Everything You Need To Know About MAT1300

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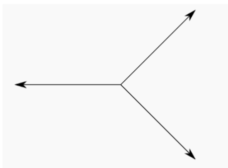
Contents

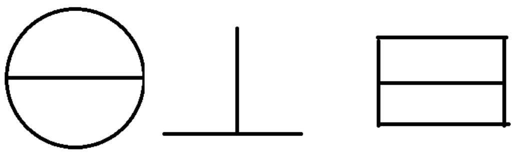
This book contains everything you need to know about differential geometry, in particular about the material covered in MAT1300. I will assume the following prerequisites:

1. Elementary set theory knowledge (elements, equivalence classes, functions, etc.)
2. Knowledge of topology, including open/closed sets, continuity, connectedness and compactness, product topologies, and how these concepts work over $\mathbb{R}^n$
3. Elementary analysis knowledge such as sup and inf, lim, sequences and series, definition of differentiability and integrability in $\mathbb{R}$ and their basic properties (ex. differentiability implies continuity)

On a high-level, this book is about studying a (topological) space X that locally looks like a base space B. The way we would achieve this is by taking copies of the space B (or open subsets of the space) and gluing them together via gluing maps. The way in which we glue them together will determine many important properties. For example, if the glue is homeomorphic, then continuous functions are preserved. If the graph of the function is closed, then the underlying space is Hausdroff. If the gluing functions are diffeomorphic, we may try to introduce some calculus. The nature of the glue will determine the nature of the object. Furthermore, the choice of base space will determine the type of object. For example, our base space is:



and our gluing was homeomorphic, we may imagine a space that is locally the above to look like any of these:



Other types of base spaces that are common are the space of solutions to polynomials (known as *varieties*) and euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$. The study of a space E that is locally B with some regularity (smooth, continuous, etc.) is called *sheaf theory*. This book will focus on the case where the base space is $\mathbb{R}^n$, and the gluing map is either a homeomorphism or a diffeomorphism. Different choices of gluing maps and base space will produce different fields of study (for example, the base space of solutions to polynomials is the beginning of defining *schemes* a field known as *algebraic geometry*). The case of the base space being $\mathbb{R}^n$ already has plenty to explore that will fill an entire course and more, and will produce many of the shapes we are familiar with. It will also be the basis of many of our intuitions for the study of many more general sheafs, including in complex analytic geometry and algebraic geometry.

Geometric Intuition

From a classical perspective, this book is about the study of shapes and space (in a "geometric sense"). The first "space" to be well-known is that of euclidean space, named after Euclid who has postulated the rules in which shapes should reside in. For euclid, this space was inherently 3-dimensional with no particular special point (i.e. no origin), however we now have a euclidean space for every dimension, and for reference we have a point at the origin. Sometimes, euclidean space is called *euclidean affine space* to make the distinction that we have no preference for any particular point like the origin, but regard it as simply a space with the properties euclid described.

Euclidean spaces, i.e. $\mathbb{R}^n$ for all $n \in \mathbb{N}$, have a great versitility of properties: it is the setting in which we can embed all our classical notion of euclidean/affine geometry, as well as all the geometrical shapes mathematicians have worked with since antiquity (ex. circles, regular n-gones, lines, planes, varieties, etc.). The name euclidean comes from the fact that the 5 axioms postulated for his theory of [now called classical] geometry (importantly, including the 5th postulate) are equivalent to the vector-space definition of $\mathbb{R}^n$ if we choose some point in euclidean space to be the origin and affix to it coordinates, i.e. a tuple of functions codify each point. While euclid did not describe his shapes in terms of a tuple functions giving points for each part of the shape, this view has now come to dominate how shapes are percieved (ex. a cirlce is usually descriged as the set of points in $\mathbb{R}^2$ such that $x^2 + y^2 = 1$, instead of the set of points that are equidistant to a single point). If $\mathbb{A}^n$ represents affine euclidean space (i.e., no coordinates) and $\mathbb{R}^n$ represents euclidean space with coordinates, then we can write:

$$\mathbb{A}^n \mapsto \mathbb{R}^n \qquad P \mapsto (x_1(P), x_2(P), ..., x_n(P))$$

This process is called *coordinization*. There is no unique way to put coordinates on euclidean space. For example, on $\mathbb{A}^2$, we can also put the coordinates $P \mapsto (r(P), \theta(P))$ for $(r(P), \theta(P)) \in (0, \infty) \times (-\pi, \pi)$. If we represent $\mathbb{A}^2$ as the standard euclidean $\mathbb{R}^2$, then

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Importantly these two spaces, though coordinitized differently, are still the same intrinsically, they are simply described differently. Part of the goal of differential geometry is to describe shapes in a coordinate free way, and to find a way to link all description's of a space through coordinates to show they are equivalent.

Another more general goal is the study of shapes that "look like" $\mathbb{R}^n$ is you zoom in enough. This will be the beginning of studying new "types" of spaces with properties different than $\mathbb{R}^n$. For example, if you zoom in at any point of a circle, it will start looking like a line. If you zoom in at a corner point of a triangle, it will always look like a mountain, and so never like $\mathbb{R}^1$. More generally, we will want to know how to zoom into a part of space and make it looks like a part of $\mathbb{R}^n$. The goal is to take euclidean properties and tools and try to apply them to these shapes. The nature in which we make sure a space is "locally euclidean" or "locally something" will determine the type of manifold we define. Usually, we will need the space to be diffeomorphic to an open subset of $\mathbb{R}^n$, since many processes in the universe happen smoothly, however we also sometimes care about weaker conditions such as a homeomorphism, or in the weakest case a bijection.

Categorical Intuition

We can think of the study of manifolds as the (co)completion of the right categories. For example, Take the category whose objects are open intervals and whose morphisms are translations. Then we can take the objects $(0, 4)$, $(0, 1)$ and $(3, 4)$ and have the morphism $(0, 1) \mapsto (3, 4)$ via translation. Consider the commutative diagram

$$\begin{tikzcd} & (0,4) & \\ (0,1) \arrow[ur, "\mathrm{id}"] \arrow[rr, "T"] & & (3,4) \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

We may ask if this diagram has a colimit in this category. Without going through the formalities, the answer to this is the circle. However, the objects of this category only has intervals, and so the colimit does not exist! Hence, we will complete this category by adding all the colimits. This idea of completing a category will eventually lead us to having locally euclidean space. Naturally, we can replace the base space with anything, for example three lines joined at a point, or $\mathbb{C}$, or solutions to polynomials, and so forth. In general, this idea of completion and thinking of this as looking at things *locally* is important. Note that the fact that the map that we used to glue $(0, 1)$ and $(3, 4)$ together was a translation. Translations are diffeomorphisms that preserve linear maps, and hence linear maps are preserved by quotienting. It also preserves the distance structure too. However, if we have a general diffeomorphism, this need no be the case. If the map was just a homeomorphism, we only preserve the topological structure, i.e. continuous maps, but are not guaranteed to preserve differentiable or smooth maps on intervals.

A quick review

In this preliminary chapter, we quickly go over elementary knowledge of differentation and integration.

0.1 Topology

Topology, compactness, connectedness, component, local connectedness, in $\mathbb{R}^n$,
secound-countable/seperable, “locally”, neigbhorhood, paracompact.

Definition 0.1.1: Refinement

Let $\mathcal{O}$ be an open cover for U. Then a *refinement* of $\mathcal{O}$ is another open cover $\mathcal{V}$ such that for every set $V \in \mathcal{V}$, there is a set $U \in \mathcal{O}$ such that $V \subseteq U$. if $\mathcal{V}$ is a refinement of $\mathcal{O}$, we say that $\mathcal{V}$ refines $\mathcal{O}$.
If B indexes the sets of $\mathcal{V}$ and A indexes the sets of $\mathcal{O}$, we will say there is a *refiniement map* $\varphi : \mathcal{V} \to \mathcal{O}$ satisfying $V_\beta \subseteq U_{\varphi(\beta)}$

Note that every subcover is trivially a refinement, but not all refinements are open covers.

Definition 0.1.2: Locally Finite Set

Let $\mathcal{O}$ be an open cover for a set U. Then $\mathcal{O}$ is said to be locally finite if for any point $x \in U$, only finitely many sets of $\mathcal{O}$ contain x.

Definition 0.1.3: Paracompact

Let X be a topological space. Then X is said to be *paracompact* if for any open cover $\mathcal{O}$ of X, there exists a refinement that is locally finite.

Definition 0.1.4: Component

Let X be a topological space. Then the component containing x, denoted $C(x)$ is defined as the union of all connected subsets of X containing x. A subset of a topological space X is called a [connected] component if it is equal to $C(x)$ for some $x \in X$.

Definition 0.1.5: Locally Connected

A topological space X is *locally connected* if it has a basis consisting of connected sets.

(equivalence between connected and path connected components in the right situation. On a manifold, connected if and only if path connected)

(Show that there is only one norm on finite dimensional vector space)

(In general, if a function takes in a function and spits out another function, it is called an *operator*)

0.2 Differentiation

Being differentiable at a point x essentially means that if we zoom in enough at the point x, we can arbitrarily well approximate an open neighborhood of that point with some linear function, i.e., the function is locally linear. In $\mathbb{R}$, we can capture this intuition with the following: the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, or equivalently if we substitute $a = x + h$, we get that $h \to 0$ become $a \to x$ and

$$\lim_{a \to x} \frac{f(a) - f(x)}{a - x}$$

exists. If it exists, we usually denote the result of this limit as $f'(x)$. In higher dimensions, this formulation of differentiability doesn't quite work (for example, the composition of differentiable functions need not be differentiable). Hence, we re-formulate this the definition to be that the function f is differentiable at x if there exists a $m = f'(x)$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - mh}{h} = 0$$

In this way, $f'(x)$ will in fact give us back linear maps, which in this case can always be represented as 1×1 matrices. In higher dimensions $\mathbb{R}^n$, we generalize this notion by taking the norm: a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $x \in \mathbb{R}^n$ if there exists a linear transformation $A_x : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|}{\|h\|} = 0$$

with the usual $\mathbb{R}^m$ norm in the numerator and $\mathbb{R}^n$ norm in the denominator. Recall (or re-prove) that the derivative is unique, so if two linear transformations A_x and B_x satisfy this same limit, they are in fact equal: $A_x = B_x$. If we choose the standard basis $\beta = \{e_1, ..., e_n\}$ for $\mathbb{R}^n$, then the matrix representation of any derivative at a, $Df(a)$, is will be an $m \times n$ matrix and is called the *Jacobian Matrix.* If $U \subseteq \mathbb{R}^n$ is an open set and if for every $x \in U$, f is differentiable at x, then we say that $f : U \to \mathbb{R}^m$ is a differentiable function. Usually, A_x is written as $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$. If f is differentiable on U, then we can say that $Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a map that takes in a point $x \in U \subseteq \mathbb{R}^n$ and gives a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Since $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{nm}$ as a vector-space, then it is equivalent to consider $Df : \mathbb{R}^n \to \mathbb{R}^m$, meaning we can consider the differeniability of Df. If Df is differentiable at a in $\mathbb{R}^n$, then we would write $D(Df)(a)$, or more succiently $D^2f(a)$. If D^2f is differentiable at every $x \in \mathbb{R}^n$, then we would get a map $D^2f : \mathbb{R}^n \to \mathbb{R}^{n^2m}$, or in terms of linear transformation $D^2f : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$. It is a useful fact to remember that $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ is isomorphic to $\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m)$ which is the set of all bilinear linear transfomrations from $\mathbb{R}^n$ to $\mathbb{R}^m$, that is all functions $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that $f(rx, y) = f(xr, y) = rf(x, y)$ and f is linear in the first and second commponent. We can repeat this process to define D^nf, and if D^nf exists for all $a \in \mathbb{R}^n$, then $Df^nf : \mathbb{R}^n \to \mathcal{L}^n(\mathbb{R}^n, \mathbb{R}^m)$ exists. The linear function in the codomain of DF can be "realised" as mappings between tangent spaces from U to $\mathbb{R}^m$. Recall that $\mathbb{R}^n_a$ is the vector-space centered at a. Then $Df(a)$ can be thought of having domain and codomain

$$Df(a) : \mathbb{R}^n_a \to \mathbb{R}^n_{f(a)}$$

This will become very important once we start trying to define a notion of integrating manifolds.

The composition of two differnetiable functions is differentiable: If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are two functions such that f is differentiable at a and g is differentiale at $f(a)$, then $g \circ f(a)$ is also differentiable and

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

given the jacobian matrix representation of these linear transofmrations, we can see this as multiplying matrices $[Dg(f(a))]_\beta \cdot [Df(a)]_\beta$. Using the chain rule, we can see that addition of two differentiable functions is differentiable (since $f + g = s(f, g)$ where $s(x, y) = x + y$ and $(f, g)(a) = (f(a), g(a))$), and $D(fg)(a) = Df(a)g(a) + f(a)Dg(a)$.

Finding the jacobian matrix $[Df(a)]_\beta$ is made easy using patial derivatives. If β is the standard basis (in particular, it is ordered), $f : \mathbb{R}^n \to \mathbb{R}$ is a function, and $a \in \mathbb{R}^n$, then f has a *ith partial derivaitve at a* if

$$\lim_{h \to 0} \frac{f(a_1, a_2, ..., a_i + h, a_{i+1}, ..., a_n)}{h}$$

exists and is denoted $D_if(a)$ of $\frac{\partial f(a)}{\partial x_i}$. In particular, notice that this is just a function from $\mathbb{R}$ to $\mathbb{R}$ in disguise: letting $g(x) = f(a_1, a_2, ..., a_{i-1}, x, a_{i+1}, ..., a_n)$, we see that $g'(a_i) = D_if(a)$. Hence, all standard notions and tecniques of differentiation are already available to us. As with differentiation of $f : \mathbb{R}^n \to \mathbb{R}^m$, if $D_if(a)$ is defined for all $x \in \mathbb{R}^n$, then we have a function $D_if : \mathbb{R}^n \to \mathbb{R}$. Hence, we can take the partial derivative of D_if in any direction: $D_j(D_if)$, which we usually denote $D_{i,j}f$. We will soon cover how $D_{i,j}f = D_{j,i}f$ (If $D_{i,j}f$ and $D_{j,i}f$ are continuous) after we cover how we can use partial derivative information to find $[Df]_\beta$.

If Df exists, then each D_if exists. However, the converse is not true: every D_if can exist but Df might not exist. In fact, if we generalize our notion form partial derivative to directional derivative (where instead of $a_i + h$ we take $a_i + \mathbf{u}h$ for some unit vector $\mathbf{u}$), then it is possible that *every* directional derivative exists but Df still doesn't exist. However, if each D_if is *continuous*, then this in fact implies that Df exists. It would be nice if Df existing implied that each D_if was continuous, however that is not the case. Thus, since we are comfortable with working with D_if, we usually don't look at differentiable functions, but the stronger class of functions with continuous partial derivatives, denoted $C^1(U, \mathbb{R}^n)$. If $n = 1$, we usually denote this as $C^1(U)$. If the partial derivatives are r-times differnetiable with the partia of $D^r f$ being continuous, we denote this as $C^r(U, \mathbb{R}^n)$. If f is in C^r for all $r \geq 0$, then we say that f is *smooth* and is in C^∞. Furthermore, if f is bijective (hence invertible), then if f^{-1} is also smooth, we say that f is a *diffeomorphism*. Theorem 0.2.1 shows that if f is C^∞ and invertible, then f^{-1} is automatically C^∞, giving us a simpler criterion for finding the "isomorphissms" in the appropriate category (which we'll define soon).

One important trick to remember is that:

$$D_x f(p) = \frac{d}{dt}\bigg|_{t=0} f(p+tx)$$

With partial derivatives defined, we can finally find $[Df(a)]_\beta$ using them. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a functions where $f = (f^1, ..., f^m)$ where $f^i : \mathbb{R}^n \to \mathbb{R}$, then

$$[Df(a)]_\beta = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{pmatrix}$$

i.e., if $M = [Df(a)]_\beta$, then $M_{ij} = D_j f^i$. We see that this matrix is indeed the derivative by first taking $m = 1$, then taking $h(x) = (a_1, ..., x, ..., a_n)$ with x in the jth place, and then noticing that $D_j f(a) = (f \circ h)'(a_j)$ so that

$$\begin{aligned} (f \circ h)'(a_j) &= f'(a) \cdot h'(a_j) \\ &= f'(a) \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

We end this review with a reminder of the differentiability of f^{-1} whenever it is defined for a differentiable function f and a theorem which let's us convert "relations" into functions, in particular it is used to easily go back and forth between different coordinates:

Theorem 0.2.1: Inverse Function Theorem

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. $a \in U$ and $\det Df(a) \neq 0$. Then there exists an open set $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ where $a \in V$, $f(a) \in W$, such that $f : V \to W$ has a differentiable inverse $f^{-1} : W \to V$, and for every $y \in W$:

$$D(f^{-1})(y) = [Df(f^{-1}(y))]^{-1}$$

One of the great results of the inverse function theorem is that it suffices that f is C^∞ and invertible to be a diffeomorphism. First, we prove a lemma:

Lemma 0.2.1: Bounding On Closd Rectangles

Let $A \subseteq \mathbb{R}^n$ be a closed rectangle and let $f : A \to \mathbb{R}^n$ be continuously differentiable. If M exists such that for all $x \in \operatorname{int} A$, $|D_i f^i(x)| \leq M$, then:

$$|f(x) - f(y)| \leq n^2 M |x - y|$$

for all $x, y \in A$

Proof **:**
We first create a tellescoping sum:

$$|f^i(y) - f^i(x)| = \sum_{j=1}^{n} f(y^1, y^2, ..., y^j, x^{j+1}, ..., x^n) - f(y^1, y^2, ..., y^{j-1}, x^j, ..., x^n)$$

Applying the man value to each term on the right hand side, we get:

$$\begin{aligned} f(y^1, y^2, ..., y^j, x^{j+1}, ..., x^n) - f(y^1, y^2, ..., y^{j-1}, x^j, ..., x^n) &= (y^j - x^j) D_j f^i(z_{ij}) \\ &\leq |y^j - x^j| M \end{aligned}$$

for appropraite z_{ij}, thus since $|y^i - x^i| \leq |x - y|$ by the reverse triangle inequality, $|f^i(y) - f^i(x)| \leq nM|y - x|$. Finally, with some norm manipulation and the triangle inequality, we get:

$$|f(y) - f(x)| \leq \sum_{i=1}^{n} |f^i(y) - f^i(x)| \leq n^2 M |y - x$$

completing the proof

We know proceed with the proof for the Inverse Function Theorem:

Proof **:**
We will start by showing that an inverse exists, which will use the fact that $\det Df(a) \neq 0$ to find a neigbhorhood around a that is invertible. Showing that it's continuous will almost immediately come out of the proof of bijectivity. Then,

For notational simplicity, let λ be the linear transformation $Df(a)$. Then λ is non-singular since $\det Df(a) \neq 0$. Now, consider

$$D(\lambda^{-1} \circ f)(a) = D(\lambda^{-1})(f(a)) \circ Df(a) = \lambda^{-1} \circ Df(a) = \mathrm{id}$$

Notice that if we prove the theorem for $\mathrm{id} = \lambda^{-1} \circ f$, it is certainly true for f (since λ is non-singular, it would be the composition of two invertible functions). Thus, without loss of generality, we may assume λ is the identity.

This will now force injectivity on a local level. Let's say $f(a + h) = f(a)$. Then:

$$\frac{|f(a + h) - f(a) - \lambda(h)}{|h|} = \frac{|h|}{|h|} = 1$$

However,

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\lambda(h)}{|h|}=\frac{|h|}{|h|}=0$$

Thus, there must exist a clsoed rectangle, U, containing a in its interior such that $f(x)\neq f(a)$, $x\in U$ when $x\in A$. Since det is a continuous function and f is continuously differentiable, we can also assume that:

$$\det Df(x)\neq 0 \quad x\in U \tag{1}$$

$$|D_jf^i(x)-D_jf^i(a)|\leq\frac{1}{2n^2} \quad \text{for all } i,j \text{ and } x\in U \tag{2}$$

The partiular choice of bound on the partial derivaties is to employ the following trick; we see to make sure that f grows faster than the distance between two points so that we can prove that we can find an inverse. To that end, we made the choice in the 2nd bullent point so that if we combine the last point with lemma 0.2.1 and apply it to $g(x)=f(x)-x$, we get:

$$|f(x_1)-x_1-(f(x_2)-x_2)|\leq\frac{1}{2}|x_1-x_2|$$

Using the reverse triangle inequality, we get:

$$|x_1-x_2|+|f(x_1)-f(x-2)|\leq|f(x_1)-x_1-(f(x_2)-x_2)|\leq\frac{1}{2}|x_1-x_2|$$

Thus we also get a bound of:

$$|x_1-x_2|\leq 2|f(x_1)-f(x_2)| \tag{3}$$

for all $x_1,x_2\in U$, showing us that f must grow faster than the linear distance between two points.

Now, since ∂U is compact, so is $f(\partial U)$. By the the fact that $f(x)\neq f(a)$ for all $a\neq x\in U$, $f(a)\notin f(\partial U)$. Therefore, there exists a $d>0$ such that $|f(x)-f(a)|\geq d$ for all $x\in\partial U$. Let $W=\{y\mid |y-f(a)|\geq d/2\}$. Note that for any $y\in W$, the distnce between y and $f(x)$ where $xin\partial U$ must always be greater than the distance between y and $f(a)$:

$$|y-f(a)|<|y-f(x)| \tag{4}$$

With this, we can show that for all $y\in W$, there exists a unique $t\in\int(U)$ such that $f(t)=Y$ (showing the existence of an inverse on W). To show existence, we will take advantage of the determinant information we established in equation (1) and (2). In particular, define $g:U\to\mathbb{R}$,

$$g(x)=|y-f(x)^2|^2=\sum_{i=1}^{n}(y^i-f^i(x))^2$$

Since g is continuous on U and U is bounded, it's minimum must be achieved somehwere. Let's say the minimum as at t. Since g is differentiable, we can find the minimum using the derivative. We know that the minimum of g is achieved. Importantely, by equation (4), $g(a)<g(x)$ for all $x\in\partial U$, so the minimum is not achieved on the border (which will be importance for f^{-1} being differentiable). Then:

$$\sum_{i=1}^{n}2(y^1-f^i(t))\cdot D_if^i(t)=0$$

however, $\det Df(x)\neq= 0$ for all $x\in\int U$! Therefore, the only possibility left for making this equation is if $y^i-f^i(t)=0$ for all i, i.e. $y=f(t)$. For uniqueness, if two points $x_1\neq x_2$ had $f(x_1)=f(x_2)$, then this would violate equation (3).

Thus, let $V = \int U \cap f^{-1}(W)$, so that $f : V \to W$ has inverse $f^{-1} : W \to V$. To show that f^{-1} is continuous, notice that equation (3) also implies:

$$|f^{-1}(y_1) - f^{-1}(y_2)| \leq 2|x_1 - x_2| \tag{5}$$

which immediately implies continuity of f^{-1}

Finally, we show that f^{-1} is differentiable., in particular if $\mu = Df(x)$ $(x \in V)$, then $D(f^{-1})(x) = \mu^{-1}$. Similarly to how we proved the chain rule, let

$$f(x_1) = f(x) + \mu(x_1 - x) + \varphi(x_1 - x)$$

where the error term φ tends to zero faster than linearly

$$\lim_{x_1 \to x} \frac{|\varphi(x_1 - x)|}{x_1 - x} = 0$$

Manipulating the equation around, we get the equivalent equation of:

$$\mu^{-1}(f(x_1) - f(x)) = x_1 - x + \mu^{-1}(\varphi(x_1 - x))$$
$$x_1 = x + \mu^{-1}(f(x_1) - f(x)) - \mu^{-1}(\varphi(x_1 - x))$$

Since f is invertible, we can re-write $x_1 = f^{-1}(y_1)$ and $x = f^{-1}(y)$, and re-shuffle to get:

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1}(\varphi(f^{-1}(x_1) - f^{-1}(x)))$$

and so it suffices to show that the error term goes to zero faster than linearly:

$$\lim_{y_1 \to y} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{y_1 - y} = 0$$

Using the operator norm on μ^{-1}, we get that if this limit exists, then

$$\begin{aligned}
\lim_{y_1 \to y} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{y_1 - y} &\leq \lim_{y_1 \to y} \frac{M|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{y_1 - y} \\
&= M \lim_{y_1 \to y} \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{y_1 - y}
\end{aligned}$$

which will go to zero if and only if $\lim_{y_1 \to y} \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{y_1 - y}$ goes to zero. With this, we can do the same trick as we've done with the chain-rule:

$$\begin{aligned}
&\frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{y_1 - y} \\
&\qquad = \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{f^{-1}(y_1) - f^{-1}(y)} \cdot \frac{|f^{-1}(y_1) - f^{-1}(y)|}{y_1 - y} \\
&\qquad \leq \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{f^{-1}(y_1) - f^{-1}(y)} \cdot 2 && \text{by equation(5)}
\end{aligned}$$

Since f^{-1} is continuous, as $y_1 \to y$, $f^{-1}(y_1) \to f^{-1}(y)$, in particular limits are preserved when taking composing with continuous functions, and hence the first factor goes to 0, and so the whole limit goes to zero. Since this is true for any $y \in W$, f^{-1} is indeed differentiable with derivative μ^{-1}, as we sought to show.

Note that it is possible that f^{-1} exists even if $\det Df(a) = 0$, (ex. x^3), howver f^{-1} cannot be differentiable at the point where $\det Df(a) = 0$

The next theorem is useful in identifying when we can write a set of equations in terms of different parameters. In particular, if we have a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and take the collection of points where $F(x^1, ..., x^n, y^1, ..., y^m) = 0$ (i.e. $F^{-1}(0)$), then under appropriate conditions, there exists an $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $F(x^1, ..., x^n, f(x^1, ..., x^n)) = 0$

This theorem is particularly useful if you recall that a manifold can be defned as the zero locus of a C^1 function with Df having constant rank.

Theorem 0.2.2: Implicit Function Theorem

Let $f : U \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuously differentiable function and $(a, b) \in U$, $f(a, b) = 0$. Let M be an $m \times m$ matrix such that

$$M_{ij} = D_{n+j} f^i(a, b) \qquad 1 \le i, j \le m$$

If $\det M \neq 0$, then there exists an open set $A \subseteq \mathbb{R}^n$ containing a and an open set $B \subseteq \mathbb{R}^m$ containing b such that for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. Furthermore, the function g is differentiable.

If you are visual, then M is

$$\begin{pmatrix} D_{n+1}f^1(a,b) & D_{n+2}f^1(a,b) & \cdots & D_{n+m}f^1(a,b) \\ D_{n+1}f^2(a,b) & D_{n+2}f^2(a,b) & \cdots & D_{n+m}f^2(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ D_{n+1}f^m(a,b) & D_{n+2}f^m(a,b) & \cdots & D_{n+m}f^m(a,b) \end{pmatrix}$$

***Proof* :**
We will be using the Inverse Function theorem to prove this result. Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, $F(x, y) = (x, f(x, y))$. Since $\det M \neq 0$, then by construction of F, if $0 = f(a, b)$ $\det DF(a, b) = \det M \neq 0$. Thus, by the Inverse Function theroem, there exists an $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and an open set $V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing (a, b), which we see can be easily written as $A \times B$, such that $F : A \times B \to W$ has a differentiable inverse function $F^{-1} : W \to A \times B$. It should be clear that by the form of F, $F^{-1}(x, y) = (x, k(x, y))$ for appropriate differentiable function k.

With this function defined, letting $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the natural projection function $\pi(x, y) = y$, we get the following chain of equalities:

$$\begin{aligned} f(x, k(x, y)) &= f \circ F^{-1}(x, y) \\ &= ((\pi \circ F) \circ F^{-1})(x, y) \\ &= (\pi \circ (F \circ F^{-1}))(x, y) \\ &= \pi(x, y) \\ &= y \end{aligned}$$

Thus, $f(x, k(x, y)) = y$, in particular $f(x, k(x, 0)) = 0$. Letting $g(x) = k(x, 0)$, we get $f(x, g(x)) = 0$, completing the proof.

In my mind, the key use of the Implicit function theorme is that it let's you relate a set of "inputs" to some desired output

Example 0.1: Implicit Function Theorem

Let $p : \mathbb{C} \to \mathbb{C}$, $p(z) = a_0 + \cdots + a_{n-1}z^{n-1}z$. Let's say all the roots of p are distinct and r is a root, $p(r) = 0$. If $\mathbb{C}^n$ is the set of all possible coefficients, is there a smooh way of relating the coefficents to the root? That is, is there a function: $f_r : \mathbb{C}^n \to \mathbb{C}$ such that given the coefficients, we can find a smooth function from $\mathbb{C}^n$ to the roots.

Going one way is easy: if $p(z) = (z - \alpha_1) \cdots (z - \alpha_n)$ are the roots, then multiplying up we get $p(z) = z^n + \sigma_{n-1}z^{n-1} + \cdots + \sigma_1 z = \sigma_0$, where σ_i are the symmetric functions. Findin an inverse of this function is really hard. However, we can find it "implicitely" by defining:

$$F(a_0, ..., a_{n-1}, z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$$

Then $f(a_0, a_1, ..., a_{n-1}, r) = 0$, and since all the roots are distinct we have that $\partial F(a, r)/\partial z \neq 0$, and hence we can find a C^1 inverse!

The key here is that there was some "input" (i.e. coefficients) that we wanted to relate to some output (in this case a root).

(A word on why it has to one root, namely the need for bijectivity).

Finding the derivative of $g(x)$ is feasible as well. Given that $f(x, g(x)) = 0$, $f^i(x, g(x)) = 0$. Taking the derivative D_j on both sides, we get:

$$D_j f^i(x, g(x))) = D_j f^i(x, g(x)) = \sum_{\alpha=1}^{m} D_{n_\alpha} f^i(x, g(x)) \cdot D_j g^\alpha(x)$$

and since $\det M \neq 0$, these equations can be solved fof $D_j g^\alpha(x)$. The answer will depend on $D_j f^i(x, g(x))$, and hence ultimately on $g(x)$. Unfortuantely, there is no more real simplification we can do here, namely since $g(x)$ is not necessarily a unique answer. For example, if we consider $f(x, y) = x^2 + y^2 - 1$, then we get $g(x) = \pm\sqrt{1 - x^2}$ as possible solutions to $f(x, g(x)) = 0$. If we differentiate f at 0, at $f(x, g(x)) = 0$, we get:

$$(D_1 f(x, g(x)) + D_2 f(x, g(x))) \cdot g'(x) = 0$$

i.e.

$$2x + 2g(x) \cdot g'(x) = 0 \quad \Leftrightarrow \quad g'(x) = \frac{-x}{g(x)}$$

which works for both $\pm$.

(when you get to Clairuts theorem show how it is a simple task to generalize form when we can swap any two indices).

0.3 Integration

Integrating over $\mathbb{R}$ and $\mathbb{R}^n$ will be taken as known (in particular, integrable functions, zero measures sets, and Fubini a). We will be reviewing what is important for understanding integrating manifolds.

The first important concept to remember is that integrals are invariant under diffeomorphism as lon as the domain of integration changes properly as well:

Theorem 0.3.1: 1 Dimensional Change Of Variables

Let $u = g(x)$. Then:

$$\int_a^b f(g(x))g'(x)dx = \int_{a'}^{b'} f(u)du$$

Proof **:**
proved in general case later

Theorem 0.3.2: Change Of Variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. If $u = g(x)$, then:

$$\int \cdots \int_a^b f(g(x)) \det(Dg(x))dx = \int \cdots \int_{a'}^{b'} f(u)du$$

Proof **:**
In many texts

Example 0.2: Change Of Variables
put here all the common change of var's.

Theorem 0.3.3: Subordinate Partitions Of Unity

Let $A \subseteq \mathbb{R}^n$ and let $\mathcal{O}$ be an open cover for A. Then there exists a collection Φ of C^∞ functions φ defined defined in an open set containing A such that

1. For each $a \in A$, $0 \leq \varphi(x) \leq 1$ for all $\varphi \in \Phi$
2. For each $a \in A$, there exists an open neighborhood V_a such that $a \in V_a$ and only finitely many $\varphi \in \Phi$ are nonzero at a
3. For each $a \in A$, $\sum_{\varphi \in \Phi} \varphi(a) = 1$ (by the previous condition, this is always a finite sum).
4. For each $\varphi \in \Phi$, there is an open set $U \in \mathcal{O}$ such that $\varphi = 0$ outside some closed set V containing U.

If we just have the first 3 conditions, we call Φ a partition of unity of A, and if we add the 4th condition we say that the partition of unity is subordinate to $\mathcal{O}$. The proof follows a similar strategy to many of its kind: first we prove it on compact sets. Let U be an open set contianing A. Since A is compact, a we can choose a finite open cover automatically, satisfying the 2nd condition. Then we can take the closure of these sets, or try to refine our choice after applying compactness so that each $U_i \subseteq A$ (such an open cover is caleld admisible) and repeat a slightly more elaborate but elementary construction (Spivak calculus on Manifolds p.63). Let's label the compact sets whose interior cover A in an admissible $\{D_i\}_i$. way Then we can choose some C^∞ functions ψ_i so that they are 0 outside of D_i. Then we can define $\varphi_i = \frac{\psi_i}{\sum_i \psi_i}$. Finally, choosing some C^∞ function $f : U \to [0,1]$ that is zero outisde of A. Then $\Phi = \{f \circ \varphi_i\}_{i=1}^n$, completing the compact case. Then we expand this in a similar way we would do for a σ-algebra, and then to general open sets by taking advantage of the fact that any open set has an "open set exaustion", and finally for arbitrary sets by taking the union of all open sets of the cover of A to be the open set containing A, then applying the previous step.

Part I

Manifold Theory

1

Manifolds

In this chapter, we will define many different notions of manifolds which will depend on how we define both the "regularity" of the locality, and the "type" of locality. We will see that given any type of manifold, we will always have to make sure that the local pieces behave well when the local pieces intersect. We then will see how we will relate manifold: the key property we will want to preserve is that the function "locally" will have the property of the manifold (ex. if the manifold is locally $\mathbb{R}^n$ with the regularity of C^∞, then a function $f : M \to N$ between two such manifolds will have to locally be a smooth function between euclidean space). The key of what types of functions will be well-defined on the manifold will be the type and strength of the gluing maps.

The define the type of "regularity" of a manifold, we would have differnet maps: If $x \in X$ has a neighborhood $x \in U \subseteq X$ and a map f such that $f : U \to \mathbb{R}^n$ is either:

1. homeomorphic
2. C^k with C^k-inverse
3. diffeomorphic (C^∞ with C^∞-inverse)

This map can be thought as the amount of "regularity" we are expecting our space to have. The choice of the type of map will define the type of manifold we choose, in particular for these three type of maps we would get:

1. a topological manifold
2. a C^k manifold
3. a smooth manifold

The other type of structure we will sometimes see that has the word manfold inside is is a *manifold with borders* and a *manifold with corners*, which are spaces that are locally respectively either:

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_1 \geq 0\} \qquad \text{or} \qquad \mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\}$$

To make sure there is some regularity between component, we will have to make sure that if $U, V \subseteq M$ are two subsets of M with functions $f_U : U \to \mathbb{R}^n$ and $f_V : V \to \mathbb{R}^n$ such that $U \cap V \neq \emptyset$, then f_U and f_V must be compatible in an appropriate way (which will mean $f_V \circ f_U^{-1}$ is a diffeomorphism or a homeomorphism). This will insure that the information on the manifold is *consistent* in some important ways that we will explore soon.

Note how this differs from the property functions usually preserve in algebra (the image $f(G)$ is another algebraic obect) or topology (we can study the local properties of codomain N via the local propeties of the codomain M). One last type of manifold we will present is an *orientable manifold* which will add add one more (global) regularity condition on the transition maps.

After having defined a nice range of manifolds, we will the maps between them. These maps will have many nice properties we expect from morphisms in categories, however they will in other ways be lacking in important ways due to their “local” nature. The introduction of maps will allow us to ask questions like what are the subobjects of our category, the product objects, and the quotient objects. All of these will be explored in this chapter.

We will end the chapter by introducing partitions of unity for manifolds. Partitions of unity will be invaluable for finding the existence of many extensions of functions and to develop the theory of integration later on.

1.1 Defining a Manifold

The goal of this section is to define a *manifold*. In essense, a manifold will be a set X that is “locally euclidean” and respects certain geometric properties (these will be elaborated in example 1.1 and under this example). We start by defining a manifold on the high-level first by letting a manifold X be a topological space with a list of properties, then defining them on a low-level by showing what type of object do we get from the right type of construction from the base space (or copies of the base space) will be considered a manifold.

1.1.1 Topological Manifold

Definition 1.1.1: Locally Euclidean Space

Let M be a topological space. Then M is said to be *n-dimensional locally euclidean* if for every $x \in M$, there exists an open set U_x containing x and a map $\varphi : U_x \to \mathbb{R}^n$ that is an embedding.

Sometimes we will say that a space M with this term is called *locally homeomorphic to* $\mathbb{R}^n$. However, this is a bit confusing since we will later define a local homeomorphism be be a map $f : M \to N$ that can be restricted to a homeomorphism at any given point. (Note sure if locally euclidean and locally homeomorphic imply each other TBD)

The embedding from the definition is given a name:

Definition 1.1.2: (Topological) Chart

Let M be a topological space. Then a map φ that embeds an open subset $U \subseteq M$ into $\mathbb{R}^n$ is called a *chart.*

We sometimes label with boldface letters $\mathbf{x}$ to indicate that we will think of $\mathbf{x}$ as defining coordinates on M. If $p \in U$ for some chart $(U, \mathbf{x})$, then if $\mathbf{x}(p) = 0$, we say that $(U, \mathbf{x})$ is *centered at* p.

Definition 1.1.3: Coordinate Functions

Let $(U, \mathbf{x})$ be a chart and let let $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi(a^1, ..., a^i, ..., a^n) = a^i$. Then $x^i = \pi_i \circ \mathbf{x}$ is called the ith *coordinate function* for the chart $(U, \mathbf{x})$. Thus,

$$\mathbf{x} = (x^1, ..., x^n)$$

Since we require every part of a manifold M to be locally euclidean, we can define a collection of charts that proves it for us:

Definition 1.1.4: (Topological) Atlas

Let M be locally euclidean space. Then a (topological) atlas is a collection of charts covering M.

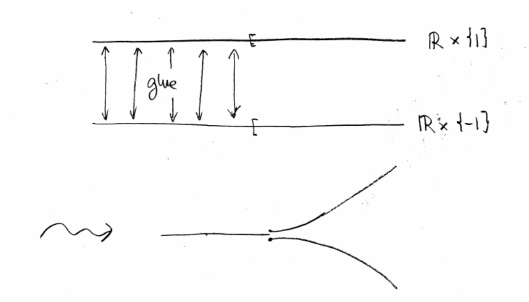
Given the intuition presented at the beginning, we might want to say that a manifold is a locally euclidean space. However, there are 2 more conditions we want to add in order for manifolds to behave in a "geometric" and "analysis" way. First, locally euclidean space is not necessarily *Hausdorff*:

Example 1.1: Locally Euclidean not Hausdorff

Let X be the disjoint union of two copies of $\mathbb{R}$. To represent X, let's say $X = \mathbb{R} \times \{-1\} \cup \mathbb{R} \times \{1\}$. Define $\sim$ on X to be

$$(x, -1) \sim (x', 1) \quad \Leftrightarrow \quad x = x' < 1$$

and let $M = X/\sim$. Roughly, the following is the picture:



The second part of this picture is in fact inaccurate since never do the lines bend or are somehow "close" to each other, however it is also innacuare to draw any of the following:

Since $\pi_{-1} : \mathbb{R} \times \{-1\} \to M$ $\pi_1 : \mathbb{R} \times \{1\} \to M$ are both continuous quotient maps, meaning the image is connected. This strangeness produces the following results.

First, let $\pi : X \to M$ be the projection map. Then:

$$U = \pi(\mathbb{R} \times \{1\}) \qquad V = \pi(\mathbb{R} \times \{-1\})$$

The map $X \to \mathbb{R}$ mapping $(x, \pm 1)$ to x is constant, and so $f : X \to M$ is a well-defined quotient map. Next, take the restrictions $\varphi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$. Then:

$$\varphi(U) = \psi(V) = \mathbb{R} \qquad \varphi(U \cap V) = \psi(U \cap V) = \mathbb{R}_{<0}$$

the transition map is clearly the identity, and φ, ψ are clearly the projection maps, which are homeomorphisms onto $\mathbb{R}$, and hence these two maps along with U and V form an atlas.

We want to understand how some more topological properties of M. In particular, the seperation of the points

$$p = \varphi^{-1}(0) \qquad q = \psi^{-1}(0)$$

these two points are not equal since $p = (0, 1)$ and $q = (0, -1)$. However, they are actually arbitrarily close to each other! If we take $A, B \subseteq \mathbb{R}$ to be two open sets containing zero, then no matter what open set they are: $\varphi^{-1}(A) \cap \psi^{-1}(V) \neq \emptyset$, i.e. M is *not* Hausdorff. This is rather odd in the world of euclidean geometry. We hence want to avoid this by adding the Hausdroff condition.

Secondly, we will require that M be *secound countable*, i.e. M as a topological space has a countable basis. This is analytic condition as well as a geoemtric condition. Without a countable basis, we:

1. cannot define partitions of unity (and hence no integration and differential forms)
2. we cannot embed manifolds into $\mathbb{R}^n$ for large enough n
3. We cannot (always) define a Riemannian metric (a way to define a metric on manifolds)
4. Sard's theorem fails (the set of critical points has measure 0)

All of these problems arise if we don't allow for a countable basis. Essentially, a lot of our geometric and analytic understanding require countability conditions, and many tools or structures we like to work with don't work without these conditions. In other words, the notion of "geometry", with concepts of integration, distance, curvature (TBD), and so forth, are intrinsically countable (being "uncountable" is a quality in it of itself here), and we can create some very "non-geometric" properties (or equivalently, loose geometric concepts)

Since we are focused on the local level, we will ask for our space to be *paracompact*, that is for any open cover $\mathcal{O}$ of M, there exists a refinement of $\mathcal{O}$ that is locally finite (every point $x \in M$ is covered by finitely many points). For the topologically inclined, there are many equivalent conditions[1]:

[1] note that component means largest connected set

1. Every component of M is σ-compact
2. Every component is 2nd countable
3. M is metrizable (using Urysohn's Metrization theorem)
4. M is paracompact
5. There exists a C^0 partition of unity subordinate to any open cover

With these two extra conditions, we are ready to define a manifold is:

Definition 1.1.5: Topological Manifold

Let M be a topological space. Then M is called an *n-dimensoinal topological manifold* if it is a paracompact hausdroff space that is locally euclidean.

Note that it suffices to show that M can be covered by countably many charts to show that it is second-countable. Every manifold has an atlas, and most of the time, an atlas will only need finitely many charts. *Any atlas covering a manifold will henceforth assumed to be countable.* Notice that the M in example 1.1 is not a topological manifold. if we remove the hasudroff condition, the resulting class of sets is usually called *non-hausdorff manifolds*; these will not be studied in these notes. Note that since M is Hausdorff and paracompact, M is in fact *normal*, and hence it is *metrizable*. Putting a metric on M will be explored in chapter 9 when we try to define the Riemann metric. Since Hausdrofness, paracompactness, and locally euclidean are all topological invariants, topological manifolds form a category with the morphisms being continuous maps. If $\mathcal{A}$ is an atlas for M and $\mathcal{B}$ is an atlas for N, then $f : M \to N$ is a continuous map if and only if for some collection of charts $A \subseteq \mathcal{A}$ covering M and $B \subseteq \mathcal{B}$ covering N, $b \circ f \circ a^{-1}$ is continuous. Hence, we can check the continuity of a map between manifolds locally.

Example 1.2: Topological Manifold

This set of examples contains some of the most common examples and the most common way of coming up with new manifolds given old ones

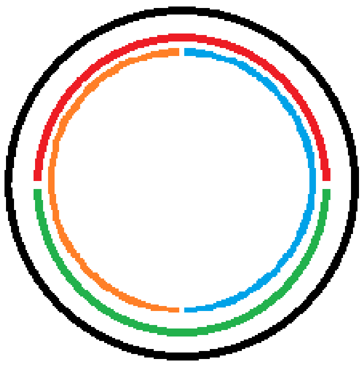
1. $\emptyset$ is a topological manifold of every dimension. This is the only Manifold of this form. Every point $\{p\} \in \mathbb{R}^n$ for any n is a 0-manifold.
2. $\mathbb{R}^n$ is the trivial n-dimensional topological manifold, since for any point $p \in \mathbb{R}^n$, the map $\mathrm{id}_p : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.
3. If X is a topological manifold and $U \subseteq X$ is an open subset, then U is also a topological manifold. Many famous examples of manifolds and knots are constructed by taking a closed subset $C \subseteq X$ and considering the open subset $X \setminus C$. For example, $\mathrm{GL}_n(\mathbb{R})$ is an open subst of $\mathbb{R}^{n^2}$ and hence a n^2-manifold (Recall the determinant function is continuous, being a polynomial).
4. If M_1 and M_2 are topological manifolds of dimension n_1 and n_2, then $M_1 \times M_2$ with the product topology is a topological manifold, since it is still hausdorff (easy exercise) and paracompact (since $M_1 \times M_2$ is metrizable), and for any $(p, q) \in M_1 \times M_2$, $\varphi_p \times \psi_q : U_p \times U_q \to V_{\varphi(p)} \times V_{\psi(q)} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a homeomorphsm. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are charts for M_1 and M_2 respectively, then $CM_1 \times CM_2$ forms a chart for $M_1 \times M_2$.

5. The disjoint union of two n-manifolds $X \sqcup Y$ is an n-manifold with the atlas being $A_1 \sqcup A_2$ (note the importance of the dimensions being the same). Notice that this shows that manifolds can be disconnected.

6. This is our first proper example of a non-trivial manifold: Take

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\}$$

with the subspace topology. Then S^1 is clearly Hausdorff and paracompact (being compact). To show it's locally euclidean, divide S^1 into four open parts



Then for any point in one of those regions, define the projection map into $\mathbb{R}$. All of these maps are embeddings, and so S^1 is a 1-manifold. These for maps are all charts and form an atlas for S^1, though not the unqiue atlas (we'll cover another example later). Note that S^1 cannot be homeomorphic to $\mathbb{R}$ since if we remove a point from S^1 it remains connected, while $\mathbb{R}$ does not, hence S^1 is a non-trivial example of a manifold.

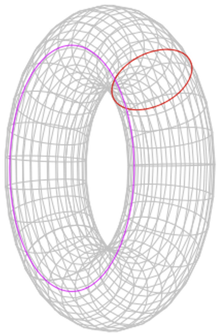
7. The fact that cartesian products of manifolds is a manifold, we see that

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}} = T^n$$

is a manifold. The resulting manifold is called an *n-torus*. If $n = 2$, it is sometime just called a torus. The 2-torus is naturally embedded into $\mathbb{R}^4$, but there is a famous embedding into $\mathbb{R}^3$, using the following paramaterization (a coordinization that also have a notion of direciton): Define $x(\theta, \psi)$, $y(\theta, \psi)$ to be:

$$x = (R + r\cos(\psi))\cos(\theta) \qquad y = (R + r\cos(\psi))\sin(\theta) \qquad z = \sin(\psi)$$

which looks like this:



It is challenging, but possible, to prove that any point $x \in T^2$ has a neighborhood $U \subseteq T^2$ homeomorphic to an open neigbhorhood of the points consiting of the parameteriation given above (in the subspace topology).

8. The triangle is homeomorphic to the circle, and hence is a 1 dimensional topological manifold with the induced topology form the homeomorphism.

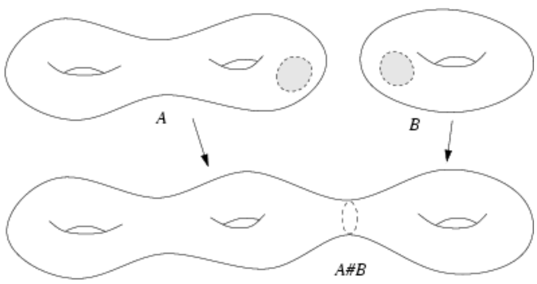
9. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be a continuous function. Then the graph of f:

$$\Gamma(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \;\middle|\; f(x) = y\right\}$$

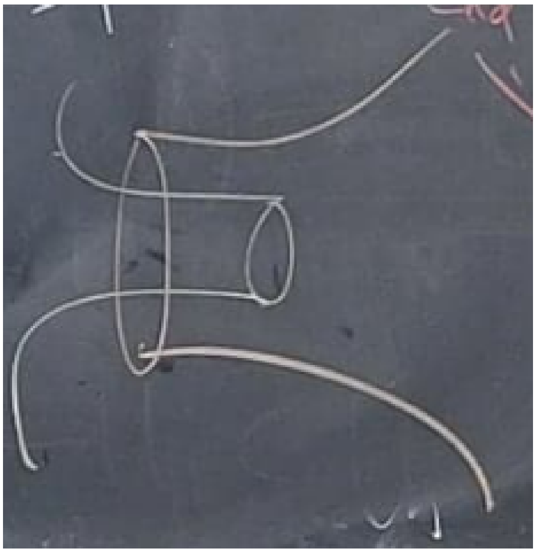
with the subspace topology of $\mathbb{R}^n \times \mathbb{R}^k$ is an n dimensional topological manifold. To see this, take $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ projeting the first n coordinates. Restricting to $\Gamma(f)$, define $\pi|_{\Gamma(f)} = \varphi$. Then φ is continuous in the subspace topology, being continuous in the parent space. Furthermore, φ^{-1} is clearly also continuous: it exists since we have resricted the pre-image to every point to be $\varphi(x)^{-1} = (x, f(x))$, and $\varphi(x)^{-1} = (x, f(x))$ is continuous, being the product of two continuous maps. Hence φ is a homeomorphism. In fact, we've just shown that $\Gamma(f)$ is homeomoprhic to U, showing the graph fits into a single chart.

More generally, if we have a space M, if for every $x \in M$, there exists an open set U containing M that is the graph of a function, then M is a topological manifold. Is the converse true? If M is a topological manifold, then any point $x \in M$ is inside an open set $x \in U \subseteq M$ that is the graph of a continuous function?

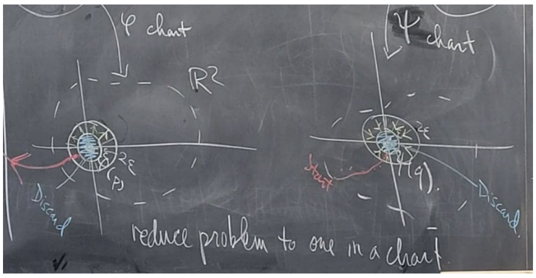
10. Let M_1 an M_2 be n-manifolds. We will define a new operation called the *direct sum* at $p \in M_1$ and $q \in M_2$ which we'll denote $M_1 \# M_2$ (assuming the two points are implcit information). This will look like so:



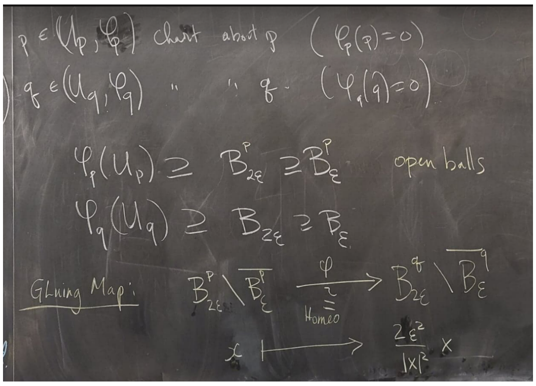
The idea is that around those two points, we will take some small neighborhood around them, delete that stretch that neighrohood out, delete a small part of it, and glue the two manifolds there. This will look like so:



The actual gluing can be visualized after applying a chart, in particular:



and the concrete maps being:



Giving us a definition of: test

$$X\#Y = \left(X \setminus \overline{\varphi_p^{-1}(B_\epsilon)}\right) \bigsqcup \left(Y \setminus \overline{\varphi_q^{-1}(B_\epsilon)}\right) / (\varphi_p^{-1}(B_{2\epsilon}) \ni x \sim \varphi_q^{-1}(\psi(\varphi_p(x))) \in \varphi_q^{-1}(B_{2\epsilon}))$$

It can be proven that the map is independent of the choice of points p, q and the choice of charts, but it may depend on the orientation of the gluing map. The proof of this would require a theorem called the *Annulus Theorem*.

One powerful result we shall not yet prove but state is that all connected 2-manifolds is an iteration of direct sums of S^1 with either T^1 or $\mathbb{R}\mathrm{P}^2$ known as the real projective space.

11. This image is not a topological manifold:

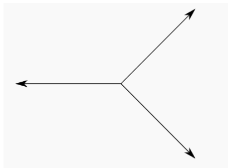


Figure 1.1: point of intersetion not homeomorphic to $\mathbb{R}^n$ for any n

Note that the notion of dimension for a topological manifold is well-defined, i.e. it is not possible that an n-dimensional manifold is homeomorphic to an m dimensional manifold (hence why the particular euclidean space was not mentioned in the definition): given a connected component and then n would be constant on the connected component. This result is due to Brouwer, and requires some DeRham cohomology theory to prove, and so we simply record the result for now so that we know not to be bothered by such a problem:

Theorem 1.1.1: Invariance Of Domain

The image of an open set $U \subseteq \mathbb{R}^n$ of an injective continuous map $f : U \to \mathbb{R}^n$ is open and f is a homeomorhpism from U onto $f(U)$. It follows that if $U \subseteq \mathbb{R}^n$ is homeomorphic to $V \subseteq \mathbb{R}^m$, then $m = n$. More generally, if M be a nonempty m-dimensional manifold and N be a nonempty n-dimensional manifold, then if $f : M \to N$ is a homeomorphism, $m = n$

Proof **:**
see theorem ref:HERE.

If the manifold has a differentiable structure (as we'll define in section 1.1.5), it is much easier to show that M must a constant dimension n using this structure (as we'll soon show).

1.1.2 Constructing Manifolds Using Quotient

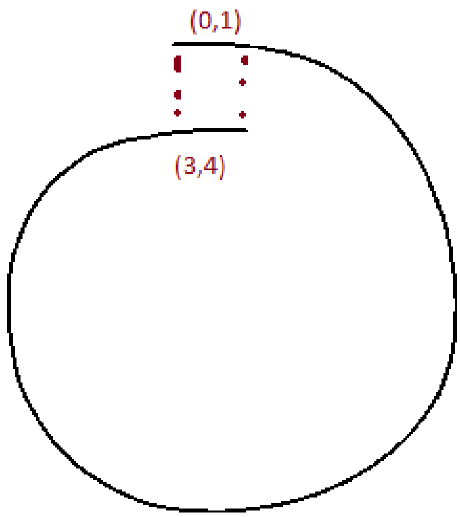
We'll now show how we would build a manifold from the ground up. The way we will do this is by taking a (at most countasble number of) set of open of $\mathbb{R}^n$, and define gluing maps between them or parts of them. Before proceeding to the general example, take:

$$B = (0, 4) \qquad (0, 1) \subseteq B,\ (3, 4) \subseteq B$$

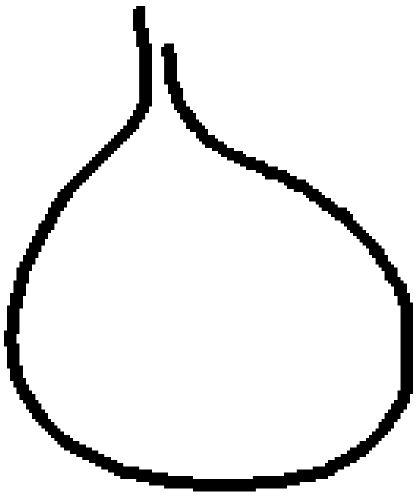
And the map $\varphi : (0,1) \to (3,4)$, $\varphi(x) = x + 3$. This map is a homeomorphism. Consider now

$$M = B/\{(x \sim \varphi(x))\}$$

that is, we identify the point of the interval $(0,1)$ with the points of the interval $(3,4)$. We may visualize this as:



Which we can see looks like a circle, and indeed if we give M the quotient topology this space is homeomrophic to S^1. I personally like to visualize a dot moving around in B, and everytime it goes a glued region, multiple dots appear (depending on the number of points that are glued) which in M would be a single point. We may think that making the map φ a homeomorphism was enough to guarantee that M is a manifold, but that is not the case. Take instead the map $x \mapsto 4 - x$. Then we would instead get an image like so:



In this map, the points 1 and 3 are infinitely closed to each other. If you imagine a dot moving around in B as described earlier, then the dot would move infinitely close to both 1 and 3 at the same time. In M, I like to maginge 1 and 3 somehow being part of an infinitesimally small "hub" where we can choose to be in one of the points or the other. Needless to say that we have broken Hausdorffness. Intuitively, what we need to happen for the gluing is as we exit the region (or limit) from one of the map, we enter the region of a new map (think of this with the original M that defined the circle). We will give a more rigorous definintion of this shortly.

Coming back now to the general construction, we see that the gluing will hapen via quotienting, where the equivalence relation is defined via gluing maps. In the case $B/(x \sim \varphi(x))$, we had a single map, but as we've shown in example 1.2 there can be multiple maps that overlap, and in general a manifold will be covered by an atlas. In order of this collection of maps to still induce the quotient topology, they need to have transition conditions to sastify transitivity: we need to make sure that when we glue maps together where they overlap, we are still moving in one direction and not splitting into two points. We thus need to keep track of the particular maps we use to glue the different components of B.

Proposition 1.1.1: Gluing Construction of Manfiold

Take the following data:

- $\mathcal{A} = \{U_i\}_{i \in I}$ is a countable collection of open subsets of $\mathbb{R}^n$
- For each i, take a finite of open subsets collection $\{U_{ik}\}_{k \in I}$ where $U_{ik} \subseteq U_i$
- for each pair (i, j), a homeomorphism $\varphi_{ij} \to U_{ji}$

for which the the following conditions hold on the maps:

1. (reflective) φ_{ii} is in the collection ($\varphi_{ii} = \text{id}$ suffices)
2. (symmetric) $\varphi_{ij} = \varphi_{ji}^{-1}$
3. (transitive) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$

with the added constraints that the graph $\Gamma_{\varphi_{ij}}$ is *closed*[a]. Then

$$M = \bigsqcup_{i \in I} U_i / (U_{ij} \ni x \sim \varphi_{ij}(x) \in U_{ji})$$

is a (topological) n-manifold with a distinguished atlas $\mathcal{A}$

[a]This mimics the idea that the diagonal $\Delta \subseteq X \times X$ is closed if and only if X is Hausdorff

For those who know some algebraic geometry, this is an example of a *sheaf*.

Proof **:**
follows from above discussion

Hence, quotienting under appropriate conditions constructs manifold. It is due to these extra conditions that quotient manifolds where not defined in the set of examples of topological manifold.

Example 1.3: Constructing Manifolds

We'll construct a circle S^1 gluing two copies of $\mathbb{R}$. Take the stereographic projection charts of S^1. Then the transition mapping becomes:

$$x \mapsto \left(\frac{2x}{1+x^2}, \pm \frac{1-x^2}{1+x^2} \right) \mapsto \frac{x}{x^2} = x^{-1}$$

This map is defiend for $\mathbb{R}^\times$, hence we have a gluging map of $\varphi_{ij} : \mathbb{R}_1^\times \to \mathbb{R}_2^\times$, $x_1 \mapsto x_2^{-1}$. Thus we have:

$$S^1 = (\mathbb{R}^1 \sqcup \mathbb{R}^1)/(\mathbb{R}_1^\times \ni x_1 \sim x_2^{-1} \in \mathbb{R}_2^\times)$$

A common way of quotienting is by defining an action of a topological group G on a manifold $\Gamma \curvearrowright M$ and taking the set of orbits M/Γ with the quotient topology[2]. Note all actions produce a manifold. It was your homework to show the following is the case:

[2]Note that every group can be equipped with the discrete topology, and so this does not limit what groups can act on manifolds, but rather the particulars of the action would have to be considered

Proposition 1.1.2: Quotient Manifolds And Group Actions

Let Γ be a group, and give it the discrete topology. Suppose Γ acts continuously on the topological n-manifold M, meaning that the action map

$$\begin{aligned} \rho : \Gamma \times M &\to M \\ (h, x) &\mapsto h \cdot x \end{aligned}$$

is continuous. Suppose also that the action is *free*, i.e. the stabilizer of each point is trivial. Finally, suppose the action is *properly discontinuous*, meaning that each $x \in M$ has a neighbourhood U such that $h \cdot U$ is disjoint from U for all nontrivial $h \in \Gamma$, that is, for all $h \neq 1$.

1. Show that the quotient map $\pi : M \to M/\Gamma$ is a local homeomorphism, where M/Γ is given the quotient topology. Conclude that M/Γ is locally homeomorphic to $\mathbb{R}^n$.
2. Show that π is an open map.
3. Let $f : M \to N$ be a continuous map such that
   $$f(h \cdot x) = f(x) \text{ for all } (h, x) \in \Gamma \times M. \tag{1.1}$$
   Show there is a unique continuous map $\bar{f} : M/\Gamma \to N$ such that $\bar{f}(\pi(x)) = f(x)$ for all $x \in M$.
4. Give an example where M/Γ is not Hausdorff.
5. Prove that M/Γ is Hausdorff if and only if the image of the map
   $$\begin{aligned} \Gamma \times M &\to M \times M \\ (g, x) &\mapsto (gx, x) \end{aligned}$$
   is closed in $M \times M$.

***Proof* :**

1. For any $x \in M$, choose an open set such that $x \in U \subseteq M$ and for any $e \neq g \in \Gamma$, $U \cap gU = \emptyset$. We'll show that π restricted to U is homeomorphic. From now on, we will say that π is locally [something] if π is [something] on an open set U satisfying the above condition.

   First, since M/Γ has the quotient topology, it is certainly a continuous map and hence locally continuous. We next show π is a locally open map. To see this, we'll need a lemma: the map $x_g : M \to M$ defined by $x \mapsto g \cdot x$ is a homeomorphism. To show it's bijective, notice that $x_{g^{-1}}$ is a two-sided inverse:
   $$x_g \circ x_{g^{-1}}(x) = x_g(g^{-1}x) = gg^{-1}x = x = g^{-1}gx = x_{g^{-1}}(gx) = x_{g^{-1}} \circ x_g(x)$$
   Hence, for any $V \subseteq M$, there exists a U such that $gU = V$. To show x_g is continuous for any g, we take advantage of continuity of ρ. Since ρ is continuous, so is $\rho_g : \{g\} \times M \to M$ with the subspace topology. Taking any open $V \subseteq M$, we know that $\rho_g^{-1}(V)$ is open. Furthermore,

we know that there exists a U such that $V = gU$. Hence,

$$\rho_g^{-1}(V) = \rho_g^{-1}(gU) = \{g\} \times U$$

Since the set $\{g\} \times U$ is open in the product topology, U is open in M. But then, $x_g^{-1}(V) = U$, showing the preimage of open sets is open and hence x_g is continuous. The argument holds also for $x_{g^{-1}}$ and hence x_g is a homeomorphism. In particular, gU is open for all $g \in \Gamma$.

Now, we'll show that π is locally open. For any $x \in M$, pick an open set such that $x \in U \subseteq M$. Consider $\pi(U)$. By definition of the quotient topology, this set is open if and only if $\pi^{-1}(\pi(U))$ is open. By definition of the quotient space, $x \sim y$ if and only if $x = gy$ for some $g \in \Gamma$. Hence,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in \Gamma} gU$$

Since U is open, gU is open, and so the arbitrary union of gU is open, showing the map is indeed open. Since π is continuous, it is certainly continuous on any restriction. what's left to show is π is locally bijective which will complete the proof. Recall that locally bijective means we need to show that π is bijective on open sets U that are disjoint by a non-trivial group action. Such a map is clearly surjective on any U so what's left to show is that it's injective. For any U, consider $x, y \in U$ where $\pi(x) = \pi(y)$, meaning $[x] = [y]$. By definition of the topology, $x \sim y$ if and only if there exists a g such that $x = gy$. By definition on of U, the only value of g for which the equation is satisfied is e, and so $x = e \cdot y = y$, showing injectivity. Thus, π is a local homeomorphism, completing the proof.

To show that M/Γ is locally homeomorphic to $\mathbb{R}^n$, notice that the composition of locally homeomorphic functions is locally homeomorphic and the inverse of a local homeomorphism is a local homeomorphism. The inverse being locally homeomorphic is immediate, so we'll take a moment to show the composition preserves local homeomorphism. To see this, if $f : A \to B, g : B \to C$ are locally homeomorphic, then if $x \in A$, we have $f|_U : U \to f(U)$ which is a homeomorphism. then choosing a neighbourhood V such that $f(x) \in V \subseteq B$ we have that $g : V \to g(V)$ is a homeomorphism. Since $f(U) \cap V$ is open, $f^{-1}(f(U) \cap V)$ is open, and we can take this set to be our domain of the homeomorphism $g \circ f$. Since x was arbitrary, this is true for any point and hence the composition of local homeomorphism is a local homeomorphism.

Since M is an n-manifold, for any $x \in M$, there exists an open set containing x, $x \in W \subseteq M$, such that $\varphi(W)$ is homeomorphic to an open subset of $\mathbb{R}^n$. If we intersect W with any other open set V containing U, $W \cap U$ the restriction map is still a homeomorphism. With that in mind, for any $[x] \in M/\Gamma$, $[x] \in \pi(U)$ for some U in M containing x, $x \in U \subseteq M$. Then by definition there is a neighbourhood U containing x that is homeomorphic to $\mathbb{R}^n$.

2. We'll define an action $C_2 \times C_2 \curvearrowright \bigsqcup_{i=1}^4 \overline{\mathbb{R}_i}$ where $\overline{\mathbb{R}_i}$ is the one-point compactification of $\mathbb{R}$ (equivalently, we could have chosen S^1 since the two are homeomorphic). Letting $M = \bigsqcup_{i=1}^4 \overline{\mathbb{R}_i}$ and $\Gamma = C_2 \times C_2$, we'll show that M/Γ is not Hausdorff.

Example 1.4: Quotient Manifold

1. Take $M = \mathbb{R}^{n+1} \setminus \{0\}$. Define the group action $\mathbb{R}^\times \curvearrowright M$ to be $r \cdot x = rx$ (i.e., to be scalar multiplication). Verify that $M/\mathbb{R}^\times$ is a manifold. For the Hausdorff condition, is the

common zero's of the continuous functions:

$$f_{ij}(x,y) = (x_i y_j - x_j x_i)$$

and since there are finitely many of these closed sets, the graph is closed. We will soon call this manifold the *real projective space.* we can also replace $\mathbb{R}$ with $\mathbb{C}$.

2. Let M be a topological manifold and $\varphi : M \to M$ a homeomorphism. Then define

$$M_\varphi = (M \times \mathbb{R})/\mathbb{Z}$$

where $k \in \mathbb{Z}$ acts via

$$k(p,t) = (\varphi^k(p), t+k)$$

This is called the *mapping torus* of φ (we will soon call this a *fibre bundle* over $\mathbb{R}/\mathbb{Z} \cong S^1$ with fibers M). If we take for example $M = \mathbb{R}$ and φ the map $x \mapsto -x$, we get the usual Möbius band.

1.1.3 Topological Manifold With Border

We often work with shapes that only varry in dimension on it's border, for example: $D^2 = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2+y^2 \leq 1\right\}$.

Definition 1.1.6: Manifold With Boundary

Let M be a paracompact hausdroff space. Then M is said to be a *topological n-manifold with boundary* if is locally homeomorphic to the upper half plane

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1, ..., x_n \geq 0\}$$

Note that it need not be that we choose ≥ 0, it can really be any value.

Example 1.5: Manifold With Boundary

1. Let $D^2 = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2+y^2 \leq 1\right\}$. Then D^2 is a filled in disk, which is locally homeomorphic to $\mathbb{H}^n$. It's boundary is S^1. More generally, D^n is a manifold with boundary with boundary S^{n-1}.

2. The set $[0,1]$ with the subspace topology of $\mathbb{R}$ is a manifold with boundary.

3. Note that in general, the product of two manifolds with boundary is not a manifold with boundary (consider $[0,1] \times [0,1]$). However, the product of a manifold with boundary and a manifold without boundary is a manifold with boundary: if X is boundaryless and Y has boundary then

$$\partial(X \times Y) = X \times \partial Y$$

Note that a topological boundary and the boundary of a manifold are different: the topological boundary contains the manifold boundary, but not necessarily the other way around (take $M = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 1 \leq x^2+y^2 < 2,\ 2 \leq x^2+y^2 \leq 3\right\}$ doesn't have $x^2+y^2 = 2$ in the manifold boundary, while it is in the topological boundary). Can you see why the boundary of a manifold is itelf always a manifold with a dimension lower then the manifold with boundary?

1.1.4 *Cobordism

(see this resource: here)

This section is a prelude to future studies. Compact $(n+1)$ manifolds with boundary provide us with a natural equivalence relation between compact n-manifolds known as *cobordism*

Definition 1.1.7: Cobordism

Two compact n-manifolds M_1, M_2 are called *cobordant* when there exists N, a compact $n+1$ compact manifold with boundary, such that ∂N is isomorphic to $M_1 \sqcup M_2$. All manifolds cobordant to M form the *cobordism class of* M. We say that M is *null-cobordant* if $M = \partial N$ for N a compact $n+1$ manifold with boundary

It is important to assume compactness, or else all manifolds are null-cobordant by taking the Cartesian product with the noncompact manifold with boundary $[0, 1)$. All n-manifolds that can be embedded into $\mathbb{R}^{n+1}$ are null-cobordant. AN example of a non nullcobordant manifold is $\mathbb{R}\mathrm{P}^2$. Let Ω^n be the set of cobordism classes of compact n-manifolds, including $\emptyset$. Then notice that:

$$[M_1] + [M_2] = [M_1 \sqcup M_2]$$

defines a binary operation on Ω^n that makes it an abelian group (with identity $[\emptyset]$). In fact, the additive inverse of any $[M]$ is $[M]$, making it 2-torsion:

Proposition 1.1.3: Additive Torsion

for any $x \in \Omega^n$, $x + x = 0$

Proof **:**
For any manifold $M<$ the manifold with boundary $M \times [0, 1]$ has boundary $M \sqcup M$. Hence $[M] + [M] = [\emptyset] = 0$, as required.

If we take the direct sum $\Omega^\bullet = \bigoplus_{n \in \mathbb{N}} \Omega^n$, then we can endow this structure with multiplication:

$$[M_1] \cdot [M_2] = [M_1 \times M_2]$$

this makes $\Omega^\bullet$ a commutative ring known as the *cobordism ring* with multiplicative unit $[*]$, the class of 0-manifold consisting of a single point. Naturally, this ring is graded by dimension.

Example 1.6: Cobordism

1. The n-sphere S^n is null-cobordant (i.e. cobordant to $\emptyset$) since $\partial B_{n+1}(0, 1) \cong S^n$.
2. Any oriented compact 2-manifold is null cobordant: we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary

We may wonder what is the ring structure of the cobordism ring. To get some regularity, we will require our manifolds to be *smooth*. For now, we shall state the result by Thom:

Theorem 1.1.2: Cobordism Ring

The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$, with generators in every dimensin $n \neq 2^k - 1$, that is

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, ...]$$

Proof **:**
see (this) paper

1.1.5 Charts and Smooth Structures

One incredible result achieved in mathematics is the invention of calculus and it's generalization to $\mathbb{R}^n$. We would like to be able to do calculus on manifolds (ex. diferentiate functions on a circle, locally invert or paramaterize functions, take the "volume" of a manifold, and so forth). In order to do this, we must be able to lift our differentiable maps on $\mathbb{R}^n$ to differentiable maps on the manifold, or equivalently perserve differentiability when gluing pieces together. It will turn out that the way we glue our pieces together will make a difference on the way we define calculus on manifolds, hence the atlas will be a vital part of the information when defining a manifold (where it was optional for the topological case).

We now continue onto adding a differentiable structure onto manifolds, or more generally a set. This will be a new type of structure which can be defined on a set (just like we can define algebraic structures, topological structure, measurable structures, etc.). A differentiable structure on a set will be characterized by being a structure that tries to desribe a set *locally*, with a suitable notion of "compatability" between local representations that let's us "glue" two reprsentation's togetether appropriately. Naturally, the name was inspired by how a function being differentiable means the function is locally linear.

The way we'll do this is by applying "coordinates systems" to patches of the set which will usually be a manifold. The charts we have been using earlier already apply coordinates to M, since for each $U \subseteq M$ mapping homeomorphically (and hence injectively) to $\varphi(U) \subseteq \mathbb{R}^n$, each point of U is represented by some tuple $(x^1, ..., x^n) \in \varphi(U)$. We will start with defining the notion more broadly, making it independent of a topology[3]

[3]For example, affine space (the space Euclid worked in, so to speak) is typically just seen as a collection of points, no topology or vector-space structure, so we might want want to think of this just being a set

Definition 1.1.8: (Set) Chart

Let M be a set. Then a *chart*, or *coordinate map*, on M is a subset $U \subseteq M$ bijectively onto an open subset of $\mathbb{R}^n$ for some n. To specify the dimension, we might say that an n-chart maps $U \subseteq M$ bijectively onto an open subset $\mathbb{R}^n$ for n in particular. Usually, we write the domain and the chart as a tuple: $(U, \mathbf{x})$ with $\mathbf{x} : U \to \mathbf{x}(U)$

Given a chart $(U, \mathbf{x})$, the domain U is usually called the *patch*, while the function is usually called the *coordinate map* or *coordinate function*.

Next, we have the problem that a chart in it of itself simply represents a part of a set. We need to establish a structure on the set that cover all of the set, and we would want the charts to be somehow "compatible" with each other:

Definition 1.1.9: C^r Atlas

Let M be a set, and let $\mathcal{A} = \{(U_\alpha, \mathbf{x}_\alpha)_{\alpha \in A}\}$ be a collection of n-charts. Then $\mathcal{A}$ is called an *n-atlas* of class C^r (or *C^r-atlas* for short) if:

1. $\bigcup_{\alpha \in A} U_\alpha = M$
2. For any $\alpha, \beta \in A$, $\mathbf{x}_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ (which includes the case where $\alpha = \beta$)
3. for any $\alpha, \beta \in A$, if $U_\alpha \cap U_\beta$ is non-empty, then:

$$\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1} : \mathbf{x}_\alpha(U_\alpha \cap U_\beta) \to \mathbf{x}_\beta(U_\alpha \cap U_\beta)$$

   is a C^r diffeomorphism

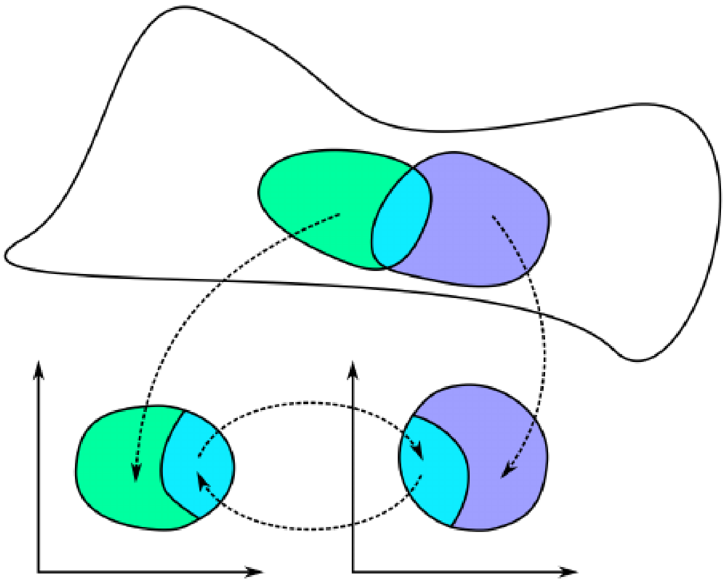
The maps $\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$ are called *transition maps*, *change of coordinate* or *overlap maps*. The charts are sometimes called C^r charts to emphasize that the transition maps formed betweeen them are C^r. If $r = \infty$, then the atals is sometimes called a *smooth atlas*, and the charts *smooth charts*.

Note that changing the condition on the transition maps changes the type of atlas we are working over. If we require the transition maps to be analytic, then the resulting atlas would be an *real analytic atlas*, often abbreviated to C^ω atlas. If we have an even dimension $\mathbb{R}^{2n}$ (so that $\mathbb{R}^{2n}$ is homeomorphic to $\mathbb{C}^n$) then if we require the transition maps to be complex analytic, then the resulting atlas a *complex analytic atlas*, often abbreviated to $\mathbb{C}^n$ atlas.

Notice that we have simplifed the notation in the definition:

$$\mathbf{x}_\beta|_{U_\alpha \cap U_\beta} \circ \mathbf{x}_\alpha^{-1}|_{\mathbf{x}_\alpha(U_\alpha \cap U_\beta)}$$

This will constantly be done, since the notation can become very overwhelming. The transition maps can be visualized like so:



In a sense, looking at coordinates and making sure the overlap is compatible is at the heart of working with atlases. For that reason, there are a few notational simplifications. Consider two charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ and $\mathbf{y} \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap V) \to \mathbf{y}(U \cap V)$. Then:

$$y^i(p) = y^i \circ \mathbf{x}^{-1}(p)(x^1(p), ..., x^n(p))$$

for any $p \in U \cap V$. Often, the (p) and $\circ \mathbf{x}^{-1}$ is often dropped and we write:

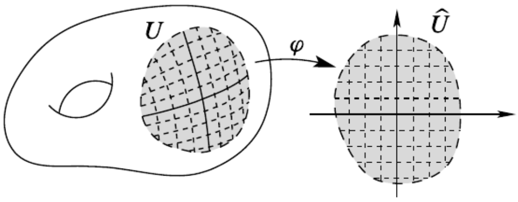
$$y^i = y^i(x^1, ..., x^n) \tag{1.2}$$

which leaves ambiguous whether the x^i's are numbers or functions. This ambiguity is sort of purposeful to emphasize how we can think of $(x^1, ..., x^n)$ as a coordinate representation of p (in a similar way we think of representing a vector with a basis). In particular, if the expression in equation (1.2) is true for all $p \in U \cap V$, it is true if we consider any point $(x^1, ..., x^n) \in \mathbf{x}(U \cap V)$. We have already seen this ambiguity when we worked with coordinates for the torus in example 1.2; recall that we wrote

$$x = (R + r\cos(\psi))\cos(\theta) \qquad y = (R + r\cos(\psi))\sin(\theta) \qquad z = \sin(\psi)$$

where x and y are the coordinate functions of a larger map (x, y) and ψ and θ are the coordinate functions of the larger map (ψ, θ).

More generally, when thinking of a chart $\mathbf{x} : U \to \hat{U}$, we can think of $\mathbf{x}$ as temporarily identifying U with $\hat{U}$, allowing us to work with points in U with it's coodinate representation from $\hat{U}$. We can imagine taking the grid in $\mathbb{R}^n$, and just putting it on our set. Now, picking a point on our grid is like picking a point on our set:



Example 1.7: Atlas's on Sphere

Let $M = S^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Usually we consider S^2 as having the subspace topology, however in this case consider S^2 as simply a set. Then we can split S^2 into 4 overlappings pieces, the upper-part, lower-part, left-part, and right-part, which project down to open intervals:

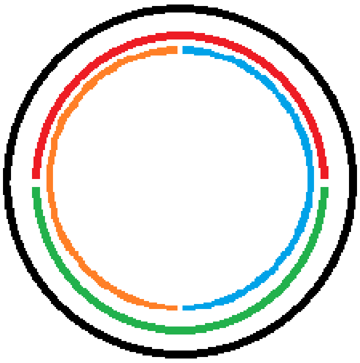


Figure 1.2: 4 colours representing the 4 altases

If we label these for parts as U, D, L, R (for up, down, left, right, respectively), then the maps:

$$\begin{aligned} \varphi_U &: U \to \mathbb{R}, \ \varphi_U(x,y) = x \\ \varphi_D &: D \to \mathbb{R}, \ \varphi_D(x,y) = x \\ \varphi_L &: L \to \mathbb{R}, \ \varphi_L(x,y) = y \\ \varphi_R &: R \to \mathbb{R}, \ \varphi_R(x,y) = y \end{aligned}$$

Then these 4 maps are all injective projection maps, and so clearly homeomorphic. Furthermore, $U \cap D = \emptyset$ and $L \cap R = \emptyset$, so checking transition maps comes down to checking $U \cap L$, $U \cap R$, and so on. Without loss of generality, we'll check $U \cap L$, in particular that $\varphi_U \circ \varphi_L^{-1} : \varphi_L(U \cap L) \to \varphi_U(U \cap L)$ is a diffeomorphism. The inverse map of φ_L^{-1} is

$$\varphi_L^{-1}(y) = (\sqrt{1-x^2}, y)$$

Hence:

$$\varphi_U \circ \varphi_L^{-1} = \sqrt{1-x^2}$$

Since $\varphi_L(U \cap L) = (0,1)$ and $\varphi_U(U \cap L) = (-1,0)$ are both open and well-defined on $\sqrt{1-x^2}$, then then the transition map is just the composition of polynomials, which we know to be diffeomorphisms these open domains/codomains. A similar logic works for the rest of the charts, showing they are all compatible, and hence this collection is an atlas

This process can be easily generalized for

$$S^n = \left\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_1^2 + \cdots + x_{n+1}^2 = 1\right\}$$

where we use $2(n+1)$ coordinate charts for the different parts of the sphere, and the result transition maps will be of the form x_i or:

$$\pm\sqrt{1 - \sum_{i \neq j} x_i^2}$$

Note that S^n can have many more atlases, in fact infinitely many. For example, we may subdivide the S^n into more pieces and verify he transition maps. For S^n, we can in fact also take very few charts and still have an atlas. Notably, the atlas $\{(\varphi_N, S^n - \{N\}), (\varphi_S, S^n - \{S\})\}$, containing a copy of S^2 minus the "north" and "south" pole, and φ_N, φ_S are the stereographic projections:

$$\varphi_N : S^n \to \mathbb{R}^{n-1} \quad \varphi_N(\vec{x}, t) = \frac{\vec{x}}{1-t}$$
$$\varphi_N : S^n \to \mathbb{R}^{n-1} \quad \varphi_S(\vec{x}, t) = \frac{\vec{x}}{1+t}$$

the intuition of this map coming from the image:

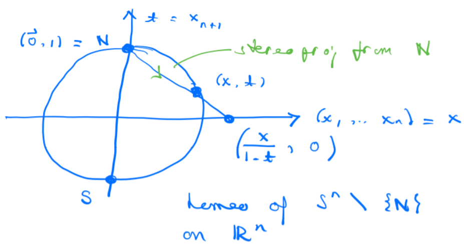


Figure 1.3: stereographic Projection

where the overlap between the two charts is all of S^n minus the north and south pole, or $\mathbb{R}^n - \{0\}$. The inverse of these maps can be calculated to be:

$$\varphi_N(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$
$$\varphi_S(u,v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2}\right)$$

and

$$\varphi_S \circ \varphi_N^{-1}(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

The transition maps show tha the two charts are in a sense "inverses" of each other. In the case of $n = 1$, the transition map is $x \mapsto x^{-1}$. These properties are important characterizations of S^2, and would be paticularly important later on in a course on complex analysis when studying the Riemann sphere.

As we have shown, a set can have more than one C^r-atlas, however, they both seem to define the same "differentiable structure" as in the transition maps between these separate atlas's form transitions maps that are diffeomorphisms. Thus, we have the following definition:

Definition 1.1.10: Compatible Atlas's

Let $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}$ be an n-dimensional atlas on M, and let (U, φ) be some chart. Then we say that (U, φ) is compatible with $\mathcal{A}$ if (U, φ) is compatible with every chart in $\mathcal{A}$, i.e., $\mathcal{A} \cup \{(U, \varphi)\}$ is an atlas. More generally, if $\mathcal{A}$ and $\mathcal{B}$ are atlases on M, then they are compatible if $\mathcal{A} \cup \mathcal{B}$ is an atlas on M.

It should be checked that compatability forms an equivalence class.

Example 1.8: Compatible

1. check that the projections charts and stereographic charts for S^1 given in the above examples are compatible.
2. Let $M = \mathbb{R}$ and take two atlases (R, id), $(\mathbb{R}, f(x))$ where $f(x) = x^3$ (verify that these are indeed atlases, note that we are still treating M as just a set). Then these two atlases are *not* compatible since $f^{-1}(x)$ is *not* a diffeomorphism, not being differentiable at x (equivalently, it is not full rank).

Note that a chart (U, φ) is compatible with an atlas $\mathcal{A}$ on M if and only if $\mathcal{A} \cup \{U, \varphi\}$ is also an atlas for M. Hence, it is no harm to take the atlas $\overline{\mathcal{A}}$ that contains all compatible atlasses:

Definition 1.1.11: Maximal Atlas

Let $\mathcal{A}$ be an atlas on M. Then $\overline{\mathcal{A}}$ is the atlas containing $\mathcal{A}$ and all compatible charts with $\mathcal{A}$.

Naturally, no larger atlas contains $\overline{\mathcal{A}}$, making it the maximal element set of atlas's on M ordered by inclusion. Thus, working with a maximal atlas makes our choice of atlas for M as "free from choice as possible" to make M as independent of an atlas as possible. Essentially, it will be akin to simply having an atlas (and hence maximal atlas) exist.

Definition 1.1.12: Differentiable Structure

Let M be a set. Then if $\mathcal{A}$ is a maximal atlas on M, we will call $(M, \mathcal{A})$ a *differentiable structure* on M. If the transition maps are C^∞, we will call this a *smooth structure*.

Notice that we can equivalently define a differentiable structure to be $(M, [\mathcal{A}])$, that is the set M together with the equivalence class of compatibles atlases. This can be desirable in proving a set has an atlas by proving it on a smaller atlas. In this way proving that S^1 has a differentiable structure will only require proving the two stareographic projection charts form an atlas. The fact that we can choose which atlas (i.e. coordinate system) to work with is "both a blessing and a curse", as John Lee puts it. On the one hand, we can choose very convenient coordinates that make computations are arguments much easier (as we'll show with examples, and we have already seen with S^1). However, as a consequence of this flexibility, we have to be careful that anything we define "globally" on a differentiable structure (i.e. on the entire differentiable structure, and not just on some or all open neigbhorhoods around each point $p \in M$) is not dependent on a particular choice of coordinate system. If we need to define something globally, we will either define it in a coordinate-dependent way and show it's independent of coordinate representation, or define it so that it is coordinate independent.

We have already seen an example where a set can have two differentiable structure (namely, example 1.8(2)). However, we will soon show that these two different differentiable structure are in some ways only "trivially" different, that is, once we introduce the category **Man** with differentiable manifolds and smooth maps between them, then we'll show they are *isomorphic*. We'll get back to this later in example 1.11. Once the notion of isomorphism of differentiable structures is introduced, we will ask if there are different differentiable structure on manifolds "up to isomorphism".

A differentiable structure is not necessarily a topological manifold since we have yet to show that M has a topology which is hausdorff and paracompact. We can at least solve the problem of M not having as topology by inducing one through the atlas:

Lemma 1.1.1: Atlas Induces Topology

Let M be a set with a C^r structure given by the atlas $\mathcal{A}$. If $(U, \mathbf{x}), (V, \mathbf{y}) \in \mathcal{A}$, then $(U \cap V, \mathbf{x}|_{U \cap V})$ and $(U \cap V, \mathbf{y}|_{U \cap V})$ are compatible with $\mathcal{A}$ (and hence, in the maximal atlas containing $\mathcal{A}$). Furthermore, if $O \subseteq \mathbf{x}(U)$ for some compatible chart $(U, \mathbf{x})$, then takign $V = \mathbf{x}^{-1}(O)$ gives the compatible chart $(V, \mathbf{x}|_V)$

It follows that the set of domains of charts (i.e. patches) from a maximal atlas forms a topological basis for a topology on M. The resulting topology is called the *topology induced by the C^r structure* on M, *topology induced by the atlas structure* on M, or the *manifold topology* on M if the C^r-structure is understood

***Proof* :**
exercsie

We may characterize open subsets $V \subseteq M$ by saying V is open if and only if $\mathbf{x}_\alpha(U_\alpha \cap V)$ is open for all charts in the atlas in any atlas for M giving the same C^r structure. Also, If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ is a subatlas, then the subatlases induce the same topology, and so if we consdier the maximal atlas $\overline{\mathcal{A}_1}$, we get the same topology for any subatlas of the maximal atlas. However, it is possiple that the different differential structures on a set induce the same topology: check that $(\mathbb{R}, [\mathrm{id}])$ and $(\mathbb{R}, [x^3])$ induce the same topology (namely, the euclidean topology), but as we know these are different differentiable structures.

Given the induced topology on a set M with a differentiable structure, then we have that each chart is an embedding, i.e. homeomorphic onto it's image by theorem 1.1.1, and so we gain that M is locally euclidean (and so M is "locally embedded" into $\mathbb{R}^n$ for appropriate n). We next see what properties do the charts need so that M is also Hausdorff and paracompact.

Proposition 1.1.4: Topology Is Manifold

Let $(M, \mathcal{A})$ be a C^r-differentiable structure. Then:

1. If for every two point $p, q \in M$, there exists two atlases $(\mathbf{x}, U)$ and $(\mathbf{y}, V)$ such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$, then the topology induced by $\mathcal{A}$ is Hausdorff
2. If $\mathcal{A}$ is countable, or has a countable subatlas, then the topology induced by the atlas is second-countable
3. If for every collection of chart domains $\{U_\beta\}_{\beta \in A} \subseteq \mathcal{A}$ and any fixed point $\beta_0 \in B$, the set $\{\beta \in B \mid U_{\beta_0} \cap U_\beta \neq \emptyset\}$ is at most countable, then the topology induced by $\mathcal{A}$ is paracompact. It follows that if M is connected, the topology induced by $\mathcal{A}$ is second countable

Proof **:**
Lee p. 14, prop 1.32

Definition 1.1.13: Differentiable Manifold

Let $(M, \mathcal{A})$ be differentiable structure with an atlas $\mathcal{A}$. Then if the topology induced by $\mathcal{A}$ is Hausdorff and paracomapct, then M is said to be a *differential manifold of class C^r of dimension n*, or a *C^r n-manifold* or simply a *C^r manifold.* We write $\dim(M)$ for the dimension of M

If $r = \infty$, then the resulting manifold is called a *smooth manifold.*

If the atlas $\mathcal{A}$ is understood, we will usually drop it and say that M is an n-manifold. If the dimension is also understood, we will simply say that M a manifold. Most of the time, we will be working with smooth manifolds. This is due to Hassler Whitney, who proved that every maximal C^r atlas contains a C^∞ atlas. Due to this, if we say that "M is a manfiold" we will mean that M is a *smooth* manifold. If instead of a differentaible or smooth structure, we have a C^ω or $\mathbb{C}^n$ structure, we would call the resulting manifold a *real analytic manifold* or *complex manifold* respectively.

It is valid to ask whether the topology of a manifold is in fact sufficient to induce a unique smooth structure, meaning we can make smoothness a topological property. This might seem the case: as we will show, $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ each have a unique smooth structure up to diffeomorphism: that is if $\mathbb{R}^1 \to \mathbb{R}^1$ is homeomorphic, it is diffeomorphic. In order to acheive this, we would have to show that if $f : M \to N$ is a homeomorphic if and only if it is diffeomorphic. This is in fact not the case: Karvaire-Milnon showed that there are 28 non-diffeomorphic structures on S^7. Kurby even found topological manifold were no smooth structure exists. Hence, smoothness is indeed a new structure on manifolds.

For reference we record the following definition:

Definition 1.1.14: Product Manifold

Let M, M' be an n- and n'-manifold with atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(U_\beta, \varphi_\beta)\}_{\beta\in B}$ respectively. Then $M \times M'$ is an $n+n'$-manifold with an atlas defined by $\{\varphi_\alpha \times \varphi_\beta, U_\alpha \times U_\beta\}_{\alpha\in A,\beta\in B}$ where $\varphi_\alpha \times \varphi_\beta : U_\alpha \times U_\beta \to \mathbb{R}^n \times \mathbb{R}^{n'}$ is given by $(\varphi_\alpha \times \varphi_\gamma)(p, q) = (\varphi_\alpha(p), \varphi_\beta(q))$

Note that the new atlas of a product manifold is not necessarily maximal (lots of open sets which are not in product form)

Now, we have defined a topological space M that can also be a topological manifold, and a set M that can have a topology induced on it that respects the conditions of being a topological manifold. What if M already had a topology? When would the two topologies match?

Proposition 1.1.5: Induced Topology Matching

Let $(M, \mathcal{T})$ be a topological space which also has a C^r atlas (i.e. $(M, \mathcal{A})$). Then if each chart domain of the atlas is open in $\mathcal{T}$ and is homeomorphic with respect to $\mathcal{T}$, then $\mathcal{T}$ has the same topology as the one induced by the C^r-structure

***Proof* :**
exercise

Example 1.9: Smooth Manifolds

It is clear that the topological manifolds we have worked with before are also smooth manifolds given the atlases we have presented in example 1.2. In fact, even the triangle can has a smooth structure put on it (which we show using proposition 1.2.8). In fact, that proposition makes the problem of finding a topological manifold that cannot have a smooth structure put on it equivalent to finding a topological manifold that is not homeomorphic to a smooth manifold. Apparently, the first example of a topological manifold on which we can't put a smooth structure on was found by Micheal Kervaire in 1960, and was a 10-dimensional compact manifold (p. 13 John M. Lee).

1. An easy source of examples of manifolds are open subsets of a manifold M with atlas $\mathcal{A}_M$. Such manifolds will be called *smooth open submanifolds of* M. Open submanifolds can sneakily appear in many cases: For example, we will show that any finite dimensional vector space is a manifold soon. There are a plethora of open subsets of finite dimensional vector spaces (given the usual norm inducing its topology) which give us interesting manifolds. More generally, the study of open submanifolds won't be done explicitely here, however they do play an important role in knot theory (Jeffrey Lee p. 16 bottom).

2. the product of two smooth manifolds is sill a smooth manifold

3. As mentioned earlier, $(\mathbb{R}, [\text{id}])$ and $(\mathbb{R}, [x^3])$ are smooth manifolds. More generally, $\mathbb{R}^n$ is certainly a smooth manifold

4. we have already proven that S^n has a chart in the previous example

5. Let G be a group. if G is also a smooth manifold where the binary operation m and the inverse map is smooth:
$$m : G \times G \to G, \qquad i : G \to G$$

then we call G a *lie group*. The quintesential examples are

(a) $\mathrm{GL}_n(\mathbb{R})$ with dimension n^2 (it's an open subset of $\mathbb{R}^2$, and the binary operation and the inverse are polynomials)

(b) $S^1 \subseteq \mathbb{C}$ with multiplication

(c) $\mathbb{R}^n$ with addition

6. Given any smooth function $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, the resulting graph $\Gamma(f)$ is a smooth manifold. We already saw that the graph of a continuous function is a topolological n-manifold (since the domain has dimension n, as we've shown in example 1.2). Furthermore, $\Gamma(f) = (U, f(U))$ is covered by a single chart $\pi : (U, f(U)) \to U$. This single chart by itself trivially forms an atlas, and hence it is a smooth manifold. More generally, we shall show in section 3.3.5 that M is locally the graph of a smooth function if and only if it is a smooth manifold.

1.1.6 Pertinent Examples of Manifolds

Single Points

As we saw, $\mathbb{R}^n$ for any $n \in \mathbb{N}$ is a manifold. Even when $n = 0$, we have a manifold: Let $M = \{m\}$ be a single point. Then we can map $\varphi : M \to \mathbb{R}^0$, $\varphi(m) = 0$. This map trivially satisfies the transition condition (being the only map, $\varphi \circ \varphi^{-1} = \mathrm{id}$), and it is trivially paracompact and Hausdorff. The fact that $\mathrm{id} : \mathbb{R}^0 \to \mathbb{R}^0$ is differentiable is vacuously true: if ψ reprsents the expression over which we are taking the limit, then it is vacuously true that:

$$\lim_{h \to 0} \psi(h) = 0$$

it is also vacuously true that the expression equals any other number, but we simply need that there exists a linear approxmiation (which is within the definition of ψ) that satisfies this property.

Vector Spaces

A fundamental result in linear algebra gives us that if V is a finite dimensional real vector space, say of dimension n, then $V \cong_{\mathbb{R}} \mathbb{R}^n$. Since $\mathbb{R}^n$ is a manifold and linear isomorphisms are diffeomorphisms, any finite dimensional real vector space is a manifold. To see this more precisely, let V be a finite-dimensional vector space over $\mathbb{R}$ with some basis $(e_1, e_2, ..., e_n)$. There is only one norm on V up to equivalence, and so only one topology induced by the norm. This topology is clearly a topological manifold (being countable and hausdroff). As is well known from linear algebra, there exits a linear isomorphsim $E : \mathbb{R}^n \to V$ defined by

$$E(x) = E(x^1, x^2, ..., x^n) = \sum x^1 e_i$$

Since this map is linear and the two topologies, match, it is easy to see that E is a homeomorphism, and so (V, E^{-1}) is a chart, and in fact an atlas for V. To find the compatible charts, let's say $(d_1, d_2, ..., d_n)$ is another basis for V with the corresponding isomorphism $D(x) = \sum_i x^i d_i$. Then the transition mapping would be:

$$D^{-1} \circ E(x) = \tilde{x}$$

where $\tilde{x} = (\tilde{x^1}, ..., \tilde{x^n})$ is determined by:

$$\sum_j = \tilde{x}^j d_j = \sum_i x^i e_i = \sum_{i,j} x^i A_i^j d_j$$

i.e. $\tilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\tilde{x}$ is an invertible linear transformation, and hence a diffeomorphism, and hence these charts are smoothly compatible. Thus, since any chart D^{-1} defined by a different basis produces a compatible chart, all smooth structure's on a vector space's are independent of basis.

We therefore have many spaces which are smooth manifolds, for example $M_{mn}(\mathbb{R})$ and $\mathrm{Hom}_{\mathbb{R}}(V, W)$. Furthermore, we know many interesting open subsets of these spaces: for example $\mathrm{GL}_n(\mathbb{R}) \subseteq M_{n^2}(\mathbb{R})$ is an open submanifold (are $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{O}(n)$ open submanifolds?[4])

1.1.7 Level sets

Let $\varphi : U \subseteq \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Then for any $c \in \mathbb{R}$, the set $M = \varphi^{-1}(c)$ is the fiber over c, also known as a *level set* of φ. If for each $a \in M$, $D\varphi(a)$ is full rank, by the implicit function theorem there is a neighbourhood U_0 of a such that $M \cap U_0$ can be expressed as a graph of an equation of the form

$$x^i = f(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n)$$

for some smooth real-valued function f defiend on an open subset of $\mathbb{R}^{n-1}$. Arguing just like for the n-sphere, we see that M is a topological manifold of dimension $n-1$, and has a smooth structure such that each of the graph coordinates charts associated with a choice of f as above is a smooth chart. We will generalize this result in section 1.3.

Real Projective Space

We saw that $\mathrm{Hom}_{\mathbb{R}}(V, W)$ is a smooth manifold since it is a vector-sapce. If we limit to linear isomorphissm from V to itself, we write $\mathrm{GL}(V)$. As we showed earlier this is an open subet of $M_{n\times n}(V)$ and so a manifold.

This shows that the symmetries of vector-space V offers a lot of geometry! Furthermore, $V \ncong \mathrm{GL}(V)$, since their dimensions don't match, showing that we can construct a different "geometry" given V. There is in fact another geometrical structure we can create using V that is (in most cases) not the same geometric structure as V: consider the set of all k-dimensional subspaces of an n-dimensional vector space V (over $\mathbb{R}$). We will show that this set is in fact a smooth manifold! It will turn out that the geometry of this set is intimately related to the geometry of "prespective" in the sense that in human vision, two parallel lines seem to converge at infinity. This type of geometry is called *projective geometry*. The details of this geometry is not covered, but know that the collection of k-dimensional subspaces has this geometry, which is why it is given the name *real projective space*.

To start with, we will define the space when we only consider 1-dimensional subspaces:

[4]It will turn out that they are not, but will be *regular submanifolds* which we shall explore soon

Definition 1.1.15: Real Projective Space

Define
$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$
where
$$x \sim y \qquad \Leftrightarrow \exists \lambda \in \mathbb{R} \text{ such that } x = \lambda y$$
and give it the quotient topology. Alternatively, let $\mathbb{R}^\times \curvearrowright (\mathbb{R}^{n+1} - \{0\})$ by left multiplication and take the orbit space $(\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times = \mathbb{RP}^n$.

Note that $\mathbb{RP}^n$ is certainly paracompact, since $\mathbb{R}^{n+1}$ is paracompact, and it can easily be proven that the image of a paracompact space is paracompact. To show it's Hausdorff, if we take the group-action, first prove that the orbit space is closed if and only if the image of the action is closed. In particular, $\rho : G \times M \to M$ is a group action, G has the discrete topology, and $\rho(G, M)$ is the image of the action, then M/G is Hausdroff if and only if $\rho(G, M)$ is closed (This is similar to how X is Hausdorff if and only if $\Delta \subseteq X \times X$ is closed). Then $\rho(\mathbb{R}^\times, \mathbb{R}^{n+1} - \{0\})$ is the pre-image of:
$$f(g, x) = \det(x - gx)$$
Equivalently, we can show that $\Gamma_\sim = \{(x, y) \in X \times X \mid x \sim y\}$ is closed, which we get by a finite union of sets of the form:
$$f_{ij} : \mathbb{R}^{n+1} \setminus \{0\} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^n \qquad f_{ij}(x, y) = (x_i y_j - x_j y_i)_{i \neq j}$$
The equivalence classes of this quotient space is sometimes denoted:
$$(x^0 : x^1 : ... : x^n) := [x] = [(x^0, x^1, ..., x^n)]$$
to emphasize that this is really an n-dimensional manifold, not an $n+1$ or $m < n$ dimensional manifold, as we'll soon show. The representation $(\cdot : ... : \cdot)$ is called the *homogeneous coordinates* of $[x]$. Yet another way of looking at $\mathbb{RP}^n$ is by identifying antipodal points on S^n; the standard atlas on S^n induces a n-dimensional smooth atlast on $\mathbb{RP}^n$

Proposition 1.1.6: Projective Space Is A Manifold

$\mathbb{RP}^n$ is a manifold

***Proof* :**
We arleady showed Hausdorffness, paracompactness, so it suffices to find an atlas. Consider the following atlas:
$$\mathcal{A} : \{(U_0, \varphi_0), ..., (U_n, \varphi_n)\}$$
where:
$$U_i = \left\{(x^0 : ... : x^n) \in \mathbb{RP}^n \;\middle|\; x^i \neq 0\right\}$$
and define:
$$\varphi_i : U_i \to \mathbb{R}^n, \ (x^0 : ... : x^n) \mapsto \left(\frac{x^0}{x^j}, ..., \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, ..., \frac{x^n}{x^j}\right)$$
Essentially, given some $[x] \in \mathbb{RP}^n$ where the ith coordinate is nonzero, we can re-scale the representative x of $[x]$ to make the ith coordinate 1, and use the remaining components as our

coordinates. Notice that:

$$\varphi_1(7:3:2) = \varphi_1\left(\frac{7}{3}:1:\frac{2}{3}\right) = \left(\frac{7}{3},\frac{2}{3}\right)$$

Then it is immediate that φ_i is bijective, and given the quotient topology, it is certainly homeomorhic.

Next, to find the tarnsition map, first note that:

$$\varphi_j^{-1}(x^1, x^2, ..., x^n) = (x^1 : \cdots x^{j-1} : 1 : x^{j+1} : \cdots : x^n)$$

Thus, we get:

$$\begin{aligned}\varphi_i \circ \varphi_j^{-1}(x^1, x^2, ..., x^n) &= \varphi_i(x^1 : \cdots x^{j-1} : 1 : x^{j+1} : \cdots : x^n) \\ &= \left(\frac{x^0}{x^i}, ..., \frac{x^{j-1}}{x^i}, \frac{1}{x^i} \frac{x^{j+1}}{x^i}, ..., \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, ..., \frac{x^n}{x^i}\right)\end{aligned}$$

These maps are certainly smooth. Hence, $(\mathbb{R}\mathrm{P}^n, \mathcal{A})$ is a smooth n-manifold.

We can imagine $U_i \subseteq \mathbb{R}\mathrm{P}^n$ to consist of all the lines which intersect the affine plane H_i:

$$H_i = \left\{x \in \mathbb{R}^{n_1} \;\middle|\; x^i = 1\right\}$$

This gives us a helpful too on understanding $\mathbb{R}\mathrm{P}^n$. In particular, since $H_i \cong \mathbb{R}^n$, we get:

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}\mathrm{P}^{n-1}$$

Continuing inductively, we get:

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0$$

Note that $\mathbb{R}\mathrm{P}^n$ somehow manages to combine all of these since it is an n-manifold (recall that a manifold has a consistent dimension, and so $\mathbb{R}\mathrm{P}^n$ is not n, $n-1$,..., 1, 0 dimensional all at once).

Complex Projective Space

Similarly to the previous example, we can do the same construction while using $\mathbb{C}$:

Definition 1.1.16: Complex Projective Plane

Define the *complex projective plane* to be

$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

where

$$x \sim y \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{C} \text{ such that } x = \lambda y$$

with the quotient topology.

Roughly, we can picture $\mathbb{C}\mathrm{P}^n$ in the following way: Identifying $\mathbb{C} \cong \mathbb{R}^2$ in the usual way, we get $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. Form this, we can take $S^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ to be the "unit sphere" of complex vectors of length $\|z\| = 1$, giving us:

$$\mathbb{C}\mathrm{P}^n = S^{2n+1}/\sim$$

where $z \sim z'$ if and only if there exists a scalar λ such that $z = \lambda z'$. In fact, λ will be unique in this case and $\|\lambda\| = 1$. Similarly to the real projective plane, it is Hausdorff and paracompact for the same reason. To show it's a smooh manifold, we can define:

$$U_i = \left\{(z^0 : ... : z^n) \in \mathbb{R}\mathrm{P}^n \;\middle|\; z^i \neq 0\right\},$$

and define:

$$\varphi_i : U_i \to \mathbb{C}^n = \mathbb{R}^{2n},\ (z^0 : ... : z^n) \mapsto \left(\frac{z^0}{z^j}, ..., \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, ..., \frac{z^n}{z^j}\right)$$

and just like for $\mathbb{R}\mathrm{P}^n$, the transition maps are smooth. In particular, (just replace x with z). Thus, $\mathbb{C}\mathrm{P}^n$ is a smooth manifold of dimension $2n$. Just like for the real case, we can decompose this manifold into:

$$\mathbb{C}\mathrm{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \mathbb{C}^-$$

We have already shown in the previous example that $S^n \subseteq \mathbb{R}^{n+1}$ which is the set

$$S^n = \left\{(x_1, ..., x_{n+1}) \;\middle|\; \sum x_i^2 = 1\right\}$$

is a manifold. As a reminder of the proof that it is a manifold:

Proof. Given the north pole to be the starting pont, we had that

$$y = \frac{x}{1-t}$$

and projection from the south gives us:

$$z = \frac{x}{1+t}$$

and the transition mapping means we have to write z in terms of y,that is:

$$z = \frac{1-t}{1+t} y$$

And when $n = 2$, we get S^2 which the set where $x^2 + y^2 = 1$, and

$$yz = \frac{x^2}{1-t^2} = 1 \quad \Rightarrow z = \frac{1}{y}$$

□

Grassmanian

In $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, we saw it was equivalent to think of them as the colleciton of all 1-dimensional subspaces of $\mathbb{R}^n$ or $\mathbb{C}^n$. We can generlize this concept:

Definition 1.1.17: Grassmannian

Let V be an n-dimensional vector space. Then for any $0 \leq k \leq n$, let $G_k(V)$ denote the set of all k-dimensional subspaces

We will show that $G_k(V)$ can be given the structure of a smooth manifold (of dimension $k(n-k)$), and so we will foten call $G_k(V)$ the *Grassmann manifold*. Often, we will work with $V = \mathbb{R}^n$ (i.e. V with the standard basis), and so we shorthand $G_k(\mathbb{R}^n) = G_{k,n} = G(k,n)$. This generlizes $\mathbb{RP}^n$ and $\mathbb{CP}^n$ since $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$ and $G_1(\mathbb{C}^n) = \mathbb{CP}^n$.

We first construct the charts. Pick a k-dimensional subspace $P \subseteq V$, and choose the complimentary subspace Q, so that $P \oplus Q = V$ and $\dim(P) = k$, $\dim(Q) = n-k$. Then the graph of any linear map $T : P \to Q$ can be identified with a k-dimensional subspace $\Gamma(T) \subseteq V$, and is equal to the set:

$$\Gamma(T) = \{x + T(x) \mid x \in P\}$$

Notice that any such subspace for any T has the property that it intersects Q trivially. Conversely, any subspace $S \subseteq V$ that intersects Q trivially is the graph of a unique linear map $T : P \to Q$. Notice that we can define π_P and π_Q to be the projection of V to P and Q respectively. Then by assumption of $S \cap Q = \{0\}$, we se that $\pi|_S$ is an ismorphism from S to P. Thus: $T : (\pi_Q) \circ (\pi_P|_S)^{-1}$ is a well-defined linear map from P to Q, and it's easy to check that S is it's graph.

Now, let $\mathcal{L}(P,Q)$ be the collection of all linear transformations from P to Q. Recall that $\mathcal{L}(P,Q) \cong \mathbb{R}^{k(n-k)}$. Let $U_Q \subseteq G_k(V)$ be collection of k-dimensional subspaces of V whose intersection with Q is trivial. Then by what we've just shown, $\Gamma : \mathcal{L}(P,Q) \to U_Q$ sending $T \mapsto \Gamma(T)$ is well-defined, and a bijection. Since it's bijective, the map $\varphi = \Gamma^{-1} : U_Q \to \mathcal{L}(P,Q) \cong \mathbb{R}^{k(n-k)}$ is well-defined. Since $\varphi(U_Q) = \mathcal{L}(P,Q)$, we see that the imge of this chart is a open.

Next, let (P', Q') be another pair of complimentary subspaces. Let $\pi_{P'}$, $\pi_{Q'}$ be the corresponding projeciton maps, and $\varphi' : U_{Q'} \to \mathcal{L}(P', Q')$ the corresponding chart. Then the set $\varphi(U_Q \cap U'_Q) \subseteq \mathcal{L}(P,Q)$ consists of all linear maps $T : P \to Q$ whose traph intersects Q' trivially. We want to show this set is open (in $\mathcal{L}(P,Q)$). For each $T \in \mathcal{L}(P,Q)$, let $I_T : P \to V$ be the map $I_T(v) = v + T(v)$, which is a bijection from P to the graph of T. Since $\Gamma(T) = \operatorname{im}(I_T)$, and $Q' = \ker(\pi_{P'})$, it follows (exercise B.22(d) in John Lee, i.e. a linear algebra fact) that that the graph of T intersects Q' trivially if and only if $\pi_{P'} \circ I_T$ has full rank. Since the matrix entries of $\pi_{P'} \circ I_T$ (with respect to any basis) depend continuously on T, the result of example(matrices of full rank are open submanifolds) shows that the set of all suh T is is open in $\mathcal{L}(P,Q)$, and hence $\varphi(U_Q \cap U_{Q'})$ is open.

Next, we need to show that $\varphi \circ \varphi'$ is smooth on $\varphi(U_Q \cap U_{Q'})$. (The proof finishes in John Lee p. 23-24, need more reason to fully understand this example)

Complex Manifolds

Finally, for those who have taken complex analysis, the notion of differentiability is also defined on $\mathbb{C}$, and $\mathbb{C}^n$ more generally. Using this, we see that we can define a *complex n-manifold* given that all the charts map to $\mathbb{C}^n$. Such a manifold is always alo a real manifold, simply because $\mathbb{C}$ is a real smooth 2-manifold! This can be accomplisehd with a single chart, namely the natural correspondence $\mathrm{id} : \mathbb{C} \to \mathbb{R}^2$ (note that it is in a sense "vacuously" differentiable, since there is a single chart covering it)! Hence all complex n-manifolds are automatically real $2n$-manifolds!

1.1.8 Smooth Manifolds with Boundary

Many surface that we deal with have a boundary. For example, a closed disk (say D^2) is the open disk (a 2-dimensional manifold), unioned with S^1. This is certainly not a manifold: any $U \subseteq D^2$ containing part of the boundary of D^2 cannot be homeomorphic to any part of $\mathbb{R}^2$ ($\varphi(U)$ is neither open or closed in $\mathbb{R}^2$). Hence, we consider this case as a new type of object.

Differentiability is a little harder to define since differentiability is defined on open subsets of $\mathbb{R}^n$, and subsets of $\mathbb{H}^n$ that touch the bounddary or not open with respect to $\mathbb{R}^n$. However, we can work around this by a natural extension of these differentiability:

Definition 1.1.18: Differentiable Over $\mathbb{H}^n$

Let $f : \mathbb{H}^n \to \mathbb{R}$ be a function. Then f will be said to be differentiable if f is differentiable on $\operatorname{int}(\mathbb{H}^n)$ and all partial derivatives extend continuously to the boundary. Equivalently, there exists a smooth extension of f that extends to $\partial\mathbb{H}^n$.

Note that such a notion of differentiability is strictly weaker than the usual one. In particular, an open subset $U \subseteq \mathbb{H}^n$ (with respect to the subspace topology on $\mathbb{H}^n$) containing boundary points cannot be diffeomorphic to a subset of $\mathbb{R}^n$ (by the inverse function theroem). Using this definition of differentiability, we can generlize the notion of differentiable between $f : \mathbb{H}^n \to \mathbb{H}^m$. Notice that $\partial\mathbb{H}^n$ must map to the boundary of $\mathbb{H}^m$, that is $f(\partial\mathbb{H}^n) \subseteq \partial\mathbb{H}^m$. Due to this, we can think of ∂M to be the set of points such that the coordinate charts map to ∂H^n. Furthermore, given $\dim M = n$, then $\dim \partial M = n - 1$.

With the notion of boundary established, we can develop further the notion of "gluing" two manifolds. Let's take $T^2 = S^1 \times S^1$, and remove some open disk from it's surface, call this manifold M. Let's take two copies of this manifold. Can we glue together these manifolds along the boundary and get a 2-manifold (without boundary)? Certainly, the two boundary need to be diffeomorphic for this to work, however we need a bit more: for any $x \in \partial M$, there exists a neighborhood containing x that is diffeomorphic to $\partial M \times [0, 1)$. We will get back to this concept in more detail soon. TBD.

1.1.9 Oriented Manifold

Definition 1.1.19: Oriented Manifold

Let $(M, \mathcal{A})$ be a smooth manifold. Then $(M, \mathcal{A})$ is a smooth manifold if all the transition maps from $\mathcal{A}$ have a jacobian matrix with a positive determinant, i.e.

$$\det(D(\psi \circ \varphi^{-1}(x))) > 0$$

There is no good reason for choosing the positive instead of the negative jacobian, we may simply compose each map in the atlas with a diffeomorphism which reverses the jacobian determinant to make them all negative. As long as they are *consistent* in orientation.

Famously, S^2 is orientable while the mobius strip is not.

1.2 Smooth map

So far, we have been working with differential structures on sets (or manifolds). As usual, we might want to ask when are two differential structure the same up to a suitable isomorphism. We would essentially want that if M and N are both manifolds with a differentiable structure, then a map between them is a topological map (hence continuous), and a map that let's us compare transition maps between their respective atlas. The topological condition is naturally not necessary if M and N are arbitrary sets, but we can always induce a topology with a differential structure and hence the definition will be made to accomodate this fact. The definition will be coordinate independent since we will require that all the patches satisfy the same condition.

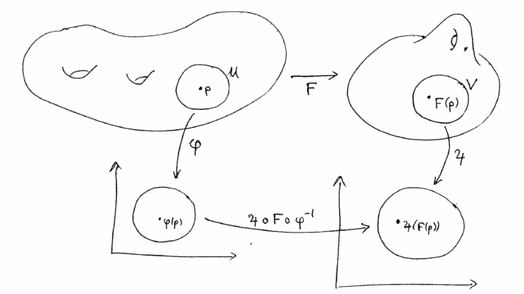
Definition 1.2.1: Smooth Map

Let M, N be two manifolds (not necessarily of the same dimension) with C^r atlas $\{(\varphi_\alpha, U_\beta)\}_{\alpha\in A}$ and $\{(\psi_\beta, V_\beta)\}_\beta$ respecitvely, and $p \in M$. Let $f : M \to N$ be function. Then f is a C^r *map at* p (resp. a *smooth map at* p) if for every chart $(\varphi_\alpha, U), (\psi_\beta, V)$ containing where $p \in U$ and $f(U) \subseteq V$, the map (called the *transition map*):

$$\psi_\alpha \circ f \circ \varphi_\beta^{-1} : \varphi_\alpha(U) \to \psi_\beta(V)$$

is C^r. If $\psi_\alpha \circ f \circ \varphi_\beta^{-1}$ is smooth for all $p \in M$, then f is said to be a *smooth map.*

Visually, this can be seen as:



Notice that as a consequence of this definition, any smooth chart is in fact a smooth map: if M is a smooth n-manifold with chart $(U, \mathbf{x})$ (where we can consider U an open submanifold), then the map $\mathbf{x} : U \to \mathbb{R}^n$ is smooth since:

$$\mathrm{id} \circ \mathbf{x} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbf{x}(U)$$

is the identity map, which is clearly smooth. Any other chart intersection U will have the same result. In effect, each patch can be considered smooth. In fact, the same proof works in reverse to show that $\mathbf{x}^{-1} : \mathbf{x}(U) \to U$ is also smooth, since:

$$\mathbf{x} \circ \mathbf{x}^{-1} \circ \mathrm{id} : \mathbf{x}(U) \to \mathbf{x}(U)$$

The result of this is that we can think of each coordinate chart are showing that a smooth manifold is *locally diffeomorphic* to $\mathbb{R}^n$, in a very similar way to how topological manifolds are locally homeomorphic to $\mathbb{R}^n$. In fact, we can think of the differential structure on M as *defining* the charts to be smooth, since *a priori* we have not a notion of smoothness on non-open sets of $\mathbb{R}^n$

Note that the definition specifies that every transition map $\psi_b \circ f \circ \varphi_a^{-1}$ where $p \in U$ and $f(U) \subseteq V$ must be smooth. However, it suffices to check for one map that contains p, since if $(U', \mathbf{x}')$ and $(V', \mathbf{y}')$ are charts where $p \in U'$ and $f(U') \subseteq V'$, then

$$\begin{aligned}
\mathbf{y}' \circ f \circ (\mathbf{x}')^{-1} &= \mathbf{y}' \circ (\mathbf{y}^{-1} \circ \mathbf{y}) \circ f \circ (\mathbf{x}^{-1} \circ \mathbf{x}) \circ \mathbf{x}' \\
&= (\mathbf{y}' \circ \mathbf{y}^{-1}) \circ (\mathbf{y} \circ f \circ \mathbf{x}^{-1}) \circ (\mathbf{x} \circ \mathbf{x}')
\end{aligned}$$

which is diffeomorphic since overlapping maps are diffeomorphic!

Definition 1.2.2: Collection Of C^r Maps

Let M and N be two C^r-manifolds. Then $C^r(M, N)$ is the collecion of all C^r maps between these two manifolds. If $N = \mathbb{R}$, we often abbreviate $C^r(M, \mathbb{R})$ to $C^r(M)$.

It is easy to see that $C^r(M, \mathbb{R})$ and $C^r(M, \mathbb{C})$ are $\mathbb{R}$-algebras and $\mathbb{C}$-algebras respectively, with addition and multiplication done point-wise, and scalar multiplication done point-wise.

As we have emphasized when working with coordinate charts and transition maps, we can think of the points of a manifold M as being identified with a open subset of $\mathbb{R}^m$. In the case of there being a smooth map $f : M \to N$, we can represent f by how f acts on it's patches:

Definition 1.2.3: Representative Maps

Let $f : M \to N$ be a C^r map, and suppose that $(U, \mathbf{x})$ and $(V, \mathbf{y})$ are admissible charts for M and N respectively. Then if $f^{-1}(V) \cap U \neq \emptyset$, the composition:

$$\mathbf{y} \circ f \circ \mathbf{x}^{-1} : \mathbf{x}(f^{-1}(V \cap U)) \to \mathbf{y}(V)$$

is called the *local representative map of f* [a]

[a] Note that we have yet to show that $f^{-1}(V) \cap U$ is open. We shall do so in proposition 1.2.1

We will often abuse language and say that f with resepct to coordintes $\mathbf{x}$ and $\mathbf{y}$ is $f(x^1, ..., x^n) = (y^1, ..., y^m)$ instead of $\mathbf{y} \circ f \circ \mathbf{x}^{-1}(x^1(p), ..., x^1(p)) = (y^1(p), ..., y^1(p))$. This abuse is very common, and emphasizes thinking of f as a function on "coordinates" on the manifolds.

We would like to find some nice properties of smooth maps that we can use repeatedly. The first thing to note is that if $f : M \to N$ is a smooth map, then this implies that f is continuous:

Proposition 1.2.1: Smooth Map Then Continuous

Let $f : M \to N$ be a smooth function. Then f is continuous.

Proof **:**
We'll show f is continuous at every point $p \in M$. Since f is smooth, there exists charts $(U, \mathbf{x})$, $(V, \mathbf{y})$ on M and N respectively such that $f(U) \subseteq V$ such that $\mathbf{y} \circ f \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbf{y}(V)$ is

smooth, and hence continuous. Since $\mathbf{x}$ and $\mathbf{y}$ are homeomorphic onto their image:

$$f|_U = \mathbf{y}^{-1} \circ (\mathbf{y} \circ f \circ \mathbf{x}^{-1}) \circ \mathbf{x} : U \to V$$

is a composition of continuous maps, and hence $f|_U$ is continuous at p, implying f is continuous at p. Since p was arbitrary, f is continuous, as we sought to show.

For this proposition to work, it is essential that $f(U) \subseteq V$. If it wasn't, the get the following scenario:

Example 1.10: Almost Smooth Map

Consider:

$$f : \mathbb{R} \to \mathbb{R} \qquad f = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then for every $x \in \mathbb{R}$, there are coordinate charts $(\mathbf{x}, U)$ containing x and $(\mathbf{y}, V)$ containing $f(x)$ where it is not necessary that $f(U) \subseteq V$ such that $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$ is smooth. It is easy to see this if $x \neq 0$ (pick $U = (x - \frac{x}{2}, x + \frac{x}{2})$, and $V = \mathbb{R}$ with smooth charts being identity). Thus, consider when $x = 0$. Pick $((1,3), \mathbf{x}_{-2})$ where $\mathbf{x}_{-2}(t) = t - 2$. Thus $\mathbf{x}_{-1}((1,3)) = (-1,1)$. For the codomain chart, pick $((0,2), \mathrm{id})$. Then:

$$\mathbf{x}_{-2}((1,3) \cap f^{-1}((0,2)) = (-2,0)$$

Thus, our map is

$$\mathrm{id} \circ f \circ \mathbf{x}_{-2} : (-2,0) \to (0,2) \quad \mathrm{id} \circ f \circ \mathbf{x}_{-2}(t) = 1$$

which is certainly smooth. However, f is not a smooth map, not being continuous (as we've shown f must be in proposition 1.2.1)

Thus, necessitating $f(U) \subseteq V$ forces continuity of f. If we ommited the condition that $f(U) \subseteq V$, then we would have to change the definition so that *all* charts on the *maximal atlas* must satisfy the need for transition maps being smooth. Note how that would break the previous counter-example since the codomain of $(-10, 10)$ would make f a non-smooth function (in fact, many textbooks give this definition). For the textbooks in which it is not necessary that $f(U) \subseteq V$, it is not necessary that f is continuous. However, f being continuous is a sufficient condition to upgrade this alternative definition to our current one. Alternatively, if $f^{-1}(V_\beta \cap U_\alpha)$ is always open, this is an equivalent definition of smoothness, since it makes sure that there exists a patch U' such that for some patch V, $f(U') \subseteq V$:

Proposition 1.2.2: Smoothness Equivalence

Define smoothness by saying that every representation must be smooth (so that it need not be that $f(U) \subseteq V$). Let $f : M \to N$. Then if one of the following are satisfied, then f is smooth (or at least C^r)

1. Let $f : M \to N$ be continuous, and let $\{(U_\alpha, \mathbf{x}_\alpha)_{\alpha\in A}$ and $\{(V_\beta, \mathbf{y}_\beta)_{\beta\in B}$ be (not necessarily maximal) C^r atlases for M and N respectively. Then f is C^r if and only if for each α and β, the representative map:
$$\bar{f} = \mathbf{y}_\beta \circ f \circ \mathbf{x}_\alpha^{-1} : \mathbf{x}_\alpha(f^{-1}(V_\beta \cap U_\alpha)) \to \mathbf{y}_\beta(V_\beta)$$
is C^r
2. For every $p \in M$, if all charts $(U, \mathbf{x})$ contianing p and $(V, \mathbf{y})$ containing $f(p)$ such that $U \cap f^{-1}(V)$ is open and the composition map
$$\mathbf{y} \circ f \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap f^{-1}(V)) \to \mathbf{y}(V)$$
is smooth, then f is smooth

***Proof* :**
easy proof that you have to do to know you're comfortable with these notions

One key fact to remember while proving a map is smooth is that we only need to check enough charts so that the entire domain is covered. What makes this possible is that any overlap is a *diffeomorphism*. Hence, if we have another pair of charts $\mathbf{x}'$ and $\mathbf{y}'$, where the assocaited patch for $\mathbf{x}'$ has the same point as $\mathbf{x}$, then:

$$\begin{aligned}\mathbf{y}' \circ f \circ \mathbf{x}'^{-1} &= \mathbf{y}' \circ (\mathbf{y}^{-1} \circ \mathbf{y}) \circ f \circ (\mathbf{x}^{-1} \circ \mathbf{x}) \circ \mathbf{x}'^{-1} \\ &= (\mathbf{y}' \circ \mathbf{y}^{-1}) \circ (\mathbf{y} \circ f \circ \mathbf{x}^{-1}) \circ (\mathbf{x}'^{-1} \circ \mathbf{x})^{-1}\end{aligned}$$

More generally, if we want to prove a property X of a map f, then if the propety is invariant under diffeomorphism, then it suffices it to prove it on the transition maps $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$, since $\mathbf{x}$ and $\mathbf{y}$ are diffeomorphisms (ex. being an open maps is an easy example of an invariant).

Next, we emphasize how smoothness is a local property:

Proposition 1.2.3: Smoothness is Local

Let $f : M \to N$ be a map. Then the following are equivalent:

1. If every point $p \in M$ has a neigbhorhood U such that $f|_U$ is smooth, then f is smooth
2. If f is smooth, then the restriction of f to any open set U, $f|_U$, is smooth.

we can use this proposition to construct a map by just defining what happens locally on a manifold, though the gluing is quite restrictive:

Corollary 1.2.1: Gluing Smooth Maps

Let M, N be smooth manifolds, $\{U_\alpha\}_{\alpha \in A}$ an open cover for M. Suppose that for each $\alpha \in A$, there exists a map $f_\alpha : U_\alpha \to N$ such that for all β such that $U_\alpha \cap U\beta \neq 0$,

$$(f_\alpha)|_{U_\alpha \cap U_\beta} = (f_\beta)|_{U_\alpha \cap U_\beta}$$

Then there exists a unique smooth map $f : M \to N$ such that $f|_{U_\alpha} = f_\alpha$.

When we discuss partition's of unity, we'll show there is a more flexible way of defining maps that can be glued together.

Proposition 1.2.4: Properties Of Smooth Maps

Let M, N, and P be smooth manifolds. Then:

1. Every constant map $c : M \to N$ is smooth
2. The identity map $\mathrm{id} : M \to M$ is smooth
3. If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota : U \hookrightarrow M$ is smooth
4. If $f : M \to N$ and $g : N \to P$ is smooth, then so is $g \circ f$.

***Proof* :**
exercsie (d is proven in John Lee p. 36)

Note that if $g \circ f$ is smooth, this does not imply either g or f is smooth (think of a non-smooth bijective function, so $f \circ f^{-1} = \mathrm{id}$, where the identity function is smooth, but neither f or f^{-1} is). This might seem trivial to point out, however there will be examples soon where there will be strong enough conditions on the composition $g \circ f$ and on f to force g to be smooth! (see theorem ref:HERE)

Since the composition of smooth maps is smooth, this gives us that the collection of smooth manifolds with smooth maps is a category, usually labeled as **Man** (Sometimes also labeled as $\mathbf{Man}^p$ or $\mathbf{Man}^\infty$ to specify whether it is a C^r or smooth manifold).

Proposition 1.2.5: Projection Map Is Smooth

Let $M_1, M_2, ..., M_k$ be a collection of manifolds and $M_1 \times M_2 \times \cdots \times M_k$ be the product manifold. Then the natural projection map:

$$\pi_i : M_1 \times M_2 \times \cdots \times M_k \to M_k$$

is smooth. Furthermore, a map $f : M \to \prod_i N$ is smooth if $\pi_i \circ f$ is smooth for each i.

***Proof* :**
exercsie (John less left it as an exercse)

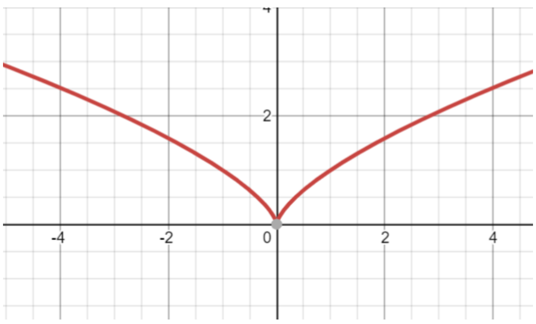
Example 1.11: Smooth Maps

1. Any map from a 0-manifold to a n-manifold (with or without boundary) is always smooth, since it's always a constant map.
2. Define $f : \mathbb{R} \to S^1$ by $x \mapsto e^{2\pi ix}$. Given the angle corodinate map θ for S^1, we have coordinate representation $\hat{f}(x) = 2\pi x + c$ for some constant c, showing this map is certainly smooth.
3. Generalizing, the map $\mathbb{R}^n \to \mathbb{T}^n$ mapping $(x_1, ..., x_n) \mapsto (e^{2\pi ix_1}, ..., e^{2\pi ix_n})$ is smooth.
4. $r_\theta : S^2 \to S^2$ where
$$r_\theta(x, y, z) = (x \cos\theta y \sin(\theta), x \sin(\theta) + y \cos(\theta), z)$$
5. The map $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth (check the north/south coordinate charts are indeed smooth).
6. The quotient map $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth.
7. Is $\mathbb{RP} \cong S^1$?
8. Show that $\mathbb{RP}^2$ is indeed diffeomorphic to the open disk and mobius strip
9. show that this starts breaking down for $\mathbb{RP}^3$ and $\mathbb{RP}^n$ for $n \geq 3$ more generally

One common property that is proved in most categories is that if X is an object of the category, then so is $f(X)$. However, one strange property C^r/smooth maps is that if $f : M \to N$ is a smooth map, then $f(M)$ need not be a submanifold of N!

Example 1.12: $f(M)$ Not Manifold

Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $t \mapsto (t^2, t^3)$. Visually, the image of this map looks like:



Then f is a smooth map, since the atlases on $\mathbb{R}$ and $\mathbb{R}^2$ are the identity atlases, so we only need that f be a C^∞ map, which is true if and only if every component is C^∞, which they certainly are.

However, $f(\mathbb{R})$ is not a manifold, namely it "drops" dimensions at the point $t = 0$. Such a point is called the *crtical* point, and will be more extensively studed later to "fix" the fact that the image of a smooth map need not be a manifold.

You might think that adding the condition that f has to be "full rank" (i.e., all transition map have to be full rank), can solve this problem, but it in fact does not. If we take $g : \mathbb{R} \to \mathbb{T}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ by $t \mapsto [(t, \alpha t)]$ for some irrational α, then $g(\mathbb{R})$ is dense in $\mathbb{T}^2$, but it's not surjective, and hence it cannot be a submanifold (from here)

If we restirct to compact manifolds, then given an injective map from a compact n-manifold $f : M \to \mathbb{R}^m$ where $n \leq m$, $f(M)$ is a manifold.

This failure of an image of a manifold being a manifold will be studied in section 3.1

Derivation

The next property we would like to explore is a notion of "differentiability" of smooth maps, afterall they are called "smooth". In euclidean space, we know that differentiability is defined at a point x for a function $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ if there exists a linear approximation $Df(p)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Df(x)(h)\|}{\|h\|} = 0$$

However, this definition doesn't carry over so well for manifolds, namely we have yet to define a norm on a manifold[5]. On the other hand, all smooth manifolds have a diferentiable structure letting us represents any patch of the manifolds in terms of coordinates in euclidean space. We essentially define a differentiable function[6] between manifolds in terms of this coordinate representation. Just like for $f : \mathbb{R}^n \to \mathbb{R}^m$, $Df(a)$ will be a linear map between tangent spaces of $\mathbb{R}^n$ and $\mathbb{R}^m$. Unfortunately, this is not so easy at this stage: we would require some notion of norm defined on the manifold, which is quite technical (we'll do this in chapter 9). If we have a map $f : M \to \mathbb{R}$, there is one notion of derivative that we can lift from our intuition on euclidean space, namely the *directional derivative*. Similarly to the usual directional derivative, we can think of this directional derivative as giving us the information of what happens at $p \in M$ when we pass it through f on an infinitesimal scase. We will generlize this notion when dealing with maps $f : M \to N$, and so understanding the case with the simplified codomain of $\mathbb{R}$ is quite insightful: (TBD – there is more to this, will come back later)

Definition 1.2.4: Derivation At Point (Directional Derivative on a Manifold)

Let M be an n-manifold and $(U, \mathbf{x})$ be a chart where $p \in U$. Write $\mathbf{x} = (x^1, ..., x^n)$. Then for $f \in C^1(M)$, define the function $\frac{\partial f}{\partial x^i} : U \to \mathbb{R}$ on U by :

$$\frac{\partial f}{\partial x^i}(p) := \lim_{h \to 0} \left[\frac{f \circ \mathbf{x}^{-1}(a^1, ..., a^i + h, ..., a^n) - f \circ \mathbf{x}^{-1}(a^1, ..., a^i, ..., a^n)}{h} \right]$$

where $\mathbf{x}(p) = (a^1, ..., a^n)$. Equivalently:

$$\frac{\partial f}{\partial x^i}(p) := \partial_i (f \circ \mathbf{x}^{-1})(\mathbf{x}(p)) = \frac{\partial (f \circ \mathbf{x}^{-1})}{\partial u^i}(\mathbf{x}(p))$$

where $(u^1, ..., u^n)$ is the standard coordinates on $\mathbb{R}^n$

[5] which, in how we proceed, we will define the norm using the derivative in something called the *metric tensor* TBD

[6] We will essentially never call such a function a differentiable function. We will always call it a *smooth map* for historical reason

Naturally, if f is C^r, then $\frac{\partial f}{\partial x^i}$ is C^{r-1}. This notion if derivation at a point makes more rigorous situations where we have functions on a manifold that give some values. For example, if we have $M = S^2$ and a function $T : S^2 \to \mathbb{R}$ that gives us the temperature at every point, then we can consider $\frac{\partial T}{\partial \theta}$ and $\frac{\partial T}{\partial \varphi}$ as being defined *on* S^2, rather than some open set in "φ, θ"-space. All the natural properties we have come to expect from derivatives work for these derivatives, namely:

$$\frac{\partial(af + bg)}{\partial x^i}(p) = a\frac{\partial f}{\partial x^i}(p) + b\frac{\partial g}{\partial x^i}(p)$$

$$\frac{\partial(fg)}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(p)g + f\frac{\partial g}{\partial x^i}(p)$$

The downside of this definition is that we require choosing a patch in order to find the directional derivative. We thus have to find a translation between the two different coordinate representations: if we have $(U, \mathbf{x})$ and $(V, \mathbf{y})$ with $p \in U \cap V$, we can use the chain rule and the above definition to do a change of variables:

$$\frac{\partial f}{\partial y^i}(p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(p)\frac{\partial x^j}{\partial y^j}(p) \tag{1.3}$$

Just like for differentiaion in $\mathbb{R}^n$, differentiaion for M is a local property. Hence, what we really care is if f is locally smooth at a point p. We consider the following notion of a map that is defined "locally". We will often be dealing with such maps:

Definition 1.2.5: Locally C^r

Let $S \subseteq M$ be an arbitrary subset of a smooth n-manifold M. Let $f : S \to N$ be a continuous map where N is a smooth m-manifold. Then the map f is said to be *locally* C^r, or just C^r if unambiguous, if for every $s \in S$, there exists an open set $O \subseteq M$ containing s and a map $\tilde{f}$ that is C^r on O such that

$$\tilde{f}\Big|_{S \cap O} = f$$

Naturally, if $S \subseteq M$ is open, then an f is C^r if it is C^r on the corresponding sub-manifold. We will later show that f is smooth on S if and only if it has a smooth extension to some open set O containing S.

Definition 1.2.6: Local Smooth Maps

Let M and N be smooth manifolds, and $p \in M$. If $\mathcal{O}_p$ is the set of all open neighborhoods of p in M, then take the collection of maps:

$$S(p, M, N) := \cup_{U \in \mathcal{O}_p} C^\infty(U, N)$$

Now, define an equivalence class on $S(p, M, N)$ by $f \sim g$ if and only if f and g agree on some open neighborhood of p. Denote:

$$C_p^\infty(M, N) := S(p, M, N)/\sim$$

Definition 1.2.7: (Smooth) Germs

Any element $[f] \in C_p(M, N)$ is called a *germ*. If $f \sim_p g$, then we say that f and g have the same germs at p.

Note that by construction, we are safe to define:

$$[f](p) = f(p)$$

and if $N = \mathbb{R}$ (resp. $\mathbb{C}$), then the resulting $C_p^\infty(M, \mathbb{R})$ (resp. $C_p^\infty(M, \mathbb{C})$) is a $\mathbb{R}$-algebra (resp. $\mathbb{C}$-algebra)

Diffeomorphisms

We introduce the isomorphisms of the cateogry **Man**. Naturally, these are defined precisely so as to make them isomorphissm in **Man**:

Definition 1.2.8: Diffeomorphism

Let M, N be two smooth manifolds. Then if there is a smooth bijective map from M to N and N to M, then M is said to be *diffeomorphic* to N and we write $M \cong N$.

Sometimes, the automorphism group $\text{Aut}_{\textbf{Man}}(M)$ is written as $\text{Diff}(M)$, and $\text{Diff}^0(M)$ denotes the group of homeomorphisms form M onto itself[7]

Proposition 1.2.6: Diffeomorphism Equivalences

let $(M, \mathcal{A}), (N, \mathcal{B})$ be two smooth manifolds with maximal atlases, and a bijective mapping $f : M \to N$. Then the following are equivalent

1. f is a diffeomorphism
2. $\psi \circ f \in \mathcal{A}$ if and only if $\psi \in \mathcal{B}$
3. A function g on $(N, \mathcal{B})$ is smooth if and only if $g \circ f$ is smooth on $(M, \mathcal{A})$

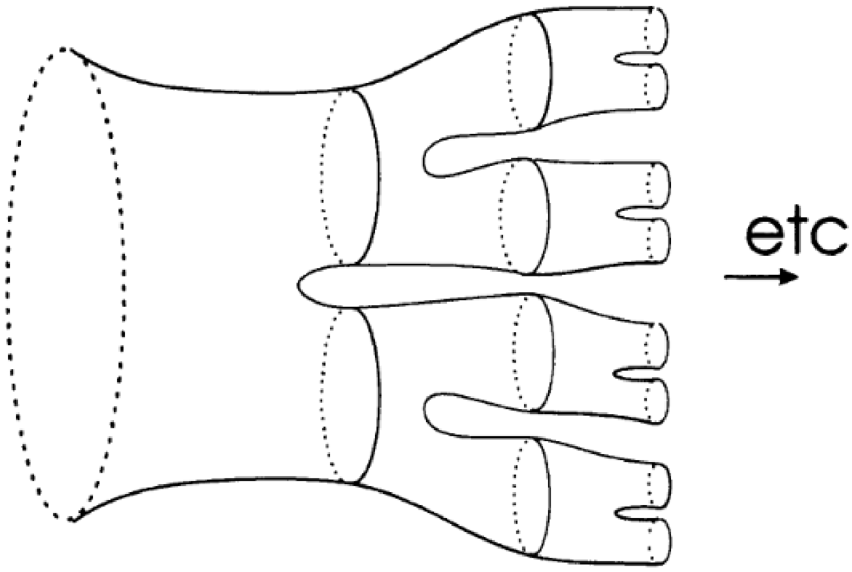
Proof **:**
simple exercise of using the chain rule and the definition

The key result of this proof is that if we know a composition of function $g \circ f$ is smooth and f is smooth, then this guarantee's that the function g is smooth, i.e., g had to be nice enough to guarantee that $g \circ f$ is smooth. Contrast this for when $g \circ f$ is continuous. It is certainly possible that f and $g \circ f$ is continuous, while g is not (for example f a step function and g the absolute value)

[7]The group $\text{Diff}(M)$ is as of writing this an object under research, not much yet is known of it!

Example 1.13: Diffeomorphisms

1. Let C be the cantor set on $[0,1] \subseteq \mathbb{R}$, an consider $\mathbb{R}$ as one of the coordinate axes of $\mathbb{R}^2$. Set $M_C = \mathbb{R}^2 \setminus C$. Then it was proven that this surface is diffeomorphic to:



2. I think we can use this to show that the triangle (edges) has a smooth structure.

3. Most $\mathbb{C}\mathrm{P}^n$ will be new manifolds that we have not yet encountered, with the exception of $\mathbb{C}\mathrm{P}^1$. Just like for $\mathbb{R}\mathrm{P}^1 \cong S^1$, we'll show that $\mathbb{C}\mathrm{P}^1 \cong S^2$. First, recall that S^2 can be described using two charts, namely stereographic projection from the north and south pole:

$$\varphi_N(x^1, x^2, x^3) = \left(\frac{x^1}{1-x^3}, \frac{x^2}{1-x^3}\right) \qquad \varphi_N(x^1, x^2, x^3) = \left(\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3}\right)$$

with the transition map:

$$\varphi_S \circ \varphi_N^{-1}(x_1, x_2) = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}\right)$$

or in terms of complex numbers, $z \mapsto \frac{1}{z}$. For $\mathbb{C}\mathrm{P}$, we will choose a slightly different transition map:

$$\varphi_0(z_0 : z_1) = \left(\frac{z_1}{z_0}\right) z_2 z_1^{-1} \qquad \varphi_1(z_0 : z_1) = \left(\frac{\overline{z_0}}{z_1}\right) = \overline{z_1 z_2^{-1}}$$

with transition map $z \mapsto \overline{z}^{-1}$, identical to that for S^2. We can go the other way and translate this information to the with the usual association of $a + bi = (a, b) \in \mathbb{R}^2$ and get the transition map to be:

$$\left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}\right)$$

The fact that these two transition maps are identical should strat giving a hint. Define:

$$f : \mathbb{C}\mathrm{P} \to S^2 \qquad f(\varphi_0^{-1}(x_1, x_2)) = \varphi_N^{-1}(x_1, x_2),\ f(\varphi_1^{-1}(x_1, x_2)) = \varphi_S^{-1}(x_1, x_2)$$

This map is well-defined since, since if $\varphi_0^{-1}(x_1, x_2) = \varphi_1^{-1}(x_1, x_2)$, then:

$$(x, y) = \varphi_S \circ \varphi_N^{-1}(x_1, x_2) = \varphi_0 \circ \varphi_1^{-1}(x_1, x_2)$$

showing the map is indeed well-defined. Similarly, we can define:

$$f^{-1} : S^2 \to \mathbb{C}\mathrm{P} \qquad f(\varphi_N^{-1}(x_1, x_2)) = \varphi_0^{-1}(x_1, x_2),\ f(\varphi_S^{-1}(x_1, x_2)) = \varphi_1^{-1}(x_1, x_2)$$

Then both of these maps are smooth (check that the local coordinate maps in terms of the 4 charts are smooth).

More generally, two manifolds are diffeomorphic if we can correspond their transition maps identically.

As a final world, we might ask whether $\mathbb{C}\mathrm{P}^n \cong \mathbb{R}\mathrm{P}^{2n}$? The answer is in fact no; the jacobian of the transition matrix for an atlas of $\mathbb{C}\mathrm{P}^n$ is always positive, while this is not the case for $\mathbb{R}\mathrm{P}^{2n}$.

4. Let $(x, y) \mapsto (x/z(x, y), y/z(x, y))$ where:

$$z(x, y) = \sqrt{1 - x^2 - y^2}$$

Then this is a diffeomorphism between $B_0(1)$ and $\mathbb{R}^2$

5. A famous theorem known as the *Manifold embedding theorem* tells us that every n-manifold is diffeomorphic to an open subset of $\mathbb{R}^{2n}$ with the appropriate subspace topology. In this way, it is feasibly possible to study manifolds strictly in terms of subspaces of $\mathbb{R}^n$. In particular, we could have defined the charts to be from $\mathbb{R}^n$ to $M \subseteq \mathbb{R}^m$, which automatically embeds M into a euclidean space (and also gives every manifold the subspace topology). However, there a many disadvantages of this approach, for example, we care about orientation and "stretching", which are not intrinsic information about a manifold but it can greatly effect the representation of the embedding, and we loose the ability to define "arbitrary differentiable structures". On the other hand, this type of embedding will become more prominent in Knot theory, precisely because of this extra information being added.

6. Recall how we had $(\mathbb{R}, \{\mathrm{id}\})$ and $(\mathbb{R}, \{x \mapsto x^3\})$ are two different smooth manifolds with the same underlying set. This examples could easily be generalized to $\mathbb{R}^n$ by taking $x_1 \mapsto x_1^3$ and keeping the rest constant. Both of these atlases will induce the same topology, so it seems that we have two homeomorphic, but not diffeomorphic sets. However, the map

$$\varphi : (x_1, ..., x_n) \to ((x_1)^3, ...x_n)$$

is in fact a diffeomorphism between these two manifolds! Note that if we took two copies of $(\mathbb{R}, \mathrm{id})$, this would certainly *not* be diffeomorphic, since φ^{-1} is not differentaible at 0.

Interestingly, up to dimension 3, a homeomorphism from a manifold to itself is equivalent to diffeomorphism (and hence have a unique maximal atlas). One might ask if the same is true for manifold of dimension 4 onwards. As it turns out, a result by HERE showed that $\mathbb{R}^4$ in fact has uncountably many differential structures that are not diffeomorphic to each other! Perhaps a bit surprisingly, $\mathbb{R}^n$ for $n \geq 5$ have only 1 differential structure up to isomorphism. This does not mean there are manifold for $n \geq 5$ which are homeomorphic but not diffeomorphic. As mentioned earlier, the 7-sphere has 28 non-isomorphic differential structures.

Proposition 1.2.7: Properties Of Diffeomrophisms

1. the composition of diffeomorphisms is a diffeomorphism
2. If $f_1 : M_1 \to N_1$, ..., $f_k : M_k \to N_k$ is a set of diffeomorphisms, then the product on the product smooth manifold is a diffeomorphism
3. Every diffeomrophism is a homeomorphism (hence it is continuous, an open map, and bijective)
4. The restriction of a diffeomorphism to an open submanifold (with our without boundary) is diffeomorphic onto it's image
5. Begin diffeomorphic is an equivalence relation on manifolds

Proof **:**
easy exercise

As we said in section ref:HERE, being a homeomorphism preserves dimension. It is much easier to prove this fact using diffeomorphisms:

Theorem 1.2.1: Invariance Of Dimension Using Diffeomorphisms

Let $f : M \to N$ be a diffeormophism between two nonempty manifolds. Then $\dim M = \dim N$

Proof **:**
Let $f : M \to N$ be a diffeomorphism and $\hat{f} : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a representative of f, which by definition of f is a diffeomorphism. $\hat{f}$ is a diffeomorphism, it has a diffeomorphic inverse $\hat{f}^{-1}$ where $\hat{f} \circ \hat{f}^{-1} = \hat{f}^{-1} \circ \hat{f} = \text{id}$. Since D_a is functorial:

$$\text{id} = D_a \,\text{id} = D_a(\hat{f}^{-1} \circ \hat{f}) = D_{f(a)}\hat{f}^{-1} \circ D_a \hat{f}$$

and similarly $D_a \hat{f} \circ D_{\hat{f}(a)} \hat{f}^{-1} = \text{id}$. Thus $D_a f$ is invertible, implying $m = n$.

Corollary 1.2.2: Invariance Of Boundary Using Diffeomorphisms

Let $f : M \to N$ be a diffeomorphism between two manifolds with boudnary. Then $f(\partial M) = \partial N$, and f restricts to a diffeomorphism between $\text{int}\, M$ and $\text{int}\, N$

***Proof* :**
exercsie with hint of thm 1.46 in John Lee

As a final word, we introduce a notion similar to weak topologies from analysis. As we've seen with weak topologies, there is a notion of defining a "weakest differentaible structure" on which a map is diffeomrophic:

Proposition 1.2.8: Transfer Of Structure

Let M be a smooth n-manifold and $\varphi : M \to X$ is a bijection. Then there exists a unique smooth structure on X that maps φ a diffeomorphism. This process is called *transfer of structure.*

If X is a topological structure and the map φ is a homeomorphism, then the topology induced by the transfer of structure is the original topology of X. This allows us to put differential structures on spaces that at first don't seem admit a smooth structure, like a triangle.

1.3 Regular-Submanifold

We next make sure that being a manifold is closed under taking "subspaces". First of all, if M is a n-dimensional topological manifold, then $S \subseteq M$ is a k-dimensional topological submanifold if S satisfies definition 1.1.5. We would naturally say that S has the subspace topology induced by M [8]. We would naturally want a k-dimensional smooth submanifold S to also have some differentiable structure preserved. A trivial example would be $\mathbb{R}^k \subseteq \mathbb{R}^n$ for some $k \leq n$: in this case the atlas that would make inclusion a smooth map is simply the intersection of the two. However, if we take $S^1 \subseteq \mathbb{R}^2$, where $\mathbb{R}^2$ has the usual atlas, then the restriction of the atlas of $\mathbb{R}^2$ to S^1 is no longer an altas on S^1 (there is no chart of S^1 containing all of S^1). Hence, a bit more casre must be put into relating these smooth spaces. The following definition gives how to relate these:

Definition 1.3.1: Smooth sub-manifold

Let $S \subseteq M$ be a subset of a n-dimension manifold M with atlas $\mathcal{A}$. Then S is a k-dimensional submanifold if for every $p \in S$, there exists a chart $(U, \varphi) \in \mathcal{A}$ around p such that

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k$$

Charts (U, φ) of M with this property are called *submanifold charts* for S

We single out submanifold charts for the following key property of submanifolds:

[8] In chatper 3, we will show a definition of a submanifold that doesn't necessarily have the sub-topology, and explain why we would care

Proposition 1.3.1: Embed Submanifold Into Manifold

Let $S \subseteq M$ be a k-submanifold of an n-manifold. and let Φ be the submanifold charts. Then the submanifold charts can be turned into a chart for S where for any $(U, \varphi) \in \Phi$, we define $(U \cap S, \pi \circ \varphi|_{U\cap S})$ where $\pi : \varphi(U) \to \mathbb{R}^k$ ($\varphi(U) \subseteq \mathbb{R}^n$) is the projection into the first k-coordinates [up to appropraite reordering]. With this, S has the subspace topology, and the inclusion map

$$i : S \to M$$

is smooth

Proof **:**
For any chart $(U \cap S, \pi \circ \varphi|_{U\cap S})$, take (U, φ) so that:

$$\varphi \circ \iota \circ (\pi \circ \varphi|_{U\cap S})^{-1}(x) = \varphi \circ \iota(\varphi^{-1}(x)) = \varphi(\varphi^{-1})(x) = x$$

This shows that regular submanifolds are indeed the subobject in the category of **Man**. From now on, submanifold will mean regular submanifold. As an exercise, show that every open submanifold is a regular submanifold. Note we could have defined (regular) submanfiolds as being the manifolds for which $\iota : S \hookrightarrow M$ is smooth with the topology of $\iota(S)$ being the subspace topology. However, this requires the use of the Constant rank theorem (theorem 3.1.2) which will allow us to pick the right charts.

Corollary 1.3.1: Smooth Maps And Regular Submanifold

Let $f : M \to N$ be a smooth map and $S \subseteq M$ a regular submanifold. Then $f|_S$ is a smooth map with respect to S and N. By definition, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ S \arrow[u, "\iota"] \arrow[ur, "f|_S"'] & \end{tikzcd}$$

Note that the converse is not true: if $f : S \to N$ is a smooth map, there does not necessarily exist an ι and an $F : M \to N$ such that the above diagram commutes (can you think of a counter-example?)

Proof **:**
By proposition 1.3.1, there is a smooth map $\iota : S \to M$. Pointwise, we have $f|_S = f \circ \iota$. Since the right hand side is the composition of smooth maps, the left hand side is smooth, completing the proof.

Example 1.14: Submanifolds

1. Let $M = \mathbb{R}^n$. Then all n-submanifolds of M are all open subset of M
2. Let $M = \mathbb{R}^n$, $k \leq n$, and $\mathbb{R}^k \subseteq \mathbb{R}^n$ where the last $n - k$ coordinates are 0. Then for any $k \geq 0$, these are all submanifolds

3. Let $M = S^n$. Then for $k \leq n$, $S^k \subseteq S^n$.

4. Let $U \subseteq M$ be an open subset of M. Then U is an submanifold as well.

We have defined regular submanifolds in such a way that it is convenient to prove many properties. There are more equivalent ways of proving a subset of a manifold is a submanifold, each with their advantages. One take's a functional approach (which will generalize well the notion of a manifold in scheme theory), some take a concrete approach using graphs, another tries to define it using only the information about S and M with as little reference to the base space as possible, and one is the definition of submanifold we just presented. Some of these approaches require the constant rank theorem (theorem 3.1.2), and so we shall show this for $\mathbb{R}^n$ for now:

Proposition 1.3.2: Submanifold Equivalences

Let $U \subseteq \mathbb{R}^n$ be an open subst. Then the following are equivalent:

1. For all $a \in U$, there is chat (φ, A) for $\mathbb{R}^n$ such that we can define $\pi \circ \varphi : M \to \mathbb{R}^{n-k}$ such that $\mathbb{R}^n \cap U = f^{-1}(0)$ and the derivative of any representative f has rank $n-k$ at all $x \in \mathbb{R}^n \cap U$.

2. Every point of $\mathbb{R}^n$ has an open neighborhood U such that $\mathbb{R}^n \cap U$ is the graph of a C^r function $z = g(y)$ where $y = (y_1, y_2, ..., y_k)$ and $z = (z_1, z_2, ..., z_{n-k})$, after a permutation of coordinates
$$(y, z) = (x_{\sigma(1)}, ..., x_{\sigma(n)})$$

3. For all $a \in \mathbb{R}^n$, there is a open neighborhood U of a, and open subset V of $\mathbb{R}^n$, and a C^r diffeomorphisms $h : U \to V$, such that
$$\begin{aligned} &= h(\mathbb{R}^n \cap U) \\ &= V \cap (\mathbb{R}^k \times \{0\}) \\ &:= \{y = (y_1, ..., y_n) \in V \mid y_{k+1} = \cdots = y_n = 0\} \end{aligned}$$

4. for all $a \in \mathbb{R}^n$, there is an open neighborhood U of a, an open subset W of $\mathbb{R}^k$, and a C^r mapping $\varphi : W \to \mathbb{R}^n$ such that φ is one-to-one, $\varphi(W) = \mathbb{R}^n \cap U$, φ has rank k at every point of W, and $\varphi^{-1} : \varphi(W) \to W$ is *continuous with respect to the subspace topology of* $M \subseteq \mathbb{R}^n$ (The latter means that, for every open subset $\Omega \subseteq W$, $\varphi(\Omega)$ is the intersection of $\varphi(W)$ with some open subset of $\mathbb{R}^n$)

***Proof* :**
In this proof, we will show the following:

$$\begin{tikzcd} a \arrow[r, Rightarrow] & b \arrow[r, Rightarrow] & c \arrow[ll, Rightarrow, bend left] \arrow[r, Leftrightarrow] & d \end{tikzcd}$$

$(a) \Rightarrow (b)$

Let $a \in M$. By (a) we know there exists a $U \subseteq \mathbb{R}^n$ containing a and a C^r mapping $f : U \to \mathbb{R}^{n-k}$ such that $M \cap U = f^{-1}(0)$ and $\text{rank}(Df(x)) = n-k$ for all $x \in M \cap U$. In particular $\text{rank}(Df(a)) = n-k$, implying that $\ker(Df(a)) = k$.

Let $F : U \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the function $F(x,y) = f(x,y)$. In particular, restricted to $M \cap U$, notice that $F(x,y) = 0$, and the $(n-k) \times (n-k)$ matrix of $D_{n+j}F^i(a)$ has rank $n-k$ after appropriate re-ordering of the coordinates[a]. Hence, by the implicit function theorem (generlized for C^r functions), given $a = (b,c)$, for appropraite open subset $A \subseteq \mathbb{R}^k$ contianing b, $B \subseteq \mathbb{R}^{n-k}$ and C^r function $g(x) : A \to B$, $F(x, g(x)) = 0$. Given an appropriate adjustement of the size of U, we see that $M \cap U$ is the C^r graph of g.

$(b) \Rightarrow (c)$

Let $g : A \subseteq \mathbb{R}^k \to \mathbb{R}^{n-k}$ be a C^r function whose graph is $M \cap U$, that is

$$M \cap U = (A, g(A))$$

for appropriate U. We need to find a diffeomorphism h such that $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$. To that end, define $h : A \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ to be:

$$h(x,y) = (x, g(x) - y)$$

Then it is clear that h is C^r, being the product and sum of C^r functions. Furthermore, for any $z = (x,y) \in U$

$$Dh(z) = \begin{pmatrix} I & \mathcal{O} \\ Dg(x) & -I \end{pmatrix}$$

showing that $Dh(z)$ has full rank for any $z \in U$, and hence invertible at every point, and so h is a diffeomorphism. Now, notice that

$$h(M \cap U) = h((A, g(A))) = (A, 0)$$

then since A is open, then letting $V = A \times (-\epsilon, \epsilon)^{n-k}$ gives us that

$$h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$$

$(c) \Rightarrow (a)$

Let $h : U \to V$ be a diffeomorphism such that $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$ and take $a \in M \cap U$. Consider $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$ to be the projection map $\pi(x,y) = y$. Recall that we've shown in MAT257 that π is smooth and an open map of rank $n-k$[b]. Thus, the composition $\pi \circ h$ is C^r and of rank $n-k$. Let $f = (\pi|_V) \circ h$. Then f is of rank $n-k$, and

$$f^{-1}(0) = h^{-1}(\pi^{-1}|_V(0)) = h^{-1}(V \cap \mathbb{R}^k \times \{0\}^{n-k}) = M \cap U$$

hence f, satisfies the condition's of part (a).

$(c) \Rightarrow (d)$

Let h be as in part (c), so that $h : U \to V$ is a C^r diffeomorphism such that $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\})$. Since h is a C^r diffeomorphism, so is $h^{-1} : V \to U$, and $h^{-1}(V \cap (\mathbb{R}^k \times \{0\})) = M \cap U$. We will use this and the fact that the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is an open mapping of rank k to construct φ. We will show that φ satisfies all the properties given in part (d), then show that φ^{-1} is continuous with respect to the subsapce topology on $M \cap U$.

Since V is open, then given the appropriate coordinates for π to project onto, we get that $V \mapsto W = V \cap (\mathbb{R}^k \times \{0\})$ Reversing, this, we can define $\iota : \mathbb{R}^k \to \mathbb{R}^n$ by $\iota(x) = (x, 0)$. Let $\varphi = h^{-1} \circ \iota\big|_W$. Restricting $h^{-1} \circ \iota$ to W does not change the fact that ι is of rank k (since

W is open and within what ι embeds into), and since h^{-1} is C^r is diffeomorphic, φ is C^r of rank k. Furthermore, since $h^{-1} \circ \iota$ is bijective, so is φ, since we're simply restricting the domain W By construction, $\varphi(W) = M \cap U$. Since $\varphi(W) \subseteq \mathbb{R}^n$, we can just make the codomain $\mathbb{R}^n$ without affecting the function, allowing us to write $\varphi : W \to \mathbb{R}^n$ as in the statement of part (d).

Next, we need to show that φ^{-1} is continuous with respect to the subspace topology of M, or equivalently that φ is an open map. As the homework suggets, we will show that for any open set $\Omega \subseteq W$, there exists an open subset $\mathcal{O}_\Omega \subseteq \mathbb{R}^n$ such that $\varphi(\Omega) = \varphi(W) \cap \mathcal{O}_\Omega$. To find $\mathcal{O}_\Omega$, first consider that $h^{-1} : V \to U$ is an open map (since it's a diffeomorphism, it's a homeomorphism) between two open sets. Since $(W, 0)$ is contained in V, and $\Omega \subseteq W$, we can extend Ω so that it is contained in an open subset of V, namely, take $\overline{\Omega} = \Omega \times (-\epsilon, \epsilon)^{n-k} \cap V$ for some $\epsilon \in \mathbb{R}$. Since $\Omega \times (-\epsilon, \epsilon)^{n-k}$ is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^n$, $\overline{\Omega}$ is open in $\mathbb{R}^n$ (or, in particular in the subspace topology of V). Thus, $h^{-1}(\overline{\Omega})$ is open. Note that $\varphi(\Omega) = h^{-1}(\pi(\Omega)) \subseteq h^{-1}(\overline{\Omega})$.

Now, let $\mathcal{O}_\Omega = h^{-1}(\overline{\Omega})$. Then by construction of $\mathcal{O}_\Omega$, $\varphi(\Omega) = \varphi(W) \cap \mathcal{O}_\Omega$. The $\subseteq$ direction is immediate since $\varphi(\Omega) = \varphi(\Omega) \cap \varphi(W) \subseteq \mathcal{O}_\Omega \cap \varphi(W)$ and the $\supseteq$ direction comes from φ being bijective onto $\varphi(W)$, thus:

$$\varphi(W) \cap \mathcal{O}_\Omega = (M \cap U) \cap \mathcal{O}_\Omega \overset{!}{\supseteq} \varphi(\Omega)$$

where $\overset{!}{\supseteq}$ comes from the fact that $\varphi(\Omega)$ is contained in all three of these sets. Thus, $\varphi : W \to \mathbb{R}^n$ is an open map onto $\varphi(W)$, and by what we've shown earlier, it is of rank k.

$(d) \Rightarrow (c)$

Define the following "padding" of φ:

$$\tilde{\varphi} : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n \qquad \tilde{\varphi}(y, z) = \varphi(y) + (0, z)$$

Then

$$D\tilde{\varphi} = \begin{pmatrix} D\varphi^{1..k} & \mathcal{O} \\ D\varphi^{(k+1)..n} & I_{n-k} \end{pmatrix}$$

Since $D\varphi$ has rank k, the above matrix is full rank. In particular, it is full rank around $(\varphi^{-1}(a), 0)$[c] that maps to $a \in M$. Thus, by the inverse function theorem, there exists an open neighborhood V and U, $(\varphi^{-1}(a), 0) \in V$ and $a \in U$, such that $\tilde{\varphi}^{-1} : U \to V$ is is a C^r inverse function. On these (open) neighborhoods, $\tilde{\varphi}$ is also C^r, meaning φ^{-1} is C^r diffeomorphic. Furthermore, since $\tilde{\varphi}$ maps $V \cap (\mathbb{R}^k \times \{0\})$ to $M \cap U$ (where U is our appropriately small choice we made while doing the inverse function theorem), by construction: $\tilde{\varphi}^{-1}(M \cap U) = V \cap (\mathbb{R}^k \times \{0\})$. Thus, letting $h = \tilde{\varphi}^{-1}$ completes the proof.

[a] As we've seen in MAT257, Re-ordering of coordinates is a combination of reflections and rotations, meaning it is a diffeomorphism, and hence if d was our function shuffling around the columns, we can take $F = d \circ f$. This trick will be repeated throughout this proof

[b] Notice that π maps the basis of open rectangles $A \times B \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ to open sets $B \subseteq \mathbb{R}^{n-k}$, and that the derivative of π at any point is represented by an $n \times n - k$ matrix with the upper $k \times n - k$ being the zero matrix, and the lower $n - k \times n - k$ matrix being the identity

[c] Recall that $\varphi^{-1}(a)$ is well-defined since φ is one-to-one to $M \cap U$

We have ommited quotient objects in our category. As we have seen, some manifolds are indeed defined through quotient, the mobius strip being an example. However, as example 1.1(2), the quotient map from a manifold M need not even be Hausdorff, therefore the quotient of even a topological manifold need not be a topological manifold. In section 1.1.2, we have seen how to construct a manifold using quotients but not a general approach for when the quotient is indeed a manifold. Unfortunately, smooth structure behaves rather badly when quotienting. In chatper ref:HERE, we will give some nice examples of manifolds and types of smooth maps on manifolds that will give nice quotient objects for the category of **Man**.

Exercise 1.3.1

1. Show that being a submanifold is transitive.

1.4 Bump Functions and Partition of unity

We will now define one of the most useful techniques for extending local information of a manifold to global informaiton of a manifold.

Definition 1.4.1: Support

Let $f : X \to \mathbb{R}$ be a function. Then the *support of f* is

$$\overline{\{f(x) \mid f(x) > 0\}}$$

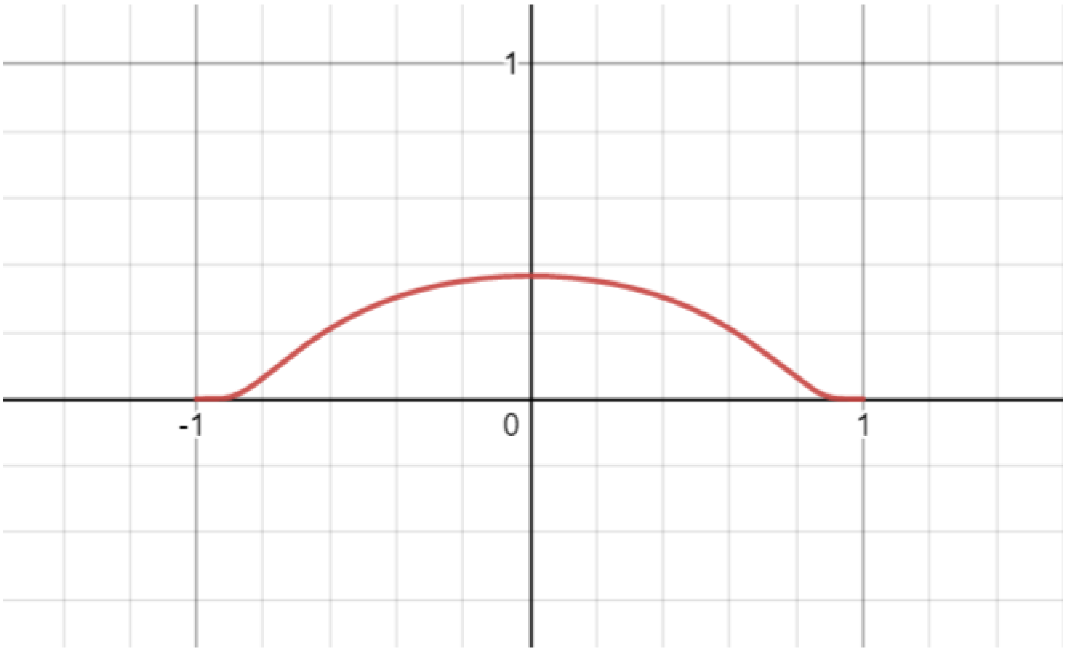
that is, it is the closure of the positive domain. It is usually denoted $\text{supp}(f)$.

Example 1.15: Smooth Compact Support On $\mathbb{R}$

Recall that the following is a smooth function with compact support (i.e. $\text{supp}(\Phi)$ is within a compact set):

$$\Phi(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

which visually looks like:



Definition 1.4.2: Subordinate Partitions Of Unity

Let M be a topological space, and $\mathcal{O} = \{U_\alpha\}_{\alpha \in A}$ an open cover for M. Then a *partition of unity subbordinate to* $\mathcal{O}$ is an indexed family of continuous function $\psi_\alpha : M \to \mathbb{R}$, $\Psi = (\psi_\alpha)_{\alpha \in A}$ such that the following properties hold:

1. For each $a \in A$, $0 \leq \psi(x) \leq 1$ for all $\psi \in \Phi$
2. $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$ for all $\alpha \in A$
3. The collection of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite: every point $x \in M$ has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many $\alpha \in A$.
4. For each $a \in A$, $\sum_{\psi \in \Phi} \varphi(a) = 1$ (by the previous condition, this is always a finite sum).

Theorem 1.4.1: Existence Of Smooth Bump Functions On Manifolds

Let M be a smooth manifold, $K \subseteq M$ a compact subset, and $O \subseteq M$ an open set containing K. Then there exists a smooth function $\beta : M \to \mathbb{R}$ that is identically 1 on K, takes on values in $[0, 1]$, and has compact support in O.

Proof **:**
We will break this down into cases

case 1: Let $M = \mathbb{R}^n$, $O = B_0(R)$ and $K = B_0(r)$ for some $0 < r < R$. Then, take:

$$\varphi(x) = \frac{\int_{|x|}^{R} g(t)dt}{\int_{r}^{R} g(t)dt}$$

where

$$g(t) = \begin{cases} e^{\frac{-1}{t-r}} e^{\frac{1}{t-R}} & \text{if } r < t < R \\ 0 & \text{otherwise} \end{cases}$$

then it is easy to see that $g(t)$ is a smooth function, and a composition with a translation gives the result for a ball centered at any point

Case 2: Let $M = \mathbb{R}$, and O, K are now arbitary open and compact sets such that $K \subseteq O$ (as in the statement of the theorem). For each $p \in K$, let U_p be an open ball centerd at P contained in O, and let K_p be a closed ball centered at p with half the radius of U_p. Then the interior of the K_p's form an open cover, and so by compactness a finite subcover exists, say $\{\int(K_p)\}_{p\in P} = \{K_i\}_{i=1}^n$.

Now, for each U_i that is concentric to K_i, define φ_i as was done in case 1 so that φ_i has support in U_i and is identically 1 on K_i. Then we can define:

$$\beta = 1 - \prod(1 - \varphi_i)$$

which if carefully examined will be the function we are looking for.

General Case: Now let M be any smooth n-manifold, with $K \subseteq M$ a compact subset and $K \subseteq O \subseteq M$ with O open and containing K. If K is contained in a chart $(U, \mathbf{x})$, then we see that we can easily define a smooth bump function through $\beta \circ \varphi$. If K is not contained in a single coordinate chart $(U, \mathbf{x})$, then by paracompactness (i.e., local finiteness) that there exists a finit number of charts $\{(U_i, \mathbf{x}_i\}$ with compact sets $K_1, ..., K_n$, $K_i \subseteq U_i$, $\cup K_i \subseteq K$ and $\cup U_i \subseteq O$. Now, define $\varphi_i \circ \mathbf{x}_i$ to be identically 1 on K_1 and identically 0 on $U_i^c = M \setminus U_i$. Then we can again define the funcion:

$$\beta = 1 - \prod(1 - \varphi_i)$$

which if carefully examined will be the function we are looking for, completing the proof.

The big use of partition's of unity is that it is a stronger verison of corrollary 1.2.1. For corrollary 1.2.1, we need that the maps agree on the intersections. This can be a strong condition to satisfy. Instead, partitionof unity allow us to be more flexible in how we define local maps.

Using this construction, we see that every representative in a germ $[f] \in C_p(M)$ (notice the codomain being $\mathbb{R}$) can be extended to βf, and is usually denoted $(\beta f)_{\text{ext}}$. In this way, every element of $C_p^\infty(M)$ has a “global” representative in $C^\infty(M)$.

Here are a few more use cases:

Corollary 1.4.1: Existence Of Smooth Bump Function

Let M be a smooth manifold (with or without boundary). For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function on A supported in U

Proof **:**
Let $U_0 = U$ and $U_1 = M \setminus A$ be an open cover. Let Ψ be the partition of unity subbordinate to this cover with functions ψ_1 and ψ_2. Then by definition, $\psi_1 \equiv 0$ on A, and $\sum_i \psi_i = 1$, so ψ_1 has the desired properties.

The next result show's how we can extend a smooth map from a closed subset to be a map on the entire manifold:

Corollary 1.4.2: Extending Smooth Function

Let M be a smooth manifold (with or without boundary), $A \subseteq M$ a closed set, and $f : A \to \mathbb{R}^k$ a smooth function. Then for any $U \subseteq M$ containing A, there exitss a $\tilde{f} : M \to \mathbb{R}^k$ such that $f|_A = f$ and supp $\tilde{f} \subseteq U$.

Proof **:**
p.45 John Lee, was also homework

Note that this fails if A is not closed (see homework).

(A word on how this fails for real-analytic manifolds and real-analytic functions)

Corollary 1.4.3: Zero Set Of Closed Subset

Let M be a smooth manifold and $K \subseteq M$ a closed subset. Then there exists a nonnegative function $f : M \to \mathbb{R}$ st $f^{-1}(0) = K$.

Proof **:**
p. 47 John Lee. long.

Corollary 1.4.4: Smooth Urysohn's lemma

if A, B are disjoint closed subsets of a smooth manifold M, then there is a smooth function $f : M \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$

Proof **:**
Since A and B are disjoint, $M \setminus A$ and $M \setminus B$ are open with $A \subseteq M \setminus B$, $B \subseteq M \setminus A$. Define a partition of unity B subordinate to an open cover of B contained in $M \setminus A$ say Φ_B (the open cover can simply contain $M \setminus A$). Let $f = \sum_{\varphi_B \in \Phi_B} \varphi_B$. This function is well-defined, since at any point $x \in M \setminus A$, φ sums at most finitely many points. Then by construction

$$f : M \to [0, 1] \qquad f|_A = 0 \quad f|_B = 1$$

as we sought to show.

2

Derivative and Tangent Structures

If $M = \mathbb{R}^n$, then as we've seen in calculus[1], when we take a smooth map $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, then for any point $p \in U$, we can define linear map $Df(p) : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ for each $p \in U$ that linearly approximates f at p. This map *is* the derivative of f. On a high-level, the derivative, if it exists, tell's us how a function austensibly behaves linearly on a small enough neighborhood at a point p. If f is differentiable at each point, we can define the function $Df : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$. f is continuously differentiable if Df is continuous. In this chapter, we try to generlize this notion of Df representing a linear approximation manifolds, that is for a smooth function $f : M \to N$ between manifolds, we want to be able to find a new linear function at every p based on f that linearly approximates f at p. The project of generlizing this notion boils down to two things:

1. finding a good way of finding the domain and co-domain for the linear map, i.e., how can we at every point $p \in M$ in a manifold find a vector space affiliated to p that maps linearly to a vector spaces affiliated to the point $f(p)$ for which we can define a linear map between these two spaces. Recall to that we have not defined as subsets of $\mathbb{R}^n$, and so we can't take affine subspaces of $\mathbb{R}^n$ as tangent spaces.

2. That this linear map represents a good linear approximation of f at p which varies smoothly, that is if we take a path on the domain M, then the equivalent of the function Df should vary smoothly along this path. In order to be "varying smoothly", we need to know on what object this smooth variation is happening (since at every point, we are in a different tangent space).

Finally, a consequence of defining linear maps is that we will have the notion of the dimension, and thus the *rank* of the lienar maps. Recall that we showed in example 1.12 that the image of a smooth map, $f(M)$, need not be a smooth manifold, not even a topological manifold, mainly due to the fact

[1]See the preliminary chapter for a refresher

that there can be points in which we dip dimensions[2]. Part of the consequence of this generlized Df will be that we can keep track of dimension information. We will take advantage of this in chapter 3.

2.1 Tangent Space

Let's start by working with the simplest manifold, $M = \mathbb{R}^n$. In the following, we will show many ways in which we can reconstruct the domain of $Df(p)$. No one definition will be "universal" or "natural", however one that will be particularly important to us will be a way to construct the tangent space *geometrically* using tangent lines (which will be easy to visualize and understand) and and to construct the tangent space *algebraically* using fundamnetal properties of the derivative (which will be easy to manipulate in a coordinate free-manner).

First, for the geometric approach, we can define any curve $c : (-\epsilon, \epsilon) \to \mathbb{R}^n$ such that $c(0) = p$, $c'(0) = q$, for any two $p, q \in \mathbb{R}^n$. We can imagine $c'(0)$ as being tangent to the curve produced by c, or as a vector starting at the point p. Formally, we can imagine $T_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n$ to be the set of derivative of curves such that $c(0) = p$. We sometimes denote an elements in $T_p\mathbb{R}^n$ with a subscript p, so $v_p := (p, v)$, and call v *principal part.* The space $T_p\mathbb{R}^n$ naturally has the same vector-space structure as $\mathbb{R}^n$, i.e. $T_p\mathbb{R}^n \cong_{\mathbf{Vect}} \mathbb{R}^n$, or more generally for any two $p, q \in \mathbb{R}^n$, $T_p\mathbb{R}^n \cong_{\mathbf{Vect}} T_q\mathbb{R}^n$. Naturally, the tangent space of a regular k-submanifold $A \subseteq \mathbb{R}^n$ at the point $x \in A$ need not be $T_x\mathbb{R}^k \times \{0\}^{n-k}$ since A might not be "flat", however it will always be isomorphic to $T_x\mathbb{R}^k \times \{0\}^{n-k}$: if we take $c : (-\epsilon, \epsilon) \to M$, then this is also a map $c : (-\epsilon, \epsilon) \to \mathbb{R}^n$, and so we can take all the values of $c'(0)$ where $c(0) = p$. These vectors are all tangent to M at p. We can tentatively write this collection as T_pM.

we make two more observations before continuing:

1. there are many curves that can have value v pasing through p. We can define a tangent vector to be the equivalence class of curves $[c]$ such that $c \sim d$ if and only if $c'(0) = d'(0)$.

2. the point (p, v) naturally relates to charts $(U, \mathbf{y})$ of the submanifold where $p \in U$. If we invert the funcion, $\mathbf{y}^{-1} : V \to U \subseteq M$, which can be thought of as a map from the ambient space $\mathbb{R}^n$ to M, then $\mathbf{y}^{-1}$ parapamterizes some of M, i.e. represent M in terms of *coordinate functions* which are also sometimes called the *coordinate curves*:

$$y^i \mapsto \mathbf{y}^{-1}(0, ..., y^i, ...) \qquad 1 \leq i \leq n$$

   Usually, we suppose $\mathbf{y}^{-1}(0) = p$ for convinience. Using these functions, we can define vectors E_i at p by taking:

$$E_i := \left(p, \frac{\partial \mathbf{y}^{-1}}{\partial y^i}(0)\right)$$

   then it can be shown that $(E_1, ..., E_n)$ forms a basis for T_pM. For another coordinate system $\overline{\mathbf{y}}$ with $\overline{\mathbf{y}}^{-1}(0) = p$, we can define another basis $(\overline{E_1}, ..., \overline{E_n})$. If

$$v_p = \sum_i a^i E_i = \sum_i \overline{a}^i \overline{E}_i$$

[2]Manifold that can have singularities are known as *orbifolds* which are manifold that are locally the quotient of an open set acted on by a finite group. Another structure similar to manifolds is *varieties*

then by the chain rule we get that:

$$\overline{a} = D(\mathbf{y} \circ \mathbf{y}^{-1})|_{\mathbf{y}(p)}(a)$$

which is classically written as:

$$\overline{a}^i = \sum_{j=1}^{n} \frac{\partial \overline{y}^i}{\partial y^j} a^j$$

so, both $(a^1, ..., a^n)$ and $(\overline{a^1}, ..., \overline{a^n})$ represent the same tangent vector, but with respect to different charts. This is a simple example of something called the *transformation law*

Tangent Spaces Via Curves

Definition 2.1.1: Tangent Curves

Let $c_0 : I \to M$ and $c_1 : I \to M$ be curves with $c_0(0) = c_1(0) = p$. Then we say that c_0 *is tangent to* c_1 *at* p if for all real valued smooth function f defined on an open neigbhorhood at p,

$$(f \circ c_0)'(0) = (f \circ c_1)'(0)$$

It is easy to see that this forms an equivalence relation where the equivalence classes represent all possible curves with the same velocity at $c'(0)$ (i.e., there can be many curves approaching at differenta direction and speeds, as long as all of them at the point p are going in the same direction and speed they are in the same equivalence class).

Definition 2.1.2: Tangent Vector [Via Curves]

Let T_pM be the collection of equivalence classes as defined by definition 2.1.1. Then any element of T_pM is called a *tangent vector at* p

The collection T_pM is then called the *tangent space at* p

Tangent vectors $[c]$ are usually denoted by a subscript and the velocity vector they represent: v_p. If $v_p = [c]$, then $c'(0) = v_p$. Though we defined equivalence locally, using bump functions, we can show the definition is equivalent for function defined globally $f : M \to \mathbb{R}$.

Lemma 2.1.1: Equivalent Condition

The curve c_1 is tangent to c_2 at p if and only if $(f \circ c_0)'(0) = (f \circ c_1)'(0)$ for all $\mathbb{R}^k$-valued functions f defiend on an open neighborhood of p

Proof **:**

Let c_0 be tangent to c_1 and consdier an $\mathbb{R}^k$ valued function $f = (f^1, ..., f^n)$. Then $(f \circ c_0)'(0) = (f \circ c_1)'(0)$ if and only if $(f^i \circ c_0)'(0) = (f^i \circ c_1)'(0)$ for $1 \leq i \leq n$, which is true if and only if c_0 is tangent to c_1.

Conversely, if $(f \circ c_0)'(0) = (f \circ c_1)'(0)$ for all $\mathbb{R}^k$ valued function, let $f = (g, 0, ..., 0)$ for any $\mathbb{R}$ valued function g. Then $(f \circ c_0)'(0) = (f \circ c_1)'(0)$ is equivalent to $(g \circ c_0)'(0) = (g \circ c_1)'(0)$, completing the proof.

The definition of T_pM gievs a nice geometric way of undersatnding the tangent space, but it does not give a "natural", so to speak, way of defining a vector space. We build-up to this in the following:

Proposition 2.1.1: Consistent Transfer Of Linear Structure

Let S be a set and $\{V_\alpha\}_{\alpha\in A}$ be a family of n-dimensional vector-spaces indexed by A with the property that $b_\alpha : V_\alpha \to S$ is a bijection. If for every $\alpha, \beta \in V$, $b_\beta^{-1} \circ b_\alpha : V_\alpha \to V_\beta$ is a linear isomrophism, then there exists a unique structure on S such that each b_α is a linear isomorphism.

Proof **:**
We simply show that we can push-forward the vector-space structure of one $\alpha \in A$, and this structure is well-defined with respect to every other $\beta \in A$, and unique by necessity.

Define $s_1 + s_2 := b_\alpha(b_\alpha^{-1}(s_1) + b_\alpha^{-1}(s_2))$. This definition is independent of α since:

$$\begin{aligned} b_\alpha(b_\alpha^{-1}(s_1) + b_\alpha^{-1}(s_2)) &= b_\alpha(b_\alpha^{-1} b_\beta b_\beta^{-1}(s_1) + b_\alpha^{-1} b_\beta b_\beta^{-1}(s_2)) \\ &= b_\alpha \circ b_\alpha^{-1} b_\beta(b_\beta^{-1}(s_1) + b_\beta^{-1}(s_2)) \\ &= b_\beta(b_\beta^{-1}(s_1) + b_\beta^{-1}(s_2)) \end{aligned}$$

Similarly, scalar multiplication is well-defined: $a \cdot s = b_\alpha(a \cdot b_\alpha^{-1}(s))$. Then with these two operations well-defined, S is clearly a vector-space since each V_α is a vector-space.

Motivated by the above, given an n-dimensional manifold M, for any chart $(U, \mathbf{x}_\alpha)$, define:

$$b_\alpha : \mathbb{R}^n \to T_pM \qquad v \mapsto [\gamma_v] \quad \text{where} \quad \gamma_v : t \to \mathbf{x}^{-1}(\mathbf{x}(p) + tv)$$

for a sufficeintly small but otherwise irrelavant interval containing 0. This map is clearly well-defined, so it remains to show that it's bijective and the "regularity condition" is satisfied:

Lemma 2.1.2: Consistent Transfer For Charts

Let $(U, \mathbf{x}_\alpha)$ be a chart containing p. Then the map $b_\alpha : \mathbb{R}^n \to T_pM$ is a bijection with $b_\beta^{-1} \circ b_\alpha = D(\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1})(\mathbf{x}_\alpha(p))$

Proof **:**
First, notice we can recover the v by doing:

$$\begin{aligned} (\mathbf{x}_\alpha \circ \gamma_v)'(0) &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{x}_\alpha \circ \mathbf{x}_\alpha^{-1}(\mathbf{x}(p) + tv) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\mathbf{x}(p) + tv) \\ &= v \end{aligned}$$

Now, consider $[\gamma_v] = [\gamma_w]$. Then by lemma 2.1.1, it is equivalent that an $\mathbb{R}^k$ valued function satisfies the equivalence relation, and hence:

$$v = (\mathbf{x}_\alpha \circ \gamma_v)'(0) = (\mathbf{x}_\alpha \circ \gamma_w)'(0) = w$$

showing that $V = w$, and hence injectivity.

For surjectivity, pick any $[c] \in T_pM$. By definition of the equivalence class, we may pick $v = (\mathbf{x}_\alpha \circ c)'(0)$. Then we know that $b_\alpha(v) = [\gamma_v]$. Now, $[\gamma_v] = [c]$, since for any smooth real value map f defined near p, we have:

$$\begin{aligned}
(f \circ \gamma_v)'(0) &= \left.\frac{d}{dt}\right|_{t=0} f \circ \mathbf{x}_\alpha^{-1}(\mathbf{x}_\alpha(p) = tv) \\
&= D(f \circ \mathbf{x}_\alpha^{-1})(\mathbf{x}_\alpha(p))(v) \\
&= D(f \circ \mathbf{x}_\alpha^{-1})(\mathbf{x}_\alpha(p))(f \circ c)'(0) \\
&= \vdots \\
&= (f \circ c)'(0)
\end{aligned}$$

completing surjectivity.

Finally, to show that the regularity condition is satisfied, we first see by lemma 2.1.1 that $[c] \to (\mathbf{x}_\alpha \circ c)'(0)$ is well-defined, and we have jusst deduced above that this map is b_α^{-1}. Thus, finsihing off the computations:

$$\begin{aligned}
b_\beta^{-1} \circ b_\alpha(v) &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}(\mathbf{x}_\alpha(p) + tv) \\
&= D(\mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1})(x_\alpha(p))(v)
\end{aligned}$$

which are all isomorphic since $x_\beta \circ \mathbf{x}_\alpha^{-1}$ is a transition map, hence a diffeomorphism (i.e. a map of constant rank).

Thus, we can put the natural $\mathbb{R}^n$ structure on T_pM. We will temporarily label this construction as $(T_pM)_{\text{Kin}}$ for *kinematic tangent space.*

Tangent Space via charts

Definition 2.1.3: Tangent Space Via Charts

Let $\mathcal{A}$ be a maximal atlas and M an n-dimensional manifold. Define:

$$\Gamma_p := \{(p, v, (U, \mathbf{x})) \in \{p\} \times \mathbb{R}^n \times \mathcal{A} \mid v \in \mathbb{R}^n,\ (U, \mathbf{x}) \in \mathcal{A}\}$$

Which is the collection of all cordinate charts containing p attached to the point p and the vector-space $\mathbb{R}^n$. Define the equivalence class $(p, v, (U, \mathbf{x})) \sim (p, w, (V, \mathbf{y}))$ if and only if:

$$D(\mathbf{y} \circ \mathbf{x}^{-1})(\mathbf{x}(p))(v) = w$$

that is v is related to w if the derivative of the change of coordinate $\mathbf{y} \circ \mathbf{x}^{-1}$ "identifies" v with w. Then

$$(T_pM)_{\text{phys}} = \Gamma_p / \sim$$

is the *tangent space via charts*.

Then $\Gamma_p / \sim$ can be given a vector-space structure as follows: For each $(U, \mathbf{x})$ where $p \in U$, define:

$$b_{(U,\mathbf{x})} : \mathbb{R}^n \to \Gamma_p \qquad v \mapsto [(p, v, (U, \mathbf{x}))]$$

To show bijectivity, notice that if $[(p, v, (U, \mathbf{x}))] = [(p, w, (U, \mathbf{x}))]$, by definition:

$$v = D(\mathbf{x} \circ \mathbf{x}^{-1})(\mathbf{x}(p))(v) = w$$

which shows that $v = w$, injectivity, and surjectivity is clear. Thus, by proposition 2.1.1, we get a vector-space structure transfered by $\mathbb{R}^n$ to $\Gamma_p / \sim$, with equivalence relations being called *tangent vectors*. Thus, we get another way of represnting the structure, which we will call the *physical tangent space*:

$$(T_pM)_{\text{phys}} = \Gamma_p / \sim$$

The subscript "phys" is inspired from physisists who commonly use this definition of a tangent space. If $v_p = [(p, v, (U, \mathbf{x}))] \in T_pM$, we commonly say that $v \in \mathbb{R}^n$ represents v_p with respect to $(U, \mathbf{x})$.

(a word on classical notation viewpoint, Lee p. 60)

Tangent Space via Derivation

(New intuition: the derivation represents algebraically the infinitesimal change, think pullback. then it makes senes that the tangent bundle is a colleciton of smooth derivations. the connection between smooth functions and derivations is similar to that of Lie groups and lie algebras!!)

(one inspiration: $\dot{\gamma}(t)(D_t(\gamma)(\frac{\partial}{\partial t})))$

The final definition of a tangent space is one that is probably the most important and the one to keep in ind the most. We will show that the generalization of directional derivative as shown in definition 1.2.4 let's us define a function so that at each point $p \in M$ in a manifold, we can use directional derivative to give us a tangent space at p. Essentially, a tangent space will be collection of linear functinals satisfying a property called the *derivation*. The advantage of this this definition is that there will be no need to reference any particular coordinate chart, however given any choice of coordinate chart we may easily represent any element of the tangent space.

To see this and gain an intuition on this definition, recall that on the garph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the domain of the total derivative can be interpreted as the tangent space at that point. If we take the gradient of f, $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ given by :

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \frac{\partial f}{\partial x^2} \\ \vdots \\ \frac{\partial f}{\partial x^n} \end{pmatrix}$$

then the gradient at any particular point p lives in the tangnt space at that point:

$$\nabla f(p) = \begin{pmatrix} \frac{\partial f}{\partial x^1}(p) \\ \frac{\partial f}{\partial x^2}(p) \\ \vdots \\ \frac{\partial f}{\partial x^n}(p) \end{pmatrix}$$

However, the gradient picks a particular point of the tangent space, that is the direction of steepest ascent with a particular slope. We would like to be able to pick any point on this tangent space. It is clear that the $\frac{\partial f}{\partial x^i}$ are somehow involved in this. To give an enlightening function, consider $f(\overline{x}) = \sum x_i$. Then:

$$\nabla f(1) = \begin{pmatrix} \frac{\partial f}{\partial x^1}(1) \\ \frac{\partial f}{\partial x^2}(1) \\ \vdots \\ \frac{\partial f}{\partial x^n}(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

we see here that each partial derivative of the function is sort of like the "basis" vectors. In as sense, we are asking to define the tangent space using curves, but in a more controlled manner than choosing curves.

The way we will use them is use the fact that there are enough functions on a manifold M so that $\frac{\partial}{\partial x^i}(f)$ will produce any velocity vector we want at any point. Thus, we will be keeping track of *functions* (i.e. maps from $M \to \mathbb{R}$) which we will evalutate at each point $p \in M$ in the manifold for which their derivative is a tangent vector in $(T_pM)_{\text{kin}}$, and at each point it will give us a point in the tangent space.

Furthermore, we want to accomplish this in a *coordinate independent* way. In particular, notice that when we worked with the gradient, we are technically working with the coordinate chart $(\text{id}, \mathbb{R}^n)$. Naturally, we want to have our definitions in such a way that they are invariant to coordinate charts. The way we will accompish this here is by defining quite an abstract space as our tangent space, and show that partial derivatives of *any* coordinate chart will form a basis for it (i.e. we can choose a particular basis for the tangent space, but we may also work with the abstract definition of the tangent space):

Definition 2.1.4: Tangent Vector's and Derivations

Let M be an n-dimensonal manifold. Then a *tangent vector* v_p *at* p is a linear map $v_p : C^\infty(M) \to \mathbb{R}$ with the *Leibniz Law*, that is for all $f, g \in C^\infty(M)$,

$$v_p(fg) = v_p(f)g(p) + f(p)v_p(g)$$

If v_p be a tangent vector at p (more generally, it is a linear map that respects the Leibniz law), then we say that v_p is a *derivation* of the algebra $C^\infty(M)$ with respect to the evaluation map ev_p at p by $\text{ev}_p(f) = f(p)$, or if it is clear from context that v_p is a *derivation* at p

We often write $v_p f$ or $v_p \cdot f$ instead of $v_p(f)$. The fact that the output of these function is one dimensional shows how they represent a single component. Note too how they can be thought of as element in the dual of $C^\infty(M)$, i.e. $(C^\infty(M))^*$. Don't forget that $C^\infty(M) = C^\infty(M, \mathbb{R})$ so a derivation takes function $f : M \to \mathbb{R}$, and gives out a real number associated to that function. The way I like to think about the fact that we are choosing smooth functions is because that is where the notion of a "graph" is most intuitively understood, and the geometric idea of a tangent space at a graph through derivation's is already understood.

With all this, we can imagine $(T_pM)_{\text{Alg}} \subseteq (C^\infty(M))^*$consiting of all elements in $(C^\infty(M))^*$ that are derivations *at* p. In this way, $(C^\infty(M))^*$ has all tangent spaces. It is clear that adding adding two tangent vectors (i.e. derivations) and scaling returns a tangent vector (the resulting maps respect the Leibniz law), and hence they form a natural vector-space. This vector space will be denoted $(T_pM)_{\text{Alg}}$ and is called the *algebraic tangent space*. Taking $r = \infty$ insead of $r \in \mathbb{N}$ is important here, since the derivations $C^r(M)$ would not give a finite-dimensional vector space (as we will see ref:HERE) and so is not a good condidate for our definition of a tangent space for an n-dimensional manifold.

Remark In fact, they are more than a vector-space since we have define a sort of "multiplication" on their domains. We can take this operation and expand further on it to define special multiplication structure. This will lead to the exploration of "Lie Algebras" in section ref:HERE

Recall that we have defined the following (definition 1.2.4): for any coordinate chart $(\mathbf{x}, U)$

$$\frac{\partial f}{\partial x^i}(p) = D_i(f \circ \mathbf{x}^{-1})(\mathbf{x}(p))$$

This is our main example of a derivation

Definition 2.1.5: Elementary Derivations

Given a chart in an atlas $(U, \mathbf{x}) \in \mathcal{A}$ for an n-dimensional manifold M and $p \in U$, define the operator

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R} \qquad \left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial f}{\partial x^i}(p)$$

Note how we chose a chart when defining these derivations, hence any elementary derivative is with respect to a chart. It is often easier to intuitively see what's going on with this operator in the following: define $c_i : (-\epsilon, \epsilon) \to M$ via $t \mapsto \mathbf{x}^{-1}(\mathbf{x}(p) + \mathbf{e}_i t)$ where $\mathbf{e}_i$ is the ith basis element of the standard basis for $\mathbb{R}^n$ (i.e. bringing in the intuition with tangent space via curves). Then:

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \lim_{h\to 0} \frac{f(c_i(h)) - f(p)}{h}$$

Given this view, we see that $\left.\frac{\partial}{\partial x^i}\right|_p$ is a derivation at a point as we've define in definition 1.2.4, and so it is an element of $(T_pM)_{\text{Alg}}$. Furthermore, each derivation thus corresponds to "lifting" the tangent space at $\mathbf{x}(p)$ of $\mathbb{R}^n$ to T_pM, precisely because we can define this curve c_i for each basis vector e_i of $\mathbb{R}^n$.

This derivation was given the name "elementary" since it is the most important type of derivation. In fact, given some chart $\mathbf{x}$, the collection $\left(\left.\frac{\partial}{\partial x^1}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p \right)$ forms a basis for $(T_pM)_{\text{Alg}}$! In this way, a derivation is *precisely* a map that behaves like a directional derivative, and vice versa a directional derivative is *precisely* a derivation! This turns the definition of a tangnet space in a purely algebraic one: Given a manifold M and a point $p \in M$, then every element in the tangnet space will be of the form:

$$v_p = \sum_i^n v_p(x^i) \left.\frac{\partial}{\partial x^i}\right|_p$$

where $x^i : M \to \mathbb{R}$ is the coordinate function. Furthermore, chart $\mathbf{x}$ will give a basis of the same dimension, and hence the definition is chart independent. We now build up to this result. We require a few properties about derivations, in particular that their behavior is defined by what happens locally:

Lemma 2.1.3: Derivation Properties

Let $v_p \in (T_pM)_{\text{Alg}}$. Then:

1. If $f, g \in C^\infty(M)$ are equal on some neigbhorhood U of p, then $v_p(f) = v_p(g)$
2. If $h \in C^\infty(M)$ is constant ons some neighborhood U of p, then $v_p(h) = 0$

***Proof* :**

1. Since v_p is a linear map, it suffices to show that if $f = 0$ on some neighborhood U of p, then $v_p(f) = 0$, since:

   $$v_p(f) = v_p(g) \quad \Leftrightarrow \quad v_p(f - g) = 0$$

   Firt of all, it is certainly the case that $v_p(0) = 0$ since v_p is linear. Next, extend f to be defined on all of M by letting β be the bump function wiht support in U and $\beta(p) = 1$, so that βf is now identically zero on M. Then:

   $$\begin{aligned} 0 &= v_p(\beta f) \\ &= \beta(p)v_p(f) + v_p(\beta)f(p) \\ &= 1v_p(f) = v_p(\beta)0 \\ &= v_p(f) \end{aligned}$$

   completing the proof

2. As we have just shown, it suffices to assume h is equal to a constant c globally on M. In the case where $c = 1$ so that $h = 1$, then:

$$v_p(1) = v_p(1 \cdot 1) = v_p(1)1 + 1v_p(1) = 2v_p(1)$$

and so $v_p(1) = 0$. If $h = c$, then

$$v_p(c) = c \cdot v_p(1) = c \cdot 0 = 0$$

completing the proof.

Notice how the above theorem show how the values of v_p operate by how a function (or two functions) behave in a neigbhorhood of p, however v_p is defined to take function from $C^\infty(M)$. This hints that derivations have properties of derivaties at a point, which only care about the local behavior of a point, i.e. derivatives act on functions $C^\infty(M)$ if they act on $C^\infty(U)$ for some open neigbhorhood U of p. However, there is nothing in the definition of a derivation that indicates that $(T_pU)_{\text{Alg}}$ can act on $C^\infty(U)$ (unless $U = M$). There is however a natural way to show that $(T_pU)_{\text{Alg}}$ identifies with $(T_pM)_{\text{Alg}}$. Later, with a "fuller understanding of tangent spaces", the following remarks will be natural and automatic.

Proposition 2.1.2: Isomorphism between T_pM and T_pU

Let $p \in U \subseteq M$ with U open. Then there exists a (natural) isomorphism $\Phi : (T_pU)_{\text{Alg}} \to (T_pM)_{\text{Alg}}$ by defining a restriction map from $C^\infty(M) \to C^\infty(U)$.

***Proof* :**
For any $w_p \in (T_pU)_{\text{Alg}}$, define

$$\tilde{w_p} : C^\infty(M) \to \mathbb{R} \qquad \tilde{w_p}(f) = w_p(f|_U)$$

it is clear that $\tilde{w_p}$ is a derivation, and hence i

$$\Phi(T_pU)_{\text{Alg}} \to (T_pM)_{\text{Alg}} \quad w_p \mapsto \tilde{w_p}$$

is a lianer map. It suffices to show that Φ is then isomorphic. Notice that we have yet to estalish that $(T_pU)_{\text{Alg}}$ or $(T_pM)_{\text{Alg}}$ is finite dimensional, and so we must show *both* injectivity and *surjectivity* with no appeal to dimension arguments. For injectivity, we'll show the kernel is trivial. To that end, suppose $\Phi(w_p) = \tilde{w_p} = 0$, that is $\tilde{w_p}(f) = 0$ for all $f \in C^\infty(M)$. Let $h \in C^\infty(U)$. Pick any bump function β with support in U so that βh extends by zero to a smooth function f on all of M that agrees with h on a neigbhorhood of p. Then by lemma 2.1.3, $w_p(h) = w_p(f|_U) = \tilde{w_p}(f) = 0$. Thus, since h was arbitrary, we see that $w_p = 0$, and so Φ has trivial kernel.

Next, we show that Φ is surjective. Pick $v_p \in (T_pM)_{\text{Alg}}$. We wish to define a $w_p \in (T_pU)_{\text{Alg}}$ such that $w_p(h) = v_p(\beta h)$ where β the same as above and βh is extended by zero to be a function in $C^\infty(M)$. This is well-defined, independent of choiec of bump function. If β' was antoher bump function, then βh and $\beta' h$ (both extended to all of M) agree on a neigbhorhood of p, and so by lemma 2.1.3, $v_p(\beta h) = v_p(\beta' h)$. Thus, w_p is well-defined. Now, thinking of $\beta(f|_U)$ as being defined on M, we have that:

$$\tilde{w_p}(f) = w_p(f|_U) = v_p(\beta\, f|_U) \stackrel{!}{=} v_p(f)$$

where the $\stackrel{!}{=}$ equality comes from $\beta\, f|_U$ aggreing on a neigbhorhood of p. Thus, $\Phi : (T_pU)_{\text{Alg}} \to (T_pM)_{\text{Alg}}$ is indeed an isomorphism.

Due to this, we usually identify any $(T_pU)_{\text{Alg}}$ with $(T_pM)_{\text{Alg}}$, and think of the derivation $\frac{\partial}{\partial x^i}\big|_p$ as being simultaneously in $(T_pU)_{\text{Alg}}$ and $(T_pM)_{\text{Alg}}$.

Finally, we require one more lemma in order to decompose smooth functions:

Lemma 2.1.4: Hadamard's Lemma

Let $a \in \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ be a star-convex open neigbhorhood of a. Let $f \in C^\infty(U)$. Then f can be written as :

$$f(x) = f(a) + \sum_{i=1}^{n} (x^i - a^i) f_i(x)$$

Proof **:**
Define $h : [0,1] \to \mathbb{R}$ by

$$h(t) = f(a + t(x - a))$$

Then by the chain rule:

$$h'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a + t(x-a))(x^i - a^i)$$

Thus, we get:

$$\begin{aligned} h(1) - h(0)a &= \int_0^1 h'(t)dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a + t(x-a))(x_i - a_i)dt \\ &= \sum_{i=1}^{n} (x_i - a_i) \int_0^1 \frac{\partial f}{\partial x^i}(a + t(x-a))dt \end{aligned}$$

Furthermore, notice that $h(1) - f(0) = f(x) - f(a)$. Substituing this and setting:

$$f_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(a - t(x-a))dt$$

we get the desired formula.

Theorem 2.1.1: Basis For Derivation

Let M be an n-dimensional manifold and $(U, \mathbf{x})$ a chart with $p \in U$ with $\mathbf{x} = (x^1, x^2, ..., x^n)$. Then the n-tulpe of vectors

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, ..., \frac{\partial}{\partial x^n}\bigg|_p \right)$$

forms a basis for $(T_pM)_{\text{Alg}}$. Furthermore, for each $v_p \in (T_pM)_{\text{Alg}}$,

$$v_p = \sum_{i=1}^{n} v_p(x^i) \frac{\partial}{\partial x^i}\bigg|_p$$

Proof **:**
Since it only matters how derivations are defined locally, we can without loss of generality assume that $\mathbf{x}(U)$ is some convex set, for example an ϵ-ball in $\mathbb{R}^n$. Then we can use a diffeomorphsim to so that $f(p) = 0$. Then by Hadamard's formula, we have that:

$$f = f(p) + \sum f_i x^i$$

If we aply the derivation $\frac{\partial}{\partial x^i}\big|_p$, we get:

$$f_i(p) = \frac{\partial f}{\partial x^i}\bigg|_p$$

Now, if we apply a general derivation v_p to f, we get

$$\begin{aligned} v_p f &= 0 + \sum v_p(f_i x^i) \\ &= \sum v_p(x^i) f_i(p) + \sum x_i(p) v_p f_i \\ &= \sum v_p(x^i) f_i(p) + \sum 0 v_p f_i \\ &= \sum v_p(x^i) \frac{\partial f}{\partial x^i}\bigg|_p \end{aligned}$$

Thus, we see that $v_p = \sum v_p(x^i) \frac{\partial}{\partial x^i}\big|_p$, and so we have a spanning set. It remains to show that this set is linearly independent. To that end assume that:

$$\sum a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

(i.e. the zero-derivation). Applying this to x^j gives us:

$$0 = \sum a^i \frac{\partial x^j}{\partial x^i}\bigg|_p = \sum a^i \delta_{ij} = a^j$$

and since j was arbirary, we get that $a_1 = \cdots = a_n = 0$, showing that these vector's are linearly independent, completing the proof.

Since we will often be working with coordinate chats to choose a basis for out tangent spaces, we are at the risk of running into confusion when representing the tangent space of $\mathbb{R}$ or $\mathbb{R}^n$. For $\mathbb{R}^n$, we usually use the coordinate chart $\mathrm{id}_{\mathbb{R}^n}$ with coordinate functions represented as $x^1, x^2, ..., x^n$. However, we often plug in the "points" $(x_1, x_2, ..., x_n)$ so that we get

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x^i}$$

Be weary that the x_i and x^i represent different things. If $n = 1$, we usually represent the usual coordinate chart by $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial u}$. The basis vector at $t_0 \in \mathbb{R}$ would then be $\frac{\partial}{\partial t}\big|_{t_0}$

Going back to this definition of tangent space, notice how much this approach was local. We could have done this entire process on germs, and indeed it is profitable to take a moment to see how this would happen. Let $\mathcal{F} = C_p^\infty(M, \mathbb{R})$ be the algebra of germs defined near p. Recall that if f is a representative of an equivalence calss of $\mathcal{F}_p$, then we may unambiguously define the value of $[f]$ at p by $[f](p) = f(p)$. Thus, we may define the evaluation map:

$$\mathrm{ev}_p : \mathcal{F}_p \to \mathbb{R}$$

Definition 2.1.6: Derivation With Germs

A *derivation* (with respect to the evaluation map ev_p as defined above) of the algebra $\mathcal{F}_p$ is a map:

$$\mathcal{D}_p : \mathcal{F}_p \to \mathbb{R}$$

such that

$$\mathcal{D}_p([f][g]) = f(p)\mathcal{D}_p[g] + g(p)\mathcal{D}_p[f] \quad \forall [f], [g] \in \mathcal{F}_p$$

The set of all derivations on $\mathcal{F}_p$ is clealry a real vector-space, and is sometimes denoted $\mathrm{der}(\mathcal{F}_p)$. Note too that we don't always make the distinction between f and $[f]$ in the context of $\mathcal{F}_p f$ and $\mathcal{F}_p[f]$ since it usually unambiguous in context.

Now, to define a tangent space from this, let M be a smooth n-manifold. Consider the set of all germs of C^∞ functions $\mathcal{F}_p$ at $p \in M$. Then the vector-space $\mathrm{Der}(\mathcal{F}_p)$ of derivations of $\mathcal{F}_p$ with respect to the evalutation map ev_p could also be taken as the definition of the tangent sapce at p, since we saw in lemma 2.1.3 and proposition 2.1.2 that the properties of a germ are completely dependent on what happens locally. The main advantage of this formulation is for a purely algebraic definition of tangnet space: Let C_a^∞ denote the ring of germs of C^∞ functions on a manifold at a point a

1. Show that C_a^∞ has a unique maximal ideal, given by $\mathfrak{m}_a = \{f \in C_a^\infty \mid f(a) = 0\}$

***Proof* :**
Note that C_a^∞ is a ring under pointwise addition and multiplication in the codomain, since if $f \in [f]$, $g \in [g]$ are two representatives of two elements in C_a^∞, then $f + g \in [f + g]$ and $fg \in [fg]$, since any representive of $[f]$ and $[g]$ must agree with f and g on some open neighorhood.

First, $\mathfrak{m}_a$ is indeed an ideal, since for any $[f] \in \mathfrak{m}_a$ and $[g] \in C_a^\infty$, $[g][f](a) = [gf](a) = 0$. Next, it is indeed unique: let's say $[f] \notin \mathfrak{m}_a$, so that $[f](a) \neq 0$. Since f is smooth on it's domain, it is smooth in some local neighborhood of a, say U. Importantly, by continuity of

f, we can shrink U so that $f|_U$ is nonzero everywhere. Then define: $(f|_U)^{-1}(x) = 1/f(x)$, which exists since $f|_U$ is nonzero on U, which also makes it smooth since taking the quotient is smooth if the denominator is nonzero. Finally, it is simply a matter of using partitions of unity to extend $f|_U^{-1}$ from U to the entire domain of f: let's label this new function simply as f^{-1}. Then $[f^{-1}] \in C_a^\infty$, and $[f][f^{-1}] = [ff^{-1}] = [1]$, and since multiplication is well-defined (i.e. not dependent on the representative) every $[f] \notin \mathfrak{m}_a$ has a multiplicative inverse in C_a^∞, and hence $[f]$ is a unit and cannot be part of any proper ideal.

Thus, $\mathfrak{m}_a$ is indeed unique. This then makes it maximal, since any larger ideal would have to contain units as we've just shown, and hence $\mathfrak{m}_a$ is the unique maximal ideal of C_a^∞

2. Prove that if $M = \mathbb{R}^n$, then $\mathfrak{m}_a$ is generated by $x_1 - a_1$, ..., $x_n - a_n$ where $a = (a_1, a_2, ..., a_n)$

***Proof* :**
Let $[f] \in \mathfrak{m}_a$, and $f \in [f]$ be any representative. Then by Hadamard's formula[a], there exists an open ball U and smooth functions $f_i \in C^\infty(U)$ such that:

$$f(x) = f(a) + \sum_{i=1}^{n}(x_i - a_i)f_i(x) = \sum_{i=1}^{n}(x_i - a_i)f_i(x)$$

where since f is smooth, so is any degree of partial differentiation or integral of partial differentiation. We see that any element has $x_i - a_i$ as a factor for approparite i, and so any f is some sum the elements $x_1 - a_1$, ..., $x_n - a_n$ times some smooth functions, meaning they generate the ideal, as we sought to show.

[a] Modifying the proof from Meinrenken a bit, take $h(t) = f(a + t(x - a))$, then taking it's derivative $h'(t)$, and then setting $h(1) - h(0) = \int_0^1 h(t)dt$, since $f(x) - f(a) = h(1) - h(0)$, Hadamard's formula immediately follows with the appropriate substitution $f_i(x)$ for the integral

3. Show that $C_a^\infty/\mathfrak{m}_a$ is a one-dimensional $\mathbb{R}$-vector space, and that, if $\dim M = n$, then $\mathfrak{m}_a/\mathfrak{m}_a^2$ is an n-dimensional $\mathbb{R}$-vector space

***Proof* :**
I claim that $[1] + \mathfrak{m}_a$ generate $C_a^\infty/\mathfrak{m}_a$ as an $\mathbb{R}$-vector-space. To see this, I'll show that if $[f](a) = p$, then in the quotient $[f] + \mathfrak{m}_a = [p] + \mathfrak{m}_a$ where p is the consant function $p(x_1, x_2, ..., x_n) = p$.

Since $f(a) = p$ and $p(a) = p$, $f(a) - p(a) = 0$, so $[p - f] \in \mathfrak{m}_a$. But then:

$$[f] + \mathfrak{m}_a = [f - f + p] + \mathfrak{m}_a = [p] + \mathfrak{m}_a$$

showing that the only information to know which coset of $\mathbb{C}_a^\infty/\mathfrak{m}_a$ we are in is where a germ $[f]$ maps a. But then, if $[f](a) = p$, then $[f] = [p] = p \cdot [1]$ (where we treat p as a real number at the end), meaning that $[1]$ is a genertor for $C_a^\infty/\mathfrak{m}_a$ as a vector-space over $\mathbb{R}$.

Next, to see that $\mathfrak{m}_a/\mathfrak{m}_a^2$ is an n-dimensional $\mathbb{R}$ vector-space, consdier the map:

$$\varphi : \mathfrak{m}_a \to \mathbb{R}^n \qquad [f] \mapsto \left(\frac{\partial f}{\partial x^1}(a), ..., \frac{\partial f}{\partial x^n}(a)\right)$$

which is well-defined since any other representative $g \in [f]$ agree's with f on a local domain, and hence on partial derivatives. The map φ is $\mathbb{R}$-linear by the properties of partial derivatives, and surjective by elementary calculus (for any $(t_1, t_2, ..., t_n) \in \mathbb{R}^n$, we can construct an element with the appropriate derivative using Taylor's Theorem). Thus,

$$\mathfrak{m}_a / \ker(\varphi) \cong \mathbb{R}^n$$

I claim that $\ker(\varphi) = \mathfrak{m}_a^2$. By the produc rule we have that:

$$\varphi([fg])_i = \frac{\partial f}{\partial x^i}(a)g(a) + f(a)\frac{\partial g}{\partial x^i}(a) = 0 + 0 = 0$$

Giving us $\mathfrak{m}_a^2 \subseteq \ker(\varphi)$. So we want to show that if $\varphi([h]) = 0$ then $h \in \mathfrak{m}_a^2$. We get this through Taylor's theorem: assuming the chart $\mathbf{x}$ is centered at a, since $h(a) = 0$ and $\frac{\partial h}{\partial x^i} = \frac{\partial(h \circ \mathbf{x}^{-1})}{\partial}(\mathbf{x}(a)) = 0$ for all x^i, we get that the first two terms of the taylors expansion are zero, and that h will be the sum of partial fractions starting at order 2, but this immdeiately gives our result. Hence $\ker(\varphi) \subseteq \mathfrak{m}_a^2$, and so equality holds, telling us that:

$$\mathfrak{m}_a / \mathfrak{m}_a^2 \cong \mathbb{R}^n$$

showing it is an n dimensional vector-space as we sought to show.

4. An alternative definition of the tangent space of M at a is the dual space $\tilde{T}_a M := (\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. Show that a C^∞ mapping $f : M \to N$ induces a linear mapping $\partial_a f : \tilde{T}_a M \to \tilde{T}_{f(a)} N$ which satisfies $\partial_a(g \circ f) = \partial_{f(a)} g \circ \partial_a f$

Proof **:**
This comes down to noticing we can define $\partial_a f$ essentially identically to our regular definition of tangent space, since both tangent spaces are defined in terms of the collection of functional on $C_a^\infty(M)$, that is, for any $v_a \in \tilde{T}_a(M)$, define:

$$(\partial_a f)(v_a)([h] + \mathfrak{m}_{f(a)}^2) = v_a([h \circ f] + \mathfrak{m}_a^2)$$

This map is well-defined: if $[h] + \mathfrak{m}_{f(a)}^2 = [h + g_1 g_2] + \mathfrak{m}_{f(a)}^2$, then the image of the map is $h \circ f + g_1 g_2 \circ f$, and by the chain rule we see that $g_1 g_2 \circ f$ have a taylor series expansion with the order 0 and 1 term still 0, showing that it is still within $\mathfrak{m}_a^2$, and so $[h \circ f] + \mathfrak{m}_a^2 = [h \circ f + g_1 g_2 \circ f] + \mathfrak{m}_a^2 = [h \circ f] + \mathfrak{m}_a^2$, showing $\partial_a f$ is well-defined.

But then, this is just the map for the tangent space! Linearity and the chain rule thus follow from the same proofs for $T_a f$.

Relating Definitions

We know show how each of definition of a tangent space are isomorphic. Let M be an n-dimensional (smooth) manifold. Consider first a tangent vector v_p as the equivalence class of curves represented by $c_p^v : I \to M$ with $c_p^v(0) = p$ and $(c_p^v)'(0) = v$. We obtain a derivation by defining:

$$v_p(f) = \left.\frac{d}{dt}\right|_{t=0} f \circ c_p^v$$

This gives us a map $(T_pM)_{\text{kin}} \to (T_pM)_{\text{Alg}}$, which can be shown to be an isomorphism (TBD). Similarly, for any $[c] \in (T_pM)_{\text{kin}}$, we obtain an element $v_p \in (T_pM)_{\text{phys}}$ by letting v_p be the equivalence class of triples $(p, v, (U, \mathbf{x}))$ where $v^i := \frac{d}{dt}\big|_{t=0} x^i \circ c$ for a chart $(U, \mathbf{x})$ with $p \in U$.

Next, we can a map from alg to phys and vice versa. for alg to phys, pick a derivation v_p at p and let $(U, \mathbf{x})$ be an admisible chat $p \in U$. Then v_p, as a tangent vector in the sense given via charts, is represented by the tripple $(p, v, (U, \mathbf{x}))$ where $v = (v^1, ..., v^n)$ given by:

$$v^i = v_p x^i$$

where v_p is acting as a derviation. Thus, this gives us a map $(T_pM)_{\text{alg}} \to (T_pM)_{\text{phys}}$ that can also be shown to be an isomorphism.

Finally, given $[(p, v, (U, \mathbf{x}))] \in (T_pM)_{\text{phys}}$ where $v \in \mathbb{R}^n$, we can obtain a derviation by defining:

$$v_p f = D(f \circ \mathbf{x}^{-1})(\mathbf{x}(p)) \cdot v$$

In other words:

$$v_p f = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p f$$

It should be checked that v_p is independent of representatives, showing this map is indeed well-defined, and then showing it's isomorphic.

Hence, when talking about the tangent space T_pM, we will freely swhitch between these 3 interpretations of the tangent space,

(Moved this seciton from John lee from *Velocity Vectors of Curves* since I think it works well here!)

Definition 2.1.7: Curve

Let $\gamma : I \to M$ be a continuos map form an interval $I \subseteq \mathbb{R}$. Then γ is called a *curve* in M. The element $\gamma'(0)$ is called the *velocity vector*, and is equal to:

$$\gamma'(t_0) = T_{\gamma(t_0)}\gamma \left(\left. \frac{\partial}{\partial t} \right|_{t_0} \right) \in T_{\gamma(t_0)}M$$

Note that the map γ it self is part of the information of what a curve is, not just $\gamma(I)$. If we need to be precise, then γ along with $\gamma(I)$ is called a *parametrized curve*. We also wrote d/dt instead of $\partial/\partial t$ due to convention from calculus when when the manifold is 1-dimenisonal. Other notations for the velocity is:

$$\dot{\gamma}(t_0) \qquad \frac{d\gamma}{dt}(t_0) \qquad \left. \frac{\partial \gamma}{\partial t} \right|_{t=t_0}$$

This has a very simple derivation, for any smooth $f : M \to \mathbb{R}$

$$\gamma'(t_0) = T_{\gamma(t_0)}\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) f = \left. \frac{d}{dt} \right|_{t_0} (f \circ \gamma) = (f \circ \gamma)'(t_0)$$

2.1.1 Regular submanifold Tangent Space

Proposition 2.1.3: Regular Submanifold Tangent Space Equivalent

Let M denote a C^r submanifold of $\mathbb{R}^n$ of dimension k. Prove that the tangent space TM_a coincides with each of the linear subspaces of $T\mathbb{R}^n_a$ (these subspaces correspond to those in proposition 1.3.2)

1. $\ker Df(a)$
2. $\{(Y, Z) \in \mathbb{R}^n_a \mid Z = Dg(b)Y\}$
   where we have written $a = (b, c)$ with respect to the coordinate (y, z) and where (Y, Z) (with $Y = (Y_1, Y_2, ..., Y_k)$, $Z = (Z_1, Z_2, ..., Z_{n-k})$) denotes the coordinate the coordinates of $\mathbb{R}^n_a$ corresponding to the coordinates (y, z) of $\mathbb{R}^n$
3. $Dh(a)^{-1}((\mathbb{R}^k \times \{0\})_{h(a)})$
4. $D\varphi(b)(\mathbb{R}^k_b)$ where $a = \varphi(b)$

Proof **:**
We want to show that $M_a = (a) = (b) = (c) = (d)$. We will start by showing $M_a = (a) = (c)$, then show $(a) = (d)$, and finish by showing $(b) = (d)$, which will then give us a chain of equalities.

We'll first show that $M_a = \ker Df(a)$, where $f : U \to \mathbb{R}^{n-k}$ and $f^{-1}(0) = M \cap U$ with rank $n - k$ at every point in $M \cap U$. Starting with $M_a \subseteq \ker Df(a)$, let $v \in M_a$, so that there is a $\gamma_v : (-\delta, \delta) \to \mathbb{R}^n$ where $\gamma_v((-\delta, \delta)) \subseteq M$ such that $\gamma_v(0) = a$ and $\gamma_v'(0) = v$. Since the domain of γ_v is open and U is open, choose δ so that $\gamma_v(x) \in M \cap U$ for all $x \in (-\delta, \delta)$. Then by the chain rule:

$$D(f \circ \gamma_v)(0) = Df(\gamma_v(0)) \circ D\gamma_v(0) = Df(a) \circ D\gamma_v(0)$$

For notational clarity on what's to come, we will write $\gamma'(0) = D\gamma_v(0)$, and treat $\gamma_v'(0)$ as the input for $Df(a)$, giving us

$$Df(a)(\gamma_v'(0))$$

Now, since $\gamma_v(x) \subseteq M \cap U$, and $f(y) = 0$ for all $y \in M \cap U$, $f(\gamma(x)) = 0$ for all $x \in (-\delta, \delta)$. Hence, $f \circ \gamma(x)$ is in fact the constant zero function on $(-\delta, \delta)$, and so

$$Df(a)(\gamma_v'(0)) = Df(a)(v) = 0$$

Since this is true for all $v \in M_a$, we have that $M_a \subseteq \ker Df(a)$.

To show $M_a \supseteq Df(a)$, we'll use the fact that $\ker Df(a)$ is a k-dimensional vector space and that M_a has k-linearliy independent vectors, implying the dimension of M_a must be at least k-dimensional, which would imply $M_a = \ker Df(a)$ (since then the inclusion map $M_a \to \ker Df(a)$ would be an inclusion map of two vector-spaces with the same finite dimension, which we know implies they are isomorphic).

To accomplish this, we first will find a basis of an appropriate tangent space $\mathbb{R}^k_{b'}$, and then we will do a "pushforward" to make it a basis of M_a. To find this basis, first recall from proposition 1.3.2 that the existence of an f in question 1 implies the existence of a C^r diffeomorphism h such that $h(M \cap U) = V \cap (\mathbb{R}^k \times \{0\}^{n-k})$, which in particular implies the existesnce of a C^r diffeomorphism

h^{-1} such that $h^{-1}(V \cap (\mathbb{R}^k \times \{0\}^{n-k})) = M \cap U$. Since $a \in M \cap U$ and h^{-1} is injective, there exists a unique b such that $h^{-1}(b) = a$. Furthermore, since V is open, $V_k = \pi_k(V \cap (\mathbb{R}^k \times \{0\}^{n-k}))$ is open in $\mathbb{R}^{k\,a}$ (given the appropriate coordinates to project onto), and hence is a submanifold of $\mathbb{R}^k$ (it can be proven by defining $\varphi : V_k \to \mathbb{R}^k$ from part (d) from proposition 1.3.2). Then it is easy to see that the tangent space of V_k at b' where $b = (b', c)$ is $\mathbb{R}^k_{b'}$, in particular

$$\mathbb{R}^k_{b'} = \left\{\beta'(0) \;\middle|\; \beta : (-\delta, \delta) \to \mathbb{R}^k,\ \beta((-\delta, \delta)) \subseteq V_k,\ \beta(0) = b\right\}$$

Since the tangent space is $\mathbb{R}^k_{b'}$, we can choose a basis $\{\beta_i'(0)\}_{i=1}^k \subseteq \mathbb{R}^k_{b'}$ consisting of k linearly independent vectors. This basis can naturally be embedded in $\mathbb{R}^n_b$ (the tangent space of V at b, for similar reasons as V_k) and form a linearly independent subset of order k in $\mathbb{R}^n_b$. We will use the same notation for this linearly independent.

Now, we want to “pushforward” this linearly independent set $\{\beta_i'(0)\}_{i=1}^k$ into M_a. To accomplish this, let

$$\gamma_i = h^{-1} \circ \beta_i$$

notice that:

1. $\gamma_i(0) = h^{-1}(\beta_i(0)) = h^{-1}(b) = a$
2. $\gamma_i((-\delta, \delta)) \subseteq M \cap U$

where we can choose an appropriately small δ since we are working with finitely many elements. Thus, the collection$\{\gamma_i'(0)\} \subseteq M_a$. Furthermore:

$$D\gamma_i(0) = D(h^{-1} \circ \beta_i)(0) \overset{!}{=} Dh^{-1}(b) \cdot \beta_i'(0)$$

where $\overset{!}{=}$ comes from the chian rule. As before, we can write this as:

$$Dh^{-1}(b)(\beta_i'(0))$$

And furthermore, $Dh^{-1}(b)$ is a *linear isomorphism* since h^{-1} has full rank! Thus, it sends linearly independent sets to linearly independent sets, meaning M_a has a linearly independent set of order k. Combining with our previous inclusion, it follows that

$$\dim M_a \leq \dim \ker Df(a) \leq \dim M_a$$

and hence the dimensions are equal, and so $M_a = \ker Df(a)$.

Note too that this shows $(a) = (c)$ since β_i' formed a basis for $\mathbb{R}^k_b = \mathbb{R}^k_{h(a)}$, and hence span $(\mathbb{R}^k \times \{0\})_{h(a)}$, and so through the previous argument we showed both that $Dh^{-1}(b)(\mathbb{R}^k \times \{0\})_{h(a)}) \subseteq \ker Df(a)$ and that the dimensions match, letting us conclude they are also equal.

Next, we prove $(a) = (d)$. First, notice that:

$$f(\varphi(W)) = 0$$

since $\varphi(W) = M \cap U$, and $f(M \cap U) = 0$. Thus, by the chain rule, for any $a \in V$:

$$D(f \circ \varphi)(a) = Df(\varphi(a)) \circ D\varphi(a)$$

Since φ maps into $M \cap U$, and $f(M \cap U) = 0$, then the derivative of any point in $M \cap U$ must be 0, hence we get

$$Df(\varphi(b)) \circ D\varphi(b) = 0$$

In particular, this means that $\operatorname{im} D\varphi(b) \subseteq \ker f(a)$. Next, by definition of φ, it has rank k on W, meanining $D\varphi(b)$ has rank k, and so it's image must be k-dimensional, and so

$$D\varphi(b)(\mathbb{R}^k_b) = \ker Df(a)$$

Finally, to show $(b) = (d)$, let

$$B := \{(Y, Z) \mid Z = Dg(b)Y\}$$

We'll show $B = D\varphi(b)(\mathbb{R}^k_b)$ (where $a = \varphi(b)$). To show this, notice that

$$\varphi(x) = (x, g(x))$$

since $\varphi(W)$ maps bijectively onto $M \cap U$, which is equal to $(W, g(W))$. Thus, the derivative at b is:

$$D\varphi(b) = \begin{pmatrix} I \\ Dg(b) \end{pmatrix}$$

and so:

$$D\varphi(b)(x) = \begin{pmatrix} I \\ Dg(b) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \sum_{i=1}^k D_1 g^i(b) x_i \\ \vdots \\ \sum_{i=1}^k D_k g^i(b) x_i \end{pmatrix}$$

With careful observation, we see that this is in fact the result we wanted, that is, plugging in any $x \in \mathbb{R}^k_b$ gives us that

$$D\varphi(b) \in \{(Y, Z) \mid Z = D\varphi(b)Y\}$$

In fact, the set on the right hand side is the image of $D\varphi(b)$, meaning they are in fact equal.

Thus, we have shown all these sets are equal, completing the proof.

[a] The projection was taken so that elements of V_k will be of the form $(v_1, ..., v_k)$ instead of $(v_1, ..., v_k, 0, ..., 0)$

2.1.2 Tangent Space of Vector Space

If V Is a finite dimensional vector space over $\mathbb{R}$, since $V \cong_{\mathbb{R}} \mathbb{R}^n$, we see that our unofficial definition of tangent space provided in the introduction works just as well for V: $\{p\} \times V \cong T_pV$. We would like to make this rigorous. We define the map $\iota_p : V \to T_pV$ which in the first definition would

$$v \mapsto [(t \mapsto p + tv)]$$

in the second definition (given a basis chart E) would be:

$$v \mapsto [(V, E^{-1}, E^{-1}(v))]$$

and in the third definition would be:

$$f \mapsto \left.\frac{d}{dt}\right|_0 f(p+tv)$$

Next, we have the natural isomorhpim $\pi_2 : \{p\} \times V \to V$, hence the map $\iota_p \circ \pi_2$ is an isomorphism from $\{p\} \times V$ to T_pV.

2.2 Tangent Map

Since we have 3 definition of tangent space, we will have 3 equivalent definitions of tangent map. The first map we will consider the main definition, and the other two will be equivalent definitions

Definition 2.2.1: Tangent Map I

Let $f : M \to N$ be a smooth map. between two manifolds and consider $p \in M$ with image $q = f(p) \in N$. Then the *tangent map* at p:

$$T_pf : T_pM \to T_pN$$

is defined as follows: Given $v_p \in T_pM$, we pick a curve c with $c(0) = p$ and $c'(0) = v$ so that $v_p = [c]$. Then:

$$T_pf(v_p) = [f \circ c] \in T_qN$$

i.e. $[f \circ c]$ is the vector represented by the curve $f \circ c$

Some authors write f_{*p} for the tangent map. Given any coordinate map, this curve is represented via it's derivations.

Definition 2.2.2: Tangent Map II

Let M be a smooth n-manifold, $f : M \to N$ a smooth map, and consider a point $p \in M$ with image $q = f(p) \in N$. Choose a chart $(U, \mathbf{x})$ containing p and a chart $(V, \mathbf{y})$ containing $q = f(p)$ so that for any $v_p \in T_pM$, it can be represented by $(p, v, (U, \mathbf{x}))$.

Then the *tangent map* $T_pf : T_pM \to T_qN$ is defined by letting by letting the representative of $T_pf(v)$ in the chart $(V, \mathbf{y})$ be given by $(f(p), w, (V, \mathbf{y}))$ where

$$w = D(\mathbf{y} \circ f \circ \mathbf{x}^{-1})(\mathbf{x}(p))(v)$$

Note that this will indeed uniquely determine $T_pf(v)$, an the chain rule guarantees this is well-defined. This definition of the tangent map is most useful when there are nice coordinate's at play. For example, using this definition, show that:

$$\iota : M \to M \times N$$

has tangent map $T_p\iota$ where $v \mapsto (v, 0)$ – this comes down to chosing the “obvious” coordinate charts. Another usefulness with this definition is that we can often work with f in coordinate representations. For example, let $f : M \to M$ for some manifold M and let's say we know that T_pf has no fixed points. Then by this definition of tangent map we have that $D(\mathbf{x} \circ f \circ \mathbf{x}^{-1})(\mathbf{x}(p))$ has no fixed points. Furthermore, if $f(v) = v$, then $f(v) = v$ in coordinate representations with the same patch, that is $\mathbf{x} \circ f \circ \mathbf{x}^{-1}(\mathbf{x}(p)) = \mathbf{x}(p)$. Now, if for any neighorhood around p there was another fixed point, say $f(p+v) = p+v$ for some $0 < |\varphi(v)| < \epsilon$, then we can see (do this as an exercise) that:

$$\frac{|f(p+v) - f(v) - D_p(v)|}{|v|} > 0$$

But then, as $v \to 0$, we get that this value is always greater than 0, but that contradicts that f is differentiable at p!

Finally, we define a definition that is "coordinate-free"[3]

Definition 2.2.3: Tangent Map III

Let M be a smooth n-manifold, $f : M \to N$ a smooth map, and consider a point $p \in M$ with image $q = f(p) \in N$. Choose a chart $(U, \mathbf{x})$ containing p and a chart $(V, \mathbf{y})$ containing $q = f(p)$ so that for any $v_p \in T_pM$, it can be represented by $(p, v, (U, \mathbf{x}))$.

Then the *tangent map* $T_pf : T_pM \to T_qN$ is defined by letting $T_pf(v_p)$ be the derivation defined so that for each $g \in C^\infty(N)$:

$$(T_pf(v_p))(g) = v_p(g \circ f)$$

We will usually shorten the notation to T_pv_p to represent the element v_p maps to.

Notice that if $g \in C^\infty(N)$, then $g \circ f \in C^\infty(M)$, and so $v_p(g \circ f)$ is indeed well-defined. There is a slight notational inconvenience here in that $T_pv_p \in T_{f(p)}N$, even though the subscripts on the left hand side are all with respect to p; this is something to watch out for. This is the definition that we will use the most often, and is in fact perhaps the most convinient for computation. We will first expand on some theoretical properties of the tangent map, and follow by how to actually find T_pf. Most of the time, we will be using the 3rd definition of tangent map, but any of the 3 definitions will work. Sometimes, we will want to make a distinction between the notation for tangent map and tangent spaces. In those cases, we will write D_pf or df_p instead of T_pf.

Proposition 2.2.1: Property of tangent map

Let $f : M \to N$ and $g : N \to P$ be smooth maps. Then:

1. $T_pf : T_pM \to T_{f(p)}N$ is linaer
2. $T_p(g \circ f) = (T_{f(p)}g) \circ T_pf$
3. $T_p \operatorname{id} = \operatorname{id}_{T_pM} : T_pM \to T_pM$
4. If f is a diffeomorphism, then T_pf is a linear isomorphism, and $(T_pf)^{-1} = T_{f(p)}(f^{-1})$

[3] I put it in quotes since it is coordinate-free in the sense that there is sort of a "natural" basis for this, but when doing most computations we will be in coordinates

Proof :

1. First:

$$\begin{aligned}(T_p v_p)(g+h) &= v_p((g+h)\circ f)\\ &= v_p(g\circ f + h\circ f)\\ &\stackrel{!}{=} v_p(g\circ f) + v_p(h\circ f)\\ &= T_p(v_p)(g) + T_p(v_p)(h)\end{aligned}$$

and:

$$\begin{aligned}T_p v_p(ch) &= v_p(ch\circ f)\\ &\stackrel{!}{=} cv(h\circ f)\\ & cT_p v_p(h)\end{aligned}$$

where $\stackrel{!}{=}$ comes from linearity of derivations in both cases.

2. Let $f : M \to N$ and $g : N \to p$. Calculting the left hand side first, that is$T_p(g\circ f)$, we get that any derivation in $T_{(g\circ f)(p)}P$ is defined by taking every $h : P \to \mathbb{R}$ to:

$$(T_p v_p)(h) = v_p(h\circ (g\circ f))$$

For the right hand side, first realise that $T_p f$ takes any derivation from $T_p M$ to $T_{f(p)}(N)$, and any image within $T_{f(p)}(N)$ is defined by taking $k : N \to \mathbb{R}$ to:

$$(T_p v_p)(k) = v_p(k\circ f)$$

Similarly, $T_{f(p)}g$ takes any derivation from $T_{f(p)}N$ to $T_{g\circ f(p)}P$, and any image within $T_{g\circ f(p)}P$ is defiend by taking every $h : P \to \mathbb{R}$ to:

$$T_{f(p)}v_{f(p)}(h) = v_{f(p)}(h\circ g)$$

Combining these results, we get that $T_{f(p)}g(T_p v_p)$ maps any $h : P \to \mathbb{R}$ to:

$$(T_{f(p)}(T_p v_p))(h) = T_p v_p(h\circ g) = v_p(h\circ g\circ f)$$

which is exactly what we got for the left hand side, showing both sides are equal, hence:

$$T_p(g\circ f) = (T_{f(p)}g)\circ T_p f$$

3. Since $\mathrm{id} : M \to M$, $T_p\,\mathrm{id} : T_p M \to T_p M$, and for any $T_p\,\mathrm{id}(v_p)$, we get:

$$(T_p v_p)(h) = v_p(h\circ \mathrm{id}) = v_p(h)$$

showing it is the identity.

4. Since $T_p f$ is always linear (as we've shown in part (1)), it suffices to check bijectivity. Since f is a diffeomorphism, there exists an inverse smooth map f^{-1} such that

$$f\circ f^{-1} = f^{-1}\circ f = \mathrm{id}$$

Thus, for any point $p \in M$:

$$\mathrm{id}_{T_pM} = T_p(\mathrm{id}) = T_p(f^{-1} \circ f) = T_{f(p)}f^{-1} \circ T_pf$$

and

$$\mathrm{id}_{T_{f(p)}M} = T_{f(p)}(\mathrm{id}) = T_{f(p)}(f \circ f^{-1}) = T_pf \circ T_{f(p)}f^{-1}$$

showing that T_pf has a two-sided inverse $T_{f(p)}f^{-1}$, thus:

$$(T_pf)^{-1} = T_{f(p)}f^{-1}$$

Note that the converse of the 4th proepry of the previous proposition is not true: take $f : \mathbb{R} \to S^1$ mapping $x \to e^{2\pi i x}$. Then f is certainly not a diffeomorphism, not being injective, but it is a linear isomorphism at every point. In fact, it is a *local diffeomorphism* at every point. The exploration of this concept will be central to us in chapter 3. One result we shall prove in that chapter and state now to demonstrate how T_pf captures information locally is the following partial converse of the 4th property:

Theorem 2.2.1: Inverse Mapping Theorem For Manifolds

Let $f : M \to N$ be a smooth map such that $T_pf : T_pM \to T_qN$ is an isomorphism. Then there exists an open neighborhood O of p such that $f(O)$ is open and $f|_O : O \to f(O)$ is a diffeomorphism. If T_pf is an isomorphism for all $p \in M$, then $f : M \to N$ is a local diffeomrophism

***Proof* :**
see theorem ref:HERE

There are a few important consequence of proposition 2.2.1. Points (1), (2), and (3) give us that T_p is a functor:

Definition 2.2.4: Tangent Functor

Let Man^* be the collection of pointed manifolds. Then we can define the *tangent functor* T that takes (M, p) to T_pM and $f : (M, p) \to (M, q)$ to the linear map $T_pf : T_pM \to T_qN$

Property 4 of proposition 2.2.1, gives us a way of showing that T_pM is an m-dimensional manifold much more rapidly:

Corollary 2.2.1: Dimension Of Tangent Space

Let M be an m-dimensional manifold. Then for each T_pM, $\dim(T_pM) = m$

***Proof* :**
Choose $p \in M$ and a chart $(U, \mathbf{x})$ containing p. As we commented, $\mathbf{x} : U \to \mathbf{x}(U)$ is a diffeomorphism (notice the restriction of the domain is important so that the map is surjective). Say $\mathbf{x}(U) = \hat{U}$. Then by proposition 2.2.1(4) $T_p\mathbf{x} : T_pU \to T_{\mathbf{x}(p)}\hat{U}$ is a linear isomrophism. Then, by

proposition 2.1.2, $T_pM \cong T_pU$ and $T_{\mathbf{x}(p)}\hat{U} \cong T_{\mathbf{x}(p)}\mathbb{R}^m$, and so by transitivity:

$$T_pM \cong T_{\mathbf{x}(p)}\mathbb{R}^m \cong \mathbb{R}^m$$

where it follwos that $\dim T_pM = m$, as we sought to show.

Note that from this, it follows that $T_pM \cong T_qM$, since they are both isomorphic to $\mathbb{R}^m$. In this way, every tangent space has identical vector-space structure. It does not follow that, $T_{f(p)}N \cong T_{f(q)}N$, as can be seen in example 1.12. On the other hand, it is not right to say that T_pM is identical to T_qM, since they might have differently "placed" in space (think of the tangent spaces on a circle). In section 2.4, we will show how to keep track of this "orientation" information. We might ask whether all the tangent spaces on a manifold can be "placed" in such a way so that they really are identical, just being a translation of each other. We will answer when this is possible in chapter 6.

Now that we have defined a tangent map, notice that we can use proposition 2.1.2 in tandem with the chain rule to show that if $U \subseteq M$ is an open submanifold, then $T_pU \cong T_pM$. From now on, we will always treat the two synonymously. Since $\mathbb{R}^n$ and $T_p\mathbb{R}^n$ are both n-dimensional vector-spaces over $\mathbb{R}$, we know from basic linear algebr that they they must be isomorphic:

$$\mathbb{R}^n \cong T_p\mathbb{R}^n$$

Similarly, since any n dimensional vector space over $\mathbb{R}$ has n dimensional tangent spaces, we have that:

$$V \cong T_pV$$

This makes linear maps translate particularly well with the tangent map:

Proposition 2.2.2: Tangent Space To Vector Space

Let V be a finite dimensional vector space with the standard smooth structure. Then $V \cong T_pV$ for each $p \in V$, and for any linear map $L : V \to W$, the following diagram commutes:

$$\begin{array}{ccc} T_pV & \xrightarrow{T_pL} & T_pW \\ \sim\uparrow & & \sim\uparrow \\ V & \xrightarrow{L} & W \end{array}$$

Proof **:**
John p. 59, it's unwinding the definition

We close of this section by mentioning that we have not mentioned the tangent map between manifolds with borders. These maps are more technical and require some care to insure the tangent spaces at the boundary are well-behaved. We shall use a theorem known as the *constant rank theorem* in our efforts to study maps between manifolds with borders, and so delay our study of them until chapter 3 in which the constant rank theorem is proven.

2.2.1 Computing Tangent Map

Let $f : M \to N$ be a smooth map with corresponding $T_pf : T_pM \to T_{f(p)}N$. Since T_pf is a linaer map, we would like a nice way of representing it. We will first start with the case wehere $N = \mathbb{R}^n$ and try to identify how $\frac{\partial}{\partial x^i}\big|_p$ acts with respect to our usual notion of partial derivative. To find this representation, we first recall from the proof of corollary 2.2.1 that we have shown that there exits an isomorphism:

$$T_p\varphi : T_pM \to T_{\varphi(p)}\mathbb{R}^n$$

and that given a chart $\mathbf{x}$ containing $\varphi(p)$, $T_{\varphi(p)}\mathbb{R}^n$ has as basis $\left(\frac{\partial}{\partial x^1}\big|_{\varphi(p)}, ..., \frac{\partial}{\partial x^n}\big|_{\varphi(p)}\right)$. In particular, we can take $\mathbf{x} = \text{id}$, and so these are simply the functionals giving the partial derivative of a function f. Since $T_p\varphi$ is an isomorphism, their pre-image under φ is a basis for T_pM:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (T_p\varphi)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = T_{\varphi(p)}(\varphi^{-1})\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right)$$

We will denote this basis:

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p\right)$$

where we are using the same $\frac{\partial}{\partial x^i}$ instead of changing coordinates to emphasize the identification between the coordintes of T_pM and it's coordinate representation Now, using the definition of the tangent map given in definition 2.2.3, we get :

$$\left.\frac{\partial}{\partial x^i}\right|_p(f) = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}(f \circ \varphi^{-1}) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(p)) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

where $\hat{f} = f \circ \varphi^{-1}$ and $\hat{p} = (p^1, p^2, ..., p^n) = \varphi(p)$. Thus, $\frac{\partial}{\partial x^i}\big|_p$ is the derivation of the coordinate represetnation of f at (the coordinate representation of) p. The vectors $\frac{\partial}{\partial x^i}\big|_p$ are called the *coordinate vectors at* p. If $M = \mathbb{R}^n$, then are simply the partial derivative operators. Thus, we may intepret elements in T_pM as sums of derivations of representations of f.

With this understanding under our belt. We would better like to undersatnd T_pf. Before we incorporate what we've just learnt about elements in T_pM for an arbitrary tangent space, it would be good to check that T_pf did in fact generlize our notion of a deivative, that is if $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, then $T_pf = D(f)(p)$. Let $(x^1, x^2, ..., x^n)$ be the standard coodinate functions for $\mathbb{R}^n$ and $(y^1, y^2, ..., y^m)$ the standard coordinate functions for $\mathbb{R}^m$. The simply computing how T_pf acts on the basis, for any $h \in C^\infty(M)$,

$$\begin{aligned}
T_pf\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(h) &= \left.\frac{\partial}{\partial x^i}\right|_p(h \circ f) \\
&= \frac{\partial(h \circ f)}{\partial x^i}(p) \\
&= \sum_i \frac{\partial h}{\partial y^i} f(p) \frac{\partial f^i}{\partial x^i}(p) && \text{chain rule} \\
&= \sum_j \left(\frac{\partial f^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{f(p)}\right)(h)
\end{aligned}$$

Thus,

$$T_p f\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_j \frac{\partial f^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{f(p)} \tag{2.1}$$

i.e., it is the Jacobian matrix:

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \cdots & \frac{\partial f^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(p) & \cdots & \frac{\partial f^m}{\partial x^n}(p) \end{pmatrix}$$

Thus, in the case of $T_p f : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$, we get the usual total derivative $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$.

We know have all the information to understand $T_p f : T_p M \to T_{f(p)} N$. If we chose coordinate patches that contains p and $f(p)$ respectively, we will get the above matrix, which we showed earlier is the coordinate representation of f at the coordinate represetnation of p. Hence, we have properly generlized the notion of derivative of f! The definition of a tangent map was specificely chosen so that we do not need to reference any basis, however it is nice to see that when a basis is chosen recover our previous results.

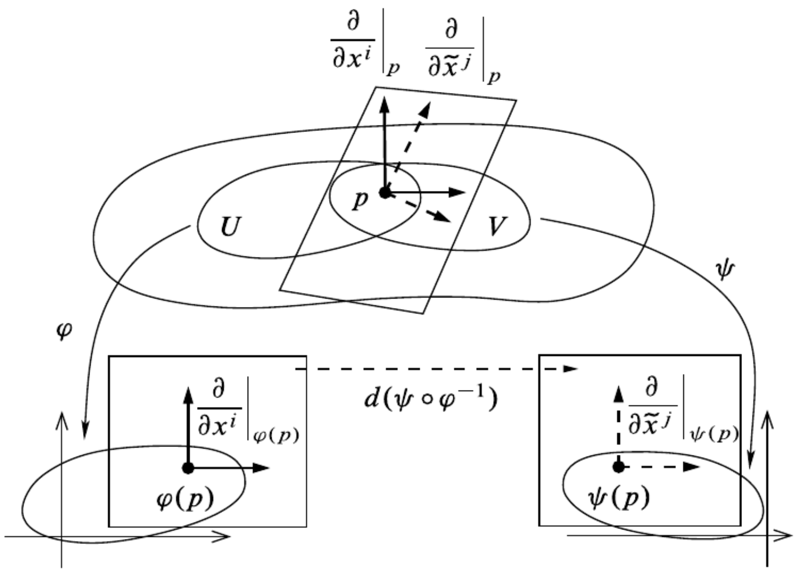
2.2.2 Change of Coordinates

The last thing we need to check is what happens in a change of coordinates. Let $(U, \mathbf{x})$ and $(V, \mathbf{y})$ be two coordinate charts where $U \cap V \neq \emptyset$. Let $\mathbf{x} = (x^1, x^2, ..., x^n)$ and $\mathbf{y} = (y^1, y^2, ..., y^n)$ be tuple of coordinate functions. Then any tangent vector at $p \in U \cap V$ (where $\mathbf{x}(p) = x$) can be represented in a basis in terms of $\left(\partial/\partial x^i|_p\right)$ or $\left(\partial/\partial y^i|_p\right)$. The transition mapping will be

$$\begin{aligned} \mathbf{y} \circ \mathbf{x}^{-1}(x) &= \mathbf{y}(\mathbf{x}^{-1}(x)) \\ &= (y^1(\mathbf{x}^{-1}(x)), ..., y^n(\mathbf{x}^{-1}(x))) \\ &= (y^1(p), ..., y^n(p)) \\ &\overset{!}{=} (y^1(x), ..., y^n(x)) \end{aligned}$$

we here have an abuse of notation: y^1 is a function from $U \cap V \to \mathbb{R}$, while $x \in \mathbf{x}(U)$. We here identify p with $\mathbf{x}^{-1}(x)$ where the last line is a typical shothand to represent the transition mapping. Hence, by equation (2.1), $T_{\mathbf{x}(p)}(\mathbf{y} \circ \mathbf{x}^{-1}) : T_{\mathbf{x}(p)}\mathbf{x}(U) \to T_{\mathbf{y}(p)}\mathbf{y}(U)$ can be represented as:

$$T_{\mathbf{x}(p)}(\mathbf{y} \circ \mathbf{x}^{-1})\left(\left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}(p)}\right) = \sum_j \frac{\partial y^j}{\partial x^i}(\mathbf{x}(p)) \left.\frac{\partial}{\partial y^j}\right|_{\mathbf{y}(p)}$$



Thus, we see that the basis transforms as:

$$
\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p &= T_{\mathbf{x}(p)}(\mathbf{x}^{-1}) \left(\frac{\partial}{\partial x^i}\bigg|_{\mathbf{x}(p)} \right) \\
&= T_{\mathbf{y}(p)}(\mathbf{y}^{-1}) \circ T_{\mathbf{x}(p)}(\mathbf{y} \circ \mathbf{x}^{-1}) \left(\frac{\partial}{\partial x^i}\bigg|_{\mathbf{x}(p)} \right) \\
&= T_{\mathbf{y}(p)}(\mathbf{y}^{-1}) \left(\sum_j \frac{\partial y^j}{\partial x^i} (\mathbf{x}(p)) \; \frac{\partial}{\partial y^i}\bigg|_{\mathbf{y}(p)} \right) \\
&= \sum_j \frac{\partial y^j}{\partial x^i}(\hat{p}) \; \frac{\partial}{\partial y^j}\bigg|_p
\end{aligned}
$$

where $\hat{p} = \mathbf{x}(p)$. Applying this to components of a vector:

$$
\begin{aligned}
v &= \sum_i v^i \; \frac{\partial}{\partial x^i}\bigg|_p \\
&= \sum_i v_i \left(\sum_j \frac{\partial y^j}{\partial x^i}(\hat{p}) \; \frac{\partial}{\partial y^j}\bigg|_p \right) \\
&= \sum_i \sum_j \frac{\partial y^j}{\partial x^i}(\hat{p}) v_i \; \frac{\partial}{\partial y^j}\bigg|_p \\
&= \sum_j \tilde{v}^j \; \frac{\partial}{\partial y^j}\bigg|_p
\end{aligned}
$$

where

$$\tilde{v}^j = \sum_i \frac{\partial y^j}{\partial x^i}(\hat{p})v^i$$

(important exercises given by John Lee p.64)

2.3 Tangents of Products

Proposition 2.3.1: Tangent Space To A Product Manifold

Let $M_1, M_2, ..., M_k$ be smooth manifolds, and let $\pi_i : M_1 \times M_2 \times \cdots \times M_k \to M_i$ be the natural projection map. Then for any $p = (p_1, p_2, ..., p_k) \in M_1 \times M_2 \times \cdots \times M_k$, the map:

$$\alpha : T_p(M_1 \times M_2 \times \cdots \times M_k) \to T_{p_1}M_1 \times T_{p_2}M_2 \times \cdots \times T_{p_k}M_k$$

defined by:

$$\alpha(v) = (T_p(\pi_1)(v), ..., T_p(\pi_k)(v))$$

is an isomorphism. The same is true if one of the spaces is a manifold with boundary.

***Proof* :**
Let M, N be C^∞ manifolds. We will show it for $M \times N$, the result generalizes naturally for any number of manifolds.

1. Show that the projection map given by $\pi_1 : M \times N$ and $\pi_2 : M \times N$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are smooth.

   Proof. we'll show that π_1 is smooth since the proof follows exactly for π_2 up to a change in notation.

   Let $(U \times V, \mathbf{x} \times \mathbf{y})$ be some chart on the product manifold $M \times N$. Notice that $\pi_1(U \times V) = U$, and so choose the chart $(U, \mathbf{x}))$ on the codomain M (this way, we only need to check there is a chart for every (p, q) and some chart containing p that form smooth representations of π_1). Then:

   $$\begin{aligned} &\mathbf{x} \circ \pi_1 \circ (\mathbf{x}^{-1} \times \mathbf{y}^{-1})(p, q) \\ &= \mathbf{x} \circ \pi_1(\mathbf{x}^{-1}(p), \mathbf{y}^{-1}(q)) \\ &= \mathbf{x}(\mathbf{x}^{-1}(p)) \\ &= p \end{aligned}$$

   showing that π_1 is the projection map from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$, which we know to be smooth. Since the charts and points where arbitrary, this works for any compatible charts, and so π_1 (hence also π_2) is smooth $\square$

2. Show that the mapping

   $$T(M \times N)_{(x,y)} \to TM_x \oplus TN_y \quad v \mapsto ((\pi_1)_{*(x,y)}(v), (\pi_2)_{*(x,y)}(v))$$

   is an isomorphism of vector-spaces

Proof. First, note that the left hand side had dimension $n+m$ and the right hand side has dimension $n+m$. Thus, it suffices to show that the map is linear and injective. Taking advantage of vhat we've shown in the previous part, if we take as basis for $T(M \times N)_{(x,y)}$ given the chart $\mathbf{x} \times \mathbf{y}$ to be the set

$$\left(\frac{\partial}{\partial z^1}\bigg|_{(x,y)}, ..., \frac{\partial}{\partial z^{n+m}}\bigg|_{(x,y)} \right)$$

then by definition of each basis element, given that we are working with the product manifold $M \times N$, and chart $\mathbf{x} \times \mathbf{y}$ we would get:

$$\frac{\partial}{\partial z^i}\bigg|_{(x,y)} = \begin{cases} \frac{\partial}{\partial x^i}\big|_{(x,y)} & 1 \le i \le m \\ \frac{\partial}{\partial y^{i-m}}\Big|_{(x,y)} & m+1 \le i \le m_+n \end{cases}$$

Thus, by part (a), notice that for any derivation $v \in \mathrm{der}(C^\infty_{(x,y)}(M \times N))$, when represented in the above basis as $(a^1, ..., a^{m+n})$, we get:

$$(\pi_1)_{*(x,y)}(v) = \begin{pmatrix} 1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 0 \end{pmatrix} \begin{pmatrix} a^1 \\ \vdots \\ a^{n+m} \end{pmatrix}$$

$$(\pi_2)_{*(x,y)}(v) = \begin{pmatrix} 0 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 1 \end{pmatrix} \begin{pmatrix} a^1 \\ \vdots \\ a^{n+m} \end{pmatrix}$$

where the first matrix is n by $n+m$ and the second matrix is m by $n+m$. Now, we show that the given map is a (linaer) isomophism. For reference, let f represent the map in the question. Clearly, f is linear, since

$$\begin{aligned} f(a+b) &= ((\pi_2)_{*(x,y)}(a+b)(\pi_2)_{*(x,y)}(a+b)) \\ &= ((\pi_2)_{*(x,y)}(a)(\pi_2)_{*(x,y)}(a)) + ((\pi_2)_{*(x,y)}(b)(\pi_2)_{*(x,y)}(b)) \\ &= f(a) + f(b) \end{aligned}$$

Next, consider $f(v) = 0$, so

$$((\pi_1)_{*(x,y)}(v), (\pi_2)_{*(x,y)}(v)) = (0, 0)$$

then by the represetnation we chose for π_1 and π_2, we see that π_1 forces the fist n variables to be 0 ,while π_2 forces the last m variables to be 0, thus $v = 0$ showing injectivity.

Since the map f is injective and linear and the dimensions match, f is bijective, and hence an isomrophism, as we sought to show. $\square$

3. For a fixed point $y \in N$, there is an embedding $i_y : M \to M \times N$ by $i_y(x) = (x, y)$. Prove that the tangent mapping

$$(\iota_y)_{*x} : TM_x \to T(M \times N)_{(x,y)} \cong TM_x \times TN_y$$

is given by $v \mapsto (v, 0)$

Proof. Using the isomorphism shown in part (b), we'll show that:

$$(\iota_y)_{*x}(v) = ((\pi_1)_{*(x,y)} \circ (\iota_y)_{*(x,y)}(v), (\pi_1)_{*(x,y)} \circ (\iota_y)_{*(x,y)}(v))$$

Computing the definition using the chain rule, we get that for any $h \in C^\infty_{(x,y)} M \times N$, we get:

$$(v(h \circ \pi_1 \circ \iota_y), v(h \circ \pi_2 \circ \iota_y)) = (v(h), v(h(y))$$

in particular, we get a constant function in the second component, which as we've shown in class implies that $v(h(y)) = 0$. Since h was arbitrary, we see that:

$$v(h \circ \pi_1 \circ \iota_y) = v(h) \quad v(h \circ \pi_2 \circ \iota_y) = 0$$

Thus:

$$v \mapsto (v, 0)$$

as we sought to show. □

4. Let $f : M \to P$ and $g : N \to Q$ be smooth mappings. Show that the mapping

$$f \times g : M \times N \to P \times Q$$

defined by $(f \times g)(x, y) = (f(x), g(x))$ is smooth, and its tangent mapping

$$(f \times g)_{*(x,y)} : T(M \times N)(x, y) \to T(P \times Q)_{(f(x),g(x))}$$

is given by $(v, w) \mapsto (f_{*x}(v), g_{*y}(w))$ under the isomorphism in b

Proof. First, $f \times g$ is clearly smooth: given coordinates an points $p \in M$ and $q \in M$, we can choose charts $(U_1, \mathbf{x}_1), (U_2, \mathbf{x}_2)$, $(V_1, \mathbf{y}_1)$, $(V_2, \mathbf{y}_2)$ where $f(U_1) \subseteq U_2$ and $g(V_1) \subseteq V_2$ which make

$$\mathbf{x}_2 \circ f \circ \mathbf{x}_1^{-1} \qquad \mathbf{y}_2 \circ g \circ \mathbf{y}_1^{-1}$$

smooth, and thus

$$(\mathbf{x}_2 \times \mathbf{y}_2^{-1}) \circ (f \times g) \circ (\mathbf{x}_1^{-1} \times \mathbf{y}_1^{-1})$$

is smooth on $p, q \in M \times N$. Since this was true for arbitray points and $(f \times g)(U_1 \times V_1) \subseteq U_2 \times V_2$, $f \times g$ is smooth.

Next, let $v \in \text{der}(C^\infty_{(x,y)}(M \times N))$. Using part (b), let $v = (v_1, v_2)$, where by part (c) we have $(\iota_y)_{*x}(v_1) = (v_1, 0)$ and $(\iota_x)_{*y}(v_2) = (0, v_2)$. Hence we can write $v = (\iota_y)_{*x}(v) + (\iota_x)_{*y}(v)$. Thus, using the isomrophism we found in part (b), we simply compute:

$$\begin{aligned}
(f \times g)_{*(x,y)}(v) &= ((\pi_1 \circ (f \times g))_{*(x,y)}(v), (\pi_2 \circ (f \times g))_{*(x,y)}(v)) \\
&= (f_{*x}(v), g_{*y}(v)) \\
&= (f_{*x}((\iota_y)_{*x}(v) + (\iota_x)_{*y}(v)), g_{*y}((\iota_y)_{*x}(v) + (\iota_x)_{*y}(v))) \\
&= (f_{*x}((\iota_y)_{*x}(v)) + f_{*x}((\iota_x)_{*y}(v)), g_{*y}((\iota_y)_{*x}(v)) + g_{*y}(\iota_x)_{*y}(v))) \\
&\stackrel{!}{=} (f_{*x}((\iota_y)_{*x}(v)) + 0, 0 + (\iota_x)_{*y}(v)) \\
&= (f_{*x}(v_1), g_{*y}(v_2))
\end{aligned}$$

where $\stackrel{!}{=}$ comes from the fact that f_* and g_* are taking in a constant functions, and hence are 0 derivations, giving us that

$$(f \times g)_{*(x,y)}(v, w) = (f_{*x}(v), g_{*y}(w))$$

as we sought to show. □

2.4 Tangent bundle

So far, we have dealt with individual tangent spaces. Each of these tangent are treated disjointly. However, as we showed when computing the tangent map with respect to a coordinate chart U, we had the smooth functions $\frac{\partial f}{\partial x^i}$ to represent the linear function from T_pM To $T_{f(p)}N$ at any given $p \in M$. In other words, there is a smooth function that gives us coordinates between tangent spaces. It then perhaps reasonable to think that we may think that the collection of tangent spaces has more structure to it. We explore this very concept in this section, and show how we may find "higher order derivatives" between smooth manifolds. We start by defining the collection of all tangent spaces for a manifold:

Definition 2.4.1: Tangent Bundle

Let M be a manifold. Then the *Tangent bundle* is the set:

$$TM = \bigsqcup_{p \in M} T_pM$$

along with a natural projection map $\pi : TM \to M$ sending $v_p \in T_pM$ to p. The collection TM is reffered to as the *tangent bundle* and the projection π is called the *tangent bundle projection map*.

Example 2.1: Tangent Bundle

let's consider the case where $M = \mathbb{R}^n$. Then TM is the collection of all $T_p\mathbb{R}^n$. As we've shown, $T_p\mathbb{R}^n$ can be identified with $\{p\} \times \mathbb{R}^n$. Thus, we may see $T\mathbb{R}^n$ as:

$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$

Any element in this cartesian product can be interpreted as (p, v), or as the deriation $D_v|_p$. Note that we cannot always do this canonically (or sometimes at all): For example, the S^1 will not have such a canonnical isomorphism, while S^2 will not at all be of this form (we show this in chatper 6)

Given a tangent bundle and a smooth map, we have a natural way of defining a map between $\bigsqcup_p T_pM$ and $\bigsqcup_{f(p)} T_{f(p)}N$:

Definition 2.4.2: Tangent Map

Let $f : M \to N$ be a smooth map. Then given a tangent map defined at every point $p \in M$, T_pf on T_pM, the map:

$$Tf : TM \to TN$$

is defiend as the map T_pf on each tangent space. The resulting map Tf is called the *tangent map* or *tangent lift* of f.

The set TM on it's own has no inherent structure to it besides being a disjoint union of sets, and so it does not (yet) make sense to think of Tf as anthing more than a set map that restricts to linear maps on each T_pM. What will let make TM into more than a set is the projection map π, since it links TM to M in a meaningful manner; in particular, similar to the notion of initial topology, we will define the "initial smooth structure" on TM given the map π which will make π smooth in a natural way.

Before proceeding to this proof, We start by showing how to naturally lift the map f to the tangent bundles (we'll show this map is smooth soon)

Example 2.2: Tangent Map

Let $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ be open subsets such that $f : U_1 \to U_2$ is smooth (or at least C^1). Then we have a tangent map $Tf : TU_1 \to TU_2$. By our discussion in example 2.1, we can view $TU_1 \cong U_1 \times \mathbb{R}^n$ and $TU_2 \cong U_2 \times \mathbb{R}^m$. Then through this identification, we can define the tangent map to be:

$$(p, v) \mapsto (f(p), Df(p)(v))$$

Proposition 2.4.1: Tangent Bundle Is Manifold

Let M be an m-dimensional manifold. Then TM has a natural topological and smooth structure induced by M making TM a $2m$-dimensional manifold. With this structure, the tangent bundle projection map $\pi : TM \to M$ is smooth.

Proof **:**

We first find an atlas for TM given the atlas for M via the natural map π. Let $(U, \mathbf{x})$ be a smooth chart for M. Then $\pi^{-1}(U) \subseteq TM$ is the set of all tangent vectors at all points of U. If $\mathbf{x} = (x^1, x^2, ..., x^n)$, then define the map $\tilde{\mathbf{x}} : \pi^{-1}(U) \to \mathbb{R}^{2m}$ by:

$$\tilde{\mathbf{x}}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (x^1(p), ..., x^m(p), v^1, ..., v^m)$$

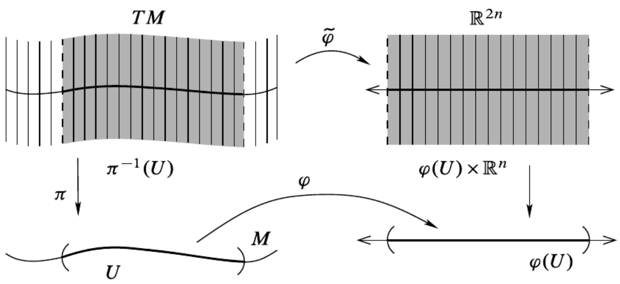


Figure 2.1: visualization of map

Notice that $\tilde{\mathbf{x}}(\pi^{-1}(U)) = \mathbf{x}(U) \times \mathbb{R}^m$, which is an open subset of $\mathbb{R}^{2m}$. This map is bijective onto it's image, since we can explicitly find:

$$\tilde{\mathbf{x}}^{-1}(x^1, ..., x^m, v^1, ..., v^m) = v^i \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}^{-1}(x)}$$

We next check these charts are compatible: Take some two charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ for M, and let

$(\pi^{-1}(U), \tilde{\mathbf{x}})$ and $(\pi^{-1}(V), \tilde{\mathbf{y}})$ be the associated charts for TM. Then:

$$\tilde{\mathbf{x}}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \mathbf{x}(U \cap V) \times \mathbb{R}^m$$
$$\tilde{\mathbf{y}}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \mathbf{y}(U \cap V) \times \mathbb{R}^m$$

are both open in $\mathbb{R}^{2m}$, and we get the transition map to be:

$$\tilde{\mathbf{y}} \circ \tilde{\mathbf{x}}^{-1} : \mathbf{x}(U \cap V) \times \mathbb{R}^m \to \mathbf{y}(U \cap V) \times \mathbb{R}^m$$

$$\begin{aligned}
&\tilde{\mathbf{y}} \circ \tilde{\mathbf{x}}^{-1}(x^1, ..., x^m, v^1, ..., v^m) \\
&\quad = \tilde{\mathbf{y}}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_{\mathbf{x}^{-1}(x)} \right) \\
&\quad \overset{!}{=} \tilde{\mathbf{y}}\left(\sum_i \frac{\partial y}{\partial x^i} v^i \left.\frac{\partial}{\partial y^i}\right|_{\mathbf{y}^{-1}(x)} \right) \\
&\quad = \left(y^1(x), ..., y^m(x), \sum_j \frac{\partial y^1}{\partial x^j}(x) v^j, ..., \sum_j \frac{\partial y^m}{\partial x^j}(x) v^j \right)
\end{aligned}$$

where $\overset{!}{=}$ comes from the chain rule, and so the map is clearly smooth, showing that TM has a smooth structure.

What remains to show that TM is a manifold is that it can be countably covered and is hausdorff. The topology induced by the atlas is the natural one: W is open if and only if $W \cap \pi^{-1}(U)$ is open for every U in a chart for M. The fact that it's countably covered comes from the fact that M is countably covered by some open cover $\{U_i\}_{i=1}^{\infty}$, and so $\{\pi^{-1}(U_i)\}_{i=1}^{\infty}$ is a countable cover of TM. To show that TM is Hausdroff, note that if we have $p \neq q$, then there exists U, V containing p and q respectively such that $U \cap V = \emptyset$, so $\pi^{-1}(U) \cap \pi^{-1}(V) = \emptyset$. If on the other hand we consider two distinct points in the same fiber $\pi^{-1}(p)$, the both of these points lie in $\mathbb{R}^m_p$, and so clearly can be separabled by open balls, and so TM is indeed a manifold.

Finally, to show that π is smooth, given any chart $(U, \mathbf{x})$ and associated chart $(\pi^{-1}(U), \tilde{\mathbf{x}})$, we get the coordinate represetnation:

$$\pi(p, v) = p$$

which is clearly smooth, completing the proof.

Corollary 2.4.1: Tangent Bundle When Manifold Covered With One Chart

Let M be a smooth m-manifold that can be covered with one chart. Then:

$$TM \cong M \times \mathbb{R}^m$$

***Proof* :**
Let $(M, \mathbf{x})$ be the chart for M. In particular M is diffeomorphic to $\mathbf{x}(M)$, say $\mathbf{x}(M) = U$ so that $U \cong \mathbf{x}(M)$ where $U \subseteq \mathbb{R}^m$ or $\mathbb{H}^m$. By what we've shown in the last proposition, TM is in bijection with $U \times \mathbb{R}^m$, and the map $\tilde{\mathbf{x}}$ is our desired diffeomorphism.

Corollary 2.4.2: Tangent Functor

Let $f : M \to N$ be a smooth map. Then given an atlas on M and N, $Tf : TM \to TN$ defined by :

$$Tf(x^1, ..., x^m, v^1, ..., v^m) = \left(f^1(x), ..., f^m(x), \sum_i \frac{\partial f^1}{\partial x^i}(x)v^i, ..., \sum_i \frac{\partial f^m}{\partial x^i}(x)v^i \right)$$

is smooth. It follows that $T : \mathbf{Man} \to \mathbf{Man}$ is a functor, called the *tangent functor*. Furthermore, $(Tf)^{-1} = T(f^{-1})$.

***Proof* :**
The fact that T is a functor follow from our previous propositions, the fact that it's smooth follows form the definition of the atlas, and $(Tf)^{-1} = Tf^{-1}$ follows from the previous proposition.

Now that we have shown TM is manifold and Tf is smooth, all the result of proposition 2.2.1 generalize quite naturally to TM:

Proposition 2.4.2: Properties of Tangent Map

Let $f : M \to N$ and $g : N \to P$ be smooth maps. Then:

1. $T(g \circ f) = (Tg) \circ Tf$
2. $T\,\mathrm{id} = \mathrm{id}_{TM} : TM \to TM$
3. If f is a diffeomorphism, then Tf is a bundle isomorphism (a linear isomorphism for each $T_p f$), and $(Tf)^{-1} = T(f^{-1})$

***Proof* :**
exercise

The map $Tf : TM \to TN$ has an additional property takes into account the additional structure, namely the map π that is given for any tangent bundle. Notice that Tf sends fibers $\pi_M^{-1}(p)$ to fibers $\pi_N^{-1}(f(p))$. This is immediate from the definition, and gives that any tangent bundle map commutes in the following diagram:

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

When a diagram like the one above commutes, we say that Tf is a *fiber bunde homomorphism*. In chapter 6, we shall explore this concept further when we generalize the notion of tangent bundle to general fiber bundles.

Just like for manifolds, we may construct a tangent bundle using quotients: if $\{(U_i, \varphi_i)\}_i$ is a countable, locally finite atlas for M, Then we define $\bigsqcup_i U_i \times \mathbb{R}^n$ and glue points via:

$$(x, v) \sim (y, w) \qquad \Leftrightarrow \qquad y = \varphi_j \varphi_i^{-1}(x) \quad w = D_x(\varphi_j \varphi_i^{-1})(v)$$

All choices of atlas's will produce naturally isomorphic manifold structures.

3

Smooth Maps and Submanifolds

Now that we have shown the link between a smooth map f and it's tangent lift Tf, we will explore what type of information we can deduce about f given properties of Tf. The reason this is useful is that Tf is just a collection of linear maps smoothly connected together (or if you want parameterized by the manifold). Hence, we can find some local information about f given the study of Tf. In this way, we may think of smooth manifolds as being easier to work with than general topological manifolds.

We start by defining an invariant on T_pf up to isomorphism, an see when this invariant holds for Tf (when it will be "constant" for all of the tangent bundle). This invariant will be called the *rank*, and if the rank is constant on Tf (i.e. f has constant rank), the map will be of constant rank. We shall see that if f has constant rank, the representative of f can be really nicely represented. Since T_pf is a linear map, injective surjective and bijective maps are all strong conditions. We shall see how T_pf being injective/surjective/bijective (along with f being constant rank) effects the nature of the map f. This exploration will also allow us to generalize our results from section 1.3 for submanifold of M given a constant rank map $f : M \to N$.

The tools we'll build from constant rank maps will allow us to return to our discussion about submanifolds. In particular, we may revisit some questions we left earlier such as when is the image of a smooth map a manifold, what equivalent conditions on a subset $S \subseteq M$ give us that S is a submanifold, as well as answer new questions such as what is the tangent space of a submanifold, and how do we find submanifolds with boundary?

Not all maps are constant rank. Fortunately, all maps are "almost" constant rank, as in they fail to be constant rank at a measure zero amount of points. This is in fact rather powerful, and we will use this result in a few important ways. Notably, we shall show that every manifolds embeds into $\mathbb{R}^N$ for sufficinetly large N, that any continuous function is uniformally approximated by smooth functions, that every continuous function is homotopic to a smooth function, and that the intersection of two manifold $M \cap N$ is almost aways a manifold, and if not there is an arbitrarily small "jiggle" we may do to insure the interseciton is a submanifold.

These results will allow us to better understand the behavior of intersecting manifolds, giving way to an introduction to *morse theroy* and *intersection theory*.

3.1 Rank

The key property that we can define on Tf (independent of it's basis) is it's *rank* at every point. In fact, the rank is the *only* property that distinguishes different linear maps, since the basis number is the unique feature that defines a vector-space up to isomorphism, and the rank is invariant under isomorphism:

Definition 3.1.1: Rank

Let f be a smooth function. Then the *rank* of f at p is the rank of T_pf. If f has the same rank at each point, then we say that f has *constant rank*. If f has the largest possible rank at p (i.e. $\operatorname{rank} T_pf = \min(\dim(M), \dim(N))$), then f is said to be *full rank* a p. If f is full rank everywehre, we say that f is *full rank* (i.e. largest possible constant rank p for every $p \in M$)

Note that "rank" is a diffeomrophism-invariant proeprty, meaning the rank of a map can be studied through representatives. Where T_pf is injective or surjective will make a big difference in the behavior of f, and so we give such maps a name:

Definition 3.1.2: Immersion and Submersion

Let $f : M \to N$ be a smooth map between manifolds. Then:

1. f is called an *immersion at p* if $T_pf : T_pM \to T_{f(p)}N$ is an injection. If f is an immersion at every point, we call f an *immersion*, or is said to have *constant rank*
2. f is called an *submersion at p* if $T_pf : T_pM \to T_{f(p)}N$ is a surjection. If f is an submersion at every point, we call f an *submersion*.

An easy mistake to make is to imagine that an immersion does not produce any self-intersections, does not (possibly periodically) retrace themselves, or approach a part of itself in some limiting way. All of these are possible, as we'll see in the example, since T_pf is a *local property*. We will require some global restriction (i.e. that the map is a homeomorphism) in order to make sure the the global image of f also behaves nicely. For the proceeding theorem's, we will require the following important result:

Lemma 3.1.1: Full Rank Matrices

Let $M(m,n)$ denote all $m \times n$ matrices and $M(m,n;k)$ the set of all $m \times n$ matrices of rank k. Then $M(n,n;k)$ is an sub submanifold of $M(m,n)$ of dimension $k(m+n-k)$. in particular, if $k = \min(m,n)$, $M(m,n;k)$ is an open submanifold.

***Proof* :**

1. For each $X_0 \in M(m, n; k)$, there are permutations matrices P and Q such that

$$PX_0Q = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

   Where A_0 is $k \times k$ nonsingular

   Proof. This is simply an appeal to linear algebra. Since the matrix is of rank k, there exists elementary matrices which when multiplied X_0 from the right shifts k linearly independent columns to the left, and similarly there exists elementary matrices which shift k linearly independent rows to the top. Let P and Q represent these elementary matrices. Then PX_0Q is in the form we desire. □

2. There exists $\epsilon > 0$ such that A is nonsingular whenever all entries of $A - A_0$ are $< \epsilon$

   Proof. We take advantage of the fact that $\det : M(k, k) \to \mathbb{R}$ is continuous (namely, it is a polynomial). Since $\det(A) = a > 0$, There exists an open neigbhorhood, say an ϵ ball, around A such that $\det(B_\epsilon(A)) \in (a - \delta, a + \delta)$ for some $\delta < a$. By definition, for any $A_0 \in B_\epsilon(A)$, $A - A_0 < \epsilon$, and each A_0 are nonsingular. □

3. If

$$Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(m, n)$$

   where A is $k \times k$ nonsingular, then Y has rank k if and only if $D = CA^{-1}B$ [Hint: If I_k denotes the $k \times k$ identity, then

$$\begin{pmatrix} I_k & 0 \\ Z & I_{m-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ ZA + C & ZB + D \end{pmatrix}$$

   Proof. Notice that multiplying by the matrix does not change the rank of the resulting matrix since the matrix on the left hand side has full rank. Let $Z = -CA^{-1}$ so that we get:

$$\begin{pmatrix} A & B \\ -CA^{-1}A + C & -CA^{-1}B + D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix}$$

   Now, first assuming that Y is rank k, since A is $k \times k$ nonsingular, we can do row and column operations to reduce it to the identity I_k:

$$\begin{pmatrix} I_k & B' \\ 0 & -CA^{-1}B + D \end{pmatrix}$$

   we can then use this matrix to eliminate B:

$$\begin{pmatrix} I_k & 0 \\ 0 & -CA^{-1}B + D \end{pmatrix}$$

   Since Y is of rank k, it must be that $-CA^{-1}B + D = 0$, i.e. $CA^{-1}B = D$

Conversely, if $D = CA^{-1}B$, then:

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

and since the rank of the matrix is limited by the fact there are k linearly independent rows, it must be that Y is of rank k (since A is $k \times k$ nonsingular). □

4. $M(m, n; k)$ is a submanifold of $M(m, n)$ of dimension $k(m + n - k)$, for all $k \leq m, n$

Proof. For matrix $X \in M(m, n; k)$, choose an open ball $B_\epsilon(X) \subseteq M(m, n; k)$ (which we have established exists by part (b)) and a function

$$f : B_\epsilon(X) \to \mathbb{R}^{(m-k)(n-k)} = \mathbb{R}^{mn-k(m+n-k)}$$

such that after transforming X so that A is $k \times k$ non-singular (which is a linear transformation, and hence a composition by a diffeomorphism), we get that:

$$f(X) = f\begin{pmatrix} A & B \\ C & D \end{pmatrix} = CA^{-1}B$$

Notice that P is simply a matrix of polynomial functions, and hence smooth. Furthermore, by part (c), $f^{-1}(0) = M(m, n; k) \cap B_\epsilon(X)$, and f has rank $mn - k(m + n - k)$ for each matrix in $M(m, n; k) \cap B_\epsilon(X)$. Thus, by our first homework, $M(m, n; k)$ is regular submanifold of $M(m, n)$, completing the proof. □

In general, it is hard to guarantee that f has constant rank on a open neighborhood. Fortunately, there is two examples in which it is easy to deduce this result:

Proposition 3.1.1: Immersion or Submersion Then Local Constant Rank

Let $f : M \to N$ be a smooth map and $p \in M$. Then

1. If $T_p f$ is injective, then there exists a neighborhood of p, say U, such that $f|_U$ is an immersion (i.e. $T_p f$ is injective on a neighborhood U)
2. If $T_p f$ is surjective, then there exists a neighborhood of p, say U, such that $f|_U$ is a submersion (i.e. $T_p f$ is surjective on a neighborhood U)

Proof **:**
Notice that both cases satisfy the definition of full rank. Then we have shown in in the above proposition that the set of $M_{mn}(\mathbb{R})$ matrice of full rank is an open subset of $M_{mn}(\mathbb{R})$. Thus, by continuity of f, f has full rank on some open set containing p.

Example 3.1: Immersions and Submersions

1. Take $T^2 = S^1 \times S^1$ with the usual manifold structure. Paramaterize T^2 through the use of angles, that is a point in T^2 can be represented as $(e^{i\theta_1}, e^{i\theta_2})$. The torus can famously be

immersed into $\mathbb{R}^3$ through the following map:

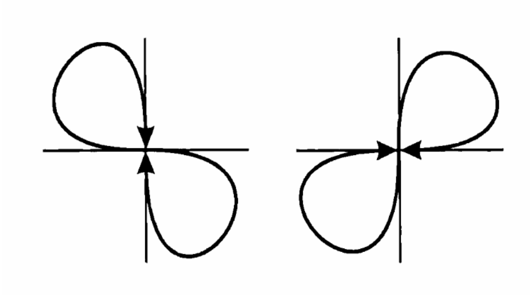
$$(e^{i\theta_1}, e^{i\theta_2}) \mapsto (x(e^{i\theta_1}, e^{i\theta_2}), y(e^{i\theta_1}, e^{i\theta_2}), z(e^{i\theta_1}, e^{i\theta_2}))$$

where:

$$\begin{aligned} x(e^{i\theta_1}, e^{i\theta_2}) &= (a + b\cos(\theta_1))\cos(\theta_2) \\ y(e^{i\theta_1}, e^{i\theta_2}) &= (a + b\cos(\theta_1))\sin(\theta_2) \\ z(e^{i\theta_1}, e^{i\theta_2}) &= b\sin(\theta_1) \end{aligned}$$

for any $a, b \in \mathbb{R}$. What conditions on a and b do we need so that this map is $1-1$?

2. more here TBD



3. More on p. 78 John Lee

Maps where $T_p f$ is a linear isomorphism will be special. Given f is a diffeomorphism around a neighborhood of p, then $T_p f$ is certainly a linear isomorphism. In the next section, we shall show the converse too. For now, we define the concept for future reference

Definition 3.1.3: Local Diffeomorphism

Let M and N be two smooth manifolds of the same dimension. Then a map $f : M \to N$ is called a *local diffeomorphism* if and only if for every point $p \in M$, there exists an open neighborhood $p \in U \subseteq M$ such that $F(U) \subseteq V$ is contained in an open set and $f|_U : M \to N$ is a diffeomorphism. Similarly if we replace smooth with C^r diffeomorphism.

Example 3.2: Local Diffeomorphism

The map $\pi : S^2 \to \mathbb{RP}^2$ given by taking (x, y, z) to the line through this point and the orgin is a local diffeomorphism. It is not a global diffeomorphism, since it is a 2-1 map, while diffeomorphisms must be injective (1-1).

Proposition 3.1.2: Properties Of Local Diffeomorphisms

Let $f : M \to N$ be a smooth function.

1. the composition of local diffeomorphissm is a local diffeomorphism
2. Every finite product of local diffeomorphissm between smooth maps is a local diffeomorphism
3. Every local diffeomorphissm is a local homeomorphism and an open map
4. The restriction of a local diffemorphism to an open submanifold (with or wihtout boundary) is a local diffeomorphism
5. Every diffeomorphism is a local diffeomorphism
6. Every bijective local diffeomorphism is a diffeomorphism
7. a map $f : M \to N$ between two smooth manifolds is a local diffeomorphism if and only if for each $p \in M$, there exists a coordinate represetantion that is a local diffeomorphism

***Proof* :**
All of thse are easy exercises. We will prove that a bijective local diffeomorphism is a diffeomrophism since it is an interseting example of local information providing global information. Since f is bijective, there exists a f^{-1} such that $f \circ f^{-1} = f^{-1} \circ f = \text{id}$. It remains to that f^{-1} is smooth. Since f is bijective, the patches $f(U)$ cover N. Since f is a local diffeomorphism, $f^{-1}(f(U)) \subseteq U$, and so $f^{-1}|_{f(U)} : f(U) \to U$ is smooth. It follow's by proposition 1.2.3 that f^{-1} is smooth, and hence f is indeed a diffeomorphism.

You may wonder if the condition that a bijective local diffeomrophism implies diffeomorphism can be weakend to saying a bijective *smooth map* is a diffeomorphism. Perhaps take a moment to think about it, the answer will be given in the next section.

Proposition 3.1.3: Equivalent Condition Of Local Diffeomorphism

Let $f : M \to N$ be a map. Then

1. $f : M \to N$ is a local diffeomorphism if and only if it is both a smooth immersion and smooth submersion
2. If $\dim(M) = \dim(N)$ and f is either a smooth immersion or smooth submersion, then f is a local diffeomorphism

***Proof* :**
These are direct results of proposition 3.1.1 and f is a local diffeormophism near p if and only if $T_p f$ is an isomorphism.

3.1.1 Rank Theorem

When working with linear maps of rank k, we may always row reduce it to get a get a $k \times k$ identity matrix I_k. This following theorem show that there in fact exist reprasentatives of f that not only represent f as a linear map, but as either an inclusion or a projection! This theorem will become the cornerstone for all the following major results in this section. We first show a simplified result that is easier to state in the case where $T_p f$ is bijective.

Theorem 3.1.1: Inverse Function Theorem For Manifolds

Let $f : M \to N$ be a smooth map and $p \in M$. Then if $T_p f$ is invertible, then there exists a connected neighborhood $U_0 \subseteq M$ containing p and $V_0 \subseteq N$ containing $f(p)$ such that $f|_{U_0} : U_0 \to V_0$ is a diffeomorphism

Proof **:**
Let $T_p f$ be invertible for some smooth map $f : M \to N$ and $p \in M$. Take $(U, \mathbf{x})$ and $(V, \mathbf{y})$ such that $p \in U$ and $f(p)$ and consider $\mathbf{y} \circ f \circ \mathbf{x}^{-1} = \tilde{f}$. Since $\mathbf{x}, \mathbf{y}$ are diffeormophism, the derivative of the representative is still a linear isomorphism, namely it is nonsingular. Furthermore, f has a set-inverse: since $\mathbf{y} \circ f \circ \mathbf{x}$ is bijective and $\mathbf{x}$ and $\mathbf{y}$ are bijective, necessarily f must be bijective.

By the usual inverse function theorem, there is $\tilde{U_0} \subseteq U$ and $\tilde{V_0} \subseteq V$ such that $\tilde{f}$ restricted is a diffeormophism on these sets, that is $(\mathbf{x}^{-1} \circ f \circ \mathbf{y})^{-1}$ exists and is a diffeomorphism on the restriction. Then:

$$\begin{aligned} (\mathbf{x}^{-1} \circ f \circ \mathbf{y})^{-1} \circ \mathbf{x} \circ f \circ \mathbf{y} &= \mathrm{id} \\ \Leftrightarrow\ (\mathbf{x}^{-1} \circ f \circ \mathbf{y})^{-1} &= \mathbf{y}^{-1} \circ f^{-1} \circ \mathbf{x} \end{aligned}$$

Since $\mathbf{y}$ and $\mathbf{x}$ are diffeomorphisms, namely they are smooth, and the composition is a smooth, necessarily f must be smooth. Finally, taking $U_0 = \varphi^{-1}(\tilde{U_0})$ and $V_0 = \psi^{-1}(\tilde{V_0})$ gives that $f|_{U_0}$ is a diffeormophism, completing the proof.

The above theorem may be sligthly generalized if we assume $T_p f$ is injective or surjective and we do some restrictions or projection before applying the inverse function theorem (or applying the implicit function theorem). However, the next theorem (the Constant Rank Theorem) that will be presented takes care of these and more cases, and in fact the above theorme will be a special case of the next theorem. It was still worth presenting since the proof is quite easy as compared to the next one and give an enlighetning connectiong between $T_p f$ and f that is straightforward and required little manipulation.

(this theorem can fail for manifolds with boundary)

The next theorem doesn't assume $T_p f$ is injective, surjective or bijective, but simply that $T_p f$ says the same rank in a neighborhood. Without loss of generailty, we may assume for the theorme that f is constant rank everywhere and apply the result locally if needed:

Theorem 3.1.2: Constant Rank Theorem

Let $f : M \to N$ be a smooth map of constant rank r between manifolds of dimension m and n respectively. Then for each $p \in M$, there exists a smooth chart $(U, \mathbf{x})$ for M centerd at p and a smooth chart $(V, \mathbf{y})$ for N centered at $f(p)$ such that $f(U) \subseteq V$, in which the coordinate representation is of the form:

$$f(x^1, ..., x^r, x^{r+1}, ..., x^m) = (x^1, ..., x^r, 0, ..., 0)$$

In particular, if f is a smooth submersion, then

$$f(x^1, ..., x^n, x^{n+1}, ..., x^m) = (x^1, ..., x^n)$$

and if f is a smooth immersion, then

$$f(x^1, ..., x^m) = (x^1, ..., x^m, 0, ..., 0)$$

***Proof* :**
Choose charts so that without loss of generailty suppose M is an open subset of $\mathbb{R}^m$ and N an open subset of $\mathbb{R}^n$. Since $\text{rank}(Df) = k$ at p, we may choose a $k \times k$ minor of $Df(p)$ with nonzero determinant. we may reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so tha the minor is in the top left (this is a linear isomorphism) and translate so that $f(0) = 0$. Label the coordinates $(x_1, ..., x_k, y_1, ..., y_{m-k})$ on the domain and $(u_1, ..., u_k, v_1, ..., v_{n-k})$. Then re-write our function as

$$f(x, y) = (Q(x, y), R(x, y))$$

where Q projects onto $u = (u_1, u_2, ..., u_k)$ and R projects onto $v = (v_1, v_2, ..., v_{n-k})$. Note that $\frac{\partial Q}{\partial x}$ is nonsingular by our re-arrangement and assumption of rank of Df. Consider now $\varphi(x, y) = (Q(x, y), y)$. Which has derivative in coordinates:

$$D\varphi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

Thus, $D\varphi(0)$ is non-singular since $\frac{\partial Q}{\partial x}(0)$ is nonsingular, and so there is a local inverse $\varphi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse we must have

$$(x, y) = \varphi(\varphi^{-1}(x, y)) = Q((A, B), B)$$

implying $B(x, y) = y$. Thus, we have $f \circ \varphi^{-1} : (x, y) \mapsto (x, S = R(A, y))$ and is still of rank k. Now, we look at the derivative of this map:

$$D(f \circ \varphi^{-1}) = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \end{pmatrix}$$

Since $D(f \circ \varphi^{-1})$ must stil have rank k, we conclude that $\frac{\partial S}{\partial y} = 0$, and so we must have y-independence letting us write:

$$f \circ \varphi^{-1} : (x, y) \mapsto (x, S(x))$$

We may finally compose with $\sigma : (u, v) \mapsto (u, v - S(u))$ to get

$$\sigma \circ f \circ \varphi^{-1} : (x, y) \to (x, 0)$$

which is the desired representative, completing the proof.

The constant rank theorem can be stated in a slightly different way that is enlightening: If f is constant rank, than f "behaves" like the linear map $T_p f$ at that point

Corollary 3.1.1: Linear Representation implies Constant Rank

Let $f : M \to N$ be a smooth map and suppose M is connected. Then the following are equivalent:

1. For each $p \in M$, there exists smooth charts containing p and $f(p)$ in which the coordinate representation of f is a linear map.
2. f has constant rank

Proof **:**
If f has constant rank, then the constant rank theorem shows that it has a linear coordinate representation in a neighborhood of each point. Conversely, if at a neighborhood of each point it has a linear represetnation, then the representation must have constant rank in that neighborhood, and by connectedness it will have the constant rank on all of M.

Theorem 3.1.3: Local Constant Rank Theorem

Let $f : M^m \to N^n$ be a smooth map. Then:

1. if $T_p f$ is injective, there exists a neighborhood $U \subseteq M$ such that $f|_U$ is injective
2. if $T_p f$ is surjective, then there exists a neighborhood $U \subseteq M$ such that $f|_U$ is surjective.
3. if $T_p f$ is bijective, then there exists a neighborhood $U \subseteq M$ such that $f|_U$ is bijective

Proof **:**
Point (3) follows from point (1) and (2). We shall prove (1) since (2) follows a similar argument. Since $T_p f$ is injective, by proposition 3.1.1 it is injective on some neighborhood U, and hence constant rank in U. By the constant rank theorem, we may represent f as

$$f(x_1, x_2, ..., x_m) = (x_1, x_2, ..., x_m, 0, ..., 0)$$

Then we immediately see that the representation of f is injective on $\hat{U}$ (or some restriction), and hence f is injective on U (or some restriction). A similar argument shows that if $T_p f$ is surjective, there is a neighborhood of U such that f is surjective. Combining these results shows that there is a neighborhood of U where f is bijective, completing the proof.

A big mistake that can be made is concluding that if $f|_U$ is injective that f is globally injective on U. Consider $f : S^1 \to \mathbb{R}^2$ represented in angle coordinates that parameterizes the leminscate:

$$f(t) = \left(\frac{a\sqrt{2}\cos(t)}{1 + \sin^2(t)}, \frac{a\sqrt{2}\cos(t)\sin(t)}{1 + \sin^2(t)} \right)$$

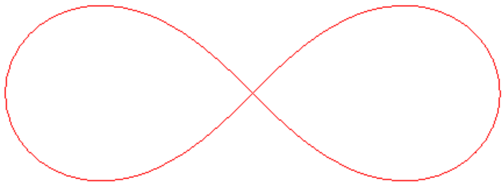
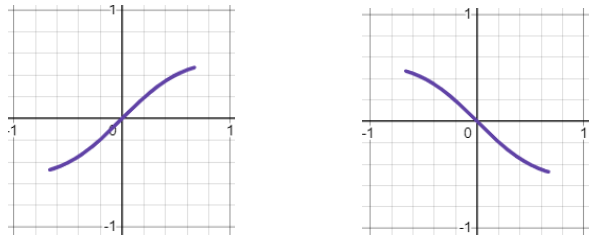


Figure 3.1: Leminscate visual

Then taking $U \subseteq S^1$ which is represented by angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, or $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$, we would get the regions:



(a) $\hat{f} : (\pi/4, 3\pi/4) \to \mathbb{R}^2$ (b) $\hat{f} : (5\pi/4, 7\pi/4) \to \mathbb{R}^2$

however f is certainly *not* injective since $f(\pi/2) = f(3\pi/2)$. This same problem does not occur for surjectivity: if $f|_U$ is surjective (in particular surjective onto $f(U)$), then certainly f is globally surjective on U (more points might map onto $f(U)$). Hence, surjectivity is "better behaved", while injectivity requires some more conditions. We shall explore this further in section 3.1.2.

The constant rank theorem is a purely local property. However, the theorem shows us that we get nice behavior if our map is constant rank. If our map is globally constant rank, we get the converese of theorem 3.1.3

Theorem 3.1.4: Global Constant Rank Theorem

Let $f : M \to N$ be a smooth map of constant rank. Then:

1. If f is injective, then f is a smooth immersion (i.e. $T_p f$ is injective for all $p \in M$)
2. If f is surjective, then f is a smooth submersion (i.e. $T_p f$ is surjective for all $p \in M$)
3. If f is bijective, then f is a diffeomorphism

***Proof* :**
Let $\text{rank}(D_p f) = r$ for all p, $\dim(M) = m$, $\dim(N) = n$.

1. For the sake of contradiction, assume f is not a smooth immersion, meaning $T_p f$ is not injective and so so $r < m$. By the constant rank theorem, choose a representative on U, V such that f is an injection $f(x_1, x_2, ..., x_r, x_{r+1}, ..., x_m) = (x_1, x_2, ..., x_r, 0, ..., 0)$. Then we may choose some $(0, \epsilon) \in U$ such that

$$f(0, ..., 0, \epsilon) = f(0, ..., 0)$$

but then f is not injective

2. For the sake of contradiction assume f is not a smooth submersion, meaning T_pf is not surjectiv eand so $r < n$. By the constant rank theorem, we may for each $p \in M$ choose a appropraite smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $f(p)$ such that $f(U) \subseteq V$ and the coordinate representation is of the form $f(x_1, x_2, ..., x_k) = (x_1, x_2, ..., x_r, 0..., 0)$. Shrinking U if necessary, we may assume it is a coordinate ball (i.e. a ball when represented in coordinates) and $f(\overline{U}) \subseteq V$. Thus, $f(\overline{U})$ is a comapct subset of

$$\left\{y \in V \mid y^{r+1} = \cdots y^n = 0\right\}$$

so it is clsoed in N and contains no open subset of N, and so it is nowhere dense in N. Since every open cover of a manifold ahs a countable subcover, choose coutnably many charts $\{(U_i, \varphi_i)\}$ covering M with corresponding charts $\{(V_i, \psi_i)\}$ covering $f(M)$ (since f is surjective). Since $f(M)$ is equal to a countable union of nowhere dense sets $f(\overline{U}_i)$, by Baire Category theorem $f(M)$ has empty interior in N, showing f cannot be surjective, a contradiction

3. By the first two parts, a biejctive map (bieng injectiv eand surjectve), and a smooth submersion and immersion, so f is a local diffeomorphism, and since it's bijctive it's a diffeomorphism, completing the proof.

Note that this answers the question posed in last section: is a bijective smooth map a diffeomorphism? The answer is yes if the map is constant rank. Once the map is not constant rank, there are examples of smooth bijecive maps that are not diffeomorphissm, even on non-measure zero sets.

Example 3.3: Smooth Bijective But Not Constant Rank

For a measure zero example, taking $\mathbb{R}$ with the atals (id) and the map $x \mapsto x^3$ is a smooth map that is bijective, but not diffeomorphic. For a non-measure zero example:

link of an example here

For a less cheaky but more complicated example:

this link talks about that

The following exmple shows that it is not even diffeomorphic a.e.

here

3.1.2 Embeddings

As we have pointed out in example 3.1, immersion need not be regular submanifolds, and that $f|_U$ being injective does not imply f is injective on $f(U)$. We require that the map is also a *homeomorphism onto its image*, or an *embedding* for that to be the case. This makes sure that the global topological structure is preserved:

Definition 3.1.4: Smooth Embedding

Let $f : M \to N$ be a smooth map. Then if f is an smooth immersion that is a homeomorphism onto it's image, then f is called an *embedding*.

It is immediately clear that smooth embeddings are topological embeddings, and it is easy to see that the image is a topological manifold. In the next section, we will go see that it is also a smooth submanifold. To make it easier to recognize when a smooth immersion is in fact an embedding, there are a copule of condition's on either f or M that guarantee that the image of an injective smooth immersion respects has the subspace topology:

Proposition 3.1.4: Smooth Embedding Sufficient Condition's

Let $f : M \to N$ be an injective smooth immersion. Then if any of the following are satisfied, f is a smooth embedding

1. f is an open or closed map
2. f is a proper map
3. M is compact
4. M has empty boundary and $\dim(M) = \dim(N)$

Proof **:**
If f is open or closed, since f is continuous, then it's image is a topological embedding. If f is proper or f is compact, then f is a closed map. For the last condition, first prove that $f(M) \subseteq \text{int}(N)$, and so by proposition 3.1.3 f is a local diffeomrophism, and so it's an open map. Hence f is the map $M \to \text{int}(N) \hookrightarrow N$.

f is a smooth immersion if and only if it is locally a smooth embedding. The use of this is to generalize the notion of smooth immersion to *topological immersion*, in which a map $f : M \to N$ is a topological immersion if it is locally a topological embedding.

Theorem 3.1.5: Local Embedding Theorem

Let $f : M \to N$ be a smooth map. Then f is a smooth immersion if and only if for every $p \in M$, there is a neighborhood $U \subseteq M$ containing p such that $f|_U : U \to N$ is a smooth embedding.

Proof **:**
p. 87 John Lee

3.1.3 Submersions

if $f : M \to N$ is a surjective continuous map, then if $g : N \to P$ is a continuous map that is constant on the fibers of g, then there is a map $\overline{g}$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[d, "f"] \arrow[dr, "\overline{g}"] & \\ N \arrow[r, "g"] & N \end{tikzcd}$$

The condition necessary for upgrading this universal property to smooth maps is that $f : M \to N$ be a surjective *smooth submersion*.

Definition 3.1.5: Local Section

Let $\pi : M \to N$ be any smooht map. Then a *section* of π is a smooth right inverse, that is $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{id}_N$:

$$\begin{tikzcd} M \arrow[d, "\pi"] \\ N \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

Theorem 3.1.6: Local Section Theorem

Let M and N be smooth manifolds and $\pi : M \to N$ a smooth map. Then π is a smooth submersion if and only if for every point $p \in M$ is in the image of a smooth local section π

***Proof* :**
John M. Lee p. 88

Proposition 3.1.5: Smooth Submersion Then Open Map

Let $f : M \to N$ be a smooth submersion. Then f is an open map

Note that if f is also surjective, then it is a topological quotient map.

***Proof* :**
By the constant rank theorem, make a represetnation of f that is a projeciton (i.e. $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$), which we know is an open map. Then when appropriately restricting, since $\mathbf{x}$ and $\mathbf{y}$ are diffeomorphissm, they are both open, and so f must also be open, completing the proof.

Corollary 3.1.2: Smooth Submersion Of Compact Manifold

Let $f : M \to N$ be a smooth submersion, M compact and N connected. Then f is surjective

Proof **:**
$f(M)$ is open since f is an open map. since M is compact $f(M)$ is closed. Since N is connected, it has no non-trivial clopen subsets, and so $f(M) = N$, completing the proof.

Proposition 3.1.6: Characteristic Property Of Surjective Smooth Submersions

Let M and N be smooth manifolds, and $\pi : M \to N$ a surejctive smooth submersion. Then for any smooth manifold P with or without boundary, a map $f : N \to P$ is smooth if and only if $f \circ \pi$ is smooth, that is the following diagram commuts

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "\overline{f}"] & \\ N \arrow[r, "f"] & P \end{tikzcd}$$

Proof **:**
John M. Lee p. 90

Theorem 3.1.7: Universal Property Of Quotient

Let M and N be smooth manifolds and $\pi : M \to N$ a surjective smooth submersion. If P is a smooth smooth manifold (with or without boundary) and $f : M \to P$ is a smooth manifold that is constant on the fibers of $\pi<$ then there exists a unique smooth map $\tilde{f} : N \to P$ such that $\tilde{f} \circ \pi = f$, that is the following diagram commutes:

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "f"] & \\ N \arrow[r, dashed, "\overline{f}"] & P \end{tikzcd}$$

Proof **:**
John M. Lee p.90

It should be checked that the map is unique.

3.1.4 Covering Map

(This looks really important)

Definition 3.1.6: Smooth Covering Map

Let E and M be connected smooth manifolds with our withotu boundary. Then a map $\pi : E \to M$ is called a *smooth covering map* if π is smooth and surjcetive, and each point in M has a neighorhood U such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π.

In the above context, we would say that U is *evenly covered*, and that M is called the *base of the covering*, while E is caleld the *covering manifold* of M. If E is simply connected, it is claled the *niversal ocvering manifold* of M.

(upgrading the topological universal covering theorem to the smooth case)

3.2 Submanifolds

So far, we have studied the properties of maps and how they relate. We have yet to talk about the image $f(M)$ of a smooth function, or more generally when a subset $S \subseteq M$ is a smooth manifold. When the image $f(M)$ is a manifold, we shall call such a manifold a *regular submanifold*. For ease of reference, here is the definition and the appropraite results we gave that naturally let's us see regular submanifolds are subobjecst of **Man** in the universal sense

Definition 3.2.1: Regular sub-manifold

Let $S \subseteq M$ be a subset of a n-dimension manifold M with atlas $\mathcal{A}$. Then S is a k-dimensional submanifold if for every $p \in S$, there exists a chart $(U, \varphi) \in \mathcal{A}$ around p such that

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k$$

Charts (U, φ) of M with this property are called *submanifold charts* for S. A submanifold of codimension 1 (i.e. of dimension $n-1$) is called a *hypersurface*.

For a more algebraic interpretation, another way of writing the theorem is that $\varphi(U \cap S)$ maps into $V(x_1, ..., x_{m-k})$ where V is the vanishing set of the coordinate functions of usual map $\mathbb{R}^m$. We naturally want the above collection of charts to form induce a collection of charts on S that makes S a smooth manifold and the inclusion map $\iota : S \hookrightarrow M$ be a smooth embedding. The next lemma will show us exactly that. We take a moment to define pertinent terms for the statment of the lemma:

Definition 3.2.2: Local Slice

Let $U \subseteq \mathbb{R}^n$ be an open set. Then

1. a *k-slice* of U is any subset of the form
$$S = \left\{(x^1, ..., x^k, x^{k+1}, ..., x^n) \in U \;\middle|\; x^{k+1} = c^{k+1}, ..., x^n = c^n\right\}$$
usually, though not necessarily, $c^i = 0$, and the positions of the constants can be re-arranged at will.

2. If (U, φ) is a smooth chart and $S \subseteq U$ such that $\varphi(S)$ is a k-slice of $\varphi(U)$, we say that S is a k-slice of U.

3. Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S satisfies the *local k-slice condition* if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U.

3b. If S satisfies the local k-slice condition, then the respective chart is called a *slice chart* for S in M, and the corresponding coordinates $(x^1, x^2, ..., x^n)$ the *slice coordinates*. A collection of slice charts that covers S is a *slice atlas*.

It should be evident that k-slices of an open set U are homeomorphic to an open subset of $\mathbb{R}^k$. an n-slice of S $U \subseteq \mathbb{R}^n$ is simply U, i.e. $S = U$. When we defined a submanifold in definition 3.2.1, notice that the submanifold atlas is a slice atlas. In the following, we shall show that this induces an atlas for S, that the inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding, and that furthermore finding an inclusion map that is a smooth embedding $\iota : S \hookrightarrow M$ for some $S \subseteq M$ is in fact a sufficient condition for S to be a regular submanifold

Lemma 3.2.1: Local Slice Condition

Let M be a smooth m-manifold. If $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subsapce topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional regular submanifold of M and making the inclusion map $\iota : S \to M$ a smooth embedding.
Furthermore, if $\iota : S \to M$ is an inclusion map that is a smooth embedding, then S satisfies the local k-slice condition.

***Proof* :**
Suppose S satisfies the local k-slice condition. Give S the subspace topology, which will inheret the Hausdroff and second-countable properties. We first show S is locally euclidean by constructing an atlas, namely we will take the slice atlas (which is a sub-atlas of M) and use it to make an atlas for S.

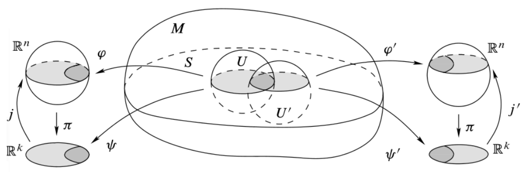
Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k coordinates. Let (U, φ) by any slice chart for S in M. Define:
$$V = U \cap S \qquad \hat{V} = \pi \circ \varphi(V) \qquad \psi = \pi \circ \varphi|_V : V \to \hat{V}$$
By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with some k-slice $A \subseteq \mathbb{R}^n$ by setting $x^{k+1} = c^{k+1}, ..., x^n = c^n$. Hence $\varphi(V)$ is open in A. Sinec $\pi|_A : A \to \mathbb{R}^k$ is a diffeomrophism, it follows that $\hat{V}$ is open in $\mathbb{R}^k$. Furthermore, ψ Is a homeomrophism since it has a continuous

inverse given by $\varphi^{-1} \circ j|_{\hat{V}}$ where

$$j(x^1, ..., x^k) = (x^1, ..., x^k, c^{k+1}, ..., c^n)$$

Thus, S is a topological k-manifold, and the inclusion map $\iota : S \hookrightarrow M$ is a topological embedding.

For S to be a smooth manifold, we need to verify the charts constructed are smoothly compactible. To that end, suppose $(U, \varphi), (U', \varphi')$ are two slice charts for S in M and let $(V, \psi), (V', \psi')$ be the corresponding charts for S respectively. The transition mappins is then $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$ which is a composition of 4 smooth maps



Thus, S is a smooth manifold with respect to the induced atlas we've just constructed.

Finally, to show $\iota S \hookrightarrow M$ is a smooth embedding, given a slice chart (U, φ) for M and the corresponding chart (V, ψ) for S, the inclusion map $S \hookrightarrow M$ has coordinate representation of the form

$$(x^1, x^2, ..., x^k) \mapsto (x^1, x^2, ..., x^k, c^{k+1}, ..., c^n)$$

which is a smooth immersion. Since the inclusion is a smooth immersion and a topological embedding, by definition it is a smooth embedding, and hence $\iota : S \to M$ is the desired inclusion map.

Furthermore, let's say we have a $S \subseteq M$ such that $\iota : S \to M$ is an inclusion map that is a smooth embedding. We'll show using the constant rank theorem that S satisfies the local slice condition. Take the smooth embedding $\iota : S \hookrightarrow M$. Since it's an immersion, by the constant rank theorem for any $p \in S$, we may choose a charts (U, φ) of S and (V, ψ) of M such that $\iota|_U$ is

$$(x^1, x^2, ..., x^k) \mapsto (x^1, x^2, ..., x^k, 0, ...0)$$

To insure we have a single slice, choose $\epsilon > 0$ small enough so that $B_\epsilon(p) \subseteq U$ and $B_\epsilon(\iota(p)) \subseteq V$. Then $\iota(B_\epsilon(p))$ is a single slice of $B_\epsilon(\iota(p))$. Since S has the subspace topology and $B_\epsilon(p)$ is open, there is an open subset $W \subseteq M$ such that $B_\epsilon(p) = W \cap S$. Letting $V' = B_\epsilon(\iota(p)) \cap W$, then we get a smooth chart $(V', \psi|_{V'})$ for M containing p such that $V' \cap S = B_\epsilon(p)$, which is a slice of V'.

Note now that we may wonder if the smooth structure on S is unique since when choosing $\iota : S \to M$ we need S to have a smooth structure. We shall show that the topology that is induced is indeed unique in section ref:HERE.

There is often another nice equivalent definition that is good to work with, especially for it's parallels to algebraic geometry and complex geometry:

Proposition 3.2.1: Regular Submanifold Equivalent Condition

Let $S \subseteq M$ be a regular k-submanifold. Then is equivalent to say that there exists an $f : U \to \mathbb{R}^{m-k}$ (namely $f = \pi \circ \varphi$ with $\pi : \varphi(U) \to \mathbb{R}^{m-k}$ projecting onto the last $m-k$ coordinates $(x_{m-k+1}, ..., x_m)$) such that

$$U \cap S = f^{-1}(0)$$

where the manifold has dimension k, and hence codimesion $m-k$

Proof **:**
Assuming $\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k$, notice that

$$\begin{aligned} f^{-1}(0) &= (\pi \circ \varphi)^{-1}(0) \\ &= \varphi^{-1}(\pi^{-1}(0)) \\ &= \varphi^{-1}(\varphi(U) \cap \mathbb{R}^k) \\ &= \varphi^{-1}(\varphi(U \cap S)) \\ &= U \cap S \end{aligned}$$

where if $x \in \varphi(U) \cap \mathbb{R}^k$ then

$$x = (x_1, x_2, ..., x_k, 0, ..., 0)$$

Conversely, if $f^{-1}(0) = U \cap S$ then

$$\begin{aligned} U \cap S &= f^{-1}(0) \\ &= \varphi^{-1}(\pi^{-1}(0)) \\ &= \varphi^{-1}(\varphi(U) \cap \mathbb{R}^k) \end{aligned}$$

Since φ is a diffeomrophism, we get

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k$$

The codomain of the map f has dimension $m-k$. It is for this reason that we shall talk about the *codimension* of the manifold.

Corollary 3.2.1: Smooth Maps And Regular Submanifold

Let $f : M \to N$ be a smooth map and $S \subseteq M$ a regular submanifold. Then $f|_S$ is a smooth map with respect to S and N. By definition, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ S \arrow[u, "\iota"] \arrow[ur, "f|_S"'] & \end{tikzcd}$$

***Proof* :**
see corrollary 1.3.1

Now that we have the notion of a smooth embedding and the constant rank theorem, we may find that $f(M)$ of is a manifold:

Theorem 3.2.1: Image of Smooth Embedding is Manifold

Let $f : M \to N$ be an immersion (i.e. an immersion at every point) where f is also a homeomoprhism onto its image $f(M)$. Then $f(M)$ is a regular submanifold of N.

***Proof* :**
Let $\dim(M) = k$ and $\dim(N) = n$. Since f is a homeomorphism, it is injective, so for a point $q \in f(M)$, there is a unique $p \in M$ such that $f(p) = q$. By theorem 3.1.2, there exists a chart $(U, \mathbf{x})$ containing p and chart $(V, \mathbf{y})$ containing $f(p)$ such that the coordinate representation of f is:

$$(x^1, ..., x^k) \mapsto (x^1, ..., x^k, 0, ..., 0) \in \mathbb{R}^n$$

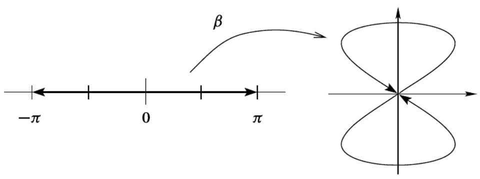
Since f is a homeomoprhism, $f(U) \subseteq f(M) \subseteq V$ is open in the relative topology on $f(M)$, there exists an open set $O \subseteq V$ such that $f(U) = f(M) \cap O$. Now, notice that the chart $(O, \mathbf{y}|_O)$ is a chart with the regular submanifold property! Since p was arbitrary, this is true for every p, and so $f(M)$ is a regular submanifold.

If one of these conditions is dropped, the image need not be a manifold:

Example 3.4: non-Embeddings

1. In example 1.12, the map $t \mapsto (t^2, t^3)$ is an embedding but not an immersion since the rank of the map at $t = 0$ is 0. Note that this also shows that a smooth embedding is not just a topological embedding that is a smooth map, the immersion is neccessary.

2. Let $\beta : (-\pi, \pi) \to \mathbb{R}^2$ by $\beta(t) = (\sin(2t), \sin(t))$, which looks like this:



   which is an immersion, but not am embedding (the image is compact in the subspace topology, however $(-\pi, \pi)$ is not compact).

3. (John Lee p. 86) The dense curve on a torus example: $\gamma : \mathbb{R} \to \mathbb{T}$, $\gamma(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$ for some irrational α.

On the otherhand, here are some examples of smooth embeddings:

Example 3.5: Smooth Embeddings

1. Let $U \subseteq M$ be an open submanifold. Then U is naturally seen as a regular submanifold.
2. If $\iota : M \to N$ is an embeding, then $D\iota : TM \to TN$ is also an embeddings (and hence higher order derivatives are embeddings, $D^k\iota : T^kM \to T^kN$). Thus, in this case, TM is a submanifold of TN
3. (slice of product manifold) Take the product manifold $M \times N$ for some smoooth manifolds M and N. The $M \times \{p\}$ for some $p \in N$ is a regular submanifold of $M \times N$ that is diffeomorphic to p, namely $x \mapsto (x, p)$ is a smooth embedding.
4. (Graphs as submanifolds) Suppose M is a smooth m-manifold (without boundary), and N is a smooth n-manifold with or wihtout boundary, $U \subseteq M$ is open, and $f : U \to N$ a smooth map. Let $\Gamma(f) \subseteq M \times N$ dentoe the graph:

$$\Gamma(f) = \{(x, y) \in M \times N \mid x \in U, y = f(x)\}$$

then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$. To see this, let $\gamma_f : U \to M \times N$ be defiend by

$$x \mapsto (x, f(x))$$

This map is certainly smooth, given a chart (W, φ) for M, choose $W \times Y, \varphi \times \psi)$ for the codomain so that the representative map is $\gamma_f(x) = (x, \tilde{f})$ for the representative $\tilde{f}$. then the derivative is certainly injective being $I_{m\times m}$ in the top half of the matrix and $D\tilde{f}$ on the bottome (namely, the top have shows it's injective). Since this is true for any point ,γ_f is a smooth immersion. It is also certainly a homeomorphism onto its image since the map $\pi : M \times N \to N$ restricted to $\Gamma(f)$ is the continuous ineverse of it, and hence it is a homeomrophism. Thus, $\gamma_f(U) = \Gamma(f)$ is regular submanifold of $M \times N$.

Due to the prevalance of compact manifolds, it is useful to take a moment to look at smooth proper embeddings (shortened to proper embedding), that is smooth embeddings that are also proper maps. In this case, it follows that S is properly embedded if and only if S is a closed subset of M. It follows that compact embedded submanifolds are properly embedded. Furthermore, if $f : M \to N$ is a smooth map, then $\Gamma(f)$ is properly embedded in $M \times N$ (can you see why?)

In section 1.1.7, we showed how level sets can be used to define smooth manifolds in the specail case where our manifold is $\mathbb{R}^n$. We may generalize this result for more arbitrary manifolds. Recall that level sets are not always submanifolds. For example:

$$f_1(x, y) = x^2 - y^2 \qquad f_2(x, y) = x^2 - y^3$$

We showed in example ref:HERE if $f : M \to \mathbb{R}$, then *every* closed suset of M can be expressed as the zer set of some smooth real valued function. We thus require an upgrade:

Proposition 3.2.2: Constant-Rank Level Set Theorem

Let $f : M \to N$ be a smooth map of manifolds. Then if D_pf has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subseteq M$ is a regular submanfold

Proof **:**
Let $x \in f^{-1}(q)$. Then there exists charst ψ, φ such that

$$\psi \circ f \circ \varphi^{-1} : (x_1, ..., x_m) \mapsto (x_1, ..., x_k, 0, ..., 0)$$

since $f(x) = q$, we may without problem choose ψ so that it is recentered at $\psi(q) = 0$. Then

$$\psi \circ f \circ \varphi^{-1}(\varphi(x)) = (0, 0, ..., 0)$$

so $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence

$$\begin{aligned}(\pi \circ \varphi)^{-1}(0) &= \varphi^{-1}(\pi^{-1}(0)) \\ &= \varphi^{-1}(\varphi(U) \cap \mathbb{R}^{m-k}) \\ &\overset{!}{=} U \cap f^{-1}(q)\end{aligned}$$

where the $\overset{!}{=}$ inequality comes from the fact that it's the points in $f^{-1}(q)$ that map to zero in the representation: if $y \in \varphi(U) \cap \mathbb{R}^{m-k}$ then y is represented as

$$y = (0, 0, ..., 0, x_{k+1}, ..., x_m)$$

Hence, we get that $f^{-1}(q)$ is a codimension k submanifold,completing the proof.

Corollary 3.2.2: Submersion Level Set Theorem

Let $f : M \to N$ be a smooth map and let $Df(p)$ has constant *full* rank on $f^{-1}(q)$. Then $f^{-1}(q)$ is a regular submanifold of M.

Proof **:**
Since the rank is maximal along $f^{-1}(q)$, it must be maximal in the open neighborhood $U \subseteq M$ containing $f^{-1}(q)$, hence $f : U \to N$ is of constant rank, and we apply the previous proposition

This corrollary can be thought of as the generalization of the result from linear algebra, where if $f : \mathbb{R}^m \to \mathbb{R}^r$ is a surjective linear map, $\ker(f)$ is a lienar subsapce of codimension r by rank-nullity.

The above corrollary can in fact be made much stronger since we only need to check the submersion condition on the level set of interest. We thus require vocabulary for surjection of tangent maps only at points.

Definition 3.2.3: Critical, Regular Points and Values

Let $f : M \to N$ be a C^r-map and $p \in M$. We say that p is a *regular point* if $T_p f$ is a *surjection*, and a *critical* or *singular* point otherwise.

A point $q \in N$ is called *regular value* if every point in $f^{-1}\{q\}$ is a regular point (this includes the case where $f^{-1}\{q\}$ is empty), and is a *critical value* otherwise.

If f is a submersion, then every point is regular, and if $\dim(M) < \dim(N)$, then every point is critical.

Note that the set of regular points is always open by proposition 3.1.1. Note that if $f^{-1}(c) = \emptyset$, then c is a regular value. The set $f^{-1}(c)$ is a *regular level set* if c is a regular value of f.

Corollary 3.2.3: Regular Level Set Theorem

Let $f : M \to N$ be a smooth map. Then every regular level set of f is a properly embedded submanifold whose codimension is $\dim(N)$

Proof **:**
let $f : M \to N$ be a smooth map and $c \in N$ a regular value. Then by proposition 3.1.1 for every $x \in f^{-1}(c)$, there is a set U_x of M containing $f^{-1}(p)$ such that $\operatorname{rank} T_p f = \dim(N)$. Let $U = \bigcup_x U_x$. It follows then that $f|_U : U \to N$ is a smooth submersion, and hence by corrollary 3.2.2 $f^{-1}(c)$ is an embedded submanifold of U. Finally, since the compositions of embeddings is an embeddings, we have $f^{-1}(c) \hookrightarrow U \hookrightarrow M$ is a smooth embedding. It is closed by continuity, showing that $f^{-1}(c)$ is a properly embedded, completing the proof.

As we've already mentioned, not all manifold are a regular level set of a function (ex. $\mathbb{C}\mathrm{P}^1$) or it is really hard to represent them as a regular level set of a function (ex. $\mathbb{T}^2$ when embedded into S^3). However, every embedded set is *loaclly* a regular level set. This is a converse for proposition 3.2.1, since we may not assume our function $f : U \to \mathbb{R}^{\operatorname{codim}(k)}$ has the form $f = \pi \circ \varphi$ since that is assuming there are charts that satisfy the regular submanifold property.

Theorem 3.2.2: Manifold is Locally Regular Level Set

Let $S \subseteq M$ where M is a smooth m-manifold. Then S is an embedded k-submanifold of M if and only if every point $x \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is a level set of a smooth submersion $f : U \to \mathbb{R}^{\operatorname{codim}(k)}$

Proof **:**
If S is an embedded submanifold, then by corrollary 3.2.1 we may choose $f = \pi \circ \varphi$ with π, φ defined as in corrollary 3.2.1. Conversely, suppose that for every $p \in S$ there is a neighborhood U and a smooth submanifold $f : U \to \mathbb{R}^{m-k}$ such that $S \cap U$ is a level set of f. Then by the submersion theorem, $S \cap U$ is an embedded submanifold of U, so it satisfies the local slice condition, (lemma 3.2.1). It follows that S is itself an embedded submanifold of M.

Definition 3.2.4: Defining Map and Local Defining Function

Let $S \subseteq M$ be an embedded submanifold. Then a smooth map $\varphi : M \to N$ such that S is a ergular level set of φ is called a *defining map* for S. In the case of $N = R^{m-k}$ so that φ is a real-valued or vector-valued function, it is sometiems called the *defining function.*
If $\varphi : U \to N$ is a smooth map such that $S \cap U$ is a regular set of φ, then φ is called a *local defining map* (or function) for S.

A famous example of a defining function would be $f(x) = |x|^2$ for the sphere S^n. Every embedded submanifold admits a local defining function in a neighborhood around its points.

3.2.1 Immersed Submanifolds

(I'd like to add this section later. Basically, it's locally smoothly embedded manifolds)

3.2.2 Tangent Space of Regular Submanifold

Let $S \subseteq M$ be an immersed or embedded submanifold so that we may define a smooth inclusion map $\iota : S \hookrightarrow M$. Then $D_p\iota : T_pS \to T_pM$ is clearly injective. It is insight to see what this means in terms of derivations: if $v \in T_pS$ and $\overline{v} = \iota(v)$, then

$$\overline{v}(f) = D_p\iota(v)(f) = v(f \circ \iota) = v(f|_S)$$

that is, it acts on functions f restricted to the submanifold S. In the following proposition's, we will characterize the tangent space of a submanifold given the definitions of tangent space we have:

Proposition 3.2.3: Tangent Space Of Submanifolds via paths

Let M be a smooth manifold with or wihtout boundary, $S \subseteq M$ a regular or immersed submanfiold, and $p \in S$> A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : I \to M$ whose image ins contained in S, and which is also smooth as a map int S, such that $0 \in i$, $\gamma(0) = p$ and $\gamma'(0) = v$.

Proof **:**
exercise

Proposition 3.2.4: Tangent Space Of Submanifolds Via Derivations

Let M be a smooth manifold, $S \subseteq M$ a regular submanifold, and $p \in S$. Then, as a subpsace of T_pM, the tangent sapce T_pS is characterized by

$$T_pS = \{v \in T_pM \mid v(f) = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\ \}$$

Proof **:**
John M. Lee p.116

Proposition 3.2.5: Tangent Space Of Submanifolds Via Defining Map

Let M be a smooth manifold a $S \subseteq M$ a regular submanifold. if $\varphi : U \to N$ is a locally defining map for S, then

$$T_pS = \ker(T_p\varphi)$$

for each $p \in S \cap U$.

***Proof* :**
Identify T_pS with the subsapce $D_p\iota(T_pS) \subseteq T_pM$ where $\iota : S \hookrightarrow M$ is the inclusion map. Since $\varphi \circ \iota$ is constant on $S \cap U$, $(D_p\varphi \circ D_p\iota) : T_pS \to T_{\varphi(p)}N$ is the zero map, and so $\mathrm{im}(D_p\iota) \subseteq \ker(D_p\varphi)$. For the other direciton, $D_p\varphi$ is surjective by definition of a defining map, and so by rank-nullity we have

$$\dim \ker(D_p\varphi) = \dim T_pM - \dim T_{\varphi(p)}N = \dim_p S = \dim(D_p\iota)$$

showing that $\mathrm{im}(D_p\iota) = \ker(D_p\varphi)$, completing the proof.

When the defining map is $\mathbb{R}^k$-valued, we have a more precise result

Corollary 3.2.4: Tangent Space For Vector-Valued Defining Map

Let $S \subseteq M$ be a level set of a smooth submersion $\varphi = (\varphi^1, \varphi^2, ..., \varphi^k) : M \to \mathbb{R}^k$. A vector $v \in T_pM$ is tangent to S if and only if

$$v(\varphi^1) = \cdots = v(\varphi^k) = 0$$

***Proof* :**
exercise

3.2.3 Maps and Immersed Submanifolds

Let $f : M \to N$ be a smooth map and $S \subseteq M$ a (regular or smooth) submanifold. Then since $\iota : S \to M$ is smooth, the map $f|_S : S \to N$ is smooth and hence restrictions of smooth maps to submanifolds is a smooth map. If $S \subseteq N$ is a regular submanifold and $f(M) \subseteq S$, then since S has the subsapce topology, we may restrict the codomain to S. This also works for tangent maps: if $S \subseteq M$ and we are considering $f|_S$, then we first find T_pS for some $s \in S$, and then we compute $D_pf|_S = D_pf \circ D_p\iota$.

If the codomain is an immersed submanifold, we have to be more careful (for later).

(consequence of the above result is the ability to show uniqueness of induced smooth structure John Lee p.114)

(finally, usin partitions of unity there is some extension of smooth maps we can do, p. 115 John Lee)

3.2.4 Manifolds With Boundary

Proposition 3.2.6: Manifolds with boundary And Cobordim

Let M be a smooth n-manifold and $f : M \to \mathbb{R}$ a smooth and proper real-valued function, and let a, b, $a < b$ be regular value sof f. Then $f^{-1}([a, b])$ is a cobordism between the $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.

Proof **:**
$f^{-1}((a,b))$ is oepn and hence an open submanifold of M. Since p is regular for all $p \in f^{-1}(a)$ we may be the constant rank theorem find charts such that f is given near p by the lienar map

$$(x_1, ..., x_m) \mapsto x_m$$

Possibly replacing x_m by $-x_m$, we thus get a chart near p for $f^{-1}([a,b])$ into $\mathbb{H}^n$, as requried. Similarly for $p \in f^{-1}(b)$.

Example 3.6: Using Cobordisms
Marco notes 2 p. 24

(for the next proposition, perhaps look into John Lee, I think I saw the exact same one in submanifolds with boundary section)

Proposition 3.2.7: Regular Submanifold With Boundary

Let $f : M \to N$ be a smooth map form a manifold with boundary to the manfiold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is its intersection with ∂M

Proof **:**
Marco ntoes 2 p.24

3.3 Critical Points and Sard's Theorem

In this section, we gain a better intuition on smooth maps by finding out "how much" of a function will fail to be full rank. This turns out to be a measurable value, and it will always be negligable in terms of measure. The theorem which will give us this result is called *Sard's Theorem*. There will be many important consequences of Sard's theorem, all of whihc where outlined in the introduction of this chapter. For future reference, our discussion of zero-measure sets will be of use when defining integration on manifolds, while Whitney's Approximation theorem will become important when discussing De Rham cohomology.

3.3.1 Measure Zero

We will show that most points of a map have regular value. We require a little bit of measure theory for this, in particular the notion of being zero measure:

Definition 3.3.1: Measure Zero

A subset $A \subseteq \mathbb{R}^n$ is said to be *measure zero* if for any $\epsilon > 0$, there exists a sequence of cube s$\{W_i\}$ such that $A \subseteq \cup W_i$ and $\sum_i \text{vol}(W_i) < \epsilon$.

Lemma 3.3.1: Image Of Zero Measure Set

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a C^1-map. If $A \subseteq U$ has zero measure, then $f(A)$ has zero measure.

If you have done analysis before, this is because every differentiable map is *locally Lipschitz*, and it is a property of lipchitz maps that they map zero measure sets to zero measure sets

Proof **:**
Since $\mathbb{R}^n$ is second-countable, A is contained in some countable union of compact balls, hence we may as well assume that A is compact, and so, we might as well assume that A is contained in some ball $B_r(0)$. Thus, by the Mean Value Theorem, there is a constant c depending on f such that for all $x, y \in B_r(0)$,

$$\|f(x) - f(y)\| \leq c\|x - y\|$$

Let $\epsilon > 0$ be given. Since A has measure zero, there is a sequence of balls $B_{\epsilon_i}(x_i)$ such that $A \subseteq \bigcup_i B_{\epsilon_i}(x_i)$ and

$$\sum \text{vol}(B_{\epsilon_i}(x_i)) < \frac{\epsilon}{2^n c^n}$$

Thus,

$$f\left(B_{\epsilon_i}(x_i)\right) \subseteq B_{f(x_i)}(2c\epsilon_i) \quad \Rightarrow \quad f(A) \subseteq \bigcup_i B_{f(x_i)}(2c\epsilon)$$

Thus:

$$\begin{aligned} \text{vol}\left(\bigcup_i B_{f(x_i)}(2\epsilon_i)\right) &\leq \sum \text{vol}\left(B_{f(x_i)}(2\epsilon_i)\right) \\ &\leq \sum_i B_1(0)(2c\epsilon_i)^n \end{aligned}$$

Given this, we can define zero measure sets on manifolds:

Definition 3.3.2: Zero Measure On Manifolds

Let M be a manifold (that is second-countable). A subset $A \subseteq M$ is said to be *measure zero* if for every admissible chart $(U, \mathbf{x})$, the set $\mathbf{x}(A \cap U)$ has measure 0.

Naturally, we can now generlize the previous lemma to manifolds.

Proposition 3.3.1: Image Of Zero Measure On Manifolds

Let M be a (second-countable) manifold with a fixed atlas $\mathcal{A}$. Then $A \subseteq M$ has measure zero if and only if $\mathbf{x}_\alpha(A \cap U)$ has measure zero.

Proof **:**
p. 76 Lee: naturally generlizes

Of course, we may show that showing that only some subcollection of charts that cover a zero-measure set A has the property that $\mathbf{x}(A) = 0$, since the transition maps will all guarantee that any part of A will still have measure 0 in any other chart containing a part of (or all of) A.

Proposition 3.3.2: Zero Measure Set With Smooth Maps

Let $f : M \to N$ be a smooth map and $A \subseteq M$ a zero measure set. Then $f(A)$ is a zero measure set

Proof **:**
The result essentially follows from lemma 3.3.1.

3.3.2 Sard's Theorem

Theorem 3.3.1: Sard's Theorem

Let $f : M \to N$ be a smooth map between (second countable) n and m dimensional manifold respectively. Then the set of critical values has measure 0

Proof **:**
p. 129 John Lee. Very long

Corollary 3.3.1: Regular Values Dense

Let M, N be second countable manifolds. Then the set of regular values of the smooth map $f : M \to N$ is dense in N

Proof **:**
John Lee p. 128 prop 6.8

Corollary 3.3.2: Embedding Manifold

Let $f : M \to N$ be a smooth function where $\dim(M) < \dim(N)$. Then $f(M)$ has measure 0.

Proof **:**
Define $\tilde{f} : M^n \times \mathbb{R}^{p-n} \to N^p$ to be the map $f \circ \pi$ where $\pi : \mathbb{R}^p \to \mathbb{R}^n$, $(x_1, ..., x_n, ..., x_p) \mapsto (x_1, ..., x_n)$. Then $D\tilde{f} = D_{(\pi(x))} f \circ D_x \pi$. Then it is clear that for any $x \in M$, $D_x \tilde{f}$ does not have full rank (it can have rank at most n), i.e. x is in the critical points of $\tilde{f}$. Thus, by Sard's theorem, $\tilde{f}(M)$ maps to a zero-measure set (i.e. the set of critical values), and hence has measure 0. Since $f(M)$ maps to the exact same set as $\tilde{f}(M)$, we see that $f(M)$ maps into a zero-measure set, and hence has measuer 0, completing the proof.

As an interesting consequence of this, space filling curves *cannot* be C^1.

Morse Lemma

(cool for the beginning of Morse Theory)

3.3.3 Whitney's Embedding Theorem

(lots of info here. Unfortunately have to skip for now. I'll just write down the statement of the theorem)

Theorem 3.3.2: Whitney's Embedding Theorem

Every smooth manifold M (with or wihtout boundary) admits a proper smooth embedding into $\mathbb{R}^{2n+1}$

Proof **:**
p. 134 John Lee

Corollary 3.3.3: Corollary name

Every smooth n-manifold (with or without boundary) is diffeomorphic to a properly embedded submanifold (with or wihtout boundary) of $\mathbb{R}^{2n+1}$.

Proof **:**

(Lots of cool and important things)

3.3.4 Whitney's Approximation Theorem

(I want will add this here for future purposes)

Theorem 3.3.3: Whitney's Approximation Theorem

Let N be a smooth manifold with or without boundary, M a smooth manifold (without boundary), and $f : N \to M$ a continuous map. Then F is homotopic to a smooth map. If F is already smooth on a closed subset $A \subseteq N$, then the homotopy can be taken to be relative to A.

Proof **:**
John M. Lee p.141

3.3.5 Transversality

Recall that if $f : M \to N$ is a smooth map and $n \in N$ is a regular value, then $f^{-1}(n)$ is a submanifold of M. We shall now study the generalization of this notion: if $W \subseteq N$ is a submanifold, when is it possible that $f^{-1}(W)$ a submanifold? In the special case where $\iota : S \to N$ is an inclusion map for $S \subseteq N$, then $\iota^{-1}(W)$ is $S \cap W$, and so the study of this question also answers the more particular question of when the intersection of two manifolds is a submanifold. After some comfort with finding when the pre-image of a manifold is a manifold, we shall do a slight generalization: if $f : M \to N$ is a smooth map and $W \subseteq N$, we can interpret W as $\iota : W \hookrightarrow N$ to be the smooth inclusion map. Notice we know have the diagram:

$$\begin{tikzcd} W \arrow[dr, hook, "\iota"'] & & M \arrow[dl, "f"] \\ & N & \end{tikzcd}$$

then it is natural to ask whether the limit of the above diagram exists, that is whether fibered products exist in Man? This will indeed be the case under the right condition. For the above case, we get:

$$\begin{tikzcd} & f^{-1}(W) \arrow[dl] \arrow[dr] & \\ W \arrow[dr, hook, "\iota"'] & & M \arrow[dl, "f"] \\ & N & \end{tikzcd}$$

where the map $f^{-1}(W) \to W$ is the map that maps the fibers $f^{-1}(w)$ to w. The above diagram is universal in the sense that if there is any other K that makes the diagram commutes, then there is a unique map $K \to f^{-1}(W)$ such that

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ W \arrow[dr, hook, "\iota"'] & & M \arrow[dl, "f"] \\ & N & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & K \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & f^{-1}(W) \arrow[dl] \arrow[dr] & \\ W \arrow[dr, hook, "\iota"'] & & M \arrow[dl, "f"] \\ & N & \end{tikzcd}$$

Then we may naturally generlize the above by making ι an arbitrary map g and it's codomain an arbitrary manifold. This will lead us to the notion of *fibered product*.

3.3.6 Intersection and Preimages of Manifolds

We shall now study the case of when two manifolds $A, B \subseteq M$ intersect nicesely enough so that $A \cap B$ is a submanifold. When working with vector spaces, when $U, W \subseteq V$ are subspaces then their intersection is a subspace. In general, the intersection of two manifolds need not be a manifold (think of a sphere in a cube that only touches the top but intersects along a cirlce segment on the botton; this is not a manifold with boundary).

Let $U, W \subseteq V$ be subspace. Then if $U + W = V$, we say that they *transverse* V. The linear combination need not be unique (ex. $\langle (x, y, 0) \rangle$ and $\langle (0, y, z) \rangle$). We get that $\dim(V) = \dim(U) + \dim(W) - \dim(U \cap W)$. Given $\operatorname{codim}(K) = \dim(V) - \dim(K)$, we have

$$\operatorname{codim}(U \cap W) = \operatorname{codim}(U) + \operatorname{codim}(W)$$

If every tangent space of two intersecting submanifolds satisfy this conditions, it wouldn't be surprising the intersction is a submanifold.

Definition 3.3.3: Transverse Regular Submanifold

Let $K, L \subseteq M$ be regular submanifolds such that for every point $p \in K \cap L$,

$$T_pK + T_pL = T_pM$$

Then K, L are said to be *transverse* submanifolds and write $K \pitchfork L$

Proposition 3.3.3: Transverse Submanifold Is Manifold

Let $K, L \subseteq M$ such that $K \pitchfork L$. Then $K \cap L$ is a submanifold (possilby empty) of dimension $\operatorname{codim} K + \operatorname{codim} L$

My intuition for codimension is that it tells you how many constraints you have (given your ambient space). Codimension is strictly a concept defined given an ambient space. For example, two spheres in $\mathbb{R}^3$ that intersect at a cirlce. Each sphere has two degrees of freedom, hence of dimension 2, but 1 degree of restraint, hence codim 1. When intersecting, you are adding more restrictions (think of how when you intersect subspaces, you are adding the equations together). Thus, you add the number of restrictions together, i.e. you add the codimensions!

***Proof* :**
We want to check that $K \cap L$ is a submanifold, which by theorem 3.2.2 is equivalent to showing that $K \cap L$ is locally a regular level set of a map. Let $p \in K \cap L$. Then by definition there are neighborhood U, V of p such that $K \cap U = f^{-1}(0)$ for a regular value 0 of a function $f : U \to \mathbb{R}^{\operatorname{codim}(K)}$ and $L \cap V = g^{-1}(0)$ for a regular value 0 of a function $g : V \to \mathbb{R}^{\operatorname{codim}(L)}$ (the points are regular values by theorem 3.2.2, and we may choose $f = \pi_k \circ \varphi$ and $g = \pi_l \circ \psi$ for appropriate charts φ, ψ of U, V respectively if we want).

Let $(f, g) : U \cap V \to \mathbb{R}^{\operatorname{codim}(K)+\operatorname{codim}(L)}$, $(f, g)(x) = (f(x), g(x))$. Then by some set-theory maniuplation we see that $(f, g)^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap U \cap V$ in particular

$$\begin{aligned}
&\Leftrightarrow x \in (f, g)^{-1}(0, 0) \\
&\Leftrightarrow (f, g)(x) \in (0, 0) \\
&\Leftrightarrow f(x) = g(x) = 0 \\
&\Leftrightarrow x \in f^{-1}(0) \cap g^{-1}(0) \\
&\Leftrightarrow x \in (K \cap L) \cap (U \cap V)
\end{aligned}$$

If 0 must be a regular value of $(f, g) : U \cap V \to \mathbb{R}^{\operatorname{codim}(K)+\operatorname{codim}(L)}$ (i.e. (f, g) is a submersion for each point in the level set $(f,g)^{-1}(0))$, then since we may do this for any p by theorem 3.2.2 $K \cap L$ is a regular submanifold.

The map f and g are defining maps. Thus, by proposition 3.2.5, the domain of the derivative $T_q(f,g)$ for $q \in (f,g)^{-1}(0,0)$ is the kernel of its derivative at q, which is the $\ker D_q f \cap \ker D_q g$, which by definition is $T_q K \cap T_q L$. By Transversality, it has codimension $\operatorname{codim} K + \operatorname{codim} L$. The derivative then becomes:

$$\begin{pmatrix} D_q f \\ D_q g \end{pmatrix}$$

By the regular value assumption, this map is full rank. Since this was true for any $q \in (f,g)^{-1}(0)$, 0 is a regular value of (f,g). Since this is true for all $p \in K \cap L$, we get that $K \cap L$ is submanifold of M, as we sought to show. Since $T_p(K \cap L) = T_p K \cap T_p L$, we get that $K \cap L$ has codimension $\operatorname{codim}(L) + \operatorname{codim}(K)$.

Example 3.7: Transversiality And Exotic Spheres

Consider the following intersection in $\mathbb{C}^5 \setminus 0$:

$$S_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_4|^2 = 1\}$$

This is a transverse intersection. Consider the complex vector field

$$\frac{1}{2} z \partial_{z_1} + \frac{1}{2} z_2 \partial_{z_2} + \frac{1}{2} z_3 \partial_{z_3} + \frac{1}{3} z_4 \partial_{z_4} + \frac{1}{6k-1} z_5 \partial_{z_5}$$

Then this vector field is tangent to the first submanifold, but not the second (which is why the second one is a hypersuface). If you're more comfortable with real coordaintes, just consider $z = x + iy$ and $\partial_z = \frac{\partial}{\partial z}$ and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

for $k \in \{1, ..., 28\}$, the intersection is a smooth manifold homeomorphic to S^7, but are not diffeomorphic! These are the *exotic 7-spheres*, and where constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may generalize this result. Instead of $K, L \subseteq M$, we may take embeddings k, l of K, L into M. Then the above case becomes the special case of when $\iota : K \to M$ and $\iota : L \to M$ are the embedding maps, and the general case is when we have two general maps $f : N_1 \to M$ and $g : N_2 \to M$.

Definition 3.3.4: Transversality Of Maps

Let $f : K \to M$ and $g : L \to M$ of manifolds are called *transverse* when $\operatorname{im}(D_a f) + \operatorname{im}(D_b g) = T_p M$ for all a, b, p such that $f(a) = g(b) = p$.

Note that this is well-defined since if $f(a) = g(b) = p$, then $D_a f : T_a K \to T_p M$ and $D_b g : T_b L \to T_p M$ both have the same codomain. The most common form of transversality is when $f : N \to M$ is a smooth map and $S \subseteq M$ so that $\iota : S \to M$ is a smooth map. We shall for clarity prove a special case before moving onto the general case. The result may be seen as under what condition is the pre-image of an embedded submanifold is an embedded submanifold.

Let $f : M \to N$ be a smooth map, $Z \subseteq N$, and consider $f^{-1}(Z)$ to be a submanifold. This means that for every $x \in f^{-1}(Z)$, there is a neighborhood $U \subseteq M$ such that $f^{-1}(Z) \cap U$ is a manifold. We thus have a local condition, and so assume that Z is simply the zero set of a regular function. Consider $y = f(x) \in Z$. Then around y, we know there is some $g = g(g_1, ..., g_l)$ where $g^{-1}(0) = Z$ and l is the codimension of Z. Then near x (say in a neighborhood W), $f^{-1}(Z) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$, i.e. $f^{-1}(Z)$ is the zero set of the functions $g_1 \circ f, ..., g_l \circ f$. We thus have that $f^{-1}(Z)$ is a manifold if the map $(g \circ f) : W \to \mathbb{R}^l$ has 0 as a regular value. Since 0 is a regular value, we must have the surjective map

$$D_x(g \circ f) = D_y g \circ D_x f \qquad D_x(g \circ f) : T_x X \to \mathbb{R}^l$$

The map $D_x(g \circ f)$ is surjective if and only if $D_y g$ $D_y g(\text{im}(D_x f)) = \mathbb{R}^l$. By assumption g is surjective with kernel $T_y(Z)$. Thus $D_x(g \circ f)$ is surjective if and only if $T_y Z$ and $\text{im}(D_x f)$ span the domain of $D_x g$, i.e. $T_y Z$. In other words, $g \circ f$ is a submersion at $x \in f^{-1}(Z)$ if and only if

Transversality Criterion

$$\text{im}(D_x f) + T_y(Z) = T_y Y$$

Note that when $Z = \{y\}$ is a single point, then the tangent space is the zero space of $T_y(Y)$, an so f is transversal to y if $D_x f(T_x X) = T_y Y$ for all $x \in f^{-1}(y)$, that is y is a regular value of f. Thus, transveraslity in a special case is regularity.

Proposition 3.3.4: Transverse, then Pre-image of Submanifold is Submanifold

Let N, M be smooth manifolds and $S \subseteq M$ an embedded submanifold. Then if $f : N \to M$ is a smooth map transverse to S (i.e. transverse to $\iota : S \to M$), then $f^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S.

***Proof* :**
Let $\dim(M) = m$ and $k = \text{codim}(S)$ where $S \subseteq M$. Given $x \in f^{-1}(S)$ since S is a submanifold we can to find a neighborhood U of $f(x)$ in M, $f(x) \in U \subseteq M$, and a map $\varphi : U \to \mathbb{R}^{\text{codim}(S)}$ such that $S \cap U = \varphi^{-1}(0)$ (note that φ is not a chart of M). If we can show that 0 is a regular value of $\varphi \circ f$ (i.e. $\varphi \circ f$ is a submerstion for each $p \in (\varphi \circ f)^{-1}(0)$), then the theorem will be proved since

$$\begin{aligned}(\varphi \circ f)^{-1}(0) &= f^{\text{Pre}}(\varphi^{\text{Pre}}(0)) \\ &= f^{\text{Pre}}(U \cap S) \\ &= f^{\text{Pre}}(U) \cap f^{\text{Pre}}(S)\end{aligned}$$

that is $f^{-1}(S) \cap f^{-1}(U)$ is the zero set of $(\varphi \circ f)|_{f^{-1}(U)}$. t

Let $z \in T_0\mathbb{R}^{\text{codim}(k)}$ and $p \in (\varphi \circ f)^{-1}(0)$. Since 0 is a regular value of φ means there is a vector $y \in T_{f(p)}U \cong T_{f(p)}M$ such that $D_{f(p)}\varphi(y) = z$. Since f is transverse to S, $T_{f(p)}M = T_{f(p)}S + \text{Im}(D_p f)$, thus $y = y_0 + D_p f(v)$ for some $y_0 \in T_{f(p)}S$ and $v \in T_p N$. Since φ is constant on $S \cap U$ (namely $\varphi(S \cap U) = 0$), I claim that $D_{f(p)}\varphi(y_0) = 0$. To see this, notice that $(S \cap U) \hookrightarrow U$ is a embedded submanifold. Since $U \subseteq M$ is an open subset, then when computing it's tangent space:

$$\ker D_{f(p)}\varphi = T_{f(p)}(S \cap U) \overset{!}{=} T_{f(p)}S$$

The $\overset{!}{=}$ can be seen in many ways, but to practice transversality notice S and U intersect transversally and $\operatorname{codim}(U) = 0$ (quick exercise). Thus, we get $D_{f(p)}\varphi(y_0) = 0$. Combining this all together we get:

$$\begin{aligned} D_p(\varphi \circ f)(v) &= D_{f(p)}\varphi(T_p f(v)) \\ &= D_{f(p)}\varphi(y_0 + T_p f(v)) \\ &= D_{f(p)}\varphi(y) \\ &= z \end{aligned}$$

showing surjectivity. Since this is true for arbirary p, by theorem 3.2.2, $f^{-1}(S)$ is an embedded submanifold of dimension k, completing the proof.

Notice that of $f : N \to M$ is a submersion and $S \subseteq M$, then $T_{f(p)}M = T_{f(p)}s + \operatorname{im}(D_p f) = T_{f(p)} = T_p M$, and so a submersion is always transversal leading to the following Corollary

Corollary 3.3.4: Lifting Submanifolds through Submersions

Let $S \subseteq M$ be an embedded manifold and $f : N \to M$ be a submersion. Then f is transverse to S, namely $f^{-1}(S)$ is an embedded submanifold.

Corollary 3.3.5: Tangent Space Of Pre-Image Of Manifold

Let $f : M \to N$ be a smooth map and $S \subseteq N$ a submanifold transverse to f so that $W = f^{-1}(S)$ is a submanifold Then for any $p \in W$

$$T_p W = (D_p f)^{-1}(T_p S)$$

where $D_p f : T_p M \to T_{f(p)}N$ is the derivative at p.

Proof **:**
By definition of a tangent space by using a defining map, $T_W p = \ker(D_p(g \circ f))$. Then:

$$\ker(D_p(g \circ f)) = \ker(D_{f(p)} \circ g \circ D_p f) = (D_p f)^{-1}(\ker(D_{f(p)} g)) = (D_p f)^{-1}(T_p S)$$

For the general case, there is no place to embedd our manifold. Hence, we need to create a new manifold that will serve as the optimal structure to hold our manifold. To see where it comes from, notice that if $f : K \to M$ and $S \subseteq M$, we can re-write $f^{-1}(K)$ in the following (isomorphic) way:

$$\{(x, y) \in K \times S \mid f(x) = y\}$$

This strange notation bears fruit when S is no longer included into M but is more generally another manifold L and $g : L \to M$ is some smooth map. Then we write

$$K\,{}_f\!\times_g L = \{(x, y) \in K \times L \mid f(x) = g(y)\}$$

This set is the natural set on which we may "lift" the transversal intersection:

Proposition 3.3.5: Transversality Of Maps

Let $f : K \to M$ and $g : L \to M$ be transverse smooth maps. Then $K_f \times_g L = \{(a, b) \in K \times L \mid f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps

$$\begin{tikzcd}
K \times L \arrow[rrd, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] & & \\
& K_f \times_g L \arrow[ul, "\iota"] \arrow[r] \arrow[d] \arrow[dr, "f\cap g"] & L \arrow[d, "g"] \\
& K \arrow[r, "f"'] & M
\end{tikzcd}$$

where ι is the inclusion map and $f \cap g : (a, b) \mapsto f(a) = g(b)$. The object $K_f \times_g L$ is called the *fibered product* of K, L with respect to f and g. If the maps are clear, we sometimes write $K \times_M L$.

***Proof* :**
Consider the graphs $\Gamma_f \subseteq K \times M$ and $\Gamma_g \subseteq L \times M$. We may impose the restriciton $f(k) = g(l)$ by taking the following diagonal

$$\Delta = \{(k, m, l, m) \in K \times M \times L \times M\}$$

and intersecting it with $\Gamma_f \times \Gamma_g$:

$$\Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta$$

We'll first show that Γ is transverse. Let $f(k) = g(l) = m$ so that $x = (k, m, l, m) \in \Gamma$ Then:

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))) \mid v \in T_k K,\ w \in T_l L\}$$

and

$$T_x(\Delta) = \{((v, m), (w, m)) \mid v \in T_k K, w \in T_l L,\ m \in T_p M\}$$

By transversality of f, g, any tangent vector $m_i \in T_p M$ may be written as $Df(v_i) + Dg(w_i)$ for some $v_i, w_i)$ with $i \in \{1, 2\}$. In particular, we may decompsoe a general tangent vector to $M \times M$ as

$$(m_1, m_2)$$
$$=$$
$$(Df(v_2)Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1))$$

which shows transversality of the above sapces. Thus, Γ is a submanifold of $K \times M \times L \times M$.

Next, the projection map $\pi : K \times M \times L \times M \to K \times L$ takes Γ bijectively onto $K_f \times_g L$. Since $T_x(\Gamma_f \times \Gamma_g)$ is a graph, we get that $\pi|_\Gamma : \Gamma \to K \times L$ is an injective immersion. Since the projection $|pi$ is an open map, it follows that $\pi|_\Gamma$ is a homeomrophism onto its image, and hence an embedding. THsu, $K_f \times_g L$ is a submanifold of $K \times L$, completing the proof.

Example 3.8: Transversality Of Maps

1. Let $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$ and consider $\pi_1 K_1 \to M$ and $\pi_2 : K_2 \to M$ These are surjective submersion, so certainly these maps are transversal. In our diagram, we would get:

$$\begin{tikzcd}
& (M \times Z_1) \times_M M \times Z_2 \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] \arrow[dd, "\pi_1 \cap \pi_2"] & \\
M \times Z_1 \arrow[dr, "\pi_1"'] & & M \times Z_2 \arrow[dl, "\pi_2"] \\
& M &
\end{tikzcd}$$

where the fiber product is isomorphic to $M \times (Z_1 \times Z_2)$, which motives the name fibered product.

2. Consider the Hopf map $p : S^3 \to S^2$ given by composing $S^3 \subseteq \mathbb{C}^{\setminus}\{0\}$ with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is certainly transverse to itself, hence we have the fiber product:

$$S^3 \times_{S^2} S^3$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$. This is a first nontrivial example at something called a *fiber bundle* which we shall cover later.

3. Take $S^1, S^{1\prime} \subseteq \mathbb{R}^3$ withere S^1 is a circle in the xy-plane, $S^{1\prime}$ is a circle in the yz-plane. Then notice that wherever $S^1, S^{1\prime}$ intersect, we either have codimension 0, 1, or 2, never 3. Therefore, these two manifold can *never* intersect transversally in $\mathbb{R}^3$, even though $S^1 \cap S^{1\prime}$ is a submanifold. This shows that transversal inersections is *ambient space dependent*. There is a generalization known as a *clean intersection* (see here) which addresses some of the limitations of transversal intersections. As we shall see soon, there is a reason to stick to the more restrictive case of transversal intersection, namely it is a rather stable and generic property.

We can use transversality to generalize the implicit function theorem

Theorem 3.3.4: Global Characterization Of Graphs

Let M, N be smooth manifolds and $S \subseteq M \times N$ an embedded manifold[a]. Let π_M, π_N be the projection maps of $M \times N$ to M, N respectively. Then the following are equivalent:

1. S is the graph of a smooth map $f : M \to N$
2. $\pi_M|_S$ is a diffeomorphism from S onto M
3. for each $p \in M$, the submanifodls S and $\{p\} \times N$ intersect transversaly in exactly one point

If these conditions hold, then S is the graph of a map $f : M \to N$ defined by $f = \pi_N \circ (\pi_M|_S)^{-1}$.

[a] Lee does Immersed, I might later too

Proof **:**
John Lee left is a problem 6-15

Since all submanifold charts locally respect the above result, we immediately get another characterization of submanifolds:

Corollary 3.3.6: Local Characterization Of Graphs

Let M, N be smooth manifolds, $S \subseteq M \times N$ an immersed submanifold, and $(p, q) \in S$. If S intersects the submanifolds $\{p\} \times N$ transversaly at (p, q), then there exists a neighborhood U of p in M and a neighborhood V of (p, q) in S such that V is the graph of a smooth map $f : U \to N$

Proof **:**
John Lee p.145

One nice thing about transversality is that it is a "stable" condition. That is, for small pertubations (of are submanifolds or our maps), transversality holds. Transversality will have another condition called genericity which will show that being transversal is the generic case. In the right topology, stability implies transversal maps from an open set, while genericity implies this open set is dense in the space of maps. We shall not define this topology, for we would be assuming functional analysis background, but we shall instead show openness by showing that there for any transversal function there always a non-trivial path from our function in a "neighorhood" by using homotopy, and use Sard's Theorem to show densness.

Definition 3.3.5: Homotopy Of Smooth Maps

Let M, N be manifolds, $f_0, f_1 : M \to N$ smooth maps. Then a smooth map

$$F : M \times [0,1] \to N$$

is a *smooth homotopy* from f_0 to f_1 if $f_t = F \circ j_t$ and $j_t : M \to M \times [0,1]$ is the embeding (x,t).

Decomposing the definition, we see that each slice $M \times \{t\}$ is a smooth map smoothly paramaterized by $[0,1]$ with $j_0 = f_0$ and $j_1 = f_1$. For our purposes we want that being transversal is stable if no matter what homotopy we choose, in a small enough neighorhood transversality is constant. There are in fact a few concepts we've come accross that have this stability property and so we generalize to the following definition:

Definition 3.3.6: Stability

A property of a smooth map $f : M \to N$ is *stable* under pertubations when for any smooth homotopy f_t with $f_0 = f$, there exists an $\epsilon > 0$ such that the propety holds for all f_t with $t < \epsilon$

Proposition 3.3.6: Stability Of Immersed Compact Manifolds

Let M be a comapct manifol.d Then the property of $f : M^m \to N^n$ being an immersion (resp. submersion) is stable under pertubations.

Proof **:**
We shall deal with the case where f is a smooth immersion, the smooth submersion case being nearly identical. Let $f : M \to N$ be a smooth immersion and consider any homotopy F such that $f_0 = f$. Then for any $p \in M$, $D_p(f_0)$ has a basis representation where the upper-left $m \times m$ matrix has nonzero determinant (is non vanishing). This value is independent of coordinate change, hence we have a well-defined continuous map $F : M \times [0,1] \to \mathbb{R}$, $(p,t) \mapsto \det_m(D_p(f_t))$. Since $F(p,0) \neq 0$, there is an open neighborhood around $(p,0)$ where $D_p(f_t)$ has nonvanishing $m \times m$ derivative. Since $M \times \{0\} \cong M$ is compact, there is a finite subcover, and hence a some $[0,\epsilon)$ for which $M \times [0,\epsilon)$ is contined in the cover. But then f_δ for $\delta < \epsilon$ is an immersion, completing the proof.

In general, many results are stable for compact manifolds, for example local diffeomorphisms and embeddings. Transversality for compact manifolds is also easier, and so we shall prove it as a special case first:

Theorem 3.3.5: Stability Of Transversality For Compact Manifolds

Let K be compact, and let $f : K \to M$ be transverse to the closed submanifold $L \subseteq M$ (meaning f is transvserse to the embedding $\iota : L \to M$), then the transversality is stable under pertubations of f

***Proof* :**
Let $F : K \times [0,1] \to M$ be any homotopy where $f = f_0$. We'll construct an open cover of K in which f_t will be transverse for a small interval, and then use compactness.

First, for the points that do not intersect L ($L \cap f^{-1}(K) = \emptyset$), since L is closed and F is continuous $F^{-1}(M \setminus L)$ is open in $K \times [0,1]$ and containd $(K \setminus f^{-1}(K)) \times \{0\}$. Thus, for each $p \in K \setminus f^{-1}(K)$, we can choose an open neighorhood $U_p \subseteq K$ and a small interval $I_p = [0, \epsilon_p)$ such that $F(U_p \times I_p) \cap L = \emptyset$.

For the points that intersect, let $p \in f^{-1}(L)$. Then like in proposition 3.3.4, we consider $f(p)$ and by the fact that L is a submanifold we can pick open $U \subseteq M$ and φ such that $U \cap L = \varphi^{-1}(0)$ and φ is a regular value at 0. By transversality, $\varphi \circ f$ is a regular value at 0 (see proposition 3.3.4 for details). In particular, at p, $\varphi \circ f$ is a submersion ($D_p(\varphi \circ f)$ is surjective). We thus have a map $\det_m : K \times [0,1] \to \mathbb{R}$ taking $(k,t) \mapsto \det_m(D_k(\varphi \circ f_t))$ where $\det_m$ is the determinant of the $m \times m$ upper block (for an arbitrary choice of basis that arranges so that the linearly independent columsn are all on the top left) which is continuous and $\det_m(p,0) \neq 0$, and hence we may choose open neighorhood around $(p,0)$ of the form $U_p \times I_p \subseteq K \times [0,1]$ where $I_p = [0, \epsilon_p)$ and I_p is replaced with the smaller interval.

Now, $\{U_p\}_{p \in K}$ is an open cover for K, and so by compactness there is a finite subcover. Choosing the smallest ϵ_p from the associated U_p and V_p of the finite subcover gives us an interval I_{ϵ_p} for which the maps f_t, $t < \epsilon_p$ is transverse, completing the proof.

Note that the transversality of two maps $f : M \to N$ and $g : M' \to N$ can be expressed in terms of transversality of $f \times g : M \times M' \to N \times N$ to the diagonal $\Delta_N \subseteq N \times N$. Thus, if M and M' are compact, we get stability for transversality of f, g under pertubations of both f and g.

We next tackle genericity. We will require a notion slightly stronger than homotopy

Definition 3.3.7: Smooth Family Of Maps

Let $F : M \times S \to N$ be a smooth map such that $F(\cdot, s) = F_s(\cdot)$ is a smooth map for each s. Then the collection $\{F_s \mid s \in S\}$ is called a *smooth family of maps.*

If S is connected, namely in the topology of manifolds this implies path connected, then we may recover a homotopy with a choice of points $s_1, s_2 \in S$ and a path from one to the other. For the next result, a set $B \subseteq S$ contains *almost every element of S* if it's compliment is measure 0.

Theorem 3.3.6: Genericity Transversiality Theorem

Let X and Y be smooth manifolds, $g : X \to Y$ a smooth map, and $\{F_s \mid s \in S\}$ a family of smooth maps from X to Y, and only Y may have a boundary. If $F : X \times S \to Y$ is transverse to g, then for almost every $s \in S$, the map $F_s : X \to Y$ is transverse to g

***Proof* :**
By Transversliaty, the fiber product $W = (X \times S) \times_Y Z$ is a submanifold (with boundary) of $X \times S \times Z$ and projects onto S via the usual projection map:

$$X \times Y \times Z \xhookleftarrow{\iota} (X \times S) \times_Y Z \xrightarrow{\pi} Z$$

We'll show that for any $s \in S$ which is a regualr value for both $\pi : W \to S$ and the boundary map $\partial\pi$ gives rise to a F_s which is transverse to g. By Sard's theorem, the s which fail to be regualr in this way form a set of measure zero.

Let $s \in S$ be a regular value of π. Supose $f_s(x) = g(z) = y$, and we want to show that f_s is transverse at this point. Since $F(x,s) = g(z)$ and F is transverse to g, we know that

$$\operatorname{im}(D_{(x,s)}F) + \operatorname{im}(D_z g) = T_y Y$$

Thus, for any $a \in T_y Y$, there exists a $b = (w,e) \in T(X \times S)$ with $D_{(x,s)}F(b) - a$ in the image of $D_z g$. Since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)}W$. Hence, we get:

$$\begin{aligned}(D_s f)(w - w') - a &= D_{(x,s)}F[(w,e) - (w',e)] - a \\ &= D_{(x,y)}F(b) - a) - DF_{(x,s)}(w', e)\end{aligned}$$

where both terms on the right hand side lie in $\operatorname{im}(D_z g)$, since $(w', e, c') \in T_{(x,y,z)}W$ means $D_z g(c') = D_{(x,y)}F(w', e)$.

This exact same argument works for showing s is regualr for $\partial\pi$ to show ∂f_s is transverse to $g$by replacign X with ∂X and F with ∂F, completing the proof.

To use this theorem, we need to find F's that are smooth. If we are working with transversal maps to $\mathbb{R}^n$, then it is easy to see that $f : M \to \mathbb{R}^n$ may be parameterized as a family of maps

$$F : M \times \mathbb{R}^n \to \mathbb{R}^n$$

via $F(x,s) = f(x) + s$. In particular, F is certinaly a submersion and hence transverse to any smooth map $g : Z \to \mathbb{R}^n$. For arbitrary codomain Y, we require a little more work to show this is possile

Theorem 3.3.7: Transversality Homotopy Theorem

Let M, N be smooth manifolds and $X \subseteq M$ an embeded submanifold. Then every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ that is transverse to X.

***Proof* :**
I like the proof John Lee does, but he uses Whitney's Approximation theorem, which I've yet to fully internalize and type up.

We finish this section with a special case of the parametric transversality theorem

Proposition 3.3.7: Parametric Transversality For Embedded Submanifolds

Let N, M be smooth manifolds, $X \subseteq M$ an embedded submanfiold, and $\{F_s \mid s \in S\}$ a family of smooth maps from N to M. If the map $F : N \times S \to M$ is transverse to X, then for almost every $s \in S$, the map $F_s : N \to M$ is transverse to X

***Proof* :**
Since F is transverse to X, $W = F^{-1}(X)$ is an embedded submanifold of $N \times S$. Take $\pi : N \times S \to S$ and restrict to $\pi|_W : W \to S$. By Sards' theorem, a.e. s is a regular value. We'll show that for a regular value s of $\pi|_W$, F_s is transverse to X.

Let $s \in S$ be a regular value of $\pi|_W$. Pick any $p \in F_s^{-1}(X)$ and let $q = F_s(p) \in X$. We need to show that
$$T_qM = T_qX + D_pF_s(T_pN) \tag{3.1}$$
Let's first tally what we know. Since F is transvserse:
$$T_qM = T_qX + D_{(p,s)}F(T_{(p,s)}(N \times S)) \tag{3.2}$$
Since s is a regular value and $(p, s) \in W$ by definition, we ahve
$$D_{(p,s)}\pi(T_{(p,s)}W) = T_sS \tag{3.3}$$
Furthermore, notice that $(D_{(p,s)}F)^{-1}(T_qX) = T_{(p,s)}W$ and so
$$D_{(p,s)}F(T_{(p,s)}W) = T_qX \tag{3.4}$$
With this, we will show equation (3.1). Let $w \in T_qM$ be any element. We need to find $v \in T_qX$ and $y \in T_pN$ such that
$$w = v + D_sF(y)$$
by equation (3.2), there eixts a $v_1 \in T_qX$ and $(y_1, z_1) \in T_pN \times T_sS \cong T_{(p,s)}(N \times S)$ such that
$$w = v_1 + D_{p,s}F(y_1, z_1) \tag{3.5}$$
By equation (3.3), there is a $(y_2, z_2) \in T_{(p,s)}W$ such that $D_{(p,s)}\pi(y_2, z_2) = z_1$. Since π is a projection, $z_2 = z_1$. By linearity, we have
$$D_{(p,s)}f(y_1, z_1) = D_{(p,s)}F(y_2, z_1) + D_{(p,s)}F(y_1 - y_2, 0)$$
Equation (3.4) implies that $D_{(p,s)}F(y_2, z_1) = D_{(p,s)}F(y_2, z_2) \in D_{(p,s)}F(T_{(p,s)}W) = T_qX$. On the other hand, if $\iota_s : N \to N \times S$ Is the map $\iota_s(p') = (p', s)$, then we have$F_s = F \circ \iota_s$ and $(D_p\iota_s)(y_1 - y_2) = (y_1 - y_2, 0)$, and so $(D_{(p,s)}F)(y_1 - y_2, 0) = (D_{(p,s)}\iota) \circ (D_pF_s)(y_1 - y_2) = (D_pF_s)(y_1 - y_2)$. By equation (3.5) we have that the values v and y that will satisfy our equation is
$$v = v_1 + D_{(p,s)}F(y_2, z_1) \qquad y = y_1 - y_2$$
completing the proof.

3.4 Intersection Theory

in Marco's Notes

4

Vector Fields

I currently have two ways of seeing vector fields: one practical and one more platonic:

1. Vector fields are a generalization of ODE's. As ODE's of nth order can be borken down into a system of n 1st order ODE's which represent the "flow" of any point (or intervals) given by the ODE, we will be able to define the flow of points on a manifolds. This is the beginning of studying things like *Ricci flow*, which was pivotal in solving problems like the poincaré conjecture.

2. As infinitesimal symmetries of a manifold. There is a dual between the symmetries of a manifold $\text{Diff}(M)$ and the "infinitesimal symmetries" $\mathfrak{X}(M)$ (the set of all vector fields on M). This dual is particularly poignant when M Is a lie group so that every point $p \in M$ represents a symmetry of the manifold. Then we shall see that the infinitesimal symmetries form something known as the *lie algebra*.

4.1 Vector Fields

The following definition is useful since it allows us to define more general notion of fields on bundles later on:

Definition 4.1.1: Section Map

Let $\pi : M \to N$ be a smooth map. Then a (global) *section* of π is a map: $\sigma : N \to M$ such that $\pi \circ \sigma = \text{id}$. If σ is only defined on an open subset $U \subseteq N$ where $\pi \circ \sigma = \text{id}|_U$, then σ is called a *local section*. If σ is a smooth (or C^r) map, then σ is called a *smooth section* or C^r *section*.

$$\begin{tikzcd} M \arrow[r, "f"'] & N \arrow[l, bend right, dashed, "\pi"'] \end{tikzcd}$$

Sections maps are prevelant in many area of mathematics (for exmaple, they appear a lot in homological algebra). For our purpoess, we will use them in the following context:

Definition 4.1.2: Vector Field

Let M be a manifold with a correpsonding tangent bundle TM. Then a smooth map $X : M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$ is caleld a *vector field*. Alternatively, a vector field on M is a *smooth section* on the tangent bundle $\pi : TM \to M$. Often, we denote $X(p)$ by X_p.

The use of defining a vector-field in terms of section is for further generalization down the line, when we generlize from tangent bundles to vector-bundles, and when we might want different types of properties other than smoothness (ex. continuity, measurability, etc). More generally, any section can also be considered a vector field (sometimes, such vector-fields are called *rough vector fields* to emphasize the lack of any smootheness smoothness condition). For future reference, the *support* of a vector field X is the closure of the set $\{p \in M \mid X_p \neq 0\}$. X is said to have *compact support* if the support is compact.

Of course, since we are working with TM, it is important to know how to interpret vector-fields in terms of coordinates. Often, we shall define a vector field on charts and then check that the transition maps glues well together.

Definition 4.1.3: Component Function Of Vector Field

Let M be a smooth n-manifold (with or without boundary), $X : M \to TM$ a rough vector field, and $(U, \mathbf{x})$ a smooth chart for M. Then any value of X at any $p \in U$ can be written in terms of coordinate basis vectors:

$$X_p = \sum_i X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

with n functions $X^i : U \to \mathbb{R}$ where $x^i(p) = X^i(p)$ which are called the *component functions* of X in the given chart. More generally, we write:

$$X = \sum_i X^i \frac{\partial}{\partial x^i}$$

Proposition 4.1.1: Smoothness Criterion For Vector Fields

Let M be a smooth manifolds and $X : M \to TM$ a rough vector field. Then if $(U, \mathbf{x})$ is any smooth chart on M with $\mathbf{x} = (x^1, x^2, ..., x^n)$, then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth

Proof **:**
Let (x^i, v^i) be the natural coordinates on $\pi^{-1}(U) \subseteq TM$ associated with the chart $(U, \mathbf{x})$. By definition of the natural coordinates, the coordinate representation of $X : M \to TM$ is

$$\widehat{X}(x) = (x^1, ..., x^n, X^1(x), ..., X^n(x))$$

where X^i is the ith component function of X in x^i-coordinates. It follows immediately that smoothness of X in U is equivalent of the smoothness of the component functions.

Lemma 4.1.1: Vector Fields Through Charts

Let M be a manifold, (U, φ) and (V, ψ) two chart for M such that $U \cap V \neq \emptyset$, and X_U a vector on U. Then we may define a vector field on $U \cap V$ to be

$$X_V(v) = D(\psi \circ \varphi^{-1}) \circ X_U \circ (\psi \circ \varphi^{-1})^{-1}(v)$$

This is usually called the *pushforward* of X_U onto X_V via $(\psi \circ \varphi^{-1})$ and is denoted

$$X_V = (\psi \circ \varphi^{-1})_* X_U$$

Proof **:**
exercise

We can phrase the above lemma more succinctly: if U_i, U_j are two coordinate charts with vector fields with $U_i \cap U_j$ with vector fields X_i, X_j such that:

$$X_i = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x^k} \qquad X_j = \sum_{k=1}^{n} b_k \frac{\partial}{\partial y^k}$$

then if φ_{ij} is the gluing map that is the change of coordinates is:

$$D(\varphi_{ij}(p))(X_i(p)) = X_j(\varphi_{ij}(p))$$

The particular notation and vocabulary will come back soon when we deal with vector fields between manifolds. For now, this lemma is crucial to be able to define some interesting vector fields on manifolds. In particular, we shall see that sometimes, we may extend a vector field from a chart on the manifold to the entire manifold, while other times it is not possible (more on this in section 4.6)

Example 4.1: Vector Fields
John Lee gives many exmaples! p. 175

1. (coordinate vector fields) Let $(U, (x^i))$ be any smooth chart on M. Then the function:

$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

determines a vector field on U, known as the *ith vector field*, and is denoted $\partial/\partial x^i$. It is a smooth vector field since its component functions are constant. Notice this vector field is defined on U, an not the entire manfiold, making this vector field a *local section.*

2. (Euler vector fields) Define a vector field V on $\mathbb{R}^n$ whose values $x \in \mathbb{R}^n$ is:

$$B_x = x^1 \left.\frac{\partial}{\partial x^1}\right|_x + \cdots + x^n \left.\frac{\partial}{\partial x^n}\right|_x$$

The vector field V is smooth since its component functions are linear. It vanishes at the origin, and points raidally outward everywhere. It is called *euler's vector field* since it appears in Euler's homogeneious function theorem.

3. (angle coordinate vector field on Circle) (here)

4. (Angel coordinate vector fields on Tori)

5. Let $S^1 = U_0 \sqcup U_1 / \sim$ with the usual identification. The transition map is $\varphi(x) = x^{-1}$. Define the vector field X_0 on U_0 to be

$$X_0 = \frac{\partial}{\partial x}$$

Then $D(\varphi)_x : T_x\mathbb{R} \to T_{x^{-1}}\mathbb{R}$ in matrix representation is:

$$[D\varphi_x] = \left[\left.\frac{\partial y}{\partial x}\right|_x \right] = [-x^{-2}]$$

where the middle term is another common representation that many geometer's use. Thus, we get:

$$(D\varphi_x) : \frac{\partial}{\partial x} \mapsto (-x^{-2})\frac{\partial}{\partial y} = -y^2 \left.\frac{\partial}{\partial y}\right|_{y=\varphi(x)}$$

Hence, the compatible vector-field is:

$$X_1 = -y^2 \frac{\partial}{\partial y}$$

(Visual here)

6. Let $M = S^2$ and take the two charts φ_N and φ_S, with $\varphi_{NS} = \varphi_S \circ \varphi_N^{-1}$ and $\varphi_{SN} = \varphi_N \circ \varphi_S^{-1}$ being the transition maps. We'll define a vector-field on the domain of φ_N and show we can push it forward to complete it to a vector field for all of S^2. Letting (s, t) be the coordinates for the codomain of φ_N, Define $X_N = \frac{\partial}{\partial s}$. Then we may define a vector field X_s on the codomain of φ_S by taking $X_S = (\varphi_{NS})_* X_n$

$$X_S = D\varphi_{NS} \circ X_N \circ \Phi_{NS}^{-1}$$

which we easily see through this diagram:

$$\begin{tikzcd} S^2 \setminus \{N\} \arrow[r, "D\varphi_{NS}"] & S^2 \setminus \{S\} \\ \mathbb{R}^2_{s,t} \arrow[u, "X_N"] \arrow[r, "\varphi_{NS}"] & \mathbb{R}^2_{u,v} \arrow[u, "X_S"'] \end{tikzcd}$$

In particular, we have

$$\begin{aligned} X_S(u,v) &= D\varphi_{NS} \circ X_n \circ \varphi_{NS}^{-1}(u,v) \\ &= D\varphi_{NS} \circ X_n \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \\ &= D_{\varphi_{NS}^{-1}(u,v)}\varphi_{NS} \left(\frac{\partial}{\partial s} \right) \end{aligned}$$

The derivative of φ_{NS} is

$$D_{(s,t)}\varphi_{NS} = \frac{1}{(s^2+t^2)^2} \begin{pmatrix} t^2 - s^2 & -2st \\ -2st & s^2 - t^2 \end{pmatrix}$$

Hence, when $(s,t) = \varphi^{-1}(u,v)$, we get:

$$D_{\varphi_{NS}^{-1}(u,v)}\varphi_{NS} = \begin{pmatrix} v^2 - u^2 & -2uv \\ -2uv & u^2 - v^2 \end{pmatrix}$$

Thus, when writing the output of this function in matrix representation, we get:

$$D_{\varphi_{NS}^{-1}(u,v)} \left(\frac{\partial}{\partial s} \right) = \begin{pmatrix} v^2 - u^2 \\ -2uv \end{pmatrix}$$

Hence $X_S(u,v) = (v^2 - u^2)\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v}$. Notice that this is well-defined when $(u,v) = 0$, and hence the vector field can be extended to a vector field on all of S^2

7. Do the same this as the above but this time let $X_N = s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}$. You should get $X_N = -u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$.

Notice in the last example that both vector field had at least one fixed point. Theorem ref:HERE will show that *any* vector field on S^2 has a fixed point (this is known a the *hairy ball theorem*).

Notice that the domain on which the vector-field is defined on can be a little bit flexible: since T_pU is naturally identified with T_pM, it is natural to identify the tangent bundle TU with an open subset $\pi^{-1}(U) \subseteq TM$. Thus, a vecor-field on U can be thought of as a map $U \to TU$ or $U \to TM$, whichever is more convenient. Conversely, if X is a vector field on M, $X|_U$ is a vector field on U, and is smooth of X is smooth.

(prop 8.6-8.7 allows for a POU like extension of a vector field, p. 177)

(prop 8.8 – smooth vec field make a C^∞-module)

Thus, for any point in TM, we can define a vector-field, and any vector field can be expanded from an closed subset U to M.

Definition 4.1.4: Collection Of Vector Spaces

Let M be a smooth manifold. Then denote $\mathfrak{X}(M)$ the collection of all vector fields on M.

The zero element is the zero vector field assignemnt for each $p \in M$ $0 \in T_pM$.

Proposition 4.1.2: Vector Fields Form Module

Let $\mathfrak{X}(M)$ be the collection of vector-fields on M. Then $\mathfrak{X}(M)$ is a $C^\infty(M)$-module with the action $f \cdot X$ defined as:

$$(fX)_p = f(p)X_p$$

It is easy to verify this proposition, namely checking that

$$(aX + bY)_p = aX_p + bY_p$$

and for any $f, g \in C^\infty(M)$, $fX + gY$ is still a smooth vector field if X and Y are.

4.2 Local and Global Frames

Notice that the set of coordinate vector fields $\left(\frac{\partial}{\partial x_i}\right)_{i=1}^n$ gives a basis for each T_pM. we give a name to any collection of vector fields that share this property

Definition 4.2.1: Local And Global Frame

Let M be a smooth manifold and $U \subseteq M$. Then an ordered set of vector-fields $(X_1, X_2, ..., X_n)$ is said to be *linearly inpendent* on U if every $(X_1|_p, ..., X_n|_p)$ is linearly independent on T_pM for each $p \in U$, and is said to *span the tangent bundle U* if $(X_1|_p, ..., X_n|_p)$ spans T_pM for each $p \in U$.

If an ordered set of vector-fields has both of these properties for some $U \subseteq M$, then it is called a *local frame* on M. If $U = M$, then it is called a *global frame*. If each X_i is smooth, then it is called a *smooth frame*.

Often, a local or global frame $(X_1, X_2, ..., X_n)$ is abbreviated to (X_i) with the indexing implied.

Example 4.2: Local And Global Frames

1. The standard coordinate vector fields on $\mathbb{R}^n$ form a smooth global frame for $\mathbb{R}^n$
2. If $(U, (x^i))$ is a smooth coordinate chart on M (ith or ihtout boundary), then the coordinate vector fields $\left(\frac{\partial}{\partial x^i}\right)$ form a smooth local frame on U. This fame is usually called the *coordinate frame* of U. Thus, every $p \in M$ is contained in the domain of a local frame.
3. example with circel and torus from earlier

The following proposition shows how to find local frames relatively easily

Proposition 4.2.1: Completion Of Local Frames

John Lee p. 178

Proof **:**
exercise 8.5

Naturally, since we are orking over $\mathbb{R}^n$, and by hitneys' embedding theorme we have that every manifolds is embeded into $\mathbb{R}^N$ for appropriate N, we can define *orthogonal frames* (skipped for no)

Though local frame are relatively easy to find, global frames are not. If a manifold admits a smooth global frame, e give it a name:

Definition 4.2.2: Parallelizable

Let M be a smooth manifold. Then M is calle parallelizable if it admits a smooth global frame

As e've seen, $\mathbb{R}^n$, S^1, and T^n each have smooth global frames. (John lee gives exercises shoing S^3 and S^7 are parallelizable. Hoever S^t is not, due to the Hairy-ball theorem. In fact, only the there mentioned sphere's are parallelizable, all others are not (something that is proven is more advanced algebraic geometry). Later, e'll sho all Lie groups are parallelizable. Hence, only these three sphere's have a lie group on it. e'll later sho hy being parallelizable is a really poerful condition hen talking about tangent and vector bundles. Essentially, a manifold is parallelizable if it can fit into a single chart (I think)).

John Lee defines moving frame on p. 178, Jeffrey Lee on p.120

4.3 Vector Fields as Derivation of Smooth functions

One important property of vector fields is that they define an operator on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$, then $\varphi_X : C^\infty(M) \to C^\infty(M)$ is a function such that

$$(Xf)(p) = X_p f$$

recall that $X_p \in T_pM$ is a map $X_p : C^\infty(M) \to \mathbb{R}$. Hence, $X_p(f)$ is well-defined and real-valued. We claimed that it maps to $C^\infty(M)$, but we should show this is indeed the case. Since the action of a tangent vector on a function is determined by the values of the function that are only on an arbitrarily small neighborhood, it follows that Xf is locally defined. In particular, for any open subset $V \subseteq U$

$$(Xf)|_V = X(f|_V)$$

We may use this to our advantage in proving a vector field is smooth:

Proposition 4.3.1: Smooth Vector Field Equivalent Condition

Let M be a smooth manifold with or without boundary and let $X : M \to TM$ be a rough vector field. Then the following are equivalent:

1. X is smooth
2. For every $f \in C^\infty(M)$, the function Xf is smooth on M
3. For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function Xf is smooth on U

***Proof* :**
p. 180 John M. Lee

Hence, the map $f \mapsto Xf$ is indeed a map from smooth functions to itself. It is clearly linear and homogeneous over $\mathbb{R}$. By the product rule, we have that

$$X(fg) = (Xf)g + f(Xg)$$

Thus, them ap $X : C^\infty(M) \to C^\infty(M)$ is a *derivation*. In fact, the converse is true too: all derivations of $C^\infty(M)$ can be identified with smooth vector fields:

Proposition 4.3.2: Derivations Of C^∞ And Vector Fields

Let M be a smooth manifold with or wihtout boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in FX(M)$

***Proof* :**
p. 181 John Lee

4.4 Vector Fields with Smooth Maps

Let $f : M \to N$ be a smooth map and X a vector field on M. Then for each $p \in M$, we get a vecotr $T_pf(X_p) \in T_{f(p)}N$. This is not necessarily a vector-field (or a vector-field on a subset). If f is not injective, then some points of N may get multiple different vector's mapped to it. In order for the vector-field to be pushed-forward, we would need that for every $p \in M$, $T_pf(X_p) = Y_{f(p)}$. In this case, we would call two vector fields *f-related*.

Proposition 4.4.1: Diffeomorphism and f-related

Let M and N be smooth manifold (with or without boundary), and let $f : M \to N$ be a diffeomrophism. For every $X \in \mathfrak{X}(M)$, there exists a unique vector field on N that is f-realted to X.

Proof **:**
exercise

The more natural direction is to go the other way around. For that, we need to define functor that will be of great use:

Definition 4.4.1: Pullback

Let $f : M \to N$ be a smooth map. Then the map

$$f^* : C^\infty(N) \to C^\infty(M) \quad f^*(h) = h \circ f$$

Is called the *pullback* of f.

It should be verified that $C^\infty(-)$ is a contravariant functor from **Man** to $\mathbb{R}$-**Algebra** (the map f^* is a homomorphism). For the opposite, we would usually want f to be a diffeomorphism:

Definition 4.4.2: Pushforward

Let $f : M \to N$ be a diffeomorphsm. Then the map

$$f_* : \mathfrak{X}(M) \to \mathfrak{X}(N) \qquad f_*(X) = Df \circ X \circ f^{-1}$$

is called a *push forward*.

(I'll need to figure out what more to put here)

4.5 Integral Curves

Definition 4.5.1: Integral Curve

Let X be a vecotr field on M. Then an *integral curve* of X is a differnetiable curve $\gamma : I \to M$ whose velocity at each poitn is equal to the value of X at that point:

$$\gamma'(t) = X_{\gamma(t)}$$

If $0 \in I <$ $\gamma(0)$ is usually called the *starting poing* of γ.

Example 4.3: Integral Curves

1. Let (x, y) be the standard coordinates on $\mathbb{R}^2$, $V = \partial/\partial x$ be the first coordinate vector field. Then the integral curves of V are exactly the straight lines parllel to the x-axis, with parameterizatoins of the form $\gamma(t) = a + t, b)$ for some constant a, b.
2. Let $W = x\partial/\partial y - y\partial/\partial x$ on $\mathbb{R}^2$. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve $\gamma(t) = (x(t), y(t))$. Then

for $\gamma'(t)W_{\gamma(t)}$, we would need

$$x'(t)\left.\frac{\partial}{\partial x}\right|_{\gamma(t)} + y'(t)\left.\frac{\partial}{\partial y}\right|_{\gamma(t)} = x(t)\left.\frac{\partial}{\partial y}\right|_{\gamma(t)} - y(t)\left.\frac{\partial}{\partial x}\right|_{\gamma(t)}$$

We seee that this is equivalent to the following system of ODE's:

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{aligned}$$

which have solutions:

$$x(t) = a\cos t - b\sin t \qquad y(t) = a\sin t + b\cos t$$

for arbitrary constant a, b, and so we get

$$\gamma(t) = a\cos t - b\sin t, a\sin t + b\cos t)$$

is an integral curve of W. If $(a, b) = (0, 0)$, this is simply the consntat curve $\gamma(t) \equiv (0, 0)<$ othise it is a circle going counterclockwise. Since $\gamma(0) = (a, b0)$, there is a unique integral curve starting at each $(a, b) \in \mathbb{R}^2$, hence the images of many integarl curves are either identical or disjoint.

As we see, finding integral curves comes down to solving a system of ODE's in a smooth chart.

Let X is a smooth vector filed on M and $\gamma : I \to M$ a smooth curve. On a smooth coordinate domain $U \subseteq m$, we can write γ in local coordinates as $\gamma(T) = (\gamma^1(t), \gamma^2(t), ..., \gamma^n(t))$. Then the condition $\gamma'(t) = X_{\gamma(t)}$ making Γ into an integral domain can be wrriten as

$$\dot{\gamma}^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = X^i(\gamma(t))\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

which redcues to the following autonomoous system of ordinary differential equations:

$$\begin{aligned} \dot{\gamma}^1(t) &= X^1(\gamma^1(t), ..., \gamma^n(t)) \\ &\vdots \\ \dot{\gamma}^1(t) &= X^1(\gamma^1(t), ..., \gamma^n(t)) \end{aligned}$$

The important result about such a system is the existence, uniqueness, and smoothness theorem that is proven in a course on ODE's (the solving of this systems is where the name "integral curves" comes from, since solving a system of ODE's is often refferd to as "integrating" the system). We will later get some very important consequences of these theorems, for now we prove a couple of important preliminary results

Proposition 4.5.1: Existence Of Integral Curve

Let X be a smooht vector field on a smooth manifold M. For each point $p \in M$, there eixtst $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of X starting at p

Proof **:**
This is simply the existence theorem for ODE's applied to the coordinate representation of X.

The next two are how affine reparamaterizations affect integral curves

Lemma 4.5.1: Rescaling Lemma

Let X be a smooth vector field on a smooth manifold M, $I \subseteq \mathbb{R}$ an interval, and $\gamma : I \to M$ an integral curve on X. For any $a \in \mathbb{R}$,the curve $\tilde{\gamma} : \tilde{I} \to M$ defined by $\tilde{\gamma}(t) = \gamma(at)$ is an integarl curve of the vectorfield aX where $\tilde{I} = \{t \mid at \in I\}$

Proof **:**
We can directly prove this by using the chain rule in local coordinates. Somewhat more invariantly, we can examien the aciotn of $\tilde{\gamma}'(t)$ on a smooth real valued function f defined i na neighborhood of a point $\tilde{\gamma}(t_0)$. By the chain rule and the fact that γ is an integarl curve of X,

$$\begin{aligned}\tilde{\gamma}(t_0)f &= \frac{d}{dt}\bigg|_{t=t_0}(f \circ \tilde{\gamma})(t) = \frac{d}{dt}\bigg|_{t=t_0}(f \circ \gamma)(at) \\ &= a(f \circ \gamma)'(at_0) = a\gamma'(at_0)f = aX_{\tilde{\gamma}(t_0)}f\end{aligned}$$

Lemma 4.5.2: Translation Lemma

Let X be a smooth vector field on a smooth manifold M, $I \subseteq \mathbb{R}$ an interval, and $\gamma : I \to M$ an integral curve on X. For any $b \in \mathbb{R}$, the curve $\hat{\gamma} : \hat{I} \to M$ deifned by $\hat{\gamma}(t) = \gamma(t+b)$ is also an integral curve, where $\hat{I} = \{t \mid t + b \in I\}$

Proof **:**
Lee left as exercise

Proposition 4.5.2: Naturality Of Integral Curves

Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are f-related if f takes integral curves of X to integarl curves of Y, meaning that for each itnegral curve γ of X, $f \circ \gamma$ is an integral curve of Y.

Proof **:**
If X and Y are f-related and $\gamma : I \to M$ is an integarl curve of X, define $\sigma : I \to N$ by $\sigma = f \circ \gamma$. Then

$$\sigma'(t) = (f \circ \gamma)'(t) = T_{\gamma(t)}f(\gamma'(t)) = T_{\gamma(t)}f(X_{\gamma(t)}) = Y_{f(\gamma(t))} = Y_{\sigma(t)}$$

hence σ is an integral curve of Y. Conversely, suppose f takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma(-\epsilon, \epsilon) \to M$ be an integarl curve of X starting at p. Since $f \circ \gamma$ is an integral curve of Y starting at $f(p)$, we have

$$Y_{f(p)} = (f \circ \gamma)'(0) = T_p f(\gamma'(0)) = T_p f(X_p)$$

which shows that X and Y are f-related, completing the proof.

4.6 Flow

Given a vector field $V \in \mathfrak{X}(M)$ for a manifold M, we may ask where does it carry all the points of the manifold. For a moment, we will assume that all integral curves are defined for all of $\mathbb{R}$, then deal with the case where they are not. If any integral curve is defined for all time, then we may define a smooth $\mathbb{R}$-action on M as follows. For each $p \in M$, there is a unique intergral cruve starting at p and defined for all time: $\theta^{(p)} : \mathbb{R} \to M$. For each $t \in \mathbb{R}$, we may define a map $\theta_t : M \to M$ be sending p to where the point will land after t amount of time has passed:

$$\theta_t(p) = \theta^{(p)}(t)$$

Thus, each θ_t "slides" the manifold along the integral curves. By the translation lemma (lemma 4.5.2), $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at $q = \theta^{(p)}(s)$. Since the integral curves are defined for all time, we may use uniqueness to get $\theta^{(q)}(t) = \theta^{(p)}(t+s)$. Translating this into θ_t notaiton, we get:

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$$

Furthermore, $\theta_0(p) = \theta^{(p)}(0) = p$, and hence is the identity. It follows that $\theta : \mathbb{R} \times M \to M$ is an action $\mathbb{R} \curvearrowright M$. This motivates the following generalization:

Definition 4.6.1: Global Flow

Let M be a manifold. Then we define a *global flow* on M (also caleld a one-parameter group aciton) to be a continuous left $\mathbb{R}$-acion on M. To be mor precise, we define a continuous map $\theta : \mathbb{R} \times M \to M$ such that for all $s, t \in \mathbb{R}$ and $p \in M$

$$\theta(t, \theta(s, p)) = \theta(s+t, p) \qquad \theta(0, p) = p$$

It is natural to define $\theta_t : M \to M$ to be $\theta_t(p) = \theta(t, p)$. Each of these maps are naturally homeomorphisms, and if the [global] flow is smooth then θ_t is a diffeomorphism[global] flow is smooth then θ_t is a diffeomorphism. By how we've defined it, we have omited any vector field. In the next proposition, we'll show that every smooth global field is derived from the integral curves of some smooth vector field (that is, the converse of what we've done at the beginning of this section holds).

Definition 4.6.2: Infinitesimal Generator

Let θ be a global flow on M. Define a vector field $X \in \mathfrak{X}(M)$ by

$$X_p = (\theta^{(p)})'(0)$$

Then the assignement $p \mapsto X_p$ is a (rough) vector filed on M and is called the *infinitesimal generator* of θ.

Proposition 4.6.1: Infinitesimal Generators And Global Flow

Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. Then the infinitesimal generator X of θ is a smooth vector filed on M, and each curve $\theta^{(p)}$ is an integrl curve of V.

Proof **:**
By proposition 4.3.1, to show V is smooth, it suffices to show that Vf is smoot hfor every $f \in C^\infty(M)$ on an open subset $U \subseteq M$. For any such f and $p \in U$,

$$Vf(p) = V_p f = (\theta^{(p)})'(0)f = \frac{d}{dt}\bigg|_{t=0} f\left(\theta^{(p)}(t)\right) = \frac{\partial}{\partial t}\bigg|_{(0,p)} f\left(\theta(t,p)\right)$$

since $f\left(\theta(t,p)\right)$ is smooth function of (t,p) by composition, so is its partial derivative with respect to t. Thus, $Vf(p)$ depends smoothly on $p<$ so V is smooth.

Next, we need to show that $\theta^{(p)}$ is an integral cuve of V, meaning we have to show that $(\theta^{(p)})'(t) = V_{\theta^{(p)}(t)}$ for all $P \in M$ and all $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$, and set $= \theta^{(p)}(t_0) = \theta_{t_0}(p)$ so that we are showing $(\theta^{(p)})'(t_0) = V_q$. By the laws of a group aciton, we get that for all t:

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t(\theta_{t_0)}(p)) = \theta^{(p)}_{t+t_0}(t+t_0)$$

Thus, for any smooth real valued functio nf defined in a neighborhood of q:

$$V_q f = (\theta^{(q)})'(0)f = \frac{d}{dt}\bigg|_{t=0} f\left(\theta^{(q)}(t)\right) = \frac{d}{dt}\bigg|_{t=0} f\left(\theta^{(p)}(t+t_0)\right) = (\theta^{(p)})'(t_0)f$$

as we sought to show.

Example 4.4: Global Flow

The vector fields in example 4.3 are defined for all of $\mathbb{R}$, and so they generate a global flow. To describe them explicitly:

1. The flow $V = \partial/\partial x$ in $\mathbb{R}^2$ is the map $\tau : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\tau_t(x,y) = (x+t, y)$$

2. The flow $W = x\partial/\partial y - y\partial/\partial x$ is the map $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\theta_t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

We know drop the restriction that integral curves are defined for all of time. We have shown that every global flow gives rise to a vector-field. The converse now is not necessarily true:

Example 4.5: Vector Fields Not Defining Global Flows

1. Let $M = \mathbb{R}^2 \setminus \{0\}$ with the standard coordinates (x,y) and let $V = \partial/\partial x$. The unique integral curve of V starting at $(-1,0) \in m$ is $\gamma(t) = (t-1, 0)$. We can immediately see that

this vector field cannot be extended past $t = 1$ since there is a hole. To prove this more rigorously, suppose $\tilde{\gamma}$ is any continuous extension of γ past $t = 1$. Then $\gamma(t) \to \tilde{\gamma}(1) \in \mathbb{R}^2 0$ as t increases to 1> But we can also consider γ as a map into $\mathbb{R}^2$ by composing wtih the inclusion $M \hookrightarrow \mathbb{R}^2$, and it is obvious from the formula that $\gamma(t) \to (0, 0)$ as t increases to 1. Since limits in $\mathbb{R}^2$ are unique, this is a contradiciton

2. A more interesting example would be $M = \mathbb{R}^2$ with $W = x^2 \partial/\partial x$. To find the integral curve starting at $(1, 0)$, we can reduce this equation to the ODE:

$$y' = y^2 \qquad y(0) = c$$

Then the solution to this separable ODE IVP is:

$$y(t) = \left(\frac{c}{1 - ct}, 0 \right)$$

which blows up as $t \to 1/c$. Thus, $u(t, x) = -\log(1 - ct)$ is an integral curve but $u \to \infty$ as $t \to 1/c$

Hence, we need to limit what is our domain.

Definition 4.6.3: Flow Domain and (local) flow

Let M be a smooth amnifold. Thnen the open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\}$ is an open interval containing 0. A *local flow* on M is a continuous map $\theta : \mathcal{D} \to M$ where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, satisfying the following properties:

1. for all $p \in M$, $\theta(0, p) = p$
2. For all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$

$$\theta(t, \theta(s, p))\theta(t + s, p)$$

Notice how in the second property, we only choose the points to sum tha are well-defined, and hence this is not a goup action. For those algebraically defined, we may call this a groupoid action. The following image can be seen as a flow domain:

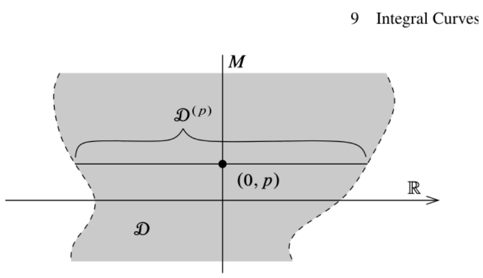


Figure 4.1: flow domain

Just like for group actions, we may write $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$ whenever $(t, p) \in \mathcal{D}$. Since the flow is now limited, we may define:

$$M_t = \{p \in M \mid (t, p) \in \mathcal{D}\}$$

giving us:

$$p \in M_t q \quad \Leftrightarrow \quad t \in \mathcal{D}^{(p)} \quad \Leftrightarrow \quad (t, p) \in \mathcal{D}$$

If θ is smooth, the *infinitesimal generator* of θ is defiend by $V_p = (\theta^{(p)})'(0)$

Proposition 4.6.2: Infinitesimal Generators And Local Flow

Let $\theta : \mathcal{D} \to M$ be a smooth flow. Then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V

Proof **:**
p. 212 John M. Lee

This leads us to the main result of this section.

Definition 4.6.4: Maximal Integral Curve And Flow

1. a *maximal integral curve* is one that cannot be extended to an integral curve on any larger open interval,
2. a *maximal flow* is a flow that admits no extension to a flow on a larger flwo domeain

Theorem 4.6.1: Fundamental Theorem Of Flows

Let V be a smooth vector field on a smooth manifold M. Then there is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ wohose infinitesimal generator is V. This flow has the following properties:

1. For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p
2. if $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \left\{t - s \mid t\ in \mathcal{D}^{(p)}\right\}$
3. For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_T \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}

Proof **:**
long proof p. 213 John Lee

Definition 4.6.5: Flow Generated By Vector Field

Let V be a vector field. Then the flow given by the above theorem is called the *flow generated by* V or just the *flow of* V.

Proposition 4.6.3: Naturality Of Flows

Let M, N be smooth manifolds, $f : M \to N$ be smooth maps, $X \in FX(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are f-related, the nfor each $t \in \mathbb{R}$, $f(M_t) \subseteq N_t$ and $\eta_t \circ f = f \circ \theta_t$ on M_t:

$$\begin{tikzcd} M_t \arrow[r, "f"] \arrow[d, "\theta_t"'] & N_t \arrow[d, "\eta_t"] \\ M_{-t} \arrow[r, "f"'] & N_{-t} \end{tikzcd}$$

Proof **:**
p.214 John M. Lee

Corollary 4.6.1: Diffeomorphism Invariance Of Flow

Let $f : M \to N$ be a diffeomrophism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow f_*X is $\eta_t = f \circ \theta_t \circ f^{-1}$, with domain $N_t = f(M_t)$ for each $t \in \mathbb{R}$

Proof **:**
exercise

4.7 Complete Vector Fields

Definition 4.7.1: Complete Vector Field

A smooth vector field is *complete* if it generates a global flow, or equivalent if each of its maximal integral cruves is defined for all $t \in \mathbb{R}$.

We have alraedy seen examples of complete and incomplete vector fields in the previous 2 example environments. It is not always easy to determine by just looking at a vector field if it is complete or not (as we've seen). If you can sovle the ODE explicitely to find all of the integral curves, and they all exist for all time, then we have a compelte vector field. If, on the other hand, we can find a singel integral curve that cannot be extended for all of $\mathbb{R}$, then it is not complete. However, it is often quite difficult to explicitley solve an ODE, so it is useful to have some general criteria for determining when a vector field is complete.

In the special case of compactly supported smooth vector fields, and hence smooth vector fields on compact manifold, they will be complete. We require the following lemma:

Lemma 4.7.1: Uniform Time Lemma

Let V be a smooth vector feild on a smooht manifold M, and let θ be its flow. Suppose there is a postiive number ϵ such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\epsilon, \epsilon)$. Then V Is complete.

Proof **:**
This is a simple matter of contradiction: take $\sup(\mathcal{D}^p)$, and by assumption expand further than the point. Use the translatoin lemma to show that this expansion is an integral curve starting at p.

Theorem 4.7.1: Compactly supported Vector Field then Complete

Every compactly supported smooth vector field on a smooth manifold is compelte

Proof **:**
P. 216 John LEe

Corollary 4.7.1: Complete Vector Field On Compact Manifold

Let M be a comapct smooth manifold. Then every smooth vector field is compelte

Proof **:**
here

Left-invariatn vector fields on Lie groups are another important class of vector fields that are always complete

Theorem 4.7.2: Lie Group Then Complete Vector Fields

Every left-invairnat vector field on a lie group is complete

Proof **:**
p.216 John Lee

We have this final useful lemma

Lemma 4.7.2: Escape Lemma

Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : I \to M$ is a maximal integral cruve of V whose domain I has a finite least upper bound b, then for any $t_0 \in I <\gamma([t_0, b))$ is not contained in any compact subset of M

Proof **:**
exercise, problem 9-6 in John Lee

(Famously, $\dot{x}(t) = x^2 \frac{\partial}{\partial x}$ is incomplete, but it *is* complete on the one-point compactification, that is on S^1. Though this is nice, in general there is no natural or easy way of extending a smooth manifold to eliminate problems of singularities: singularities may get quite ugly)

Theorem 4.7.3: Vector Fields On Compact Manifolds

Let M be a compact manifold. Then any vector field on M is complete

Proof **:**
Let (U, Φ) be a maximal local 1-parameter group diffeomorphisms generated by X. For the sake of contradiction let's say $x \in M$ such that $U \cap (R \times \{x\})$ is an open interval with finite upper limit ω (a similar case happens for the lower limit). By compactness, there is an accumulation point y for $\Phi(T, x)$ as t appraoches ω. We will then use the flow defined near y to extend $\Phi(t, x)$ in such a wa ythat will contradict maximality of Φ.

Let $\delta > 0$ and pick a neighborhood W of y sufficinetly small so that $(-\delta, \delta) \times W \subseteq U$ and let $\tau \in (\omega - \delta, \omega)$ such that $\varphi_\tau(x) \in W$. Then we can find a neighborhood V of x with the property that $\{\tau\} \times V \subseteq U$ and $\varphi_\tau(V) \subseteq W$. Then if we enlarge U to $U \cup ((\omega - \delta, \omega + \delta) \times mV)$, we can extend Φ by

$$\Phi'(t, x) = \Phi(t - \tau, \Phi(\tau, x))$$

for $(t, x) \in (\omega - \delta, \omega + \delta) \times V$, contradicting maximality.

4.8 Flow out

Part II

Bundles and Integration

In this part, we will start exploring another local Structure. So far, we have structure that are locally $\mathbb{R}^n$ with regularity C^k for $k \in \{0, 1, ..., n, ..., \infty\}$ that are also Hausdorff and paracompact. Equivalently, we studied structures that can be constructed by taking open subsets, and gluing parts of them via C^k maps ($k \in \{0, 1, ..., n, ..., \infty\}$) in such a way that the spaces remain Hausdorff. In this section, we will now take a look at spaces that are locally the Cartesian product of an open subset of our manifold M and a *fiber*. We will get to a formal definition in the next chapter, we will for now simply give the high-level intuition on this new type of structure. Take for example $E = S^1 \times (-1, 1)$. This space comes equipped with a natural projection $\pi : E \to S^1$. Then for any open $U \subseteq S^1$ and consider $\pi^{-1}(U)$, we want it to be isomorphic to $U \times (-1, 1)$ in such a way that fibers of U remain above U, that is we want this diagram to commute:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr] & & U \times (-1,1) \arrow[dl] \\
& U \arrow[ul] &
\end{tikzcd}$$

The isomorphism in question will be part of the regularity information. The space E is called the *fiber bundle*. The type of fiber will determine the type of fiber bundles we are working with: for example they may be vector-spaces (giving us vector-bundles), or groups (giving us group bundles), or circle (giving us Hopf bundles), and so on. Fiber bundles will turn out give us a lot of information about manifolds and allow us to not only define many new manifolds, but define many concepts on manifolds like length and distance, measure, curvature, angle, and so forth.

5

Fiber Bundles

Definition 5.0.1: Fiber Bundle

A *fiber bundle* is a 4-tulpe (B, F, E, π) consisting of

1. a Manifold B (the base space)
2. a fibre type F (some topological space)
3. E, the *total space* of the fibre bundle (a topological space)
4. $\pi : E \to \pi$ the *bundle projection*

where the bundle projection satisfies the following commutative diagram: for each $p \in B$, there exists a $U_p \subseteq B$ containing B st

$$\begin{tikzcd} \pi^{-1}(U_p) \arrow[rr, "\Psi_p"] \arrow[dr, "\pi"'] & & U_p \times F \arrow[dl, "\pi_1"] \\ & U_p & \end{tikzcd}$$

with the added condition that if $U, V \subseteq B$ such that $U \cap V \neq \emptyset$, then there exists a transition map such that

$$\begin{tikzcd} (U \cap V) \times F \arrow[rr, "\varphi_{UV}"] \arrow[dr, "\pi_1"'] & & (U \cap V) \times F \arrow[dl, "\pi_1"] \\ & U \cap V & \end{tikzcd}$$

commutes and $\varphi_{UV}(p \in U \cap V) : F \to F$ is an automomorphism (i.e. there is a continuous map $\varphi_{UV} : U \cap V \to \mathrm{Aut}(F)$)
the set $\pi^{-1}(p) = E_p$ are called the *fiber over* p. A collection (U_p, Ψ_p) that covers E is called a *bundle Atlas*, and the maps Ψ_p are called *local trivializations*.

Just like manifold, we can construct fibre bundles:

Proposition 5.0.1: Constructing Fiber Bundles

here, in Marco Notes

Proof **:**

6

Vector Bundles

We saw that when we have two manifolds M_1 and M_2, we can define the product manifold $M_1 \times M_2$ for which the universal property of products hold. These shapes are particularly easy to study, as their tangent bundle can be done component-wise, and so we reduce the study of manifolds to the study of it's components. However, there are manifolds that are not products of manifolds, however they *locally* look like product of manifolds. Perhaps the most famous example is the *mobius band*, which is $S^1 \times (0, 1)$, but with a twist. The fact that it's a twist means we can simply identify the mobius band with $S^1 \times (0, 1)$, however we can define a structure on the mobius band such that the mobius band is covered in a special atlas that shows that the mobius band is always *locally isomorphic* to an open subset of $S^1 \times (0, 1)$! Thus, the mobius band is locally a product manifold (up to diffeomorphism).

In this chapter, explore this idea in a more general context, and show how this concept of being "locally a product" differs from that of a manifold.

6.1 Vector Bundles

Definition 6.1.1: Vector Bundles

Let M be a topological space. Then a (real) *vector bundle of rank k over M* is a topological space E together with a surjective homomorphism $\pi : E \to M$ satisfying

1. For each $p \in M$, $\pi^{-1}(p) = E_p$ is a k-dimensional real vector space
2. For each $p \in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ satisfying:
   (a) $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection map
   (b) For each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k$

   The map Φ is called the *local trivialization*

Usually, E is called the *total space*, M the *base space*, and π the *projection*

As usual, if we make the vector spaces $\mathbb{C}^k$, thne we will call such a bundle a *complex vector bundle*. If π is smooth, we call the resulting bundle a *smooth vector-bundle*. We will sometimes say "E is a vector-bundle over M", or "$E \to M$ is a vector bundle", or "$\pi : E \to M$ is a vector bundle". Soon, we will define what it means to have morphisms between local trivializations, which will allow us to *bundle atlases*.

The condition on the bundle map can be written as the diagram:

$$\begin{tikzcd} \pi^{-1}(U_p) \arrow[r, "\Psi_p"] \arrow[d, "\pi"] & U_p \times F \arrow[dl, "\pi_I"] \\ U_p & \end{tikzcd}$$

The diagram can be thought of as representing the fact that Ψ_p is a *fiber-preserving map*. in particular, if we just had the diagram

$$\begin{tikzcd} \pi^{-1}(U_p) \arrow[r, "\Psi_p"] \arrow[d, "\pi"] & U_p \times F \\ U_p & \end{tikzcd}$$

Then we can imagine that there could be homeomorphisms that do not send fibers to fibers and yet is a homeomorphism. That extra map closing the diagram forces this type of mapping. (Another cool think Marco said: You can think of this map sending verticals to verticals in the cartesian product, but on the gluing region it does not sent horizontals to horizontals, hence the non-cartesian nature of the entire fibre bundle)

To make condition (a) and the diffeomorphism condition more intutiive, I like to think about (a) as making sure that the tangent bundle patch $\pi^{-1}(U)$ "stays at the same place". For exmaple, you can imagine take $\pi^{-1}(S^1)$, and making all the tangent spaces "orthogonal" so that you get a big cylander. In the process of doing so, we must make sure not to "rotate" the circle, since if we do then $\pi_U \circ \Phi \neq \pi$. The diffeomorphism condition insures that when we "straighten it out", we do so smoothly (i.e. can't rip or bend or do so in an non locally linear way).

If there exists a local trivialization for all of M (called the *global trivialization*), then E we give E a special name:

Definition 6.1.2: Trivial Bundle

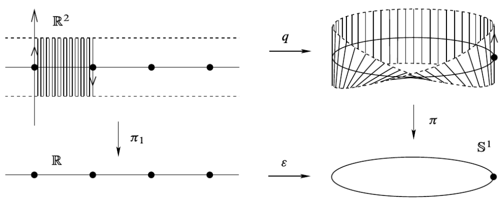
Let E be a vector bundle over M. Then E is called a *trivial vector bundle* there exists a local trivialization over all of M. Such a trivialization is called the *global trivialization*. We say that E is *smoothly traivial*

The image of E that is a trivial bundle I keep in mind is a circle with $\mathbb{R}$ attached at every point that is wiggled a bit. The tiny wiggling doesn't change that E is just the infinite cylander $S^1 \times \mathbb{R}$ up to diffeomorphism.

Example 6.1: Vector Bundles

1. For any topological space M, $E = M \times \mathbb{R}^k$ with the usual projection map is a vector-bundle. This is called the *product bundle*, and is always a trivial bundle (the identity map being the global trivialization). If M is a smooth manifold (with or without boundary) then $M \times \mathbb{R}^k$ is smooth trivial

2. We will define the mobius strip and show it is the total space for a vector bundle over S^1.



3. $\mathbb{R}^{n+1}$ is a vector bundle over $\mathbb{RP}^n$, mapping a line in $\mathbb{R}^{n+1}$ to the appropriate equivalence class in $\mathbb{RP}^n$. The map $\pi : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ sending $x \mapsto [x]$ is the desired projection. If we

Proposition 6.1.1: Tangent Bundles Are Vector Bundles

Let M be a smooth n-manifold (with or without boundary), and let TM be its tangent bundle. Given the standard projection map $\pi : TM \to M$, the natural vector space structure of each T_pM, and the manifold sructure on TM, TM is a smooth vector bundles of rank n over M

***Proof* :**
Let $(U, \mathbf{x})$ be an admissible chart for M. Using this, define the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by:

$$\Phi\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (p, (v^1, v^2, ..., v^n))$$

This map is naturally linear on fibers (being the identity map with respect to the appropriate basis), and $\pi_1 \circ \Phi = \pi$. It remains to show that this map is a diffeomorphism. Since $\mathbf{x}$ is a diffeomorphism, the map:

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\mathbf{x}\times \mathrm{id}_{\mathbb{R}^n}} \mathbf{x}(U) \times \mathbb{R}^n$$

is equal to $\tilde{\mathbf{x}}$ that we have constructed in proposition 2.4.1. Since $\tilde{\mathbf{x}}$ and $\mathbf{x}\times\mathrm{id}_{\mathbb{R}^n}$ are diffeomorphisms, so is Φ by proposition 1.2.6. Thus, Φ is a smooth local trivialization, as we sought to show.

Naturally, most interesting vector bundles are non-trivial, and so will require more than one local trivialization. The following lemma tells how the composition of two local trivialization behave. Essentially, we will show that there is a *transition function*[1] that is satisfied by the local trivializations. This will motivate the definition of a *bundle atlas* momentarily:

Lemma 6.1.1: transition maps for bundle atlas

Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth localizations with E with $U \cap V = \emptyset$. Then there exists a smooth map $\tau : (U \cap V) \to \mathbb{R}^k$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \to \mathbb{R}^k \to (U \cap V) \to \mathbb{R}^k$ has the form:

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$

***Proof* :**
Essentially, notice that the following diagram commutes:

$$\begin{tikzcd} U \cap V \times \mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \cap V \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U \cap V & \end{tikzcd}$$

with the appropriate restriction's of the maps. It follows that $\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1$, i.e.

$$\Phi \circ \Psi^{-1}(p, v) = (p, \sigma(p, v))$$

where $\sigma : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ is some appropriate smooth map.

Now, for each fixed $p \in U \cap V$, the map $v \mapsto \sigma(p, v)$ is bijective linear map $\mathbb{R}^k$ to itself. Thus, there is a nonsingular matrix $\tau(p)$ such that $\sigma(p, v) = \tau(p)v$. It remains to show that $\tau : U \cap V \to \mathrm{GL}_k(\mathbb{R})$ is smooth. Treating $\mathrm{GL}_k(\mathbb{R})$ as an open subset of $\mathbb{R}^{n^2}$, it suffices to assume that $U \cap V$ is the domain of the chart (φ, U) when the chart is restricted. With this, we want to show that

$$\tau \circ \varphi^{-1} : A \subseteq \mathbb{R}^k \to B \subseteq \mathbb{R}^{n^2}$$

is smooth, which is equivalent to showing every component function is smooth, which comes down to showing that every component function of τ is smooh. However, this is evident: Since the image of any $\tau(p)$ is a matrix, we simply combine τ with the appropriate inclusion and projection maps (which are both smooth maps) to get the appropriate component maps τ_i^j to be smooth. But then $\tau_i^j \circ \varphi^{-1}$ would be smooth for each i, j, and so τ is smooth, completing the proof.

[1] The use of the word "function" here instead of map is due to historical legacy

Example 6.2: Bundle Transition Map

The easiest example of τ would be in the case of the tangent bundle TM over M. Then if Φ and Ψ are two local trivializations of TM, each associated assocaited to two different charts $(\pi^{-1}(U), \tilde{\mathbf{x}})$ and $(\pi^{-1}(V), \tilde{\mathbf{y}})$ of TM, the transition function between them is the jacobian matrix of the coordinate transition map, $\tau = D(\varphi \circ \psi^{-1})$:

$$\Psi \circ \Phi^{-1} : U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n$$

$$\begin{aligned}\Psi \circ \Phi^{-1}&(p_1, p_2, ..., p_n, v^1, ..., v^n) \\ &= \left(p_1, p_2, ..., p_n, \sum_j \frac{\partial \tilde{x}^1}{\partial x^j}(x(p))v^j, ..., \sum_j \frac{\partial \tilde{x}^n}{\partial x^j}(x(p))v^j \right)\end{aligned}$$

which is clearly smooth and has the form given in the previous lemma with $\tau(p)$ being the Jacobian matrix at p

If two local trivializations respect above property, we will call them compatible. Using this, we have the following definnition:

Definition 6.1.3: Bundle Atlas

Let $\pi : E \to M$ be a vector bundle. Then the collection of compatible local trizializations for E is called the *bundle atlas* for E.

We have shown that tangent bundle's are manifolds, but what about general vector bundles? If we want a vector bundle E to be a smooth vector bundle, we would require defining a smooth structure on E that induces the manifold topology (or give a manifold topology and then show we can create a smooth structure on it), and then find local trivializations for every point (i.e. some collection of local trivializations). The next lemma tells us that we just need to construct the local trivializations as long as they overlap with smooth transition functions

(can also call lemma Vector Bundle Chart Lemma)

Lemma 6.1.2: Bundle Atlas inducing Manifold

Let M be a smooth manifold (with or without boundary), and for each $p \in M$, let E_p be real vector space of some fixed dimension k. Let $E = \coprod_{p \in M} E_p$, and $\pi : E \to M$ the map mapping any E_p to p. Then if we have the following "data":

1. There exists an open cover $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ for M
2. For each $\alpha \in A$, there exists a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $(\{p\} \times \mathbb{R}^k) \cong \mathbb{R}^k$
3. For each $\alpha, \beta \in A$ where $U_\alpha \cap U_\beta \neq \emptyset$, there exists a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ such that $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself is of the form:

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta} v)$$

Then E has a unique topology and smooth structure making it into a smooth manifold (with or without boundary) and a smooth rank k vector bundler over M, with π as projection and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ as bundle atlas

Proof **:**
p. 253 John Lee

Using this lemma, we can constrct some interesting examples

Example 6.3: Further Example Of Vector Bundles
p. 254

1. (Whitney Sums)
2. (restriction of a vector bundle)

6.2 Local and Global sections of Vector Bundles

As we've seen in section 4.2, we can define a *frame* on a tangent bundle. This concept readily generlizes to vector bundles:

Definition 6.2.1: Local And Global Frame

Let $\pi : E \to M$ be a vector bundle and $U \subseteq M$ and open subset. Then an ordered set of local sections $(\sigma_1, \sigma_2, ..., \sigma_n)$ is said to be *linearly inpendent* on U if every $(\sigma_1|_p, ..., \sigma_n|_p)$ is linearly independent on E_p for each $p \in U$, and is said to *span the vector bundle* U if $(\sigma_1|_p, ..., \sigma_n|_p)$ spans E_p for each $p \in U$.

If an ordered set of vector-fields has both of these properties for some $U \subseteq M$, then it is called a *local frame* on M. If $U = M$, then it is called a *global frame*. If each σ_i is smooth, then it is called a *smooth frame*.

Notice a terminological difference between frames defined using vector-fields and on vector-bundles: A frame on a tangent bundles with the old definition would be "a frame on M", while in our new definition it would be "a frame on TM". We will use both interchangibally with the undersatnding that these to definition match $E = TM$.

Just like for local frames on manifolds, local farmes on vector bundles are relatively easy to find:

Proposition 6.2.1: Completion Of Local Frames On Vector Bundles

p. 258 John Lee

Proof **:**
Given as on the spot exercise in John Lee

The easiet example of a vector bundle admitting a global frame is the prouct bundle $E = M \times \mathbb{R}^n \to \mathbb{R}^n$, with standard basis $(e_1, e_2, ..., e_n)$ for $\mathbb{R}^n$ an a frame $(\tilde{e}_i)$ for E defined by:

$$\tilde{e}_i(p) = (p, e_i)$$

If M is a smooth manifold, this frame is smooth.

Example 6.4: Local Frames and Local Trivialization

Let $\pi : E \to M$ be a vector bundle and $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ a local trivialization. Then we can use this local trivialization to construct a local frame on U. Inspired by the case for the product bundle, let

$$\sigma_i : M \to E \qquad \sigma_i(p) = \Phi^{-1} \circ \tilde{e}_i(p)$$

If M is smooth, then σ_i is the composition of smooth maps, and hence smooth. Furthermore:

$$\pi \circ \sigma_i(p) = \pi(\pi^{-1}(p)) = p$$

and hence each σ_i is indeed a section. They are naturally a frame, being a basis at every p, and so (σ_i) is a local frame.

Thus, given a local trivialization on U, we can construct a local frame on U. This frame is called the *local frame associated with* Φ, since $\tilde{e}_i$ is the cannonical basis for $\mathbb{R}^k$

In the above example, we took a local trivialization and found a smooth local frame. The converse is alos possible:

Proposition 6.2.2: Local Frame Gives Local Trivialization

Let $\pi : E \to M$ be a [smooth] vector bundle and (σ_i) a [smooth] local frame on U. Then (σ_i) is associated with a [smooth] local trivialization Φ.

Proof **:**

($\Rightarrow$) Let E be trivial, so that $\Phi : \pi^{-1}(M) \to M \times \mathbb{R}^k$ is a global trivialization. Define the section maps:

$$s_i : M \to E \qquad s_i(p) = \Phi^{-1} \circ \overline{e}_i(p)$$

where $\overline{e}_i : M \to M \times \mathbb{R}^k$ maps $p \mapsto (p, e_i)$. Since Φ^{-1} is smoooth and each $\overline{e}_i$ is smooth (being in inclusion map), each s_i is smooth. Furthermore, Since Φ^{-1} is a diffeomorphism, the elements $(s_1(p), s_2(p), ..., s_k(p))$ form a linearly independent set for E_p for all $p \in M$.

($\Leftarrow$) Let the smooth section maps $s_1, s_2, ..., s_k$ be everywhere linearly independent. Since E is of rank k, $(s_1(p), ..., s_k(p))$ forms a basis for each E_p. Define the map:

$$\Phi : \pi^{-1}(M) \to M \times \mathbb{R}^k \qquad (v_i s_p(p)) \mapsto (p, v_1, ..., v_k)$$

This map is bijective since each $\Phi|_p$ is bijective, where we see that each $\Phi|_p$ is bijective since $(s_1(p), ..., s_k(p))$ form a basis for $\pi^{-1}(p)$. Clearly, $\Phi|_p$ is a linear isomorphism, so it remains to show that Φ is a diffeomorphism, or in particular since Φ is bijective, by the constant rank theorem, it suffices to show it's a local diffeomorphism.

The key is to take advantage of the fact that E is a smooth vector-bundle, since this implies E is covered in smooth local trivializations: there exists a subset $U \subseteq M$ containing p such that $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a diffeomorphism. Thus, given that Ψ is a diffeomrophism, if we show that $\Psi \circ \Phi^{-1}|_{U \times \mathbb{R}^k}$ is a diffeomorphism (from $U \times \mathbb{R}^k$ to itself), Then $\Phi^{-1}|_{U \times \mathbb{R}^k}$ would be a diffeomrophism (by homework 1), and so Φ on the approparíate domain would be a diffeomorphism. From now on, we will assume the domain is appropriately restricted.

We show now that $\Psi \circ \Phi^{-1}$ is smooth: since Ψ is smooth, so is $\Psi \circ \sigma_i$ for each $1 \leq i \leq k$. If we let

$$\sigma_i^j = \pi^j \circ \Psi \circ \sigma_i|_U : U \to \pi^{-1}(U) \to U \times \mathbb{R}^k \to \mathbb{R}$$

represent the components of σ_i with respect to to the coordinates given by Ψ, we get that:

$$\Psi \circ \Phi^{-1}(p, x_1, ..., x_k) = \left(p, \sum_i x_i \sigma_i^1(p), ..., \sum_i x_i \sigma_i^k(p)\right)$$

which is certainly smooth. Next, to show that $(\Psi \circ \Phi^{-1})^{-1}$ is smooth, note that we could have represented the above map as:

$$\Psi \circ \Phi^{-1}(p, x) = (p, S(p)x)$$

where $x = (x_1, x_2, ..., x_k)$ and

$$S(p) = \begin{pmatrix} \sigma_1^1(p) & \cdots & \sigma_k^1(p) \\ \vdots & \ddots & \vdots \\ \sigma_1^k(p) & \cdots & \sigma_k^k(p) \end{pmatrix}$$

Notice that $S(p)$ is invertible at each p since $\sigma_i(p)$ is linearly independent at each p (even better, they form a basis at each E_p since E_p is k-dimensional). Hence, $S(p) \in \mathrm{GL}_k(\mathbb{R})$. By the previous homework, the function ι^{-1} mapping $A \mapsto A^{-1}$ is smooth. Then the form of $(\Psi \circ \Phi^{-1})^{-1}$ is given by replacing S with S^{-1}, and so is certainly smooth.

Thus, we showed that Φ is a local diffeomorphsim, and so as a consequence of the constant rank theorem a diffeomorphism, and so combining with the fact that the image of Φ is $M \times \mathbb{R}^k$ and Φ_p is a linear isomorphism, Φ is a global trivalization, completing the proof.

Corollary 6.2.1: Global Frame Then Global Trivialization

Let $\pi : E \to M$ be a [smooth] vector bunle and (σ_i) a [smooth] global frame. Then (σ_i) is associated with a [smooth] global trivialization Φ.

***Proof* :**
This is a direct result of the last proposition.

6.3 Bundle Homomorphisms

It is perhaps clear that every vector bundle E over S^1 of rank 1 that is a trivial bundle simple wiggles the lines a bit – these are all equivalent bundles, we would really want to treat these as the same. This motivates the following definition

Definition 6.3.1: Bundle Homomorhpism

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be two vector bundles. Let $F : E \to E'$ be a continuous map. Then F is called a *bundle homomorphism* if there exists a map $f : M \to M'$ such that $\pi' \circ F = f \circ \pi$, i.e. the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

with the property that for each $p \in M$, the restriction $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map. If F is a bundle homomorhpism and f is the induced map between M and M' satisfying the aforementioned condition, we say that F *covers* f.

Naturally, id is a bundle homomorphism, and if there exists a bundle homomorphism G such that $F \circ G = G \circ F = \mathrm{id}$, then F is said to be a *bundle isomorphism* and $G = F^{-1}$.

Example 6.5: Bundle Homomorphisms

1. Let M be a smooth manifold and $M_{p\times n}(\mathbb{R})$ be the set of $p\times n$ matrices. Let $T : M \to M_{p\times n}(\mathbb{R})$

be a smooth map between these spaces. Then

$$\begin{tikzcd} M \times \mathbb{R}^n \arrow[r] \arrow[d] & M \times \mathbb{R}^p \arrow[d] \\ M \arrow[r] & M \end{tikzcd}$$

where $(m, v) \mapsto (m, T(m)(v))$ is a bundle homomorphism. It is a bundle isomorphism if $p = n$.

2. TS^1 is a trivial bundle. To see this, notice this diagram commutes:

$$\begin{tikzcd} TS^1 \arrow[rr, "F"] \arrow[dr, "\pi"'] & & S^1 \times \mathbb{R} \arrow[dl, "\pi"] \\ & S^1 & \end{tikzcd}$$

where $(a, t_{v_a}) \mapsto (a, t)$ (t_{v_a} being the element t with respect to the basis element v_a). As an interesting remark, TS^2 is not a trivial bundle. In fact, the only other two trivial sphere bundles are TS^3 and TS^7 (which we don't prove here)

3. If G is a Lie group, then TG is always a trivial bundle (it's in your homework, but as a hint remember that the binary operation is invertible...)

6.4 Oriented Bundle

We first recall what it means for a vector-space to be oriented. The idea of orientation start when it was realised that in $\mathbb{R}^3$, there is a notion of "left-handedness" and "right-handedness", where a something like your hand or a molecule cannot simply be rotated to create the other object. In chemistry, this is called *chirality*, and can have major effects on many chemical processes (ex. one chirality can be harmless to the neuro-chemsitry of the brain, while the other is cocaine). We can with not too much difficulty imagine the analogous scenario happening in $\mathbb{R}^2$ or $\mathbb{R}$, where in $\mathbb{R}^2$ we can turn anti-clockwise or clockwise (with many mathematicians have the ingrained preference of going anti-clockwise when choosing a basis), while in $\mathbb{R}$ there is a preference for going positive, which is usually going on the right of a real number line.

In the case of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, there is a natural way of choosing a basis that satisfies these conditions: for $\mathbb{R}$, it is (1), for $\mathbb{R}^2$, it is usually $((1,0),(0,1))$, while for $\mathbb{R}^3$ it is $((1,0,0),(0,1,0),(0,0,1))$. In terms of orientation, the first one is going to the positive, the second one going anti-clockwise on the plane to get to the second one, and for he first one, if we have a right-hand that curls it's fingers from the first basis to the second, the thumb points towards the third basis vector. We will call all of these the *positive orientation*. To generlize this pattern into $\mathbb{R}^n$, we will define a basis to be *positivly oriented* if the determinant of the basis is positive (remember that a basis is ordered, and that the sign of the determinant is sensitive to changes in order).

This same principle can be generlized for any vector space V. If $\dim(V) \geq 1$, given an (ordered) basis for V, the basis can have a positive or negative orientation depending on it's determinant. In fact, we can define an equivalence class of basis where two basis are in the same equivalence class if and only if the transition matrix between them has positive determinant. If that is the case, we will say that the two bases are *consistenly oriented*. Since the determinant of a transition matrix can only either be positive or negative, there are only 2 possible equivalence classe. These two equivalence classes on the bases of V are called the *orienations of* V. Because of this, if $[E_1, E_2, ..., E_n]$ is an orientation of V, then we can unambiguously denote the opposite orientation of V by $-[E_1, E_2, ..., E_n]$. A vector-sapce V along with an orientation is called a *oriented vector-space*, and we say that V is *oriented*. With this terminology, if V is oriented and $(E_1, ..., E_n)$ is a basis for V, then if $(E_1, E_2, ..., E_n)$ is in the orienation of V, we would say that $(E_1, E_2, ..., E_n)$ is *positively oriented*, and *negatively oriented* otherwise. If $\dim(V) = 0$, then we will assign either ± 1 to be it's orientation. With this new way of phrasing orientation, we will say that the orienation of $[e_1, e_2, ..., e_n]$ for $\mathbb{R}^n$ is the *standard orientation*.

When we had the change of basis transformation $B : V \to V$, we were taking the determinant of of the change of basis matrix B. We can do a similar process for arbitrary maps: a linear map between oriented vector spaces $T : V \to W$ is called *orientation preserving* if for some choice of basis in the orietnation of V and W, $\det(T) > 0$, and *orientation-reversing* if $\det(T) < 0$.

Given oriented vector spaces (V, μ), (W, ν), we call a linear transformation $T : V \to W$ *orientation preserving* if $[T(v_1), ..., T(v_n)]\nu$ where $\mu[v_1, v_2, ..., v_n]$.

We now consider vector bundles. We can have each vector-space in a vector bundle have an orientation. This is not too intereting, since these orientation can vary wildely. We will want a restriction on a vector-bundle so that the orienation's varry consistenly:

Definition 6.4.1: Oriented Vector Bundle

Let E be a vector bundle over M. Then E is called an *oriented vector bundle*, if there exists a collection of orientations $[\mu_x]$ for each $x \in M$ (collectively labeled $[\mu]$) such that for a local trivialization:

$$\Phi : E|_U \to U \times \mathbb{R}^k$$

If $\mathbb{R}^k$ is given the standard orientation of $[e_1, e_2, ..., e_k]$, each Φ_x is orientation preserving on each $x \in U$

Note that thse if we consider a trivial vector bundle $E = M \times \mathbb{R}^n$ and a bundle isomorphism from E to itself, then E is an oriented vector bundle if and only if it is *connected* (use Intermediate Value Theorem?). Note that if E has an orientation μ, it also has an inverse orientation by considering $-\mu = \{-\mu_x\}$; however, if it has no orientation, then E also cannot have an inverse orientation. A bundle is called *orientable* if it has an orientation.

Definition 6.4.2: Oriented Manifold

A manifold is called an *oriented manifold* if TM is orientable as a vector bundle. If μ is an orientation of TM, then μ is also called the orientation of M.

This condition can be simplified: If TM is orientable, then for any two local trivializations Φ, Ψ induced by charts $(U, \mathbf{x})$ and $(\mathbf{y}, V)$, we have:

$$\begin{tikzcd}
\mathbf{x}(U \cap V) & U \cap V \times \mathbb{R}^k \arrow[l, "\mathbf{x}\circ\pi_1"'] \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d, "\pi"] & U \cap V \times \mathbb{R}^k \arrow[r, "\mathbf{y}\circ\pi_1"] \arrow[dl, "\pi_1"] & \mathbf{y}(U \cap V) \\
& & U \cap V & &
\end{tikzcd}$$

where every map on top of the diagram is a diffeomrophism. Thus, we have the diffeomophism from $\mathbf{x}(U \cap V)$ to $\mathbf{y}(U \cap V)$, which by definition of Φ and Ψ is the transition map. Let's say the maps from $\mathbf{x}(U \cap V)$ to $\pi^{-1}(U \cap V)$ is labeled as φ^{-1} and the maps from $\pi^{-1}(U \cap V)$ to $\mathbf{y}(U \cap V)$ is labeled as ψ. These two maps are Then since TM is orientable:

$$\det D(\psi \circ \varphi^{-1})(x) > 0$$

Example 6.6: Oriented Manifold

1. S^n is orientable.
2. The mobius band is not orientable.

7

Cotangent Bundle

Some of the most important constructs in mathematics are maps on dual spaces, the integral being perhaps the most famous such example. In particular, if a function $f : \mathbb{R} \to \mathbb{R}$, then we can take $\int_a^b \_\_\_ : \hom_{\mathbf{Meas}}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, where $f \mapsto \int_a^b f(t)dt$.

When integrating in higher dimension, we need to keep track of two pieces of information: the dimension of the ambient space we are in (which in elementary calculus is always $\mathbb{R}^n$) and the dimension of the object we are integrating (ex. a curve, a plane, etc.). As an example of integrating over 1-dimensional objects in an arbitrary dimensional $\mathbb{R}^n$, we can define some curve $\gamma : [a, b] \to \mathbb{R}^n$, and then have $f : \mathbb{R}^n \to \mathbb{R}$. We might wish to define $\int_\gamma f(t)dt$ which would give the "length" of the curve. it if it is a straight line, it should be the same as just measuring it with a ruler, and if it is curved, it should be the same as taking a piece of string and measuring out the length, and then pulling it straight and then measuring it with a ruler. You may have seen before that:

$$\int_\gamma f(s)ds = \int_a^b f(\gamma(t))t\sqrt{1 + \gamma'(t)^2}dt$$

The intuitive formula was given by the fact that the squareroot information inside the integral was a mini-pythagoras theorem. But how would we come to this conclusion more readily?

More generally, what if we had a general n-dimensional M instead of $\mathbb{R}^n$, how would we find the area along a curve? what about if we are trying to integrate something more complicated than a curve? The more general discussion for objects that are not curves will be done in chatper 11, which will be a generalization of our discussion of the case for curves.

7.1 Dual Space Reminder

Recall that $V^* = V^\vee = \mathrm{Hom}_V(V, k)$ is the *dual vector space* of V, and is isomorphic to V.

Proposition 7.1.1: Basis For Dual Sapce

Let V be a vector-space with basis $(e_1, e_2, ..., e_n)$. Then the collection $(f_1, f_2, ..., f_n)$ defined by:

$$f^i(e_j) = \delta^i_j$$

where δ^j_i is the kronecker delta is a basis for V^*

Proof **:**
exercise

Thus, if $v \in V$, then $v = \sum_i f^i(v_i)$. Since a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ would have a matrix representation as a row, we sometimes write f^i in a matrix representation as:

$$f^1 = (1\ 0\ \cdots\ 0) \quad f^2 = (0\ 1\ 0\ \cdots\ 0) \quad \cdots \quad f^n = (0\ 0\ \cdots\ 1)$$

If $\omega \in V^*$, we can write it in terms of the basis as:

$$\omega = \omega_i f^i \qquad \omega(e_i) = \omega_i$$

If $f : V \to W$, then $f^* : W^* \to V^*$ is defined by taking the function $x \in W^*$ to the function $f^*(x) \in V^*$ defined by:

$$f^*(x) = x \circ f$$

Furthermore:

$$(g \circ f)^* = f^* \circ g^* \qquad (\mathrm{id}_V)^* = \mathrm{id}_{V^*}$$

showing that $(\_\_\_)^*$ is a functor from $\mathbb{R}$-**Vect** to itself (more generally, any field k will work). Furthermore, the map $(\_\_\_)^{**}$ is a natural isomorphism.

(If you want, add a word on angle bracket notation like John Lee doesp. 274)

7.2 Cotangent Bundle

Definition 7.2.1: Cotangent Bundle

Let M be a smooth manifold and TM its tangent bundle. Then define:

$$T^*_p(M) = (T_pM)^* \qquad T^*M = \bigsqcup_{p \in M} T^*_pM$$

The space T^*_pM is called the *cotangent space* of M at p, and T^*M is called the *cotangent bundle* of M. An element $v \in T^*_pM$ is called a *tangent covector at p* or just *covector at p*.

Given a chart $(U, \mathbf{x})$, if $\left(\frac{\partial}{\partial x^i}\Big|_p \right)$ are coordinates for T_pM, there exists a dual basis:

$$(\lambda^i\big|_p)$$

and as we've just seen, if $\omega \in T_p^*M$, then $\omega = \omega_i \lambda^i|_p$, where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \in \mathbb{R}$$

We now have to deal with what happens if we have another chart $(V, \mathbf{y})$ containing p. We would thus get another basis $(\mu^i|_p)$ for T_p^*M. Then we know from our computation in equation (1.3) on page 41 that:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_j \frac{\partial y^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_p$$

Thus, writing ω in both coordinates: $\omega = \omega_i \lambda^i|_p = \tilde{\omega}_i \tilde{\lambda}^i|_p$, we get:

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \omega\left(\sum_j \frac{\partial y^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^i}\right|_p\right) = \sum_j \frac{\partial y^j}{\partial x^i}(p)\tilde{\omega}_j$$

Hence, we have the following two equataions: one is the components of a vectors after change of variables on a charts of a tangent bundle, another is a change of variables on the charts of a cotangent bunde:

$$\tilde{v}^i = \sum_j \frac{\partial y^i}{\partial x^j}(p)v^j \qquad\qquad w_i = \sum_j \frac{\partial y^j}{\partial x^i}(p)\tilde{\omega}_j$$

Notice how the indices are a bit different, that is, the change of variables for cotangent bundles has the "dual" effect as the change of variables for the tangent bundle.

(got to build-up these computations earlier to motivate the terms *covariant vector* and *contravariant vector*)

We now turn out attention to T^*M and covector-fields. Given any chart $(U, \mathbf{x})$ where $\mathbf{x} = (x^1, x^2, ..., x^n)$, we would like as with vector fields to make a "covector field" using $x^1, x^2, ..., x^n : U \to T^*M$ as *coordinate covector fields.* To do so, we must make sure that T^*M has a vector-bundle structure. The proof of this is essentially the same for all dual of vector-bundles, and so we prove the slightly more general result and show as as corrollary that cotangent bundle's are vector-vundles

Lemma 7.2.1: Dual Bundles Are Vector Bundles

Let $\pi : E \to M$ be any smooth vector bundle. Thenthe dual E^* is a well-defined vector bundle over M i.e., there exits bundle charts and transition mappings corresponding to those for E.

***Proof* :**
First, since all our vector bundles are over the reals, they are inner products so each E_p is naturally isomorphic to E_p^*. Using this, define $\pi^* : E^* \to M$ to be the map sending

$$\sum_i c_i e_{i,p}^* \mapsto p$$

where $e_{i,p}^*$ is the dual basis of E_p. Since each E_p is naturally isomorphic to E_p^* (given the inner product), then bundle atlas for E can be made to a bundle atlas for E^*, as we'll now demonstrate.

It is clear that for any Φ in the bundle atlas of E, then Φ^* sending:

$$\Phi^* : \pi^{-1}(U) \to U \times \mathbb{R}^n \qquad \Phi^*\left(c_i e^*_{i,p}\right) = (p, c_1, ..., c_n)$$

is still smooth and induces a linear isomorphism for each E^*_p. It remains to show that the transition functions are well-defined and smooth. For E, we have that for any patches U, V of the vector-bundle, there exists a $\tau : U \cap V \to \mathrm{GL}(n)$ such that

$$\Psi^{-1} \circ \Phi : U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n \qquad (p, v) \mapsto (p, \tau(p)v)$$

where $\tau(p)$ is the change of basis matrix and τ varries smoothly. Thus, we just need to find how a change of basis works for dual spaces. If $v_p \in E_p$, then represented in two different bases, we get:

$$v_p = \sum_i c_i e_i^* = \sum_i d_i f_i^*$$

Computing the ceofficients, we get:

$$\begin{aligned} c_i &= v(e_i) \\ &= v\left(\tau(p)^{-1} f_i\right) \\ &= v\left(\sum_j \tau(p)^{-1}_{ji} f_j\right) \\ &= \sum_j \tau(p)^{-1}_{ji} v(f_j) \\ &= \sum_j \tau(p)^{-1}_{ji} d_i \end{aligned}$$

Thus, we get that

$$(\Psi^*)^{-1} \circ \Phi^* : U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n \qquad (p, v*) \mapsto (p, (\tau^{-1})^t(p)v*)$$

since τ^{-1} is smooth with smooth inverse (as we've shown the homework on Lie Groups) it is a diffeomorphism, and since the transpose is a linear transormation, $(\tau^{-1})^t$ is also a diffeomorphism. Hence:

$$\tau^* = (\tau^{-1})^t$$

is the desired transition function, showing that $\pi^* : E^* \to M$ is still a vector-bundle, as we sought to show.

Corollary 7.2.1: Cotangent Bundle Is Vector-Bundle

Let M be a n-manifold and T^*M be the associated cotangent bundle. Then there is a unique topological and smooth structure on T^*M making it a smooth vector bundle of rank n for which all coordinate vectors are smooth local sections.

***Proof* :**
This comes from the fact that T^*M is the dual of TM, and so we apply the previous lemma.

If $(U, \mathbf{x})$ is a chart for M, then we can define a chart from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ by:

$$(\omega_i \lambda^i|_p) \mapsto (x^1(p), ..., x^n(p), \omega_1, ..., \omega_n)$$

This chart is called the *natural coordinate chart* for T^*M with respect to $\mathbf{x}$, just as it was for TM.

Definition 7.2.2: Covector-Fields: Differential 1-Forms

Let M be a manifold and T^*M be its cotangent bundle. Then a section map $\sigma_i : M \to T^*M$ is called a *covector field* or a *differential* 1*-form* (or just *1-form* for short).

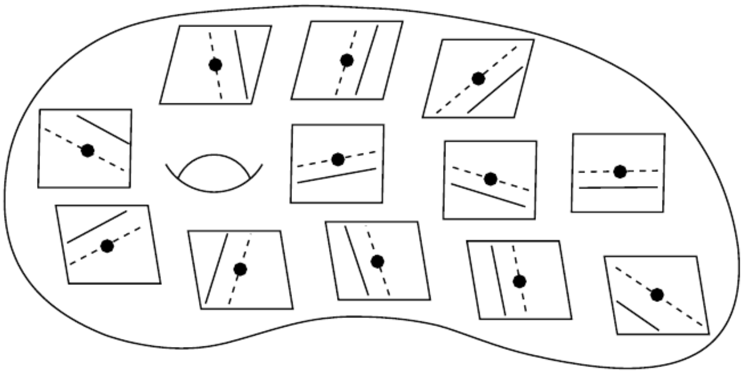
Proposition 7.2.1: Smoothness Criterion For Covector Fields

Let M be a smooth manifold and ω a rough covectorfield. then the following are equivalent:

1. ω is smooth
2. In every coordinate chart, each component function is smooth
3. each point of M is contained in some coordinate chart in which ω has smooth coordinate functions
4. For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M
5. For every open subset $U \subseteq M$ and every smooth vector-field X on U, the function $\omega(U) : U \to \mathbb{R}$ is smooth on U.

***Proof* :**
was left as exercise

Vector-fields are relatively easy to picture, since one of their equivalent constructions was as tangent vectors to curves. Covector fields are not so easy to visualize, however one does still exist:



where for for any $\omega_p \in T_p^*M$, imagine taking it's kernel (which will be 1-dimensional), and the fiber above $\omega_p^{-1}(1)$, which is parallel to the kernel since ω_p is linear. The point of the kernel is to see the "direction", so to speak, that we care about, while the fiber above 1 tell's give us an idea of the "size" of ω_p: when the covector-field is "small" (which I'll take as the values of $\omega_p(e_i)$ are small), then lines of points that map to $\omega_p(v) = 1$ becomes far from the kernel (since if $\omega_p(e_i) \ll 1$, by linearity there would have to be a $r \in \mathbb{R}$, $1 \ll r$ such that $\epsilon_p(r \cdot e_i) \gg 1$). If we have a covector-field then, these hyperlines vary continuously, or smoothly if the covector-field is smooth.

(coframes)

Definition 7.2.3: Collection Of Covector Fields

Let M be a manifold. Then $\mathfrak{X}^*(M)$ is the collection of all smooth covectorfields on M Then $\mathfrak{X}^*(M)$ is the collection of all smooth covectorfields on M

As with $\mathfrak{X}$, $\mathfrak{X}^*$ is naturally a $C^\infty(M)$-module with the action of $f \cdot \omega$ being defined pointwise as:

$$(f\omega)_p = f(p)\omega_p$$

7.3 Differential of Functions

Recall in elementary calculus that we have defined the gradient of a function. In the current notation, we would write it as:

$$\nabla f = \sum_i^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

In other words, we have a map $\nabla : C^\infty(\mathbb{R}^n) \to TM$, or $\nabla f : \mathbb{R}^n \to TM$ (i.e. a vector-field!). This definition worked well when we were not concerned about being coordinate independent. However, this will now pose a problem for us:

Example 7.1: Gradient In Different Coordinates

Let $f(x, y) = x^2$ on $\mathbb{R}^2$ and let X be the vector space defined by:

$$X = \nabla f = 2x\frac{\partial}{\partial x}$$

Computing X in polar coordinates (use the change of variables formula, we see that it will *not* be equal to:

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}$$

Although partial derivative of a smooth function cannot be intepreted in a coodinate-independent way as the components of a vector field, it turns out they *can* be interpreted in a coordinate-*independent* as the components of a *covector field*. The following takes the example of ∇f and gives us how we can generlize it to give us a covector field:

Definition 7.3.1: Differential Of Smooth Function

Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M (with or without boundary). Then we define the covector field $df : M \to T^*M$ where $df(p) \in T_p^*M$ as:

$$df(p) : T_pM \to \mathbb{R} \qquad df(p)(v) = v(f)$$

The covector field df is called the *differential* of f.

Recall that $T_pM = \text{der}(C_p^\infty(M)) \subseteq (C_p^\infty(M))^\vee$, and so $v : C_p^\infty(M) \to \mathbb{R}$, and hence $df_p(v) = v(f)$ is well-defined. We will often write df_p and $df(p)$ interchangeably. We claimed that df is a smooth covector field, we should actually prove it:

Proposition 7.3.1: Differential Is Smooth Covector Field

Let $f : M \to \mathbb{R}$ be a smooth function and df the differential of f. Then df is a smooth covectorfield.

***Proof* :**

It is straight-forward to see that for any $p \in M$ and $v_p \in T_pM$ that $df_p(v_p)$ is indeed a linear functional, and so $df_p(v) \in T_p^*M$. For smoothness of df, by proposition 7.2.1, we have that for any smooth vector-field X on M, $df(X)$ is smooth, since Xf is smooth, and so df is smooth, as we sought to show.

To see that df is the proper generlization of the gradient, consider $(U, \mathbf{x})$ to be a chart on M, so $x = (x^1, x^2, ..., x^n)$. Let (λ^i) be the corresponding coframe on U. Then we can write

$$df_p = \sum_i A_i(p)\lambda^i|_p$$

for appropriate functions functional $A_i : U \to \mathbb{R}$ representing the coeficients as p varries, where

$$A_i(p) = df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Note that if the covector field is smooth, then each $A_i \in C^\infty(U)$. We get that df in coordinates is represented as:

$$df_p = \sum_i \frac{\partial f}{\partial x^i}(p)\lambda_i|_p \tag{7.1}$$

Hence, the component functions of df in any smooth coordinate chart are the partial derivaties of f with respect to those coordinates. Notice how this is eactly the formula for the gradient, but translted into a co-vectorfield. Hence df is indeed the appropraite generlization of the gradient.

Applying equation (7.1) to the special case where $f = x^i$ is a component function of some chart $\mathbf{x}$, we get

$$dx^j|_p = \sum_i \frac{\partial x^j}{\partial x^i}\lambda^i|_p = \sum \delta_i^j \lambda^i|_p = \lambda^j|_p$$

hence, $dx^i|_p = \lambda^i|_p$! We can thus re-write equation (7.1) as:

$$df_p = \sum_i \frac{\partial f}{\partial x^i}(p)dx_i|_p$$

so, as an equation between covector *fields*, we get:

$$df = \sum_i \frac{\partial f}{\partial x^i}(p)dx_i$$

and in the 1-dimensional case, we get:

$$df = \frac{\partial f}{\partial x}(p)dx$$

Which is now *exactly* the definition of the gardient with $\frac{\partial}{\partial x^i}$ replaced with dx^i. From now one, we replace λ^i with dx^i.

Example 7.2: Differential Of Function

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 \cos(x)$. Then Given any chart on $\mathbb{R}^2$ (ex. the identity chart), we get:

$$\begin{aligned} df &= \frac{\partial x^2 y\cos(x)}{\partial x}dx + \frac{\partial x^2 y\cos(x)}{\partial y}dy \\ &= (2xy\cos(x) - x^2 y\sin(x))dx + x^2\cos(x)dy \end{aligned}$$

Proposition 7.3.2: Properties Of Differential

Let M be a smooth manifold and $f, g \in C^\infty(M)$. Then:

1. for any constants $a, b \in \mathbb{R}$, $d(af + bg) = ad(f) + bd(g)$
2. $d(fg) = fd(g) + d(f)g$
3. $d(f/g) = (fd(g) + d(f)g)/g^2$ on the set where $g \neq 0$
4. If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)df$
5. If f is constant, then $df = 0$

Proof **:**
John left as exercise. For (4), use coordinate representation.

Since df generlizes ∇f, we would hope that the information that ∇f gave us is still given by df, in paricular the information given when $\nabla f = 0$. The following proposition shows us that it does:

Proposition 7.3.3: Functions With Vanishing Differential

Let $f : M \to \mathbb{R}$ be a smooth real valued function, then $df = 0$ if and only if f is constant on each component of M

Proof **:**
It suffices to check in the case where M is connected. If f is constant, then by proposition 7.3.2 $df = 0$

Conversely, suppose $df = 0$. Let $p \in M$ be some pont, and consider $\mathcal{C} = \{q \in M \mid f(q) = f(p)\}$. Choose some $q \in M$, an let U be some smooth coordinate ball centered at q with coordinate chart $\mathbf{x} = (x^1, x^2, ..., x^n)$. Since

$$0 = df = \frac{\partial f}{\partial x^i} dx^i$$

we get that $\frac{\partial f}{\partial x^i} \equiv 0$ on U, then it is not difficult using elementary caluculs that f is constant on U. This shows that $\mathcal{C}$ is open. Furthermore, $\mathcal{C} = f^{-1}(p)$, and hence is clsoed by continuity, making $\mathcal{C}$ clopen. Hence, $\mathcal{C} = M$, since M is connected, as we sought to show.

Just like how ∇f represent the small change with respect to the "independent" variables x^i, we get a similar meaning for df with the right interprtation. (df_p best linaer approximation of ∇f at p)

You may have noticed that we have already encounted something like df_p before: If $f : M \to \mathbb{R}$, then $T_p f : T_p M \to T_p \mathbb{R}$, and since $T_p \mathbb{R} \cong \mathbb{R}$ we have $T_p f : T_p M \to \mathbb{R}$, so $T_p f \in T_p^* M$! In fact, thse are indeed identical: notice how they are exactly the same in coordinate represetnations! The usefulness of distinguishing $T_p f$ and df_p with two different symols will come from the role df_p will play in integrating functions over manifolds!

(motivation for the following)

Proposition 7.3.4: Derivative Of A Function Along A Curve

Let M be a smooth manifold, $\gamma : [a, b] \to M$ a smooth curve, and $f : M \to \mathbb{R}$ a smooth function. Then the derivative of $f \circ \gamma : [a, b] \to \mathbb{R}$ is given by:

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\left(\gamma'(t)\right)$$

***Proof* :**
This is definition unwinding, for any $t_0 \in [a, b]$:

$$\begin{aligned} df_{\gamma(t_0)}(\gamma'(t_0)) &= \gamma'(t_0)f && \text{definition of } df \\ &= d\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right)f && \text{definition of } \gamma'(t) \\ &= \left.\frac{d}{dt}\right|_{t_0}(f \circ \gamma) && \text{definition of } d\gamma \\ &= (f \circ \gamma)'(t_0) && \text{definition of } d/dt|_{t_0} \end{aligned}$$

Visually, what we are working towards is:

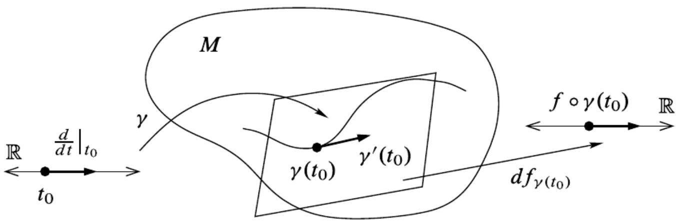


Fig. 11.3 Derivative of a function along a curve

7.4 Pullback Of Covectors

Given $f : M \to N$, we get $T_p f : T_p M \to T_{f(p)} N$, the dual of which is $T_p f^* : T_{f(p)} N^* \to T_p M^*$, the collection of which is $T f^* : TN^* \to TM^*$. We give a name to this function:

Definition 7.4.1: Pullback of tangent Space

Let $f : M \to N$ be a smooth map. Then the map Tf^* is called the *pullback* of f, or the *cotangent map* of f. Unravelling the definition, we get $T_p f^*(\omega) \in T_p^* M$:

$$T_p f^*(\omega)(v) = \omega(T_p f(v))$$

Naturally, T_p^* is a contravariant functor from the category of pointed smooth manifolds to the category of real vector spaces, while T^* is a contravariant functrom from the cateogry of smooth manifolds to itself. It is thus pretty inconvinient to call the elemnts of $T_p^* M$ “covariant vectors”, but it is now a convention entrenched in differential geometry.

(a word on covector field pullback. Won’t be able to do yet because haven’t discussed vector field pushforward because that requries lie group homomorhpisms to do successfully)

Definition 7.4.2: Pullback Of Covector Field

Let $f : M \to N$ be a smooth map, $Tf^* : TN^* \to TM^*$ be the pullback of f, and $\omega : N \to T^*N$ a covector-field on N (i.e. a 1-form). Then the *pullback* of ω by f is a rough covector field $(f^*\omega) : M \to T^*M$ given by:

$$(f^*\omega)_p(v) = \omega_{f(p)}(T_pf(v))$$

which we can represent as:

$$(f^*\omega)_p = T_pf^*(\omega_{f(p)})$$

We do not yet know that $(f^*\omega)$ is continuous or that it is smooth if ω is smooth. The following proposition gives us that:

Proposition 7.4.1: Properties of Pullback of Covector Fields

Let $f : M \to N$ be a smooth map, $u : N \to \mathbb{R}$ a continuous real-valued function on N, $\omega : N \to T^*N$ a covector field on N. Then:

$$f^*(u \cdot \omega) = (u \circ f) \cdot f^*\omega$$

where $(u \cdot \omega)_p = u(p)\omega$ is the action defined on $\mathfrak{X}^*$. If u is smooth, then:

$$f^*du = d(u \circ f)$$

***Proof* :**
To prove the first equation, we see that:

$$\begin{aligned}
(f^*(u \cdot w))_p &= T_pf^*((u \cdot \omega)_{f(p)}) && \text{pullback definition}\\
&= T_pf^*(u(f(p))\omega_{f(p)}) && \text{module action}\\
&= u(f(p))T_pf^*(\omega_{f(p)}) && \text{Linearity of } T_pf^*\\
&= u(f(p))(f^*\omega)_p && \text{pullback definition}\\
&= (u \circ f) \cdot (f^*\omega)_p && \text{module action}
\end{aligned}$$

Givign equality. If u is furthermore smooth, then du is a covector field on N. For any $v \in T_pM$,

$$\begin{aligned}
(f^*du)_p(v) &= T_pf^*(du_{f(p)})(v) && \text{pullback definition}\\
&= du_{f(p)}(T_pf(v)) && \text{evaluate } T_pf^*\\
&= T_pf(v)(u) && \text{evaluate } du\\
&= v(u \circ f) && \text{evaluate } T_pf(v)\\
&= d(u \circ f)_p(v) && \text{pullback definition}
\end{aligned}$$

Proposition 7.4.2: Continuous And Smooth 1-Forms

Let $f : M \to N$ be a smooth map and $\omega : N \to T^*N$ a 1-form on N. Then $f^*\omega$ is a continuous 1-form n M. If ω is smooth, so is $f^*\omega$.

Proof **:**
Let $f : M \to N$ be smooth, $p \in M$ be arbitrary, and choose smooth coordinate charts (y^j) for N in a neighborhood V of $f(p)$. Let $U = f^{-1}(V)$ which is a neighborhood of p. Writting ω as $\omega = \omega_j dx^j$ for continuous functions ω_j on V, and using the above proposition twice, (on $f|_U$), we get:

$$f^*\omega = f^*(\omega_j dx^j) = (\omega_j \circ f)f^*dy^j = (\omega_j \circ f)d(y^j \circ f)$$

Often, the above formula is written as:

$$f^*\omega = (\omega_i \circ f)d(f^i) \tag{7.2}$$

where f^i is the component function of f.

Example 7.3: Pullback
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be $(u, v) = f(x, y, z) = (x^2y, y \sin z)$. Let $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ be:

$$\omega = udv + vdu$$

Then $f^*\omega$ would be:

$$\begin{aligned} f^*\omega &= (u \circ f)d(v \circ f) + (v \circ f)d(u \circ f) \\ &= (x^2y)d(y \sin z) + y \sin(z)d(x^2y) \\ &= 2xy^2 \sin z dx + 2x^2y \sin z dy + x^2y^2 \cos z dz \end{aligned}$$

(1-form restriciton to submanifold)

7.5 Line Integrals

As mentioned, covector-fields (which are 1-forms) allow us to define integration on manifolds, in particular line integrals (hence the name 1-form).

In the simplest case, we can take $[a, b] \subseteq \mathbb{R}$ and consider a smooth covector field $\omega : [a, b] \to T^*[a, b]$ on $[a, b]$ (smooth here means that it extens to a smooth covector field to some open neighborhood of $[a, b]$). If we let t denote the standard coordinate on $\mathbb{R}$ (i.e. the coordinate chart $(t, \mathbb{R})$), Then we can represent ω in terms of coordinates as: $\omega_t = f(t)dt$ for some smooth function $f : [a, b] \to \mathbb{R}$. Using this, we can finally define the *integral of* ω *over* $[a, b]$ to

$$\int_{[a,b]} \omega = \int_a^b f(t)dt$$

This seems to be dependent on the particular choice of basis to represent ω and hence not well-defined; the following proposition shows that is not the case *given* that we preserve orientation!

Proposition 7.5.1: Diffeomrophsim Invariance Of The Integral

Let ω be a smooth covector-field on a compact interval $[a, b] \subseteq \mathbb{R}$. If $\varphi : [c, d] \to [a, b]$ is an increasing diffeomorphism, then

$$\int_{[c,d]} \varphi^* \omega = \int_{[a,b]} \omega$$

Proof **:**
Lettting s denote the standard coordinate on $[c, d]$ and t on $[a, b]$, then by proposition 7.4.1, we see that the pull-back $\varphi^*\omega$ has the coordinate expression $(\varphi^*\omega)_s = f(\varphi(s))\varphi'(s)ds$. Putting this into the definition of the ordinary integarl and using change of variables, we get:

$$\int_{[c,d]} \varphi^* \omega = \int_c^d f(\varphi(s))\varphi'(s)ds = \int_a^b f(t)dt = \int_{[a,b]} \omega$$

Note (or prove) that if φ was a decresing diffeomorphism, we would get:

$$\int_{[c,d]} \varphi^* \omega = -\int_{[a,b]} \omega$$

Thus, we see that it is essentially well-defined up to a minus sign. Thus, we will keep track of this information in order to be able to define integration that change the curve even if it introduces a negative:

Definition 7.5.1: Reparamaterization

Let $\gamma : [a, b] \to M$, $\tilde{\gamma} : [c, d] \to M$ be curves into a smooth manifold. If $\tilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi : [c, d] \to [a, b]$, then if φ is an *increasing* function, it is called a *forward re-paramaterization*, and if φ is a decreasing function, it is called a *bacwkards reparamaterization.*

Proposition 7.5.2: Parameter Independent Line Integral

p. 290 John lee

Proof **:**
here

(now gies a thing about piecwise line integral results, skipped for now, p. 288, I believe it is to show linaerity of integral. Make sure to amend the definition of reparamaterization to be aware of piecewise segemnts)

Definition 7.5.2: Line Integral On Manifold

Let $\gamma : [a, b] \to M$ be a smooth curve segment and ω a smooth covector field on M. Then we define the *line integral* of ω over γ to be the real number:

$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega$$

Proposition 7.5.3: Properties of Line Integral

Let M be a smooth manifold (with or without boundary). Suppose $\gamma : [a, b] \to M$ is a piecewise smooth curve-segment and $\omega_1, \omega_2 \in \mathfrak{X}^*(M)$. Then:

1. For any $c_1, c_2 \in \mathbb{R}$:
$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2$$
2. If γ is a constant map, then:
$$\int_\gamma \omega = 0$$
3. If $\gamma_1 = \gamma|_{[a,b]}$ and $\gamma_2 = \gamma|_{[c,d]}$ with $a < c < b$, then:
$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$
4. If $f : M \to N$ is any smooth map and $\eta \in \mathfrak{X}^*(M)$, then:
$$\int_\gamma f^*\eta = \int_{f\circ\gamma} \eta$$

Proof **:**
exercise

Example 7.4: Computing Line Integral
Let $M = \mathbb{R}^2 \setminus \{0\}$ given by
$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$
with $\gamma : [0, 2\pi] \to M$ with $\gamma(t) = (\cos(t), \sin(t))$. The answer should be 2π. Check John Lee p. 290 if you're stuck.

This next proposition gives another way we can see the line integral

Proposition 7.5.4: Concrete Line Integral

Let $\gamma : [a, b] \to M$ be a piecewise smooth curve segment. Then the line integral of ω over γ can be expressed as:

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt$$

***Proof* :**
unwinding definition, John Lee p.291

Theorem 7.5.1: Ftc For 1-Forms

Let M be a smooth manifold (with or without boundary). Suppose $f : M \to \mathbb{R}$ is smoot hand $\gamma : [a, b] \to M$ is a piecewise smooth curve segment in M. Then:

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

***Proof* :**
By proposition 7.4.1 and proposition 7.5.4, we get:

$$\int_\gamma df = \int_a^b df_{\gamma(t)}(\gamma'(t))dt = \int_a^b (f \circ \gamma)' dt$$

which by the normal fundamental theorem of calculus is equal to:

$$f(\gamma(b)) - f(\gamma(a))$$

If we have a curve, the mid-points cancel.

7.6 Conservative Covector Field

The FTC shows that integarls of forms of the form df are particularly easy once f is known.

Definition 7.6.1: Exact Differnetial

Let ω be a 1-form. Then ω is called *exact* if there exists a f such that

$$\omega = df$$

In this case, f is called the *potential* of ω.

Thus, it is meaningful to wonde which forms are exact, if not all. The FTC provides a hint: the integral of a form should be $f(\gamma(b)) - f(\gamma(a))$. Thus, any *closed curve* (so $\gamma(a) = \gamma(b)$) should integrate to zero. We give a name to form with this proeprty:

Definition 7.6.2: Conservative Form

Let ω be a 1-form. Then ω is called *conservative* if every line integral over a piecewise smooth closed curve is 0.

Proposition 7.6.1: Line Integral Independent On Coservative Form

A smooth 1-from ω is conservative if and only if its line integrals are path independent in the senes that if $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$ whenever γ and $\tilde{\gamma}$ are piecewise smooth curve-segments with the same starting and ending point

Proof **:**
Remember that we defined integration on *piece-wise* curve segments, making this proposition relatively easy!

8

Tensors

Make this chapter a reivew of tensor algebra. Define symmetric and alternating tensor.

8.1 Multilinear Algebra

Recall the following in the context of vector-spaces:

Definition 8.1.1: Multilinear Map

Let $V_1, V_2, ..., V_k$ and W be vector-spaces. Then $f : V_1 \times V_2 \times \cdots \times V_k \to W$ is called *multilinear* if it is linear in each component.

Example 8.1: Multi-Linear Maps

1. the dot product from $\langle \cdot, \ \cdot \rangle : \mathbb{R}^n \to \mathbb{R}$ is bilinear map that is interpreted as the "length" of vectors along with some orientation them
2. the cross product on $\cdot : \mathbb{R}^3 \to \mathbb{R}$ is a bilinear map interpreted as computing the area of a parallelogram, or finding an orthogonal vector to two vectors which with priority given to "right-handedne".
3. The determinant $\det(\_\_\_) : \mathbb{R}^{n^2} \to \mathbb{R}$ as a n-linear function giving us a notion of volume
4. The lie bracket for a lie algebra $\mathfrak{g}$ is a bilinear map (not sure how to interpret it yet)
5. The Euler form on root lattices from Lie Algebra representation is another example.

We see how multilinear maps are pretty common in geometry, and many represent geometric notions like volume or length (idk what lie brackets yet represent TBD). Recall from geometry that given a R-module M, we can define a new R-module

$$\mathcal{T}(M) = \bigoplus_{i \in \mathbb{N}} \mathcal{T}^k(M)$$

where $\mathcal{T}^k(M) = \bigotimes_i^k M$. In our case, we will let $M = V^*$ be the dual of a vector-space. Then for any $f, g \in V^*$, we define their tensor-product to be:

$$f \otimes g(a, b) = f(a)g(b)$$

notice that $f \otimes g$ is bilinear, and so by the universal property of tensors we are jusitifed in writing $f \otimes g$. We will say that $f \otimes g \in \mathcal{L}^2(V, V, R)$, or since the vector-spaces are the same, we will write $\mathcal{L}^2(V, \mathbb{R})$. Furthermore, $\mathcal{L}^2(V, \mathbb{R})$ is still a vector-space.

This process easily generalized: Let $F \in \mathcal{L}^n(V_1, V_2, ..., V_n, \mathbb{R})$ and $G \in \mathcal{L}^m(W_1, W_2, ..., W_m, \mathbb{R})$ be an n-linear and m-linear map respectively. Then:

$$F \otimes G(v_1, ..., v_n, w_1, .., w_m) = F(v_1, ..., v_n, w_1, ..., w_n)$$

is an $n + m$-linear map and belongs in $\mathcal{L}^{n+m}(V_1, V_2, ..., V_n, W_1, W_2, ..., W_m, \mathbb{R})$ (recall that tensor-products are associative, and hence this is indeed well-defined).

Recall that a basis for $\mathcal{L}^n(V_1, V_2, ..., V_n, \mathbb{R})$ is simply the k-tensor product of the basis for each V_i^*. Due to this, we have:

$$V_1^* \otimes \cdots V_k^* \cong_{k\text{-}\mathbf{Vect}} \mathcal{L}^k(V_1, ..., V_k, \mathbb{R})$$

and so since $V_1 \cong V_1^{**}$ cannonically, we also have:

$$V_1 \otimes \cdots V_k \cong_{k\text{-}\mathbf{Vect}} \mathcal{L}^k(V_1^*, ..., V_k^*, \mathbb{R})$$

giving us a characteriztion of tensor products in terms of a k-linear map on *dual spaces*.

8.2 Covariant and Contravariant tensors on Vector Spaces

The last thing we showed is that we can take the tensor product of V_i's or V_i^*'s. These behave differently, and so we will give these a name:

Definition 8.2.1: Covariant And Contravariant Tensor

Let V be a finite dimensional vector space and $k \in \mathbb{N}_{>0}$. Then a *covariant* k-tensor on V is an element of

$$V^* \otimes \cdots \otimes V^* = \mathcal{T}^k(V^*)$$

and a *contravariant* k-tensor on V is an element of

$$V \otimes \cdots \otimes V = \mathcal{T}^k(V)$$

Example 8.2: Covariant And Contravariant Tensors

1. Any linear functional $\omega : V \to \mathbb{R}$ is linear, and so a *covariant 1-tensor*. We also call such a function a covector, an element of $T^1(V^*)$.
2. The dot product, or more generally the inner product, is a *covariant 2-tensor*. Covariant 2-tensors are also called *bilinear forms*.
3. The determinant is a covariant n-tensor.
4. I will see if I can find a good way of adding the contravariant tensors here TBD

it is sometimes useful to mix these two:

Definition 8.2.2: Mixed Tensors

Let V be a finite dimensional vector space, and $k, l \in \mathbb{N}_{>0}$. Then ω is a *mixed tensor* of type (k, l) if it is in:

$$T^{(k,l)}(V) := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ times}}$$

8.3 Symmetric And Alternating Tensors

The most important tensors products for us are those that have more geometric meaning. We present the two important ones here:

8.3.1 Symmetric Tensors

Definition 8.3.1: Symmetric Tensor

Let V be a finite dimensional vector space. Then a covariant k-tensor on V is said to be *symmetric* if it's value is unchanged if we interchange any pairs of arguments

The collection of all symmetric k-tensors is denoted $\Sigma^k(V)$

(eventually will fill)

8.3.2 Alternating Tensors

Definition 8.3.2: Alternating Tensor

Let V be a finite dimensional vector space. Then a covariant k-tensor on V is said to be *alternating* if it's sign changes if we interchange any pairs of arguments

The collection of all alternating k-tensors is denoted $\Lambda^k(V)$

Proposition 8.3.1: Properties Of Alternating Tensorsq

Let α be a covariant k-tensor on a finite dimensional vector-space V. Then the following are equivalent:

1. α is alternating
2. $\alpha(v_1, ..., v_n) = 0$ whenever the k-tulpe $(v_1, v_2, ..., v_n)$ is linearly independent
3. If two arguments are the same, then:

$$\alpha(v_1, ..., w, ..., w, ..., v_n) = 0$$

Proof **:**
p.350 John Lee

8.4 Tensor Bundle

Definition 8.4.1: Tensor Bundle

Let M be a smooth manifold, TM it's tangent bundle, and $\mathcal{T}^k T_p M$ the k-tensors on $T_p M$. Then:

$$\mathcal{T}^k(T^*M) = \bigsqcup \mathcal{T}^k(T_p^*M)$$

is called the *[covariant] tensor bundle* of M. Analogously, we define:

$$\mathcal{T}^k(TM) = \bigsqcup \mathcal{T}^k(T_p M)$$

to be the *[contravariant] tensor bundle* of M, and

$$\mathcal{T}^{(k,l)}(TM) = \bigsqcup \mathcal{T}^{k,l}(T_p M)$$

to be the *mixed tensor bundle* of M of type (k, l)

Notice that the tangnet and cotangent bundles are both tensor bundles, and hence they are all special cases.

Definition 8.4.2: Tensor Field

Let M be a smooth manifold. Then ω is a (covariant, contravariant, mixed) *tensor field* if it is a section map from M to the (covariant, contravariant, mixed) tensor bundle. A tensor field is *smooth* in the same way a section map for a vector-bundle is smooth

Thus, contravariant 1-tensor fields are vector-fields, covariant 1-tensor field are covector-fields. Since 0-tensors are scalars (i.e. $\mathbb{R}$), 0-tensors fields are continuous real valued functions. The space of smooth sections on these tensor bundles, which we'll denote $\Gamma(\mathcal{T}^k(T^*M)), \Gamma\left(\mathcal{T}^k(TM), \Gamma(\mathcal{T}^{(k,l)}(TM)\right)$ is an infinite dimensional vector-space over $\mathbb{R}$, and a module over $C^\infty(M)$. For any smooth coordinate chart $(U, \mathbf{x})$, these bundles can be written as:

$$A = \begin{cases} A_{i_1 \dots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}, & A \in \Gamma\big(T^k T^* M\big); \\ A^{i_1 \dots i_k} \dfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \dfrac{\partial}{\partial x^{i_k}}, & A \in \Gamma\big(T^k TM\big); \\ A^{i_1 \dots i_k}_{j_1 \dots j_l} \dfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \dfrac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}, & A \in \Gamma\big(T^{(k,l)} TM\big). \end{cases}$$

The most important tensor fields for us are *covariant k-tensor* fields, and so we single-out a notation for them:

$$\mathfrak{T}^k(M) = \Gamma(\mathcal{T}^k(T^*M))$$

(some useful lemmas and propositions)

(define symmetric and alternating tensor fields)

8.5 Pullback Of Tensor Fields

Just like covector fields can be pulled back along smooth maps, so can covariant tensor fields (which should be excitting, since we were able to define line integrals on manifolds due to this). The fact that we can pull back like this is probably the main reason covarian tensor fields are prioritized in study

Definition 8.5.1: Tensor-Field Pullback

Let $f : M \to N$ be a smooth map. Then for any $p \in M$ and any k-tensor $\alpha \in \mathcal{T}^k(T^*_{f(p)}N)$, define $T_p f^*(\alpha) \in \mathcal{T}^k(T^*_p M)$ to be:

$$(T_p f^*(\alpha))(v_1, ..., v_k) = \alpha(T_p f(v_1), ..., T_p f(v_k))$$

for any $v_1, v_2, ..., v_k \in T_p M$. This is called the *pointwise pullback* of α by f at p. If $A : N \to \mathcal{T}^k(T^*_{f(p)}N)$ is a covariant k-tensor field on N, define the rough k-tensor field f^*A on M, called the *pullback* of A by f by:

$$(f^*A)_p = T_p f^*(A_{f(p)})$$

where this tensor fields acts on vectors $v_1, v_2, ..., v_k \in T_p M$ by:

$$T^*_p(f^*A)(v_1, ..., v_k) = \alpha(T_p f(v_1), ..., T_p f(v_k))$$

Proposition 8.5.1: Properties Of Pullbacks

Let $f : M \to N$, $g : N \to P$ be smoth maps, A, B covariant tensor fields on N, and $\tau : N \to \mathbb{R}$ a real-valued function on N. Then:

1. $f^*(\tau B) = (\tau \circ f) f^* B$
2. $f^*(A \otimes B) = f^* A \otimes f^* B$
3. $f^*(A + B) = f^* A + f^* B$
4. $f^* B$ is a *continuous* tensor field, and is smooth if B is smooth
5. $(g \circ f)^* B = f^*(g^* B)$
6. $(\mathrm{id}_N)^* B = B$

Proof **:**
John left as exercise, since it repeats many previous things we've seen in covector fields

Note that if τ is a 0-tensor field an B a k-tensor field, if we interpret $\tau \otimes B$ as fB $f^*\tau$ as $\tau \circ f$, then (1) in the previous propostion is just a special case of (b).

Corollary 8.5.1: Computing Pullbacks

Let $f : M \to N$ be a smooth map an B a covariant k-tensor field on N. Picking $p \in M$, if $(V, \mathbf{y})$ is a coordinate chart containing $f(p)$ wit $\mathbf{y} = (y^1, y^2, ..., y^n)$, then we can write $f^* B$ as:

$$
\begin{aligned}
&f^* \left(B_{i_1 \dots i_n}) dy^{i_1} \otimes \dots \otimes dy^{i_k} \right) \\
= &(B_{i_1 \dots i_n} \circ f) d(y^{i_1} \circ f) \otimes \dots \otimes d(y^{i_k} \circ f))
\end{aligned}
$$

Proof **:**
exercsie

Example 8.3: Pullback Of Tensorfield
p. 321 John Lee

9

Riemann Metric

The first concept we will explore is what information symmetric tensor bundles will give us, in particular symmetric 2-forms which are positive-definite, that is an inner product, allowing us to define notions of angles, lengths, distance, and so forth (notions of size will be left for chatper 11). In essense, we are defining *geometric concepts*. As a reminder:

1. The *length* or *norm* of a vector $v \in V$ is defined to be

$$\|v\| = \langle v, v\rangle^{1/2}$$

2. The *angle* between two nonzero vectors $v, w \in V$ is the unique $\theta \in [0, \pi]$ satisfying

$$\cos(\theta) = \frac{\langle v, w\rangle}{\|v\|\|w\|}$$

3. two vectors are said to be *orthogonal* if $\langle v, w\rangle = 0$

Just like inner products give these geometric notions on vector-spaces, a positive-definite symmetric 2-form (i.e. a smoothly varying inner producton each tangent space) will allow us to generlize these notions by defining them locally. The main tool in this chapter is the above mentioned tensor-field, and so we give it a name: *Riemann metric*, and a manifold with a Riemann metric is called a *Riemann manifold*. (TBD more here after chapter written)

9.1 Riemann Manifold

Definition 9.1.1: Riemann Metric and Manitolfd

Let M be a smooth manifold (with or without boundary). Then

1. A *Riemannian metric* on M, usually denoted g, is a smooth symmetric covariatn 2-tensor field on M that is positive deifnite at each point (i.e. is an inner product at each point). If g is a Riemannian metric, then we usually denote the inner product on T_pM at g_p and write $g_p(v, w)$ or if it is understood that $v, g \in T_pM$, $\langle v, w \rangle_g$. The norm of a tangent vector $v \in T_pM$ is denoted $\|v\|_p$ or $|v|_p$.

2. A *Riemannian manifold* (resp. Riemannian manifold with boundary) is a pair (M, g) where M Is a smooth manifold (resp. smooth manifold with boundary) and g is a Riemannian metric on M. If g is understood from context, one sometimes simply says "Let M be a Riemannian manifold" (resp. Riemannian manifold with boundary)

Note The Riemannian metric is not a metric, however the two concepts are related, as we'll soon show. In the context of Riemannian manifolds, we will usually calls metrics in the metric space sense "distance functions", while using the word metric for a Riemannian metric.

Given any smooth local coordinates (x^i), the Riemann metric is written as:

$$g = g_{ij} dx^i \otimes dx^j$$

where (g_{ij}) is a symmetric positive definite matrix of smooth functions. By the symmetries of g, we can re-write g in terms of symmetric products:

$$\begin{aligned}
g &= g_{ij} dx^i \otimes d^x \\
&= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i dx^j) && g_{ij} = g_{ji} \\
&= \frac{1}{2}(g_{ij} dx^i \otimes dx^j + g_{ij} dx^j dx^i) && \text{change of var in 2nd term} \\
&= g_{ij} dx^i dx^j && \text{symmetric product definition}
\end{aligned}$$

Example 9.1: Riemannian Metric

1. (Euclidean Metric) Let $M = \mathbb{R}^n$. The simplest metric we can define is the *Euclidean metric*, usually denoted $\overline{g}$, whihc given the standard coordinates on $\mathbb{R}^n$ is defined by:

$$\overline{g} = \delta_{ij} dx^i dx^j$$

where δ_{ij} is the kronecker delta. If we abbrevaite the symmetric product $\alpha\alpha$ to $\alpha\alpha^2$, then the Eucildean metric is often simplified to:

$$\overline{g} = (dx^1)^2 + \cdots + (dx^n)^2$$

To see that this is indeed the usual inner product on $\mathbb{R}^n$ (i.e. the dot product), take $v, w \in T_p\mathbb{R}^n$. Then:

$$g_p(v,w) = \delta_{ij}v^i w^j = \sum_{i=1}^{n} v^i w^i = v \cdot w$$

A word on notation: it was warned tha texpression involving the euclidean dot prodcut are likely to violate our index conventions and therefore to requrie explicit summation signs. This can usually be aoivded by writing the metric foeccicients explicitly, as in $\delta_{ij}v^i w^j$.

2. (Product Metric). Given (M,g) and $(\tilde{M},\tilde{g})$ are Riemannian manifolds, we can define a Riemannian metric $\hat{g} = g \oplus \tilde{g}$ on the product manifold $M \times \tilde{M}$, caleld the *product metric*, by doing:

$$\hat{g} = ((v,\tilde{v}),(w,\tilde{w})) = g(v,w) + \tilde{g}(\tilde{v},\tilde{w})$$

for any $(v,\tilde{v}),(w,\tilde{w}) \in T_pM \oplus T_q\tilde{M} \cong T_{(p,q)}(M \times \tilde{M})$. Given any local coordinaets $(x^1,x^2,...,x^n)$ for M and $(y^1,y^2,...,y^m)$ for $\tilde{M}$, we obtain local cordinates

$$(x^1,x^2,...,x^n,y^1,y^2,...,y^m)$$

for $M \times \tilde{M}$, and the product metric is represented locally by the block diagonal matrix:

$$\hat{g}_{ij} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \tilde{g_{ij}} \end{pmatrix}$$

As an exercise, verify the euclidean metric on $\mathbb{R}^{n+m}$ is the same as the metric determined by the Euclidean metric on $\mathbb{R}^n$ and $\mathbb{R}^m$

3. (Lorenzian Metric) here

Proposition 9.1.1: Existence Of Riemannian Meric

Let M be a smooth manifold (with or withtou boundary). Then M admits a Riemannian metric.

Proof **:**
John Lee p. 329

Note how much choice is made in the construction of this metric, it is far from being natural or cannonical. We can thus come up with some very different geometries

Example 9.2: Four Riemannian Metrics On $\mathbb{R}^2$
problem 13-20 in John Lee

Now that we have an inner product on each tangent space, we have the notion of orthonormality

Definition 9.1.2: Orthonormal Frame

Let (M, g) be a Riemannian manifold. Then a (local or global) frame $(E_1, E_2, ..., E_n)$ for M is an *orthonormal* frame if the vectors $(E_1|_p, ..., E_n|_p)$ form an orthonormal basis for each T_pM, i.e.

$$\langle E_i, E_j \rangle_g = \delta_{ij}$$

Example 9.3: Orthonormal Frames

1. John lee p. 329

(given smoothlocal frame on Riemannian manifold, can find smooth orthornomal frame)

(cor existence of local orth frame)

(important remark..?)

9.2 Pullback Metric

We as usual want to know how to induce a metric given a smooth map $f : M \to N$ between two smooth manifolds, with particular interest when $N = \mathbb{R}^n$.

10

Differential Forms

(motivation: alternating tensors are universal with respect to determinants, so they naturally capture volume informatin. An alternating tensor field thus captures volume information locally. The goal of this section is to find a way to use that information to integrate on manifolds)

The first thing to realize is that there is no way of integrating functions without being coordinate depenent[1]. For example, Let $C \subseteq \mathbb{R}^n$ be a closed ball, $f : C \to \mathbb{R}$ be $f(x) \equiv 1$. The intuitively:

$$\int_C f dV = \mathrm{Vol}(C)$$

However, this is clearly dependent on some choice of coordinates, which we know for arbitrary manifolds there is no natural choice. On the other hand, we showed that a covector field could be integrated over curves in a natural way. Let's look at the geometry of covector fields a little more closely. Given a smooth manifold M, a covector field $\omega : M \to TM^*$ assigns a number to *each* tangent vector in TM (contrast this to how a vector-field simply chooses *a* tangent vector at every point). Since $\omega_p : TM \to \mathbb{R}$ is a linaer function, scaling the tangent vectors has the effect of scaling the output by the same scalar. Due to this, we only need to think of the magnitude and direction of $\omega_p(e_i)$. Thus, ω assigned a sort of "signed length meter" to each 1-dimensional subspace of each tangent space T_pM. When computing the line integral of a co-vector field, we are in effect assigning a "length" to a curve by using this varying measure of scale along points in a curve.

We know see a generlization of this, a sort of "field" that can be integrated on a k-submanifold of an n-manifold. The value at each point of this field whould give a sort of "signed volume meter" on k-dimensional subspaces of the tangent space. John Lee phrases it this way:

> It is a machine ω that accept any k tangent vectors $(v_1, v_2, ..., v_k)$ and returns a number $\omega(v_1, v_2, ..., v_k)$ that we can think of as a "signed volume" of the parrallelepiped spanned by those vectors, measured according to the scale determined by ω

[1] At least without adding any extra structure like a Riemann Metric TBD

The most famous such tool we've leanrt in linear algebra to be det on $\mathbb{R}^n$. For example, if $n = 2$ and $v_1, v_2 \in \mathbb{R}^2$, then $\det(v_1, v_2)$ gives the area of the square these two vectors give, along with an orientation determined by whether the determinant is positive or negative. The function det in higher dimensions give signed parrallelepipeds.

Let us now consider what properties such a "signed k-dimensional volume meter" ω should have. First, we know if we multiply any edge of an n-dimensional volume, we should scale the volume of the entire area by that scalar, and if we add two parrallelepipeds where all the edges are of the same length, except two, then we should add the two parrallelepipeds together i.e. ω should be multi-linear given the edges as input,

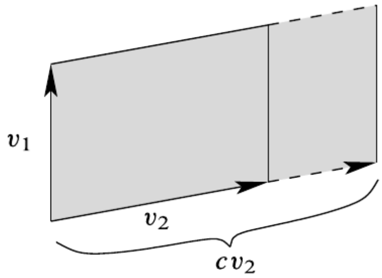


Fig. 16.2 Scaling by a constant

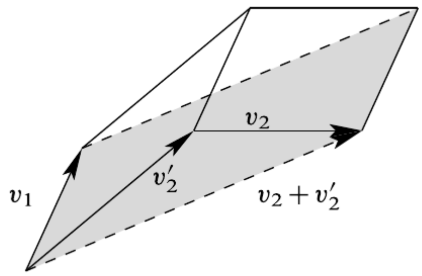


Fig. 16.3 Sum of two vectors

Finally, if two of the vectors of a parrallelepipeds are the same, then in the given dimension we would expect the resulting volume to be zero. These tools in this chapter

10.1 Algebra of Alternating Tensor

Lemma 10.1.1: Alternating Tensor Equivalence

Let α be a covariant k-tensor on a finite dimensional vector space V. Then the following are equivalent:

1. α is alternating
2. $\alpha(v_1, v_2, ..., v_k) = 0$ whenever the k-tuple $(v_1, v_2, ..., v_k)$ is linearly depenent
3. α gives a zero whenver 2 argumetns are equal:

$$\alpha(v_1, ..., w, ..., w, ..., v_k) = 0$$

Proof **:**
p.350 John

We can defin the *alternization* to be the map Alt : $T^k(V^*) \to \Lambda^k(V^*)$,

$$\alpha \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)({}^\sigma \alpha)$$

Example 10.1: Alternating Tensor

A 1-tensor is always alternating. For a general alternating 2-tensor or 3-tensor from a general tensor (do it as an exercsie)

If α is already alrernating, then $\operatorname{Alt}(\alpha) = \alpha$.

To keep track of computations of alternating tensors the following convention is useful: If $I = (i_1, i_2, ..., i_k)$ is a collection of positive integers, it is called a *multi-index* of length k. If I is such a multi-index, and $\sigma \in S_k$, then:

$$I_\sigma = (i_{\sigma(1)}, ..., i_{\sigma(k)})$$

Note that with this definiton that $(I_\sigma)_\tau = I_{\sigma\tau}$.

Definition 10.1.1: Elementary Alternating Tensor

Let V be a finite dimensional vector space, and $(\epsilon^1, \epsilon^2, ..., \epsilon^n)$ is any basis for V^*. For each multi-index set $I = (i_1, i_2, ..., i_k)$ where $1 \leq i_1 \leq \cdots \leq i_k \leq n$, define the *alternating covariant k-tensor* $\epsilon^I = \epsilon^{i_1 \cdots i_k}$ by:

$$\epsilon^I(v_1, v_2, ..., v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} (v_1)^{i_1} & \cdots & (v_k)^{i_1} \\ \vdots & \ddots & \vdots \\ (v_1)^{i_k} & \cdots & (v_k)^{i_k} \end{pmatrix}$$

To streamline computations, it is nice to generlize the Kronecker delta to:

$$\delta_I^J = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \cdots & \delta_{j_k}^{i_1} \\ \vdots & \ddots & \vdots \\ \delta_{j_1}^{i_k} & \cdots & \delta_{j_k}^{i_k} \end{pmatrix}$$

In particular, we have that:

$$\delta_I^J = \begin{cases} \operatorname{sgn} \sigma & \text{neither } I \text{ nor } J \text{ has repeated index and } J = I_\sigma \text{ for some } \sigma \in S_k \\ 0 & \text{either } I \text{ or } J \text{ has repeated index or } J \text{ is not a permutation of } I \end{cases}$$

Proposition 10.1.1: Properties Of Elementary k-Tensors

Let (E_i) be a basis for V and (ϵ^i) a basis for V^*. Let ϵ^I be an elementary k-tensor. Then:

1. If I has a repeated index , then $\epsilon^I = 0$
2. If $J = I_\sigma$, then $\epsilon^I = (\operatorname{sgn}\sigma)\epsilon^J$
3. The result of evaluating ϵ^I on a sequence of basis vectors is:

$$\epsilon^I(E_{j_1}, ..., E_{j_k}) = \delta_I^J$$

Proof **:**
exercsise

This proposition suggests that to form a basis for alternating k-tensors, we should be looking at *increasing* multi-indexes. We will start using the following notation to say we are summing over all *increasing* multi-inexes:

$$\sum_I{}' \alpha_I \epsilon^I = \sum_{I: i_1 < \cdots < i_k} \alpha_I \epsilon^I$$

Proposition 10.1.2: Basis For $\Lambda^k(V^*)$

Let V be an n dimensional vector space, V^* the corresponding dual with a basis (ϵ^i). Then for each positive integer $k \leq n$, the collection of k-covectors:

$$\mathcal{E} = \left\{\epsilon^I \;\middle|\; I \text{ is an increasing multi-index of length } k\right\}$$

forms a basis for $\Lambda^k(V^*)$. Thus:

$$\dim(\Lambda^k(V^*)) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof **:**
p. 353-354 Jonn Lee

Next, we define tensor product that keeps alternating tensors alteranting, that is $v \otimes w \in \Lambda^k(V^*)$

Definition 10.1.2: Wedge Product

Let V be a finite dimensional vector space, $\omega \in \Lambda^K(V^*)$ and $\eta \in \Lambda^l(V^*)$. Then the *wedge product* or *exterior product*, as deifne as:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta)$$

The complicated coefficient is for the following purpose:

Proposition 10.1.3: Wedging Elementary Alternating Tensors

Let V be an n-dimensional vector space, (ϵ^i) a basis for V^*. Then for any multi-indexes I, J:

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$$

Proof **:**
p.355 John Lee

Proposition 10.1.4: Properties Of Wedge Product

Let $\omega, \omega', \eta, \eta'$, an ζ be multicovectors on a finite dimensional vector space V. Then: Go over bilinearity, anti-commutativity, associativity, and

1. (Bilinearity) for all $a, a' \in \mathbb{R}$:

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$$
$$\eta \wedge \eta(a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega)$$

2. (Associativity)

$$\omega \wedge (\eta \wedge \zeta) = (\omega \wedge \eta) \wedge \zeta$$

3. If (ϵ^i) is any basis for V^*, and $I = (i_1, i_2, ..., i_k)$, then:

$$\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} = \epsilon^I$$

4. For any covectors $\omega^1, \omega^2, ..., \omega^k$ and vectors $v_1, v_2, ..., v_k$,

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, ..., v_k) = \det(\omega^j(v_i))$$

Proof **:**
p. 356

Note that not all k-covectors η can is the wedge product of covectors:

$$\eta = \omega^1 \wedge \cdots \wedge \omega^k$$

If it is, then η is called *decomposable*. Any k-covector is the lienar combination of decoposable k-covectors.

Corollary 10.1.1: Uniqueness Of Wedge Product

Show that if $f : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V^*)$ is associative, bilinear, and anti-commutative, then $f \equiv \wedge$.

Proof :
exercise

Definition 10.1.3: Exterior Algebra

Define:

$$\Lambda(V^*) = \oplus_{k=0}^{n} \Lambda^k(V^*)$$

Then this, along with the wedge product, is an anti-commutative, graded, associative algebra, called the *extior algebra* or *Grassmanian Algebara*. Naturally,

$$\dim \Lambda(V^*) = 2^n$$

10.2 Interior Multiplication

I ommited for now John Lee p. 358

10.3 Differential Forms on Manifolds

Now that we have built-up the properties we desire for our "volume metric", we next show how to attach this to a manifold:

Definition 10.3.1: Alternating Tensor Bundle

Let M be a manifold and $T^k(TM)$ be the k-tensor bundle on M. Then the subbundle consisting of all alternating k-tensors is denoted:

$$\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$$

Definition 10.3.2: Differential Form

Let M be a manifold and $\Lambda^k T^*M$ the alternating tensor bundle on M. Then any section $\omega : M \to \Lambda^k T^*M$ is called a *differential k-form.*

Definition 10.3.3: Collection of k-Forms

Let M be a manifold. Then the collection of all k-forms on M is denoted:

$$\Omega^k(M)$$

which is a vector-space over $\mathbb{R}$. We will consider $\Omega^0(M) = C^\infty(M)$.

We can define the wedge product between a k-form and l-form pointwise:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$$

which produces a $(k+l)$-form. If f is a zero form, then we interpred $f \wedge \omega$ as $f\omega$. We define:

$$\Omega^*(M) = \bigoplus_{k \in \mathbb{N}} \Omega^k(M)$$

Then the wedge product on $\Omega^*(M)$ turns it into an associative, anticommutative graded algebra. Given a smooth chart, we can write ω as:

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I{}' \omega_I dx^I$$

In terms of (this lemma), we get:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_n} \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) = d_J^I$$

Example 10.2: k-form

1. A 0-form is just a continuous real valued function $f : M \to \mathbb{R}$, an a 1-form is a co-vector field as we've seen them before. These give you a notion of "size" at any point. Thus, given any vector in T_pM, a linear functional will tell you the "volume metric" at the moment for that vector.

2. If $M = \mathbb{R}^3$, here are 2 examples of smooth 2-forms:

$$\omega = (\sin xy) dy \wedge dz$$
$$\eta = dx \wedge dy + dx \wedge dz + dy \wedge dz$$

These two functions both give different "2d" volume metrics on a 3 dimensional subspace. For example:

$$\omega(\pi, 1, 1) = \sin(xy) dy \wedge dz(\pi, 1, 1) = -\pi$$

3. Every 3-form on $\mathbb{R}^3$ is a real-valued continuous functions times $dx \wedge dy \wedge dz$. This is the most "elementary" example. Notice how:

$$dx \wedge dy \wedge dz = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = dx \left(\frac{\partial}{\partial x} \right) dy \left(\frac{\partial}{\partial y} \right) dz \left(\frac{\partial}{\partial z} \right) = 1$$

telling us that a 1 by 1 by 1 cube should have volume 1 in this metric.

Like with co-vector fields, the pull-back will play a central role in being able to integrate on manifolds independent of coordinate chart:

Definition 10.3.4: Differential Form Pullback

Let $f : M \to N$ be a smooth map and ω a k-form on N. Then $F^*\omega$ is a differential form on M defined by:

$$(f^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(T_pf(v_1), ..., T_pf(v_k))$$

Proposition 10.3.1: Properties Of k-Form Pullback

Let $f : M \to N$ be a smooth map. Then:

1. $f^* : \Omega^k(N) \to \Omega^k(M)$ is a linear map
2. $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$
3. In any smooth chart:

$$f^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I{}'(\omega_I \circ f)\left(d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f)\right)$$

Proof **:**
left as exercise

Example 10.3: Pullback

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(u, v) = (u, v, u^2 - v^2)$ and let ω be the 2-form, $\omega = ydx \wedge dz + xdy \wedge dz$ on $\mathbb{R}^3$. Then the pullback $f^*\omega$ is:

$$\begin{aligned} f^*(ydx \wedge dz + xdy \wedge dz) &= vdu \wedge d(u^2 - v^2) = udv \wedge d(u^2 - v^2) \\ &= vu \wedge (2udu - 2vdv) = udv \wedge (2udu - 2vdv) \\ &= -2v^2du \wedge dv + 2u^2dv \wedge du \end{aligned}$$

   which simplies to:

$$f^*\omega = -2(u^2 + v^2)du \wedge dv$$

2. Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. We will show how change of coordinates is naturally captured in pullbacks: thinking of $x = r\cos(\theta)$ and $y = r\sin(\theta)$ as the change of coordinates from cartiain to polar coordinates, we get:

$$\begin{aligned} dx \wedge dy &= d(r\cos\theta) \wedge (r\sin\theta) \\ &= (\cos(\theta)dr - r\sin\theta(d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\ &= rdr \wedge d\theta \end{aligned}$$

   Notice that this is exactly what we would put in the change of variable substituion!

The following result generlizes the 2nd example in the above, in particular the change of variables requires that the dimensions of the domain and codomain match:

Proposition 10.3.2: Pullback Formula for Top-degree form

Let $f : M \to N$ be a smooth map between smooth manifolds that are each n-dimensional. If (x^i) and (y^i) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$ respectively, and $u : V \to \mathbb{R}$ is a real continuous valued function on V, then the following holds on $U \cap f^{-1}(V)$:

$$f^*(u\,dy^1 \wedge \cdots \wedge dy^n) = (u \circ f)(\det Df)(dx^1 \wedge \cdots \wedge dx^n)$$

Proof **:**
Since the fiber of each M in $\Lambda^n T^*M$ is spanned by $dx^1 \wedge \cdots \wedge dx^n$, it suffices to show that both sides of the equatoin in the proposition give the same result when evaluated on $(\partial/\partial x^1, ..., \partial/\partial x^n)$. By proposition 10.3.1:

$$f^*(u\,dy^1 \wedge \cdots \wedge dy^n) = (u \circ f)(df^1 \wedge \cdots \wedge df^n)$$

and using properties of the wedge product, we have that:

$$df^1 \wedge \cdots \wedge df^n \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \right) = \det \left(df^j \left(\frac{\partial}{\partial x^i} \right) \right) = \det \left(\frac{\partial f^j}{\partial x^i} \right)$$

Thus, the left hand side gives $(u \circ f) \det(Df)$ when applied to $(\partial/\partial x^i, ..., \partial/\partial x^n)$. On the right hand side, we get the same thing since:

$$dx^1 \wedge \cdots \wedge dx^n \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \right) = 1$$

completing the proof

Corollary 10.3.1: Change Of Variables

Let M be a smooth manifold and $(U, \mathbf{x})$, $(V, \mathbf{y})$ be overlapping patches. Then the following identity holds on $U \cap V$):

$$dy^1 \wedge \cdots \wedge dy^n = \det \left(\frac{\partial y^j}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n$$

Proof **:**
Let $f = \mathrm{id}$ in proposition 10.3.2 on $U \cap V$

(a word about interior multiplication)

10.4 Exterior Derivative

(John Lee p.362)

We've seen in section ref:HERE not all 1-forms arise as the differential of a function, $df = \omega$, and so we surmized properties necessary for this to happen, in particualr we found that for any coordinate chart:

$$\frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j} = 0$$

and called a form that satisfies this property a *closed form* (which we showed was coordinate independent in proposition ref:HERE). How would we then ask of such a property for k-forms? The key to the generlization comes down to the antisymmetric properties of indices i and j in a k-form, so "localizing" at any two indices, we may imagine them as ij-components of 2-form. Defining a 2-form $d\omega$ "locally" in each smooth chart, we would get:

$$d\omega = \sum_{i<j}\left(\frac{\partial w_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j$$

and with this, we would say that ω is closed if $d\omega = 0$. TBD

Definition 10.4.1: Exterior Derivative on $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be an open subset. Then if $\omega \in \Omega^k(U)$ is represented as:

$$\omega = \sum_J{}' \omega_J dx^J$$

Then define the *exterior derivative* $d : \Omega^k(U) \to \Omega^{k+1}(U)$ to be:

$$d\omega = d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J$$

Essentially, we take the differential of the function in front, and then wedge the result. Breaking down the definition, we get:

$$d\left(\sum_J{}' \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^1 \wedge dx^{j_1} \cdots dx^{j_k}$$

If ω is a 1-form, then:

$$\begin{aligned}
d(\omega_j dx^j) &= \sum_{i,j} \frac{\partial \omega^j}{\partial x^i} dx^i \wedge dx^j \\
&= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\
&= \sum_{i<j}\left(\frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j}\right) dx^i \wedge dx^j
\end{aligned}$$

and if ω is a 0-form, so that $\omega = f$ for some smooth real valued function, then:

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

showing that the exterior derivative does indeed generalize our notion of the derivative! We would like to transfer this definition to manifolds, so we need it to satisfy certain properties:

Proposition 10.4.1: Properties Of Exterior Derivative On $\mathbb{R}^n$

Let $M = \mathbb{R}^n$, $\Omega^k(\mathbb{R})$ be the differential k-forms over $\mathbb{R}^n$, and d the exterior derivative. Then:

1. d is linear over $\mathbb{R}$
2. If on $U \subseteq \mathbb{R}^n$, ω is a smooth k-form and η is a smooth ℓ-form, then:
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
3. $d \circ d \equiv 0$
4. d commutes with pullbacks: if $U \subseteq R^n$, $V \subseteq \mathbb{R}^m$, $f : U \to V$ a smooth map, and $\omega \in \Omega^k(V)$, then:
$$f^*(d\omega) = d(f^*\omega)$$

***Proof* :**

1. by definition of the wedge product, d is $\mathbb{R}$-linear
2. It suffices to show this on "simple tensors", or "simple k-forms", so let $\omega = udx^I \in \Omega^k(M)$ and $\eta = vdx^J \in \Omega^l(M)$. The first thing we'll show is that $d(udx^I) = du \wedge dx^I$ even for an *arbitrary* multi-index set I. If I has repeated index, then it is clear that $d(udx^I) = 0 = du \wedge dx^I$. If not, let σ be a permutation sending I to an increasing multi-index set J. Then:
$$d(udx^I) = (\operatorname{sgn} \sigma) d(udx^J) = (\operatorname{sgn} \sigma) du \wedge dx^J = du \wedge dx^I$$
Using this, we may compute:
$$\begin{aligned} d(\omega \wedge \eta) &= d((udx^I) \wedge (vdx^J)) \\ &= d(uvdx^I \wedge dx^J) \\ &= (vdu \wedge udv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (vx^J) + (-1)^k (udx^I) \wedge (dv \wedge dx^J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \end{aligned}$$
3. Do it for the 0-form case first, then go through the computation's, it will work out
4. It is sufficeint to show it for simple k-forms, $\omega = udx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Computing the left hand side:
$$\begin{aligned} f^*(d(udx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= f^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= d(u \circ f) \wedge d(x^1 \circ f) \wedge \cdots \wedge d(x^{i_i} \wedge f) \end{aligned}$$
and right hand side:
$$\begin{aligned} d(f^*(udx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= d((u \circ f) d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_k} \circ f)) \\ &= d(u \circ f) \wedge d(x^1 \circ f) \wedge \cdots \wedge d(x^{i_i} \wedge f) \end{aligned}$$
completing the proof

We will now show that we can show there exists a unique map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ where we now replaced $\mathbb{R}^n$ with M. The part (4) of this question can be in the special case of f being a diffeomorphism: Then this says that the exterior derivative is "invariant" under pullback!

Proposition 10.4.2: Exterior Derivative On Manifold

Let M be a smooth manifold (with or without boundary). Then there exists a unique operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k such that

1. d is linear over $\mathbb{R}$
2. If $\omega \in \Omega^k(M)$ an $\eta \in \Omega^l(M)$, then:
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
3. $d \circ d \equiv 0$
4. For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

such a map for any k is called the *exterior derivative*.

Proof **:**
p.365

(word on anti-derivation)

Proposition 10.4.3: Commuting With Pullback

Let $f : M \to N$ be a smooth map. Then for each k, the pullback map $f^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d, that is for all $\omega \in \Omega^k(N)$

$$f^*(d\omega) = df^*(\omega)$$

Proof **:**
p.366

10.4.1 Exterior Derivative and Vector Calculus in $\mathbb{R}^3$

10.5 Invariant formula for the External Derivative

(Requires Lie Brakets)

10.6 Orienting Manifolds Using Forms

chap 15 John Lee

11

Integration on Manifolds

We have now constructed our "volume metric" that will give information at each point on a manifold. We can now show how this leads to integrating manifolds!

11.1 Integration of Differential Forms

Just like for line integarls, we will first treat the case of $\mathbb{R}^n$. We will say that $D \subseteq \mathbb{R}^n$ is a *domain of integration* if it's a bounded set with a measure zero boundary.

Definition 11.1.1: Integration Of Differential Form

Let $D \subseteq \mathbb{R}^n$ be a domain of integration an ω a (continuous) n-form on $\overline{D}$, which in terms of coordinates can be written as $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for some (continuous) function $f : \overline{D} \to \mathbb{R}$. Then we define the *integral of* ω *over* D to be

$$\int_D \omega = \int_D f dV$$

which in terms of coordinates can be written as:

$$\int_D f dx^1 \wedge \cdots \wedge dx^n = \int_D f dx^1 \cdots dx^n$$

Essentially, we can think of integrating an n-form in $\mathbb{R}^n$ by erasing the wedges! A slight generalization of the above is if $U \subseteq \mathbb{R}^n$ (or $\mathbb{H}^n$) is an open subset and ω is compactly supported n-form on U. Then we define:

$$\int_U \omega = \int_D \omega$$

where $D \subseteq U$ is any domain of integration containing $\operatorname{supp}\omega$ (ex a rectangle), and ω is extended to 0 on the compliment of it's support.

Naturally, we must check that this definition is invariant of the domain of integration:

Proposition 11.1.1: Domain Independent

Let D and E be open domains of integration in $\mathbb{R}^n$ (or $\mathbb{H}^n$). Let $G : \overline{D} \to \overline{E}$ be a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism. Then if ω is an n-form on $\overline{E}$, then:

$$\int_D G^*\omega = \begin{cases} \int_E \omega & G \text{ is orientation preserving} \\ -\int_E \omega & G \text{ is orientation reversing} \end{cases}$$

***Proof* :**
Let $(y^1, y^2, ..., y^n)$ be the standard coordinates on E and $(x^1, x^2, ..., x^n)$ the standard coordinates on D. Let's first let G be orientation-preserving. Putting ω into coordinate form $\omega f dx^1 \wedge \cdots \wedge dx^n$, we can use the change of variables formula along with ref:HERE for pullback of n-forms to get:

$$\begin{aligned} \int_E \omega &= \int_E f dV \\ &= \int_D (f \circ G)|\det DG| dV \end{aligned}$$

$$\int_D (f \circ G)(\det DG) dV$$

$$\int_D (f \circ G)(\det DG) dx^1 \wedge \cdots \wedge dx^n$$

$$\int_D G^*\omega$$

If G is orientation-reversing, we do the exact same computation, except a negative sign will be introduced.

We next want to extend this theorem to compactly supported n-forms defiend on open sets. Since arbitrary open subsets or arbitrary compact subsets are not guaranteed to be domains of integration, we need the following lemma:

Lemma 11.1.1: Finding Domain Of Integration

Let $U \subseteq \mathbb{R}^n$ (or $\mathbb{H}^n$) be an open subset and let $K \subseteq U$ be a compact subset. Then there exists an open domain of integration D such that

$$K \subseteq D \subseteq U \qquad \overline{D} \subseteq U$$

***Proof* :**
p. 403

Proposition 11.1.2: Domain Independence Extended

Let $U, V \subseteq \mathbb{R}^n$ (or $\mathbb{H}^n$) be open subsets, $G : U \to V$ an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then:

$$\int_V \omega = \pm \int_U G^* \omega$$

where it is positive if G is orientation-perserving, and negative otherwise.

Proof **:**
p. 404

11.1.1 Integration on Manifolds

Now that we know how to integarte a form on $\mathbb{R}^n$, we look at integrating a form in one chart:

Definition 11.1.2: Integration On Manifolds on One Chart

Let M be an oriented smooth n-manifold, ω a smooth n-form on M. Suppose that ω is compactly supported in the domain of one smooth chart (U, φ) that is either positively or negatively oriented. Then the *integral of* ω *over* M is:

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

where the sign depends on the orietnation of φ

Proposition 11.1.3: Chart Independent

Let ω be a n-form that is defined as in definition 11.1.2. Then $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains $\operatorname{supp} \omega$

Proof **:**
p. 405

Using this, we can now extend the definition of integration using partitions of unity:

Definition 11.1.3: Integartion On Manifolds

Let M be an oriented smooth n-manifold, an ω a compactly suppored smooth n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts. Let $\{\psi_i\}$ be a subordinate smooth partition of unity. Then the *integral of* ω *over* M is:

$$\int_M \omega = \sum_i \int_M \psi_i(\omega)$$

The reason we allow for negatively oriented charts is that it may not be possible to find positively oriented boundary charts on a 1-manifold with boundary, as noted in the proof of proposition ref:HERE. Since for each i, the n-form $\psi_i(\omega)$ is compactly supported in U_i, each of these terms well-defined by our definition for one chart. The final key step to do is show the independence of choice of open cover and parition of unity:

Proposition 11.1.4: Integration On Manifolds Well-Defined

The definition $\int_M \omega$ given does not depend on the choice of open cover or partition of unity

Proof **:**
p. 405

Thus, we now have a way to (at least theoretically) have a notion of integration on a manifold! For the special case of a of a compactly supported 0-form (i.e, a real-valued function) f over an oriented 0-manifold M, we would define:

$$\int_M f := \sum_{p \in M} \pm f(p)$$

Since f is compactly supported, there are only finitely many nonzero terms, and hence the sum is well-defined without any requirement for convergene conditions.

If $S \subseteq M$ is an oriented immerse k-submanifold (with or without boundary), and ω is a k-form on M, whose restriction to S is compactly supported, then we can interpret $\int_S \omega$ as $\int_S \iota_S^* \omega$ where $\iota_S^* : S \hookrightarrow M$ is the natural inclusion map. If M is also compact n manifold with boundary and ω is an $(n-1)$-form on M, then we can interpret $\int_{\partial M} \omega$ unambiguously as $\int_{\partial M} \iota_{\partial M}^* \omega$, with ∂M having the induced (Stokes) orientation.

It should be noted that we can extend this definition to non-compactly suppored forms, and this can be useful in many cases, hower we would need to pay attention to convergence issues (i.e. define an analogous improper integral), and so such a notion will be left for later

Proposition 11.1.5: Properties Of Integrating Forms

Suppose that M and N are non-empty oriented smooth n-manifolds (with or wihtout boundary), and ω, η are compactly supported n-forms on M.

1. If $a, b \in \mathbb{R}$, then:
$$\int_M a\omega = b\eta = a\int_M \omega + b\int_M \eta$$
2. If $-M$ denotes the opposite-oriented manifold, then:
$$\int_{-M} \omega = -\int_M \omega$$
3. If ω is a positively-oriented form, then
$$\int_M \omega > 0$$
4. (Diffeomorphism Invariance) If $f : M \to N$ is an orietnation-preserving or orietnation-reversing diffeomorphism, then:
$$\int_M \omega = \begin{cases} \int_N f^*\omega & f \text{ is orientation-preserving} \\ -\int_N f^*\omega & f \text{ is orientation-reversing} \end{cases}$$

Proof **:**
p. 408 John Lee

Though this is nice theoretically, this would be a huge hassle to actually integrate (imagine trying to find a partition of unity which can be nicely integarble). It is much better to chop-op the manifold and integarte in each peace. The following gives a way of doing this through parameterization using a pullback:

Proposition 11.1.6: Integration Over Parameterization

Let M be an oriented smooth n-manifold (with or without boudary), and let ω be a compactly supported n-form on M. Suppose that $D_1, D_2, ..., D_k$ are open domains of integration in $\mathbb{R}^n$, and for $i = 1, ..., k$, we are given smooth maps $f_i : \overline{D_i} \to M$ satisfying:

1. f_i restricts to an orientation-preserving diffeomrophism from D_i onto an open subsets $W_i \subseteq M$
2. $W_i \cap W_j = \emptyset$ when $i \neq j$
3. $\operatorname{supp} \omega \subseteq \overline{W_1} \cup \cdots \cup \overline{W_k}$

Then:

$$\int_M \omega = \sum_{i=1}^{k} \int_{D_i} f_i^* \omega$$

Proof **:**
As before, it is enough to assume that ω is compactly supported in the domain of one oriented smooth chart (U, φ). Restricting to sufficeintly nice maps, we may assume that U is pre-compact (closure is compact), $Y = \varphi(U)$ is a domain of integration in $\mathbb{R}^n$ (or $\mathbb{H}^n$), and φ extends to a diffeomorphism from $\overline{U}$ to $\overline{Y}$

(the rest is on p.409)

Example 11.1: Integrating On Manifold
We will use this result to compute the integral of a 2-form over S^2, oriented as the boundary of $\overline{D}^2$. Let ω be the two form:

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

Let $D = (0, \pi) \times (0, 2\pi)$ be an open rectangle, and let $f : \overline{D} \to S^2$ be the spherical coordinate parameterization $f(\psi, \theta) = (\sin\psi \cos\theta, \sin\psi \sin\theta, \cos\psi)$. In example ref:HERE, we showed that f is orientation preserving, and thus it satisfies the hypotheses of proposition 11.1.6. Next, note that:

$$\begin{aligned}
f^* dx &= \cos\psi \cos\theta d\psi - \sin\psi \sin\theta d\theta \\
f^* dy &= \cos\psi \sin\theta d\psi - \sin\psi \cos\theta d\theta \\
f^* dz &= -\sin\psi d\psi
\end{aligned}$$

Thus:

$$\begin{aligned}\int_{S^2}\omega &= \int_D (-\sin^3\varphi\cos^2\theta d\theta\wedge d\varphi + \sin^3\varphi\sin^2\theta d\varphi\wedge d\theta)\\ &\quad + \cos^2\varphi\sin\varphi\cos^2\theta d\varphi\wedge d\theta - \cos^2\varphi\sin\varphi\sin^2\theta d\theta\wedge d\varphi)\\ &\stackrel{!}{=} \int_D \sin\varphi d\varphi\wedge d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}\sin\varphi d\varphi d\theta\\ &= 4\pi\end{aligned}$$

where the $\stackrel{!}{=}$ comes from repeated use of the $\sin^2+\cos^2=1$ identity.

(comment on how the hypothesis can be slightly relaxed and still work well, p. 410)

11.2 Stoke's Theorem

One of the big results we learn in calculus is that that:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

considering f as a form, we can convert this into:

$$\int_{[a,b]} df = \int_{\partial[a,b]} f$$

This is one of the key powerful results of calculus: since df holds all the variation information within a space, then f can be re-interpreted as holding this "cumulative" information! Perhaps the most important first result in differential geometry is that this same idea can be massively generalized for arbitrary n-froms!

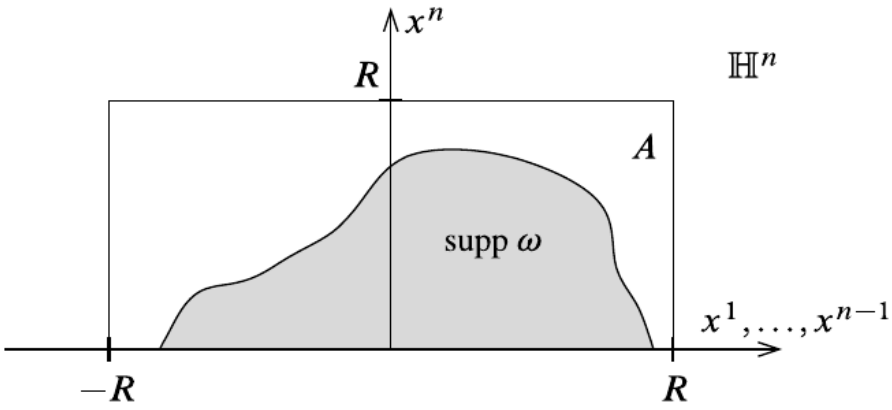
Theorem 11.2.1: Stokes Theorem

Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then:

$$\int_M d\omega = \int_{\partial M}\omega$$

where ∂M has the induced Stokes orientation

Proof **:**
We start with a special case: suppose that $M = \mathbb{H}^n$. Since ω has compact support, there exists an $R > 0$ st $\operatorname{supp}\omega$ is contained in $A = [-R, R]^{n-1}\times[0, R]$:



Since $M = \mathbb{H}^n$, we only need one chart. Hence, write ω in standard coordinates:

$$\omega = \sum_{i=1}^{n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where $\widehat{dx^i}$ is the ommited component. Thus, computing $d\omega$:

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} \wedge dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \end{aligned}$$

Then, computing the integral:

$$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^{n} (-1)^{i-1} \int_A \frac{\partial w_i}{\partial x^i} dx^1 \wedge \cdots \wedge \quad = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^{R} \int \cdots \int_{-R}^{R} \frac{\partial w_i}{\partial x^i}(x) dx^1 \cdots dx^n$$

We can now by Fubini-Tonelli's theorem, we can switch the order of the integrals, and so we will move the dx^i to the front

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \int \cdots \int_{-R}^R \frac{\partial w_i}{\partial x^i}(x) dx^1 \cdots dx^n \\
&= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \int \cdots \int_{-R}^R \frac{\partial w_i}{\partial x^i}(x) dx^i \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \int \cdots \int_{-R}^R [w_i(x)]_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= 0
\end{aligned}
$$

where the last line comes from the fact that we have chosen R so that $\omega = 0$ when $x^i = \pm R$. The only term which might not be zero is the one for which $i = n$. For that term, we have:

$$
\begin{aligned}
&= (-1)^{i-1} \int_{-R}^R \int \cdots \int_{-R}^R \int_0^R \frac{\partial w_i}{\partial x^i}(x) dx^n \cdots \cdots dx^{n-1} \\
&= (-1)^{i-1} \int_{-R}^R \int \cdots \int_{-R}^R [w_i(x)]_{x^n=0}^{x^i=R} dx^1 \cdots \cdots dx^{n-1} \\
&= (-1)^{i-1} \int_{-R}^R \int \cdots \int_{-R}^R [w_i(x)]_{x^n=0}^{x^i=R} dx^1 \cdots \cdots dx^{n-1} \\
&= (-1)^{i-1} \int_{-R}^R \int \cdots \int_{-R}^R \omega_n(x^1, ..., x^{n-1}, 0) dx^1 \cdots \cdots dx^{n-1}
\end{aligned}
$$

since $\omega_n = 0$ when $x^n = R$.

Let's now compare this to the other side:

$$\int_{\partial \mathbb{H}^n} \omega = \sum_i \int_{A \cap \partial \mathbb{H}^n} \omega_i(x^1, ..., x^{n-1}, 0) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

since x^n vanishes on $\partial \mathbb{H}^n$, the pullback of dx^n to the boundary is identically 0 (John Lee give sthis as an exercsie 11.30). Thus, the only term that is non-zero is the one for which $i = n$, which becomes:

$$\int_{\partial \mathbb{H}^n} \omega = \int_{A \cap \partial \mathbb{H}^n} \omega_n(x^1, ..., x^{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}$$

Since the coordinates $(x^1, x^2, ..., x^{n-1})$ are positively oriented for $\partial \mathbb{H}^n$ when n is even and negatively when n is odd, see that the left hand side is equal to the right hand side.

The next special case is for $M = \mathbb{R}^n$. In this case, we have that $\operatorname{supp} \omega \subseteq [-R, R]^n$. Doing the same computation, we see now that that the terms of $i = n$ both vanish, and so the left hand side is zero. Since there is no boundary, the right hand side is zero as well.

Now, let M be an arbitrary smooth manifold with boundary, but ω is an $(n-1)$-form that is compactly supported in the domain of a single positively or negatively oriented smooth chart (U, φ). Assuming that φ has a positively oriented boundary chart, the definition gives us:

$$\int_M d\omega = \int_{(\mathbb{H}^n)} (\varphi^{-1})^* d\omega = \int_{\mathbb{H}^n} d\left((\varphi^{-1})^* \omega\right)$$

But then, by the computation we've just done, this gives:

$$\int_{\partial\mathbb{H}^n} (\varphi^{-1})^*\omega$$

with $\partial\mathbb{H}^n$ having the induced orientation. Since $d\varphi$ take outwards pointing vectors on ∂M to outwards pointing vectors on $\mathbb{H}^n$ (by proposition 5.41 on p. 118 of John Lee), it follows that $\varphi|_{U\cap\partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U)\cap\partial\mathbb{H}^n$, and thus the above equation is equal to:

$$\int_{\partial M} \omega$$

For negatively oriented smooth boundary charts, the same argument applies where we would have negative signs show up on both sides. For interior charts, we will replace $\mathbb{H}^n$ with $\mathbb{R}^n$ and get the same result as before.

Finally, let ω be an arbitrary compactly supported smooth $(n-1)$-form. Choose some cover of $\operatorname{supp}\omega$ by finitely many domains of positively or negative oriented smooth charts $\{U_i\}_{i=1}^n$, and choose a subordinate smooth partition of unity $\{\psi_i\}$. Then we can apply the preceeding argument to $\psi_i\omega$ for each i to obtain:

$$\begin{aligned}
\int_{\partial M} \omega &= \sum_i \int_{\partial M} \psi_i\omega \\
&= \sum_i \int_M d(\psi_i\omega) \\
&= \sum_i \int_M d\psi \wedge \omega + \psi_i d\omega \\
&= \int_M d\left(\sum_i \psi\right) \wedge \omega + (\sum_i \psi_i) d\omega \\
&= 0 + \int_M d\omega \\
&= \int_M d\omega
\end{aligned}$$

since $\sum_i \psi_i \equiv 1$

Example 11.2: Special Case: Line Integrals

Let M be a smooth manifold and let $\gamma : [a,b] \to M$ be a smooth embedding so that $S = \gamma([a,b])$ is an embedding 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(M)$, Stoke's Theorem tells us that:

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a))$$

Thus, we get back the Fundamental theorem for line integarls! When we take $\gamma : [a,b] \to \mathbb{R}$, then Stok'es theorem is just our ordinary fundamental theorme of calculus.

Corollary 11.2.1: Integrals Of Exact Forms

Let M be a compact oriented smooth manifold without boundary. Then the integral of every exact form over M is zero:

$$\int_M d\omega = 0 \quad \text{if } \partial M = 0$$

Proof **:**
Let M be a compact manifold where $\partial M = \emptyset$. Since η be an exact form so that $\eta = d\omega$ for some ω. Then:

$$\int_M \eta = \int_M d\omega = \int_{\partial M} \omega = \int_{\emptyset} \omega = 0$$

Corollary 11.2.2: Integarls Over Closed Form With Boundary

Let M be a smooth compact oriented manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:

$$\int_{\partial M} \omega = 0 \qquad \text{if } d\omega = 0 \text{ on } M$$

Proof **:**
Let ω be a closed form so that $d\omega = 0$. Then:

$$\int_{\partial M} \omega = \int_M d\omega = \int_M 0 = 0$$

Notice that if ω is a closed form that is also exact (so $\omega = d\eta$), we would get:

$$\int_{\partial M} \omega = \int_{\partial M} d\eta = \int_M d(d\eta) = \int_M 0 = 0$$

Thus, if $\int_S \omega \neq 0$ for some manifold S, we can conclude these two powerful results:

Corollary 11.2.3: Characterizting Exact Forms

Let M be a smooth manifold (with or without boundary), $S \subseteq M$ is a compact smooth k-submanfiold *without boundary*, and ω any closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:

1. ω is not exact on M
2. S is not the boundary of an oriented compact smoth submanifold with boundary in M

Proof **:**
If ω was exact, then $\int_S \omega = 0$. Furthermore, if $S = \partial M$ for some manifold, then:

$$\int_S \omega = \int_{\partial M} \omega = \int_M d\omega = \int_M 0 = 0$$

Example 11.3: Non Exact Form
p. 415 with $\mathbb{R}^2 \setminus \{(0,0)\}$

We naturaly get back classical results from vector-calculus:

Theorem 11.2.2: Green's Theorem

Suppose that D is a compact regular domain on $\mathbb{R}^2$, and P, Q are smooth real valued functions on D. Then:

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} Pdx + Qdy$$

Proof **:**
Apply Stokes theorem to any 1-form $Pdx + Qdy$.

11.3 Divergence Theorem

The divergence theorem applies to an n-dimensional manifold with boundary in $\mathbb{R}^n$. You can imagine that on the boundary, there is an outward facing normal unit vector. Recall that if F is a vector-field in $\mathbb{R}^n$, then:

$$\text{div}(F) = \frac{\partial f}{\partial x^1} + \cdots + \frac{\partial f}{\partial x^n}$$

Theorem 11.3.1: Divergence Theorem

Let F be a smooth (or C^1) vector-field and $M \subseteq \mathbb{R}^n$. Then:

$$\int_M \text{div}\, F dV_{\mathbb{R}^n} = \int_{\partial M} \langle F, n \rangle dV_{\partial M}$$

where $dV_{\mathbb{R}^n}$ is a volume element of $\mathbb{R}^n$, i.e. $dx_1 \wedge \cdots \wedge dx_n$.

Proof **:**
not covered??

The other volume form (i.e. $n-1$-form on ∂M) presented is a bit ambiguous, i.e. $dV_{\partial M}$. We define it so that:

$$dV(x)(v_1, ..., v_{n-1}) = 1$$

if $(v_1, v_2, ..., v_{n-1})$ is in ∂M and $[v_1, v_2, ..., v_{n-1}] = \partial\mu_x$. Thus, this differential form is always just the volume of an $n-1$ parrallelepiped spanned by $v_1, v_2, ..., v_{n-1}$. Then, since we have coordinates, we can compute it directly:

$$dV(x)(v_1, v_2, ..., v_{n-1}) = \det \begin{pmatrix} n(x) \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = (-1)^{n-1} \det \begin{pmatrix} v_1 \\ \cdots \\ v_{n-1} \\ n(x) \end{pmatrix} = (-1)^n \langle (v_1, ..., v_{n-1}), n(x) \langle$$

(Now I'm not sure what's going on..) By Riesz's Representation Theorem, if $z = (v_1, ..., v_{n-1})$, then:

$$\det \begin{pmatrix} v_1 \\ \cdots \\ v_{n-1} \\ n(x) \end{pmatrix} = \langle w, z \rangle$$

actually, he just goes on the prove the theorem, which atm I have no interest in doing so.