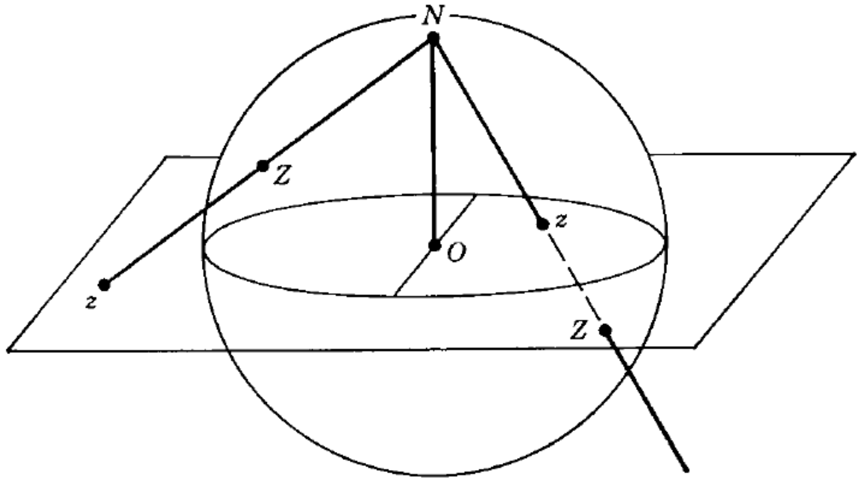
$$z = \frac{x_1 + ix_2}{1 - x_3}$$

By taking the square of the modulus ($|z|^2$), we can re-arrange that equation to get

$$x_1 = \frac{z + \overline{z}}{1 + |z|^2} \quad x_2 = \frac{z - \overline{z}}{i(1 + |z|^2)} \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{1.5}$$

By letting the pole $(0,0,1)$ correspond to the point at infinity, we get that the sphere is homeomoprhic to $\mathbb{C}_\infty$ (and hence, we get a nice geometric object representing $\mathbb{C}_\infty$ without needing to "distinguish" a point). This process is called *stereographic projection*, and visually looks like so:

[7]Note that it is not closed for any type of sequence, for example for oscillating sequences



Note that the points where $x_3 < 0$ corresponds to the disk $|z| < 1$, and when $x_3 > 0$ the points correspond to $|z| > 1$ while if $x_3 = 0$ then $|z| = 1$. We may also stereographically project from the south-pole, giving us

$$z' = \frac{x_1 - ix_2}{1 + x_3}$$

Note that z an z' are related by the equation $zz' = 1$ unless z or z' is a pole N or S.

For future purposes, we will quickly say a word on how circles on the Riemann sphere map to $\mathbb{C}_\infty$, namely since this gives a nice visualization of how to think about $\mathbb{C}_\infty$. Any circle *not* passing through the pole maps a circle on the plane. If a point of a circle is at the north pole, then the cirlce maps to a *line* in $\mathbb{C}_\infty$. To see this, notice that a circle on the Riemann sphere lies in the plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$ where we can assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \leq \alpha_0 < 1$. From equation (1.5), we can re-write this previous equation as:

$$\alpha_1(z + \overline{z}) + \alpha_2 i(z - \overline{z}) = \alpha_3(|z|^2 - 1) = \alpha_0(|z|^2 + 1)$$

or:

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0$$

When $\alpha_0 \neq \alpha_3$, this is the equation of a circle, and when $\alpha_0 = \alpha_3$, this is the equation of a straight line. Conversely, the equation of a cricle and a line can always be written in this form.

Finally, on S^2, it is easy to calcualte the distance between the two points (i.e. the geodesic) and so have a finite metric on $\mathbb{C}_\infty$: If $z = (x_1, x_2, x_3)$ and $z' = (x_1', x_2', x_3')$, we first get the equation of the geodesic:

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 = (x_3 - x_3')^2 = 2 - 2(x_1 x_1 + x_2 x_2' + x_3 x_3')$$

Metrics on Riemann Sphere

The rest of this section can be skipped till it is needed. The *chordal distance* between two points on the Riemann sphere (considered as the unit sphere in $\mathbb{R}^3$) means the length of the line segment in $\mathbb{R}^3$

joining the points. The chordal distance induces a metric $d(z, w)$ on $\mathbb{C}$; i.e., if $z, w \in \mathbb{C}$, then $d(z, w)$ is defined as the chordal distance between the points of the Riemann sphere corresponding to z, w by stereographic projection.

Theorem 1.2.1: Chordal metric on Rimeann Sphere

The following:

$$d(z, w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

Is a well-defined metric. For $w = \infty$, the corresponding formula is

$$d(z, \infty)\frac{2}{\sqrt{(1+|z|^2)}}$$

Proof **:**
First, the mapping from $\mathbb{C}$ to the Riemann sphere is:

$$z \mapsto \left(\frac{z+\overline{z}}{|z|^2+1}, \frac{z-\overline{z}}{i|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

Hence, the chordal distance between two points z, w is:

$$\begin{aligned}&= \left(\frac{z+\overline{z}}{|z|^2+1} - \frac{w+\overline{w}}{|w|^2+1}, \frac{z-\overline{z}}{i|z|^2+1} - \frac{w-\overline{w}}{i|w|^2+1}, \frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1}\right)\\ &= (x-x', y-y', t-t')\end{aligned}$$

Then taking square of the distance we get:

$$(x^2 - y^2 + t^2) - 2(xx' - yy' + tt') + ((x')^2 - (y')^2 + (t')^2)$$

Solving the first (resp third) term in the above equation, we get

$$\frac{z^2 + 2|z|^2 + \overline{z}^2) - (z^2 - 2|z|^2 + \overline{z}^2) + |z|^4 + 1}{(|z|^2+1)^2} = 1$$

Simplifying the equation to:

$$2 - 2(xx' - yy' + tt')$$

Replacing and simplifying again, we get

$$\frac{4|z-w|^2}{(|z|^2+1)(|w|^2+1)}$$

square rooting, we get:

$$d(z, w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

If we take $w = \infty$ in the extended $\mathbb{C}$, then $\infty \mapsto 0$ giving us:

$$d(z, \infty)\frac{2}{\sqrt{(1+|z|^2)}}$$

as we sought to show.

This is not to be confused with the geodesic metric which takes the smallest distance between the two points *on* the Riemann sphere:

Definition 1.2.2: Geodesic Metric

$$\rho(z,w) = \arccos(z \cdot w)$$

Finally, in chapter 6, we shall be taking the derivative of many functions on the Rimeann sphere, and shall need the following result:

Corollary 1.2.1: Spherical Derivative

Let f be a function defined on the Riemann sphere. Then:

$$f^{\#}(z) := \lim_{z \to w} \frac{|d(f(z), f(w))|}{|z-w|} = \frac{2|f'(z)|}{1+|f(z)|^2}$$

Proof **:**
here

1.3 Complex Functions

In this section, we will go over how to think about complex functions and emphasize geometric intuitions. For convenience, we will use z, w as complex numbers and x, y as real numbers. a *real function* is a function with codomain $\mathbb{R}$, while a *complex function* has a codomain $\mathbb{C}$. Whether the domain is real, complex, or anything else (ex. a manifold, a vector space, a topological space) is usually deduced from context.

1.3.1 Complex Linear Transformations

Since a function is differentiable if it is locally linear, it is worth taking a moment to understand complex linear functions. Let's start with the simplest $\mathbb{C}$-linear transformation: a map $f : \mathbb{C} \to \mathbb{C}$ where

$$f(z) = az \qquad a \in \mathbb{C}$$

representing $z = x + yi$ and $a = a + bi$, we get:

$$(a+bi)(x+yi) = (ax - by) + (bx + ay)i$$
$$\Rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}$$

Notice that this linear translation is a rotation and a scaling. In particular, it *preserved the angles*. A function that is angle-preserving is called *homothetic*. Furthermore, we cannot have any mirror symmetry. For example

$$z \mapsto \overline{z}$$

is homothetic, but it is *not* $\mathbb{C}$-linaer. The way I like to think about it is that if you are in one "mirror-space", so to speak, then you will always measure the angle going from a fixed choice of either left or right. If you change mirror spaces, then you change which side you measure your angle from. Thus, when we think of angle preserving we will think of angle preserving with respect to a fixed side. From this perspective, $z \mapsto \overline{z}$ maps an angle θ to $2\pi - \theta$, which will not be considered "angle preserving"

Lemma 1.3.1: Real Angle Preserving Then Complex

Let $T : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear function that perserves angles. Then there exists an $a \in \mathbb{C}$ such that $T(z) = az$ or $T(z) = a\overline{z}$

Proof **:**
Since T is homothetic, it must be bijective. Let S be a homothetic function such that $S^{-1}T(1,0) = (1,0)$. $S^{-1}T$ is also homothetic, so it must be that Since $S^{-1}T(0,1) = (0,c)$ for some $c \neq 0$. Since $S^{-1}T$ is linear, $S^{-1}T(1,1) = (1,c)$. Since $S^{-1}T$ is angle preserving, it must be that $c = \pm 1$. depending on which this is, we get $T(z) = az$ or $T(z) = a\overline{z}$, completing the proof.

1.3.2 Polynomial and Rational Functions

Polynomial functions (with real coefficients) are easy example of real differentiable functions. So too will complex polynomials. It will turn out that a slight-generalization of these polynomials give a huge example of complex differentiable functions, in particular for allowing the *quotient* of polynomials. Functions that are polynomials over polynomials are called *rational polynomials* or *rational functions.* In this section, we give some common names to notions for rational functions. By the fundamental theorem of algebra, we get

$$P(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

for n not necessarily distinct roots (we will prove this theorem in section ref:HERE). For now, we will assume differentiation of complex polynomials is identical to that of real polynomials (which will turn out to be the case, see section 2.1). If $P(\alpha_i) = 0$, we call α_i a zero or root of P. Since the roots need not be distinct, we call the order of a zero the number of roots that are equal. Note that if α has order h for P, then $P^{(h-i)}(\alpha) = 0$ for $1 \leq i \leq h-1$, and $P^{(h)}(\alpha) \neq 0$. Note that if the order of an α is 1, then $P(\alpha) = 0$ but $P'(\alpha) \neq 0$. We call a zero of order 1 a simple zero.

Definition 1.3.1: Rational Function

Let $p(x)$ and $q(x)$ be polynomials over $\mathbb{C}$ where $\gcd(p(x), q(x)) = 1$ (i.e., they don't have any similar roots). Then

$$R(x) : \mathbb{C}_\infty \to \mathbb{C}_\infty, \; R(x) = \frac{p(x)}{q(x)}$$

is called a *rational function* or a *Fracitonal Linear Transformation* over $\mathbb{C}_\infty$.

For more information, see `https://encyclopediaofmath.org/wiki/Fractional-linear_mapping`. We let the codomain be the extended complex numbers in order that the denominator be able to accept 0 values. In particular, we'll define

$$R(\alpha) = \frac{p(\alpha)}{q(\alpha)} = \frac{n}{0} = \infty$$

This is another reason to allow only one point of compactification, to allow this extension to be well-defined. This also motivates the name pole: these points get mapped to the north pole of the Riemann sphere. It also allows us to rigorously capture the idea by doing:

$$R(\infty) = \lim_{z\to\infty} R(z)$$

and so R defined on $\mathbb{C}_\infty$ is indeed well-defined. Since it is possible to plugin $R(\infty)$, it would be convenient to find a way to find zeros or poles at infinity. To find these, we will define

$$R_1(z) = R(1/z)$$

And define $R(\infty) = R_1(0)$. From this, we can define the order of zeros (i.e. roots) or poles at ∞ to be the zeros or poles of the origin of R_1. Now, given some R:

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

Then

$$R_1(z) = z^{m-z} \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}$$

From this we, can see the order of zero's or poles at ∞. If $m > n$, then the zeroes at infinity has order $m - n$. Conversely, if $m < n$, then the poles at infinity have order $n - m$. If $n = m$, Then we either get

$$R(\infty) = a_n/b_m \neq 0 \quad \text{or} \quad \infty$$

with all the zeros and poles established, we can count the number of zeros or poles a given rational function has, including the zero's or poles at ∞; namely by the observations in the above paragraph we have that the number of zeroes/poles is always $\max(n, m)$. Since this is a consistent number, we call this number the *order* of the rational function. The order of a rational function is quite stable. The function $R - a$ has the same number of poles and zeros and hence the same order (meaning any $R(z) = a$ of order p has p roots). Furthermore, $R - a$ (called a *parrallel transition*) keeps the point ∞ stable, while the rational function $1/z$ (called the *inversion*) interchanges 0 and ∞.

Treating differentiation of rational functions as a formal operation [8], we also see that the poles are stable under differenitation, which we can see using the quotient rule. As an exercise, check that if a pole of $R(z)$ has order k, then a pole of $R'(z)$ has order $k + 1$.

We may also use differentiation and the inverse function to find the order of a pole: Using $R_1(z)$ to flip the poles and the roots, we can then take the formal derivative until we get a nonzero value.

1.3.3 Möbius Transformation

(see this website: here)

A rational function of particular interest which we will focus on in section ref:HERE is those of of order 1. Due to their importance, they are given a name for reference:

[8] think of it as a new algebraic operation on the ring of rational functions over $\mathbb{C}$

Definition 1.3.2: Möbius Transformatino

Let f be a rational function of degree 1

$$S(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \alpha\delta - \beta\gamma \neq 0$$

Then f is called a *Möbius transformation*.

The reason we have $\alpha\delta - \beta\gamma \neq 0$ is so that there is 1 pole and 1 root. Let's say that

$$S(-b/a) = \frac{0}{c\frac{-b}{a} + d} = 0$$

If the denominator was 0 we would get $0/0$, that is the root would not be well defined. That would happen if:

$$c\frac{-b}{a} + d = 0 \quad \Leftrightarrow \quad -cb + ad = 0$$

Hence, we will require sthat $ad - bc \neq 0$. If $ad - bc = 0$, then $c = \frac{ad}{b}$, which substituing in the 1st order FLT we get:

$$\frac{az+b}{\frac{ad}{b}z + d} = \frac{az+b}{\frac{d}{b}(az+b)} = \frac{b}{d}$$

and hence has no poles. Manipulating the equation, we get that the inverse is:

$$S^{-1}(w) = \frac{dw - b}{-cw + a}$$

showing the inverse is also a first order rational function. The key result that we care about these functions is that the pole is sent to a single point. Once we define complex differentiation, we will see that Möbius transformations are the diffeomorphissm of $\overline{\mathbb{C}}$, making them worthwhile to study. Already we can see that such a function is certainly a homeomorphism, and hence they belong to $\mathrm{Aut}_{\mathbf{Top}}(\mathbb{C})$.

Given Möbius transformations are such nice function, they should be relatively simple to imagine. Indeed, if $\gamma = 0$, then we have $\frac{\alpha z}{\delta} + \frac{\beta}{\delta} = az + b$ for some constants a and b showing that this möbius transformation is just scaling and transformation, while if $\gamma \neq 0$ then

$$\begin{aligned}
\frac{az+b}{cz+d} &= \frac{az}{cz+d} + \frac{b}{cz+d} \\
&= \frac{a}{c} + \frac{\frac{-ad}{c}}{cz+d} + \frac{b}{cz+d} \\
&= \frac{a}{c} + \frac{b-ad}{cz+d} \\
&= \frac{a}{c} + \frac{(1/c)(b-ad)}{z+d/c} \\
&= \alpha + \frac{\beta}{z+\gamma}
\end{aligned}$$

showing that in this case a möbius transformation is a scaled by β, translated by α, recipricated around γ function. This is nice to keep in mind, though when it comes to manipulating the original möbius transformation it might be hard to see what's going on. Another way we can decompsose a Möbius transformation is given if $c \neq 0$:

$$f_1(z) = z + d/c \quad f_2(z) = 1/z \quad f_3(z) = \frac{bc - ad}{c^2}z \quad f_4(z) = z + a/c$$

then

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{az + b}{cz + d}$$

The following proposition may help visualize the images of this type of function:

Proposition 1.3.1: Properties Of Möbius Transformations

Let $f : \mathbb{C} \to \mathbb{C}$ be a möbius transformation. Then:

1. f circle's and lines to circles and lines
2. f is uniquely determined by where it takes three distinct points of $\mathbb{C}$ to any three distinct points in $\mathbb{C}$.
3. f fixes 1, 2, or all points.

Proof **:**

1. in Alhors
2. Suppose z_1, z_2, z_3 are distinct points in $\mathbb{C}$ and w_1, w_2, w_3 are distinct points in $\mathbb{C}$. We'll start by showing there exists a map such that $z_1 \mapsto 1$, $z_2 \mapsto 0$ and $z_3 \mapsto \infty$. First, invert about any cirlce centered at z_3, which takes z_3 to ∞. The points z_1, z_2 get mapped to z_1', z_2', neither can be ∞, Next, translate z_2' to 0, which keeps z_1 mapped to ∞ and maps z_3' to z_3''. Finally rotate and dilate about the origin in such a way that $z_3'' = 1$. This keeps 0 and ∞ fixed and hence works. In order for this to be a Möbius transformation, relfect across the real axis: this keeps $1, 0$, and ∞.

   The last steps of the proof are left as an exercise.

We next analyze the global properties of Möbius transformations. These will be important for us in section ref:HERE, where we prove two important theorems (note that biholomorphic means complex diffeomorphic)

1. *Riemann mapping theorem*: If U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f from U onto the open unit disk
2. *Uniformization theorem*: Let R be a Riemann surface (a complex 1-manifold) and $U \subseteq R$ a simply connected connected open subset of R. Then U is biholomorphic to one of the following:
   (a) Riemann sphere (which we'll eventually show is $\mathbb{C}\mathrm{P}^1$)

(b) $\mathbb{C}$

(c) $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$

In other words, we will use Möbius transformation in classifying all simply-connected one-dimensional complex manifolds, for which there is surprisingly only 3 up to isomorphism! These following theorems give us the appropriate global mapping properties that will be lemma's for the above two theorems.

Proposition 1.3.2: Möbius Transformation On Upper Half Plane

Let f be a Möbius transformtion with domain $\mathbb{H}$. Then:

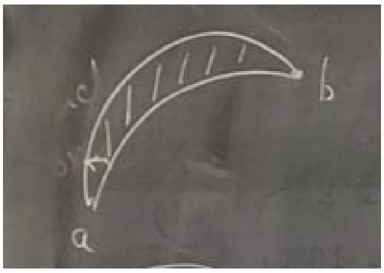
1. if $f(\mathbb{R}) \subseteq \mathbb{R}$, then $a, b, c, d \in \mathbb{R}$
2. if $f : \mathbb{H} \to \mathbb{D}$, then f must be of the form:

$$\eta \frac{z - \alpha}{z - \overline{\alpha}}$$

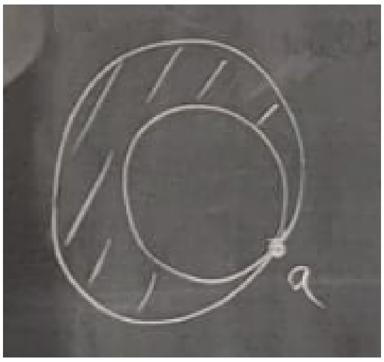
with $|\eta| = 1$ and $\operatorname{im} \alpha > 0$.

***Proof* :**
exercise, or look at your notes

To better understand such maps, we will see how they map certain shapes to either $\mathbb{H}$ or $\mathbb{D}$. These are commonly occuring shapes in complex analysis and so knowing where they map is useful, in particular they will be useful when constructing Riemann surfaces via gluing. A circular wedge is a shape of one of these two forms:



(a) distinct endpoints



(b) same endpoints

Figure 1.6: cirtulcar wedges

For figure (a), map a to 0 and b to infinity likeso:

$$\frac{z - a}{z - b}$$

This will stretch out (a) so that it looks like so

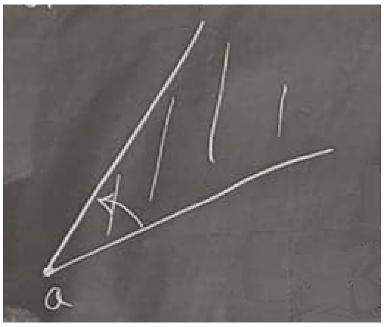


Figure 1.7: apply $\frac{z-a}{z-b}$

Now just rotate so that one of the wedge lie on the x-axis, and scalr by an appropriate z^α so that it maps to $\mathbb{H}$. For figure (b), map it by $\frac{1}{z-a}$ so that it looks like so:

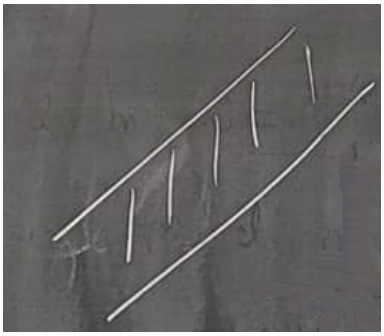
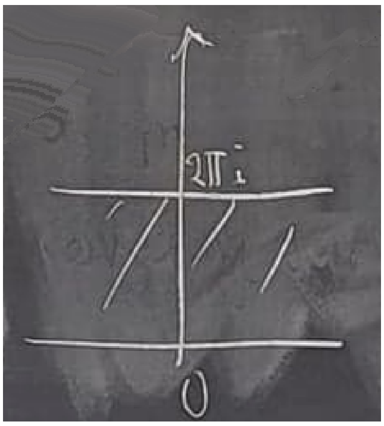


Figure 1.8: apply $\frac{1}{z-a}$

then rotate it and dilate it so that we get:



and finally exponentiate it to get $\mathbb{H}$. The last shape we will care to map if $\mathbb{C} - I$ for some open interval I. Without loss of generailty say $I = (-1, 1)$. First apply the map $\frac{z+1}{z-1}$ to map $-1 \mapsto 0$, $1 \mapsto \infty$ and $0 \mapsto 0$, then apply the square root function $z^{1/2}$ for your choice of square root function, which will create a $\mathbb{H}$ but with the y-axis, then map that to the disk $\mathbb{D}$ so that we may map it to $\mathbb{H}$. this will make us end up with the equation $w = z - \sqrt{z^2 - 1}$.

1.3.4 Partial Fractions

Every rational function can be represented as a *partial fraction* which splits the rational function as a sum of two rational function with the degree of the numerator being smaller than the degree of the denominator. In order for this to be the case, we need that $R(z)$ has a pole at infinity. Such a decomposition is well-defined since polynomial division is well-defined[9].

Definition 1.3.3: Partial Fraction Decomposition

Let $R(z)$ be a rational function. Then the resulting partial fraction decomposition will be of the form

$$R(z) = G(z) + H(z)$$

where $G(z)$ is a *polynomial* without a constant term, and $H(z)$ is finite at ∞. The degree of G is the order of the pole at ∞. The polynomial G is called the *singular part of* R *at* ∞.

Example 1.1: Partial Fraction Decomposition
here

We can manipulate this equation further to get some nice results that allow for easier integration of more complex functions. Let $b_1, ..., b_n$ be the poles of $R(z)$. Define a new function $R'_j(\zeta) = R(B_j + 1/\zeta)$. R' has a pole at ∞. By definition:

$$R'_j(\zeta) = R\left(b_j + \frac{1}{\zeta}\right) = G_j(\zeta) + H_j(\zeta)$$

Or, with some simple change of variables:

$$R(z) = G_j\left(\frac{1}{z - b_j}\right) + H_j\left(\frac{1}{z - b_j}\right)$$

Notice how now $R(z)$ is written in a different from: the polynomial G_j has now $\frac{1}{z-b_j}$ as its indeterminate. The polynomial G_j is called the *singular part of* $R(z)$ *at* b_j. By construction, $H_j\left(\frac{1}{z-b_j}\right)$ is finite for $z = b_j$. Now, consider the expression

$$R(z) - G_j\left(\frac{1}{z - b_j}\right) + H_j\left(\frac{1}{z - b_j}\right) = 0$$

(words in Althors boringly explaining here on page 32 why we get)

$$R(z) = G(z) + \sum_{i=1}^{q} G_j\left(\frac{1}{z - b_j}\right)$$

which is the expression used in calculus to simplify integrals.

[9] If you've taken algebra, recall that $\mathbb{C}[x]$ is a Euclidean Domain

2

Complex Differentiation

Now that we've studied the arithemtics, geometry, and analytical/topological properties of $\mathbb{C}$, we will move on to studying *differentiable complex functions*. Since $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, we see that continuity of $\mathbb{C}$ works identicaly to that of $\mathbb{R}^2$. We will find that differentiability will produce some very different results, which will in fact greatly restrict which functions are differentiable and give *much* nicer results than the theory of differentiability of real numbers. We will in fact rather quickly answer lot of the "usual" questions that we had to build up a lot of work for in real analysis rather quickly in complex analysis, and we'll see that complex-differentiable function are more like the "infnite analog" or "analytlic analogue" of polynomials[1]

After defining and expanding and exploring complex-differentiable functions and finding some necessary and sufficient conditions for a functions to be complex-differentiable, we will start building up a repertoire of complex-differentiable function. We will show that complex-linear functions are complex-differentiable (as we hope they should be). Then since the sum, product, and quotient of complex-differentiable functions is complex-differentiable, we will get that all polynomials and rational functions are complex-differentiable. The next step would be to explore infinite polynomials, that is power-series. Naturally, power-series need not converge everywhere and so a power-series will only be defined up to domain of convergence which will be well defined by a radius of convergence (with the exception of the border, for which we have to be more careful). Using power-series, we can show that many of our common functions from real analysis are complex-differentiable, most importantly $e^x, \sin, \cos, \log$. Using the technics we've built up, we can show many properties of these functions, like periodicity or the algebra of the exponential ($e^{x+y} = e^x e^y$).

[1]Note that this is not true for real-differentiable functions

2.1 Build-up

let f be a real or complex function. We may ask whether some point a in the codomain satisfies:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-Df(a)(h)}{\|h\|}=0$$

The answer will partly depend on the domain and codomain. Let's say $f:\mathbb{C}\to\mathbb{R}$ is a differentiable function (i.e. the limit above exists for all $z\in\mathbb{C}$). Since we're working in "one-dimension" over $\mathbb{C}$, let's use the usual definition of the derivative:

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(h)}{h}$$

Remember that the value $f'(z)$ must be independent of how h approaches 0. Let's first approach $h\to 0$ as only real numbers (i.e., immediately make the imaginary component 0 and keep approaching that way). Then $f'(z)$ is the quotient of two real numbers, and hence will be real. On the other hand, if we $h\to 0$ from the complex component (i.e. immediately make the real component 0 and keep approaching that way), then $f'(z)$ is the quotient of a real number by an imaginary number. As we know

$$\frac{1}{i}=-i$$

and so the resulting number must be imaginary. The only number to be both real and imaginary is 0, and hence if f is indeed differentiable, then it's derivative must be 0 (or f is not differentiable). As a consequence of this, by what we know of integration, the resulting function must be *constant*. This makes sense if we recall $\mathbb{C}$-linear functions preserve angles, and projecting to $\mathbb{R}$ would certainly not perserve angles locally.

If we switch the order of the domain and co-domain and consider $f:\mathbb{R}\to\mathbb{C}$, then as the codomain has the same topology as $\mathbb{R}^2$, this case can be thought of as $f:\mathbb{R}\to\mathbb{R}^2$. Consider that any such function can be represented as $f(t)=x(t)+iy(t)$ for appropriate functions $x:\mathbb{R}\to\mathbb{R}$ and $y:\mathbb{R}\to\mathbb{R}$:

$$f(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

The element i will act simply as a scalar (or the vector component $(0,1)$) since it is in the numerator, and so

$$f'(t)=x'(t)+iy'(t)$$

and so f is differentiable if x and y are differentiable. Thus, the case of $f:\mathbb{C}\to\mathbb{C}$ is truly different than the real case, so the rest of this section comes to figuring the structure of such a function

2.2 Holomorphic Functions

To find out the conditions for which a function $f:\mathbb{C}\to\mathbb{C}$ must be complex-differentiable (which $\mathbb{R}^2$-differentiable function can be interpreted as $\mathbb{C}$-differentiable functions), we will define a function to be differentiable, and then find then necessary condition's that this will impose onto f. We will give a special name to functions that are complex-differentiable:

Definition 2.2.1: Holomorphic (Analytic) Function

let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a function with Ω open. Then if for every point $z \in \mathbb{C}$, $f'(z)$ is defined, that is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(h)}{h}$$

then f is said to be *holomorphic* or *analytic*

You might recall from calculus that an analytic function is a function such that the function can be represented as a power-series in a local neighborhood at every point. Soon, we will show how this is true for all Holomorphic functions, and in fact is both a necessary and sufficient condition for a holomorphic function, and so the words Analytic and Holomorphic are used interchangeably by many mathematicians.[2].

The key example of holomorphic functions are:

Example 2.1: Holomorphic Functions

1. the function $f(z) = c$ for some $c \in \mathbb{C}$ is a holomorphic function
2. the function $f(z) = z$ is a holomorphic function
3. the function $f(z) = az$ for some $a \in \mathbb{C}$ is a holomorphic function (in other words, $\mathbb{C}$-linear functions ought to be complex-differentiable).

Holomorphic functions work well under our usual binary operations, and have many properties we would expect of them:

[2]Note how this is *not* true for real-differentiable functions, consider $f(x) = e^{-1/x}$ when $x > 0$ and 0 otherwise (the derivative grows "too fast", see cite:REAL-ANALYSIS

Proposition 2.2.1: Properties of Holomorphic Functions

Let $f : \mathbb{C} \to \mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions. Then

1. $f + g$, fg, $g \circ f$, and zf (for some $z \in \mathbb{C}$) are holomorphic functions (hence, with the result from example 2.1, all polynomials are holomorphic) with the same derivative formula as the real equivalent. Similarly, f/g is an holomorphic function, provided that $g(z) \neq 0$ for all $z \in \mathbb{C}$ (i.e. g does not vanish). Hence, all well-defined rational polynomials are holomorphic functions. Sometimes, we abuse notation and say f/g is holomorphic even if $g(z) = 0$ for some points by excluding those points from the domain.[a]

2. f is continuous.

3. f is real-differentiable (interpreting f as $f : \mathbb{R}^2 \to \mathbb{R}^2$)

4. There exists functions u, v such that f can be written as $f(z) = u(z) + iv(z)$. Both u and v are continuous. In particular, if $z = x + iy$, then we have $f(z) = u(x, y) + iv(x, y)$. This is called the *standard representation of f*.

[a]This shall be less of an abuse once we define the derivative between $\mathbb{C}_\infty$

Proof **:**

1. This is the same breaking down the limit trick as seen in elementary calculus

2. Let
$$f(z+h) - f(z) = \frac{h}{h}(f(z+h) - f(h)) = h \cdot \left(\frac{f(z+h) - f(h)}{h} \right)$$
and, evaluating the limit, we get
$$0 \cdot f'(z) = 0$$
and so
$$\lim_{h \to 0} f(z+h) - f(z) = \Leftrightarrow \lim_{h \to 0} f(z+h) = f(z)$$
and since this is true for all z, f is continuous

3. Since f is holomorphic, then f', exists, that is for any $z \in \mathbb{C}$, the limit
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$
exists. Take the absolute value of the right hand side:
$$\left| \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \right|$$
since $|\cdot|$ is continuous, it passes through limits:
$$\lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} \right| = \lim_{h \to 0} \frac{|f(z+h) - f(z)|}{|h|}$$
which, since the topology of $\mathbb{C}$ is identical to that of $\mathbb{R}^2$, this is equivalent to saying that f, if interpreted as $\mathbb{R}^2 \to \mathbb{R}^2$, is real-differentiable!

4. Recall that every element of $\mathbb{C}$ can be written of the form $a + bi$. Thus, the output of any function $f(z)$ will be of the form $a + bi$. The real and imaginary part are dependent on both a and b, and $f(a + bi) = u(a, b) + iv(a, b)$ is a well-defined function where we have $u, v : \mathbb{R}^2 \to \mathbb{R}$. Furthermore, since f is holomorphic, it is continuous, and so restricting to only going through the real or imaginary part must also be continuous since projections are continuous, and hence u and v are continuous. Conversely, if u and v are continuous, then so so is adding them (mutiplying by i is equivalent to specifying a basis element), and so f is continuous.

   Notice that since the projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, both u and v are real-differentiable when interpreted as $u, v : \mathbb{R}^2 \to \mathbb{R}$. Note that they *cannot* be interpreted as complex-differentiable, since all functions of the fomr $u, v : \mathbb{C} \to \mathbb{R}$ are constant, and u, v are certainly not always constant.

The collection of holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is sometimes denoted $C^\omega(\mathbb{C})$. By the above proposition, $C^\omega(\mathbb{C})$ is a $\mathbb{C}$-algebra, and is almost a division $\mathbb{C}$-algebra[3]. We have yet to establish if we can put any properties on u, v to make f holomorphic; we will give sufficient conditions section 2.2.2. The following proposition we will outline a necessary condition that needs to be put on u, v:

Proposition 2.2.2: Cauchy-Riemann Condition

Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be the standard representation of f, so that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are well-defined. Then

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Or equivalently, if we represent f as $f(z) = u(z) + iv(z)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations are called the *Cauchy-Riemann Equations.*

Since we can take the limit any way we want, using the Cauchy-Riemann equation, we have that:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

making it relativly easy to find the complex-derivaive of f by simply using real-differentiation. If $a = \frac{\partial u}{\partial x}$ and $b = \frac{\partial u}{\partial y}$, then in matrix form these equations are:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which should strongly remind the reader of complex-linear function from section 1.3.1.

[3] In section 2.5.2, we shall show how we may loosen the definition a bit to make it into a field

***Proof* :**
Recall that the limit at z must be the same regardless of how h approaches 0. If we approach strictly from the real part, then we will get the partial derivative

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

On the other hand, if we approach from a purely imaginary number, we get an intriguing result: let $h = ik$ for some $k \in \mathbb{R}$. Then

$$f'(z) = \lim_{ik \to 0} \frac{f(z+ik) - f(ik)}{ik} = \lim_{ik \to 0} -i \cdot \frac{f(z+ik) - f(z)}{k} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since f' must be equal no matter which direction we approach from, it must be that:

$$f'(x) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

or equivalently, since the real and imaginary part of these equations must agree, we get that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

as we sought to show

Thus, for a function to be holomorphic this partial differential equation *must* be satisfied! Note that the converse is not true: If we have two functions u, v satisfying the Cauchy-Riemann condition, it does not imply that $f = u + iv$ is holomorphic (as we'll show in example 2.2). On the other hand, with some added regularity to f (or to it's partials or u, v), the resulting function is complex-differentiable. We will explore more on what regularity condition's we can pose. For now, the simplest we can give is that f is real-differentiable and satisfies C.R. then it is Holomorphic

Proposition 2.2.3: C.R + Real-Differentiable Implies Complex Differentiable

Let $f : \mathbb{C} \to \mathbb{C}$ be a real-differentiable[a] functions satisfying the Cauchy-Riemann equations. Then f is holomorphic.

[a]Sometimes called Fréchet differentiable due to it's generalization to normed vector spaces

***Proof* :**
Since f is real-differentiable, there exists a matrix A such that

$$F((x,y) + h) = F(x,y) + Ah + o(|h|) \tag{2.1}$$

as $h \to 0$ with

$$A = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}$$

By the Cauchy-Riemann equations, the matrix A represents the multiplication by the complex number

$$\lambda = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

and so we can re-write equation (2.1) in complex terms with $z = x + iy$ as

$$f(z + h) = f(z) + \lambda h + o(|h|)$$

as $h \to 0$. But this implies f is holomorphic, completing the proof.

As we know, a function is differentiable if it is locally linear. Thus, we would expect that a holomorphic function is locally $\mathbb{C}$-linear. As we know from multi-variable calculus, a function f real-differentiable at a produces a tangent map df_a which is $\mathbb{R}$-linear between tangent spaces. We will show that if $f = u + iv$ is holomorphic, then df_a is $\mathbb{C}$-linaer.

Proposition 2.2.4: Holomorphic, Then Locally Complex Linear

Let f be holomorphic, $a \in \mathbb{C}$, and define df_a to be the tangent map of a real-differentiable function as defined above and $f = u + iv$ be the standard represetnation. Then u, v satisfy the Cauchy-Riemann condition if and only if df_a is $\mathbb{C}$-linear, and:

$$\frac{\partial f}{\partial z}(a)(\zeta) = df_a(\zeta) \qquad \forall \zeta \in \mathbb{C}$$

***Proof* :**
Let $d_p f : \mathbb{C} \to \mathbb{C}$ be the tangent map at $p \in \mathbb{C}$. Then as we showed in proposition 2.2.3, the linear approximation satisfies the Cauchy-Riemann equations, which forces our matrix to be of the form representing complex multiplication. But then $d_p f$ is $\mathbb{C}$-linear.

Example 2.2: Holomorphic and non-Holomorphic Functions

1. Let
$$f(x + yi) = \sqrt{|x||y|}$$
Then f is Cauchy-Riemann, but is no Holomorphic (approach from $h = (h_1, h_2) = (t, t)$ and $h = (h_1, h_2) = (t, -t)$). Notice that f is not real-differentiable, showing satisfying the Cauchy-Riemann condition is insufficient for f to be complex-differnetiable (which should make sense, recall we needed C^1 for existence of partials implying existence of total derivative)

2. You should check that $f(z) = \overline{z}$ (i.e. the conjugation function) does not satisfy the Cauchy-Riemann. However, $f(z)$ is real-differentiable. Thus, though it looks very complex-differentiable (it's real-differentiable), it cannot be!

3. The function $f(z) = x^2 - y^2 - i2xy = z^2$ will satisfy the Cauchy-Riemann equations and is real-differentiable.

4. More generally, any polynomial
$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$
satisfies the Cauchy-Riemann equations and are holomorphic, with derivative
$$f'(z) = n a_n z^{n-1} + (n-1) z_{n-1} z^{n-1} + \cdots + a_1$$

5. $R = f/g$ be a ratioanl function. Let $\mathbb{C} - P$ be the domain of R, where P is the set of poles. Then R is holomorphic. R is almost holomorphic on all of $\mathbb{C}$, only failing on a finite set of points. A function which is holomorphic on all but a discrete set of points is called a *meromorphic* functions (from the greek *mero* meaning "part"). We shall show in section 2.5.2 that poles and zeros of a holomorphic functions cannot accumulate[a], making this concept more than a 1-off.

6. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then we see in the proof of the C.R. equations that $f' = \frac{\partial f}{\partial x}$. Thus, the notion of real and complex differentiation are compatible. If we take $f|_{U\cap\mathbb{R}} : U \cap \mathbb{R} \to \mathbb{C}$ to be the restriction of f to the real line, then the real-derivative of $f|_{U\cap\mathbb{R}}$ exists and is equal to the restriction of $f : U \to \mathbb{C}$ to the real line.

7. All of these functions are not Holomorphic since non satisfy the Cauchy Riemann equations:

$$|z|, \qquad \operatorname{Re} z, \qquad \operatorname{Im} z, \qquad \operatorname{Arg}(z), \qquad \overline{z}$$

   This can also be seen since any real holomorphic function must be constant (by the observations we've made earlier), but none of these are constant.

[a] there always exits an ϵ-ball that contains infinitely many points. It shall imply the entire function is 0

Before continuing onto further explorations of holomorphic functions, a comment must be made: there is often a game in mathematics about how weak of a condition can you impose on your building blocks to make our original condition hold. In this case, we have that $f = u + iv$, and we may ask how weak of conditions can be impose on u and v (or f) to make f Holomorphic. We saw that if f is real-differentiable and satisfies the Cauchy condition, then f is complex-differentiable. In fact, we can assume even more weakly that f is continuous and u, v each have first partial derivatives. This is the *Looman-Menchoff Theorem*, which we shall not prove[4]. We will take a moment to prove a stronger claim than proposition 2.2.3 to practice working with partial derivatives:

Theorem 2.2.1: C^1 + C.R. Then Holomorphic

Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be conjugate harmonic functions (or more weakly, u, v are both C^1 and satisfy the Cauchy-Riemann equations). Then:

$$f = u + iv$$

is a holomorphic function.

Note We will later be able to say if and only if to the above theorem by showing that f is Holomorphic, then $f \in C^\infty(\mathbb{C})$.

***Proof* :**
Let $u, v \in C^1(\mathbb{R}^2)$ where u, v satisfy the Cauchy Riemann equations. As we know, the derivative

[4] see ref:HERE for a proof

is a linear approximation and so gives rise to the following equation learnt in elementary calculus:

$$u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1$$

$$v(x+h, y+k) - v(x,y) = \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2$$

where ϵ_1 and ϵ_2 tend to zero faster than h and ik, and so in particular more rapidly than $h+ik$, i.e.

$$\frac{\epsilon_1}{h+ik} \to 0 \qquad \frac{\epsilon_2}{h+ik} \to 0$$

Using the Cauchy-Riemann equations, we can re-write $f = u + iv$ as:

$$f(z + (h+ik)) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h+ik) + \epsilon_1 + i\epsilon_2$$

dividing by $h+ik$ and taking the limit as $(h+ik) \to 0$, we get:

$$\lim_{(h+ik)\to 0} \frac{f(z+(h+ik)) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

showing the limit exists. Since u and v also satisfy the Cauchy-Riemann condition, f must be holomorphic, as we sought to show

2.2.1 Relation to Conjugate

Consider a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. We can interpret such a function as a real-differentiable function $f : \mathbb{R}^2 \to \mathbb{C}$ that satisfies the Cauchy-Riemann equations: $f(x,y)$. We will analyze such a map and see that it will give us some useful relations between $f(z)$, $f(\overline{z})$ and $\overline{f}(z)$. Since

$$z = x + iy \qquad \overline{z} = x - iy$$

we can isolate x and y and get:

$$x = \frac{z+\overline{z}}{2} \qquad y = \frac{z - \overline{z}}{2i}$$

Then applying the chain rule imagning we can to real-differentiation, we get the following equations that are important enough to be given a name:

Definition 2.2.2: Wirtinger Derivatives

Let f be a $\mathbb{R}^2$-differentiable function (*not* necessarily complex-differentiable). Then define the following operations:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

We may think of these as operator's that we may apply to any real differentiable function with complex codomain:

$$\frac{\partial}{\partial z}(f) = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

Let's say f is a holomorphic function. Applying the Cauchy-Riemann conditions, we see that:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = f' \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}} = 0$$

This shows that we can think of a holomorphic function f as being a function of *a complex variable*, rather than a function of *two real variables* with a complex output (in other word, you can think that the imaginary part of a holomorphic function is essentially determined, up to a constant, by the real part, and vice-versa). We can also think of holomorphic functions as being independent of their conjugate, meaning if we interpret a complex function as being $f(z, \overline{z})$ then we would get $f(0, \overline{z}) = c$ for some constant $c \in \mathbb{C}$ (so we cannot have any conjugate terms in a complex function). Taking the above definition, we have that:

$$\frac{\partial}{\partial z} z = 1 \qquad \frac{\partial}{\partial z}\overline{z} = 0$$
$$\frac{\partial}{\partial \overline{z}} z = 0 \qquad \frac{\partial}{\partial \overline{z}}\overline{z} = 1$$

Using this, we see that if

$$f(z) = \sum_{n,m=0}^{k} a_{m,n} z^m \overline{z}^n$$

Then f is holomorphic if and only if $a_{m,n} = 0$ for all $n \neq 0$, hence no complex polynomial which is holomorphic has $\overline{z}$ as a variable. If we have a holomorphic function f, then we can define $\overline{f}$ to be $\overline{f}(z) = \overline{f(z)}$. Applying Wirtinger's derivative we get:

$$\frac{\partial \overline{f}}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial \overline{f}}{\partial \overline{x}} = \frac{\partial f}{\partial x} = f'$$

We'll show $\frac{\partial \overline{f}}{\partial z} = 0$, leaving the other identity as an exercise. Computing:

$$\begin{aligned}
0 &= \overline{0} \\
&= \overline{\frac{\partial f}{\partial \overline{z}}} \\
&= \frac{1}{2}\overline{\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial \overline{z}}\right)} \\
&= \frac{1}{2}\left(\overline{\frac{\partial f}{\partial x}} - i\overline{\frac{\partial f}{\partial y}}\right) \\
&= \frac{1}{2}\left(\frac{\partial \overline{f}}{\partial x} - i\frac{\partial \overline{f}}{\partial y}\right) \\
&= \frac{\partial \overline{f}}{\partial z}
\end{aligned}$$

showing that taking the conjugate of a function "flips" the behavior of the Wirtinger derivative.

In terms of differential forms, if we take $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, then $dz = dx + idy$ and $d\overline{z} = dx - idy$. If we re-write, we get

$$dx = \frac{1}{2}(dz + d\overline{z}) \qquad dy = \frac{1}{2i}(dz - d\overline{z})$$

giving us df in terms of dz and $d\overline{z}$ to be

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\overline{z}$$

we can then shorten this by putting in the Wirtinger derivative to get:

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

and thus we get

$$df = \frac{\partial f}{\partial z}dz$$

which recovers our original notion of derivative of differential form.

2.2.2 Harmonic Functions

Let $f = u + iv$ be a holomorphic function. Due to the Cauchy-Riemann equations, u and v have some interesting analytical properties. The two equalities of the partial derivatives of u and v can be re-written in many forms:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{2.2}$$

which is the Jacobian. This should make sense, as f when interpreted as a real-differentiable function would require the jacobian to be nonzero in order for $f'(z) \neq 0$.

For the next part, we shall be adding some assumption that are not technically needed. Since we will eventually show that f will be C^∞, the following discussion is in fact broader then the restrictions we shall put it in, but for now since we have yet to prove f being differentiable once implies infinitely differentiable, we will explicitly assume that the partials of f are at least twice differentiable, meaning the partials of u and v are at least twice differentiable. Then we have:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial x \partial y} - \frac{\partial v}{\partial x \partial y} = 0$$

and similarly for v: $\triangle v = 0$. Note that we can switch the order of the mixed derivative since we are integrating strictly in the real direction each time, and so the results of real differentiation applies. Thus, if f is holomorphic, we have that $\Delta u = \Delta v = 0$. For future reference we box this result:

Lemma 2.2.1: Holomorphic Then Harmonic

Let f be an holomorphic functions and represent it as $f(x+iy) = u(x,y) + iv(x,y)$ whose partials are twice real-differentiable. Then u and v must be harmonic:

$$\Delta u = 0 \qquad \Delta v = 0$$

Proof **:**
This was proved in equation (2.2).

Functions u, v satisfying the above criterion are actualy quite important in PDE's, and so we will label them for future reference:

Definition 2.2.3: Laplace Operator And Harmonic Functions

Let f be a function that is at least twice differentiable. Then the *Laplace operator* on f is defined as:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

If $\Delta f = 0$, then f is said to be *harmonic*.

The intuition that I have for harmonic functions is given any point $f(z)$, the "average" of all of it's neighbouring points is zero. Think about this in terms of the 2nd derivative for a function from $\mathbb{R} \to \mathbb{R}$. If the 2nd derivative is zero, then you have a line (as you can check). At any point on the line, the "average" of its neighbouring point is zero. If f'' was not zero, then if you look at point's around it (say for the function x^2 at 0 where $f''(0) > 0$) then the average of the points around it are not zero (in this example, they are positive)! Thus, if f is holomorphic and u, v are obtained via the standard represetnation, we have $\Delta u = \Delta v = 0$. This gives us a stronger necessary condition for a function f to be holomorphic.

The converse is also true: if u, v satisfy the Cauchy-Riemann equations and are each Harmonic, then we can in fact construct a holomorphic function f via $f := u + iv$. Note that being harmonic does not imply the Cauchy-Riemann equations: $z \mapsto \overline{z}$ is harmonic, but not holomorphic. However, since being harmonic implies being at least C^1, by proposition 2.2.3 the function is holomorphic.

Definition 2.2.4: Harmonic Conjugate Function

If u is a harmonic function and v is the harmonic function that makes $f = u + iv$ an holomorphic function, then v is called the *harmonic conjugate of* u. In particular, u, v are harmonic functions satisfying the Cauchy-Riemann equations

Example 2.3: Finding Conjugate harmonic Function

Let $u(x, y) = x^2 - y^2$. This is a harmonic function, and so we can proceed in finding it's harmonic compliment. Notice that

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

Thus

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

We can integrate one or the other equation to solve for v. To make an arbitrary choice, let's pick the first one and integrate it with respect to x to get

$$v = 2xy + \varphi(y)$$

where $\varphi(y)$ is some functions based on y. Deriving with respect to y gives us

$$\frac{\partial v}{\partial y} = 2x + \varphi'(y) = 2x$$

telling us that $\varphi'(y) = 0$ so $\varphi(y) = 0$. Thus we get

$$f(z) = f(x + iy) = x^2 + y^2 + i(2xy) = z^2$$

You can now verify that $f(z) = x^2 + y^2 + i2xy$ is indeed holomorhpic. Check that this function is actually $f(z) = z^2$. What's the derivative? Remember that you can write it in terms of partial derivatives of u and v. Can you write it in terms of z?

For reference, here is a list of harmonic conjugates

u	v	$u + iv$
x	y	z
x	$y+1$	$z+i$
y	$-x$	$-iz$
$x^2 - y^2$	$2xy$	z^2
$e^x \cos(y)$	$e^x \sin(y)$	e^z
$\frac{x}{x^2+y^2}$	$\frac{-y}{x^2+y^2}$	$\frac{1}{z}$

Note that for the last example we must exclude the origin from the domain. Being harmonic gives us some important regularity on functions maxima's and minima's:

Theorem 2.2.2: Maximum Principle Of Harmonic Functions

Let U be an open subset of $\mathbb{C}$ and let $u : U \to \mathbb{R}$ be a harmonic function. Let K be a compact subest of U, and let ∂K be the boundary of K. Then:

$$\sup_{z \in K} u(z) = \sup_{z \in \partial K} u(z)$$

and

$$\inf_{z \in K} u(z) = \inf_{z \in \partial k} u(z)$$

Proof **:**
We'll prove it for sup, since inf follows similarly (note that $-u$ is also harmonic). It is always the case that $\sup_{z \in K} u(z) \geq \sup_{z \in \partial K} u(z)$, so for the sake of contradiction let's say

$$\sup_{z \in K} u(z) > \sup_{z \in \partial K} u(z)$$

As u is continuous and K is compact, u achieves it's maximum at some point, say $z_0 \in K$. By our assumption, z_0 is in the interior. Since z_0 is a local maximum of u, and u is twice differentiable, we must have:

$$\frac{\partial^2 u}{\partial x^2}(z_0) \leq 0 \qquad \frac{\partial^2 u}{\partial y^2}(z_0) \leq 0$$

This almost contradicts the harmonicity of u, but it is still possible that both of these partial derivatives vanish. To fix this, we wiggle around an epsilon amount of room to add some convexity. Letting $\epsilon > 0$ be a small number we'll choose later, let $u_\epsilon : U \to \mathbb{R}$ be the modified function

$$u_\epsilon(x + iy) := u(x + iy) + \epsilon(x^2 + y^2)$$

Since K is compact, the function $x^2 + y^2$ is bounded on K. If ϵ is small enough, then by our assumption on u we must also have on u_ϵ the relation:

$$\sup_{z \in K} u_\epsilon(z) > \sup_{z \in \partial K} u_\epsilon(z)$$

By the same argument u_ϵ achieves its maximum at some interior point z_ϵ of K, thus:

$$\frac{\partial^2 u}{\partial x^2}(z_\epsilon) \leq 0 \qquad \frac{\partial^2 u_\epsilon}{\partial y^2}(z_\epsilon) \leq 0$$

Sine u is harmonic, we must have:

$$\frac{\partial^2 u_\epsilon}{\partial x^2} + \frac{\partial^2 u_\epsilon}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + 2\epsilon + \frac{\partial^2 u}{\partial y^2} + 2\epsilon = 4\epsilon > 0$$

on U. But that's a contradiction, completing the proof.

Corollary 2.2.1: Maximum Principle For Holomorphic Functions

Let $f : U \to \mathbb{C}$ be a continuously twice differentiable holomoprhic function on an open set U an K a compact subset of U. Then

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|$$

This reuslt is also known as the *maximum modulus principle*

Proof **:**
Use the fact that $|w| = \sup_{\theta \in \mathbb{R}} \mathrm{Re}(we^{i\theta})$.

There is naturally a *minimum modulus principle* which has the same result but with inf.

Proposition 2.2.5: Consequences Of Harmonic Functions

Let $U \subseteq \mathbb{C}$ be an open subset.

1. If $f : U \to \mathbb{C}$ is twice continuously differentiable

$$\nabla f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial f}{\partial z}$$

2. If f is a complex polynomial:

$$f(z) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} c_{n,m} z^n \overline{z}^m$$

   then f is harmonic on $\mathbb{C}$ if and only if $c_{n,m}$ vanishes whenever n and m are both positive (f only contains terms $c_{n,0}z^n$ or $c_{0,m}\overline{z}^m$)

3. if $u : U \to \mathbb{R}$ is a real polynomial

$$u(x+iy) = \sum_{\substack{n,m\geq 0\\ n+m\leq d}} a_{n,m} x^n y^m$$

   then u is harmonic if and only if it is the real part of a complex polynomial

$$f(z) = \sum_{n=0}^{d} c_n z^n$$

Proof **:**
exercise

We leave with a generalization of the notion of harmonic conjugate. We saw that every harmonic polynomial has at least one harmonic conjugate (up to constant). We may ask whether this holds for more general harmonic functions. If $U = \mathbb{C}$ is the entire plane, this is indeed the case:

Proposition 2.2.6: Entire Harmonic Conjugate On $\mathbb{C}$

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function. Then there exists a harmonic conjugate $v : \mathbb{C} \to \mathbb{R}$ of u. Furthermore, this harmonic conjugate is unique up to constants: if v, v' are two harmonic conjugate of u, then $v' - v$ is a constant function

Proof **:**
proposition 28 here

This proof generalizes for some other domains like rectangles, but is left for when we have built up the notion of contour integrals. In some cases (in particular, when U is not simply connected), harmonic functions will not have harmonic conjugates!

2.2.3 Properties of Holomorphic Functions

In this section, we upgrade some classical results of real-differentiability to complex differentiability taking advantage of the fact that f must satisfy the C.R. Equations. We then show a very important geometrical property of holomorphic functions that we mentioned in section 1.3.1, namely that they must be *conformal*.

Proposition 2.2.7: Derivative Zero Then Constant

Let f be holomorphic and $f' = 0$. Then f is constant

Proof **:**
Since f is holomorphic and $f' = 0$:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = 0$$

Thus, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Now the standard real-differentiable argument shows that f is constant.

(Ahlfors has a more elaborate proof that doesn't use the real result on p.72)

Proposition 2.2.8: Equivalent to Derivative Zero

Let f be holomorphic. Then:

1. If $|f|$ is constant, then f is constant
2. If $\operatorname{Re} f$ is constant, then f is constant

Note how this contrasts to a real-differentiable function $f : U \subseteq \mathbb{R}^2 \to \mathbb{C}$, if $|f|$ is constant, this does not imply f is constant (think $U = \mathbb{R}^2 \setminus (0,0)$ and $(x,y) \mapsto (x,y)/\sqrt{x^2+y^2} = (x,y)/\|(x,y)\|$), and similarly for $\operatorname{Re} f$ (which we can think of as taking the function $\pi \circ f$ with any nonconstant function f)

Proof **:**

1. $|f|^2 = f\overline{f}$. Then taking the Wirtinger derivative on both sides and noting that $\frac{\partial \overline{f}}{\partial z} = 0$, we get

$$0 = \frac{\partial f}{\partial z} \cdot \overline{f}$$

   If $\overline{f} = 0$ at some point, say p then $|f(p)| = 0$, but $|f|$ is constant so $|f| = 0$, and so $f = 0$ is constant. Thus, assume it's nonzero everywhere, so $\frac{\partial f}{\partial z} = 0$. But then by proposition 2.2.7 f is constant.

2. We have $\operatorname{Re} f = \frac{1}{2}(f + \overline{f})$ where the left hand side is constant. Then

$$0 = \frac{1}{2}\frac{\partial f}{\partial dz}dz + \frac{\partial \overline{f}}{\partial z}dz$$

cancelling terms out we get:
$$0 = \frac{\partial f}{\partial z} dz$$
completing the proof.

As an exercise show that if $\log|f|$ or $\arg(f)$ is constant, then f is constant.

One important result that is "invisible" in $\mathbb{R}$ but becomes apparent in $\mathbb{C}$ is that (real and complex) differentiable functions preserve angles! Being complex differentiable will also mean the function perserves orientation:

Definition 2.2.5: Conformal Maps

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Then f is said to be *conformal* for every z_0 in the domain, f preserves the angles between two curves passing through z_0 and perserves orientation. If f does not perserve orientation, it is called *homothetic* or *angle-preserving*.

A conformal map may be interpreted as preserving the shape of any sufficiently small figure, while possibly rotating and scaling (but not reflecting) it. If f is holomorphic, we will says it's conformable if $Df(a)$ interpreted as a real 2 by 2 matrix is an angle-preserving map and orientation preserving. If it is simply angle-preserving, then *a priori* we don't know if if f will filp-flop bebetween orientations. If f is additionally at least real C^1, then f *has* to strictly stay within one orientation:

Proposition 2.2.9: C^1 and Angle Preserving Functions

Let $\Omega \subseteq \mathbb{C}$ be a conencted open set, $f : \Omega \to \mathbb{C}$ a real C^1 function with a nonzero Jacobian determinant at every point. Then if f preserves angles at every point of Ω (that is, if it is homothetic), then either:
$$\frac{\partial f}{\partial z} = 0 \qquad \text{or} \qquad \frac{\partial f}{\partial \overline{z}} = 0$$
where both can't be zero (or else the jacobian's determinant is zero). In the second case f is holomorphic, in the first case f is anti-holomorphic (locally $\overline{\mathbb{C}}$-linear)

***Proof* :**
Since f is $\mathbb{R}$-differnetiable, Df exists. Since it is angle preserving, by lemma 1.3.1 Df is either of the form:
$$Df(x)(p) = ap \qquad \text{or} \qquad Df(x)(p) = a\overline{p}$$
that is, it is either a rotation, or a rotation and a reflection. Consider the sets:
$$\left\{ z \in \Omega \;:\; \frac{\partial f}{\partial z}(z) = 0 \right\} \qquad \left\{ z \in \Omega \;:\; \frac{\partial f}{\partial \overline{z}}(z) = 0 \right\}$$
These two sets are disjoint, since if both $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0$, then f a zero Jacobian, contradiction the assumption of the quesiton. Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, the sets
$$\left\{ \frac{\partial f}{\partial z} = 0 \right\} \qquad \left\{ \frac{\partial f}{\partial \overline{z}} = 0 \right\}$$

are closed sets. By what we've shown they are disjoint and their union is Ω. By connectivity of Ω, one of these sets must be empty.

This compelte the proof. If the second one is empty, then Df is $\mathbb{C}$-linear and hence f is holomorphic. If the first one is empty, then Df is anti $\mathbb{C}$-linear (that is $Df_p(az) = \overline{a}Df_p(z)$) and f is called *anti-holomorphic*.

We can re-state the theorem a bit by saying that the Jacobian determinant is nonzero and f is a C^1 angle-preserving function if and only if f is either (strictly) holomorphic or anti-holomorphic. In the above theorem, we have assumed that being locally $\mathbb{C}$-linear is sufficient to be conformal, and indeed it is. However, we there is a nice proof that goes about this more directly that I think is worth showing to show that it is in fact a very simple result:

Proposition 2.2.10: Holomorphic Then Conformal

Let f be a holomorphic function. Then f is a conformal map.

***Proof* :**
let $\gamma : (-\epsilon, \epsilon) \to U \subseteq \mathbb{C}$ be some differentiable cruve with $\gamma(0) = z_0$ and $f : U \to \mathbb{C}$ a holomorphic function. We can think of $\gamma'(0)$ as being the velocity vector of a particle passing through z_0. Take $f \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{C}$. By the chain rule:

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$$

Representing $f'(z_0)$ in polar coordinates we get $f'(z_0) = re^{i\theta}$. Thus, we get:

$$(f \circ \gamma)'(0) = re^{i\theta}\gamma'(0)$$

that is, f transforms the velocity vector by multiplying the speed by a factor of r and rotating it counte-clockwise by a fixed angle θ. If we know consider two trajectories γ_1, γ_2 passing through z_0 at $t = 0$, then the map f will preserve the angle between the velocity vector $\gamma_1'(0)$ and $\gamma_2'(0)$ as well as their orientation, completing the proof.

This gives us another way of eliminating non-holomorphic functions. For example $f(x + iy) = x + i(x + y)$ preserves orientation, but not angle, while $f(z) = \overline{z}$ preserves angle, but not orientation. Note too how we in fact have the same angle-preserving phenomena happening for real-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ for functions of the form $f : I \to \mathbb{R}$, but this is much less interesting since the only two possible angles between velocity vectors are 0 and π; this shows the 2-dimensionality of the complex plane makes conformality a much more "rigid" property for complex differentiable functions.

The final result we shall state shall build upon our earlier observation's that the derivative of f resemble the Jacobi identity, making it tempting to say that holomorphic implies continuously holomorphic. This may be jumping the gun, but for the next theorem we shall assume that this is indeed the case (in section 3.2 we shall prove it). We shall use this to show the complex version of the inverse function theorem. This theorem naturally applies holomorphic functions since a holomorphic function is already real-differentiable, but we would want the inverse to also be complex differentiable:

Theorem 2.2.3: Holomorphic Inverse Function Theorem

Let f be holomorphic in a neighbourd of z_0 and $f'(z_0) \neq 0$. Then there exits a U, v, $z_0, f(z_0) = w_0$ such that $f|_U$ is a homeomoprhism onto V with inverse $g : f(U) \to U$. Furthermore, g is holomorphic and

$$g'(w) = \frac{1}{f'(g(w))} = \frac{1}{f'(z)}$$

where $w = f(z)$

Proof **:**
With our assumption that holomorphic implies continuously holomorphic, the real version of the inverse function gives us all but g being complex-differentiable. Sicne f is holomorphic, we have that in the standard basis we may represent $f'(z)$ as:

$$[f'(z)] = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then we compute:

$$g'(w) = [f'(z)]^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which shows g satisfies the Cauchy-Riemann equations, and thus is holomorphic, completing the proof.

For the Implicit function theorem, we have the following weaker version.

Theorem 2.2.4: Holomorphic Implicit Function Theorem

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a C^1 function that is separately holomorphic (holomorphic given either fixed x or y). Then if $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, There exists a holomorphic function y defined on a sufficinetly small interval where $y(x_0) = y_0$) (similarly if we take $\frac{\partial f}{\partial x} \neq 0$)

Proof **:**
This essentially is just about reducing it to the real case and then verifying our result is holomorphic. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$, $f(x, y) = z$, $z = z_1 + iz_2$. Then for fixed x:

$$dz = \frac{\partial f}{\partial y} dy \qquad d\bar{z} = \frac{\partial \bar{f}}{\partial \bar{y}} d\bar{y}$$

Hence:

$$dz \wedge d\bar{z} = \left| \frac{\partial f}{\partial y} \right|^2 dy \wedge d\bar{y}$$

Now, if we expand $dy = dy_1 = idy_2$ and $d\bar{y} = dy_1 - idy_2$, we get

$$dz \wedge d\bar{z} = \left| \frac{\partial f}{\partial y} \right|^2 dy_1 \wedge d\bar{y}_2$$

So
$$\det\left(\frac{\partial(z_1,z_2)}{\partial(y_1,y_2)}\right)(x_0,y_0)\neq 0$$
Thus, by the real implicit function theorem, there exists a C^1 function $y(x)$ such that on a small enough region
$$f(x,y(x))=0$$
Taking the derivative with respect to to x, we get
$$0=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial x}dx+\frac{\partial y}{\partial \bar{x} d\bar{x}}\right)$$
and so, by linear independence, we get $\frac{\partial y}{\partial \bar{x}}=0$, showing y is holomorphic.

2.2.4 Polynomial and Rational Functions

As we saw in example 2.1, constant and linear functions are holomorphic. By proposition 2.2.1, the sum and product of holomorphic functions is holomorphic. Therefore, all polynomials are holomorphic
$$P(z)=a_0+a_1z+\cdots+a_nz^n$$
with derivative
$$P'(z)=a_1+2a_2z+\cdots+na_nz^{n-1}$$
To not repeat the same restrictions in the following theorems, if we write out a polynomial as we have done for $P(z)$, we will assume $a_n\neq 0$, and 0 is not a polynomial [5].

Found this video (commented to compile) :

A defining feature of polynomials is that they are defined by their roots (up to a constant) along with one more point. Having theorems on how the roots behave under differentiation will give us some good insights on polynomial. The following two theorems give geometric insight on the roots:

Theorem 2.2.5: Roots Of Derivative Within Convex Hull (Gauss-Lucas Theorem)

Let P be a polynomial. Then the roots of P' are within the convex hull of the roots of P

***Proof* :**
The proof is broken down into steps:

1. Suppose that $P(z)$ has degree n and zeros $b_1,...,b_n$ (each zero listed as many times as multiplicity). Show that
$$\frac{P'(z)}{P(z)}=\sum_{k=1}^{n}\frac{1}{z-b_k}$$

   Proof. This is simply a matter of computation. First, by the fundamental theorem of algebra:
$$P(z)=c(z-b_1)\cdots(z-b_n)$$

[5] for formal reasons we will soon see, it's degree will have to be $-\infty$, see ref:HERE

then:

$$P'(z) = c \sum_{i=1}^{n} \prod_{k \neq i} (z - b_k)$$

Then dividing, we get:

$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - b_k}$$

as we sought to show $\square$

2. Show that if $P'(z) = 0$, then

$$\left(\sum_{k=1}^{n} \frac{1}{|z - b_k|^2}\right)\overline{z} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$

Proof. If $P'(z) = 0$, then:

$$0 = \sum_{k=1}^{n} \frac{1}{z - b_k}$$

Now, for each term in the summand, multiply it by $\frac{\overline{z} - \overline{b_k}}{\overline{z} - \overline{b_k}}$ and the summand in two:

$$0 = \sum_{k=1}^{n} \frac{\overline{z}}{|z - b_k|^2} - \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$

Moving the values around, we get:

$$\left(\sum_{k=1}^{n} \frac{1}{|z - b_k|^2}\right)\overline{z} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z - b_k|^2}$$

as we sought to show $\square$

3. Deduce that if $P'(z) = 0$, then z lies within the convex hull of the points b_k.

Proof. Let $P'(z_0) = 0$. Then by part (b):

$$\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right)\overline{z_0} = \sum_{k=1}^{n} \frac{\overline{b_k}}{|z_0 - b_k|^2}$$

Conjugating both sides and isolating z_0, we get:

$$z_0 = \frac{\sum_{k=1}^{n} \frac{b_k}{|z_0 - b_k|^2}}{\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right)} = \sum_{k=1}^{n} \frac{\frac{b_k}{|z_0 - b_k|^2}}{\left(\sum_{k=1}^{n} \frac{1}{|z_0 - b_k|^2}\right)} = \left(\sum_{k=1}^{n} \frac{\frac{1}{|z_0 - b_k|^2}}{\left(\sum_{j=1}^{n} \frac{1}{|z_0 - b_j|^2}\right)} b_k\right)$$

where the value in the summand is non-negative. Notice here that the coefficients of the b_k's add up to 1. Since the b_k's form a convex hull, this is a convex combination, and so z_0 must be in the convex hull, as we sought to show. $\square$

A rational function of particular interest is the Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

Note that since

$$f^{-1}(w) = \frac{dw - b}{-cw + a}$$

then f is in fact *biholomorphic*, meaning it is in fact a symmetry of $\hat{\mathbb{C}}$. In fact, $\mathrm{Aut}(\hat{\mathbb{C}})$ is the collection of all Möbius transformation. If you did not take differential geometry, then the following discussion may be skipped and the statement may be taken for granted for now. Recall that $S^2 \cong \mathbb{C}\mathrm{P}^1$ as smooth manifolds. Thinking of S^2 as the Riemann sphere $\hat{\mathbb{C}}$, we see that finding the symmetries of S^2 is finding the symmetries of $\mathbb{C}\mathrm{P}^1$, i.e. the automorphism group of complex diffeomorphisms. Recall that $\mathrm{GL}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})$ and let $\mathrm{GL}(\mathbb{C}) \curvearrowright \mathbb{C}$ in a natural way. We may descend this action to $\mathrm{PGL}(\mathbb{C}) \curvearrowright \mathbb{C}\mathrm{P}^1$ by taking $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/Z(\mathbb{C})$ where $Z(\mathbb{C})$ is the center of $\mathrm{GL}_2(\mathbb{C})$, i.e. the collection of all diagonal matrices. To see what this descension looks like, take

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(\mathbb{C})$$

Then:

$$[z, 1] \begin{pmatrix} a & c \\ b & d \end{pmatrix} = [az + b, cz + d] = \left[\frac{az+b}{cz+d}, 1\right]$$

which is exactly a Möbius transformation! Thus, we see that the collection of Möbius transformations, which we may label as $\mathrm{PGL}(\mathbb{C})$, are all the automorphisms (i.e. the biholomorphic functions) of $\hat{\mathbb{C}}$.

2.3 Power Series

Since polynomials are holomorphic functions, it is natural to ask whether the completion of polynomials, power series[6], are also holomorphic functions. The answer was revealed earlier to be yes when we said that another common name for holomorphic functions are analytic functions (since an analytic function is locally given by a convergent power-series).

Definition 2.3.1: Formal Power Series

Let z_0 be any complex number. Then a *formal power series* with complex coefficients around the point z_0 is a formal series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some complex numbers $a_0, a_1, \ldots$, with z an indeterminate

These are called “formal” since we require to establish an actual domain and codomain in order for them to be functions, at which point we usually call them power series. Another way of thinking about power series is that they are the completion of the ring of polynomials $\mathbb{C}[z - z_0]$, and hence they

[6] in particular $\varprojlim (k[x]/(x)^n) = k[[x]]$

are the "formal" result of this completion. We can naturally try and define an evaluation function by replacing z with some value in $\mathbb{C}$ and see if it converges. The following theorem gives us a systemic way of finding which values can be plugged in:

Theorem 2.3.1: Abel's Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be a power series. Then there exists a number R, $0 \le R \le \infty$, called the *radius of convergence*, with the following properties:

1. The series converges absolutely for every z with $|z| < R$. If $0 \le \rho < R$, the convergence is uniform for $|z| \le \rho$.
2. If $|z| > R$, the terms of the series are unbounded, and the series thus diverges
3. In $|z| < R$, the sum of the series is a holomorphic function. The derivative is obtained by term-wise differentiation, and the derived series has the same radius of convergence.

Note that when $|z| = R$, the result depends on the power-series[7]. This circle is called the *circle of convergence*.

Proof **:**
Without loss of generality, let $z_0 = 0$ (this is simply shifting the power series over to 0, and doesn't effect convergence). Recall the important formula by *Hadamards* that relates R to given the coefficients of a power-series:

$$1/R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

If $|z| < R$, we can find a ρ such that $|z| < \rho < R$. Then $1/\rho > 1/R$, and by the definition of the limit superior, there exists an n_0 such that $|a_n|^{1/n} < 1/\rho$, meaning

$$|a_n| < \frac{1}{\rho^n} \qquad \forall n \ge n_0$$

Thus:

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \qquad \forall n \ge n_0$$

and for any fixed z, the right hand side is a geometric series which for $\rho > 1$ converges by Hadamards formula. To show uniform convergence for $|z| \le \rho < R$, we take advantage of Weiestrass M-test. Pick a ρ' with $\rho < \rho' < R$. Then we can get for large enough n_0:

$$|a_n z^n| \le \left(\frac{\rho}{\rho'}\right)^n \qquad n \ge n_0$$

Since the majorant (the power series given by the right hand side) is convergent and has constant terms, by the Weiestrass's M test the power series is uniformly convergent. If $|z| > R$, then we can do the same manipulation we've done before but instead choose $R < \rho < |z|$ to get

$$|a_n z^n| > \left(\frac{|z|}{\rho}\right)^n \qquad \forall n \ge n_0$$

[7]This problem is related to the same tricky problem of asking when a Fourier series converges

Thus, fixing an n_0, we get that

$$|a_{n_0} z^{n_0}| + |a_{n_0} z^{n_0}| < |a_{n_0} z^{n_0}| + |a_{n_0+1} z^{n_0+1}| + \cdots$$

showing the right hand side is unbounded, and all $|z| > R$ diverge.

Next, we will show that the derived series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence. First, we will show that $\sqrt[n]{n} \to 1$

Proof. Set $\sqrt[n]{n} = 1 + \delta_n$. Then $\delta_n > 0$, and by the binomial theorem

$$n = (1 + \delta_n)^n > 1 + \frac{1}{2} n(n-1)\delta_n^2$$

Manipulating, we get $\delta_n^2 < 2/n$, and so $\delta_n 0 > 0$. □

We now continue the proof. For $|z| < R$, decompose the power-series two:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = s_n(z) + R_n(z)$$

where

$$\begin{aligned} s_n(z) &= a_1 + a_1 z + \cdots + a_{n-1} z^{n-1} \\ R_n(z) &= \sum_{k=n}^{\infty} a_k z^k \end{aligned}$$

write the "desired" derivative of f as

$$f_1(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \lim_{n\to\infty} s_n'(z)$$

we want to show that $f'(z) = f_1(z)$. To that end, write:

$$= \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \tag{2.3}$$

$$= \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0) \right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0} \right) \tag{2.4}$$

where we naturally assume that $z \neq z_0$ and both $|z|, |z_0|$ are $< \rho < R$. Recalling that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$, the last term in the above can be rewritten as

$$\sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_) + \cdots + z z_k^{k-1} + z_0^{k-1})$$

since $|z|, |z_0| < \rho$ we have:

$$\left| \frac{R_n(z) - R_n(z_0)}{z - z_0} \right| \leq \sum_{k=n}^{\infty} k |a_k| \rho^{k-1}$$

the right hand side of the above expression is the remainder term of a convergent series, and so we can find a large enough n_0 such that

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| < \frac{\epsilon}{3}$$

The same reasoning can be applied For the second term in equation (2.4) for some $n \geq n_1$. For the 1st term, choose some $n \geq n_0, n_1$. Then by definition of the derivative there exists a $\delta > 0$ such that for $0 < |z - z_0| < \delta$ implies

$$\left|\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right| < \frac{\epsilon}{3}$$

Combining all these equations, we get that:

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0)\right| < \epsilon$$

when $0 < |z - z_0| < \delta$. But then that is exactly the condition we need to show that $f'(z_0)$ exists and is equal to $f_1(z_0)$, completing the proof.

The last reasoning can be repeated indefinitely, meaning a power series with positive radius of convergence has derivatives of all order with the same radius of convergence. Now that we have a radius of convergence, we can define the *power series* of a formal power series to be the function $F : B_R(z_0) \to \mathbb{C}$ such that

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

We will most the time not make a distinction between formal power series and power series and call them both power series unless the situation calls for nuance. At this point, it is a good idea to recall some tests to check whether a power series converges:

1. **root test**: if $\limsup_{n\to\infty}(a_n)^{1/n} < 1$, then the series converges, if > 1 it diverges, and if $= 1$ it is inconclusive.
2. **Ratio test** : if $\lim_{N\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series converges. if > 1 it diverges, and $= 1$ it is inconclusive.
3. **Comparison test**: If $\sum_{n=0}^{\infty} a_n z^n$ converges and $b_n < a_n$, then $\sum_{n=0}^{\infty} b_n z^n$ converges
4. **Integral test**: If $f[1, \infty) \to \mathbb{R}_+$ is a non-negative monotonically decresing function such that $f(n) = a_n$, then if
$$\int_1^{\infty} f(x)dx < \infty$$
then the series also converges, and if it diverges so does the series
5. If $\sum_i^{\infty} |a_n||z^n|$ converges for some z, so does the power series. If $a_n \nrightarrow 0$, then the $\sum_n^{\infty} a_n z^n$ for any z does not converge.

We mentioned that what happens at the border is not determined. We in fact get some subtle behavior that is worth going into. Take the power sries $\sum_{n=0}^{\infty} z^n$. Then it has radius of convergence of 1. if $z \in B_1(0)$, then since the series converges uniformally we have:

$$\begin{aligned} z \sum_{n=0}^{\infty} z^n &= \sum_{n=0}^{\infty} z^{n+1} \\ &= \sum_{n=1}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} z^n - 1 \end{aligned}$$

Thus, we some algebraic manipulation we get the closed form of:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

It is evident to see that as long as $z \in B_1(0)$ the function $z \mapsto \frac{1}{1-z}$ does not extend continuously to the boundary point $z = 1$ of the disk. however, it *does* extend continuously, even smoothly, to the rest of the boundary. In fact, even more remarkably (though perhaps not so remarkable if we take a moment to think about it), it can be *holomorphically extended* to all of $\mathbb{C} \setminus \{1\}$. However, $\sum_{n=0}^{\infty} z^n$ diverges at every point in the boundary (when $|z| = 1$, the coefficeints of z^n of the series do not converge to zero), and evidentily it diverges outside the unit ball. Thus, we see that the function that matches the power series can be well-defined much beyond the radius of convergence of the power series. We'll prove in the next proposition (proposition 2.3.1) that the power series matching $1/(1-z)$ is unique and hence we can't find a power series that somehow fixes this.

We can play around with the fact that $\sum_{n=0}^{\infty} z^n$ and $\frac{1}{1-z}$ must be uniquely identified by *formally* identifying the two on all of $\mathbb{C} \setminus \{1\}$. For example, if we plug in $z = 2$, then this identification would lead us to say that:

$$1 + 2 + 2^2 + 2^3 + \cdots = -1$$

which under our current understanding of convergence would certainly be absurd. However, there is a way of interpreting this beyond the classical notion of convergence, leading to concepts like the generalised summation methods ilke the zeta function regularisation. These will be further discussed in ref:HERE.

(see also Tao's blog here and here (example 11))

Proposition 2.3.1: Taylor Expansion Is Unique

Let F, G be two power series centered at z_0 that agree on some neighbourd U. Then the coefficients of each term are equal. In particular, the Taylor series expantion is unique.

Note that if the point around which the power series is centered is different, we can no longer compare coefficients easily. For example, we can see that both $\sum_{n=0}^{\infty} z^n$ and $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(z+1)^n$ both converges to $\frac{1}{1-z}$ on $B_1(0)$ but have diferent coefficients.

Proof **:**
exercise?

(in this article, exercise 18 gives a way to go back and forth betwen two power series centered around different points)

As we have seen, the power series (as a function) can be well behaved as one approaches the boundary of the disk of convergence, while being divergent at the boundary. However the converse of this, where the power series converges at the boundary but does not behave well (as a function) as one approaches the boundary, does not occur:

Theorem 2.3.2: Abel's Limit Theorem

If $\sum_0^\infty a_n$ converges, then $f(z) = \sum_0^\infty a_n z^n \to f(1)$ as z approaches 1 in such a way that

$$\frac{|1-z|}{1-|z|}$$

remains bounded (sometimes known as the *Stolz angle*) or *non-tangent angle approach.*

Geometrically, this condition can be interpreted to mean that as we approach 1, we do so in such a way that the angle our approaching curve is taking is not tangent to the point 1, see the following link here.

Proof **:**
We amy assume $\sum_0^\infty a_n = 0$ by shifting over the sequence and have $\sum_0^\infty a_n$ be a_0. Writing the partial sum $s_n = a_0, a_1, ..., a_n$, we get

$$\begin{aligned} s_n(z) = a_0 + a_1 z + \cdots + a_n z^n \\ &= s_) + (s_1 - s_0)z + \cdots + (s_n - s_{n-1})z^n \\ &= s_0(1-z) + s_1(z - z^2) + \cdots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\ &= (1-z)(s_0 + s_1 z + \cdots + s_{n-1}z^{n-1}) + s_n z^n \end{aligned}$$

Since $s_n z^n \to 0$, we get the representation

$$f(z) = (1-z)\sum_0^\infty s_n z^n$$

Next we are assuming that $|1-z| \leq K(1-|z|)$ for some choice $K \in \mathbb{R}$. Since $s_n \to 0$, choose m large enough so that $|s_n| < \epsilon$ for $n \geq m$. The remainder of the series $\sum s_n z^n$ form $n = m$ onwards, is the ndomianted by the geometric sers

$$\epsilon \sum_m^\infty |z|^n = \epsilon \frac{|z|^m}{1-|z|} < \frac{\epsilon}{1-|z|}$$

It follows that

$$|f(z)| \leq |1-z| \left| \sum_0^{m-1} s_k z^k \right| + K\epsilon$$

The term on the right hand side can be made arbitrarily small by choosing z sufficinetly close to 1, and so we can conclude that $f(z) \to 0$ as $z \to 1$ subject to our given Stolz angle restriction.

We continue now to explore the properties of power series. Important operations we are used to on our functions is adding, scaling, multiplying, inverting, or composing them. We shall see how these operations interact with power series. One important comment must be made: the problem of convergence of the composition of power is one that is still being investigated. In particular, sufficient condition's for convergence is an area of research, I found this paper (here) . Hence, to avoid such problems when they arise, we may treat our power series as formal power series, the derivative as a formal operation, and define $f(0) := a_0$ (since we are not treating f as a function).

Proposition 2.3.2: Composing Power Series

Let f, g be formal power series. Then $g(f(z))$ is well-defined if $a_0 = 0$.

***Proof* :**
When composing power series, we get:

$$a_1(b_0 + b_1 z + b_2 z^2 + \cdots) + a_2(b_0 + b_1 z + b_2 z^2 + \cdots)^2 + \cdots$$

where each coefficient is an infinite sum, and so we must ask for their convergence conditions. One thing we can do to guarantee this is by setting $a_0 = 0$, which will guarantee that all the coefficients are finite sums and hence well-defined. Note that as we are working with formal power series, there is no issue in re-arranging terms.

Theorem 2.3.3: Inverse Function Theorem On Formal Power Series

Let $f(z)$ be a formal power series. Then there is a formal power series $g(z)$ such that $b_0 = 0$ and $f \circ g = \mathrm{id}$ (where id is the power series $\mathrm{id}(z) = z$) if and only if $a_0 = 0$ and $f'(0) = 0$. In this case, g is unique and $g \circ f$ is also the identity. If f has positive radius of convergence, so does g

***Proof* :**
We want to a g such that $f(g(z)) = z$. We want:

$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z = b_2 z^2 + \cdots)^2 = z$$

expanding and using the method of undetermined coefficients expanding the first two terms we get:

$$a_0 = 0 \qquad a_1 b_1 = 1$$

This shows us that $a_0 = 0$ and $a_1 = f'(0) \neq 0$ are necessary conditions. To show they are sufficient conditions, we can see that we may deduce all other coefficeints given these conditions.For example, we may deduce the coefficient of z^n as $a_0 + a_1 g(z) + \cdots + a_n g(z)^n$, so

$$a_1 b_n = P_n(a_2, ..., a_n, b_1, ..., b_{n-1})$$

Thus, we would start by doing $b_1 = 1/a_1$, and $b_2, b_3, ...$ are defiend recursively (this resulting polynomials is called *Bells polynomial*). With our construction, we have that g satisfies $b_0 = 0$ and $b_1 \neq 0$. We can do the same thing to find the other-side inverse, that is $g(f_1(w)) = w$. These inverses are equal:

$$f_1 = \mathrm{id} \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1) = f \circ \mathrm{id} = f$$

where we assumed the associativity of composition, something that can be checked. Finally, for the radius of convergence, we shall delay such considerations for now, noting that when we show that holomorphic functions are analytic we can use the inverse function theorem to estimate the values.

Proposition 2.3.3: Convergence On Operations Of Powerseries

Let f, g be convergent power series with radius of convergence $R(f)$ and $R(g)$. Then $g \circ f$ converges too. In particular, if we take some $r > 0$ so that $\sum_{n=0}^{\infty} |a_n| r^n < R(g)$ (i.e. is less then the absolute power series), then

1. $R(g \circ f) \geq r$
2. $|f(z)| < R(f)$ if $|z| < r$
3. $(g \circ f)(z) = g(f(z))$

***Proof* :**
This is simply finding good bounds using the real power series. If we have

$$(g \circ f)(z) = \sum_p^{\infty} b_p \left(\sum_n^{\infty} a_n z^n \right)^p = \sum_k^{\infty} c_k z^k$$

then taking the aboslute value and using the natural generalization of the triangle inequality:

$$\left| \sum_p^{\infty} b_p \left(\sum_n^{\infty} a_n z^n \right)^p \right| \leq \sum_p^{\infty} |b_p| \left(\sum_n^{\infty} |a_n| |z|^n \right)^p = \sum_k^{\infty} \gamma_k |z|^k$$

where we evidently see by the use of the triangle inequality that $|c_k| \leq \gamma_k$. Thus, using the limi comparison test, we get that $(g \circ f)$ convreges absolutely if $|z| < r$, telling us that the radius of convergence of $(g \circ f)$ Is at lesat r and that $|f(z)| < R(g)$. The final fact now becomes the fact that we may re-arrange the terms of an absolutely convergent series.

Proposition 2.3.4: Operations On Power Series

Let f, g be power series with radius of convergence R_1 and R_2:

$$\sum_{n=1}^{\infty} a_n z^n \qquad \sum_{n=1}^{\infty} b_n z^n$$

Then:

1. $f + g$ is a power series given by term-wise addition with radius of convergence of at lesat $\min(R_1, R_2)$ (though it may be much larger)
2. fg is a power series with
$$\left(\sum_{k}^{\infty} a_n (z-c)^k\right)\left(\sum_{k}^{\infty} b_k (z-c)^k\right) = \sum_{k}^{\infty}\left(\sum_{i}^{\infty} a_i b_{k-1}\right)(z-c)^k$$
3. If $a_0 \neq 0$, then there is a unique power series such that $f(z)g(z) = 1$. If f has a positive radius of convergence, so does g

Proof **:**

1. here
2. here
3. Without loss of generailty we may assume $a_0 = 1$ by dividing by a_0. Then we may write $f(z) = 1 - h(z)$ where $h(z)$ is now a power series satisfying $h(0) = 0$. Then it is a classical result that
$$\frac{1}{1-w} = 1 + \sum_{n=1}^{\infty} w^n$$
substituting $w = h(z)$, we get $g(z) = (1 - h(z))^{-1}$. For the radius of convergence, we may use the same trick as used in the inverse function theorem for power series.

2.4 Exponential, Trigonometric, and Logarithmic Function

We know explore 3 important examples of power series which will come up again and again. They will all be based on the single power series which will be known as the *exponential function.*

There are many reasons on why the exponential function is a very nice function, some of them being:

1. It is smooth and defined on all of $\mathbb{C}$
2. It's derivative is itself (even when defined over $\mathbb{C}$)
3. it has no roots; though it's an "infinite polynomial", there is no value of $z \in \mathbb{C}$ such that $e^z = 0$.

4. The real part of e^z is injective, while e^z shall surject onto $\mathbb{C}^\times$ (as we shall soon show)
5. The exponential shall encapsulate the "rotational" behavior of complex numbers, and shall give us new coordinates in which we may work in (disregarding 0)

While all these facts may already be familiar, I want to draw attention to the 3rd fact, for though it may seem innocuous, it will be the first, and essentially only[8] "infinite polynomial" with no root, a very non-polynomial behavior as polynomials (over $\mathbb{C}$) can be defined by their roots. We shall also see much later that there will be "infinite rational polynomials" polynomials with *transcendental* roots (roots of elements that are not algebraic over a field), the reader already knows these as sin and cos.

For reader that shall be interested in modular forms much down the line, sin and cos are interesting in their own right as being the simplest example of functions that are invariant under a group operation, in this case $\mathbb{Z}$. Thinking of $\mathbb{Z}$ as $\mathrm{GL}_1(\mathbb{Z})$, we will later see that modular forms are invariant under $\mathrm{SL}_2(\mathbb{Z})$ (not $\mathrm{GL}_2(\mathbb{Z})$ for reasons that would be clear once the modular theory is studied) along with some extra natural conditions.

Exponential Function

The exponential function is the function that satisfies the solution to the ODE

$$f'(z) = f(z) \qquad f(0) = 1$$

We can solve this by setting

$$\begin{aligned} f(z) &= a_0 + a_1 z + \cdots + a_n z^n + \cdots \\ f'(z) &= a_1 + 2a_2 z \cdots + n a_n z^{n-1} + \cdots \end{aligned}$$

which requires that $a_{n-1} = n a_n$ with $a_0 = 1$. Using induction, we get that

$$a_n = \frac{1}{n!}$$

This solution is usually denoted as either e^z or $\exp(z)$

Definition 2.4.1: Exponential Function

Let f be the function such that $f = f'$. Then the solution to this ODE is called the *exponential function*:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

it is usually denoted as either e^z or $\exp(z)$.

Notice that $\sqrt[n]{n!} \to \infty$, and hence e^z converges on the entire complex plane. the exponential function satisfies some nice properties. On immediate inspection, it is clear that the exponential is its own derivative. We also get a homomorphism $\exp : \mathbb{C} \to \mathbb{C}^\times$ via

$$e^{a+b} = e^a \cdot e^b$$

[8] up to multiplication by a constant, see ref:HERE

To see this, notice that

$$D(e^z e^{c-z}) = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$$

and so $e^z \cdot e^{c-z}$ is a constant. Note that we have not yet proved the exponential law for complex functions, so we couldn't just take $e^z e^{c-z} = e^c$. In fact, form this we *can* conclude that $e^z \cdot e^{c-z} = e^c$. Letting $z = a$ and $c = a + b$ gives us our desired result. Another way of doing this is by taking the partial sums of e^{a+b} and doing some algebraic manipulation to get the result, and concluding it works as $n \to \infty$. We can thus see that we have a homomorphism $r \mapsto e^{ir}$ mapping $\mathbb{R} \to S^1$. This homomorphism, when restricted to $\mathbb{R}$, has kernel $2\pi\mathbb{Z}$ so $\mathbb{R}/2\pi\mathbb{Z} \cong S^1$. This is in fact a topological homomorphism, and we can consider S^1 having the quotient topology. As an exercise, this topology can be verified to be the same as the subspace topology of $S^1 \subseteq \mathbb{C}$. By our observation that $\mathbb{C}^\times \cong S^1 \times (0, \infty)$, and that the $e^z = e^{x+iy} = e^x e^{iy}$, we in fact get that

$$\exp : \mathbb{C} \to \mathbb{C}^\times$$

is a surjective homomorphism from $\mathbb{C}$ with addition to $\mathbb{C}^\times$ with multiplication which has the same kernel, and hence the exponential map links addition to multiplication homomorphically! As an aside, This is in fact another way of characterizing the exponential function. As the kernel is $2\pi\mathbb{Z}$, we have that:

$$e^z = e^{2\pi i k} \qquad k \in \mathbb{Z}$$

Since $e^z \cdot e^{c-z} = e^c$, $e^z \cdot e^{-z} = e^{-z} \cdot e^z = 1$, implying e^z must have *no roots*, even the nth partial summand always has n roots (hence, an infinite polynomial needn't have roots[9]). Another way of seeing this is that the real component of e^z can never be zero, e^z is *never zero*. Choosing $z = x \in \mathbb{R}$ to be real, notice that $e^x > 0$ for $x > 0$. Since e^x and e^{-x} are recipricals, this means that $0 < e^x < 1$ for $x < 0$. Since the series has real ceofficients, $e^{\overline{z}}$ is the complex conjugate of e^z. Thus,

$$|e^{iy}|^2 = |e^{iy} \cdot e^{-iy}| \qquad |e^{x+iy}| = e^x$$

2.4.1 Trigonometric Functions

Since the addition, composition, multiplication, and dividions of holomorphic functions is holomorphic, we have the following two holomorphic functions:

Definition 2.4.2: Trigonometric Functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{2.5}$$

By doing some substitution and computation, we get:

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

[9] and it also may have infinitely many as we shall see when defining $\sin(z)$ and $\cos(z)$

Using equation (2.5), There is a natural way of linking these formulas to e^z. First, since $e^{x+iy} = e^x e^{iy}$, we only need to expand for e^{iy}:

$$e^{iy} = 1 + iy - i^2 \frac{y^2}{2!} + i^3 \frac{y^3}{3!} - \cdots$$

Since e^{iy} is absolutely convergent, we can re-arrange the terms, getting us:

$$e^{iy} = (1 - \frac{y^2}{2!} + \cdots) + i(y - \frac{y^3}{3!} + \cdots)$$

where the right term on the right hand uses the fact that we can factor constants form converging series. We thus get:

$$e^{iz} = \cos(z) + i\sin(z)$$

Be mindful that for a complex number z, we usually get

$$z = |z|(\cos(\arg(z)) + i\sin(\arg(z)))$$

while for the expression e^{iz} we put z dirrectly into cos and sin. Using this formula, there is an easy way of remembering the definitions of sin and cos: take $e^{iz} = \cos(z) + i\sin(z)$, and take $e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z)$. Add these two together and divide by 2 to get the definition of cos. Similarly for sin. By doing some algebraic manipulations, we can get:

$$\cos^2(z) + \sin^2(z) = 1$$

Similarly, we get that

$$D(\cos(z)) = -\sin(z) \qquad D(\sin(z)) = \cos(z)$$

and we also get the identities

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b) \end{aligned}$$

Any other trigonometric function and their properties can now be quickly derived using these (for example, tan, cot, and so forth). An important and counter-intuitive fact to point out is that cos and sin are defined on all of $\mathbb{C}$ (since the radius of convergence is ∞) and that the solution to the equation $\cos(z) = w$ has a solution for any $w \in \mathbb{C}$, not just $w \in [-1, 1]$!

Besides these two functions standing out for their importance in trigonometry, they also stand out as being the simplest "infinite polynomials" with an infinite number of roots. Focusing on $\sin(z)$ (since $\cos(z)$ is a translation), we shall show in section 4.2.3 that :

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

where the 1 in the summation is to insure the elements in the product converges to 1. The reader may say that this is not the simplest possible infinite polynomial (with roots at the integers), wouldn't it be z/n? The fact that we are considering z^2/n^2 to be the term is because:

$$\prod_{n=1}^{\infty} \left(1 \pm \frac{z}{n}\right)$$

does not converges (as we shall explore later), and requires some correcting terms[10]. In fact, we may consider the πz in multiplying the right hand side and the π term in the sin to be correction terms to insure we may decompose sin into an infinite product.

[10] This function, along with the correction terms, shall be the gamma function

2.4.2 Periodicity

A function is periodic if there exists a (nonzero) c such that $f(z+c) = f(z)$ for all $z \in \mathbb{C}$. We shall show that e^z has a period, that in particular there exists a nonzero c such that $e^{z+c} = e^z e^c = e^z$. From the properties of e^z, we immeditately get if the period exists then $c = iw$ for some real number w. Since

$$D \sin y = \cos(y) \leq y \qquad \sin(0) = 0$$

we get by the mean value theorem that $\sin(y) < y$ for $y > 0$. Similarly

$$D\cos(y) = -\sin(y) > -y \qquad \cos(0) = 1$$

which gives

$$y > 1 - \frac{y^2}{2}$$

which in turn gives

$$\sin(y) > y - y^3/6$$

and so

$$\cos(y) < 1 - y^2/2 + y^4/24$$

Using this, we get that

$$\cos(\sqrt{3}) < 0$$

Thus, by the Mean value theorem there exists a y_0 between 0 and $\sqrt{3}$ such that $\cos(y_0) = 0$. Since

$$\cos^2(y_0) + \sin^2(y_0) = 1$$

we have that $\sin y_0 = \pm 1$ and that $e^{iy_0} = \pm i$. Hence $e^{4iy_0} = 1$, giving us a period of $4y_0$.

This period is in fact the smallest possible priod. Take $0 < y < y_0$¿ Since

$$y > y(1 - y^2/6) > y/2 > 0$$

wwe have that $\cos(y)$ is strictly decreasing. Since $\sin(y)$ is positive and $\cos^2(y) + \sin^2(y) = 1$, it follow that $\sin(y)$ is strictly increasing, and hence $\sin(y) < \sin(y_0) = 1$¿ The double inequality $0 < \sin(y) < 1$ guarantees that e^{iy} is niether ± 1 nor $\pm i$. Thus, $e^{4iy} \neq 1$, and so $4y_0$ is indeed the smallest positive period. Let's say $\omega_0 = 4y_0$.

Let's now say ω is an arbitrary period. We'll show that there exists an n st

$$n\omega_0 \leq \omega < (n+1)\omega_0$$

If ω is not equal to $n\omega_0$, then $\omega - n\omega_0$ would be a positive period $< \omega_0$. But we just showed this is impossible, hence every periodmust be an integral multiple of ω_0.

We will denoted the smallest positive period of e^{iz} by 2π. Important consequence's of the proof we've just done is the following identities:

$$e^{\pi i/2} = i \quad e^{\pi i} = -1 \quad e^{2\pi i} = 1$$

2.4.3 Logarithm

The (compositional) inverse of the exponential function is called the *logarithm*, denoted $\log(z)$. In particular we would want that $e^{\log(z)} = z$. Since e^z is never 0 (it's image is $\mathbb{C}^\times$), the number 0 has no logarithm. If we write $z = \log(\omega)$ and choose $\omega \neq 0$, the equation $e^{x+iy} = \omega$ is equivalent (by euler's formula) to

$$e^x = |\omega| \quad e^{iy} = \frac{\omega}{|\omega|}$$

the first equation has a unique solution $x = \log|\omega|$, that is the real logarithm of the positive number $|\omega|$. The second equation is a complex number with absolute value 1, and thus it has only one solution in the interval $0 \leq y < 2\pi$. It is also satisfied by all y that differ form this solution by an integer multiple of 2π. Thus, every complex number has infinitely many logarithms which differ from each other by multiples of $2\pi i$. The imaginary part of $\log(\omega)$ is called the *argument* and is sometimes denoted $\arg(\omega)$. It is usually seen as the angle (measuring from the right side) between the x axis and $\omega/|\omega|$. By definition, the argument has infinitely many answers, for example:

$$\arg(i) = \{\pi/2, 5\pi/2, 9\pi/2, ...\}$$

This reflects the periodic nature of e^z, for

$$e^i = e^{i+2\pi} = e^{i+4\pi} = \cdots$$

Overall, the logarithm can be decomposed into the family of equations:

Definition 2.4.3: Logarithm

Given $\log|z|$ and $\arg(z)$, we get the following countable family of functions

$$\log(\omega) = \log|\omega| + i\arg(\omega)$$

Naturally, we wish to work with functions. For that, we must choose over which interval we consider log too properly define the inverse:

Definition 2.4.4: Branch

Let $f(z)$ be a continuous function defined on a connected set Ω. then f is said to be a *branch* of log if

$$e^{f(z)} = z$$

that is $\exp \circ f = \text{id}$

Notice that once a branch is chosen log must be a *right inverse*; it cannot be a left inverse for it must *choose* a particular angle to which it maps to (or equivalently, as e^x is surjective but not injective on $\mathbb{C}$, it has a right-inverse but not a left-inverse). For future purposes, we will box the following result:

Lemma 2.4.1: Difference Between Brancehs

Let $f(z)$ be a branch of $\log(z)$ in a connected open set Ω. Then any other branch will have ht form $g(z) = f(z) + 2k\pi i$ for some $k \in \mathbb{Z}$. Conversely, for all $k \in \mathbb{Z}$, $f(z) = 2k\pi i$ is a branch

Proof **:**
Let f, g be two branches and let
$$h(z) = \frac{f(z) - g(z)}{2\pi i}$$
Since h Is continuous on a connected set, we know it's image is connected as well. The image of h is in $\mathbb{Z}$ and thus is only one point in the image, implying h must be constant, completing the proof.

When we choose $f(z)$ for which $\arg(1) = 1$, we shall call $f(z)$ the *principal branch*, and denote the argument function as Arg to emphasize we are now using the continuous function with a fixed choice of angle.

2.4.4 Complex Exponentiation

Using log and exp, we may define z^α.

Definition 2.4.5: General Exponent

Let $a, b \in \mathbb{C}$. Then:
$$a^b := \exp(b \log a)$$

When $z \in \mathbb{R}$, then by convention we will take the real logarithm unless stated otherwise. If a is restricted to positive numbers, then $\log(a)$ shall be real and a^b has a single value. Otherwise, we would consider $\log(a)$ to be the complex logarithm with a^b having infinitely many values, each different from one another by $e^{2\pi inb}$. There is only a single value if and only if b is an integer n and a^b can be interpreted as a power of a or a^{-1}. If b is a rational number with reduced form p/q, then a^b has exactly q values that can be represented as $\sqrt[q]{a^p}$. We also have the equality of sets (remember these sets are infinite)
$$\log(z_1 z_2) = \log z_1 + \log z_2$$
$$\log(z_1 z_2) = \arg z_1 + \arg z_2$$

Using this, we can also find inverse sin and cos by solving the equation
$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) = \omega$$

Manipulating, we get
$$e^{iz} - 2w + e^{iz} = 0 \qquad \Leftrightarrow \qquad (e^{iz})^2 - 2we^{iz} + 1 = 0$$

This a quadratic equation with e^{iz} as the root $e^{iz} = \omega \pm \sqrt{\omega^2 - 1}$, which implies $iz = \log(\omega \pm \sqrt{\omega^2 - 1})$. Thus:
$$z = \arccos(\omega) = -i \log(\omega \pm \sqrt{\omega^2 - 1})$$
Since $\omega + \sqrt{\omega^2 - 1}$ and $\omega - \sqrt{\omega^2 - 1}$ are reciprocals, we can also write:
$$z = \arccos(\omega) = \pm i \log(\omega + \sqrt{\omega^2 - 1})$$

The inveres of $\sin(z)$ is now easily seen as:
$$\arcsin(\omega) = \frac{\pi}{2} - \arccos(\omega)$$

Proposition 2.4.1: Derivative Of Logarithm

Let $f(z)$ be a branch of $\log(z)$ on a domain Ω. Then $f(z)$ is holomorphic and

$$f'(z) = \frac{1}{z}$$

Proof **:**
We simply compute:

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \\
&= \lim_{h\to 0} \frac{f(z+h)-f(z)}{e^{f(z+h)}-e^{f(z)}} \\
&= \lim_{w\to f(z)} \frac{w-f(z)}{e^{w}-e^{f(z)}} \\
&\overset{!}{=} \frac{1}{e^{f(z)}} \\
&= \frac{1}{z}
\end{aligned}$$

where the $\overset{!}{=}$ equality comes from the fact that

$$\lim_{w\to f(z)} \frac{e^{w}-e^{f(z)}}{w-f(z)}$$

exists since e^z is differentiable, and so the reciprocal exists too with limit $1/e^z$.

We should also find the power series of the log:

Proposition 2.4.2: Log Power Series

For $|z| < 1$, the power series

$$f(z) := \sum_{n}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

converges and is equal to the principal branch of $\log(1+z)$

Proof **:**
It can directly be checked that $f(z)$ and $g(w) = \sum_n^\infty \frac{w^n}{n!}$ (which is the series expansion of $e^w - 1$) are inverses of each other, and hence $g(f(z)) = z$. Thus, $e^{f(z)} = z + 1$, which by definition makes $f(z)$ the branch of $\log(z+1)$. Evaluating f at 0 shows that this is a principal branch.

Finally, we take a moment to think about a Riemann surface that will make log single-valued. Consider the covering

$$X = \left\{(z, w) \in \mathbb{C}^2 \ : \ z = e^w\right\}$$

Then log would just be a mapping to the second coordinate. By the periodicity of log, we see that for any small neighbourd around $z \in \mathbb{C}$, we have countably many open sets laying above it in X. Furthermore, if we let $\log(1) = 0$, then as we go around counter-clockwise on S^1 with the assumption that log is continuous, we will get that we would get back to $\log(1) = 2\pi$, and everytime we go around we get another 2π factor: $\log(1) = 2k\pi$. This motivates a visual like so:

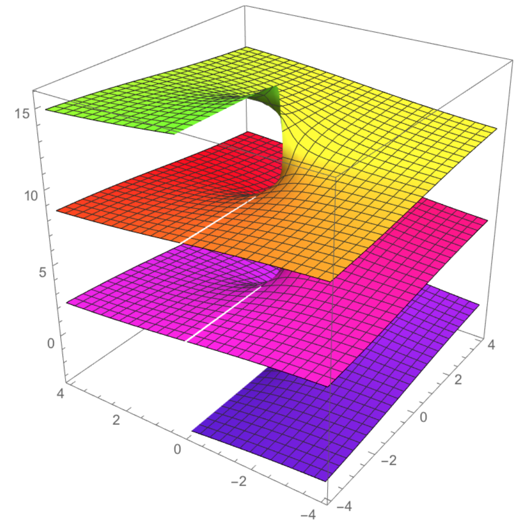


Figure 2.1: Branch Covering for complex log

which we have already briefly mentioned at the end of section 1.2.2.

2.5 Analytic Functions

Since differentiation is a local property, it is natural to talk about functions which are locally a power series. The following is exactly this:

Definition 2.5.1: Analytic Function

Let A be a function on an open set Ω. Then f is said to be *analytic* if every $f(z)$ can be written as a convergent power series centered at some $z_0 \in \Omega$. In particular, for every $z_0 \in \Omega$ there is a convergent power series centered at z_0 such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where z satisfies $|z - z_0| < r$ and r is less than the radius of convergence of the above power series.

Analytic functions have many of the same nice properties as power series. For example, they are infinitely differentiable, and their integral is easy to find:

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$$

which has the same radius of convergence by the root test. The easiest example of analytic functions are power series

Proposition 2.5.1: Power Series Are Analytic

Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a convergent power series with radius of convergence R. Then $f(z)$ is analytic in $|z - z_0| < R$

Proof **:**
Without loss of generality we may assume $z_0 = 0$ since we are simply translating the entire function to be around 0. Pick any $|r_0| < R$. We shall convert the power series $\sum_n^{\infty} a_n z^n$ into one centered at z_0. Computing:

$$\begin{aligned} f(z) &= \sum_n^{\infty} a_n(z_0 - (z - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z - z_0)^k \end{aligned}$$

Since the z in the above is within the radius of convergence at the point 0, we get that the above power series is absolute continuous and hence we may re-arrange the terms:

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k$$

If this re-arrangement looks foreign, think of the layer-cake representation for double integrals. This power series evidently respects $|z - z_0| < r$ for any $r < R - |z_0|$, and hence is a valid power series representation, completing the proof.

2.5.1 Principle of Analytic Continuation

Given a domain $\Omega \subseteq \mathbb{C}$ on which a polynomial is defined, we may enlarge Ω to a larger open set, and the polynomial will uniquely be enlarged set, namely as coefficients remain unchanged. Naturally, this generalizes to power series too. What is great is that this uniqueness of extension is true for analytic functions! The key idea is that if all derivatives at a point are zero for a polynomial and a power series, then it is zero for a connected neighbourhood:

Theorem 2.5.1: Principle of Analytic Continuation

Let $f(z)$ be an analytic function in a connected open set Ω and $z_0 \in \Omega$. Then the following are equivalent:

1. $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$
2. f is identically 0 in a neighbourhood of z_0
3. f is 0 in Ω

Proof **:**
(3) $\Rightarrow$ (1) is immediate, and (1) $\Rightarrow$ (2) comes from Taylor's Theorem. We only require to show (2) $\Rightarrow$ (3).

Let $\omega' = \{z \in \Omega \ : \ f \equiv 0$ in a neighbourhood of z in $\Omega\}$. The set Ω' is nonempty since z_0 is in Ω'. it is also certainly open by definition. If we show Ω' is closed, then it is clopen and hence must be equal to Ω. Let $z \in \overline{\Omega'}$. Since the derivative is continuous, it's values can be extended uniquely to it's border and hence $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}$. Then, by (1) $\Rightarrow$ (2), f is identically 0 in a neighbourhood of z, and so in fact $z \in \Omega'$. But then $\Omega' = \overline{\Omega'}$, showing it's closed. But then by connectedness $\Omega' = \Omega$.

Note that there are smooth real functions with derivatives zero that are only 0 at a single points, e^{-1/x^2} is an example, and hence this result cannot extend to general smooth functions.

Corollary 2.5.1: Analytic Continuation Of Functions

Let f, g be analytic in a connected open domain Ω and $f = g$ in a neighbourhood of some point. Then $f = g$ in Ω

Proof **:**
Simply take the function $f - g$ and 0.

Corollary 2.5.2: Analytic On $\mathbb{C}$, Then Power Series

Let f be an entire analytic function (defined on all of $\mathbb{C}$). Then f is a power series.

Proof **:**
As f is analytic on $\mathbb{C}$, it is analytic at 0, so take $f(0) = \sum a_n z^n$ on some open neighborhood, U. Represent f on U as g. Then since $f - g$ is 0 in the neighborhood of $\mathbb{C}$, g extends to all of $\mathbb{C}$, completing the proof.

Hence, the extension of an analytic function must be *unique*, we only need two analytic functions to agree on a open neighbourhood, or even better non-isolated set of points (see exercise ref:HERE), and analytic function whose domain is all of $\mathbb{C}$ is a power series[11];! This is in stark contrast with general smooth functions which may have multiple different possible extensions on their entire domain. For example take

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x \geq 0 \end{cases}$$

then f on $(-2, -1)$ agrees with the zero function, but the two will certainly not extend to the same function! Note to that it is possible that the sequence of roots converge on the boundary; for example

$$f(z) = \sin\left(\frac{\pi}{z}\right)$$

has $1/n$ as a sequence of zeros, but is holomorphic on $B_1(1)$[12].

One interesting algebraic fact we get from this is the following

Corollary 2.5.3: Ring Of Analytic Functions

The ring of analytic functions on an open domain Ω, $\mathcal{A}(\Omega)$ forms an integral domain

Proof **:**
Let $f, g \in \mathcal{A}(\Omega)$ and consider $fg = 0$. If $f \neq 0$, then f must be nonzero in some open domain. Hence g must be zero in some open domain. But then g is identically 0 in Ω, and hence $g = 0$. But then $\mathcal{A}(\Omega)$ is an integral domain, completing the proof.

This further reflects how analytic functions are like the completion of polynomial rings, and later we shall see more similarities between these rings.

Finally, let us look at the poles and roots of analytic functions. Let's say f is analytic in a neighbourhood of z_0. Then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ given a z in a close enough neighbourhood of z_0. Let's say $f(z_0) = 0$ but f is not identically 0. Let k be the smallest integer such that $f^{(k)}(z_0) \neq 0$ (which is equivalent to saying $a_k \neq 0$). Then we may write:

$$f(z) = (z - z_0)^k g(z)$$

Then in a small enough neighbourhood, $g(z) \neq 0$ and $(z - z_0)^k \neq 0$ if $z \neq z_0$, and hence $f(z) \neq 0$ for $0 < |z - z_0| < \epsilon$ for appropriate $\epsilon > 0$. Hence, the root's of f must be isolated, just like for polynomials. The value k is called the *order* or *multiplicity* of the root z_0 at f. Doing a coordinate change on the above makes f into quite a simple function: if $\zeta := (z - z_0)g(z)^{1/k}$, then

$$f(z(\zeta)) = \zeta^k$$

This also means that any compact subset of the domain can only contain finitely many zeros.

[11] This shall further motivate studying functions with *poles* or *essential singularities* on $\mathbb{C}$, as these will be the only analytic functions that are not power series

[12] this does not contradict exercise ref:HERE as the point 1 is not in the domain, and hence is not an accumulation point

2.5.2 Meromorphic Functions

Let's now generalize to a quotient of analytic function: $\frac{f(z)}{g(z)}$ where g is not identically 0. If $g(z_0) \neq 0$, then $\frac{f(z)}{g(z)}$ is well-defined and analytic in a neighbourhood of z_0 which we see by taking fg^{-1} in proposition 2.3.4. If $g(z_0) = 0$, then either g is identically zero in a neighbourhood around z_0, in that case the coefficients in the Taylor expansion around z_0 of g are all zero, or it is not identically zero, in which case there is a k such that $g^{(n)}(z_0) = 0$ for $n \leq k$.

Then we may use an limit argument on polynomials to show that z_0 is a root of f and g, in particular we can re-write $f(z) = (z - z_0)^k f_1(z)$ and $g(z) = (z - z_0)^l g_1(z)$ where $f_1(z_0), g_1(z_0) \neq 0$. Notice how this is not necessarily possible if f is not analytic, take for example a smooth bump function with compact support and try factoring out $(x - x_0)^n$ outside the support. Since we can do this for analytic functions, we have:

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l} \frac{f_1(z)}{g_1(z)}$$

If $k \geq l$, then $\frac{f}{g}$ extends to be analytic at z_0. If $k < l$, then z_0 is a pole of $\frac{f}{g}$ of order $l - k$, and we get

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$$

Notice that the above limit is for any path[13], and hence it is natural to extend $\frac{f}{g}$ as a function on the Riemann sphere. Due to this non-analytic behavior on only a discrete set (or similarly, analytic behavior on S^2) we label them:

Definition 2.5.2: Meromorphic Functions

A *meromorphic function* in an open set Ω is a function that is well-defined and analytic in the compliment of a discrete set and expressible in a neighbourhood of any point in Ω as a quotinet of analytic function $\frac{f}{g}$ where g is not identically 0.

It is clear that meromorphic functions from a *field*: they are the field of fractions of the analytic functions which form an integral domain. The key here was the fact that analytic functions have roots, we shall see in chapter 3, section 3.3.1 that the poles can be thought of as determining most of the "important" information.

*Analytic Continuation

The next concept we will tackle is known as *analytic continuation.* Say we have a real differentiable function $f : (0, 1) \to \mathbb{R}$. If I ask you to tell me what is $f(-1)$, then you'll say this is obviously undefined. We may extend f smoothly to have it defined at -1, but there are certainly many ways of doing this. In the complex case, we will get that if f is determined to be complex-differentiable in some connected region, then it automatically determines uniquely on any larger connected open set, that is there is always a unique holomorphic extension to a larger connected open set. The most infamous analytic continuation is that of the *Riemann-zeta function*

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{s^k}$$

[13] This shall be more nuanced when working in $\mathbb{C}^n$, see chapter 8

As of writing this, there is a million dollars to them that could show that all the zero's of $\zeta(s)$ are either real or have real part 1/2:

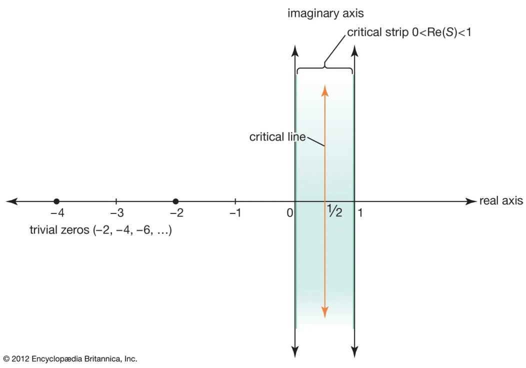


Figure 2.2: Visualizing where the critical points lie

Exercise 2.5.1

1. Show that the ring of analytic functions is a Bézout domain, namely all finitely generated ideals are principle.
2. Show that the ring of analytic functison is a GcD domain (may require infinite products for this)
3. Let f, g be analytic functions on U that agree a non-isolated set of points S. Then $f = g$ on U

3

Complex Integration

As we mentioned earlier, we can already define the Lebesgue integral on $\mathbb{C}$ (by the Borel σ-algebra given by the euclidean topology on $\mathbb{R}^2$) giving us notion of integration of the 2D regions in $\mathbb{C}$. Thus, we may instead study the case of a path γ when the complex 1-form (real 2-forms) has a holomorphic coefficient, that is:

$$\omega = f(z)dz = f(z)(dx + idy) \qquad f \in C^\omega(\Omega)$$

In studying these, we shall find that such forms have the property of being *closed* or *conservative* (the complex counter-part of the Fundamental Theorem of Calculus, FTC), which equivalently shall show us that the integral of any path is path independent (on connected open sets).

Once we show that $f(z)dz$ is conservative given f is holomorphic, we shall show the important consequence that if f is once complex-differentiable, it is infinitely complex differentiable, and even better it is analytic. Furthermore, using proposition 2.3.1 it can be deduced that analytic functions are holomorphic, giving us the following important equivalence:

$$\text{Holomorphic} \iff \text{Analytic}$$

This is completely false for real-differentiable function: consider any "pathological" (Legesgue) integrable function f. Then $F(x) = \int_0^x f(t)dt$ is (real) differentiable by the FTC. However, it can be far away from being twice differentiable. The fact that the Cauchy-Riemann equations force this relation is thus at first hand certainly quite a surprising result, but it will come down to the fact that this PDE equation forces enough regularity on f so that it is *harmonic*[1].

After undersatnding the case of $f(z)dz$ where f is holomorphic, we shall look at the case where f is meromorphic. For example, $1/z$ is meromorphic, or if we eliminate the singularities it is holomorphic on $\mathbb{C} \setminus \{0\}$. It will turn out that in this case, $f(z)dz$ is not as well-behaved, in particular fdz does not always have a primitive and f is not always conservative. However, f will still have many nice

[1]As a consequence, all harmonic function are indeed analytic

properties and will be relatively simple to integrate using residue theory. Residue theory will give us many integration tools allowing us to:

1. evaluating $\int_0^\infty \frac{\sin(x)}{x} dx$ and get $\frac{\pi}{2}$
2. Showing that
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

3.1 Integration Review

(Tao's blog can also be helpful: here)

As mentioned, we need to specify a path on which we are integrating. A path (or curve) will be a continuous function $\gamma : [a, b] \to U \subseteq \mathbb{C}$. It is differentiable if γ is differentiable. The notatation $-\gamma$ will represent reversing the path, that is

$$-\gamma(x) = \gamma\left(\frac{b - x}{a}\right)$$

Now consider $f : U \to \mathbb{C}$. Just like we've done for Riemann integrals, we can take any path γ in a plane and split it up via linear connections, then take their sum:

$$\sum_{i=1}^{n} f(z_c)(z_{i+1} - z_i)$$

where $z_i \in \gamma([a, b])$. Intuitively, we can then take the limit of this value[2] to define $\int_\gamma f(x)dz$. This limit would exist in cases were, say, f is continuous. If you remember your theorem's from calculus, when the integral exists we can also think of this as taking the supremum over all possible path approximations. Overall, this notion gives a good intuition on what is the integral, but not a good way of computing it. We can find a good computational tool if the path γ is C^1. In this case, we can perform a change or variables. By letting $dz = \gamma(x)$, then $dz = \gamma'(x)dx$, which is saying that

$$z_{i+1} - z_i \approx \gamma'(x)(x_{i+1} - x_i)$$

and this approximation becomes better as the partition gets finer. Then the integral becomes:

$$\int_a^b f(\gamma(x))\gamma'(x)dx$$

But now, we have this integral back down along the real axis, and so like we mentioned earlier we can split it along the real and imaginary parts. This integral is also well-defined since γ' is continuous. Sometimes, we will want γ to have some sharp points, which in derivative language means it has critical points. This is fine, since it will be only a zero-measure set amount of points, and so we would split γ into a piece-wise differentiable path and define the integral to be the sum over the pieces.

[2]which can be shown to be independent of partition

Another way of looking at this is by coming back to vector-valculus. Given $dz = dx + idy$ and $f = u + iv$, we get

$$\int_\gamma f(z)dz = \int_\gamma (u + iv)(dx + idy)$$
$$= \int_\gamma (udx - vdy) + \int_\gamma (vdx + udy)$$

This will be important when we use Green's theorem in a later proof (TBD).

The integral is independent of the re-paramaterization up to change of direction: if $\gamma' : [a, b] \to U$ defines the same path, the integarl is the same, but if $\gamma' = -\gamma$, then that would change the sign of the integral. Here are some important properties we will want to reference later

Proposition 3.1.1: Properties Of Line Integrals

1. $\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \cup \gamma_2} f(z)dz$
2. $\int_\gamma f + gdz = \int_\gamma fdz + \int_\gamma gdz$
3. $\left|\int_\gamma f(z)dz\right| \leq M|\gamma|$ where M is an upper bound for $|f|$ and $|\gamma|$ is the length of γ.

Proof **:**
These are all proved in a calculus class. If you're re-proving them, notice that (3) is the generalization of the usual result of $\left|\int_a^b f(x)dx\right| \leq M|b - a|$.

Lemma 3.1.1: Paths In Connected Set

Any two points in a connect set $\Omega \subseteq \mathbb{R}^2$ can be joined by a piece-wise C^1 curve

Proof **:**
This is inherently just a connectedness problem. Let $a \in \Omega$. Let

$$E = \{b \in \Omega \;:\; a, b \text{ can be joined by a piece-wise curve}\}$$

Clearly $a \in E$, and for any point in E, any open neighborhood around a point can be joined by a straight line and hence E is open. Let $b \in \overline{E}$ and any neighborhood around b. This neighborhood certainly intersects E by assumption of b being border point, and hence a path in E connects to b by a straight line. But then $b \in E$, so $E = \overline{E}$, and hence E is clopen and so $E = \Omega$ since by definition Ω has no nontrivial clopen subsets.

Many of our results will necessitate the property of being simply connected, which says our space does not have a hole (or more technically that all closed paths are nullhomotopic). We shall give an example of what can go wrong to see why we care:

Example 3.1: Complex Integration not Simply Connected

1. We'll integrate z^n for different n's and different paths where $\gamma(0) = \gamma(1) = 1$. Naturally, if the path is constant the $\int_\gamma z^n = 0$. Let's instead take the path $\gamma(\theta) = e^{2\pi i\theta}$ so that it forms a circle around 0. Then, by also changing $z^n = e^{inx}$ and doing a change of variables we get:

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^{2\pi} e^{inx} ie^{ix} dx \\ &= i\int_0^{2\pi} e^{i(n+1)x} dx \\ &= i\left(\frac{e^{i(n+1)x}}{n+1}\bigg|_0^{2\pi}\right)\end{aligned}$$

we now get to a scenario where the value we get depends on n. If $n \neq -1$ we get that the integral is equal to 0. If $n = -1$, then we in fact get that the integral is $2\pi i$! Looking back at our function, we see that our integral is $\int_\gamma 1/z dz$. This is essentially exactly a definition of the real $\log(x)$. In the complex case, notice that $1/z$ is not defined at $z = 0$, meaning the domain of this function is $\mathbb{C} \setminus \{0\}$, in particular this is *not a simply connected set*. This is a good example of what goes wrong if your set is not simply connected.

Differential Forms Review

Differential forms are the natural tool to use to hold geometric information when integrating, which in our case would be paths. In this section, a map $\gamma : [a, b] \to \Omega$ where $\Omega \subseteq \mathbb{R}^2$ will be C^1 or piecewise C^1, and $\gamma(t) = (x(t), y(t))$. A 1-form will be written as

$$\omega = Pdx + Qdy$$

with P, Q being (real or complex) continuous functions on Ω. Then define:

$$\int_\gamma \omega = \int_a^b \gamma^*\omega = \int_a^b F(t)dt$$

where $F(t) = P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)$, which we get from the pullback:

$$\gamma^*(Pdx + Qdy) = \gamma^*(P)\gamma^*(dx) + \gamma^*(Q)\gamma^*(dy)$$

where $\gamma^*(P) = P \circ \gamma$ and $\gamma^*(dx) = d(\gamma^* x) = d(x \circ \gamma) = d(x(t)) = x'(t)$. As usual, reparametirization of the curves does not affect the value of the curve up to a sign: if $u : [c, d] \to [a, b]$ is a strictly positive diffeomorphism so that we have a new path $t \mapsto \gamma(u(t))$, then if this new path is γ', then:

$$\int_{\gamma'} \omega = \int_c^d f(\gamma(u(t)))u'(t)dt = \int_a^b f(u)du = \int_\gamma \omega$$

If it is strictly negative then $\int_{\gamma'} \omega = -\int_\gamma \omega$. a diffeomorphism can only be stricIty positive or negative since if it's derivative is 0 at any point is is not a diffeomrophism. If we have a piecewise curve, then we define:

$$\int_\gamma \omega = \sum_{i=1}^n \int_{\gamma_i} \omega$$

If $\gamma(a) = \gamma(b)$ (if the endpoints are the same), we shall say that γ is a closed curve. In this case, we may start by integrating at any point (show this if you're not comfortable with it). One of the nicest 1-forms are those of the form $\omega = dF$. These are particularely nice to compute. By the FTC:

$$\int_\gamma dF = \int_a^b F'(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a))$$

which we can see visually as thinking of the graph F defining the surface for which gravity "pulls down" on, and then dF represents the "potnetial" energy given by walking up or down this surface. Then the overall potential energy is only dependent on your height, giving us the above formula. If we return to the same point, then we get zero net potential energy, hence

$$\oint_\gamma dF = 0$$

Since we are trying to define integration of 1-forms $f(z)dz$, it would be nice if they always have an antiderivative just like the real 1-forms $f(x)dx$. We lead up to this by developing the theory for general complex 1-forms:

Definition 3.1.1: Primitive

given a 1-form ω, a *primitive* of ω is a C^1 function $F : \Omega \to \mathbb{C}$ such that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Outside of complex, such forms are usually called *exact*.

Note that this gives another quick proof that if Ω is connected and dF is zero then F is constant (which in the gravity interpretation can be thought as never gaining any potential energy). Using this, we see that if $\omega = dG$ for some other function, then $d(F - G) = 0$, showing that F and G differ by a constant and hence primitives are unique up to a constant. Not all complex 1-forms have a primitive (as we shall see after the next proposition). The next proposition gives the exact property needed for complex 1-forms to have a primitive:

Proposition 3.1.2: Criterion for Primitive Existing

Let ω be a 1-form on an open set $\Omega \subseteq \mathbb{C}$. Then ω has a primitive if and only if $\oint_\gamma \omega = 0$ for every piecewise C^1 closed curve γ

Proof **:**
If ω has a primitive so that $\omega = dF$, then certainly

$$\int_\gamma \omega = F(\gamma(b)) - F(\gamma(a)) = 0$$

since $\gamma(a) = \gamma(b)$. Conversely, fix some $p = (x_0, y_0) \in \Omega$ and define

$$F(x, y) = \int_\gamma \omega$$

where γ is a path from (x_0, y_0) to (x, y). F is well-defined since if δ is another path from (x_0, y_0) to (x, y) then going along γ then back down δ forms a closed curve, by assumption has value 0, and hence since we can split integrals over piece-wise paths the integral over both paths are equal. I claim that $dF = \omega$. Letting $\omega = Pdx + Qdy$, then taking a straight path (since it is path independent) we get:

$$F(x+h, y) - F(x, y) = \int_x^{x+h} P(t, y)dt$$

Then

$$\lim_{h \to 0} (F(x+h, y) - F(x, y)) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} P(t, y)dt = P(x, y)$$

where the least equality comes form the FTC. But now we do the same for Q, giving us our final result and completing the proof.

It would be nice if we can check a subset of curves. For example, if we can only check all rectangles paths are zero, we would have a much easier time computing. In general this is not possible: we will require our domain to be *simply connected*. On a simply connected surface, like the open disk, this is a sufficient condition

Proposition 3.1.3: Existence Of Primitive On Disk

Let D be an open disc. If $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle contained in D with sides parallel to the axes, then ω has a primitive in D

The key is that every rectangle must be defined, and hence our space cannot have any holes.

***Proof* :**
Let (x_0, y_0) be the center of the disk and $(x, y) \in D$ be any point of D. Then we have two paths γ_1, γ_2 each starting at (x_0, y_0) and ending at (x, y) which each are two sides of the rectangle (one going horizontal then vertical, the other first goes vertically then horizontally). Then by assumtpion $\int_R \omega = \int_{\gamma_1 + \gamma_2} \omega = 0$ and so $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ We may now define

$$F(x, y) = \int_{\gamma_1} \omega$$

where we arbitrarily chose one of the two intervals. We an repeat the same argument to show that

$$\frac{\partial F}{\partial x} = P \qquad \frac{\partial F}{\partial y} = Q$$

where $\omega = Pdx + Qdy$, completing the proof.

Thus, instead of checking on all curves, it suffices to check for all γ that are boundary of rectangles, or in particular sufficiently small rectangles, if Ω is an open disk.

As we mentioned, this works since there are no holes on open disks. On spaces that have holes, the above argument breaks down, and we can find a form ω that does not have a primitive (dz/z on $\mathbb{C} \setminus \{0\}$ is an example of this by proposition 3.1.2). For forms of the form $f(z)dz = f(z)(dx + idy)$, we will fortunately still have that it has *local primitives* for any simply connected subset (this a version of Poincaré's lemma). We thus define the following:

Definition 3.1.2: Closed From

Let $\omega = Pdx + Qdy$ be a 1-form on an open set Ω. Then we say that ω is *closed* if for any point $z \in \Omega$, there is an open neighbourood in which ω has a primitive.

Another more common definition in the world of geometry is that a form is a closed form if $d\omega = 0$. We can always assume we can pick a small enough neighbourood around $z \in D$ to make it a disk, hence a form ω is closed if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle contained in a disk D.

In general, a closed differential form on a disk has a primitive, but a closed differential form on Ω needn't have a primitive as we saw in example 3.1. In the example, we had the form

$$\frac{dz}{z} = \frac{dx + idy}{x + iy} = \frac{xdx + ydx}{x^2 + y^2} + i\frac{xdy - ydx}{x^2 + y^2}$$

Let's consider a new form which is the imaginary part of this form:

$$w = \frac{xdy - ydx}{x^2 + y^2}$$

This is a closed form in the plane with the origin excluded, and has no primitive since if γ is the unit circle then

$$\int_\gamma \frac{xdy - ydx}{x^2 + y^2} = 2\pi$$

If we inspected this form more closely, we would see that it's equal to $d\arctan\left(\frac{y}{x}\right)$, which is a *many-valued function*, showing us how ω fails to have a primitive.

If ω is a closed form, it only has local primitives. We may wonder if instead of finding a global primitive, we find a primitive given a chosen path γ. The following makes precise what this may mean:

Definition 3.1.3: Primitive Along A Path

Let $\gamma : [a, b] \to \Omega$ be a path contained in an open set Ω and let ω be a closed form in Ω. A continuous function $f : [a, b] \to \Omega$ is called a *primitive of* ω *along* γ if for any $\tau \in [a, b]$, there exists a primitive F of ω in a neighbourood of the point $\gamma(\tau) \in \Omega$ such that

$$F(\gamma(t)) = f(t)$$

for t near enough τ.

To understand the above intuitively, we shall give an example. Recall that $\int_\gamma dz/z$ has is nonzero if γ is a circle, hence it is not a primitive form (however it is closed since it is locally a primitive). Letting $\omega = dz/z$, if we take $f(t) = \int_0^t \gamma^*\omega$ where $\gamma(t) = e^{2\pi i t}$, then $f(0) = 0$ and $f(2\pi) = 2\pi i$. If ω was a primitive, we would have $f(0) = f(2\pi) = 0$, i.e. we would be on a single surface, or the graph of a function, thus f would represent the "height" at the point $f(t)$. However, this is not the case, as $\gamma(0) = \gamma(2\pi)$. This is probably some foreshadowing, but we may see f as hinting at the existence of a *Riemann surface* on which dz/z does have a primtive in some correct generalization.

Note that if ω had a global primitive, then a loop γ would have $f(\gamma(0)) = f(\gamma(1))$, while this is not necessarily the case for local primitive functions.

Theorem 3.1.1: Primitive Along A Path

For any path γ, such that $f(t) = \int_0^t \gamma^*\omega$ is continuous, f is a primitive, always exists, and is unique up to addition of a constant.

Proof **:**
Cartan p. 58, doable proof

Primitive of $\log(z)$ primitiveIntegerValueRmk

We may use this theorem to show things like any closed path which does not pass throug the origin for the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

is an integer, since the primitive along the path will be a branch of $\log(z)$, and so we will get $f(b) - f(a)$, which will be the different between two branches of $\log(z)$ at $\gamma(a) = \gamma(b)$, which we showed in lemma 2.4.1 is $2k\pi i$, and hence we have an integer.

The form dz/z is rather important as we saw it does not have a primtive, but in a real sense it is the "only" complex form that does not have a primitive. It is reasonable to think that any other form that doesn't have a primitive stems from this form.

Relation to Homotopies

Since holes are the main problem in defining a primitive, we shall take some time showing that up to being "stuck" or "wound" around a hole, the value of the ingral is path independent. The natural concept to introduce then is that of homotopy. Let $I = [0, 1]$. In this section, we show that integrals of paths in closed forms are homotopy invariant.

Definition 3.1.4: Homotopy

Let $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ be two path with the same initial and final points. Then these paths are *homotopic* (in D) with fixed endpoints if there exists a continuous map $(t,u) \to \delta(t,u)$ of $I \times I$ onto D st

$$\begin{cases} \gamma(t,0) = \gamma_0(t) & \gamma(t,1) = \gamma_1(t) \\ \delta(0,u) = \gamma_0(0) = \gamma_1(0) & \delta(1,y) = \gamma_0(1) = \gamma_1(1) \end{cases}$$

Similarly, we have a homotopy between two closed paths γ_0, γ_0 if there is a continuous maps $(t,u) \to \delta(t,u)$ of $I \times I$ into D st

$$\begin{cases} \delta(t,0) = \gamma_0(t) & \delta(t,1) = \gamma_1(t) \\ \delta(0,u) = \delta(1,u) & \text{for all } u \end{cases}$$

we say a closed path is null homotopic if it is homotopic to a constant function.

We shall show that on a closed form, two homotopic paths (with fixed points or that are closed) have the same value. We first upgrade theorem 3.1.1 to homotopies

Definition 3.1.5: Primitive Along Homotopies

Let $(t,u) \to \delta(t,u)$ be a continuuous mapping of a rectangle

$$a \leq t \leq b \qquad a' \leq u \leq b'$$

into the open et D and let ω be a clseod form in D. A *primitive of* ω *following the mapping* δ is a continuous function $f(t,u)$ in the rectangle satisfying the property that for any point (τ,ν) of the rectangle, there exists a primitive F of ω in a neighbourood of $\delta(\tau,\nu)$ such that

$$F(\delta(t,u)) = f(t,u)$$

at any point (t,u) sufficinetly near (τ,ν)

Lemma 3.1.2: Primitive Along Homotopies

There always exists an f as given in the above definition and it is unique up to addition of a constant.

***Proof* :**
I will skip writing this proof down for now.

Theorem 3.1.2: Closed Forms And Homotopic Paths

Let γ_0, γ_1 be two homotopic paths of D with fixed end points or two homotopic closed paths. Then if ω is any closed from in D, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

Proof **:**
ibid.

Definition 3.1.6: Simply Connected

Let D be a set. We say that D is *simply connected* if D is connected and any closed path in D Is nullhomotopic.

Simply connected sets are the exact condition needed to guarantee a closed form is a primitive:

Theorem 3.1.3: Closed Form Has Primitive In Simply Connected Set

Let ω be a closed differential form on a simply connected open set D (i.e. ω has local primitives). Then ω has a global primitive in D

Proof **:**
By theorem 3.1.2, we have $\int_\gamma = 0$ for any closed paths contained in D, which by proposition 3.1.2 means that ω has a primitive in D.

This means that the closed form dz/z on any simply connected open set of $\mathbb{C}$ not containing zero has a primitve. This primitive would be a branch of $\log(z)$.

Winding Number of Closed Path

We keep bringing up the 1-form dz/z as our example of a closed non-primitive form. We may slightly expand this to a family of such 1-forms by translating:

$$\frac{dz}{z-a}$$

The reason for this particular closed (but not primitive) form is two fold:

1. they show up in the limit definition:

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

   We shall in proceeding sections integrate the above value to great effect, and so understanding the behavior of the denominator will be very important.

2. We shall show that all holomorphic functions are locally equal to series of the form $\sum_{n\in\mathbb{Z}} a_n z^n$. These fail to be primtiive exactly when $n = -1$, hence due to the rigidity of holomorphic functions, they only fail represent primitive forms whenever $a_{-1} \neq 0$.

The value the integral
$$\int_\gamma \frac{dz}{z-a}$$
Is dependnet on the path γ and where how often it "winds"

Definition 3.1.7: Winding Number Of Closed Path

Let γ be a closed path in $\mathbb{C}$ and let $a \in \mathbb{C}$ such that a is not in the image of γ. Then the *winding number* of γ with respect to a (sometimes also called the *index* of γ with respect to a) is defiend to be the value of the integral
$$I(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

By remark primitiveIntegerValueRmk, $I(\gamma, a)$ is an integer.To find the winding number, we must find some continuous complex-valued function $f(t)$ defined for $0 \leq t \leq 1$ st
$$e^{f(t)} = \gamma(t) - a$$
which then gives us
$$I(\gamma, a) = \frac{f(1) - f(0)}{2\pi i} = n$$
where n depends on the number of times γ winds. Some notable properties of the winding number are

1. If a is fixed, $I(\gamma, a)$ remains constant when the closed path γ is continuosly diffformed without passing through a.
2. If we fixed the closed path γ and varry a in the compliment of the image of γ, then $I(\gamma, a)$ is a locally constant function with respect to a. Hence $I(\gamma, a)$ is a function of a which is constant in each connected component in the compliment of the image of γ
3. If $\gamma(I)$ is conatined in a simply connected open set Ω where the image doens't contain a, then $I(\gamma, a) = 0$.
4. If γ is a cirlce described in the positiv esense (so that $I(\gamma, 0) = 1$), then $I(\gamma, a) = 0$ for a outside the circle, and $I(\gamma, a) = 1$ for a inside the circle.

The winding number has a few intersting mapping consequences

Proposition 3.1.4: Image Of Cirlce

Let $f : \mathbb{D}_r \to \mathbb{R}^2$ be a coninuous function from $\mathbb{D}_R = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq r^2\right\}$ and let γ be the restriction of f to $x^2 + y^2 = r^2$. If a point a of the plane does not belong to the image of Γ and if $I(\gamma, a) \neq 0$, then f takes the value of a at at least one point in the open disk $x^2 + y^2 < r^2$

***Proof* :**
For the sake of contradiction, let's say this was not the case. Then there is a continuous homotopogy from γ to the concentric circle of radius 0. Thus, $\int_\gamma \frac{dz}{z-a} = 0$, contradicting our hypothesis, as we sought to show.

(Product of paths, p. 64 Cartan)

(idk if I should add information about oriented boundaries here)

3.2 Integrating Holomorphic Functions: Cauchy's Theorems

Recall that in 1-dimensional real integration (whether it is signed definite, or unsigned definite like Lebesuge, or antiderivative giving a function), there is always a simple choice for the area of integration, with only two choices of orientation if we required such information. For example, given the usual Riemann integration, if we take

$$\int_a^b f(x)dx$$

there is only one direction we can go from a to b[3]. In 2 dimensions, there are many possible paths we can take between two points. We've seen in calculus that the natural setting in which to think about integrating such paths are integration over differential forms:

$$\int_\gamma \omega$$

where $\omega = f_1 dx + f_2 dy$ for some (at least continuous, usually C^1) function f and a (at least piece-wise continuous, usually piece-wise C^1) path γ. We saw that under the right condition, $\int_\gamma \omega$ is path independent (on a connected set), and other times there exists an F such that $\omega = dF$. If f has the first property, it is said to be *conservative*, and if f has the second property it is said to have a primitive. If f is holomorphic, then we are *guaranteed* that fdz ($dz = dx + idy$) is conservative, and if the domain is simply connected then it even has a primitive! In a sense, this can be seen as a feature that would be greatly lacking if it wasn't the case: By the FTC this is true of f is real differentiable, and so we would certainly expect a complex equivalent of the FTC. On $\mathbb{R}$, a non simply-connected set is simply not a connected set, which makes the introduction of requiring a simply connected set a novelty in the case of $\mathbb{C}$ which is not "visible" in the case of $\mathbb{R}$.

We now bring our discussion to 1-forms of the form $\omega = f(z)dz$ where f is a *holomorphic function.* The first fundamental result is that on simply connected open sets, ω has a primitive, hence generalizing the FTC for complex integration of curves. Since this is the generalization of FTC, this result can also be seen as a special case of the generalized Stokes theorem.

[3]Granted, we may change the speed at which we get there, but the change of variables theorem tells us the value does not change

Theorem 3.2.1: Cauchy's (Integral) Theorem

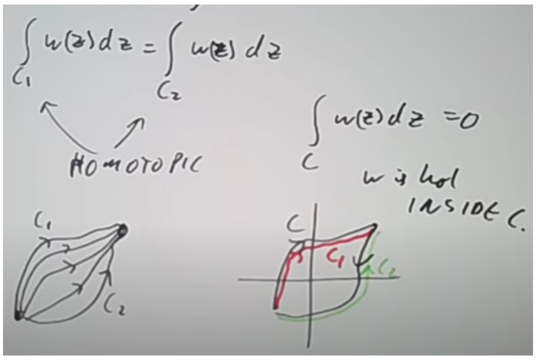
Let $U \subseteq \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $\gamma : [a, b] \to U$ be a smooth closed curve. If γ is homotopic to a constant curve, then

$$\int_\gamma f(z)dz = 0$$

Equivalently, if γ_1 and γ_2 are homotopic, then:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Another way of stating this theorem is that if $f(z)$ is holomorphic in an opn set D of the complex plane, then $f(z)dz$ is a closed form in D (where $dz = xdx + iydy$). To see why we may either take a closed form to be zero or two paths with the same fixed point, you may stare at this image:



There are many proofs of this, depending on how strong of assumptions we have. Here is a quick intuitive proof if we also have access to Green's Theorem. Recall that Green's theorem states that for a closed path γ:

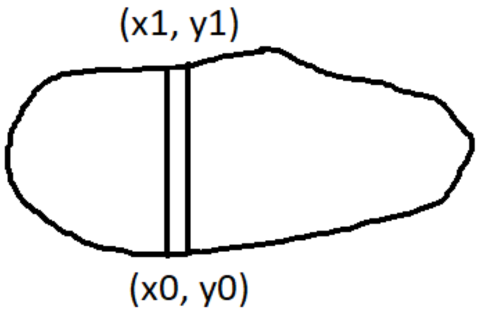
$$\int_\gamma f(x,y)dx + g(x,y)dy = \int\int_{\text{int}(\gamma)} \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dxdy$$

that is, we may integrate over the boundary of γ, or we integrate over the interior region. We'll recall the sketch of the proof of Green's theorem on a convex region for simplicity. We may simpilfy further by just showing

$$\int_\gamma f(x,y)dx = \int\int_{\text{int}(\gamma)} -\frac{\partial f}{\partial y} dxdy$$

since we would do the same proof for the g component and sume the two. Take some convex region:

Take some vertical strip, and select two points on the bottom and top border:



Now, by the fundamental theorem of calculus, we get that:

$$\int_{y=y_0}^{y_1} \frac{\partial f}{\partial y} dy = f(y_1) - f(y_0)$$

Now, take the above equation, and integrate both sides with respect to x:

$$\int\int_{\text{int}(\gamma)} \frac{\partial f}{\partial y} dy = \int_x \int_{y=y_0}^{y_1} \frac{\partial f}{\partial y} dy = -\int_\gamma f(x,y)dx$$

where the integarl is negative because we integrated anti-clockwise, if we integrated clockwise we'd get a positive. But this is the result. Then for most cases of Cauchy's theorem, we would have a finite union of convex regions, which suffices for our purposes.

Proof **:**
Let γ be a closed path and f a holomorphic function. With Green's Theorem, we simply compute:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(dx+idy)\\ &= \int_\gamma udx - vdy + i\int_\gamma udy + vdx\\ &= \int\int_{\text{int}(\gamma)} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + \int\int_{\text{int}(\gamma)} \left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy\\ &\overset{!}{=} 0\end{aligned}$$

where the $\overset{!}{=}$ inequality comes by using the Cauchy Riemann equations!

Using Wirtinger derivatives, we may compress the entire theorem into the following one-liner:

$$\int_\gamma f(z)dz = \int\int_{\text{int}(\gamma)} \frac{\partial f}{\partial \bar{z}} d\bar{z}dz = 0$$

where the first equality is green's theorem and the rest is the compressed version of what we just did.

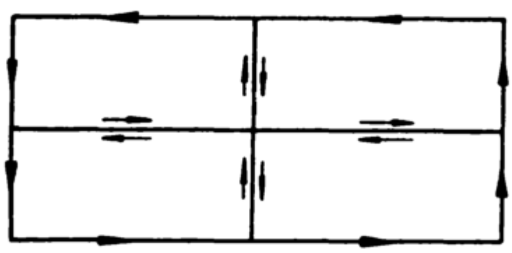
see this video for where the proof comes from: (Richard Brocherd in here)

The theorem can be proven by just assuming f is holomorphic without yet knowing the partials are continuous:

Proof **:**
We want to show $f(z)dz$ is closed, so we must prove that the integral $\int_\gamma f(z)dz$ is zero along the boundary γ of any rectangle R contained (with its interior) in D. Let's say that γ is the boundary of R and

$$\int_\gamma f(z)dz = p(R)$$

We want to show $p(R) = 0$. Subdivide R into 4 equal sided rectangles, labeled R_i and define paths γ_i like so:



It is easy to see that

$$\int_\gamma f(z)dz = \sum_i^4 f(z)dz = \sum_i^4 p(R_i)$$

Thu,s there must be at least one rectangle such that

$$|p(R_i)| \geq \frac{1}{4}|p(R)|$$

Let' say this recantlges is $R^{(1)}$. Then we may subdivide $R^{(1)}$ into four equal rectangles, and again at least one of which, say $R^{(2)}$ will satisfy the condition $|p(R^{(2)})| \geq \frac{1}{4^2}|p(R)|$. We can repeat this indefinitely, and from it define a sequence sequence that converges to $z_0 \in D$ (in particular f is holomorphic at z_0). Thus

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)|z - z_0| \tag{3.1}$$

where the error term approaches 0 as $z \to z_0$, that is $\lim_{z \to z_0} \epsilon(z) = 0$.
Now, if $\gamma(R^{(k)})$ is the orineted boundary of the rectangle $R^{(k)}$, then we get

$$\left| \int_{\gamma(R^{(k)})} f(z)dz \right| \geq \frac{1}{4^k}|p(R)| \tag{3.2}$$

Computing the left hand side using equation (3.1), we get

$$\int_{\gamma(R^k)} f(z)z = f(z_0) \int_{\gamma(R^{(k)})} dz + f'(z_0) \int_{\gamma(R^{(k)})} (z - z_0) + \int_{\gamma(R^{(k)})} \epsilon(z)|z - z_0|dz$$

on the right hand side of the above equation, the first two integrals are zero, and the third becomes negligible with the area of the rectagnle $R^{(k)}$ as k increases indefinitely, and hence is negligible as compared with $\frac{1}{4^k}$. Thus, when comparing with equation (3.2), we get that $p(R) = 0$. Hence, we get that

$$\int_\gamma f(z)dz = 0$$

completing the proof.

Corollary 3.2.1: Local Primitive For Holomorphic Functions

Let f be a holomorphic function in an open neighbourood $\Omega \subseteq \mathbb{C}$. Then f locally has a primitive which is holomorphic.

***Proof* :**
By Cauchy's Theorem, $f(z)dz$ is a closed form, and hence has a local primitive. The local primitive is holomorphic since the complex derivative exists (the differentiation being path-independent and equalling $f(z)$)

For technical purposes and that of our next theorem, we shall temporarily require a slightly more general result to Cauchy's Theorem

Corollary 3.2.2: Cauchy Theorem On All But line of Points

Let $f(z)$ be a continuous function in an open set D which is holomorphic at every point of D except perhaps at the points of a line Δ parallel to the real axis. Then the form $f(z)dz$ is closed. In particular, if f is holomorphic at any point of D except perhaps at some isolated points, then the form $f(z)dz$ is closed

***Proof* :**
This is simply an epsilon argument bounding away the bad points with a negligible extra area.

We want to show that $\int_\gamma f(z)dz$ is zero for the boundary γ of any rectangle contained in D. This is obvious if the rectangles does not intesect the line Δ, so suppose it has a side contained in Δ. Let $u, u+a, u+ib, u+a+ib$ be the four corners of the rectangle, and $u, u+a$ being on the line Δ; a and b are real and without loss of generailty we may assume $b > 0$. Let $R(\epsilon)$ be the rectangle with corners

$$u + i\epsilon, u + a + i\epsilon, u + ib, u + a + ib$$

with $\epsilon > 0$. The integral $\int f(z)dz$ is zero on the boundary of $R(\epsilon)$, and as $\epsilon \to 0$, this integral tends to the integral round the boundary γ of the rectangle R. Certainly the limit passes through since we are dealing with compect sets, and hence $\int_\gamma f(z)dz = 0$.

Finally, let's say Δ intersects the rectangle without containing the entire boundary (i.e. it intersects it at two points). Then the line Δ splits R into two rectangles R', R'', and we may repeat the previous argument to show that $\int f(z)dz$ is zero around the boundaries of both R' and R'', and so the sum of these rectangles is equal to the integarl $\int_\gamma f(z)dz$, completing the proof.

This next theorem combines many results of holomorphic functions that at first seems to state a fact that should be generally true, but is in fact quite particular to our current build-up:

Theorem 3.2.2: Cauchy's Integral Formula

Let f be a holomorphic function in an open set Ω. Let $a \in \Omega$ and let γ be a closed path of Ω which does not pass through a and which is homotopic to a point in Ω. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = I(\gamma, a)f(a)$$

where $I(\gamma, a)$ denotes the winding number of the closed path γ with respect to a. In particular, if γ winds around a once, then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a)$$

***Proof* :**
Let $g : \Omega \to \mathbb{C}$ be the function in Ω defiend by

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(z) & z = a \end{cases}$$

which is continuous by the definition of the derivative. Then g is holomorphic at any point of Ω except the point a. By corollary 3.2.2 we thus have for any path γ that is nullhomotopic:

$$\int_\gamma g(z)dz = \int_\gamma \frac{f(z) - f(a)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}dz - \int_\gamma \frac{f(a)}{z-a}dz = 0$$

which implies:

$$\int_\gamma \frac{f(z)}{z-a} = \int_\gamma \frac{f(a)}{z-a}$$

For the right hand side, by definition:

$$\int_\gamma \frac{f(a)}{z-a}dz = f(a)\int_\gamma \frac{1}{z-a}dz = 2\pi i I(\gamma, a)f(a)$$

Thus, substituting back in we get and moving terms around we get:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = I(\gamma, a)f(a)$$

If γ winds around a once, then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a)$$

completing the proof.

Some important observations are in order:

1. There is no analogy for this in $\mathbb{R}$. Any path in $\mathbb{R}$ that is closed will cannot go around the singularity $x-t$. Even if we tried to fix this somehow, we would get $0 = \int_0^0 f(t)/(x-t)dt = f(x)$, which is simply not true, there are many smooth non-constant functions.

2. We cannot generlize this for $\mathbb{R}^n$ differentiation since we cannot divide ($\mathbb{R}^n$ is not a field[4]). In fact, the only complete division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, and only $\mathbb{R}$ and $\mathbb{C}$ are fields, hence Cauchy's Integral theorem is a particularity of the structure of $\mathbb{C}$

3. Notice how γ can be *any* path as long as it's nullhomotopic, while $f(a)$ is fixed. This shows the rigidity the global nature of holomorphic functions!

4. if a function g is of the form:

$$\int_\gamma g(z)dz = \int_\gamma \frac{f(z)}{z-z_0}dz$$

   Then we can immediately evaluate $g(z)$:

$$\int_\gamma g(z)dz = f(z_0)$$

   We shall return to this soon (see section ref:HERE)

[4]In particular, there is not field extension from $\mathbb{R}$ that has degree greater than 2

The key result we may conclude with this theorem is that in an open disc, any holomorphic function can be expanded as a power series

Theorem 3.2.3: Taylor Expansion Of Holomorphic Function

Let $f(z)$ be a holomorphic function in the open disc $|z| < \rho$. Then f can be expressed as a power series in this disk. In particular, there exists a power series $p(x) = \sum_i^\infty a_n x^n$ whose radius of convergence is $r \geq \rho$ and whose sum $p(z)$ is equal to $f(z)$ for $|z| < \rho$.

Proof **:**
Let $r < \rho$. We shall find a power series which uniformally convergent to $f(z)$ for $|z| \leq r$. By the uniquness of the power series expansion, this power series is indepednent of our choice of r.

Choose an r_0 such that $r < r_0 < \rho$. Then by theorem 3.2.2, by taking γ to be the radius of the circle r_0 centered at 0 going counter-clockwise (i.e. in the positive direction), then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t - z} dt$$

for $|z| \leq r$. The function $1/(t - z)$ can be expanded to a series for $|z| < |t|$, in particular

$$\frac{1}{t - z} = \frac{1/t}{1 - z/t} = \frac{1}{t}(1 + \frac{z}{t} + \frac{z^2}{t^2} + \cdots + \frac{z^n}{t^n} + \cdots)$$

Thus, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}} dt$$

The series converges uniformally for $|z| \leq r$ and $|t| = r_0$. Thus, we may integrate term by term an we obtain a uniformally convergent series for $|z| \leq r$:

$$\begin{aligned} f(z) &= \int_\gamma \frac{1}{2\pi i} \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}} dt \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t^{n+1}} dt \right) z^n \\ &= \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

where the coefficient are given by the integral formula

$$a_n = \frac{1}{2\pi i} \int_{|t| = r_0} \frac{f(t)}{t^{n+1}} dt$$

as we sought to show.

Corollary 3.2.3: Holomorphic, Then Analytic

Let f be a holomorphic function on a domain Ω. Then f is analytic

***Proof* :**
For any point $a \in \Omega$ of the domain of the holomorhpic function f, by theorem 3.2.3 we may write f as a power series, and hence it is locally a power series, and hence it is analytic.

Corollary 3.2.4: Entire, Then Power Series

Let f be holomorphic on $\mathbb{C}$. Then f is a power series.

***Proof* :**
In theorem 3.2.3, we may take $r \in \mathbb{R}_+$ since $\rho = \infty$. Note that we could have established this as well by using analytic continuity.

As a consequence, every holomorphic function is smooth, and every derivative is itself holomorphic. As we an recover the coefficients of $f(z) = \sum_n^\infty a_n(z - z_0)^n$ by taking the derivative:

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

we have:

$$\frac{1}{2\pi i} \int_{|t|=r_0} \frac{f(t)}{t^{n+1}} dt = \frac{1}{n!} f^{(n)}(z_0)$$

Using the fact that holomorphic functions must be analytic, we can prove the converse of Cauchy's theorem, showing us that all closed forms on $\Omega \subseteq \mathbb{C}$ are holomorphic:

Theorem 3.2.4: Morera's Theorem

Let $f(z)$ be a continuous function in an open (connected) set $\Omega \subseteq \mathbb{C}$. If the differential form $f(z)dz$ is closed, then the function $f(z)$ is holomorphic in D

The connected condition is simply to make the proof nicer.

***Proof* :**
$f(z)dz$ has a holomorphic primitive g locally, and by definition $g'(z) = f(z)$. Since $f = g'$ is the derivative of a holomorphic function, it is itself holomorphic. Since Ω is connected, we see that $g' = f$ on Ω. Hence, f is holomorphic.

Thus, we have characterized the 1-forms ω on $\mathbb{R}^2$ that are closed, namely they are of the form $\omega = f(z)dz$ for holomorphic $f(z)$. From this perspective, the Cauchy-Riemann equations are a geometric property: they guarantee that $d\omega = 0$.

Corollary 3.2.5: Holomorphic On All But Some Points, Then Holomorphic

Let f be continuous on D and holomorphic at all points of D except perhaps at points of some lie Δ. Then f is holomorphic on all points of D

Proof **:**
Without loss of generailty we may suppose Δ is parallel to the real axis by rotating if necessary. Then we have by corollary 3.2.2 that $f(z)dz$ is closed. Then by Morera's theorem f is holomorphic at all points of D, completing the proof.

From this perspective, corollary 3.2.2 only seemed like a generalization, however it was important to establish for technical reasons. As a consequence, if f fails to be holomoprhic at an isolated point, f cannot be continuous at that point. let's say x is the point at which f is not holomorphic and not continuous. We shall see in proposition 3.3.3 that if f is bounded around x, then x holomorphically extends to x, and hence it has to be that f is *unbounded* at x, i.e it has to be undefined. We shall explore the properties of such points in section 3.3.

Using corollary 3.2.5, we may holomorphically extend a function in the way that we are about to describe, this type of extension is known as *Schwarz' principle of symmetry.*

Proposition 3.2.1: Schwarz Principle Of Symmetry

Let D be a non-empty, connected, open set D which is symmetric with respect to the real axis. Let D^+ be the intersection of D with the half-plane $y \geq 0$ and D^- be the intersection of D with the half plane $y \leq 0$. Let's say $f(z)$ is continuous on D^+, and holomorphic on the points of D^+ where $y > 0$. Then we may extend f to all of D

Proof **:**
Consider $g(z)$ defined on D^- like so

$$g(z) = \overline{f(\overline{z})}$$

By our discussion in section 2.2.1, g is holomorphic on all D^- where $y < 0$. Then the function $h(z)$ which is $f(z)$ on D^+ and $g(z)$ on D^- is continuous on D and holomorphic on all of D except perhaps the points on the real axis. Thus, by corollary 3.2.5, it is holomorphic on all of D.

3.2.1 Growth of Holomorphic Functions

Let f is holomorphic in an open set Ω. Then as we just saw, $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in some ball of radius r around z_0. Then any power series is determined by its coefficient, which can partly be studied by looking at their growth which is the goal of this section.

We know that the coefficients of this power series can be recovered by taking derivatives of f, namely

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

We shall show two very important results:

1. that the growth of these coefficient is bounded in an important way. If you know some real analysis, then since f has been shown to be analytic, it is "obvious" since a smooth function which has bounded derivative growth rate is analytic

   (see, for examples, Pughs real analysis or EYNTKA real Analysis I)

2. that $f(z)$ is always the average of a the integral of a circle of points (it satisfies the man value property, the integral equivalent of Harmonic)

These are analytical results, and shall even have connection to harmonic analysis (which we shall touch on). Hence, this section can be considered analyzing holomorphic functions from an analysis lens. We start by how the coefficients of the Taylor series are bounded:

Theorem 3.2.5: Cauchy Inequalities

Let f be a holomorphic function in Ω, and pick any $z_0 \in \Omega$ from which we take a power series representation around z_0: $f(z) = \sum a_n(z - z_0)^n$ with radius R. Then for each coefficient we have the bound, for each $r < R$:

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| = |a_n| \leq \frac{\sup_\theta(f(z_0 + re^{i\theta}))}{r^n} = \frac{M(r)}{r^n} \qquad n \in \mathbb{N}$$

where $M(r) = \sup_\theta \left(z_0 + f(re^{i\theta})\right)$. These inequalities are called the *Cauchy inequalities*.

A key observation is that this bound works for all $n \in \mathbb{N}$, that is, the bound works for all derivatives. If we fix some r so that we get a fixed $M/r^n = M'$, we can re-write the equation as:

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| \leq M' \qquad \forall n \in \mathbb{N}$$

Proof **:**
Pick $z_0 \in \Omega$ and write the Taylor series expansion at z_0

$$f(z) = \sum_n a_n(z - z_0)^n$$

with radius of convergence $|z - z_0| < R$. Then for $r < R$, the input may be written as: $z = z_0 + re^{i\theta}$. giving us:

$$f(z_0 + e^{i\theta}) = \sum_n a_n r^n e^{in\theta} \tag{3.3}$$

Fixing r and allowing θ to vary, $g_r(\theta) = f(z_0 + re^{i\theta})$ is a periodic function with respect to θ and the right hand side converges uniformally when θ varies and r is fixed (since the circle is compact). In fact equation (3.3) is the fourier expansion of f. Since the series converges uniformally, we may integrate term by term the following usual integral from fourier analysis:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} f(z_0+re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n\geq 0} a_n r^n e^{i(n-p)\theta}d\theta \\
&= \sum_{n\geq 0}\frac{1}{2\pi}\int_0^{2\pi} a_n r^n e^{i(n-p)\theta}d\theta \qquad \text{abs. convergent} \\
&= \sum_{n\geq 0}\frac{1}{2\pi}a_n r^n\int_0^{2\pi} e^{i(n-p)\theta}d\theta \\
&\overset{!}{=} a_p r^p
\end{aligned}$$

for the $\overset{!}{=}$ equality on the right hand side, we see that integrating against $n \neq p$ gives 0, while integrating against $p = n$ gives 2π, which cancels with $1/2\pi$, and hence we get a single constant giving us:

$$a_p r^p = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} f(z_0+re^{i\theta})d\theta \tag{3.4}$$

which holds for all $r < R$. If this holds for $r = 1$, we get a simple formula for the coefficients:

$$a_p = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} f(z_0+e^{i\theta})d\theta \left[\text{ if } z_0 = 0\text{: } = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} f(e^{i\theta})d\theta\right]$$

Note that we could have deduced this from theorem 3.2.3 by using

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} z^n \frac{f(t)}{t^{n+1}}dt$$

since

$$a_n = \frac{1}{2\pi i}\int_{|t|=r_0}\frac{f(t)}{t^{n+1}}dt$$

which we leave as an exercise. With equation (3.4), we can upper bound the value of a_n: Let $M(r) = \sup_\theta |f(z_0 + re^{i\theta})|$ where r is fixed. Then the right hand side of equation (3.4) becomes

$$|a_p r^p| = \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} f(z_0+re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|e^{-ip\theta}\right| M(r)d\theta = \frac{2\pi}{2\pi}M(r) = M(r)$$

Hence, we get:

$$|a_p| \leq \frac{M(r)}{r^p} = \frac{\sup_\theta(f(z_0+re^{i\theta}))}{r^p} \qquad p\in\mathbb{N},\ \forall r < R$$

Letting $M(r) = \sup_\theta(f(z_0 + re^{i\theta}))$ we get:

$$|f^{(n)}(z_0)| \leq \frac{M(r)n!}{r^n},\ \forall r < R$$

Due to their importance, we shall put a box around the definition:

Definition 3.2.1: Cauchy Inequalities

Let f be a holomorphic function and $f(z) = \sum a_n z^n$. Then for each coefficient we have the family of bounds:

$$|a_n| \leq \frac{M(r)}{r^n} = \frac{\sup_\theta(f(re^{i\theta}))}{r^n} \qquad n \in \mathbb{N}, \ \forall r < R$$

and similarly:

$$|f^{(n)}(z_0)| \leq \frac{M(r)n!}{r^n} = \frac{\sup_\theta(f(re^{i\theta}))}{r^n} \qquad n \in \mathbb{N}, \ \forall r < R$$

which vary with for all $r < R$. These inequalities are called the *Cauchy inequalities*.

The Cauchy inequalities can be used to great effect as they force the bounds on the growth of f, in particular each derivative $f^{(n)}$ is bounded in growth by values of f, and the limitation becomes "weaker" as n gets bigger. Note that if is just f smooth, it is not guaranteed that these bounds are finite, in fact if the bounds are finite as they are in Cauchy's inequalities then a real smooth function is analytic, showing that the real limitor behind analyticity is rate of growth (see Pugh's Real Analysis). As practice, it should be shown that if f is holomorphic in a disk $|z| < R$, and that if $|z| \leq r < R$, then:

$$|f^{(n)}(z)| \leq \frac{M(r)n!}{(R-r)^n}$$

This limitations on the growth of coefficients puts lots of important restraint on holomorphic functions, perhaps the most famous such example is the following theorem:

Theorem 3.2.6: Louville's Theorem

Let f be a holomorphic function on $\mathbb{C}$. Then if f is bounded, f is constant

Notice how this is certainly not true for $\mathbb{R}$: $\sin, \cos, e^{-1/x^2}$ are all smooth on $\mathbb{R}$ and bounded but certainly not constant. Notice further how this shows the importance of the domain being $\mathbb{C}$: e^{1/x^2} can be complexified to e^{1/z^2} which is unbounded on $\mathbb{C}$, but *is* bounded on a strip $[-i, i] \times \mathbb{R}$. We shall also find functions that are bounded on the upper half plane $\mathbb{H}$ ($\frac{1}{z+i}$ is an example), showing that it is not simply that the function is unbounded.

***Proof* :**
Since f is holomorphic in $\mathbb{C}$, it is a power series, hence express f in terms a of a power series around 0. By the discussion above, each coefficient a_n are bounded by Cauchy's inequalities.

$$|a_n| \leq \frac{M(r)}{r^n}$$

Since f is bounded, $M(r) \leq M$ for some positive M independent of r, and so we have

$$|a_n| \leq \frac{M}{r^n}$$

for any $r \in \mathbb{R}$. But then that means means that the above inequality holds true for all r, and so it must be that $a_n = 0$. since this is true for all $n \geq 1$, we see that f is in fact constant

One famous consequence of this is that any complex polynomial must have a root

Theorem 3.2.7: D'Alembert's Theorem

Let $p(x)$ be a non-constant complex polynomial. Then $p(x)$ has at least one root.

Proof **:**
For the sake of contradiction, assume $p(x)$ is non-constant and and $p(z) \neq 0$ for any $z \in \mathbb{C}$. Then $1/p(z)$ is holomorphic on the whole plane. It is bounded, since

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n = z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \right)$$

for $a_n \neq 0$ tends to infinity as $|z|$ tends to inifnity, so there is compact disc outside of which $|1/p(z)|$ is bounded. On the other hand $|1/p(z)|$ is bounded inside the compact disc since it is a continuous function. Hence $1/p(z)$ is bounded in the whole plane and hence is constant by Louiville's theorem. But that implies $p(z)$ is constant, contradicting our hypothesis, and completing the proof.

Using the Cauchy inequalities, we see that there are relatively few holomorphic functions in each growth class ($O(f(n))$). The following gives two simples examples of this:

Proposition 3.2.2: Bounding Growth Of Holomorphic Function

Let $f(z)$ is an entire function (i.e. holomorphic in $\mathbb{C}$). Then:

1. If $|f(z)| < 1 + |z|^{1/2}$, f is constant
2. If $|f(z)| < 1 + |z|^n$, f is a polynomial of degree $\deg(f) \leq n$

Proof **:**

1. By the taylor expansion of Holomorphic functions, let

   $$f(z) = \sum a_n z^n$$

   where

   $$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt$$

   Since f is entire, f agrees with this power series on all of $\mathbb{C}$ (we may use analytic continuity or directly use the Taylor expansion theorem). Then, bounding the coefficients, we get:

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt \right| \\
&\leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(t)}{t^{n+1}} \right| dt \\
&\leq \frac{1}{2\pi} \int_{|z|=r} \frac{1+t^{1/2}}{t^{n+1}} dt \\
&= \frac{2\pi}{2\pi} \frac{r(1+r^{1/2})}{r^{n+1}} \\
&= \frac{1+r^{1/2}}{r^n}
\end{aligned}$$

since this holds as $r \to \infty$, we see that $|a_n| = 0$, hence f is constant. Since $f(0) = a_0$, we get that $|f(0)| \leq 1$, so f is a constant function in with image in $[-1, 1]$, completing the proof.

2. By the taylor expansion of Holomorphic functions, let

$$f(z) = \sum a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt$$

Then

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t^{n+1}} dt \right| \\
&\leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(t)}{t^{n+1}} \right| dt \\
&\leq \frac{1}{2\pi} \int_{|z|=r} \frac{1+t^k}{t^{n+1}} dt \\
&= \frac{2\pi}{2\pi} \frac{r(1+r^k)}{r^{n+1}} \\
&= \frac{1+r^k}{r^n}
\end{aligned}$$

Hence, if $n > k$, we see that the limit as $r \to \infty$ goes to 0, and so we see that the Taylor expansion of $f(z)$ must be

$$f(z) = \sum_{k=0}^{n} a_n z^n$$

and so it's a polynomial. Since f is entire, by analytic continuity it must be that f is equal to this polynomial everywhere, hence f is a polynomial completing the proof.

Recall that holomorphic functions are harmonic, and that harmonic functions satisfy the maximum modulus principle (theorem 2.2.2) which allows us to upgrade the bounds given by the Cauchy inequality to a bound on the growth of f. If $n = 0$, then the Cauchy inequality along with the maximum modulus principle tells us that:

$$|a_0| = |f(z_0)| \leq \frac{\sup_{z\in\partial\overline{D_{z_0}}} |f(z)|}{r^0} = \sup_{z\in\partial\overline{D_{z_0}}} |f(z)| = \sup_{z\in\overline{D_{z_0}}} |f(z)|$$

or written differently:

$$|f(z)| \leq \sup_{\theta} |f(z_0 + re^{i\theta})| \qquad |z| \leq r < R$$

The maximum modulus principle will extend this result to the entire disk:

Corollary 3.2.6: Cauchy Inequalities And Maximum Modulus Principle

Let f be holomorphic at z_0 and $M(r) = \sup_\theta |f(z_0 + re^{i\theta})|$. Then

$$|f(z)| \leq M(r) \qquad \forall |z| \leq r$$

Proof **:**
By the maximum modulus principle, the maximum of $|f(z)|$ in some compact neighbourood is achieved on the boundary. let $f(z_0)$ be the point such that $|f(z_0)|$ is the maximum, then $|f(z_0)| = M(r)$, hence

$$|f(z)| \leq M(r) \qquad |z| \leq r$$

More generally,

$$|f^{(n)}(z_0)| \leq \frac{\sup_\theta(f(re^{i\theta}))}{r^n} \qquad n \in \mathbb{N},\ \forall r < R$$

completing the proof.

3.2.2 Mean Value Property and Schwarz' Lemma

We know examine the bahavior of $f(z)$ instead of the growth of its derivative, which corresponds to the case where $n = 0$. We get the following family of equalities:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \qquad r < R$$

which varies for all all r less than the radius of convergence: $r < R$. Since $z^0 = 1$, this can equivalently be written as:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

This means that $f(z_0)$ is the average (the mean value) on the any circle centered at z_0 of radius r (reflecting the harmonic property of holomorphic function discussed in section 2.2.2). Thus, if S_{z_0} is a closed disk centered at z_0 that is contained in contained in an open set D_{z_0} centered at z_0 and f is holomorphic in the interior of S_{z_0}, the value of f at the centre of S_{z_0} is its mean value of the boundary of S_{z_0}. This is a rather powerful property, and so we distinguish it in hopes of finding it in other sorts of functions[5]

Definition 3.2.2: Mean Value Property

Let f be a complex-valued continuous function. Then f is said to respect the *mean value property* if for all $r \in \mathbb{R}$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta$$

As mentioned in section 2.2.2, all harmonic functions have this property, and and in fact all functions satisfying the mean value property are harmonic, but not all harmonic functions are holomorphic (easy counter-example is $z \mapsto \operatorname{Re} z$). We shall explore more this generalization in chapter 7. For our purposes with holomorphic functions, the mean value property shall allow us to upgrade Cauchy's inequalities to hold true for as set of points, not just the single point at which the power series is centered. For this, we require the following key propety of funcitons satisfying the mean value property:

Theorem 3.2.8: Maximum Modulus Principle via Averaging

Let f be a continuous complex-valued function in an open set D of the plane $\mathbb{C}$. If f has the mean value property and $|f|$ has a relative maximum at a point $a \in D$ (i.e. $|f(z)| \leq |f(a)|$ for any z sufficinetly close to a), then f is constant in some neighbourood of a

***Proof* :**
If $f(a) = 0$, then the result is immediate, so suppose that $f(a) \neq 0$. By multiplying f by a compelx constant if necessary, we can reduce the theorem to the case where $f(a)$ is real and > 0. Define:

$$M(r) = \sup_\theta |f(a+re^{i\theta})|$$

by maximality hypothesis, we have that for sufficicently small $r \geq 0$, we have $M(r) \leq f(a)$. By the mean value property,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta \leq M(r) \tag{3.5}$$

thus $f(a) \leq M(r)$, and so $f(a) = M(r)$. Thus, the following function is always ≥ 0 for sufficinetly small $|z-a| = r$

$$g(z) = \operatorname{Re}(f(a) - f(z))$$

and $g(z) = 0$ if and only if $f(z) = f(a)$. By equation (3.5), the mean value of $g(z)$ on the circle $|z-a| = r$ is zero. Since g is continuous and ≥ 0, this means that g is identically zero on this circle, and so $f(z) = f(a)$ when $|z-a| = r$ when r is sufficinetly small. Since this is true for all r smaller, we have that f is constnat on a sufficiently small neighbourood, completing the proof.

If D is a connected set, then we can conclude that there is no functions satisfying the mean value property that is locally constant without being constant:

[5] see chapter 7

Corollary 3.2.7: Harmonic and max Attained in Interior, then Constant

Let D be a bounded, connected, open subset of $\mathbb{C}$ and f be a complex-valued continuous function defined on $\overline{D}$ having the mean value property in D. Then if M is the upper bound of f and $|f(a)| = M$ at some point $a \in D$, then f is constant

Proof **:**
Recall by theorem 2.2.2 that $\sup_{z\in\overline{D}} |f(z)| = \sup_{z\in\partial\overline{D}} |f(z)| = M$. Let's say that there is a $a \in D$ such that $|f(a)| = M$. Then by theorem 3.2.8 f is constnat in some neighbourood of a. This set is certainly open, and by doing a limit argument we see that it is also certainly closed, and it contains a meaning it's non-empty, and so it must be all of $\overline{D}$. Hence $f(z) = f(a)$ for all $z \in \overline{D}$

As we see, the Maximum modulus principle puts more strong constraints on the growth of holomorphic functions. The following theorem is one application of this, and shows how it can even force linearity of a function in the right situation (this can be thought of as an extension of proposition 3.2.2)

Theorem 3.2.9: Schwarz' Lemma

Let f be a holomorphic function in the disk $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Assume that:

$$f(0) = 0 \qquad |f(z)| < M \qquad \forall z \in \mathbb{D}$$

Then

1. $|f(z)| \leq M|z|$ for all $z \in \mathbb{D}$
2. if for some $z_0 \neq 0$, we have $|f(z_0)| = |z_0|$ and $|f(z)| < 1$, then in fact
$$f(z) = \lambda z$$
for some λ such that $|\lambda| = 1$

Proof **:**
By Taylor's expansion theorem, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in some neighborhood of 0, and so the coefficient a_0 is zero since $f(0) = 0$. Thus, $f(z)/z$ is holomorphic in $\mathbb{D}$ as well. Since $|f(z)| < M$ by assumption, we have

$$\left|\frac{f(z)}{z}\right| < \frac{M}{|z|} = \frac{M}{r} \qquad |z| = r$$

By the maximum modulus principle, this holds true for all $|z| \leq r$. Fixing $z \in \mathbb{D}$, we get $|f(z)| < M|z|/r$ for any $r \geq |z|$ and $r < 1$. In the limit as $r \to 1$, we get

$$|f(z)| \leq M|z|$$

which establishes (1).

If $|f(z)| < 1$ and $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then the holomorphic function $f(z)/z$ attains the upper bound of its modulus at a point in the interior $|z| < 1$, thus by corollary 3.2.7

$$f(z)/z = \lambda$$

for some λ such that $|\lambda| = 1$. But then $f(z) = \lambda z$, completing the proof of (2).

This theorem is in fact more powerful than it first may seem. Theorem 4.3.4 in section 4.3 will show that all simply connected proper subets of $\mathbb{C}$ are biholomorphic to the unit disk. Hence, this theorem gives us a blueprint for bounding conditions on a larger larger family of functions!

Exercise 3.2.1

1. Show that *not* every power series has a root
2. Let φ be a continuous function, γ and oriented boundary of a compact set K, and $D = \mathbb{C} \setminus \{\Gamma\}$ and define on D:
$$f(z) = \int_\Gamma \frac{\varphi(t)}{t - z} dt$$
Show that for $a \in D$, f can be expanded as a taylor series with radius $\rho = \inf_{\gamma \in \Gamma} |\gamma - a|$, making f analytic. In this way, the above formula is a transformation from the set of continuous function to a set of analytic functions!

3.3 Meromorphic Functions and Laurent's Expansions

So far, we've been studying holomorphic function's without much concern for singularities (for example, $f(z) = 1/z$ is holomorphic except at $z = 0$ where it has a singularity). Furthermore, our integration techniques where all up to homotopy, and Cauchy's Theorem applies to simply connected domains. This leaves out a many important functions, as we've shown that rational functions are (up to the roots of the denominators) holomorphic functions. As we've seen with $1/z$, the behavior of holomorphic functions around singularities changes. When it comes to studying the singularities, Taylor series start getting limited. We thus extend the notion of Taylor series in effect in insure a good tool for studying singularities, namely we shall add negative terms to our power series to deal with poles:

Definition 3.3.1: Laurent Series

A *Laurent series* is a formal infinite series of the form:

$$\sum_{n \in \mathbb{Z}} a_n x^n$$

where n can vary accros all integers.

Any Laurent series is assocaited with two formal power series

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=-1}^{-\infty} a_n x^{-n}$$

If ρ_1 was the radius of convergence of the first and $1/\rho_2$ was the radius of convergence of the second one, then the original laurent series would have an *annulus of convergence* $\rho_2 < z < \rho_1$.

Proposition 3.3.1: Laurent Series Is Holomorphic

Let f be a Laurent series convergent in $\rho_2 < z < \rho_1$. Then f is holomorphic and the laurent series converges uniformally in the anulus $\rho_2 \leq z \leq \rho_1$ and absolutely in $\rho_2 < z < \rho_1$.

***Proof* :**
Let's say that we have convergent Laurent series:

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for} \qquad |z| < \rho_1 \tag{3.6}$$

$$f_2(z) = \sum_{n=-1}^{-\infty} a_n z^n \qquad \text{for} \qquad |z| > \rho_2 \tag{3.7}$$

We start by showing that f_2 is holomorphic with respect to z. Putting $z = \frac{1}{u}$, we get

$$g(u) = \sum_{n=0}^{\infty} a_{-n} u^n$$

which is holomorphic for $|u| < 1/\rho_2$ and it's derivative is given by

$$g'(u) = \sum_{n=0}^{\infty} n a_{-n} u^{n-1}$$

Then differentiating and using the chain rule, we get

$$f_2'(z) = \frac{-1}{z^2} g'(1/z) = \sum_{n=-1}^{-\infty} n a_n z^{n-1}$$

Thus, the series f_2 is differentiable term by term for $|z| > \rho_2$, and hence holomorphic. From now on in the proof, assume $\rho_2 < \rho_1$. Then the sum of the series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

is holomorphic in the annulus $\rho_2 < |z| < \rho_1$ and it's derivative $f'(z)$ is the sum of the series $\sum_n a_n z^{n-1}$ given by differentiating term by term. Note that Laurent series converge uniformally in the annulus $r_2 \leq |z| \leq r_1$ where

$$\rho_2 < r_2 < r_1 < \rho_1$$

Finally, for uniqueness, in the uniformally convergent part of the annulus, if we integrate

$$f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n r^n e^{in\theta}$$

with respect to θ, we get just like before the integral formula

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta$$

and thus given a function f, the coefficients a_n of the Laurent expansion (when it exists) is determined uniquely by the above relation.

Definition 3.3.2: Laurent Expansion

Let f be a function. Then f is said to have a *Laurent expansion* in the annulus $\rho_2 < |z| < \rho_1$ if there is a Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$ which converges in this annulus and that is equal to $f(z)$ at each point in the annulus.

Naturally, by our previous comment, if f is locally a Laurent series, f is holomorphic in the annulus and uniformally converges in $r_2 \leq |z| \leq r_1$ where $\rho_2 < r_2 < r_1 < \rho_1$.

Theorem 3.3.1: Laurent Expansion Of Holomorphic Functions

Let f be a holomorphic function. Then given ρ_2, ρ_1, in an annulus $\rho_2 < |z| < \rho_2$, f is representable as a Laurent expansion

Proof **:**
Pick ρ_1, ρ_2 and choose r_1, r_2 such that

$$\rho_2 < r_2 < r_1 < \rho_1$$

By our earlier observations of the uniqueness of the Laurent expansion, the Laurent series we will find will not depend on the choice of r_1 and r_2, and so the Laurent series will converge on $\rho_2 < |z| < \rho_1$. Choose r_1', r_2' such that

$$\rho_2 < r_2' < r_2 < r_1 < r_1' < \rho_1$$

and consider the compact annulus $r_2' \leq |z| \leq r_1'$ whose oriented boundary is the difference of the circles γ_1 with the center 0 and the radius r_1' in the positive direction, and the circle γ_2 with the center 0 and radius r_2' in the positive direction. By Cauchy's integral formula, we have for $r_2 \leq |z| \leq r_1$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)}{t-z}dt$$

For the first integral on the right hand side, we have $|t| = r_1'$ and $|z| \leq r_1 < r_1'$, so we can write the series expansion

$$\frac{1}{t-z} = \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}$$

which converges uniformally when t describes the circle of center 0 and radius r_1'. We may replace $\frac{1}{t-z}$ in the first integral by this series as we've done before and by uniform convergence integrate term by term and we again get

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t-z}dt = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)}{t^{n+1}}dt$$

for $n \geq 0$. For the second integral, we have

$$|t| = r_2' \qquad |z| \geq r_2 > r_2'$$

and so we have

$$\frac{1}{t-z} = -\frac{1}{z}\frac{1}{1-t/z} = -\sum_{n=-1}^{-\infty} \frac{z^n}{t^{n+1}}$$

replacing $1/(t-z)$ in the second integral by this series and then using uniform convergence to integrate term by term, we get

$$-\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{t-z} = \sum_{n=-1}^{-\infty} a_n z^n \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{t^{n+1}}dt$$

for $n < 0$. combining these two, we get

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

for $r_2 \leq |z| \leq r_1$ which converges uniformally, completing the proof.

Proposition 3.3.2: Holomorphic Function Decomposition In Annulus

Let f be a holomorphic function in the annulus $\rho_2 < |z| < \rho_1$. Then there exists holomorphic functions $f_1(z)$ in the disk $|z| < \rho_1$ and a holomorphic function $f_2(z)$ for $|z| > \rho_2$ such that

$$f(z) = f_1(z) + f_2(z)$$

Furthermore, this decomposition is unique if f_2 tends to 0 as $|z|$ tends to infinity.

Proof **:**
Let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be the Laurent expansion of f and let

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \qquad f_2(z) = \sum_{n=-1}^{-\infty} a_n z^n$$

Certainly we have found the desired decomposition and $|f_2(z)|$ tends to zero as $|z|$ tends to infinity. Now, suppose that

$$f(z) = g_1(z) + g_2(z)$$

is another such decompoition where g_2 decays to 0 as $|z| \to \infty$. We'll show that $f_1 = g_1$ and $f_2 = g_2$. Let h be the holomorphic function which is equal to $f_1 - g_1$ for $|z| < \rho_1$ and equal to $g_2 - f_2$ for $|z| > \rho_2$. This function is holomorphic on $\mathbb{C}$ and tends to 0 as $|z| \to \infty$. By the maximum modulus principle or Louville's theorem, the function h must be identically 0, but then we get $f_1 = g_2$ and $f_2 = g_2$, showing uniqueness and completing the proof.

An immediate use of Laurent expansion is showing that e^{-1/z^2} is *not* analytic. Indeed, the Laurent series converges everywhere except $z = 0$. There is this nice visual from wikipedia for this:

`https://en.wikipedia.org/wiki/Laurent_series#/media/File:Expinvsqlau_GIF.gif`

Note that by using the above decomposition, it is easy to see that the Cauchy inequlaities expand to the Laurent expansion, namely

$$|a_n| \leq \frac{M(r)}{r^n} \qquad n \in \mathbb{N}$$

3.3.1 Study of Singularities

So why did we just make sure the laurent series is well-defined? The power of Laurent series is that we can expand laurent series around singularities. For exmaple, we cannot center a power series around 0 for $1/z$, but we can "center" a laurent series around 0. Let's now use our new local expansion of f in terms of Laurent series to study singularities. Let f be a holomorphic on the punctured disk $\mathbb{D}$ punctered at 0. When can f be holomorphically extended so that f is holomorphic on all of $\mathbb{D}$? In $\mathbb{R}^2$, it is certainly the case that there exists smooth function that cannot be extended, consider $\|x\|$ on the punctured disk. Such an extension is certainly unique (by analytic continuation, or simply because any continuous function is uniquely extended to its boundary), and it certainly has to be bounded in a neighbourood of the puncture. For complex function's being bounded is actually the extact condition needed for the extension to be holomorphic:

Proposition 3.3.3: Holomorphic Extension To Singularities

Let f be a holomorphic function on $\mathbb{D} \setminus \{0\}$. Then f extends holomorphically to 0 if and only if f is bounded in some neighbourood of 0

***Proof* :**
It is clearly a necessary condition, hence we show it's sufficient. In the punctured disk $0 < |z| < \rho$, take the Laurent expansion of the holomorphic function f:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

In an appropriately small neighbourood, there exists a number M for which $|f(z)|$ is bounded above for $|z| = r$ for sufficinetly small r. By Cauchy inequalities, we get

$$|a_n| \leq \frac{M}{r^n}$$

for small r and $n < 0$. In particular, for negative n as $r \to \infty$ we see that $a_n = 0$. Thus, the Laurent expansion of f reduces to a Taylor series. Then the Taylor series defines the natural required extension, and by uniqueness of the extension we have our desired result, completing the proof.

Corollary 3.3.1: Holomorphic Cannot be Dominated

Let f, g be entire holomorphic functions. If $|f| \leq |g|$ on $\mathbb{C}$, then it must be that $f = cg$ for some constant c (in other words, there is no holomorphic functions that "dominate").

***Proof* :**
Consider $\frac{f}{g}$. Then $\left|\frac{f}{g}\right| = \frac{|f|}{|g|} \leq 1$. If h is entire, we may apply Louisville's Theorem. For all points $g^{-1}(0)$, the inequality just established shows that its bounded, hence a removable discontinuity and hence we may holomorphically extend h to it. But then h is entire, and hence by Louisville theorem constant: $f/g = \lambda$ showing $f = \lambda g$, completing the proof.

If we cannot extend, we define the following:

Definition 3.3.3: Isolated Singularity

Let $f(z)$ be a holomorphic function in the punctured disk $0 < |z| < \rho$. The origin 0 is said to be an *isolated singularity* of f if the function f cannot be extended to a holomorphic function on the entire disk $|z| < \rho$.

Note that the singualrity is isolated since analytic functions only have isolated singularities. Certainly, if not all negative coefficient of the laurent expantion of f are zero, then 0 is an isolated singularity. This splits into two cases

1. There are only finitely many $n < 0$ such that $a_n \neq 0$. In this case, there is a positive n such that $z^n f(z) = g(z)$ is a holomorphic function in some neighbourood around the origin. Thus, $f(z) = g(z)/z^n$ is *meromorphic* (a quotient of holomorphic functions) in some neighbourood of the origin. Hence, in a neighorhood of the singularity, f is holomorphic on the Riemann sphere.
2. There are infinitely many negative integers such that $a_n \neq 0$. in this case, $f(z)$ is not a meromorphic function in a neighborhood around the singularity.

These lead to the following definition:

Definition 3.3.4: Pole And Essential Singularities

If f cannot be extended to a holomorphic function at the origin but is meromorphic, then 0 is a *pole* of f. If f is not meromorphic on $\mathbb{D}$, then 0 is an *essential singularity* of f

Naturally, since poles are involved, this leads us to talk about the Riemann sphere, and define holomorphic functions on the Riemann sphere. The way we define a function f to be holomorphic on S^2 is how we do it for [complex] manifolds: let $f : S^2 \to \mathbb{C}$, and take the coordinate charts eliminating the north and south pole respectivey by:

$$z = \frac{x+iy}{1-u} \qquad z' = \frac{x-iy}{1+u}$$

Then if $D \subseteq S^2$ is an open subset, f is holomorphic on D if and only if for any $p \in D$ that is not the north pole, f is holomorphic in coordinates z, and for any $p \in D$ that is not the south pole, f is holomorphic in coordinates z' (i.e., f represented in our two charts is holomorphic). We will say that $f : \mathbb{C} \to \mathbb{C}$ is "holomorphic at infinity" if by doing a change of variables $z' = 1/z$ and a neighborhood $|z| > r$, f is holomorphic at 0 in $|z'| < 1/r$. Similarly f is meromorphic at infinity and has an isolated essential singularity at infinity if the function $f(1/z)$ is meromorphic or has an isolated essential singularity at 0.

3.3.2 A Word on Essential Singularities

Notice that for a or zero pole, either $\lim_{z\to a} f(z)$ or $\lim_{z\to a} 1/f(z)$ exists, while this is *not* the case for essential singularities. Essential singularities can be thought of as the "bad" case. Their behavior will be a good deal more complicated than poles. However, they do exist prominently: every meromorphic function on the Riemann sphere that is *not* a rational function has an essential singularity at infinity.

Essential singularities can be characterized by One way to detect if we have an essential singularity is to check that small enough neighborhood around the origin map to small enough neighbors around infinity, where a small neighborhood of infinity means compliment large neighbors around 0. This result can be thought of as a "lack of convergence" when switching over to $z' = 1/z$ coordinates:

Theorem 3.3.2: Weiestrass Theorem

If 0 is an isolated essential singularity of a holomorphic function f on the punctured disk $0 < |z| < \rho$, then for any $\epsilon > 0$, the image of the punctured disk $0 < |z| < \epsilon$ under f is everywhere dense in the plane $\mathbb{C}$

Note that the case where z_0 is an essential singularity is naturally reduced to the case when $z_0 = 0$ by replacing z by $z - z_0$. To check if it is at infinity we take $f(1/z)$.

Proof **:**
Suppose for the sake of contradiction that there exists a disc centered at a of radius $r > 0$ which is outside the image of the punctured disk $0 < |z| < \epsilon$ under f. Then

$$|f(z) - a| \geq r \qquad 0 < |z| < \epsilon$$

Take the function $g(z) = \frac{1}{f(z)-a}$. Then g is holomorphic and bounded in the punctured disc $0 < |z| < \epsilon$. By proposition 3.3.3 this function can be extended to a holomorphic function on $|z| < \epsilon$; we shall denote the extension by the same letter, g. Re-arranging g, we get

$$f(z) = a + \frac{1}{g(z)}$$

showing that f is meromorphic on $|z| < \epsilon$, being the sum of meromorphic functions, which contradicts the hypothesis that 0 is an essential singularity of $f(z)$.

This should show why $\lim_{z\to 0} f(z)$ cannot be well-defined, even on the Riemann sphere: no matter how small we make the neighboured a neighbors around 0, we will never force a convergent sequence in the codomain as the image of the open neighborhood is always almost all of $\mathbb{C}$[6]. In fact, we can get a stronger result known as Picard Theorem, but we shall delay it until we have more tools:

Theorem 3.3.3: Picard's Theorem

Let 0 be an isolated essential singularity of the holomorphic function f. Then the image of f of any punctured disk $0 < |z| < \epsilon$ is either the whole plane $\mathbb{C}$ or the plane $\mathbb{C}$ with one missing point

Proof **:**
see theorem 4.3.8

One might like to have an example of a holomorphic function with an essential singularity:

[6] The following visual shows how this is not the case for poles: demos graph

Example 3.2: Essential Singularity

1. Take $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$ which is holomorphic in the punctered plane $z \neq 0$ and has an isolated essential singularity at the origin since the coefficeint of $1/z^n$ is nonzero for all $n \geq 0$. This function never take the value 0. Any other value *is* taken for any z in $0 < |z| < \epsilon$.

2. On the Riemann sphere, we can as usual do a change of coordiantes $z' = 1/z$ to bring ∞ to 0 and work with it. Then f has a pole singularity at infinity if $f(z) = \sum_{n}^{\infty} a_n z^n$ in $|R| > z$ and only finitely many coefficients are nonzero.

(put the result you found that doesn't require open mapping theorem, but then also put the later the open-mapping theorem proof sicne it's simply elegant)

3.4 Residue Theorem and Argument Principle

Cauchy's theorem required that two paths be homotopic. Sometimes, we can "fix" the fact that two paths are not homotopic by extending the domain to be simply connected. By proposition 3.3.3, analytic continuity, and the fact that singularities of analytic functions are isolated, we have a generally good idea of when this is possible, that is we may always extend f to be holomorphic up to removable singularities, leaving us with poles and essential singularities. We would now like to find the integral of a function even if two paths are not homotopic, that is they are impeded by a singularity. This leads us to *Residue Theory*, the subject of most of the remainder of this chapter. Residue theory can be thought of extending the result of the integral of $1/(z - z_0)$ as far as possible:

Theorem 3.4.1: Cauchy's Theorem On An Annulus

Let $f(z)$ be holomorphic in an annulus $\rho_2 < |z| < \rho_1$ centered at the origin. Then if γ is a closed path contained in the annulus, then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = I(\gamma, 0)a_{-1}$$

where $I(\gamma, 0)$ is the winding number of the path γ with respect to to origin 0 and a_{-1} is the coefficient of $1/z$ in the Laurent expansion of f

***Proof* :**
f has a Laurent expansion, and so we may write:

$$f(z) = \frac{a_{-1}}{z} + g(z) \qquad g(z) = \frac{a_{-1}}{z} + \sum_{n\neq 1} a_n z^n$$

Note that $g(z)$ is certainly holomorphic in the anulus. Thus, by uniform convergence:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \frac{a_{-1}}{z} + \sum_{n\neq 1} a_n z^n dz \\ &= a_{-1}\int_\gamma \frac{1}{z}dz + \sum_{n\neq 1}\int_\gamma a_n z^n dz \\ &= a_{-1}\int_\gamma \frac{1}{z}dz + \sum_{n\neq 1} 0\end{aligned}$$

Thus, since the integral is the winding number, we get:

$$\int_\gamma f(z)dz = 2\pi i a_{-1} I(\gamma, 0)$$

In particular, if γ winds around once in the positive sense, we get

$$\int_\gamma f(z)dz = 2\pi i a_{-1}$$

completing the proof.

Note that if f is holomorphic in $|z| < \rho_1$, then $a_{-1} = 0$ (since f has a Taylor expansion), and hence this theorem properly generalizes Cauchy's Theorem.

Definition 3.4.1: Residue

Let f be a function defined on $0 < |z| < r$ with an isolated singularity at 0 (which may be either a pole or an essential singularity). When γ is a closed path in some neighbourood of 0 which does not pass throguh 0, then the coefficient a_{-1} of the laurent expansion is called the *residue* of the function f at the singular point 0, and is written as

$$\text{Res}(f, 0) := a_{-1}$$

For a residue at the point at infinity, let $f(z)$ be holomorphic for $|z| > r$ and let $z = \frac{1}{z'}$. Then

$$f(z)dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'$$

Definition 3.4.2: Residue At The Point At Infinity

Let f be holomorphic in $|z| > r$. Then the residue of the point at infinity is the residue of the function $-\frac{1}{z'^2} f\left(\frac{1}{z'}\right)$ at the point $z' = 0$. If $\sum_{n\in\mathbb{Z}} a_n z^n$ is the Laurent expansion of $f(z)$ in a neighbourood of the point at infinity, the residue of f at infinity is $-a_{-1}$, and is written as

$$\text{Res}(f, \infty) = a_{-1}$$

see here

The following theorem generalizes theorem 3.4.1 for the general case of integrating a holomorphic functions with multiples poles or essential singularities:

Theorem 3.4.2: Residue Theorem

Let $\Omega \subseteq S^2$ be an open subset of the Riemann sphere and let f be a holomorphic function in Ω except perhaps at isoalted points which are singularities of f. Let Γ be the oriented boundary of a compact subset $A \subseteq \Omega$ and suppose that Γ does not pass through any singularities of f or the point at infinity. Then only a finite number of singularities z_k are contained in A, and

$$\int_\gamma f(z)dz = 2\pi i \left(\sum_k \operatorname{Res}(f, z_k) \right)$$

where $\operatorname{Res}(f, z_k)$ denotes the residue of the function f at the point z_k. The summation extends over all singularities $z_k \in A$ including the point at infinity if it qualifies

***Proof* :**
We shall split the proof into two cases: When the point at infinity is in A, and when it is not.

If the point at infinity is not in A, then A is a compact set of $\mathbb{C}$. Each singular point z_k is the center of a closed disc S_k in the interior of A, and we can choose small enough radii so that these discs are disjoint (since there are finitely many of them). Let γ_k be the boundary of the disk S_k described in the positive sense and let $\delta_k = S_k \setminus \gamma_k$. Let

$$A' = A - \bigcup_k \delta_k$$

the boundary of A' is the difference between Γ and the circles γ_k, that is $\Gamma \bigcup -\gamma_k$. Now, since f is holomorphic in an open neighbourood of A',

$$\int_\Gamma f(z)dz = \sum_k \int_{\gamma_k} f(z)dz$$

and for each integral:

$$\int_{\gamma_k} f(z)dz = 2\pi i \operatorname{Res}(f, z_k)$$

Thus:

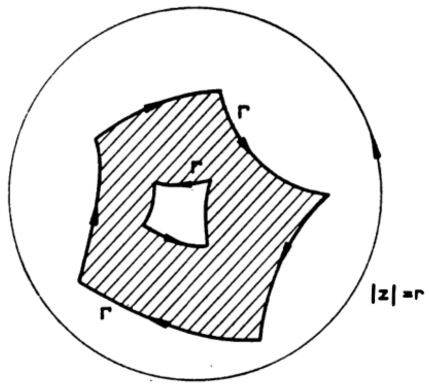
$$\int_\Gamma f(z)dz = \sum_k 2\pi i \operatorname{Res}(f, z_k)$$

completing the proof in the case of A not containing the point at infinity.

Let's now let A contain the point at infinity. Let $r \in \mathbb{R}_{>0}$ such that $|z| \geq r$ is a neighbourood of the point at infinity which does not intersect Γ and such that $f(z)$ is holomorphic in this neighbourood (the point at infinity being excluded). Let

$$A' = A - \{z \in \mathbb{C} \ : \ |z| > r\}$$

This will leave us with a region like so where the shadded region is the compliment of A



The oriented boundary of A' is the sum of the oriented boundary of Γ and of $|z| = r$ described in the positive sense now. Now, f is holomorphic in an open neighbourood of A', and so applying what we've proved in the first case we get:

$$\int_\Gamma f(z)dz + \int_{|z|=r} f(z)dz = 2\pi i \sum_k \operatorname{Res}(f, z_k)$$

where the sum on the right hand side is over all singularities z_k in A *except* the point at infinity. Now, by definition of the residue at infinity, we get

$$\int_{|z|=r} f(z)dz = -2\pi i \operatorname{Res}(f, \infty)$$

Thus, substituting and moving around we get

$$\int_\Gamma f(z)dz = 2\pi i \left(\operatorname{Res}(f, \infty) + \sum_k \operatorname{Res}(f, z_k) \right)$$

which is exactly the relation we were seaking, completing the proof.

We shall show how to use the residue theorem to compute many integrals without the explicit need to find anti-derivatives. Before we move on to giving many useful tools, we immediately point out on consequence of the Residue theorem when $A = S^2$ is the entire Riemann sphere. Then $\Gamma = \emptyset$ and so $\int_\Gamma f(z)dz = 0$, giving us:

$$\sum_k \operatorname{Res}(f, z_k) = 0$$

For example, rational functions are holomorphic at all but a finite number of isolated singularities, and so the sum of residues of a rational function (including the residue at infinity) is zero.

3.4.1 Calculating Residues

Let f have a simple pole z_0 (pole of multiplicity 1) that is not at infinity. Then we may write

$$f(z) = \frac{1}{z - z_0} g(z)$$

where g is holomorphic in a neighborhood of z_0 and $g(z_0) \neq 0$. Let

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be its Taylor expansion at z_0. Then the Laurent expansion of f at z_0 is (by uniqueness):

$$f(z) = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}$$

Since $a_0 = g(z_0)$, we see that $b_{-1} = g(z_0)$ at $f(z)$, $z \neq z_0$. Thus:

$$\text{Res}(f, z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} (z - z_0) f(z) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} g(z) = g(z_0)$$

Hence, we simply have to find g and plug in z_0. Let's now consider $f = P/Q$ where P and Q are holomorphic in a neighbourood of z_0 and z_0 is a simple zero of Q with $P(z_0) \neq 0$. Then:

$$f(z) = \frac{1}{z - z_0} \frac{P(z)}{Q_0(z)}$$

Now $\frac{P(z)}{Q_0(z)}$ is holomorphic in a neighbourood of z_0, so it has power series:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Hence:

$$f(z) = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}$$

Hence

$$\text{Res}(f, z_0) = a_0 = \frac{P(z_0)}{Q_0(z_0)} = \frac{P(z_0)}{Q'(z_0)}$$

Example 3.3: Application I

Let $f(z) = \frac{e^{iz}}{z^2+1}$, which has two simple poles at $z = \pm i$. By our above observations, we see that we must find $P/Q' = \frac{1}{2z} e^{iz}$ so that:

$$\text{Res}(f, i) = -\frac{i}{2e} \qquad \text{Res}(f, -i) = \frac{ie}{2}$$

Let's now examine the case of multiple poles so that

$$f(z) = \frac{1}{(z - z_0)^k} g(z)$$

where $g(z)$ is holomorphic in an open neighbourood of z_0 where $g(z_0) \neq 0$. Repeating the above computations for finding the Laurent series of f, we get:

$$f(z) = \frac{a_0}{(z - z_0)^k} + \cdots + \frac{a_{k-1}}{z - z_0} + a_k + \sum_{n=k+1}^{\infty} a_n (z - z_0)^{n-k}$$

Hence, $\mathrm{Res}(f, z_0) = a_{k-1}$ where a_{k-1} is the coefficient of the Taylor expansion of g. Hence, to find the residue, we simply have to partly do the Taylor expansion of g at z_0.

Example 3.4: Application II

Let $f(z) = \frac{e^{iz}}{z(z^2+1)^2}$. We'll find $\mathrm{Res}(f, i)$. Then:

$$f(z) = \frac{1}{(z-i)^2} \frac{e^{iz}}{z(z+i)^2} = \frac{1}{(z-i)^2} g(z)$$

For computational simplicity, let $t = z - i$ so that $z = i + t$ and we expand around 0 instead of i. We shall find the coeficient of t in the Taylor expansion of:

$$g(t) = \frac{e^{i(i+t)}}{(i+t)(2i+t)^2}$$

To so do so, we write down a partial Taylor expansion of the 3 terms in $h(t)$. In our case, it suffices to find the degree 1 term:

$$e^{i(i+t)} = e^{-1}(1 + it + \cdots)$$
$$(i+t)^{-1} = -i(1 - it)^{-1} = -i(1 + it + \cdots)$$
$$(2i + t)^{-2} = -\frac{1}{4}\left(1 - \frac{i}{2}t\right)^{-2} = -\frac{1}{4}(1 + it + \cdots)$$

Hence, multiplying all the power series we get:

$$g(t) = \frac{i}{4^e}(1 + 3it + \cdots)$$

Thus, taking the 2nd order element we get:

$$\mathrm{Res}(f, i) = -\frac{3}{4e}$$

3.4.2 Argument Principle and Consequences

The above examples demonstrate how important roots and poles of meromorphic functions are in finding their integrals. We can reverse this, and use integrals to find the roots and poles of a meromorphic, since the integral is nonzero if there is a root in the denominator. If we are clever with our function, the value of the integral will be the multiplicity of the roots:

Theorem 3.4.3: Argument Principle

Let $f(z)$ be a meromorphic function which is not constant in an open set D and let Γ be the orinted boundary of a compact set K contained in D. Suppose that the function f has no poles on Γ and does not take the value α on Γ. Then:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z - P \tag{3.8}$$

where Z denotes the sum of the orders of multiplicity of the roots contained in K of the equation

$$f(z)-\alpha = 0 \qquad \text{i.e.} \qquad f(z) = a$$

and P denotes the sum of the orders of multiplicity of the poles of f at α contained in K.

Proof **:**
Let f be meromorphic in an open neighborhood of z_0. Since it is meromorphic, then any root has finite degree so:

$$f(z) = (z-z_0)^k g(z)$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$. If f is holomorphic, $k \geq 0$, and if z_0 is a pole then $k < 0$. Taking the logarithmic derivative of f, we get

$$\frac{f'}{f} = \frac{k}{z-z_0} + \frac{g'}{g}$$

Thus:

$$\text{Res}(f'/f, z_0) = k$$

showing that the residue of f'/f is the order of the zeros and poles at z_0. It is not hard to see that this is shift invariant and hence:

$$\text{Res}\left(\frac{f'}{f-\alpha}, z_0\right) = Z - P$$

Thus, by the Residue theorem, for Γ as described in the proposition:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z - P$$

completing the proof.

If f is holomorphic, then the above equation (3.8) becomes:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-\alpha}dz = Z$$

Hence, this integral both finds zeros of polynomials and their multiplicity. This result gives us another way of showing the roots of a holomorphic function cannot be dense in a more analysis-flavoured proof:

Proposition 3.4.1: Stability of Roots

Let z_0 be a root of order k of the equation $f(z) = a$, f being a non-constant holomorphic function in some neighborhood of z_0. For any sufficiently small enough of U of z_0, and for any b sufficiently near to a and not equal to a ($b \neq a$), the equation $f(z) = b$ has exactly k simple solutions in V.

Proof **:**
Let Γ be a circle centered at z_0 with sufficiently small radius so that z_0 is the only solution to the equation $f(z) = a$ contained in the closed disc bounded by γ. Furthermore, suppose that the radius of γ is sufficiently small so that $f'(z) \neq 0$ at any point of the disc except the center z_0. Consider the integral:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz$$

Then the above remains constant when b varies in a connected component of the compliment of the image of γ under f. Thus, for any b sufficiently near to a, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}dz = k$$

Thus, the equation $f(z) = b$ has exactly k roots in the interior of γ, if each root is counted with its order of multiplicity. But for b sufficiently near to a but $\neq a$, the root of the equation $f(z) = b$ are all simple because the derivative of $f'(z) \neq 0$ t any point of z sufficiently near to z_0 and $\neq z_0$, completing the proof.

Corollary 3.4.1: Augment Argument Principle

Let $f(z)$ be a meromorphic function which is not constant in an open set D and let Γ be the oriented boundary of a compact set K contained in D. Suppose that the function f has no poles on Γ and does not take the value α on Γ. Then if $p(z)$ is a polynomial:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)p(z)}{f(z)-\alpha}dz = \sum_{\substack{w \\ \text{zero or pole}}} n_w p(w) \tag{3.9}$$

where w denotes the zero or pole, n_w the multiplicity of the zero or pole of w.

Proof **:**
exercise

The following theorem gives another way of finding roots of a function in the case of one polynomial bounding another. By the harmonicity of holomorphic functions, it will suffice to check on the boundary of the region in which we are checking for zeros:

Theorem 3.4.4: Rouchés Theorem

Let f, g be holomorphic function on K with closed countour ∂K (that has no self-intersection). Then if on ∂K

$$|g(z)| < |f(z)|$$

Then f and $f+g$ have the same number of roots in K, including multiplicity.

Proof **:**
By doing some substitution, it is equivalent to prove that if $|f(z) - g(z)| < |f(z)|$ on ∂K, then $f(z)$ and $g(z)$ have the same number of roots in K. Since $|f(z)|$ must be strictly greater than $|f(z) - g(z)|$ on ∂K, it follows that $|f(z)| > 0$ on the boundary. Thus dividng both sides of the previous equation by $f(z)$ we get that on ∂K:

$$\left|1 - \frac{g(z)}{f(z)}\right| < 1$$

Thus, the function $F(z) := \frac{g(z)}{f(z)}$ takes values in $|z-1| < 1$ to the unit disk centered at 1. Consider now the argument principle:

$$\frac{1}{2\pi i}\int_{\partial K} \frac{F'}{F} dz = Z - P$$

Since $F = g/f$, we see that the number of zeros is the number of zeros of g, an the number of poles is the number of zeros of f (namely, since f and g are holomorphic in Ω). Thus, if we show this integral is zero, we have completed the proof. To show this, first notice that by definition ∂K is a disjoint union of closed curves, and hence the above integral splits into the sum of closed curvse γ of ∂K. Next, we may do the substitution $w = F(z)$ to get

$$\frac{1}{2\pi i}\int_{\gamma} \frac{F'}{F} dz = \frac{1}{2\pi i}\int_{F\circ\gamma} \frac{dw}{w}$$

which is the integral of the index, hence to find the value of the above integral it suffices to find how many times $F \circ \gamma$ winds around 0. Since $F \circ \Gamma$ lies in $|z-1| < 1$, the path *never* winds around the origin, and hence:

$$\frac{1}{2\pi i}\int_{F\circ\gamma} \frac{dw}{w} = 0$$

hence, the number of zeros of f and g are the same in K, as we sought to show.

The next result we shall prove using the Argument principle is that all holomorphic functions are in fact open maps.

Theorem 3.4.5: [Complex] Open Mapping Theorem

Let $f : U \to V$ be a non-constant holomorphic function in a connected open set U. Then f is an open map.

Not to be confused with the functional analysis open mapping theorem for Linear operators on Banach spaces. In particular, f *need not be surjective.*

Proof **:**
Let $z_0 \in U$, $y_0 inf(z_0)$. We shall find an open neighborhood for y_0 in $f(U)$. First, as $f(z_0) - y_0 = 0$, we have that under the right arrangement:

$$\int_\gamma \frac{f'(z)}{f(z) - y_0} dz = \rho \geq 1$$

Consider now:

$$g(w) = \int_\gamma \frac{f'(z)}{f(z) - w} dz$$

Then g is certainly holomorphic, and by proposition 3.4.1 in a small enough open neighborhood of y_0 it is constant, and hence $g(w) = \rho$, but that implies the equation $f(z_1) = w$ is satisfied in a sufficiently small open neighborhood of y_0 showing it is open, as we sought to show.

It is very informative to see that there is an alternative more elementary proof requiring only the inverse function theorem:

Proof **:**
It suffices to show that the image of every sufficiently small open ball is open. Let $f : U \to V$ be a non-constant holomorphic function and pick $z \in U$. Then if $f'(z) \neq 0$, then by the holomorphic inverse function (theorem 2.2.3), there exists a local inverse which tells us that the image of sufficiently small open balls are open, so say $f'(z) = 0$. Since shifting is a biholomorphism, we may without loss of generailty assume that $z = 0$. What we shall do is write $f(z)$ in a differnet form, namely:

$$f(z) = (zh(z))^n$$

where n will be the order of the root and $h(0) \neq 0$. Then the function on the inside will have an inverse in a small enough neighboured by the Inverse Function Theorem, and from there we shall be able to derive our desired result.

Let's say the order of the zero is n. Then we may expand the taylor series of f and re-write it as:

$$f(z) = cz^n(1 + f_1(z))$$

where $c \neq 0$ and f_1 is holomorphic with $f_1(0) = 0$. Next, define:

$$f_2(z) = c^{1/n}(1 + f_1(z))^{1/n}$$

where we choose some branch. Then f_2 is holomorphic at the origin and, most importantly, $f_2(0) \neq 0$. With this function defined, noticed that we may write f as:

$$f(z) = (zf_2(z))^n$$

Since $(zf_2(z))' = f_2(z) + zf_2'(z)$, $zf_2(z)$ has a nonzero derivative at 0! Thus, by the Inverse Function Theorem (theorem 2.2.3), there exists a local inverse around 0, say g. Then taking $g \circ w^{1/n}$ by choosing a branch of $w^{1/n}$, we get on a sufficiently small neighbourood:

$$g(w^{1/n}) = z$$

But this means that f is biholomorphic in some small neighorhood of 0, and hence the image of a sufficently small open ball around 0 is open, completing the proof.

Notice how this breaks in the real case since there doesn't exist an inverse $z^{1/n}$.

Corollary 3.4.2: Proper Holomorphic Map Is Surjective

Let $U, V \subseteq \mathbb{C}$ denote domains (connected open sets) in $\mathbb{C}$ and let $f : U \to V$ be a holomorphic mapping. Suppose that f is *proper* ($f^{-1}(K)$ is compact for compact K). Then $f(U) = V$

Proof **:**
If we show that f is closed, then f maps clopen sets to clopen sets, and so $f(A) = B$. Let's say $f(x_n) \to y$. Then

$$S = \{f(x_n) \ : \ n \in \mathbb{N}\} \cup \{y\}$$

is compact, hence $f^{-1}(S)$ is compact, in particular

$$\{x_n \ : \ n \in \mathbb{N}\} \subseteq f^{-1}(S)$$

By compactness, there exists a convergent subsequence say $x_{n_k} \to x$. By continuity $f(x_{n_k}) \to f(x) = y$. Hence, the image of a closed set is closed, and by connectedness we have the proof.

Since we are working with metric spaces, it is equivalent to say that $f : X \to Y$ is proper if whenever a sequence (x_n) escapes to infinity[7], then $(f(x_n))$ escapes to infinity. With this, all poylnomials are proper maps. Other examples include the trigonometric functions.

Using the open mapping theorem, we may generalize theorem 3.3.2 to open neighorhood of infinity:

Proposition 3.4.2: Neighborhood At Infinity

Let $U \subseteq \mathbb{C}$ be open and connected, and let $f : U \to \mathbb{C}$ be holomorphic with a pole at $a \in \mathbb{C}$ of arbitrary order. Then for any $\epsilon > 0$ where $P = B_\epsilon(a) \setminus \{a\} \subseteq U$, there exists $r > 0$ such that

$$\{z \in \mathbb{C} \ : \ |z| > r\} \subseteq f(P)$$

Proof **:**
Let $g(z) = \frac{1}{f(z)}$. Then $\lim_{z \to a} g(z) = 0$, hence g has a removable singularity at a and hence is defined. By the open maping theorem, any open neighborhood at infinity U of a maps to an open set $g(U) = \{1/f(z) \ : \ z \in U\})$. Hence, there exists a $\delta > 0$ such that

$$\{z \in \mathbb{C} \ : \ |z| < \delta\} \subseteq g(U)$$

Letting $r = 1/\delta$ completes the proof.

Next, Runge's approximation theorem allows us to approximate holomorphic functions (on not necessarily simply connected domains) by rational functions.

[7]For every $S \subseteq X$, only finitely many elements of (x_n) are in S

Theorem 3.4.6: Runge's Approximation Theorem

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on an open set Ω and $K \subseteq \Omega$ a compact subset. Then if A is a set containing at least one complex number from each bounded connected component of $\mathbb{C} \setminus K$, then there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of rational function which converges uniformly to f on K such that all the poles of each function r_n are in A

Proof **:**
Cool result, will put down the proof when I have time.

3.4.3 Method of Residues: Solving Real Integrals

We now introduce some integration tricks. Note that there is no general way of solving integrals, we just gained some more tools.

First type (I)

Consider an integral of the form

$$I = \int_0^{2\pi} R(\sin(t), \cos(t))dt$$

where $R(x, y)$ is a rational function without a pole on the cirlce $x^2 + y^2 = 1$. Let $z = e^{it}$ so that we can write:

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad \cos(t) = \frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Thus:

$$\int_0^{2\pi} R(\sin(\theta), \cos\theta)d\theta = -i\int_{|z|=1} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{dz}{z} = 2\pi\sum_{|z|<1}\mathrm{Res}(g(z))$$

where after our substitution we had:

$$\frac{1}{iz}R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)$$

and so we get:

$$I = 2\pi\sum\mathrm{Res}\left\{\frac{1}{z}R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right\}$$

Example 3.5: Residue Method I
Consider

$$\int_0^{2\pi} \frac{1}{a + \sin(t)}dt$$

where $a \in \mathbb{R}_{>1}$. Decomposing, we get

$$\int_0^{2\pi} \frac{1}{a+\sin t} dt = \int_\gamma \frac{1}{a+\frac{1}{2i}(z-z^{-1})} \frac{dz}{z}$$
$$= \int_\gamma \frac{2i}{z^2+2iz-1} dz$$

Hence, our residue in the unite circle is:

$$I = 2\pi \sum \operatorname{Res} \frac{2i}{z^2+2iaz-1}$$

The only pole contaiend in the unit disc is

$$z_0 = -ia + i\sqrt{a^2-1}$$

by factoring the denominator and evaluating it at the pole, we get that it's residuce is $\frac{i}{z_0+ia} = \frac{1}{\sqrt{a^2-1}}$. Thus:

$$I \int_0^{2\pi} \frac{1}{a+\sin(t)} dt = \frac{2\pi}{\sqrt{a^2-1}}$$

Second Type (II)

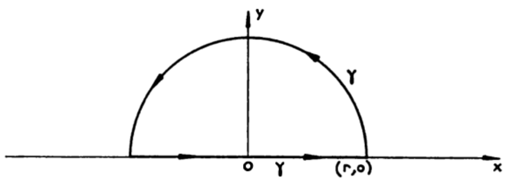
Our next type of integral will be of the form:

$$I = \int_{-\infty}^{\infty} R(x) dx \tag{3.10}$$

where R is a rational function without a real pole. Naturally, we need to assume the integral is convergent, which is equivalent (i.e. it is a necessary and sufficient condition) that the principal part of $R(x)$ at infinity is of the form $\frac{1}{x^n}$ for integer $n \in \mathbb{N}_{\geq 2}$, or stated another way if R has a zero of order at least two at $\pm\infty$. Equivalently, we must have:

$$\lim_{|x|\to\infty} xR(x) = 0$$

To calculate integrals of the form in equation (3.10), we will integrate the function $R(z)$ in the complex plane with the boundary of a half-disc centered at 0 with radius r and situated in the half-plane $y \geq 0$:



For a sufficiently large r, $R(z)$ is holomorphic on the boundary, and so by the Residue theorem $\int_\gamma R(z)dz$ is equal to the sum of the residues of the poles of R inside γ. We thus have the following breakdown of the integral:

$$\int_{-r}^{r} R(x)dx + \int_{\delta(r)} R(z)dz = 2\pi i \sum \operatorname{Res}(R(z)) \tag{3.11}$$

where $\delta(r)$ is the rest of the unit circle without the $[-r, r]$ interval, i.e. the upper half circle centered at 0 with radius r described in the positive sense. Our goal now is to show that as $r \to \infty$, the second integral in equation (3.11) tends to 0. This will leave us with

$$\int_{-\infty}^{\infty} R(x)d = 2\pi i \sum \operatorname{Res}(R(z))$$

or, if we were going in the lower half plane, we'd get

$$\int_{-\infty}^{\infty} R(x)d = -2\pi i \sum \operatorname{Res}(R(z))$$

where we sum over all the poles in the lower half plane.

Lemma 3.4.1: Residue Method Lemma I

Let $f(z)$ be a continuous function defined in the sector $\theta_1 \leq \theta \leq \theta_2$ and r, θ denote the modulus and the argument of z. Then if

$$\lim_{|z|\to\infty} zf(z) = 0 \qquad (\theta_1 \leq \arg(z) \leq \theta_2)$$

Then $\int f(z)dz$ extended over the arc of the circle $|z| = r$ contained in the sector tends to 0 as $r \to \infty$.

Proof **:**
Let $M(r)$ be the maximum of $|f(z)|$ on $|z| = r$. Then

$$\left| \int_{\delta(r)} f(z)dz \right| \leq M(r) \int_{\delta(r)} dz = r(\theta_2 - \theta_1) \xrightarrow{r\to\infty} 0$$

since we can do this on the sector from 0 to π, we see that the sector has no effect on the integral.

Notice that we have the same result when $z \to 0$ if $zf(z) \to 0$:

Lemma 3.4.2: Residue Method Lemma II

Let $f(z)$ be a continuous function defined in the sector $\theta_1 \leq \theta \leq \theta_2$ and r, θ denote the modulus and the argument of z. Then if

$$\lim_{|z|\to 0} zf(z) = 0 \qquad (\theta_1 \leq \arg(z) \leq \theta_2)$$

Then $\int f(z)dz$ extended over the arc of the circle $|z| = r$ contained in the sector tends to 0 as $r \to 0$.

Example 3.6: Residue Method II
Let
$$I = \int_0^\infty \frac{1}{1+x^6}dx$$
First, the domain of integration is not from $-\infty$ to ∞, but this is fine since the integral is even, and so we have:
$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+x^6}dx$$
The complexification $1/(1+z^6)$ has 6 poles, all on the unit circle, three in the upper half plane and three in the lower. Sticking to the upper-half plane:
$$e^{i\frac{\pi}{6}} \qquad e^{i\frac{3\pi}{6}} \qquad e^{i\frac{5\pi}{6}}$$
Calculating the residues at each pole we get:
$$\text{Res}(f, z_1) = \lim_{z\to z_1}(z-z_1)f(z)$$
$\frac{1}{6z^5} = \frac{-z}{6}$ (since $z^5 = -1$). Thus:
$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{1+x^6}dx \\ &= -\frac{\pi i}{6}\left(e^{i\frac{\pi}{6}} + e^{i\frac{3\pi}{6}} + e^{i\frac{5\pi}{6}}\right) \\ &= \frac{\pi}{6}\left(2\sin(\pi/6)+1\right) \\ &= \frac{\pi}{3} \end{aligned}$$

Third Type (III)

We next study integrals of the form
$$I = \int_{-\infty}^\infty f(x)e^{ix}dx$$
where f is holomorphic in a neighbourood of of each point of the closed half plane $y \geq 0$, except perhaps at a finite number of points. By taking the real and imaginary part, this is the same as solving:
$$I = \int_{-\infty}^\infty f(x)\cos(x)dx \qquad I = \int_{-\infty}^\infty f(x)\sin(x)dx$$
Note that it is important to convert this problem into an exponential one since sin and cos are unbounded in the complex plane. We shall split up this integral into the cases where the singularities or on the real axis and when they are not on the real axis. Let's start with the case where the singularities are *not* on the real axis. Then the integral
$$\int_{-r}^r f(x)e^{ix}dx$$

is well-defined, and as $r \to \infty$, if f does has a pole of order 2 at infinity ($zf(z) \to 0$), we naturally get

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(R(z)e^{iz})$$

if the integral is convergent. T This time, it will we will not be able to take the equivalent path in the lower half plane since e^{iz} grows quickly in the lower half plane (recall $|e^{iz}| = e^{-y}$). Due to the presence of e^{iz}, we may weaken the speed of convergence of $R(z)$:

Lemma 3.4.3: Residues Method Lemma II

Let $f(z)$ be a function defined in a sector of the half-plane $y \geq 0$. If

$$\lim_{|z|\to\infty} f(z) = 0$$

Then $\int_{|z|=r} f(z)e^{iz}dz$ contained in the sector tends to 0 as $r \to \infty$.

Proof **:**
Let $z = re^{i\theta}$ and $M(r) = \max_{\theta\in S} |f(re^{i\theta})|$ where S is the sector in the upper half plane. Then:

$$\left|\int f(z)e^{iz}dz\right| \leq M(r)\int_0^{\pi} e^{-r\sin(\theta)}rd\theta$$

I claim that the integral on the right hand side is bounded by π. Indeed, first notice that

$$\int\int_0^{\pi} e^{-r\sin(\theta)}rd\theta = 2\int_0^{\pi/2} e^{-r\sin(\theta)}rd\theta$$

Next, for $0 \leq \theta \leq \pi/2$, we have $\frac{2}{\pi} \leq \frac{\sin(\theta)}{\theta} \leq 1$. Thus:

$$\int_0^{\pi/2} e^{-r\sin(\theta)}rd\theta \leq \int_0^{\pi/2} e^{-\frac{2}{\pi}r\theta}rd\theta \leq \int_0^{\infty} e^{-\frac{2}{r}r\theta}rd\theta = \frac{\pi}{2}$$

Sinec $f(z) \xrightarrow{|z|\to\infty} 0$, $M(r) \to 0$, completing the proof.

Proposition 3.4.3: Evaluating Type III Integrals

If $\lim_{|z|\to\infty} f(z) = 0$ for $\mathrm{Im}(z) \geq 0$, then

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum \mathrm{Res}(f(z), e^{iz}) \tag{3.12}$$

where the summation is over the singularities of $f(z)$ contained in the upper half-plane $y > 0$.

Proof **:**
This is now an immediate consequence of the above lemma and our previous observations.

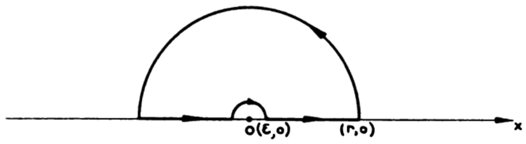
Example 3.7: Residue Method III

1. Consider
$$\int_0^\infty \frac{\cos(x)}{x^2+1}dx = \frac{1}{2}\operatorname{Re}\left(\int_{-\infty}^\infty \frac{e^{ix}}{x^2+1}dx\right)$$
Then it is equal to
$$\pi i \sum \operatorname{Res}\left(\frac{e^{iz}}{z^2+1}\right)$$
where the sumation goes over the poles in the upper half-plane. Since there is only the pole $z = i$, it is simple, and hence it's residue is $\frac{1}{2ei}$. Thus:
$$\int_0^\infty \frac{\cos(x)}{x^2+1}dx = \frac{\pi}{2e}$$

2. Using our current results, we may integrate integrals of the form $\int_{-\infty}^\infty f(x)e^{-ix}dx$. Note that we would have to integrate in the *lower* half-lane instead of the upper, for $|e^{-iz}|$ is bounded in the lower half-plane. More generally, an integral of the form
$$\int_{-\infty}^\infty f(x)e^{ax}dx$$
where $a \in \mathbb{C}$ can be evaluated in the half-plane where $|e^{az}| \leq 1$. To see this, consider $\int_0^\infty \frac{\cos(mx)}{x^2+1}$. Then it is equal to
$$\frac{1}{2}\operatorname{Re}\int_{-\infty}^\infty \frac{e^{imx}}{2x+1}dx$$
Then substituting $z = mx$, we get:
$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{imx}}{x^2+1}dx &= \int_{-\infty}^\infty \frac{e^{iz}}{(z/m)^2+1}\frac{dz}{m}\\ &= \int_{-\infty}^\infty \frac{me^{iz}}{z^2+m^2}dz\end{aligned}$$
Hence, we have a pole at $z = im$, and so
$$\operatorname{Res}\left(\frac{me^{iz}}{z^2+m^2}, im\right) = \frac{me^{-m}}{2im}$$
Hence, simplifying and multiplying by $\frac{1}{2}2\pi i$, we get
$$\int_0^\infty \frac{\cos(mx)}{x^2+1} = \frac{\pi}{2e^m}$$

3. Now, what about:
$$\int_{-\infty}^\infty f(x)\sin^n(x)dx \qquad \int_{-\infty}^\infty f(x)\cos^n(x)dx$$
For such an expression, we can convert $\sin^n(x)$ and $\cos^n(x)$ into a rational function of $\sin(mx)$ and $\cos(mx)$ be repeated use of double or half-angle identities.

Let's now say that $f(z)$ has singularities on the real axis. For simplicity, let's say it is at the origin (the proof for other points is a translation of this special case). What we shall do is modify our path of integration to bypass the singualrity:



Let's say that the smaller semi-circle has path $\gamma(\epsilon)$. Then:

Lemma 3.4.4: Residues Method Lemma III

Let $z = 0$ be a simple pole of $g(z)$. Then

$$\lim_{\epsilon \to 0} \int_{\gamma(\epsilon)} g(z)dz = \pi i \operatorname{Res}(g, 0)$$

Proof **:**
First, we have $g(z) = \frac{a}{z} + h(z)$ where h is holomorphic at the origin and $a = h(0)$. The integral $\int_{\gamma(\epsilon)} h(z)$ tends to 0 as $\epsilon \to 0$, and the integral $\int_{\gamma(\epsilon)} \frac{a}{z}dz$ tends to $\pi i a$, but that complete the proof.

Thus in general we have

$$f(z)e^{iz} = \frac{a}{z} + R_0(z)$$

where $a = \operatorname{Res}(R(z)e^{iz}, 0)$. Overall, we have:

$$\int_{-\infty}^{\infty} R(x)e^{ix} = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(R(z)e^{iz}) + \pi i \sum_{\operatorname{Im}(z)=0} \operatorname{Res}(R(z)e^{iz})$$

Example 3.8: Residue Method IV
Let

$$I = \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz$$

Then, since 0 is a simple pole, we see that the residue at 0 of the integrand on ths right hand side is 1. Hence:

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z} = 1\pi i = \pi i$$

taking the imaginary part of the above and divisind by two we get:

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2}\pi = \frac{\pi}{2}$$

Fourth Type (IV)

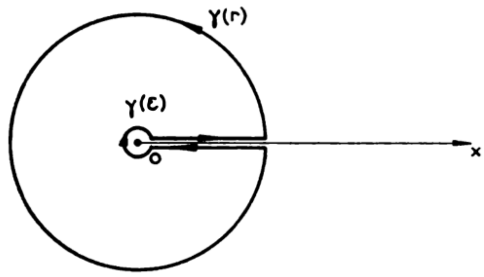
When integrating real valued function, we often come across fractional functions such that $x^{1/2}$ In the complex case, such functions are multi-valued, and hence we must take extra care when working with them. Thus, the forth type of integral we'll consdier is:

$$I = \int_0^\infty \frac{R(x)}{x^\alpha} dx$$

where $\alpha \in (0,1)$, and $R(x)$ is a rational fucntion with no poles in the positive real axis $x \geq 0$. It is clear the integral converges in the lower limit. For it to converge in the upperlimit, the principal part of $R(x)$ at infinity must be of the form $\frac{1}{x^n}$ where $n \geq 1$. In other words, it is necessary and sufficient that:

$$\lim_{x\to\infty} R(x) = 0$$

To find this integral, we naturally first complexify to $f(z) = \frac{R(z)}{z^\alpha}$ defined on the plane with positive real axis $x \geq 0$ excluded: let Ω be this set. Since $\alpha \in (0,1)$, we have to specify the branch of z^α in Ω: we shall take the branch with argument between in $(0, 2\pi)$. With this, we shall integrate on the following region:



Let's label this path $\delta(r,\epsilon)$. This path goes from $\epsilon > 0$ to $r > 0$, then circles around 0 with radius r in the positive sense, then goes down the real axis from r to ϵ, and finally circles around 0 with radius ϵ. Now, considering the integral

$$\int_{\delta(r,\epsilon)} \frac{R(z)}{z^\alpha} dz$$

we see it is equal to the sum of residues of the poles of $\frac{R(z)}{z^\alpha}$ contained in Ω, if r has been choosen to be sufficiently large and ϵ sufficiently small. Let's now show that the contour on the two circles go to zero. We first decompose the integral: since we are circling around a point that has a non-trivial index, notice that after letting $f(z) = R(z)/z^\alpha$, we can decompose this integral into:

$$\int_{\delta(r,\epsilon)} f(z)dz = \int_{\gamma(r)} f(z)dz + \int_{\gamma(\epsilon)} f(z) + (1 - e^{-2\pi i\alpha}) \int_\epsilon^r f(x)dx$$

where $\gamma(r)$ and $\gamma(\epsilon)$ are the two circles centered at 0. Now, since the argument of z is bounded, $zf(z)$ tends to 0 when $z \to 0$, and when $|z| \to \infty$. Thus, the integral over $\gamma(r)$ and $\gamma(\epsilon)$ tends to 0 as $r \to \infty$ and $\epsilon \to 0$ (this is lemma 3.4.1 and 3.4.2). Thus, at the limit, we have

$$(1 - e^{2i\pi\alpha})I = 2\pi i \sum \text{Res}\left(\frac{R(z)}{z^\alpha}\right) \tag{3.13}$$

which we can now use to calculate the value of I

Example 3.9: Residue Method V

Let

$$I = \int_0^\infty \frac{1}{x^\alpha(1+x)}dx$$

for $\alpha \in (0,1)$. Then $R(z) = \frac{1}{1+z}$ which only has one pole at $z=-1$. Using our normal method of residue computations:

$$\text{Res}(\frac{1}{z^\alpha(1+z)}, -1) = e^{-\pi i\alpha}$$

Thus, by equation 3.13 we get:

$$(1-e^{-2\pi i\alpha})\int_0^\infty \frac{1}{x^\alpha(x+1)}dx = 2\pi i \sum_{\mathbb{C}\setminus[0,\infty)} \text{Res}(\frac{1}{z^\alpha(1+z)}) = 2\pi i\frac{1}{e^{\pi i\alpha}}$$

Simplifying, we get:

$$\int_0^\infty \frac{1}{x^\alpha(1+x)}dx = \frac{2\pi i}{e^{\pi i\alpha}(1-e^{-2\pi i\alpha})} = \frac{2\pi i}{e^{\pi i\alpha}-e^{-\pi i\alpha}} = \frac{\pi}{\sin(\pi\alpha)}$$

Fifth Type (V)

The last type of integral we'll consider is of the form:

$$\int_0^\infty R(x)\log(x)dx$$

where R is a rational function with no pole on the positive real axis $x \geq 0$, and $\lim_{x\to\infty} xR(x) = 0$ so that the integral converges. Let Ω be the same as for the fourth type and let $\delta(r,\epsilon)$ be the same path of integration. The branch of $\log(z)$ we we choose takes argumetns of z between in $(0, 2\pi)$.

For reasons we shall see soon, we shall be integrating $R(z)(\log(z))^2$, this will allow us to put $R(z)$ and $R(z)\log(z)$ in relation. Just like before, the integrals along $\gamma(r)$ and $\gamma(\epsilon)$ tend to 0 as $r\to\infty$ and $\epsilon \to 0$ by lemma 3.4.1 and 3.4.2. Whe nthe argument of z is 2π, we have

$$\log(z) = \log(|z|) + 2\pi i$$

Letting $x = |z|$, we have:

$$\int_0^\infty R(x)(\log(x))^2dx - \int_0^\infty R(x)(\log(x)+2\pi i)^2dx = 2\pi i\sum \text{Res}\left(R(z)(\log(z))^2\right)$$

Thus:

$$-2\int_0^\infty R(x)\log(x)dx - 2\pi i\int_0^\infty R(x)dx = \sum \text{Res}\left(R(z)(\log(z))^2\right)$$

For a general rational function, this is about as good as we could get in the general case. However, if the coefficeints of $R(z)$ are *real*, or it takes real values for real x, we can separate the real and imaginary aprts of the above equation to obtain the two realtions:

$$\int_0^\infty R(x)\log(x)dx = -\frac{1}{2}\text{Re}\left(\sum \text{Res}\left(R(z)(\log(z))^2\right)\right) \tag{3.14}$$

$$\int_0^\infty R(x)x = -\frac{1}{2}\operatorname{Im}\left(\sum \operatorname{Res}\left(R(z)(\log(z))^2\right)\right) \tag{3.15}$$

where the summation extends over all the poles of the rational function $R(z)$ in Ω

Example 3.10: Residue Method VI

We shall evaluate

$$I = \int_0^\infty \frac{\log(x)}{(1+x)^3}dx$$

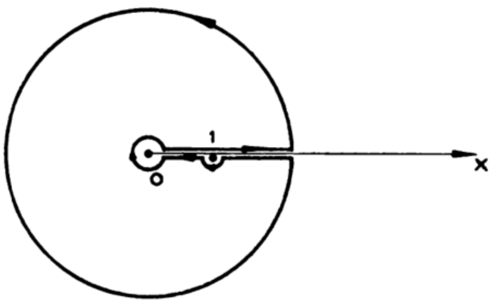
The residue of $(\log(z))^2/(1+z)^2$ at $z=-1$ is equal to the coefficeint of t^2 in the partial taylor expansion of $(i\pi + \log(1-t))^2$. Computing, we get $1 - i\pi$, and so

$$\int_0^\infty \frac{\log(x)}{(1+x)^3}dx = -\frac{1}{2}$$

Note that if we integrate $R(z)\log(z)$ on $\delta(r,\epsilon)$, then we would get

$$\int_0^\infty R(x)dx = -\sum \operatorname{Res}\left(R(z)\log(z)\right)$$

We may still get similar results if there is a simple pole on the real axis, say $x=1$ for simplicity. In this case, we can sitll make sense of $\int_0^\infty R(x)\log(x)dx$ since $\log(z)$ has a simple zero at 1. It will be necessary to modify the contouer of integration to go around 1, so tha tit looks like so:



It should be proven that on this path:

$$\int_0^\infty R(x)\log(x)dx = \pi^2\operatorname{Re}(\operatorname{Res}(R,1)) - \frac{1}{2}\operatorname{Re}\left(\sum \operatorname{Res}(R(z)(\log(z))^2)\right)$$

where the summation goes ove rall poles of $R(z)(\log(z))^2$ except $z=1$.

Exercise 3.4.1

1. Show that $\int_0^\infty \frac{\log(x)}{x^2-1} = \frac{\pi^2}{4}$.

4

Analysis of Holomorphic Functions

With the exposition of integration of holomorphic functions, we prove a couple more important results on the representation and the convergence of holomorphic functions. We shall strongly be relying on some basic notions of functional analysis in this section, see cite:NateFunctional for more information.

4.1 Topology of Holomorphic Functions

Recall the following definitions:

Definition 4.1.1: Space Of Holomorphic Functions

Let D be an open subset of $\mathbb{C}$. Then $\mathcal{H}(D)$ be the set of all holomorphic functions from D to $\mathbb{C}$.

Naturally, this forms a complex vector-space even a complex algebra, for adding and multiplying by a constant $\mathbb{C}$ gives another holomorphic function, and multiplying two holomorphic functions is again a holomorphic function. We are interested in the construction of new holomorphic functions given a sequence of holomorphic functions. As we have implicitely been using every-time we created a power-series convergence, we have a particular notion of convergence that will result in the limit being holomorphic:

Definition 4.1.2: Convergence On Compact Sets

Let $f_n \in C(D)$ be a sequence continuous functions. Then we say that $f_n \to f$ on compact sets if and only if for each compact subset $K \subseteq D$, $f_n \to f$ uniformally on K.

This is indeed a sufficient notion for convergence: By Morera's Theorem, if holomorphic functions $f_n \to f$ uniformally, then f is holomorphic (see Corollary 4.1.2). The following immediately follows:

Lemma 4.1.1: Uniform Convergence On Compact Sets

let $(f_n) \subseteq \mathcal{H}(D)$ be a sequence of holomorphic functions such that $f_n \to f$ uniformally on compact sets. Then f is holomorphic

Proof **:**
By Morera's theorem it suffices to show that $\int_\gamma f = 0$. But γ is the boundary of a compact set K, and by uniform convergence on compact sets we have

$$\int_\gamma f = \int_\gamma f|_K = \int_\gamma \lim_n f_n|_K = \lim_n \int_\gamma f_n|_K = 0$$

completing the proof.

Naturally, if we strengthen the condition to uniform convergence, we get the same result. This result is mentioned to compare to the case of real differentiation:

Lemma 4.1.2: Weiestrass Convergence Theorem

Let $f_n \to f$ be a sequence of holomorphic function converging uniformally to f on an open disk $\mathbb{D}$. Then f is holomorphic

Note how there is no mention of how the derivatives of f_n converge, which is usually required for the equivalent result for real differentiable functions to converge to real differentiable functions.

Proof **:**
Since $\mathbb{D}$ is connected, for any closed path in $\mathbb{D}$ we have

$$\int_\gamma f_n(z)dz = 0$$

Then by uniform convergence:

$$\int_\gamma f(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0$$

Thus, by Morera's Theorem f is holomorphic, completing the proof.

Uniform convergence on compact sets is weaker than uniform convergence. The reason for this weakening is necessary because uniform convergence is in fact too strong of a notion for convergence of holomorphic functions.

Example 4.1: Uniform Convergence Need Not Exist

Let $f_n = \sum_k^n z^n$. Then we know this converges pointwise to $\sum_k^\infty \frac{1}{1-z}$ on $|z| < 1$. However, this does *not* converge uniformally on $|z| < 1$. The problem is that the error term:

$$|f_n - f| = |z^n + z^{n+1} + \cdots| = \left|\frac{z^n}{1-z}\right|$$

does not uniformally shrink for $z < 1$, in fact for arbirary $z = 1 - \epsilon$ we may make the error term as large as we want. This problem has already showed up before and was dealt with: recall in Abel's Theorem (theorem 2.3.1) that we were careful to pick $r < |z|$ and then insured convergence on $\rho \leq r$. This is precisely because of this limitation.

For our case, notice that if we fix any particular r, we may in fact bound the error term in such a way that does not depend on z. For example, if we take $|z| \leq 1/2$, then

$$\left|\frac{z^n}{1-z}\right| \leq \frac{(1/2)^n}{1/2} \xrightarrow{n\to\infty} 0$$

and more generally for any $|z| \leq 1 - \epsilon$ for $\epsilon > 0$, we see that the error term tends to zero.

The fact that holomorphic functions are preserved on uniform convergence for compact sets is intuitive since being holomorhpic is a local condition[1]. In this way, we may ask if being locally uniform convergence is sufficient, that for any point we may choose a neighbourhood for which the functions converge uniformally. Since $\mathbb{C}$ is a locally compact space, this is in fact equivalent: namely any point is contained in a compact neighbourhood and hence it is covered by finitely many open neighbourhoods, giving us the equivalence.

Since we have a convergence, we may naturally ask if it comes from a topology to better understand what type of space we are working with (is it a metric space? a Fréchet space? A Banach space? what are it's compact sets? and so forth)[2]. For those who have seen some functional analysis, the following will be seen as the natural answer:

[1]Though it has a lot of fascinating global consequences, nevertheless the check for a function being holomorphic is still a point-wise check

[2]Notes that some forms of convergence, like convergence in measure, do not come from topological notions, and hence this is not immediately intuitive. On the other hand, uniform convergence comes from a topology, so uniform on compact sets seem like an intuitive generalization

Definition 4.1.3: Topology On $\mathcal{H}(D)$

Take $\mathcal{H}(D)$, and let (K, ϵ) be a pair where $K \subseteq D$ and $\epsilon > 0$. Define the set $V(K, \epsilon)$ by

$$V_g(K, \epsilon) = \left\{ f \in \mathcal{H}(D) \ : \ \sup_{z \in K} |f(z) - g(z)| < \epsilon \right\}$$

where $V(K, \epsilon) := V_0(K, \epsilon)$, The collection $V(K, \epsilon)$ for all $K \subseteq D$ and $\epsilon > 0$ form a basis for the *uniform-compact* topology on $\mathcal{H}(D)$.

These balls can equivalently be defined using the collection of norms $\rho_K(f) = \sup_{z \in K} |f(z)|$[a]. This collection of open balls is the basis for the *uniform-compact* topology on $\mathcal{H}(D)$.

[a]Note that these are norms and not semi-norms since if $\rho_K(f) = 0$ then f is zero a.e., but then since f is holomorphic, f is identically zero

Naturally, this topology can be defined on continuous functions or even L^p functions. Uniform convergence on compact sets translate to convergecne in $\mathcal{H}(D)$ in this topology by insuring that $f_n - f$ if and only if for all $\epsilon > 0$, there exists an n large enough so that

$$f_n \in V_f(K, \epsilon) = \left\{ g \in \mathcal{H}(D) \ : \ \sup_K |f - f_n| < \epsilon \right\}$$

or equivalently there exists an n large enough so that that $f - f_n \in V(K, \epsilon)$. We shall call the collection $V(K, \epsilon)$ the *fundamental neighbourhoods*. If you have done some functional analysis, you may recognize this is a Fréchet space[3], and hence the this collection of (semi)norms induced an invariant metric. we can infact explicitely figure out this metric. Note that the metric that is defined must induce the same topology, name uniform convergence on compact subspaces, and that the construction of this metric is just the special case of the same construction for general Fréchet spaces:

Proposition 4.1.1: Invariant Metric

Let $\mathcal{H}(D)$ be given the topology above. Then given a compact exhaustion $\{D_i\}$ of D we have

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-n} \min\{\max_{D_i} |f - g|\}$$

as a translation-invariant metric on $\mathcal{H}(D)$

***Proof* :**
p. 146 Cartan proves the existence of a compact exhaustion, and then defines

$$d(f) = \sum_{i=1}^{\infty} 2^{-n} \min\{1, \max_{D_i} |f|\}$$

and shows this satisfies the condition of an invariant metric.

[3]Note that any open subset of $\mathbb{C}$ has a countable compact exhaustion

To summarize, if d is the metric on $\mathbb{C}$, define

$$\delta(a,b) = \frac{d(a,b)}{1+d(a,b)}$$

then

$$\delta_k(f,g) = \sup_{z\in E_k} \delta(f(z),g(z))$$

where E_k is part of the compact exhaustion of Ω. Then, deifne:

$$\rho(f,g) = \sum_{k=1}^{\infty} \delta_k(f,g)2^{-k}$$

This metric induces the right topology, if$f_n \to f$ in the ρ-distance, then for alrge n we have $\rho(f_n,f) < \epsilon$, meaning $\delta_k(f_n,f) < 2^k\epsilon$, but this implies $f_n \to f$ uniformally on E_k, first with respect to the δ-metric, but then cetainly with respect to the d-metric. Then sicne very compact E is conatined in an E_k¡ it is uniformally convergent on E. Conersely, if f_n converges uniformally to f on compact subsets, then $\delta_k(f_n,f) \to 0$ ofr every k¡ since

$$\sum \delta_k(f_n,f)2^{-k}$$

has a convergent majorant with terms independent of n, hence by the Weiestrass m test we have $\rho(f_n,f) \to 0$.

We give this metric a name:

Definition 4.1.4: Functional Chordal Metric

The metric

$$d(f,g) = \sum_{i=1}^{\infty} 2^{-n} \min\{\max_{D_i} |f-g|\}$$

is called the *chordal metric*.

The main take-away from us is that $\mathcal{H}(D)$ is a *metric space*, meaning we have the results we usually have on metric spaces, which we shall do when finding the compact subsets of $\mathcal{H}(D)$.

Convergence of Derivatives, Injective, and non-zero Functions

Since holomorphic functions converging uniformally on compact sets is a sufficient for the limit to be holomorphic, it is perhaps intuitive that the derivative converges to the derivative of f, that is if $f_n \to f$ on compact sets, then $f_n' \to f'$, on compact sets:

Proposition 4.1.2: Convergence of Derivative Of Holomorphic Functions

Let the sequence of holomorphic functions $f_n \to f$ uniformally on compact sets. Then $f_n' \to f'$ uniformally on compact sets.

This is not true for smooth functions, for example take x^2 converging to $|x|$ on $[-1,1]$; the derivative x does not uniformally converge to $\text{sgn}(x)$. This means that this is particular to harmonic functions, in particular we must take advantage of the analytic expansion:

***Proof* :**
Let $K \subseteq D$ be a compact set, without loss of generailty a compact disk. Let $K \subsetneq C$ be a slightly larger compact disk and take $\gamma = \partial C$. Consider:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)}{t-z}dt$$

then differentiating, we get (by differentiating under the integral sign):

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)}{(t-z)^2}dt$$

Then by uniform convergence:

$$f'(z) = \lim_n \frac{1}{2\pi i}\int_\gamma \frac{f_n(t)}{(t-z)^2}dt = \lim_n f_n'(z)$$

from this, uniformness on compact sets follows with some bounding arguments.

There is another proof that directly uses Cauchy inequalities that is insightful enough to put down:

***Proof* :**
(alternative)
We'll show that if f is holomorphic on $|z| < r_0 + \epsilon$ for small ϵ and $|f(z)| \leq M$ for all $|z| \leq r_0$, then:

$$|f'(z)| \leq M\frac{r_0}{(r_0-r)^2} \qquad |z| \leq r < r_0$$

First, in $|z| \leq r_0$ we get

$$f(z) = \sum_k^\infty a_k z^k$$

by cauchy's inequaltiie,s we get $|a_k| \leq \frac{M}{(r_0)^k}$. On the other hand, differentiating the above we get

$$f'(z) = \sum_k^\infty k a_k z^{k-1}$$

for $|z| \leq r < r_0$. Subsituting, we get

$$|f'(z)| \leq \frac{M}{r_0}\sum_k^\infty \frac{kr^{k-1}}{(r_0)^{k-1}}$$

What's left to show is that $\sum_k^\infty n\left(\frac{r}{r_0}\right)^{k-1}$ converges. Since kt^{k-1} is the derivative oft^n, then

$\sum_k kt^{k-1}$ is the derivative of $\sum t^k = \frac{1}{1-t}$. Thus

$$\sum_k^{\infty} k \left(\frac{r}{r_0}\right)^{k-1} = \frac{1}{(1 - r/r_0)^2}$$

Substituing, we are left with:

$$|f'(z)| \leq \frac{M}{r_0} \frac{1}{(1 - r/r_0)^2} = M \frac{r_0}{(r_0 - r)^2}$$

But now, we have hte same bounds on the derivative, from which we can deduce the taylor expansion have the same coefficients, completing the proof.

Note that by using the Maximum modulus principle, we can reduce the proof of uniform convergence on compact sets to uniform convergence on the boundary of the compact setse, since $|f_m(z) - f_n(z)|$ would atain its maximum in a compact set K on the boundary of K.

Due to the restrictive nature of holomorphic functions, they can they have a unique analytic continuation, and by taking the log of an analytic function we may count the number of roots of a holomorphic function. These two facts lead to the following theorem:

Proposition 4.1.3: Hurwitz Theorem

Let $(f_n) \subseteq \mathcal{H}(D)$ where D is a connected open set. Then if $f_n(z) \neq 0$ for all $z \in D$ and $f_n \to f$ uniformally on compact sets, then $f(z) \neq 0$ on all $z \in D$

This is not true for the real case: consider $f_n(x) = x^2 + 1/n$.

Proof **:**
Assume first that f is not identically zero. We know that the zeros of f are isolated since f is analytic. Hence, let's say $f(z) = 0$. Then we know that for a γ sufficinetly small that contains z we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} > 0$$

that is, the multiplicity of the pole is greater than 0. Since $f_n \to f$ and $f_n' \to f'$, we have

$$0 < \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} = \lim_n \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)} \overset{!}{=} 0$$

but each $f_n(z) \neq 0$ justifying the $\overset{!}{=}$ equality, but that leads to a contradiction, completing the proof.

The key intuition is that if we extend f_n analitically, we can get at most countably many zeros which are all isolated. Then as $f_n \to f$ uniformally on compact sets, these zero's move around (they cannot suddently "apprear" like in $f_n(x) = x^2 + 1/n$ since there is a unique analytic continuation for any holomorphic function). Since D is open, if $f(z) = 0$, then for ϵ small enough there would be an n such that f_n would have a zero in the open disk (the zeros would have to follow a path). Thus, this theorem essentially tells us that the zero's of a function already "exist", even if not in D.

Natuarlly, we often work with meromorphic functions, which are functions defined on $\overline{\mathbb{C}}$, or equivalently $\mathbb{C}\mathrm{P}^1 \cong S^2$. The above metric cannot be extended to $\overline{\mathbb{C}}$ since there is no finite distance from any point of $\mathbb{C}$ to ∞, however we may use a metric on the sphere (either the chordal metric or the spherical metric) to recover a similar result:

Corollary 4.1.1: Hurwitz Theorem For Meromorphic Functions

Let $(f_n) \subseteq \mathcal{H}(\Omega)$ be a sequence of meromorphic functions such that $f_n \to f$ uniformally on compact sets with respect to the spherical metric[a]. Then the limit is a meromorphic function or identically equal to ∞. If the functions are holomorphic, the limit is holomorphic or identically equal to ∞

[a] $d(x, y) = \cos^{-1}(x \cdot y)$, where $x \cdot y$ is the dot product of x and y

***Proof* :**
Suppose $f_n \to f$ uniformally on compact sets with respect to the spherical metric d_s, that is for all ϵ and any compact set K, there exists n sufficiently large such that:

$$\sup_{z \in K} d_s(f_n(z), f(z)) < \epsilon$$

First, since f is meromophic, f is continuous in the spherical metric. If for $z_0 \in K$, $f(z_0) \neq \infty$, then by continuity $f(z)$ is bounded in a neighbourhood of z_0, say U, and hence for large enough n $f_n(U) \neq \infty$. Hence, by lemma 4.1.2 that $f(z)$ is holomorphic in a neighbourhood around z_0. If $f(z_0) = \infty$, then take $f_1(z) = 1/f(z)$. Note that $1/f_n(z) \to f(z)$ uniformally on compact sets with respect to teh spherical metric, and hence again by lemma 4.1.2 $1/f(z)$ is analytic around z_0, but then $f(z)$ is meromorphic.

If each f_n is holomorphic but $f(z_0) = \infty$, then by Hurwitz's theorem it must be thata $1/f = 0$, but then $f = \infty$, completing the proof.

Coming back to holomorphic functions, we get yet another property of functions that is preserved under the limit that is usually not preserved for non-harmonic functions, again taking advantage of the argument principle:

Proposition 4.1.4: Injective, The Limit Injective

Let $f_n \subseteq \mathcal{H}(D)$ be a sequence of injective holomorphic functions. Then if $f_n \to f$ uniformally on compact sets, f is injective or constant.

Note how this is false for the real case, imagine

$$f(x) = \begin{cases} 0 & x < 0 \\ e^x - 1 & x \geq 0 \end{cases}$$

and a locally uniform sequence of injective smooth functions $f_n \to f$ converging to f

Proof **:**
For the sake of contradiction let's say $f(z_1) = f(z_2) = a$. Then for γ containig a and $f(z) \neq a$ on γ:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} \geq 2$$

but:

$$2 \leq \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a} = \frac{1}{2\pi i}\int_\gamma \lim_n \frac{f_n'(z)}{f_n(z)-a} = \lim_n \frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)-a} \leq 1$$

a contradiction.

We summarize these results in more theoretical real-analysis language:

Corollary 4.1.2: Properties Of Holomorphic And Continuous Function

Let D be an open connected set and $C(D)$, $\mathcal{H}(D)$ the collection of [complex] continuous and holomorphic function on D. Then giving $C(D)$ the uniform-compact topology

1. $C(D)$ is a complete space
2. $\mathcal{H}(D) \subseteq C(D)$ is a closed subspace
3. The mapping $\mathcal{H}(D) \to \mathcal{H}(D)$ mapping $f \mapsto f'$ is continuous

Proof **:**
All these statments are just reformulations of what we've just shown. (1) and (2) are immediate. To see (3), if $f_n \to f$, then $f_n' \to f'$, hence if $F(f) = f'$,

$$\lim_n F(f_n) = \lim_n f_n' = f' = F(f) = F(\lim_n f_n)$$

giving continuity.

4.1.1 Compact Subsets of $\mathcal{H}(D)$

One of the important properties of the topology of $\mathcal{H}(D)$ is that it has the heine-borel property: compact subsests are closed and bounded. Equivalently, since $\mathcal{H}(D)$ is a metric space, we know that a set is compact if and only if it is sequentially compact. In this section, we shall show that closed and bounded implies compact in $\mathcal{H}(D)$. This makes finding compact subsets rather easy, and if we can find a compact subset, then we know every sequence has a convergent subsequence, letting us very easily obtain the existence of a holomorphic function (in compact sets). We start with explicitely laying out what it means for a subset of $\mathcal{H}(D)$ to be bounded in its topology:

Definition 4.1.5: Bounded Set Of Holomorphic Functions

Let $A \subseteq \mathcal{H}(D)$ be a subset. Then A is said to be *bounded* if for any neighbourhoods $V(K,\epsilon)$ of the origin, there is a finite number $\lambda > 0$ such that $A \subseteq \lambda V(K,\epsilon)$.

Notice how this is equivalent in being bounded in the functional chordal metric. To understand this bounding condition explicitly, the relation $A \subseteq \lambda V(K, \epsilon)$ can be thought of as

$$\sup_{\substack{f \in A \\ z \in K}} |f(z)| \leq \lambda\epsilon$$

Thus, a set A of holomorphic functions on D bounded if and only if if for every $K \subseteq D$, there exists a $M_A(K) \in \mathbb{R}_{>0}$ such that

$$\sup_{\substack{f \in A \\ z \in K}} |f(z)| \leq M_A(K)$$

In other words, every function in A is *uniformally bounded on compact subsets*. Now, if A is bounded, certainly $\overline{A}$ is bounded (recall that convergence in this topology is uniform convergence on compact subsets). Thus, we imediately get:

Lemma 4.1.3: Bounded Operator On Holomorphic Funitons

The map $f \mapsto f'$ from $\mathcal{H}(D)$ to itself takes bounded sets to bounded sets.

Proof **:**
Follows since $f_n' \to f'$ and f' is bounded by M where $|f(z)| < M$ for $|z| < r_0$ times some proportion of r (see the alternative proof of proposition 4.1.2, page 134), namely if f is holomorphic on $|z| < r_0 + \epsilon$ for small ϵ and $|f(z)| \leq M$ for all $|z| \leq r_0$, then:

$$|f'(z)| \leq M \frac{r_0}{(r_0 - r)^2} \qquad |z| \leq r < r_0$$

With this established, we prove the easy direction:

Proposition 4.1.5: Compact In Holomorphic, Then Closed And Bounded

Let $A \subseteq \mathcal{H}(D)$ be compact. Then A is closed and bounded

Proof **:**
As we demonstrated earlier, $\mathcal{H}(D)$ is metrizable, hence any compact subset of $\mathcal{H}(D)$ is closed by the usual topological argument. Hence, we must show that A is bounded (the metric result only guaranees *totally bounded*). Let K be a compset subset of D, and take the (continuous) norm $\mathcal{H}(D) \to \mathbb{R}$

$$f \mapsto \|f\|_K = \sup_{i \in K} |f(z)|$$

Since it's continuous, $\|A\|_K \subseteq \mathbb{R}$ is bounded, namely each $f \in A$ are uniformally bounded on compact subsets of K. Since this is true for any $K \in D$, the set A is indeed bounded in $\mathcal{H}(D)$, as we sought to show.

Note that what has been proved also works for $C(D)$ (The space of continuous functions), for we may define the exact same topology. The converse though shall use properties specific to holomorphic functions. Due to historical reasons, precompact sets are given another name:

Definition 4.1.6: Normal Family

Let $A \subseteq \mathcal{H}(D)$. Then if A is precompact (it's closure is compact), A is called a *normal family* of functions.

More generally, we may define normal families for any type of space, usually function spaces such as $C(D)$. To proceed, we require some classical result form functional analysis:

Arzela-Ascoli

First, recall a result from real analysis that in a metric space, a set K is compact if and only if every sequence has a convergent subsequence (see EYNTKA Real Analysis I). This shall be the basis of how we prove (and use) compactness. Next, recall that compact sets of continuous functions are characterized by the following:

Definition 4.1.7: Uniform Equicontinuous Family

Let $\mathscr{F} \subseteq C(D)$ be a family of (i.e. a set of) continuous functions. Then $\mathscr{F}$ is said to be *equicontinuous* on $E \subseteq D$ if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall f \in \mathscr{F},\ \forall z, w \in E,\ |z - w| < \delta \implies d(f(z), f(w)) < \epsilon$$

Note that if $\mathscr{F} = \{f\}$, then $\mathscr{F}$ is equicontinuous if and only if f is uniformally continuous, and more generally each $f \in \mathscr{F}$ in an equicontinuous family if and only if f is uniformally continuous. Since we are in a metric space, we may characterise normal families

Proposition 4.1.6: normal, exists convergent sequence

Let $\mathscr{F} \subseteq \mathbb{C}(\Omega)$. Then $\mathscr{F}$ is said to be normal in Ω if and only if every sequence $(f_n) \subseteq \Omega$ contains a uniformally convergent subsequence for every compact subset $K \subseteq \Omega$

Proof **:**
standard real analysis

Note that the limit function of the subsequence *does not* need to converge in $\mathscr{F}$ (unless $\mathscr{F}$ is compact)

Theorem 4.1.1: Bolzano-Weiestrass for Metric Spaces

A family of function $\mathscr{F}$ is normal if and only if it's closure $\overline{\mathscr{F}}$ with respect to the chordal metric is compact

Proof **:**
see EYTNKA real analysis

There is also the strenthening of the converse of the Heine-Borel property that works more generally for metric spaces:

Theorem 4.1.2: Characterizing Compactness In Metric Space

Let $\mathscr{F} \subseteq C(\Omega)$ be a family functions. Then $\mathscr{F}$ is totally bounded if and only if every compact $E \subseteq \Omega$ and every $\epsilon > 0$, there is $f_1, f_2, ..., f_n \in \mathscr{F}$ such that

$$\forall f \in \mathscr{F},\ \exists f_j,\ d_E(f, f_j) < \epsilon$$

Proof **:**
see EYTNKA real analysis

Theorem 4.1.3: Arzela Ascoli Theorem

Let $\mathscr{F} \subseteq C(\Omega)$. Then $\mathscr{F}$ is normal iff:

1. (uniform equicontinuous) $\mathscr{F}$ is equicontinuous on every compact compact set $E \subseteq \Omega$
2. (pointwise relatively compact) For any $z \in \Omega$, the value $f(z)$, for each $f \in \mathscr{F}$ (i.e. the set $A_z = \{f(z)\}$) lie in a compact subset of the codomain (in our case, $\mathbb{C}$)

Proof **:**
see EYTNKA real analysis

Since the codomain is $\mathbb{C}$ (meaning compact sets are closed and bounded), condition (2) is equivalent to $f(z)$ being bounded.

Application to $\mathcal{H}(D)$

We shall now focus on compact subsets of $\mathcal{H}(D)$. Note that the 1st codition of Arzella ascoli (for complex functions) implies that if A is normal, then the functions of A are uniformly bounded on every compact set.

This implication is in fact sufficient for holomorphic functions. The following theorem characterizes normal families in terms of sets and in terms of sequences:

Theorem 4.1.4: Montel's Theorem

Let $A \subseteq \mathcal{H}(D)$ be a family of function. Then the following are equivalent conditions for A being normal:

1. A is normal if and only if if the functions in A are uniformally bounded on every compact set $K \subseteq D$, that is they are locally uniformally bounded.
2. A sequence $(f_k) \subseteq A$ is convergent (in the topology of $\mathcal{H}(D)$) if and only if for each $n \geq 0$, the sequence of nth derivatives $f_k^{(n)}(z_0)$ has a limit.

Proof **:**

1. (Alhfors p.224)

2. Call the condition that each sequence of nth derivatives $f_k^{(n)}(z_0)$ the $C(z_0)$ condition. Certainly if a sequence f_n if convergent in $\mathcal{H}(D)$, then each $f_k^{(n)}$ is convergent in the topology by proposition 4.1.2 applied inductively, hence we'll show the converse, namely that given $C(z_0)$, the sequence (f_k) converges uniformally on any compact disk centered at z_0 (of a radius less than that of D).

   As you may imagine, the key is Cauchy's inequalities. we want to show that for each $\epsilon > 0$, there exists k sufficiently large so that:

   $$\sup_{\in B(z_0,r)} |f_k(z) - f_n(z)| \leq \epsilon$$

   Let r be the radius of D, and choose $r_0 > r$ such that $B(z_0, r_0) \subseteq D$. Since A is bounded, there exits an $M \in \mathbb{R}_{>0}$ such that

   $$|f_k(z)| \leq M \qquad \text{for} \qquad |z - z_0| \leq r_0$$

   Take the Taylor expansion of each f_k at z_0:

   $$f_k(z) \sum_{n=1}^{\infty} a_{n,k}(z - z_0)^n$$

   Then by Cauchy's inequlities:

   $$|a_{n,k}| \leq \frac{M}{(r_0)^n}$$

   Hence, for $|z - z_0| \leq r$, for all k, h:

   $$\begin{aligned} |f_k(z) - f_h(z)| &\leq \sum_{n=1}^{\infty} |a_{n,k} - a_{n,h}| r^n \\ &\leq \sum_{0 \leq n \leq \rho} |a_{n,k} - a_{n,h}| r^n + 2M + \sum_{n > \rho} \left(\frac{r}{r_0}\right)^n \end{aligned}$$

   Since $r/r_0 < 1$, we may cut off the above geometric series at a ρ sufficinetly large so that

   $$2M \sum_{n > \rho} \left(\frac{r}{r_0}\right)^n$$

   for some $\epsilon > 0$. Next, by $C(z_0)$, we know that $a_{n,k} - a_{n,h} \to 0$ as $k, h \to \infty$ for each n since:

   $$a_{n,k} = \frac{1}{n!} f_k^{(n)}(z_0)$$

   thus ,for k_0 sufficiently large:

   $$\sum_{0 \leq n \leq \rho} |a_{n,k} - a_{n,h}| r^n \leq \frac{\epsilon}{2} \qquad k, h \geq k_0$$

Thus, we get for $k, h \geq k_0$, $|z - z_0| \leq r$:

$$\sup_{z \in K} |f_k(z) - f_n(z)| \leq \epsilon$$

hence, each f_k converges uniformally on the compact disc centered ta z_0 of radius r, as we sought to show.

Since normal families are characterized by this property in the case where $X = \mathcal{H}(D)$, it is worth giving this property a name:

Definition 4.1.8: Locally Uniformly Bounded on Compact Sets

Let $A \subseteq \mathcal{H}(D)$ be a family of functions. Then A is said to be *locally uniformally bounded on compact sets* if:

$$\sup_{\substack{f \in A \\ K \subseteq D}} |f(U)| < M_K$$

if unambiguous, such a set will simply be called *locally bounded*, with the understanding that the bounding is uniform and the sets overwhich we are bounding are the compact subsets.

Both of these formulations will render themselves useful. For the 2nd, we may use this to prove the Hein-Borel theorem for holomorphic functions:

Theorem 4.1.5: Haine-Borel for Holomorphic Functions

Let $A \subseteq \mathcal{H}(D)$. Then A is compact if and only if it is closed and bounded.

***Proof* :**
proposition 4.1.5 showed one direction; we show that closed and bounded implies compact.

Sicne $\mathbb{C}$ is Lindeloff (every set has a countable cover), let D be covered by a countable sequence of open discs centered at $z_0 \in D$. For each $n \geq 0$ and each i, define:

$$\lambda_i^n : \mathcal{H}(D) \to \mathbb{C} \qquad \lambda_i^n(f) = f^{(n)}(z_i)$$

which is certainly continuous. If $(f_k) \subseteq A$ is a bounded sequence, then $(\lambda_i^n(f_k))$ is bounded by the continuity of λ_i^n. If we show that for each pair (i, n) the sequence $(\lambda_i^n(f_k))$ has a convergent subsequence, we have that A is compact. What we shall do is arrange the λ_i^n into a single sequence $\mu_1, ..., \mu_m, ...$; now we shall show that there exists a subsequence of f_k, $f_{k,l}$ such that

$$\lim_l \mu_m(f_{k,l}) \qquad \forall m \geq 1$$

If you have seen the proof for compact sets of C^0, namely bounded clsod equicontinuous sets are compact, then this is the same argument. (skip for now since I have this proof in another set of notes EYNTKA Real Analysis I, or see p.166 of Cartan)

Corollary 4.1.3: Detecting Convergence

Let $A \subseteq \mathcal{H}(D)$ be a bounded set. Then if a sequecne $(f_k) \subseteq A$ has at most one function in its closure, then the sequence is convergent

Proof **:**
classical result of compact topological spaces.

A famous result of this is akin to the *uniform boundedness principle*: if a sequence of functions pointwise converges in a non-empty, connected, bounded, open set D, then f_k converges uniformally. For, if f, g are two holomorphic function in D that are both in the closure of the sequence f_k, then $f(z) = g(z)$ at any $z \in D'$, implying that f and g are identical in D by analytic continuation.

Next, let's say f_k is a bounded sequence of functions satisfying $C(z_0)$ for some $z \in D$ where D is non-empty and connected. Then the sequence converges uniformally in any compact subset of D, since if f, g are two holomorphic function in the closure of the sequence (f_k), then $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \geq 0$, then $f = g$ by analytic continuation. Naturally, this result extends to non-discrete sets.

Finally, we show that local boundedness is preserved under the derivative, which is something that must be assumed for the real case but follows in the complex case:

Corollary 4.1.4: Local Boundedness preserved under derivative

A locally bounded family of holomorphic functions has locally bounded derivative

Proof **:**
This follows from Cauchy's representation of the derivative:

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)}{(t-z)^2}dt$$

Thus, if γ is the boundary of a closed disk in Ω of radius r, then $|f'(z)| \leq 4M/r$, in the concentric disc of radius $r/2$,and M is the bound of $|f|$ on C, showing it is locally bounded.

Normal Families of Meromorphic Functions

We now slightly extend our domain to include the point at infinity. In this way, a sequence that tends to infinity should be regarded as part of our types of sequences to consider

Definition 4.1.9: Normal Family On Riemean Sphere

A family of holomorphic functions in a region $\Omega \subseteq S^2$ is said to be normal if and only if every sequence fontaisn either a subsequecne that converges uniformally on every compact set $E \subseteq \Omega$, or a subsequence that tends uniforamlly to ∞ on every compcact set.

Note The deriative of the function in a normal family need not form a normal family; for example $f_n(z) = n(z^2 - n)$ on $\mathbb{C}$, which is normal since $f_n \to \infty$ uniformally on every compact set, however, $f_n'(z) = 2nz$ does not form a normaly family since $f_n'(z)$ tends to ∞ for $z \neq 0$ but 0 for $z = 0$.

By corollary 4.1.4, if A is locally bounded, so is A'. The converse is true too; if A' is locally bounded then A is locally bounded. In the meromorphic case, A' need not be locally bounded if A is locally bounded (making it an iff), however there is a different "type" of derivative we can take on A which can be used to check for local boundedness of A:

Theorem 4.1.6: Marty's Theorem

Let $\mathscr{F} \subseteq \mathcal{H}(\Omega)$ be a family of holomorphic or meromorphic functions. Then $\mathscr{F}$ is normal if and only if for each compact $K \subseteq \Omega$:

$$\sup_{k \in K} f^{\#}(z) = \sup_{k \in K} \frac{2|f'(z)|}{1 + |f(z)|^2} \leq M_K$$

Proof **:**
We first show it is a sufficient condition. The quantity given very naturally comes from the chordal metric on the Riemann sphere. Recall from theorem 1.2.1 that the chordal metric to be:

$$d(a,b) = \frac{2|a-b|}{\sqrt{(1+|a|)^2(1+|b|)^2}}$$

Now, if $f^{\#} \leq M$, we get

$$d(f(z_1), f(z_2)) \leq M|z_1 - z_2|$$

which immediately gives equicontinuity when $f^{\#}$ is locally bounded.

To show that it is necessary, first notice that

$$f^{\#} = (1/f)^{\#}$$

Now, assume that $\mathscr{F}$ is a family of normal meromorphic functions, and $f^{\#}$ fails to be bounded on a compact set K. Take $f_n \in ff$ such that the maximum $f_n^{\#}$ on K tends to infinity. Let $f_{n_k} \to f$ of a convergent subsequence. Then around each point of K we cna find a small closed disk contained in D, on which either f or $1/f$ is analytic. If f Is analytic, it is bounded on the closed disk, and it follows by the spherical convergence that $\{f_{n_k}\}$ has no poles in the disk for k sufficently large. Then since the uniform convergence on compact sets perserves holomorphic functions and the spherical derivative is a continuous function, $(f_{n_k})^{\#} \to f^{\#}$ uniformally on a sligthly smaller disk. Since $f^{\#}$ is continuous, $(f_{n_K})^{\#}$ is bounded on the smaller disk. If $1/f$ is holomoprhic, the same proof applies.

Now, since K is compact, it can be covered by a finite number of small dissk, hence the collection $(f_{n_k})^{\#}$ is bounded on K; a contradiction to our original assumption, and completing the proof.

The notion used here is important enough to be given a name:

Definition 4.1.10: Spherical Derivative

The quantity in in Marty's Theorem:

$$f^{\#}(z) := \frac{2|f'(z)|}{1+|f(z)|^2}$$

is called the *spherical derivative*.

4.2 Factorization of Holomorphic Functions

In this section, explore the convergence of series of holomorphic and meromorhpic functions. A driving factor in the convergence of series is generlizing the notion of characterizing polynomials by their roots. Given a polynomial, we may take it to be a product of monomials, which using partial fraction decomposition we can turn into a finite sum. We may try the same technic with meromorphic functions, identifying the roots and poles, and then defining a series, the "partial fraction decomposition", that let's us analyze the function in terms of our better known analysis techniques. Many functions, such as our usual trigonometric functions, logarithm, and exponential function shall fall under this category. However, whole new types of functions will be found that shall capture really important geometric, number theoretic, PDE, or algebraic information in ways that are still being studied today. For example, the study of modular forms (generalization of Weiestrass p-functions that will be an example of an infinite partial fraction) are closely linked to number theory, and the Riemann zeta-function is closely linked to the distribution of primes. From the persepctive of complex analysis, these are just any other functions like the trigonometric functions (granted not usually entire or holomorphic). Hence, knowing how to identify properties and work with such functions is a useful skill to posess.

I have also gained one more meaningful intuition. In many ways, this part of complex analysis is the continuiation of usual Calculus; the study of taylor series generalizes to the study of analytic, hence holomorphic, functions. Having essentially exhausted the theory of entire holomorphic functions (the study of power series), we study not necessarily entire holomorphic functions (with more lenient growth factors) and meromorphic functions (which may have "ramification" around singularities, that is new behavior such that those of $1/z$). These are naturally interpreted as infinite rational polynomials. Though it may seem like an easy step up, it turns out to still have a lot of mysteries! For example, the Riemann-zeta function will be seen to be a meromorphic function defined via meromorphic convergnece (as we'll demonstarte soon), however we have yet (as of writing these notes) to fully understand the nature of their poles! Hence, this can be thought of as a natural continuation of calculus!

One more insight that is worth mentioning is the link to number theory and modular forms. The closed form of power series hold a (countably) infinite amount of numerical information within a single compact expression. For example, if we take the sequence $(1, 1, ..., 1, ...)$, then this can be associated to

$$q + q^2 + \cdots + q^n + \cdots$$

Factoring q and noticing the rest is the usual $1/(1-q)$ when $|q| < 1$, we see that this sequence can be represented as $q/(1-q)$. If we take the sequence $(1, 2, 3, ..., n, ...)$, and take:

$$q + 2q^2 + \cdots + nq^n + \cdots$$

and again factor the q, notice that the result series is the derivative of $1/(1-q)$, and hence $q/(1-q)^2$ holds information about the sequence $(1, 2, 3, ..., n, ...)$. Let us link this to modular forms with the following insight that shall not e proved. Consider the equation $y^2 + y = x^3 - x^2$. We can take solutions to this mod p for primes p. Counting the number of solutions, for each prime, we get the sequence:

$$(4, 4, 4, 9, 10, 9, ...)$$

It shall not be proven here, but this sequence is in some sense "fudged") by the prime, and it is more accurate to "normalize"modular at p-# of solutions; this number shall also grow but hover around 0. This gives us a new sequence:

$$(-2, -1, 1, -2, 1, 4, ...)$$

Now, consider the power series given by the infinite product:

$$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2(1-q^{33})^2 \cdots$$

Expanding this, we get:

$$q - 2w^2 - q^3 + 2q^4 + q^5 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + \cdots$$

Notice that the prime powers of each of these coefficients match the normalized solutions! The modularity theorem (the theorem resulting from the Taniyama–Shimura conjecture) states that this happens for any curve that is an elliptic curve, which for all intent and purposes of this motivating section they are curves of the form $y^2 = x^3 + ax + b$[4], and links them to a special type of power series called a *modular form* . This was the key for proving the infamous *Fermat Last Theorem* (that every elliptic curve is *modular*).

4.2.1 Series of Meromorphic Functions

Starting with the definition of meromorphic convergence:

Definition 4.2.1: Meromorphic Convergence

Let (f_n) be a sequence of meromorphic function on $\Omega \subseteq \mathbb{C}$. Then $\sum_n^\infty f_n$ converges uniformally (resp. uniformally and absolutely) on $X \subseteq \Omega$ if for all but finitely many terms have no poles on X, and eliminating the terms with poles forms a uniformally (resp uniformally and absolutely) convergent series of holomorphic functions in Ω

For a sequence $\sum_n f_n$ of meromorphic functions with all but finitely many of them having poles, we may split it as:

$$\sum_{n=1}^{n \leq n_0} f_n + \sum_{n > n_0}^{\infty} f_n$$

where for all $n > n_0$, f_n is holomorphic. The left sum in the above equation is meromorphic, being a finite sum of meromorphic function, and the right sum is a holomorphic function since it is a uniformally converging series of holomorhpic function on U. Note that the meromorphic function does not deepend on the choice of n_0 by the independence of associativity.

[4]More precisely, they are curves of 'genus one', which the curve in the above example is one

Theorem 4.2.1: Convergence Of Meromorphic Function

Let (f_n) be a sequence of meromorphic functions. Then if f_n converges uniformally on compact subset of Ω, then $f = \sum_n f_n$ is meromorphic on Ω and $\sum f_n'$ converge uniformally to f' on Ω

Proof **:**
$f = \sum_n f_n$ is already meromophic since for a given division:

$$\sum_{n \leq n_0} f_n + \sum_{n > n_0} f_n$$

we have the right sum converges uniformally to some holomorhpic f_0, and the left sum is a meromorphic function, say g. Then $f = g + f_0$ is meromorphic.

For the derivative, if we consider

$$f = \sum_{n \leq n_0} f_n' + (\sum_{n > n_0} f_n)'$$

where since the left sum is a finite sum we may pass the derivative through, and for the right sum by proposition 4.1.2 we can bring the derivative in as well, but that give

$$f' = \sum_n f_n'$$

as we sought to show.

Example 4.2: Series Of Meromorphic Functions
Consider

$$\sum_{-\infty < n < \infty} \frac{1}{(z-n)^2}$$

We shall show that this series uniformally converges on all compact subsets of $\mathbb{C}$. First, it suffices to show these converge on strips:

$$S_{x_0,x_1} = \{z \in \mathbb{C} \ : \ x_0 \leq \operatorname{Re} z \leq x_1\}$$

where $x_0, x_1 \in \mathbb{R}$. Then for any such strip, there are only a finite number of integers n where $x_0 \leq n \leq x_1$; hence the series

$$\sum_{n < x_0} \frac{1}{(z-n)^2}$$

is bounded by $\frac{1}{(x_0-n)^2}$, and hence the partial series is uniformally convergent on this strip. Simiarly, the partial series

$$\sum_{x_1 < n} \frac{1}{(z-n)^2}$$

is also bounded and hence uniformally converges too. Thus, removing a suitable finite number of terms from the series, we are left with a series of holomorphic functions which uniformally converge in the strip, and hence it converges holomorphically.

What is this function? Let $f(z)$ represent this series. Then we know it is a meromorhpic function defined on all of $\mathbb{C}$. The function f has period 1, that is

$$f(z+1) = f(z)$$

since

$$\sum_n \frac{1}{(z+1)-n)^2} = \sum_m \frac{1}{(z-m)^2}$$

where $m = n+1$. The poles of f are the integers $z = n$, and they are all double poles. The residue of these poles are zero since, in some neighbouring of $z = n$

$$f(z) = \frac{1}{(z-n)^2} + g(z)$$

where g is holomorphic. From this, we shall deduce the following

Proposition 4.2.1: Important Meromorphic Series

Let $f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$. Then

$$f(z) = \left(\frac{\pi}{\sin(\pi z)}\right)^2$$

Proof **:**
Let $z = x+iy$. First, note that $f(z)$ uniformally tends to zero as $|y| \to \infty$, that is for all $\epsilon > 0$, there exists an a such that $|y| \geq a$ implies $|f(z)| < \epsilon$. To see this, let z remain in the strip $x_0 \leq x \leq x_1$ and $|y| \geq 0$ for $a > 0$. As we've seen, this series is uniformally convergent to a holomorphic function as $|y| \to \infty$, where as the term tends to 0 uniformally with respect to the x strip. Thus, the sum of the series uniformally tends to 0 as $|y| \to \infty$ with respect to the x strip. But $f(z)$ has period 1, so by applying the above argument to a strip of width at least 1, we get that $f(z)$ tends to 0 as $|y| \to \infty$ absolutely with respect to x.

Now, consider $g(z) = \left(\frac{\pi}{\sin(\pi z)}\right)^2$. It has the following properties:

1. It is meromorphic in $\mathbb{C}$ and has period 1 (immediate)
2. it has double poles at all integers $z = n \in \mathbb{Z}$ (expand the taylor series) with prinicpal parts $\frac{1}{(z-n)^2}$
3. $g(z)$ tends to 0 absolutely with respect to x as $|y| \to \infty$

To see the third property, recall that $|\sin(\pi z)|^2 = \sin(\pi x) + \sinh^2(\pi y)$, hence $|\sin(\pi z)|$ uniformally tends to infinity with respect to x as $|y|$ tends to infinity.

Finally, we shall show that $f(z) - g(z)$ is bounded with constant zero to show they are equal. The fucntion $f - g$ is holomorphic in $\mathbb{C}$ since f, g have the same poles and same principal parts. To show it's bounded, consider a strip $x_0 \leq x \leq x_1$. It is bounded for $|y| \leq a$ (since it's on a compact set), and hence $f - g$ is bounded on the whole plane by periodicity. Thus by Louiville's theorem it is constant. Since $f - g$ tends to 0 as $|y| \to \infty$, it must be that the constnat is 0, showing equality.

This proof shows a general startegy to finding when a serise is equal to a function:

1. Check for the principal part (all terms with negative exponents in the power seies expansion)
2. The convergence in the limits
3. The periodicity

A famous result from this proof is the following famous sum. Take:

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 - \frac{1}{z^2} = \sum_{n\neq 0} \frac{1}{(z-n)^2}$$

Then the right hand side is a holomorphic function in some neighbouring of $z = 0$, say it is equal to $h(z)$. Furthermore, $h(0) = \sum_{n\neq 0} \frac{1}{n^2}$. Hence, we get

$$\lim_{z\to 0} \left[\left(\frac{\pi}{\sin(\pi z)}\right)^2 - \frac{1}{z^2}\right] = 2\sum_{n\geq 1} \frac{1}{n^2}$$

here is the key result: the value on the left hand side is easily evaluted through some limit expansions:

$$\begin{aligned}
\lim_{z\to 0}\left(\frac{\pi}{\sin(\pi z)}\right)^2 &= \lim_{z\to 0} \pi^2 \left(\frac{1}{\pi z} + \frac{\pi^2 z}{6} + \cdots\right)^2 \\
&= \lim_{z-0} \frac{1}{z^2} + \frac{\pi^2}{3} + \frac{\pi^6 z^4}{36} + \cdots - \frac{1}{z^2} \\
&= \lim_{z\to 0} \frac{\pi^2}{3} + \cdots \\
&= \frac{\pi^2}{3}
\end{aligned}$$

Dividing both sides by 2, we get:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Yet another way of computing the value is by taking the Laurent series of each side and comparing coefficients; this should be done as an exercise (can you comptue $\sum_n 1/n^4$?). It is good to see a few such examples and hence we demonstrate another, this next one allows to create a new series via an old one:

Proposition 4.2.2: Tangent Function As Meromorphic Series

Let

$$f(z) = \frac{1}{z} + \sum_{0\neq n\in\mathbb{Z}} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

Then

$$f(z) = \frac{\pi}{\tan(\pi z)}$$

Proof **:**
First, it should be clear that this series uniformally converges on compact sets. It's poles are at $z = n$ for $n \in \mathbb{Z}$ which are all simple poles with residue 1. Note that:

$$f'(z) = \frac{-1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} = -\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \frac{d}{dz}\left(\frac{\pi}{\tan(\pi z)}\right)$$

Hence, $f(z) - \frac{\pi}{\tan(\pi z)}$ is constant. Since $f(-z) = -f(z)$, we see that f is odd, and hence the constant must be zero.

Note that the series can be re-arranged by pairing together the term for n and $-n$:

$$\left(\frac{1}{z-n} + \frac{1}{n}\right) + \left(\frac{1}{z+n} - \frac{1}{n}\right) = \frac{2z}{z^2 - n^2}$$

Hence, we equivalently have:

$$\frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2} = \frac{\pi}{\tan(\pi z)}$$

As a final one, it can be shown that

$$\sum_{n \in \mathbb{Z}} \frac{(-1^n)}{(z-n)^2} = \frac{\pi^2}{(\sin(\pi z))(\tan(\pi z))}$$

Using this one, we can use some differentiation to show that

$$\frac{1}{z} \sum_{n \geq 1} (-1)^n \frac{2z}{z^2 - n^2} = \frac{\pi}{\sin(\pi z)}$$

If you have done some representation theory, these functions can be thought of as a sort of "averaging" similar to waht is given in Maschkle's Theorem (recall the formula $\pi = \frac{1}{n}\sum_{g \in G} g\pi_i g^{-1}$). Though this connection is not of much use to us, this idea of somehow averaging shall come back later when look to find the class of all meromorphic functions define on $\mathbb{C}$.

4.2.2 Functioned Defined by Poles

On S^2, a meromorphic function is rational, so there are in particular only finitely many poles. In this section, we classify all meromorphic functions on $\mathbb{C}$. On $\mathbb{C}$, there can be infinitely many, there can even be a limit of poles (ex. $\sec(z)$ or $\tan(z)$). Given some presecribed list of poles (and of principal parts), can we always find a meromorphic functions with given poles? In particular, given a set of poles $\{b_k\} \subseteq \mathbb{C}$, $\lim_{k \to \infty} b_k = \infty$ and $\{p_k(z)\}$ is a polynomial without constant term, can we find a meromorphic function with poles b_k and principal parts $p_k\left(\frac{1}{z-b_k}\right)$? The answer is yes!

Theorem 4.2.2: Mittag-Leffler Theorem

Any meromorphic function on $\mathbb{C}$ can be written of the form:

$$f(z) = \sum_{n=1}^{\infty} \left(P_k \left(\frac{1}{z - b_k} \right) - p_k(z) \right) + g(z)$$

where $p_k(z)$ are the chosen so that the sum converges uniformally on compact sets and $g(z)$ is an entire function

Proof **:**
Assume $b_k \neq 0$. Then $p_k((z-b_k)^{-1})$ is holomorphic in $|z| < |b_k|$, so we can expand the taylor series at 0. Let $p_k(z)$ be the sum of the first n_k terms where n_k is chosen so that

$$\left| p_k \left(\frac{1}{z - b_k} \right) - p_k(z) \right| \leq \frac{1}{2^k} \qquad |z| \leq \frac{|b_k|}{2}$$

Then we'll show that $\sum(p_k((z-b_k)^{-1}) - p_k(z))$ converges uniformally and absolutely on $|z| \leq r$ for any r. For this, choose m such that $|b_k| > 2_r$ if $k \geq m$. Then

$$\sum_{k=m}^{\infty} (p_k((z-b_k)^{-1}) - p_k(z))$$

converges uniformally and absolutely on $|z| \leq m$ by comparison with $\sum \frac{1}{2^k}$.

Finally, any meromorphic function with these poles and principal parts differ from one another by a holomorphic function, which gives the final form and completing the proof.

4.2.3 Infinite Products

Every polynomial can be represented as a product of lienar terms. In the following we generalize this idea, leading us to the notion of defining a function to be an infinite product of holomorhpic functions. We shall decompose the trigonometric functions as product of roots, and shall also create new functions using this method. We shall show that all entire functions can be factorized into an (infinite) product of linear terms with correction factors up to an entire function.

We start with the element-wise definition of infinite product:

Definition 4.2.2: Infinite Product

Let (a_i) be a sequence. Then we say that $\prod_i^{\infty} a_i$ exists if $p_k = \prod_i^k a_i$ converge, that is

$$\prod_i^{\infty} a_i = \lim_k \prod_i^k a_i = \lim_k p_k = p$$

More generally, we say the product *almost converges* if the product converegs after removing finitely many terms.

Certainly, we require that $p_k \to 1$ if all $a_k \neq 0$. If $a_n \neq 0$ for all n we can write write $u_n = 1 + a_n$, and for n sufficiently large,

$$\sum_n \log(1 + a_n)$$

is well-defined where we take the principal branch of log. If $s_n = \sum_k^n \log(1 + a_k)$, then $p_n = e^{s_n}$ is well-defined.

Lemma 4.2.1: Converges Of Partial Series, Then Convergence Of Partial Products

Let s_n, p_n be defined as above. Then if s_n converges if and only if p_n converges.

Proof **:**
If s_n converges, then certianly p_n converges. Conversley, suppose p_n converges. The problem is that $\log(n)$ is a many-valued function, or in particular we must be careful with choosing the appropriate branch (the appropriate imaginary part). Let $p = \lim_n p_n$. Choose a branch by taking $\log p = \log|p| + i\arg(p)$. Then consider

$$\log p_n = \log|p_n| + i\arg p_n \qquad \arg p_n \in (\arg p - \pi, \arg p + \pi)$$

Then we get $s_n = \log p_n + 2\pi i k_n$, $k_n \in \mathbb{Z}$. Then:

$$\log(1 + a_{n+1}) = s_{n+1} - s_n = \log(p_{n-1}) - \log(p_n) + 2\pi i(k_{n+1} - k_n)$$

For n large enough, we have $\arg(1+a_n), \arg(p_n - p)$, $\arg(p_{n+1} - p)$ are all $< 2\pi/3$. Thus, $k_{n+1} = k_n$ since they must all be in $\mathbb{Z}$. Finally, setting $k = k_n$ for large enough n, we get

$$s_n \to \log p + 2\pi i k$$

showing convergence the other way, completing the proof.

Definition 4.2.3: Product Absoltue Convergence

Let $(a_n) \subseteq \mathbb{C}$. Then $\prod a_n$ converge absolutely if

$$\sum_n \log(1 + a_n)$$

converges absolutely.

This is equivalent to saying that $\sum a_n$ conveges absolutely, since

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1 \implies \left| \frac{|\log(1 + a_n)|}{|a_n|} - 1 \right| < \epsilon$$

hence

$$(1 - \epsilon)|a_n| < |\log(1 + a_n)| < (1 + \epsilon)|a_n|$$

showing equivalence.

Definition 4.2.4: Infinite Product of Functions

Let (f_n) be a sequence of continuous functions defined on $D \subseteq \mathbb{C}$. Then we say that $\prod_n f_n$ converges uniformally (resp. uniformally and absolutely) on compact sets $K \subseteq D$ if:

1. As $n \to \infty$, f_n uniformally tends to 1 on K. In particular, for n sufficiently large , $|f_n - 1| < 1$ on K. Hence, $\sum_n \log f_n$ converges uniformally.

2. the series is said to converge *absolutely* if $\sum_n \log f_n$ converge absolutely.

For a succint condition, let $f_n = 1 + u_n$; then condition (1) says that u_n converges uniformally to 0 in K, and when u_n is small, $\log f_n$ and u_n are equivalent to the first order, hence we see that $\sum_n u_n$ converges uniformally on K. Thus we get that $\prod f_n$ converges uniformally if and only if $\sum u_n$ converges uniformally:

$$\prod_n f_n \iff \sum_n u_n$$

Proposition 4.2.3: Uniform Product Convergence Is Holomorphic

Let $(f_n) \subseteq \mathcal{H}(D)$ be a sequence of nonzero holomorphic functions. Then if $\prod_n f_n$ converges uniformally,

$$f = \prod_n f_n$$

is holomorphic, and we can write for any $p \in \mathbb{N}$

$$f = f_1 f_2 \cdots f_p \left(\prod_{n>p} f_n \right) \tag{4.1}$$

The set of zeros of f is the union of the sets of zeros of the functions f_n, the order of multiplicity of a zero of f is equal to the sum of orders of multiplicity given by each f_n

Proof **:**

f is holomorphic since $\prod_k^n f_k$ uniformally converges to f (the product of holomorphic functions is holomorphic). The decomposition is immediate. Since $f_n \to 1$ uniformally, for n sufficiently large f_n has no zeros in U, hence the zeros of f is the union of the set of zeros of (f_n), and similarly for their multiplicity.

Proposition 4.2.4: Logarithmic Derivative Converges

Given $\prod f_n \to f$ uniformally and absolutely on D, the series $\sum_n f_n'/f_n$ of meromorphic functions converges uniformally on compact subsets of E and its sum is the logarithmic derivative of f'/f.

Proof **:**
Let U be a pre-compact (relatively compact) open subset of D. Define

$$g_p = \exp\left(\sum_{n>p} \log f_n\right)$$

which is holomorphic for sufficinetly large p (since f_n is nonzero for sufficiently large p). Then by equation (4.1) we have

$$\frac{f'}{f} = \sum_{n\leq p} \frac{f_n'}{f_n} + \frac{g_p'}{g_p} = \sum_{n\leq p} \frac{f_n'}{f_n} + \sum_{n>p} \frac{f_n'}{f_n}$$

Note that $\sum_{n>p} \frac{f_n'}{f_n}$ converges absolutely on compact subsets of D since by assumption the series $\sum_{n>p} \log f_n$ converges absolutely to $\log g_p$ meaning it's derivative converge uniformally absolutely, hence the series of derivatives of these logarithm converges (absolutely on compact subsets) to the derivative g_p'/g_p. Hence, the above equation becomes:

$$\frac{f'}{f} = \sum_n \frac{f_n'}{f_n}$$

as we sought to show.

Example 4.3: Infinite Product

1. We shall try to write $\sin(\pi z)$ as an infinite product. Let

$$f(z) = z\prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right)$$

This converges uniformally on compact subsets of $\mathbb{C}$ since $\sum_n z^2/n^2$ converges uniformally on compact sets. Hence, $f(z)$ is holomorphic on $\mathbb{C}$ and it's zeros are simple and have exist at the integers.

Taking the logarithmic derivative, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

but the term on the right hand side has already been studied, namely:

$$\frac{\pi}{\tan(\pi z)} = \frac{(\sin(\pi z)')}{\sin(\pi z)}$$

Hence, $f'/f = g'/g$, meaning taking the integral we get:

$$\frac{f(z)}{z} = c\frac{\sin(\pi z)}{z}$$

What remains is to determine c. We know that $f(z)/z \to 1$ as $z \to 0$, and since $\sin(\pi z)/z$ has π as the limit, we get $c = 1/\pi$, giving us

$$\frac{\sin(\pi z)}{\pi z} = \prod_n \left(1 - \frac{z^2}{n^2}\right)$$

2. (Γ-function) Consider

$$\begin{aligned} g_n(z) &= z(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)\frac{1}{n^z} \\ &= \frac{z(z+1)(z+2)\cdots(z+n)}{n!}n^{-z} \end{aligned}$$

For $n \geq 2$, we have

$$\frac{g_n(z)}{g_{n-1}(z)} = \left(1+\frac{z}{n}\right)\left(1-\frac{1}{n}\right)^z = f_n(z)$$

3. (Weiestrass infinite product)

Some basic classifications questions should be asked, for example can we always find a holomorphic function with certain given zeros? If there are finitely many zeros (given the sequence of zeros do not converge, since zeros of holomorhpic functions are isolated), then polynomial would be the answer, however what about countably many zeros? Can we find all such entire functions? First, if $g(z)$ is entire, then $f(z) = e^{g(z)}$ remains entire and has no zeros, and f'/f is the derivative of the entire function $g(z)$. If $a_1, a_2, ..., a_n \in \mathbb{C}$ (counted for multiplicity), then we may defien for soem $m \geq 0$ and entire holomorphic function g:

$$f(z) = z^m e^{g(z)} \prod_k^n \left(1 - \frac{z}{a_k}\right)$$

If $n = \infty$, then we can do a similar trick as when the Wiestrass $\wp$ function: take the taylor series expansion of the log

$$\log\left(1 - \frac{z}{a_k}\right) = \frac{-z}{a_k} - \frac{1}{2}\left(\frac{z}{a_k}\right)^2 - \frac{1}{3}\left(\frac{z}{a_k}\right)^3 - \cdots$$

now, we shall eliminate enough terms so that our series converges:

Theorem 4.2.3: Weierstrass Factorization Theorem

Let $(a_n) \subseteq \mathbb{C}$ where $\lim_{n\to\infty} a_n = \infty$. Then there is an entire function with zeros at each a_n. The most general form of such a function is:

$$z^m e^{g(z)} \prod_k^\infty \left[\left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_{m_k}}\right)^2}\right]$$

where $m_k \in \mathbb{Z}$ is chosen so the series convergse uniformally and absolutely o compact sets

note that the z^m term is to signify a root at the origin, the $e^{p(x)}$ is the correction factors to insure convergence, and the $e^{g(z)}$ is the "similarity" factor, that is if we find holomorphic functions with the same roots, then they are equal up to a $e^{g(z)}$

***Proof* :**
Take $(a_n) \subseteq \mathbb{C}$ where $\lim_{n\to\infty} a_n = \infty$ and let

$$p_k(x) = \frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots_{\frac{1}{m_k}} \left(\frac{z}{a_{m_k}}\right)^2$$

be the coefficeint of the exponent in the product. Then it suffices to show that

$$\prod_{k}^{\infty}\left[\left(1 - \frac{z}{a_k}\right) e^{p_k(x)}\right]$$

converges, which is equivalent to

$$\sum_{k=1}^{\infty} \log\left(1 - \frac{z}{a_k}\right) + p_k(z) = \sum_{k=1}^{\infty} = g_k(z)$$

and log is chosen so that $\log(1 - z/a_k) + p_k(z)$ is the principal branch of log of the corresponding function in the product. Choose $g_k(z)$ so that the imaginary part is between $-\pi$ and π. Taking the Taylor series of $\log(1 - z/a_k)$,we get

$$\log\left(1 - \frac{z}{a_k}\right) = -\frac{z}{a_k} - \frac{1}{2}\left(\frac{z}{a_k}\right)^2 - \cdots$$

so choose

$$p_k(z) = \frac{z}{a_k} + \left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}$$

where m_k will be chosen soon to be large enough for the series to converge. To find m_k, take $|z| \le r$ and consider a_k where $|a_k| > r$. Then

$$g_k(z) = \frac{-1}{m_k + 1}\left(\frac{z}{a_k}\right)^{m_k+1} - \frac{1}{m_k + 2}\left(\frac{z}{a_k}\right)^{m_k+2} - \cdots$$

So:

$$|g_k(z)| \le \frac{1}{m_k + 1}\left(\frac{r}{|a_k}\right)^{m_k+1}\left(1 - \frac{r}{|a_k|}\right)^{-1}$$

Now we choose m_k so that

$$\sum_{k=1}^{\infty} \frac{1}{m_k + 1}\left(\frac{r}{|a_k|}\right)^{m_k+1}$$

converges (for example, choose $m_k = k$). Then $|g_k(z)| \to 0$, in the part between $(-\pi, \pi)$ for k large enough. Thus $\sum_k^\infty g_k(z)$ is uniformally and absolutely convergent in $|z| \le r$, as we sought to show.

Thus, it was proved that we can find functions with poles and zeros at fixed locations. This allows us to construct some entire functions with many well-controled properties. The following shows that we may express any meromorphic function in a "global manner" as a quotient of two entire holomorphic functions:

Corollary 4.2.1: Meromorphic Quotient of Entire Function

Every meromorphic function on $\mathbb{C}$ is a qutotient of 2 entire functions

Proof **:**
Let $h(z)$ be a meromorphic function on $\mathbb{C}$. Then we may choose an entire function $g(z)$ whose zeros are the poles of $h(z)$ (with multiplicity). Hence, $g(z)h(z) = f(z)$ is an entire holomorphic. Thus

$$h(z) = \frac{f(z)}{g(z)}$$

completing the proof.

This next theorem shows we can be even more precise: given some [isolated] sequence of points, we may choose corresponding points in $\mathbb{C}$ (with multiplicity), and find a holomorphic function with the appropriate values:

Corollary 4.2.2: Determining Holomorphic Function given Sequence

Given $(a_k), (b_k) \subseteq \mathbb{C}$, $\lim_{k\to\infty} a_k = \infty$, $m_k \in \mathbb{N}$, then there is an entire function $f(z)$ such that every a_k is a root of order m_k of $f(z) = b_k$

Proof **:**
Near a_k, $f(z)$ looks like $b_k = (z - a_k)^{m_k} \ell(z)$ where $\ell(z)$ is something nonzero near a_k. Now, there is an entire function $g(z)$ with zeros of order $m_k + 1$ at a_k. Now take $f(z) = g(z)H(z)$ where

$$H(z) = \left(\frac{b_k}{g(z)} + h(z) - \frac{b_k}{g(z)} \right)$$

and where $h(z)$ is a meromorphic function with poles a_k with princpal part $\frac{1}{z-a_k}$ plus the principal at a_k of $\frac{b_k}{g(z)}$.

Corollary 4.2.3: Factorization Results

1. Let f denote an entire function (not identically zero). Let n be a natural number. Prove that there exists an entire function g such that $f = g^n$ if and only if every zero of f has order divisible by n

2. Denote $Z(f)$ the zet of zeros of a function f. Prove that if f_1, f_2 are entire functions such that $Z(f_1) \cap Z(f_2) = \emptyset$, then there exist entire function g_1, g_2 such that $f_1g_1 + f_2g_2 = 1$

Proof **:**

1. If $f = g^n$ on $\mathbb{C}$, then if $f(z) = 0$, $g(z)^n = 0$, showing that the order of the pole must be divisible by n.

Conversely, suppose that f has zeros with order divisible by n: $a_1, a_2, ..., a_k$ with order $n_1, n_2, ..., n_k$. Define g_1 to be an entire holomorphic function with zeros at a_i with order n_i/n. Then:

$$f/(g_1^n) = g_2'$$

is a nonzero entire function. By homework 1 question 4, we have that there exists a g_2 such that $g_2' = (g_2)^n$. Hence:

$$f = (g_1^n g_2^n) = (g_1 g_2)^n = g^n$$

as we sought to show.

2. Let f_1, f_2 be entire functions with disjoint roots. We want to find entire functions g_1, g_2 such that $f_1 g_1 + f_2 g_2 = 1$. Define h to be the entire function with roots from f_1 of the same order and roots of f_2 at 1 of the same order. Define:

$$g_1 := h/f_1$$

g_1 is entire since the multiplicity of the poles of f_1 match the multiplicity of the roots of h at each pol of f_1. Next, define:

$$g_2 := \frac{1 - f_1 g_1}{f_2} = \frac{1 - f_1(h/f_1)}{f_2} = \frac{1-h}{f_2}$$

Then g_2 is entire since for each root of f_2, the numerator has the same root with the same order. Hence, we have:

$$f_1 g_1 + f_2 \left(\frac{1 - f_1 g_2}{f_2} \right) = f_1 g_1 + 1 - f_1 g_1 = 1$$

as we sought to show.

4.2.4 Gamma Function and Riemann Zeta Function

While sin (and cos) have zeros at all the integers, we shall introduce a (rather simple) function which has zeros at all the natural numbers. With the technics we have given above, we see taht such a function sould be

$$G(z) = \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

with the correction term $e^{-z/n}$ insuring convergence, and has roots at all the negative integers. Naturally, $G(-z)$ has roots at all the positive integers. Using $G(z)$ and $G(-z)$, we rather simply construct a function with poles at all the integers, namely:

$$zG(z)G(-z) = \frac{\sin(\pi z)}{\pi}$$

By construction, G has some nice properties, for example $G(z-1)$ has the same zeros as $G(z)$ along with a zero at the origin. Thus, by using theorem 4.2.3 that

$$G(z-1) = z e^{\gamma(z)} G(z)$$

For some entrie function $\gamma(z)$.To find it, we take the logarithmic of both sides giving us:

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)=\frac{1}{z}+\gamma'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

Notice that the left hand side can be manipulated like so:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)&=\frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)\\&=\frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\end{aligned}$$

Moving the terms around, we see that $\gamma'(z)=0$, hence γ is a constant function, giving us:

$$G(z-1)=e^{\gamma}(z)$$

Defining $H(z)=e^{\gamma}G(z)$, we get:

$$H(z-1)=zH(z)$$

Riemann Zeta Function

4.3 Riemann Mapping Theorem

When working with holomorphic functions, we usually work with either:

1. a proper connected subset $\Omega \subsetneq \mathbb{C}$
2. a proper simply connected subset $\Omega \subsetneq \mathbb{C}$
3. all of $\mathbb{C}$
4. the compactification $\overline{\mathbb{C}} \cong \mathbb{C}\mathrm{P}^1 \cong S^2$

Since being differentiable is a local condition, we may focus on the simply connected cases of $\mathbb{C}$, simply connected proper subset of $\mathbb{C}$, and it's compactification. What we shall show in this section is that up to biholomorphism, the only simply connected subsets are:

1. D, the unit disk
2. $\mathbb{C}$, the complex plane
3. $\mathbb{C}\mathrm{P}^1$, the complex projective line, equivalently the Riemann sphere.

This will allow us to classify all Biholomorphisms of simply connected subsets of $\overline{\mathbb{C}}$, and lead towards an upgrade of Weiestrass's Theorem (theorem 3.3.2) to Picard's theorem (theorem 4.3.8).

4.3.1 Classifying Automorphisms

We start by showingthat $\mathbb{C}$ and D are distinct up to biholomorphism:

Theorem 4.3.1: Plane And Disk Not Biholomorhpic

The plane $\mathbb{C}$ and the open disk $D = \{z \ : \ |z| < 1\}$ are *not* biholomorphic

Proof **:**
Let $f : \mathbb{C} \to D$ be a holomorphic map. Then this is a bounded map, hence since f is entire it must be constant, thus it cannot be bijective (namely it's not injective), hence cannot be biholomorphic.

Thus, it is meaningful to classify automorphisms of $\mathbb{C}$ and D distinctly. Starting with $\mathbb{C}$:

Proposition 4.3.1: Automorphisms Of $\mathbb{C}$

$$G_{\mathbb{C}} := \mathrm{Aut}_{\mathbf{Holo}}(\mathbb{C}) = \{az + b \ : \ a, b \in \mathbb{C}, a \neq 0\}$$

Proof **:**
Consider the type of pole f has at infinity, it must either be essential or a pole, hence either:

1. f has an essential singularity at infinity
2. f is a polynomial

The first one is impossible since the function would not be injective: if f were injective, then $\operatorname{im} f$ of $|z| > 1$ does not meet the image of f under $|z| < 1$, this image is open an non-empty. Hence $|z| > 1$ is not *dense* in the whole plane, hence by Weiestars theorem (theorem 3.3.2), the point at infinity is *not* an essential singularity, thus f must be a polynomial. But then by Abel's theorem if the degree is greater than 1, then $f(z) = w$ has n distinct roots (except a small number of w), but f must be injective, hence this is impossibl. Thus, the only possibility is to linear functions, completing the proof.

Next, we shall classify the automorphism of the Riemann sphere $\mathbb{CP}^1$. We first require a group-theoretical lemma:

Lemma 4.3.1: Stabalizers And Transitive Groups

Let $D \subseteq \mathbb{CP}^1$ and $G \leqslant \mathrm{Aut}_{\mathbf{Holo}}(D)$. Then if:

1. G is transitive in D
2. there is at least one point of D whose stabiliser is contained in G

Then $G = \mathrm{Aut}_{\mathbf{Holo}}(D)$.

***Proof* :**
Let $S \in \text{Aut}_{\textbf{Holo}}(D)$ and $z_0 \in D$ be a point whose stabiliser is contained in G. Since G is tarnsiive, there is a $T \in G$ such that

$$T(z_0) = S(z_0)$$

Hence ,the transforamtion $T^{-1} \circ S \in \text{Aut}_{\textbf{Holo}}(D)$ leave the point z_0 fixed, and hence belongs in G. Thus

$$T \circ (T^{-1} \circ S) = S \in G$$

showing equality, as we sought to show.

Note that if we take the automorphisms of $\mathbb{C}$ and consider the compactification of $\mathbb{C}$, then if $a = 1$, it has a fixed point at infinity, while if $a \neq 1$ then

$$z = \frac{b}{1-a}$$

is a fixed point. Note that $G_{\mathbb{C}}$ is a *transitive group* on the plane, that is for any pair (z_1, z_2), $z_1, z_2 \in \mathbb{C}$, there exists at least one $f \in G_{\mathbb{C}}$ such that $f(z_1) = z_2$.

Theorem 4.3.2: Automorphisms Of Riemann Sphere

$$\text{Aut}_{\textbf{Holo}}(\mathbb{C}\text{P}^1) = \left\{ \frac{az+b}{cz+d} \ : \ ad - bc \neq 0 \right\}$$

***Proof* :**
First, it is clear that any fractional lienar transformation takes the Riemann sphere to itself, and since $ad - bc \neq 0$:'

$$z = \frac{dz - b}{-cw + a}$$

is the inverse. Certainly this map is holomorphic, hence it is biholomorphic. I claim that that all automorphisms of $\mathbb{C}\text{P}^1$ are of this form. Indeed, let $G \leqslant \text{Aut}_{\textbf{Holo}}(\mathbb{C}\text{P}^1)$ be the subgroup FLT's, and consider the elements of G which are FLT's leaving the point at infinity fixed, i.e. $c = 0$ and $d \neq 0$, without loss of generailty we may suppose $d = 1$ (since multiplying all the constant by a constant doesn't change the transformation). Then these elements can be written in the form $w = az + b$. Then we see that G is certainly transitive on $\mathbb{C}\text{P}^1$ and contain the stabaliser of the point at infinity, hence by the above lemma must be the entire automorphism group, as we sought to show.

Thus, our earlier study of FLT's was in fact the study of the automorphisms of $\mathbb{C}\text{P}^1$! From proposition 1.3.2, we get the followign result:

Corollary 4.3.1: Automorphisms Of Upper Half Plane

$$\text{Aut}_{\textbf{Holo}}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} \ : \ a, b, c, d \in \mathbb{R}, \ ad - bc > 0 \right\}$$

Proof **:**
proposition 1.3.2 tell us that a biolomorphic transformation from $\mathbb{H}$ to itself may have the above form, and by the above lemma it suffices to show that the gorup is transitive. For this, we see that the point i can be transformed into an arbitrary point $a + bi$ $(b > 0)$. We must thus show that the stability subgrou pof i consits of homographic tnransforamtions. if H is the stability group, then

$$\frac{z - i}{z + i}$$

is an isomorphism onto the subgroup $|w| < 1$ of $\mathrm{Aut}_{\mathbf{Holo}}(D)$ consisting of automorphisms of the disc which leave the center 0 fixed. Thus, it suffies to show that an automorphim of the disc $|z| < 1$ which fixes the origin is a rotation $z \mapsto ze^{i\theta}$ for some angle θ. If f were such an automorphisms whwerer $f(0) = 0$, then we have by Schwarz lemma that:

$$|f(z)| \leq |z|$$

for al all $|z| < 1$. But, since there exists the inverse, we may apply the Schwarz' lemma to it and get :

$$|z| \leq |f(z)|$$

Thus, we have $|z| = |f(z)|$, which again by Schwarz lemma tells us that $f(z) = cz$ for $c \in \mathbb{C}$ where $|c| = 1$. But this is excatly what we wanted to show.

Thus, by lemma 4.3.1, we have that these automorphisms are all the automorphisms of $\mathbb{H}$, as we sought to show.

For those who are curious, the stabilizer at i are automorphisms of the form:

$$z \mapsto \frac{z + \tan(\theta/2)}{z - \tan(\theta/2)}$$

This also shows we may extend automorphisms of $\mathbb{H}$ to automorphisms of $\mathbb{C}\mathrm{P}^1$, something that, before the classification, is not obvious. With the automorphisms of $\mathbb{H}$ classified, we may easily get the automorphisms of the unit disk D by composing with $\frac{z-i}{z+i}$.

Theorem 4.3.3: Automorphisms Of Unit Disk

$$\mathrm{Aut}_{\mathbf{Holo}}(D) = \left\{ e^{i\theta} \frac{z + z_0}{1 + \overline{z}_0 z} \ : \ \theta \in \mathbb{R}, \ |z_0| < 1 \right\}$$

Proof **:**
This is the automorphissm of $\mathbb{H}$ composed with $(z - i)/(z + i)$. However, this may also be directly worked out. (Cartan on p.183 explicitly works the following out from scratch)

4.3.2 Riemann Mapping Theorem

We shall now build up to to showing that all simply connected proper open subsets $U \subsetneq \mathbb{C}$ are isomorphic to the unit D.

Lemma 4.3.2: Riemann Mapping Theorem Lemma I

Let $D \subsetneq \mathbb{C}$ be simply connected. Then there exists an isomorphism of D onto a bounded open set of $\mathbb{C}$.

Proof **:**
We shall construct an explicit isomorphism. By assumption, there exists a point $a \notin D$. Take $\log(z-a)$ in D. Since D is simply connected, we may choose a branch $g(z)$. Then g is injective in D since if $g(z_1) = g(z_2)$, then:

$$e^{g(z_1)} = e^{g(z_1)} \implies z_1 - a = z_2 - a$$

giving us $z_1 = z_2$. Now, choose $z_0 \in D$. Then g takes the values of a disc E centered at $g(z_0)$ into D. Translating this disk by $2\pi i$, we get a disk which has no points in common with the image of D by g, since e^g is injective. Then:

$$f(z) = \frac{1}{g(z) - g(z_0) - 2\pi i}$$

is holomorphic, injective, and *bounded* in D. Thus, we get an isomorphism of an open such that D onto a bounded set of $\mathbb{C}$, as we sought to show.

Thus, we may assume that D is bounded. By shrinking D enough via the biholomorphism λz and translating, we may, suppose that $0 \in D$ and D is contained in the unit disk $|z| < 1$, which is something we shall assume from here on out in this section

Lemma 4.3.3: Riemann Mapping Theorem Lemma II

If two complex numbers $u, v \in \mathbb{C}$ satisfy $\text{Re}(u) < 0$ and $\text{Re}(v) < 0$, then:

$$\left|\frac{v-u}{v+\overline{u}}\right| < 1$$

Proof **:**
exercise

Lemma 4.3.4: Riemann Mapping Theorem Lemma III

Let $\Omega \subsetneq \mathbb{C}$ be simply connected and bounded, containing 0 and conatined in $|z| < 1$. Let $A \subseteq \mathcal{H}(\Omega)$ be a subset of injective functions where

1. $f(0) = 0$
2. $|f(z)| < 1$ for all $z \in \Omega$

Then $\operatorname{im} f$ is exactly the unit disc if and only if $|f'(0)|$ is maximum in the set of values which it takes as f describes A

Proof **:**
If the image of f is the unit disk, then these two conditions are certianly satisfied, so let's show that these condition's are sufficient.

For the sake of contradiction, let' say that there is a a in the unit disk such that $a \notin \text{im}(f)$. Then I claim there exists a g such that

$$|f'(0)| < |g'(0)|$$

To see this, let

$$F(z) = \log\left(\frac{f(z) - a}{1 - \overline{a}f(z)}\right)$$

This function is holomorphic and injective in Ω. The value of $f(z)$ are in the unit dis, and the avlue sof $\frac{f(z)-a}{1-\overline{a}f(z)}$ are alos in theunit disk, hence $F(z)$ has real part < 0. Naturally, we have chosen some branch of the logarithm for $F(z)$, which we can do since D is simply connected. Now, consider:

$$g(z) = \frac{F(z) - F(00)}{F(z) + \overline{F(0)}}$$

which is holomorphic and injective in $\text{im}(f)$. Then $G(0) = 0$, and furthermore by lemma II $|g(z)| < 1$. Hence, $g \in A$, and:

$$g'(0) = \frac{F'(0)}{F(0) + \overline{F(0)}} \qquad F'(0) = \left(\overline{a} - \frac{1}{a}\right) f'(0)$$

Thus:

$$\frac{|g'(0)|}{|f'(0)|} = \frac{1 - a\overline{a}}{2|a| \log\left|\frac{1}{a}\right|}$$

now, to show that $|g'(0)| > |f'(0)|$, it suffice to show that

$$\frac{1 - t^2}{t} - 2\log(1/t) > 0 \qquad 0 < t < 1$$

but this is rather easy to see: the left hand side is a functio nof t whose derivative is < 0¡ thus it is strictly decreasing on $0 < t \leq 1$, and is equalt to 0 for $t = 1$, and is > 0 for $) < t < 1$. But then we get the result.

Theorem 4.3.4: Riemann Mapping Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected proper subset of $\mathbb{C}$. Then Ω is isomorphic to the unit disc.

Proof **:**
By lemma III, we need only pove that ther exists a function $f \in A$ such that the upper bound of $|f'(0)|$ is attained. Let B be the set of all $f \in A$ such that $|f'(0)| \geq 1$. Then the set B is non-empty since $f(z) = z$ is in B. The set B Is also bounded in $\mathcal{H}(D)$ since $|f(z)| < 1$ for all $z \in D$ and $f \in B$. If we also show that B is *closed*, then we would get that that it is compact. Given B is compact, then the mapping which associates the real number $|f'(0)|$ to each $f \in B$ is a continuous mapping, which then must atain it's uuper bound, which gives the desired result,

To see that it is closed, let f be a holomorphic unciton which is a limit of $f_n \in B$. Then

$$f(0) = \lim f_n(0) = 0$$

Next, since the derivative $f_n' \to f'$, we get that

$$|f'(0)| = \lim_n |f_n'(0)| \geq 1$$

Thus, f is *not* constant on D. Next, f is the limit of injective functino, and hence is injective. Since $|f_n(z)| < 1$ for all $z \in D$, we have $|f(z)| \leq 1$. Since $|f(z)| = 1$ is impossible for any $z \in D$ (by the maximum modulus principal, since f is not constant), we have that f satisfies all the condition to be in B, that is

$$f \in B$$

but this then proves the result, completing the proof.

Corollary 4.3.2: Unique Mapping

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected proper subset of $\mathbb{C}$. Then for each $z_0 \in \Omega$, there exists a unique holomorphic function $f(z)$ over Ω where $f(z_0) = 0$ and $f'(z_0) > 0$, such that $f(z)$ is biholomorphic onto the open disk.

Proof **:**
The uniqueness comes quickly, for if f_1, f_2 where two such functions, then $S = f_1 \circ f_2^{-1}$ would be an injective map from the unit disk to itself. We clasified all biholomorphisms, hence we know what such a map would be. Given $S(0) = 0$, $S'(0) \in \mathbb{R}$, and $S'(0) > 0$, the only possiblitiy is that $S(z) = z$, i.e. $S = \text{id}$, hence $f_1 = f_2$.

Exercise 4.3.1

1. We have found how to factor all holomorphic functions on $\mathbb{C}$ and $\mathbb{CP}^1$. Using the Riemann mapping theorem, it suffices to find how to factor functions on D (develop Blaschke products).

4.3.3 Application: Schwarz-Christoffel Mappings

One consequence of the Riemann mapping theorem that seems totally implausable is that Ω is biholomorphic to regular n-gons, even though it seem like the angles may change at the corners. In fact, it is not too difficult to explicitly describe the biholomorphism, which is what we aim to achieve in this section.

Definition 4.3.1: Schwarz–Christoffel Mapping

A Schwarz-Christoffel mapping is a holomorphic function from Ω or $\mathbb{H}$ onto a simple polygon.

Consider an set $\Omega \subseteq \mathbb{C}$ that is a regular n-g with vertices $z_1, z_2, ..., z_n$ where $z_{n+1} = z_1$, and the angel between each is

$$\arg \frac{z_{k-1} - z_k}{z_{k+1} - z_k} = \alpha_k \pi \quad (0 \leq \alpha_k < 2)$$

For future purposes, we also introduce

$$\beta_k \pi = (1 - \alpha_k)\pi \qquad -1 < \beta_k < 1$$

since

$$\beta_1 + \cdots + \beta_n = 2$$

and the polynomial is convex if $\beta_i > 0$ for all i.

(intuition here)

Theorem 4.3.5: Schwarz-Christoffel Formula

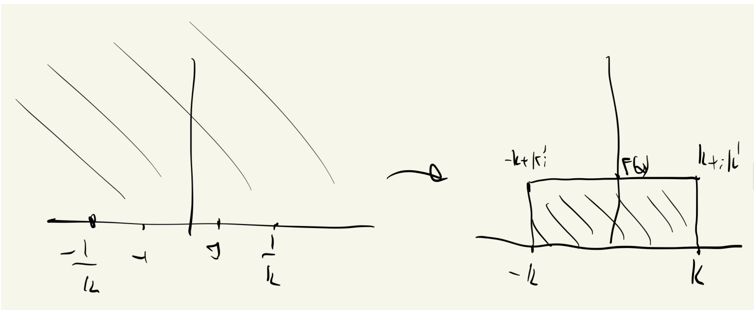
Let $F(w) = z$ be the map from $|w| < 1$ that conformally maps onto the polygons with angles $\alpha_k \pi$ (for $k \in \{1, ..., n\}$). Then they are of the form:

$$F(w) = C \int_0^w \prod_{k=1}^{n} (w - w_k)^{-\beta_k} dw + C'$$

where $\beta_k = 1 - \alpha_k$, the w_k are points on the unit circle, and C, C' are complex constants.

Proof **:**
Alfhors p.236

Example 4.4: Mapping Onto Rectangle
Take $\mathbb{H}^+$, and identiy the points $\frac{-1}{k}, -1, 1, \frac{1}{k}$.



Then:

$$F(w) = \int_0^w \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt$$

where

$$K = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt \qquad K' = \int_1^\infty \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt$$

4.3.4 Picard's Theorem

(Zoltsman lemma first published in the 1970s, at first seemed a little inconspicuous, was just a notes in the AMS plublishing when published, but has beeen generalized to higher dimensions and found many important and powerful applications)

Lemma 4.3.5: Zoltsman Lemma

A family of meromorphic functions N on a domain Ω is *not noraml* in the chordal metric if there exists a squecne $a_n \subseteq \omega$ and positive numbers $\rho_n \to 0$, and $f_n \in N$ such that $g_n(z) = f_n(a_n + \rho_n z)$ converges to a non-constnat function $g(z)$ which is meromrophic on the entire plane such that

$$g^{\#}(z) \leq 1 = g^{\#}(0)$$

Interestingly, we are giving a criterion for *failure* of normality, not by a lack of convergence condition, but *by* a convergence condition. The idea is that $g(z)$ is *not* constant, the point being that if the f_n converged, then because the ρ_n's are ending to 0, the g_n's would converge to a constant function. Hence, normality would mean convergence to a constant.

***Proof* :**
Suppose N is normal. Then any sequence would contain a convergent subsequence, say $f_n \to f$. Consider now any sequence $a_n \to a_\infty \in \Omega$ and $\rho_n \to 0$. Then define

$$g_n(z) = f_n(a_n + \rho_n z) \to f(a_\infty)$$

which converged by uniform equicontinuity by Arzela Ascoli. But then, g is constant.

Let's now assume N is not normal, or equivalently

$$N^{\#} = \left\{ f^{\#} \ : \ f \in N \right\}$$

is not locally bounded. Thus, there exists $b_n \to b_\infty \in \Omega$ and $f_n \in N$ such that $f^{\#}(b_n) \to \infty$. Without loss of generailty, we can assume $b_\infty = 0$ and $\{|\zeta| \leq r\} \subseteq \Omega$. Let

$$M_n = \max_{|\xi|=r} (r - |\zeta|) f_n^{\#}(z) = (r - |a_n|) f_n^{\#}(a_n)$$

where that last equality is because the maxmimum is certainly not on the boundary, so it works for some a_n in the open disc $|a_n| < r$. Then:

$$M_n \xrightarrow{n \to \infty} \infty$$

since $b_n \to 0$. We shall get now our desired result by taking $\rho_n = \frac{1}{f^{\$}(a_n)}$. Then

$$g_n(z) = f_n \left(a_n + \frac{z}{f_n^{\#}(a_n)} \right)$$

We have to make sure this is well-defined. To see this:

$$\begin{aligned}\left|a_n + \frac{z}{f_n^{\#}(a_n)}\right| &\leq |a_n| + \frac{|z|}{|f_n^{\#}(a_n)} \\ &\leq |a_n| + \frac{M_n}{f_n^{\#}(a_n)} \\ &\leq |a_n| + (r - |a_n|) \\ &= r\end{aligned}$$

hence, this is defined on $|z| \leq M_n$, since Now fix any big radius $R < \infty$. We shall estimate $g_n^{\#}$ ins thie disc $|z| \leq R < M_n$, for n large enough. Then:

$$\begin{aligned}g_n^{\#}(z) &= \frac{f_n^{\#} + \frac{z}{f_n^{\#}(a_n)}}{f_n^{\$}(a_n)} \\ &\leq \frac{M_n}{r - \left|a_n + z/f_n^{\#}(a_n)\right|} \cdot \frac{r - |a_n|}{M_n} && \text{using maximum} \\ &\leq \frac{r - |a_n|}{r - |a_n| - \frac{|z|}{f_n^{\#}(a_n)}} \\ &= \frac{1}{1 - |z|/M_n} \\ &\to 1 && n \to \infty\end{aligned}$$

We see thus that $|g_n|$ contains a subsequence that converges uniformally in the cordal metric. Hence, we may assume (g_n) does. Letting $g = \lim_n g_n$, we have

$$g^{\#}(z) \leq 1 = g^{\#}(0)$$

hence, g is meromorphic, and it is not constant sicne $g^{\#}(0) \neq 0$. Finally, to show $a_n \to a_\infty \in \Omega$, we can first pass to a subsequence, and since we are working in the closed disc inside Ω, which gives us our desired result.

Example 4.5: Lacking Normality

The family $N = \{f_n\}$ where $f_n(z) = z^n$ is normal on D and on $\mathbb{C}\backslash\overline{D}$, but not normal in $0 \leq |n| < 2$. T osee this, let $a_n = 1$, $\rho_n = 1/n$, and

$$f_n(a_n + \rho_n z) = \left(1 + \frac{z}{n}\right)^n \to e^z = g(z)$$

and

$$g^{\#}(z) = \frac{2|g'(z)|}{1 + |g(z)|^2} = \frac{2e^x}{1 + e^{2x}} \leq 1 = g(0)$$

Theorem 4.3.6: Montel's Big Theorem

A family of meromorphic functions on domain Ω which omits 3 distinct values in $\mathbb{C}^*$ (i.e. one of the ommited points may be infinity) is normal in the chordal metric

By Picard's theorem, this can be a family of entire holomorphic functions whihc ommit 2 distinct values.

Proof **:**
Recall we can measure normality by a covering of open discs, hence we may ssume $\Omega = D$ is an open disc. By composing with an FLT (which is biholomorphic), we can further assume the 3 values are

$$a = 0 \qquad b = 1 \qquad c = \infty$$

Hence, N is the family of *all* holomorphic functions on E which ommit the value $0, 1$.

Let $N_m = \{ f \in \mathcal{H}(D) \ : \ f \text{ omits } 0,\, e^{2\pi i k/2^m} \ k = 0, ..., 2^m - 1 \}$. Naturally,

$$N = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

Take any $f \in N_m$. It doesn't vanish, hence it has a holomorphic square root by ref:HERE: $f^{1/2} \in N_{m+1}$. Hence, none of the N_m vanish: $N_M \neq \emptyset$.

Now, suppose N is not normal, so there exists a sequence with no convergent subsequence, $(f_n) \subseteq N$. Then $(f_n^{1/2}) \subseteq N_1$ and has no convergent subsequence. Continuing inductively, this implies non of the N_m's are normal. With this, we shall apply Zoltsman's lemma to each N_m, and take the function that it constructs. Namely, for each m, take h_m which is the given function g given by Zoltsman lemma. I claim that each of these functions are in fact entire (not just meromorphic). (THIS IS BC OF A PROBLEM IN THE PS, move it here once you prove it, think Hurwitz lemma)

Next, $\{h_m\}$ is normal by Mary's lemma and the definition of "g". So, there is a subsequence that converges to some h, which is entire and not constant by the PS problem .

Now, limit h which omits in adius 2^m root sof unity for all m (by Hurwit'z lemma). Now, the 2^m roots are dense in S^1. Hence, $h(\mathbb{C})$ is connected and open, so is either inside the disc or outside the disc. In other words h is bounded or $1/h$ is bounded. So by Louiville, h is constant by louiville, but that contradicts $h^{\#}(0) = 1$, as we sought to show.

We shall now show Picard's Big Theorem:

Theorem 4.3.7: Picard's Big Theorem I

If f is meromorphic in a punctured disk

$$0 < |z - z_0| < \delta$$

and f omtis 3 values in $\mathbb{C}^*$, then f is meromorphic in $|z - z_0| < \delta$

Proof **:**
We can assume the disc is centerd at 0 and that f omits $0, 1, \infty$. Let ϵ_n be a strictly dcreasing sequence that decreases t o0. Let

$$S_n = \{f(\epsilon_n z)\}$$

This is a normal family by Montel's Theorem. Let S be a normal family on $\{0 < |z| < 2\}$. This is normal in the cordal metric by Montel again. Hence, there is a subsequence that converges on compact sets. So, we can assume that $f(\epsilon_n z)$ converge uniformally on compact sets of Ω to a holomorphic funciton (by PS5 again).

Let's consider cases: let's first consider g to be holomorphic. Take an upper bound on the unit disc

$$|g(z)| \leq M < \infty \qquad |z| = 1$$

so

$$|f(z)| \leq M + 1 \qquad |z| = \epsilon_n$$

Hence, by the maximum modulus principle

$$|f(z)| \leq M + 1 \qquad \epsilon_{n+1} \leq |z| \leq \epsilon_n, \ \forall n \geq n_0$$

so

$$|f(z)| \leq M + 1 \qquad 0 < |z| \leq \epsilon_{n_0}$$

giving us a removable singularity, meaning it extends to be holomorphic!

In the second case, $g \equiv \infty$, apply the argument to $1/(\epsilon_n z)$, so we can conlude that $1/f$ is holomorphic at 0 that is f is meromorphic, completing the proof.

Theorem 4.3.8: Picard's Big Theorem II

Let f be a holomorphic function with an essential singularity at z_0. Then there exists $\lambda \in \mathbb{C}$ such that any neighbourhood of zs_0 assumes every value except may λ (infinitely many times)

The two big picad theores are equivalent

Proof **:**
for (1) $\implies$ (2), by FLT, we can assume f omits ∞ and two other values. By(1) z_0 cannot be an essential singularity

For (2) $\implies$ (1), f is ameromorphic function in a punctured disk which omits ∞, so can't ommit 2 values $\neq \infty$, since it's not meromorphic at z_0.

Theorem 4.3.9: Picard's Little Theorem

Any non constant entire function omits at most 1 value

Proof **:**
f has either a pole or essential singularity at ∞. If pole, then f is rational so takes every value. The caes for essential singularity is given by Picaard' Big Theorem

5

Elliptic Functions and Weierstrass p-function

As we saw, periodic functions such that $\sin, \cos, \tan$ are all easy to understand holomorphic functions. Now that we are in $\mathbb{C}$, we can imagine the interest in studying doubly periodic functions and how these may behave. Originally linked to the study of ellipses (hence the name *elliptic*), these functions eventually greatly evovled and now play an important role in linking two seemingly disparate areas of mathematics

Harmonic Analysis and *Number Theory*

Due to this important connection, it is worth-while taking a moment to explore these types of functions. Unfortunately, we won't have time to explore modular forms, but this section shall start providing some of the background material for them.

5.1 Elliptic Functions

Definition 5.1.1: Doubly Periodic Functions

Let e_1, e_2 be two complex numbers that are $\mathbb{R}$-linearly independent, that is $e_0 \neq 0$ and $e_2/e_1 \notin \mathbb{R}$. Then we may form a lattice $\Gamma = \{n_1 e_2 + n_2 e_2 \,:\, n_1, n_2 \in \mathbb{Z}\}$, giving us a discrete subgroup of $\mathbb{C}$. Then we say that a function $f(z)$ defined on $\mathbb{C}$ is *doubly periodic* and has Γ as period, if:

$$f(z + n_1 e_1 + n_2 e_2) = f(z) \qquad \forall z \in \mathbb{C}$$

Using induction, this is equivalent to the condition

$$f(z+e_1) = f(z) \qquad f(z+e_2) = f(z)$$

Now, if $z_0 \in \mathbb{C}$ is any complex number, we may consider the closed parallelogram bounded by

$$z_0 \qquad z_0 + e_1 \qquad z_0 + e_2 \qquad z_0 + (e_1 + e_2)$$

which we may label as:

$$P = \{z + 0 + t_1 e_1 + t_2 e_2 \ : \ t_1, t_2 \in [0,1]\}$$

this parallelogram is called the *parallelogram of periods* or the *fundamental domain* with first verex z_0. Let's say f is a meromorphic functions on $\mathbb{C}$ that has Γ as its group of periods and we choose z_0 in such a way that f has no poles on the boundary γ of the parallelogram of periods with z_0 as its first vertex. Does such a function exist? Naturally, smooth such functions exist, so the interesting questions is are there any holomorhpic functions that are doubly periodic? First, if f holomorphic, then that would imply f would be *bounded*, and hence by Louivilles theorem a constant.

So it has to be Meromorhpic. How would we find such a function? The easier way is to start with a meromorphic function f, and take its "average", that is

$$f(z) = \sum_{n,m} g(z + ne_1 + me_2)$$

as long as the series converges absolutely (so that we have no worry of the order of summation), this is doubly periodic. To roughly see what conditions we need on g so that the function is uniformally convergent, draw out a lattice in $\mathbb{C}$, and say we have a circle of radius R and $R+1$. Then the number of point between these two rings is roughtly a a constant c times R, cR. Thus roughly, the sum of the points is:

$$\sum_{R \in \mathbb{N}} \frac{cR}{R^\alpha}$$

since R^α is an upper bound. Then this converges when $\alpha > 2$. Hence, we would like:

$$g(z) \leq \frac{C}{z^\alpha} \qquad \alpha > 2$$

Thus, we may take g to be any ratioanl function of degree ≥ -3. This means that we can easily find an elliptic function with 3 poles (ex. $g(z) = \frac{1}{z-a}\frac{1}{z-b}\frac{1}{z-c}$). The question for 1 and 2 poles shall soon be addressed. First, we give a lemma that shall simplify comparing elliptic functions:

Lemma 5.1.1: Comparing Elliptic Function Using Poles

Let f_1, f_2 be two elliptic functions. Then if the poles and zeros match, then $f_1 = f_2$ up to a constant

***Proof* :**
Take f_1/f_2. Then this is bounded, hence constant.

What can we say about f? First, since it is a meromorphic function, we are naturally interested in its roots and poles. We can consider the integral $\int_\gamma f(z)dz$:

Proposition 5.1.1: Doubly Periodic Functions

If $f(z)$ is a non-constant meromorhpic function in $\mathbb{C}$ with Γ as group of periods, then the number of zeros of this function contained in a parallelogram of periods is equal to the number of poles contained in the same parallelogram, if no zeros or poles of the function f occur on the boundary of the parallelogram

Proof **:**
We may write:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 [f(z_0+te_0) - f(z_0+e_2+te_1)]dt \\ &+ \int_0^1 [f(z_0+e_1+te_2) - f(z_0+te_2)]dt\end{aligned}$$

Then if we integrate along the border of the fundametnal domain, we get by periodictiy that the integral values cancel, hence:

$$\int_\gamma \frac{f'(z)}{f(z)} = 0$$

but then that means the number of poles and roots are the same. If it has a zero or pole on the boundary of the fundamental domain, then we can just circle around it (circling in such a way that we don't double count on each border)

There is one more important relation we must consider. Take:

$$\frac{1}{2\pi}\int_\gamma \frac{g(z)f'(z)}{f(z)-a}dz \tag{5.1}$$

where g is holomorphic and f is some meromorphic function. Integrating and solving we get:

$$\frac{1}{2\pi}\int_\gamma \frac{g(z)f'(z)}{f(z)-a}dz = \sum_{\substack{p \\ \text{poles and zeros}}} n_p \cdot g(p)$$

In the special case where $g(z) = z$, we get

$$\frac{1}{2\pi}\int_\gamma \frac{zf'(z)}{f(z)-a}dz = \sum_{\substack{p \\ \text{poles and zeros}}} n_p \cdot p$$

Now, if f is elliptic, we get:

$$\frac{-e_2}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)-a}dz + \frac{e_1}{2\pi i}\int_{\gamma_2} \frac{f'(z)}{f(z)-a}dz$$

where γ_1 denotes the side of the parallelogram starting at z_0 and ending at $z_0 + e_1$, and γ_2 denotes the side of the parallelogram starting at z_0 and ending at $z_0 + e_2$. The value of these two integrals (when omiting e_1 and e_2) are integers.

Thus, we get:

$$\sum_{\substack{p \\ \text{poles, zeros}}} pn_p = me_1 + ne_2 \qquad m, n \in \mathbb{Z}$$

where n_p is the multiplicity of the zero/pole p.

Proposition 5.1.2: Relation between function and Group

Let $f(z)$ be a non-constant meromorphic function in the whole plane $\mathbb{C}$ having Γ as its group of periods. Then for any complex number a, we have

$$\sum_i \alpha \equiv \sum_i \beta_i \mod \Gamma$$

where α_i denotes the roots of the equation $f(z) = a$ counted for multiplicity, and β_i denotes the poles counted for multiplicity, each contained in a parallelogram of periods.

Importantly, $\sum_i \alpha_i$ modulo Γ is independent of a.

Corollary 5.1.1: Single Pole Elliptic Function Does not Exit

There does not exist a elliptic function with a single pole.

Proof **:**
If there is a single pole, there is a single root, say a is the root and b is the pole. Then by the above we have:

$$a - b = ne_1 + me_2$$

But now, if a differs from b by $ne_1 + me_2$, then by periodicity since f has a zero at a, it must have a zero at b. But we just established that it has a pole at b; a contradiction.

Overall, elliptic functions must satisfy the following two properties:

$$\sum n_p = 0 \qquad \sum pn_p = ne_1 + me_2$$

These will turn out to be sufficient conditions to satisfy to be an elliptic function, meaning they are completely determiend by these algebraic limitations. We shall also find an elliptic function with two pole in the following section.

Corollary 5.1.2: Number Of Poles Reflects Mapping Property

Let f be an elliptic function with m poles. Then f is an m to 1 map

Proof **:**
The number of poles and zeros match, hence apply the argument principle to $f(z) - c$.

5.2 Weiestrass $\wp$-function

The following function can be thought of as the building block of elliptic functions: all elliptic functions will be a rational function of $\wp$ and $\wp'$ (theorem 5.2.2). Let $\Gamma = \{ne_1 + me_2 \ : \ e_1, e_2 \in \mathbb{C}\}$ where e_1, e_2 are linearly independent. We shall show we can associate to such a group a function with many intersting properties.

First, the function

$$\sum_{\omega \in \Omega} \frac{1}{(z-\omega)^2}$$

is not absolutely convergent as we saw before. Instead, we'll modify it so that it shrinks fast enough so that it *is* absolutely convergent:

Proposition 5.2.1: Weiestrass $\wp$-Function

Let $\Omega \subseteq \mathbb{C}$ be a discrete subgroup with two generators. Then the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is uniformally convergent on compact subsets of $\mathbb{C}$

We first require a lemma:

Lemma 5.2.1: Weiestrass Function Lemma

The series

$$\sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{1}{|\omega|^3}$$

converges

Proof **:**
For each $n \in \mathbb{N}_{>0}$, consider the parallelogramm P_n formed by the points $z = t_1 e_1 + t_2 e - 2$ where the real numbesr $t - 1, t_2$ respect $\sup(|t_1|, |t_2|) = n$, take for examlpe:

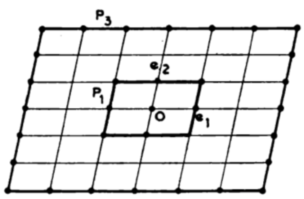


Figure 5.1: Cartan, p. 154

In each border of P_n, there are $8n$ points of Ω, and the distance between each of these points and 0 is $\geq kn$ for some fixed $k > 0$ (the smallest distance from 0 to the points of P_1). Then the sum of

$\frac{1}{|\omega|^3}$ over the points of P_n is bounded by $\frac{8n}{kn^3}$, hence:

$$\sum_{\omega \neq 0} \leq \sum_{n \geq 1} \frac{8}{k^3 n^2}$$

and since $\frac{1}{n^2}$ is convergent, by the comparison test so is our desired series.

***Proof* :**
Of Weiestrass $\wp$ Function Convergence
We'll show $\wp$ converges on $|z| \leq r$. Then $|\omega| \geq 2r$ for all but finitely many ω, thus for all but finitely many terms:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega(z-z^2)}{\omega^2(\omega-z)^2} \right| = \frac{\left|z\left(2-\frac{z}{\omega}\right)\right|}{|\omega^3|\left|1-\frac{z}{\omega}\right|^2} \leq \frac{r(5/2)}{|\omega|^3(1/4)} = \frac{10r}{|\omega|^3}$$

since $|z| \leq r$. But then, by our above lemma, we see that the series converges uniformally on the disc, and hence convverges uniformally on compact sets, as we sought to show.

Definition 5.2.1: Weiestrass Function

A *wesetrsass function* $\wp$ is a meromorphic function which is the sum of a series as in the above proposition depending on a choice of discrete group Ω with two generators.

By construction, the poles of $\wp$ are the poles of Ω, in particular they are double poles whose residue is zero. To see this, if we have some neighbourhood around $z = \omega$, then

$$\wp(z) = \frac{1}{(z-\omega)^2} + g(z)$$

Next, $\wp$ is an even function of z, since:

$$\wp(-z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where we use the fact that the sum sums over negative values. Furtheremore, by proposition 4.1.2, we have that

$$\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

Notice that the derivative is Ω-periodic and is odd:

$$\wp'(z+\omega) = \wp'(z) \qquad \wp'(-z) = -\wp'(z)$$

It is not immediately obvious, but $\wp$ is itself Ω-periodic. By linearlity, it suffices to show that $\wp(z+e_i) = \wp(z)$ for $i \in \{1,2\}$. To see this, note that:

$$\wp(z+e_i) - \wp(z) = c$$

since the derivative $\wp'(z + e_i) - \wp'(z) = 0$. So what is c? Well, if we plug in $z = e_i/2$ and take advantage of evenness, we get

$$0 = c$$

and hence it is indeed even. Overall, The Weiestrass $\wp$-function is a doubly periodic meromorphic function function with period Ω, with poles at the points of Ω, each pole having order 2 and principal part $\frac{1}{(z-\omega)^2}$. Note to by the evenness of $\wp$ we have that if $\wp(\pm a) = b$. To see this, let $f = \wp - b$ so that $f(a) = \wp(a) - b = 0$. Notice that f is even since:

$$f(-z) = \wp(-z) - b = \wp(z) - b = f(z)$$

Furthermore:

$$\wp(-a) - b = f(-a) = f(a) = 0 \implies \wp(-a) = b$$

Thus, for every b in the fundamental domain, at least 2 points map to it.

Weiestrass functions are important in many ways, one that will be omited for now but is important to know is that all doubly periodic functions are constructed out of the Weiestrass function and it's derivative. The second is that there is always an associated *elliptic curve* that is associated to a Weiestrass function:

Theorem 5.2.1: Elliptic Curve Of Weiestrass Function

Let $\wp$ be a Weiestrass p-function. Then there exists constants g_2 and g_3 such that

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

This can be intuitively seen like so: since $\wp$ has a pole of order 2, $\wp'$ has a pole of order 3. Hence, there is some relation between $(\wp')^2$ and $\wp^3$ up to some massaging:

Proof **:**
Recall that two doublye-periodic functions are equal up to a constant if they have the same poles and principal parts, and if then it may be shown that the constant is zero, the two functions are equal. This essentially means that we can construct many important identities by "cancelling out the poles" so to speak. Here is a famous example of this, consider:

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$$

Then

$$\begin{aligned} \wp(z) &= -2z^{-3} + 2a_2 z + 4a_4 z^3 + \cdots \\ \wp'(z) &= -2z^{-3} + 2a_2 z + 4a_4 z^3 + \cdots \\ \wp'(z)^2 &= 4z^{-6} + 8a_2 z^2 - 16a_4 + \wp(z)^3 & &= z^{-6} + 3a_2 z^{-2} + 3a_4 \end{aligned}$$

Thus:

$$\wp'(z)^2 - 4\wp(z)^3 = -20a_2 z^{-2} - 28a_4 + z^2(\cdots)$$

Thus, we get that the function$\wp'(z)^2 - 4\wp(z)^3 + 20a_2\wp(z) + 28a_4$ is holomorphic in some neighbourhood of the origin and is zero at the origin, hence it must be zero. We thus get:

$$\wp'(z)^2 = 4\wp(z)^3 - 20a_2\wp(z) - 28a_4$$

which can be thought of as plugging in $(\wp, \wp')$ into the aglebraic curve:

$$y^2 = 4x^3 - 20a_2 x - 28a_4$$

This curve has some nice properties, for one all it's poles are always distict, that is the equation

$$0 = 4x^3 - 20a_2 x - 28a_4$$

has 3 distinct roots. Later, we shall see that a_2 and a_4 are *modular forms* when interpreted as function on the upper half plane (see ref:HERE).

Let $r = \min\{|\omega| : 0 \neq \omega \in \Omega\}$. Let $G_n = \sum_{0 \neq \omega \in \Omega} \omega^{-n}$. Then the Laurent expansion around $0 < |\lambda| < r$ is:

$$\wp(z) = \frac{1}{z^2} \sum_n^\infty (2n+1) G_{2n+2} z^{2n}$$

It is easy to find the roots of $\wp'$: if the period is $1, \tau$ (which, up to a biholomorphism all elliptic functions have this period, where $\tau = \omega_2/\omega_1$), then the roots of $\wp'$ are

$$1/2 \quad \tau/2, (1+\tau)/2$$

However, the roots of $\wp$ are in general hard to determine. If $\wp(a) = 0$ and a is a double zero, then

$$a = \frac{\omega_1 + \omega_2}{2}$$

Besides this case, it is rather difficult to find the root, and starts going into the theory of modular forms (the modualr form $j(\tau)$ is involved). (I downloaded a pdf called "zerosOfWeiestrassFunc" in EYNTKA complex analysis if you'd like more details).

Theorem 5.2.2: Elliptic Function Rational Function Of Weiestrass

Let f be an elliptic function. Then f is a rational polynomial of the Weiestrass function and it's deriative:

$$f = R(\wp, \wp')$$

Proof **:**

1. Any *even* elliptic function $f(z)$ can be written in the form

$$f(z) = c \prod_{k=1}^n \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

(where c is a constant and $\wp(z)$ denotes the Weiestrass $\wp$-function with the same periods), provided that 0 is neither a zero nor a pole of f. Conclude that every even elliptic function f can be written $f = R(\wp)$ where R is a rational function

Proof. If f is a constant, then $f(z) = c$, so assume f is non-constant.

First, we shall find some properties of the roots in the case where $f = \wp$. In particular, if $a \not\equiv -a \mod \Omega$, then a is a simple zero of $\wp$, and if $a \equiv -a \mod \Omega$, then in the fundamental domain:

$$a \in \left\{ \frac{\omega_1}{2}, \frac{\omega_2}{2} \frac{\omega_1 + \omega_2}{2} \right\} \tag{5.2}$$

and a is a root of order 2. To see this, we need to show that $\wp'(a) = 0$. Since $\wp$ is odd and doubly periodic, we see that:

$$-\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(-\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_1}{2}\right)$$

Hence, $\wp'\left(\frac{\omega_1}{2}\right) = 0$. Similarly for $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$. Since $\wp'$ has order 3 (since $\wp$ has order 2), these must all be simple zeros of $\wp'$. Hence a is a simple within the fundamental domain if it is not equal to one of these values, and is of order 2 if it is equal to one of these values. More generally, if f is an even elliptic function, then for some root a, we may divide f by $(\wp - \wp(a))^n$ for appropriate n that will make the resulting function have the root be of order 2 or 3, at which point we repeat the argument.

Next, two elliptic functions are equal up to a constant if they share the same roots/poles: if f_1, f_2 share their roots and pole, f_1/f_2 is is holomorphic, hence bounded in $\mathbb{C}$ (due to periodicity), hence constant.

Thus, it suffices to show that the given expression f has the same number of roots/poles as the expression on the right hand side. Let f be an elliptic function with period group Ω, and let $a_1, a_2, ..., a_k$ be the roots of f within the fundamental domain (counting multiplicity, and $a_i \neq 0$ for all i) with corresponding poles $b_1, b_2, ..., b_k$ (recall the number of roots and poles match) where $b_i \neq 0$ for all i. If each a_k/b_k is a simple root/pole of $\wp - \wp(a_k)$ (resp. $\wp - \wp(b_k)$), then we may simply take:

$$g(z) = \frac{f(z)}{\prod_{k=1}^{n} \frac{\wp(z)-\wp(a_k)}{\wp(z)-\wp(b_k)}}$$

giving us a bounded holomorphic function, hence for some constant $c \in \mathbb{C}$:

$$f(z) = c \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

If one of the a_k/b_k is in the set in equation (5.2), then that means that there is an i, j such that $a_i = a_j$ (resp. $b_i = b_j$). In this case, since $\wp - \wp(a_k)$ has order 2, we don't need to divide by $\wp - \wp(a_k)$ twice (or else we would over count).

Notice that even if we change the number products in the numerator/denominator, there *must* be the same number of products in the denominator/numerator, or else $f(z)$ would have a pole/zero at 0, contradicting the given conditions of f. Thus, we once eagina end up with:

$$f(z) = c \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

as we sought to show. □

2. Show that every odd elliptic function f can be writen $f = \wp' R(\wp)$ where R is a rational function

Proof. If f is odd, then since $\wp'$ is odd, $g = f/\wp'$ is even $(f(-z)/\wp'(-z) = f(z)/\wp'(z))$. Then we may apply the above to get g in terms of a rational function of $\wp$, which gives us

$$f = \wp' R(\wp)$$

as we sought to show. □

3. show that every elliptic function f can be written $f = R(\wp, \wp')$, where R is rational

   Proof. Note that
$$f = \frac{f(z) - f(-z)}{2} + \frac{f(z) + f(-z)}{2}$$
   where the fisrt term on the right hand side is odd and the 2nd term is even. Then $f = \wp' R_1(\wp) + R_2(\wp) = R(\wp, \wp')$, as we sought to show. □

Theorem 5.2.3: Additive Law For Weiestarss Function

Let Ω be the fundamental parallelogram and $u, v \in \Omega$. Then:

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp(v)}{\wp(u) - \wp(v)}\right)^2$$

Proof **:**
First, we'll show that for all $a, b \in \mathbb{C}$, the function

$$\wp' - a\wp(z) - b$$

has 3 zeros in a period parallelogram, and their sum equals a period. Then, we'll show that if $u, v \in \mathbb{C}$, $u \pm v \not\equiv 0 \mod \Gamma$, then we can find a, b such that the function above has zeros at $u, v, -u-v$ and deducatice tha additive theorem: if $u + v + w = 0$, then

$$\det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0$$

Proof. First, note that the $\wp'$ has a pole of order 3 and $\wp$ at each $\omega \in \Omega$ and $a\wp$ has a pole of order 2 at each $\omega \in \Omega$. Since the principal parts differ, the poles don't cancel out, and hence there are three poles in the period parallelogram, meaning there are 3 zeros in the period parallelogram (by using the argument principle as we did in class). Since the sums of the poles and the zeros are 0 mod Ω, and the sums of the poles are 0 mod Ω (since they are all in Ω), we see that the sum of the zeros are 0 mod Ω, i.e. the sum of the poles aer in Ω.

Now, choose $u, v \in \mathbb{C}$. We will want to find a, b such that

$$\wp'(u) - a\wp(u) - b = 0 \qquad \wp'(v) - a\wp(v) - b = 0$$

If u, v are zeros of this equation, then since $u + v + w \in \Omega$, it must be that $w \equiv -u - v \mod \Omega$. If $-u - v \equiv 0 \mod \Omega$, then $u + v \equiv 0 \mod \Omega$, so $-u - v$ would also be a *pole*, but that contradicts the fact that it's a *root*.

Hence, let's say $u + v \not\equiv 0 \mod \Omega$. By linear-algebra manipulation, we get:

$$a = \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp'(v)} \qquad b = \wp'(v)\wp(u) - \wp(v)\wp'(u)$$

where $\wp(u) - \wp(v) \neq 0$ since . Finally, if $u + v + w = 0$, then notice that

$$\begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} \begin{pmatrix} -a \\ 1 \\ b \end{pmatrix} = 0$$

showing the matrix cannot be full rank, hence the determinant is zero. □

We now prove the main result.Recall that

$$\wp'(z)^2 = 4\wp(z)^3 - 28a_2\wp(z) - 20a_4$$

and that $a = \frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$. Since $\wp'(u) = a\wp(u) + b$, plugging it into the equation we get:

$$a^2\wp(u)^2 + 2ab\wp(z) + b^2 = 4\wp(z)^3 - 28a_2\wp(z) - 20a_4$$

$$\Leftrightarrow$$

$$0 = 4\wp(z)^3 - a^2\wp(u)^2 - 28a_2\wp(z) - 20a_4 2ab\wp(z) + b^2$$

Then we know by properties of polynomials that the the sum of the roots will be equal to $a^2/4$, in particular

$$\wp(u + v) + \wp(u) + \wp(v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

where $\wp(-u - v) = \wp(u + v)$ by evenness of $\wp$. Re-arranging, we get:

$$\wp(u + v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

as we sought to show.

Theorem 5.2.4: Elliptic Functions With Same Period

Let f, g be two ellipic functions with the same period. Then the two are linked by an algebraic relation:

$$f = P(g)$$

***Proof* :**
This is really cool! How to link two elliptic functions with the same period:

`https://math.stackexchange.com/questions/822173/an-algebraic-relation-between-any-two-elliptic-functions-with-the-same-periods`

6

Complex Manifolds and Riemann Surfaces

In this section, we take a look at how to make functions like log and z^α for fractional α well-defined by finding a good domain. This technique will require the use of manifold theory; the notion of maifolds will be taken as prequisite for this section, to see EYNTKA Geometry for a refresher, in particular the transition mappings are holomorhpic functions. We shall be focusing on 1-dimensional complex manifolds. For reference, we shall put the definition of a complex manifold

Definition 6.0.1: Complex Manifold

A space X is said to be a *complex manifold* if it is paracompact, Hausdorff, and has a complex-atlas that covers it, that it has an open covering where the transition maps betwen intersections is holomorphic.

A map $f : M \to N$ between complex manifolds is said to be holomorphic if it is continuous and each representative is holomorphic. If an holomorphic inverse exists, then it is said to be *biholomorphic* or *isomorphic* and we write $M \cong N$.

It is sometimes desirable to distinguish a holomorphic function between open sets of $\mathbb{C}$ to itself and between complex manifolds. In this case, functions between complex manifolds will be called *holomorphic maps*.

Example 6.1: Complex Manifolds

1. $\mathbb{C}$ is trivially a complex manifold
2. Let $\mathbb{C}/\mathbb{Z}$ is a complex manifold. Let the transition mappings be projections onto $\mathbb{C}$, so that

the transition mappings are polynomials along with the square root. Then we see that they are indeed holomorphic, and hence $\mathbb{C}/\mathbb{Z}$ is a complex manifold, and it is homeomoprhic to $S^1 \times \mathbb{R}$.

3. The Riemann sphere $\mathbb{C}\mathrm{P}^1$ is a complex manifold: Take the usual 2-set open covering of S^2 which has polynomial+square-root transition mappings and hence is holomorphic (you can also look at your homework to see it explicitly).

4. Let Ω be a doubly periodic group and consider the cannonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Omega$. Then C/Ω can be shown to be a compact complex manifold. This is a compact manifold which has a "hole" since it can be thought to be constructed similarly to T^2.

The first thing to see is if the results of complex functions translate to maps between complex manifold. Since holomorphic functions are analytic, their global behavior is essentially defined locally due to analytic continuation. The same is true for holomorphic maps:

Proposition 6.0.1: Analytic Continuity For Holomorphic Maps

Let $f, g : M \to N$ be holomorphic mapppings. Then the set $U \subseteq X$ of points on which f and g coincide in a neighborhood is both open and closed

***Proof* :**
By definition, U is open, hence it must be shown to be closed. This is just reducing the proof to coordinates and apply proposition 2.5.1. Let $a \in \bar{U}$ so that (by unique extension of continuous functions) $f(a) = g(a)$. Then choose representatives $\tilde{f}, \tilde{g}$ so that $\tilde{a} = z = 0 \in V$ (the domain of the representative). Then by the clasicl principle of analytic continuation, the set $E \subseteq V$ containing z on which $\tilde{f}$ and $\tilde{g}$ concides is closed. Hence, $\tilde{f}$ and $\tilde{g}$ coincide in a closed neighborhood around 0, and hecne f, g coincide in a closed neighborhood around a, as we sought to show.

Proposition 6.0.2: Maximum Modulus For Holomorphic Mappings

Let $f : M \to N$ be a holomorphic mapping on a connected complex manifold X. Then if $|f|$ has a relative maximum at a point $a \in X$, then f is constant.

***Proof* :**
exercise (translate into coordinates, apply the classical maximum modulus principal, show why this translate back)

Corollary 6.0.1: Holomorphic Maps On Compact Manifolds

Let X be a compact, connected, complex manifolds. Then any holomorphic map f on X is constant

Proof **:**
Since $|f|$ is continuous on the compact space, it attains a local maximum, and hence we apply proposition 6.0.2.

Note how this is another way of showing that holomorphic doubly-periodic functions must be constant: The map $f \mapsto f \circ \pi$ where $\pi : \mathbb{C} \to \mathbb{C}/\Omega$ is a bijection. As a final important definition, we give a meromorphic function on a manifold:

Definition 6.0.2: Meromorphic Maps

Let M be a complex manifold. Then a *meromorphic function* on M is a holomorphic map $f : M \to S^2$, in other words a meromorphci function is a continuous function which can take the value ∞ and which in a neighborhood of each $a \in X$ can be expressed as a meromorhpic function of a local coordinate in the neighborhood of a

For example, the cannonical projection $\pi : \mathbb{C} \to \mathbb{C}/\Omega$ gives a bijection between the meromorphic map. Finally, we generalize theorem the open mapping theorem for manifolds.

(intuition on ramification)

Definition 6.0.3: Ramification Index

Let $\varphi : M \to N$ be a complex mapping and $a \in M$. Then if $\tilde{\varphi}(a) = 0$ and a has multiplicity p, we say that a has *ramification index* p. If $p > 1$, we say that a is *ramified*, and *unramified* otherwise.

It is clear that the notion of ramification is independent of coordinate representation, of the derivative of the transition maps must be nonzero. If $\tilde{\varphi}(a) = 0$, then by the open mapping theorem there exists a g such that in a sufficiently small neighborhood, $\tilde{\varphi} = g^n$. By this, we see that in a sufficiently small enough neighborhood, every point near 0 will be mapped to p times, and hence every point sufficently near $\varphi(a)$ will be mapped to p times. Hence, if $p = 1$, that is if the map is injective, we get the following proper generalization of injective maps between complex manifolds

Theorem 6.0.1: Injective Maps Between Complex Manifolds

Let $f : M \to N$ be an injective holomorphic map. Then f is an isomorphism

Proof **:**
We see that the ramification index at each point is necessarily 1, hence we have a local inverse at every point, which is enough to give the desired result.

Example 6.2: Practical Example

Take $z \mapsto e^{2\pi i z}$ that maps $(\mathbb{C}, +)$ to $(\mathbb{C}^\times, \cdot)$. Taking the quotient, we get a holomorphic map $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$. This map is certainly holomorphic and simple, hence we get

$$\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$$

it can even be shown to be an isomorphism of topological groups by verifying the homomorphism condition.

A result that is worth mentioning but too complicated to be proved right now is the classification of all simply connceted complex manifolds:

Theorem 6.0.2: Fundamental Theorem

Let M be any simply connected complex manifold. Then M is isomorphic to one of hte following:

1. Riemann sphere $\mathbb{CP}^1$
2. The place $\mathbb{C}$
3. The unit disk $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$

***Proof* :**
see (this book)

Now going to integration of complex manifolds, we may define holomorphic differential forms to be those whose local coordiante representatives

$$\omega_i = f_i(z_i)dz_i$$

are holomorphic, with change of coordiante rule satisfying $f_j(z_i) = f_i(f_{ij}(z_j))f'_{ij}(z_j)$ for appropriate transition function f_{ij}. Just like before, a primive of ω, if it exists, if a function g such that $dg = \omega$. In general, a global primitive does not exist. If X is simply connceted, then any holomorphic different form on X has a primitive, and if X is not simply connected then the intgral of ω alogn a closed path of X is not always zero, and the integral has the same value for two homotopic paths. Often, the value of the intgral along such a path is called hte *period* of the integral $\int \omega$. If X is an orientable manifold, then we may take an orientable boundary Γ. Then we see that if Γ is the oriencted boundary of a compacted subset of X, then $\int_\Gamma \omega$ is zero for any holomorphic differential form ω where the boundary is nullhomotopic.

Moving on to residue theory for complex manifolds, let's say E
$subseteq M$ is a discrete subset of X and ω is holomorphic on $M \setminus E$. Let $a \in E$ and elt z be local coordiante of a which is zero at a. Then in some neighborhood of a, the form ω can be written as $f(z)dz$, where f is holomorphic in the neighborhood of 0 except parhaps at $z = 0$. Then the Laurent expansion of $f(z)$ shows that in a engih of a, the ω for can be written as

$$\omega = \omega_1 + \left(\frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right) dz$$

where ω_1 is a holomorphic differential form in a neighborhood of a (including a). If γ is a closed path isituated in a small enigh of a which does not pass through a and whose inde with respect to to a is equal to 1 (i.e. it winds around only once, then by the classical residue theorem:

$$\int_\gamma \omega = 2\pi i c_1$$

Importantly, c_1 does not deepdns on the choice of loca lcoordiantes z, which is zero at a. This is the residue of hte differential form ω at a. Then we can naturally generalize the residue theorem to the following:

Theorem 6.0.3: Residue Theorem For Complex Manifolds

Let Γ be the oriented boundary ofa compact set K which does not contain any points of the discrete clseod sbuset E in the compelx of which the differential form ω is holomorphic. Then the integral $\int_\Gamma \omega$ is equal to $2\pi i$ time the sum ofthe residues of ω at the points of E situated in K

Proof **:**
exercise

6.0.1 Riemann Surfaces

Recall when we said that we shall introduce the notion of Riemann surface to make the many-to-one functions $z^{1/2}$ and $\log(z)$ well-defined. We now cover these notions:

Definition 6.0.4: Riemann Surface

Let Y be a complex manifold. Then a *Riemann surface* spread over Y (or simply a *Riemann surface* over Y) is defiend ot be a connected compelx manifold X and a non-constant holomorphic mapping $\varphi : X \to Y$. Uusally, we consider the csae where $Y \in \{\mathbb{C}, \mathbb{C}\mathrm{P}^1\}$.

Note that if the ramification index of each $a \in X$ is 1, then φ is a an injective local biholomorphism, and if φ is either proper or surjective, it is a a biholomorphism. If the ramification index is greater than 1, it would happen at isolated pointed. Note too that φ may not be injective and have no ramifications (think $z \mapsto e^{iz}$ on S^1, or $\mathbb{C} \to \mathbb{C}^\times$ mapping $t \mapsto e^{it}$). Furthermore, if $S \subseteq X$ is the set of ramified points, then $f(S)$ need not be discrete. On the other hand, it may be that $f(S)$ maps to a single point, even if there are infinitely many S (that is, f need not be proper).

For reference purposes, we define the following:

Definition 6.0.5: Unramified Riemann Surface

An *Unramified Riemann Surface* over Y is a Riemann surface (X, φ) where the mapping φ is unramified, that is all points $x \in X$ have ramification index 1.

Example of unramified Riemann surfaces are local homeomorphisms or covering spaces. In fact, we have the following result from algebraic topology that we shall put but not prove:

Theorem 6.0.4: Universal Covering Of Open Sets

let $U \subseteq M$ be any connected open set of a complex manifold M. Then U has a simply connected covering space.

Proof **:**
See Hatcher or EYTNKA Algebraic Topology.

We now introduce our main object of interest

Definition 6.0.6: Holomorphic Function On Riemann Surface

Let (X, φ) be a Riemann surface over Y. Then a Holomorphic (resp. meromorphic) function on (X, φ) is a holomorhpic (resp. meromorphic) function on X.

Example 6.3: Holomorphic Function On Riemann Surface

1. We shall show how to consider $\log(z)$ as a single-valued function. Consider $(\mathbb{C}, \varphi)$ covering $\mathbb{C}^{\times}$ where $\varphi(t) = e^{it}$. Since φ is unramified, we see that φ can be expressed as local coordiante in a neighborhood of each $x \in X$, say coordiantes z so that $z = e^{it}$. Then any holomorhpic function $f : \mathbb{C} \to Y$ can be *locally expresed* as a holomorphic function of z. But since different points of X can be mapped by φ onto the same points of $\mathbb{C}^{\times}$, f is not in general a global holomorphic function in the variable $z \neq 0$. (finish here)
2. $(y = (1 - x^3)^{1/3})$
3. (S^2 covering $\mathbb{C}\mathrm{P}^2$, and more generally S^n covering $\mathbb{C}\mathrm{P}^n$)

6.0.2 Riemann Surfaces and Elliptic Curves

Recall that any $\wp$-function induces the following algebraic relation

$$y^2 = 4x^3 - 20a_2 x - 28a_4 \tag{6.1}$$

where $(x, y) = (\wp', \wp)$. Let $p(x)$ represent the polynomial on the right hand side.We may choose a_2, a_4 so that the right hand side has 3 distinct roots, and hence we have a smooth curve (namely since $p'(x) \neq 0$ for any value of x such that $p(x) = 0$). From this, we shall define the following manifold: Let $X \subseteq \mathbb{C} \times \mathbb{C}$ to be the pair of points $(x, y) \subseteq \mathbb{C} \times \mathbb{C}$ that satisfy equation (6.1).To show X is a complex manifold, for any point $(x_0, y_0) \in X$ where $y_0 \neq 0$, as shall take a neighborhood sufficiently small so that we may project onto the first coordinate; the x are the "local coordinates". If at a point $(x_0, y_0) \in X$ we have $y = 0$, then notice that $P'(x_0) \neq 0$ by assumption of smoothness. Hence by the implicit function theorem (theorem 2.2.4), equation (6.1) is equivalent to a relation of the form $x = f(y)$ in a sufficiently small neighborhood of $(x_0, 0)$ where $f(0) = x_0$. In such a neighborhood, we take y as the local coordinates.

We thus have a mapping $\varphi : X \to \mathbb{C}$ which maps (x, y) onto$x \in \mathbb{C}$. Certainly, this is a holomorphic map, and hence (X, φ) is a Riemann surface over $\mathbb{C}$. This surface has two "sheets" corresponding to the two values of y that correspond to x; if $p(x) \neq 0$, they are distinct. Furthermore, the map $X \to \mathbb{C}$ mapping (x, y) onto y is also holomorphic on X; we shall write these coordinates as y. We may then define the differentiable form $\omega = dx/y$ in a neighborhood of $(x_0, y_0) \in X$ such that $y_0 \neq 0$ and

$$\omega = \frac{dy}{6x^2 - 10a_2}$$

in a neighbourhood of $(x_0, 0) \in X$. Then this is a holomorphic differential form on X, hence a closed form. In particular, it has a primitive in a neighbourhood of each point of X, globally this primitive is a *many-valued* function z which is holomorphic in a neighbourhood of each point X.

We now define another Riemann surface over $\mathbb{C}\mathrm{P}^1$. First, we must make our polynomial a homogeneous polynomial so that it's well-defined in projective space. If we consider $[x, y, t] \in \mathbb{C}\mathrm{P}^1$, then

$[x/t, y/t, 1] = [x, y, t]$. Substituting these values in the equation and re-arranging, we get

$$y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3 \tag{6.2}$$

giving us a Homogeneous polynomial, hence on that is well-defined on $\mathbb{C}\mathrm{P}^1$. Hence, we shall consider the subset of points $X' \subseteq \mathbb{C}\mathrm{P}^1$ that satisfy equation (6.2). The space X' can be seen as an extension of X as follows: we can identify X as a subspace of X' by associating each $(x, y) \in X$ with a point X' whose homogeneous coordinates are $[x, y, 1]$. Then the compliment $X' \setminus X$ consists only of $[0, 1, 0]$. We shall denote this point as ∞, and take the local coordinates $x/y = x'$ since x' defines a homeomorphism of a neighborhood of the point ∞ onto a neighborhood 0 in $\mathbb{C}$ (put $t/y = t'$ in equation (6.2)). Then we get

$$t' = 4(x')^3 - 20a_2 x'(t')^2 - 28a_4(t')^3$$

in some neighbourhood $x' = 0, t' = 0$ and the implicit function theorem gives t' as a holomorphic function of x':

$$t' = 4(x')^3 - 320a^2(x')^7 + \cdots \tag{6.3}$$

The complex manifold structure on X' is now defined since we have chosen x' as the local coordinates at ∞, and the mapping φ' is defined to be equal to φ on X and take the point *ooo* of X' onto the point at infinity of $\mathbb{C}\mathrm{P}^1$. The holomorphic differential form ω defined on X extends to a holomorphic differential form on X': in a neighborhood of ∞, we use the local coordinates x' and the holomorphic function t' of x' defined in equation (6.3) to get

$$\omega = t' d(x'/t') = dx' - x'\frac{dt'}{t'} = dx' - \frac{12(x')^2 + \cdots}{4(x')^2 + \cdots} dx' = -2dx'(1 + g(x'))$$

where g is a holomorphic function in a neighborhood of $x' = 0$ and is zero for $x' = 0$. The form ω is thus defined on the compact space X', hence has a local primitive which is many-valued function on X' and serves so a local coordinate at each point of X'.

Now, if we get a_2, a_4 via a Weiestrass $\wp$-function defined given a discrete group Ω, then we have that ht meromorphic transformation

$$x = \wp(z) \qquad y = \wp'(z)$$

defines an isomorphism of the complex manifold $\mathbb{C}/\Omega$ onto the complex manifold X'. The inverse isomorphism defines z as a holomorphic many-valued function on X' whose (local) branches differ in value by a constant belonging to Ω.

(there is still more in Cartan p. 202, I'll get back to it)

(here is from lecture; I believe this is linking to the original motivation of the term "elliptic")

Recall we have the following implicit equation defining a circle: $y^2 = 1 - x^2$. Then if

$$x = \cos(\theta) = \sin'(\theta) \qquad y = \sin(\theta) \qquad dy \sin'(\theta) d\theta = x d\theta$$

Then differentiating we get

$$xdx + ydx = 0 \implies d\theta = \frac{dy}{x} = -\frac{dx}{y}$$

Then

$$\int d\theta = -\int \frac{dx}{y} = \int \frac{dy}{\sqrt{1 - y^2}}$$

We can invert $\int \frac{dy}{\sqrt{1-y^2}}$ in a neighborhood of $(1,0)$ by defining trigonometric functions by

$$\theta = \int_{(1,0)}^{(\cos(\theta),\sin(\theta))} \frac{dy}{x} = \int_0^{\sin(\theta)} \frac{dy}{\sqrt{1-y^2}}$$

Now, if $dx = \wp'(z)dz = ydz$, then $dz = \frac{dx}{y}$ where x has local coordinates (given $y \neq 0$). From the curve, we then have

$$2ydy = (12x^2 - 20a_2)dx \qquad \frac{dy}{6x^2 - 10a_2} = \frac{dx}{y} = dz$$

Then dz is an extension to all of X' of the holomorphic differential form

$$\frac{dx}{\sqrt{4x^{-}20a_2x - 28a_4}}$$

and

$$z = \wp^{-1}(x) = \int_{[0,1,0]}^{\wp(z),\wp'(z),1} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}$$

7

Harmonic Functions

Recall that all holomorphic functions are harmonic, but not all harmonic functions are holomorphic. In this section, we shall study harmonic functions, see their connection to holomorphic functions, and understand better their behavior.

7.1 Harmonic Functions and Dirichlet Problem

Recall that a (two dimensinoal) function is harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

which as a Wirtinger derivative we may write as:

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} = 0$$

which shows us that holomorphic functions are harmonic. However, not all harmonic functions are holomorphic (ex. $\operatorname{Re} z$). In this section, we shall look over general (real and complex) harmonic functions and see how they relate to holomorphic functions. First, if we have any complex harmoinc function, then by linearity of the derivative its real and imaginary parts are harmonic. Hence, looking at real harmonic functions, we shall show that they are locally equal to a holomorphic function. First, suppose g is a real harmonic function. Then by definition we have

$$\frac{\partial^2 g}{\partial z \partial \overline{z}} = 0$$

hence, $\frac{\partial g}{\partial z}$ is holomorphic, and so the differential form $\frac{\partial g}{\partial z} dz$ locally has a holomorphic primitive $f(z)$. In particular, this means that locally:

$$df = \frac{\partial g}{\partial z} dz$$

taking the conjugate of both sides, then similar to the calculation we've done on page 38, we get

$$d\overline{f} = \frac{\partial g}{\partial \overline{z}} d\overline{z}$$

Thus:

$$df + d\overline{f} = d(f + \overline{f}) = \frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \overline{z}} d\overline{z} = dg$$

Hence:

$$g = 2\operatorname{Re}(f) + c$$

for some constant c. We can in fact explicitly compute f. Since it's holomorphic, we know f has a power series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Without loss of generailty we can assume a_0 is real (add a constant to make it real since f was only defined up to a constant). Let's say this power series has radius R, so consider $r < R$. Then doing a u-substituion $z = e^{i\theta}$, we can work out the real part of $f(re^{i\theta})$ to find $g(r\cos(\theta), r\sin(\theta))$. In particular:

$$g(r\cos\theta, r\sin\theta) = \operatorname{Re}(f(z)) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n a_n (e^{in\theta} + e^{-in\theta})$$

Then integrating both sides from 0 to 2π we get

$$\frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) d\theta = a_0$$

and we can use the usual trick to get the Fourier coeffficeints by multiplying by $e^{in\theta}$:

$$\frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) d\theta e^{-in\theta} d\theta = \frac{1}{2} r^n a_n$$

where $a_n = \frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) r^{-n} e^{in\theta} d\theta$. The power series representation of $f(z)$ thus becomes:

$$\begin{aligned}
f(z) &= a_0 + \sum_{n=1}^{\infty} a_n z^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) d\theta + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta) r^{-n} e^{in\theta}\right) z^n \\
&= \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}^n\right)\right] d\theta
\end{aligned}$$

where the inner sum in the last equality is the usual geometric series. Simplifying the last line out, we get:

$$f(z) = \frac{1}{2\pi}\int_0^{\infty} g(r\cos\theta, r\sin\theta)\frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta$$

We can also re-write g to be of the above form. To see this, first note that

$$\frac{re^{i\theta} + z}{re^{i\theta} - z}\cdot\frac{re^{i\theta} - \overline{z}}{re^{i\theta} - \overline{z}} = \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} + \frac{-re^{i\theta}\overline{z} + zre^{i\theta}}{|re^{i\theta} - z|^2}$$

The frist term is real and is called the *Poisson kernel*. The second term is purely imaginary since the difference between a compelx number and its conjugate. Thus, for $|z| < r$, we get

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta$$

The reason this is calledthe poisson kernel is if take $g \equiv 1$ then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta = 1 \tag{7.1}$$

which if you've done some PDEs or or Fourier Analysis should ring some bells

7.2 Dirichlet Problems for a Disk

In mathematics, extension questions are very prevalant, for example Tietze exentsion theorem or Hahn-Banach Theorem. Another famous extension theorme comes from algebraic geometry and asks when a continuous function $f : S^1 \to \mathbb{R}$ (or $\mathbb{C}$) can be extended to a continuous function on B^1. Dirchelet problem is a question about the extensionality of a continuous function on S^1 into a harmonic function. If f is periodic and defined on a circle, then it can be extended to a harmonic function on the disk. We first need a lemma:

Lemma 7.2.1: Dirichlet Problem On Disk Lemma

Let $\eta > 0$ and $\theta_0 \in [0, 2\pi)$. Let γ denote the arc of the ircle with radius r where $|\arg(z) - \theta_0| > \eta$. Then

$$\frac{1}{2\pi}\int_\gamma \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta \xrightarrow{(z \to re^{i\theta_0})} 0$$

that is, the above integral tends to zero as $z \to re^{i\theta_0}$.

***Proof* :**

Choose $\rho < r$ and let $z = \rho e^{i\alpha}$. We want to boudn the denomiantor of the integral $|re^{i\theta} - z|$ for z close to $re^{i\theta_0}$ so that we can factor it out of the integral, leaving us with $r^2 - |z|^2 = r^2 - \rho^2$. Then clearly as $z \to re^{i\theta}$, the integral will go to zero.

First, by the triangle inequality, if $|\theta - \theta_0| > \eta$ and $|\alpha - \theta_0| < \frac{\eta}{2}$, then $|\alpha - \theta| \geq \frac{\eta}{2}$. Then by a simple geometric argument, it is easy to see that z and $re^{i\theta}$ are separatedby a distanceof d where d isthe distance between two radii spanning the sector of angle $\eta/2/$ In particualr, z is on one side of the sector and $re^{i\theta}$ is on the other. Thus, this distance is necessarily $r\sin(\eta/2)$. Thus:

$$|z - re^{i\theta}| \geq r\sin\left(\frac{\eta}{2}\right)$$

for all $re^{i\theta}$ on γ. Thus:

$$\frac{1}{2\pi}\int_\gamma \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta le \frac{1}{2\pi}\int_\gamma \frac{r^2 - |z|^2}{r^2\sin^2(\eta/2)}d\theta < \frac{r^2 - \rho^2}{r^2\sin^2(\eta/2)}$$

Thus, as $z \to re^{i\theta_0}$, $\rho \to r$, and so the integral tends to 0, as we sought to show.

Theorem 7.2.1: Dirchilet Problem On A Disk

Let $f(\theta)$ be a continuous, periodic function defined the cicle of raidus r centered at 0 with period 2π. Then there exists a function $F(z)$ that is continuous on the closed disk $|z| \leq r$ and harmonic in the interior $|z| < r$ such that

$$F(re^{i\theta}) = f(\theta)$$

Moreover, F is unique.

Proof **:**
It suffices to show this for real-valued f since harmonicity is closed under linaerity. Uniqueness is alos easy: if F_1, F_2 ar etwo harmonicextension of f, then $F_1 - F_2 = 0$ on $|z| = r$. By the Mximum Modulus principle, $F_1 - F_2 = 0$ on $|z| \leq r$, whichimplies $F_1 = F_2$.

For existence, we shall take advantage of what we found in the begining of last section, namely define

$$F(z) = \frac{1}{2i}\int_0^{2\pi} f(\theta)\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta$$

We know that F is the real part of the holomorphic function:

$$\frac{1}{2i}\int_0^{2\pi} f(\theta)\frac{re^{i\theta} + z}{re^{i\theta} - z}d\theta$$

and hence is Harmonic. Certainly $F(z) = f(z)$ on the boundary, so we need to check that it is continuous and the boundary. First, recall equation (7.1)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta = 1$$

Now, consider:

$$\begin{aligned}
&= F(z) - f(\theta_0) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta - \frac{1}{2\pi}\int_0^{2\pi} f(\theta_0)\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta \\
&= \frac{1}{2\pi}\int_{|\theta-\theta_0|\leq\eta} (f(\theta) - f(\theta_0))\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta + \frac{1}{2\pi}\int_{|\theta-\theta_0|>\epsilon} (f(\theta) - f(\theta_0))\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta
\end{aligned}$$

This can be done for any η, in particular we want to choose one so that the value of the equation goes to zero. Since f is continuous, we know $\sup_{|\theta-\theta_0|\leq\eta}|f(\theta) - f(\theta_0)|$ can be arbitrarily small by choosing η appropriately. In particular, shoose η so that the first integral is ella than $\epsilon/2$ for some $\epsilon > 0$ (that is, we will integrate over a sufficinetly small arc). Then:

$$\left|\frac{1}{2\pi}\int_{|\theta-\theta_0|>\epsilon} (f(\theta) - f(\theta_0))\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta\right| \leq M \cdot \frac{1}{2\pi}\int_{|\theta-\theta_0|>\epsilon} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta$$

Then by lemma 7.2.1, the intgral on the right hand side can be made arbitrarily small, in particualr we can make it smaller than $\epsilon/2$, hancegiving us continuity of F on the boundary, hence completing the proof.

This gives us a new way to characterize harmonic functions without needing to rely on differentiability, showing they can be defined in a much weaker context[1]

Theorem 7.2.2: Mean Value Property, Then Harmonic

Let f be a continuous function on an open set Σ that satisfies the mean value property. Then f is harmonic.

Proof **:**
It suffices to show that f is locally harmonic. Let $z \in \Omega$ be arbitrary and D a disk centered at z such that $D \subseteq \Omega$. Then $f|_{\partial D}$ is continuous and thus by the above theormethere exists a continuous F tha tagrees on f on the boundary and is harmonic on the interior of D¿ Sicne F and f satisfy the maximum modulus principle,so does $F - f$. But $F - f$ is zero on the boundary, andhence must be identically zero on D. Thus, $f = F$ is harmonic on D.

(perhaps use Harnacks inequality to prove Harnack's Theorem? When the convergence of harmonic functions is haronic, commented)

Exercise 7.2.1

1. Let $\Omega \subseteq \mathbb{C}$ be a simply connected subset of $\mathbb{C}$. Show that the Dirichlet problem can be solved Ω (use the extended Riemann mapping theorem proved in exercise ref:HERE)

[1] However,the mean value property implies smoothness, and hence it turns out that this is not a weakening of the conditions

8

Higher Dimensional Complex Differentiation

In this chatper, we shall go over the theory for higher dimensions. In many ways, this theory is not very central to complex analysis, as it ventures more into the realm of differential geometry and away from many of the interseting results afforded to complex differentiable functions. Nonetheless, many results from one dimensions naturally generlize to higher dimensions (for example, Cauchy's Theorem),while on the other hand many key results do not (are the zeros of two dimensionsl holomorphic functions still isolated? What conssequences does that bring?)

8.1 The Algebra

The formal power series on two variables is defiend to be $k[[x, y]]$ where for any element $f(x, y) \in k[[x, y]]$ we ahve

$$f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$$

Note that $k[[x]][y] \neq k[y][[x]]$. Addition, scalar multiplication, and multiplication make $k[[x, y]]$ into a k-algebra. The *order* of a foraml pwoer series is the smallest n such that

$$\sum_{i+j=n} a_{i,j} x^i y^j \neq 0$$

The order of the product of two non-zero series is the sum of the orders of the series, hence $k[[x, y]]$ is an integral domain. If $k \in \{\mathbb{R}, \mathbb{C}\}$, we may consider for any $f(x, y) \in k[[x, y]]$ the series

$$|f|(x, y) = \sum_{i,j \geq 0} |a_{i,j}(r_1)^i (r_2)^j$$

where $r_1, r_2 \geq 0$. We want to find all positive r_p that satisfy the equation. Let Γ be the set of points where $r_1, r_2 \geq 0$ where

$$|f|(x,y) = \sum_{i,j\geq 0} |a_{i,j}(r_1)^i(r_2)^j < \infty$$

Then using Γ, we can find the set of all points in $\mathbb{C}$ whwere f is an *absolutely convergent series*. The set Γ is certainly non-empty since $(0,0) \in \Gamma$. This set is given a name:

Definition 8.1.1: Domain Of Convergnce

Given a series $f(x,y) \in k[[x,y]]$, is defined to be

$$\Delta = \text{int}(\Gamma)$$

that is, it is the interior of Γ.

Note that Δ may be empty, just recall the example for the one-dimensional case, and is always open. in the one dimensional case, Δ would simply be $(0, \rho)$ where ρ is the radius of convergence.

Proposition 8.1.1: Openness Of Domain Of Convergence

If Δ is the domain of convergence, then $(r_1, r_2) \in \Delta$ if and only if there exists $r_1' > r_1$, $r_2' > r_2$ such that $(r_1', r_2') \in \Gamma$

Proof **:**
if:

$$\sum_{i,j} |a_{i,j}(r_1')^i(r_2')^j < \infty$$

then certainly the same works for (r_1, r_2), and the converse follows from continuity.

With this, we may ask when a series converges. Just like for the one dimensinoal case, we need Abels lemma

Lemma 8.1.1: Abel's Lemma Higher Dimension

If $|a_{i,j}|(r_1')^i(r_2')^j \leq M$, where M is independent of p, q, and if $r_1 < r_1'$, $r_2 < r_2'$, then the series $\sum_{i,j} a_{i,j} z_1^i z_2^j$ converges for $|z_1| \leq r_1$ and $|z_2| \leq r_2$

Proof **:**
this follows the same argument as Abel's Theorem (theorem 2.3.1). Essentially, we bound by above the aboslute value of the terms of the serise of a double geometric progression.

using this, we get that $(r_1, r_2) \in \Delta$ then $f(z_1, z_2)$ converges for $|z_1| \leq r_1$ and $|z_2| \leq r_2$ and that if $(|z_1|, |z_2|) \notin \overline{\Gamma}$, then $f(z_1, z_2)$ is diverent.

Note that we will sometiems abuse langauge and say a point (z_1, z_2) is in the domain of convergence if $(|z_1|, |z_2|)$ is in the domain of convergence. We do this all the time in 1-dimensions, where we say the domain of convergence is the open disk $|z| < \rho$ (where ρ si the radius of convergence).

Some other results that are easily carried over are

(addition and multiplying and domain of conveegence, taking partial derivatives and is like taking derivative)

8.2 Higher Dimensional Analogues

(generalization of analytic function)

(infinite differentiability of harmonic function?)

(holomorphic functions of several complex variables)

(cauchy integral formula and series expansion for several complex variables)

(Would love Hartogs Extension Theorem)

(Riemann mapping does not extend to higher dimension: see this link)