Everything You Need To Know About Algebraic Topology

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April 6, 2023

Contents

1 Combinatorial Topology and Homotopy **5**

1.1 Homotopy and Homotopy Type 5

1.2 Cell-Complex 11

1.2.1 Operations on Cell Complex 14

1.3 Homotopy Equivalence and Homotopy Extension 16

2 Fundamental Group **24**

2.1 Path Homotopy and Fundamental Group 24

2.1.1 Functorial Properties 32

2.1.2 Covering Spaces and Lifting Property 34

2.1.3 Fundamental Group Invariant Under Homotopy 43

2.1.4 Interesting Consequences 46

2.2 Products, Fibred Products, and Van Kampen's Theorem 49

2.3 Classification of Covering Spaces 58

2.3.1 Deck Transformations 61

2.4 *Bonus Material 62

3 Homology **63**

3.1 Delta-Complex 64

3.2 Singular Homology 71

3.3 Homotopy Invariance 74

3.4 Relative Homology: Quotient Complexes 77

3.4.1 Naturality 86
3.4.2 Mayer-Vietoris Sequences 86
3.4.3 Equivalence of Simplicial and Singular Homology 89
3.4.4 Retractions 90
3.5 Cellular Homology and Euler Characteristic 90
3.5.1 Degree 90
3.5.2 Cellular Homology 94
3.5.3 Euler-Characteristic 101
3.5.4 Lefschetz Fixed Point Theorem 102
3.5.5 Homology of Groups 103
3.6 Axioms of Homology 104

4 Cohomology **106**
4.1 Cohomology of Spaces 106
4.2 Cup Product 114
4.2.1 Küneth's Formula 119
4.3 Poincaré Duality 119

5 Homotopy Theory **121**

6 Spectral Sequence **122**

A Common Structures and Related Algebraic Structure **123**
A.1 Delta and Cell Complexes 123
A.2 Fundamental Groups and Covering Spaces 123
A.3 Homologies and Cohomologies 124
A.3.1 Cohomology Rings 125

Algebraic topology is about inducing algebraic structure out of topological structures (normally geometric in nature) in a functorial manner, often in terms exact functors. These days, there is a multitude of ways of accomplishing this, each algebraic notion measuring or capturing some property of the underlying shape. Some examples of these theories are:

1. K-theory
2. Homology/cohomology
3. Homotopy
4. cobordism

In this book, we shall cover *homotopy*, *homology*, and *cohomology*. Each of these theories introduce a algebraic structure $A(X)$ given a topological space X which works well functorially ($f : X \to Y \implies \varphi(f) : \varphi(X) \to \varphi(Y)$). Each of these will have their benefits that shall be explored, and by the functorial nature of the theory, we shall see that many rather difficult proofs shall become relatively simple-minded exercises in algebra. Some famous motivation examples of what shall be proven is:

1. $\mathbb{R}^n \ncong \mathbb{R}^m$ iff $n \neq m$
2. Any continuous map $f : D^2 \to D^2$ (from the closed unit disk to itself) *must* have a fixed point
3. Every complex polynomial has a complex root
4. Every vector field on the sphere must have two points on each side with the same value

Overall, algebraic topology allows many results that are often difficult to prove in a purely geoemtric or topological setting to be transalted into an algebraic problem where we have many more tools at our disposal

A core part of many startegies in translating topological spaces into geometric spaces the use of *combinatorial topology* which aims to find a way construction more complicated topological spaces from simpler building blocks. For eample, we may try to construct a space which (up to some continuous deformation) is gluing simplexes (higher dimensional triangles) together along borders (the torus, sphere, klein bottle, projective space, and circle are all examples of such constructions). Many such combinatorial theories shall be introduced which shall also naturally add a geometric flavour to our study since the simple objects that will be worked with are usually circles or triangles (and their higher dimensional analogues) glued together with certain rules. It will often be easier to translate geometrically what the induced algebraic object given to a topological space “means” by these constructions, however it will be important to show that the choice of any particular combinatorial construcrion or structure on a space is independent of the algebraic structure given to the space (or else we wouldn’t have an algebraic invariant on a space, but an algebraic invariant given a certain construction).

Antoher very natural question to ask is to what degree do we want the associated algebraic sturcture $A(X)$ to a space X be invariant? Naturally, we will want $A(X)$ to be invariant under homeomorphisms, so that if $f : X \to Y$ is a homeomorphism $A(f) : A(X) \to A(Y)$ is an isomorphism. However, it turns out that homeomorphisms are a rather strong ask to find between topological spaces, and all of the algebraic structures will be invariant under the (much) weaker notion of *homotopy equivalence*. While homeomorphisms can be thought as the right notion of isomorphism for topological spaces (namely they preserve the "lattice" of open sets bijectively), homotopy equivalence preserve the *shape* of a topological space, where by shape we usually means the properties of a space that is preserved *up to continuous deformation* (the usual idea of a topological space being invariant to"stretching and squishing" applies better to homotopies). Though homotopy equivalence is weaker, we shall see that due to their greater generality we will be able to show many spaces are homotopic equivalent, often rather complicated spaces are homotopic equivalent to very simple spaces, which will allow us to calcualte the algebraic object of the simple spaces and deduce the same algebraic structure is associated to much more complicated spaces.

1

Combinatorial Topology and Homotopy

Let X be a topological space. Combinatorial topology aims to create a new structure on X that will be thought as a way of interpreting X as build up by simpler spaces (most commonly circles and triangles, along with their higher dimensional counter-parts). How they are glued and how they are chosen to map into our space X will determine the "type" of combinatorial object we are working with. In this chapter, we shall explore the common *CW complex* that will have many important properties in the rest of this book.

We shall also explore a generalization of the notion of a continuous map known as a *homotopy*. As we shall see, a homotopy will be able to distort a shape much more drastically than a continuous map, but as added benifit will be more flexible and give many new maps between spaces that previously where unconnected. These spaces will share many properties that are invariant under the stronger deformation given by homotopies. In the following chapters, we shall see that all the algebraic structure we shall explore shall be invariant under homotopies, showing how they the right notion of morphism in algebraic topology.

1.1 Homotopy and Homotopy Type

For notational simplicity, we'll take the convention that

$$I = [0, 1]$$

Definition 1.1.1: Homotopy

Let X, Y be topological spaces, and suppose we have two maps f, f' which are continuous maps from X to Y. We say that f is ***homotopic*** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x), \qquad F(x, 1) = f'(x)$$

F is called a homotopy between f and f', and we write this as $f \simeq f'$

Thinking of the graphs of functions as shapes, if we can continuously deform one shape into another, the two shapes are homotopic! Note that the topology from which we define continuity ($X \times I$) has *both* the topology of X and the Euclidean topology to define it. The "continuous deformation" is more reliant of the euclidean portion, meaning we can use our usual intuition for the most part of deformations. In terms of I being a closed interval, we can think of this as having "one minute of time to change it, and we must do it in that time" (due to the compactness of the interval). To formalize his intuition, homotopy can be interpreted another way: using the compact-open topology on $[0, 1]$ and the fact that $[0, 1]$ is locally compact, it can be shown that a homotopy can equivalently be defined as a continuou map

$$f : X \to Y^{[0,1]}$$

which sends each point x to a path in Y. Hence, a homotopy between two maps f, g can be thought of as saying saying there is a path from every point $f(x) \in Y$ to a point $g(x) \in Y$ that continuously. Notice the importance of the domain in this definition: Even if $f(X) = \{y\}$, a homotopy will assign a path for each $x \in X$, hence we shall find that two images of different dimensions are homotopic (as we'll demonstrate).

Example 1.1: Homotopies

1. Let $f, g : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, x^3)$ and $g(x) = (x, e^x)$. Then the map $F : R \times I \to \mathbb{R}^2$,

$$F(x, t) = (x, (1-t)x^3 + te^x)$$

Is a homotopy. $F(x, 0) = (x, x^3)$, and $F(x, 1) = (x, e^x)$, the way is easily checked to be continuous. Thus

$$f \simeq g$$

2. Take $f : I \to I$ $x \mapsto x^2$. Then

$$F(x, t)((1-t)x^2 + tx)$$

is a homotopy. More generally, x^2 can be any polynomial, and this homotopy will work.

3. Take $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 0$. Then:

$$F : \{\mathbb{R}\} \times I \to \mathbb{R} \qquad F(x, t) = (1-t)f(x)$$

is a continuous function, with $F(x, 0) = f(x)$ and $F(x, 1) = 0$. Thus, every continuous function on $\mathbb{R}$ is homotopic to the constant function.

If we think of the graphs of the functions, the point $\{0\}$ with the trivial topology any shape of $\mathbb{R}$ with the euclidean topology can be continuously deformed to on another.

More generally, if F is a homotopy such that it's construction is:

$$F(x, t) = ((1-t)f(x) + tg(x))$$

then it is a Straight-line homotopy

4. Let $f, g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ where f is the identity map and $g(x) = \frac{x}{\|x\|}$. Then there is a homotopy between f and g, can you think of it? We shall later show that f is not homotopic to a constant map. However, if the domain and codomain was $\mathbb{R}^2$, then f *would* be homotopic to a consant map (the staright-line homotopy works). If the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^2 \setminus \{0\}$, if f is surjective, can it be homotopic to a constant map?

This last example hints at the importants of holes in our space. Can you repeat the same think in $\mathbb{R}^n$ by making gradually higher dimensional holes (remove a point, a line, a plane, etc)?

It should be verified that the collection of homotopies between two spaces forms an equivalence relation (exercise ref:HERE). Hence, the composition of homotopies is well-defined, motivating the following definition:

Definition 1.1.2: hTop

Let **hTop** denote the category whose objects are topological spaces and whose morphisms are homotopic equivalence classes of continuous functions.

When function's are homotopic to the constant function, we give function a special name:

Definition 1.1.3: Nullhomotopic

If f is homotopic to a constant map, we say that f is ***nullhomotopic***. An image of a function that is nullhomotopic is called *contractible*.

The idea of nullhomotopy is saying that if you deform your space in a continuous way such that you can bring it to a constant point, then we can think of the two being, in some sense, homotopically trivial [1].

Example 1.2: Nullhomotopic

Take any conected shape $U \subseteq \mathbb{R}^n$ that does not have a whole (say, a shape that is is homeomorphic to a unit ball). Then the identity map $f : U \to \mathbb{R}^n$ is homotopic to the constant map using the straight-line homotopy. Thus, all such shapes are contractible.

We now strive towards the notion of inverse homotopies and isomorphisms in **hTop**. The following can be thought of as a first step in that direction as well as an important notion in it of itself:

[1] not necessarily trivial in the most general sense, because compactness, normality, connectedness, and all the other properties might still be present in this "trivial" case, but it's trivial when it comes to trying to form to bending it around

Definition 1.1.4: Deformation Retraction

Let $f : X \to X$ be the identity map and $A \subseteq X$ a subspace. Then if there exists a homotopy $f_t : X \times I \to X$ such that:

1. $f_0 = \text{id}$
2. $f_1(X) = A$
3. $f_t|_A = \text{id}$ for all $t \in I$

Then F is said to be a *difformation retraction* of the space X to A. If $f_t : X \times I \to Y$ is a family of homotopy where for some subspace $A \subseteq X$, $f_t|_A$ is independent of t, then we call the homotopy *homotopy relative* to A, or *homotopy rel* A.

Think about taking bold-face letters or higher dimensional objects like S^n punctured at hte poles and retracting them to their thin counterparts or lower dimensional equivalences. Note that being retractble to a point is slightly stronger than being contractible since the map must restrict to the identity to a point. Every deformation retraction is homotopy relative to a subspace A.

Each deformation retraction retracts a map f_0 onto a mp f_1 called a *retraction*:

Definition 1.1.5: Retraction

Let $f : X \to X$ be continuous map. Then f is said to be a *retraction* on A if $f(X) = A$ and $f|_A = \text{id}$.

Note how the definition of retraction has no notion of a homotopy, in particular there are less conditions that need to hold for a retraciton.

Example 1.3: Retraction

1. If $X = Y \times Z$, and $A = Y \times \{z_0\}$, then $f(y, z) = (y, z_0)$ is a retraction.
2. if A is an annulus, and we have a circle in the annulus, then pinching everything to A is a retraction.
3. $X = \mathbb{R}$, $A = \mathbb{R}_{\geq 0}$, then $\sigma(t) = |t|$ is a retraction

Thus, each deformation retraction is homotopic to a retraction map. Note that for each $x \in X$, we have a retraction map to $\{x\}$, however since not all spaces are simply connected, not all spaces have a deformation retraction to a retraction (as we shall shortly see).

Example 1.4: Deformation Retraction

1. The following is three different deformation retraction of $\mathbb{R}^2$ with two puncutred holes:

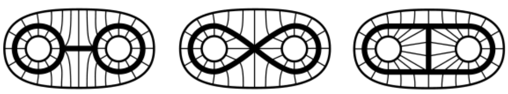


Figure 1.1: $\mathbb{R}^2 \setminus \{a, b\}$ retracted to different curves

There is a nice way of visualizing a deformation retraction. For $f_t : X \to Y$, take

$$M_f := (X \times I) \sqcup Y/[((x, 1) \sim y \text{ iff } f_1(x) = y)]$$

which we may visualize as the following:

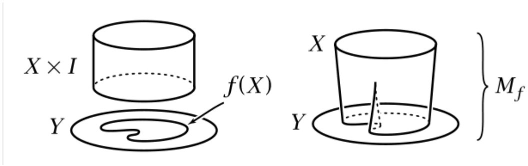


Figure 1.2: defoRetractCyl

Notice that M_f is also a deformation retraction of onto $f(X)$, namely by continuously dragging each (x, t) along $\{x\} \times I \subseteq M_f$ to $f(x)$.

2. Let $f : X \to X$ be a deformation retract onto a point $x \in X$. Show that for each open neighbourhoods of x, there exists a $V \subseteq U$ such that $V \hookrightarrow U$ is nullhomotopic. use this to show the only deformation retract of the space (called the infinite comb or pathological comb)

$$([0, 1] \times \{0\}) \cup (\{r\} \times [0, 1 - r])$$

are to point in $[0, 1] \times \{0\}$. Hence, a space like so:

Figure 1.3: Contractible, but not Deformation Retractible

is contractible, but not deformation retractible[a].

[a]Though, there is a *weak defomration retract* from the zig-zagy space above to the darker line

Retraction are useful since they allow us to generlize the notion of invertability, building us up to isomorphism in **hTop**:

Definition 1.1.6: Homotopy Equivalence

Let $f : X \to Y$ be a continuous map. Then f is called a *homotopy equivalence* if there exists a g such that $f \circ g \simeq \text{id}$ and $g \circ f \simeq \text{id}$. In this case, we shall write $X \simeq Y$ and call them *homotopy equivalenent* or that they have the same *homotopy type*. In **hTop**, homotopy equivalence are the isomorhpisms.

Example 1.5: Homotopy Equivalent

1. Let's say the shape X is contractible so that the identity $F : X \to X$ has a homotopy $F_t : X \times I \to X$ to $F_1(X) = \{x\}$. Consider the map $f : X \to \{*\}$ where $f(X) = *$ and

homotopy inverse $g : \{*\} \to X$ where $g(*)$ maps to $\{x\}$. Then the map

$$g \circ f : X \to X \qquad g \circ f(X) = \{x\}$$

is homotopy equivalent to $\mathrm{id} : X \to X$ by taking $(g \circ f)_t = F_{1-t}$.

2. A non-trivial example would be that the 3 shapes in figure 1.1 are homotopy equivalent. This also shows that homotopy equivalence need not perserve cardinality, or even dimension, and hence is certainly weaker than a homeomorphism[a].

3. Let $f : X \to X$ be a deformation retract onto A. Then $A \hookrightarrow X$ is a homotopy equivalent (this can be thought as the generalization of the contractible argument). Slightly weaker condition that still works is the following: if $f : X \to X$ has a homotopy such that $f_0 = \mathrm{id}$, $f_1(X) \subseteq A$ and $f_t(A) \subseteq A$, then f is called a *weak deformation retract* onto A. Show that $A \hookrightarrow X$ is still a homotopy equivalence.

[a] The simplest example that only requires a notion of connectivity is showing that $\mathbb{R} \not\cong \mathbb{R}^2$

As an interesting remark two compact surfaces that are homotopy-equivalent are in fact homeomorphic, showing that homeomorphisms are exactly the maps which contract exactly like a homotopy, or equivalently that homotopies add no extra flexibility when working with compact surfaces (can you think of why?).

Lemma 1.1.1: Homotopy Equivalence Space Representation

Let $X \simeq Y$. Then there exists Z such that X and Y are deformation retracts of Z

***Proof* :**
Let $Z = M_f$.

We finish off this little section with an example of how being contractible may be eaiser than being a (deformation) retractible:

Example 1.6: House With Two Rooms
Stare at this image for a bit:

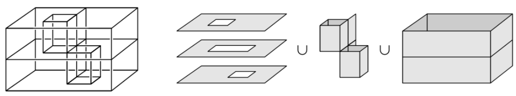


Figure 1.4: House With Two Rooms, Hatcher

If X represents this space, Let X_ϵ be this space with each point thickened with a filled in ϵ-ball sufficinetly small so that X_ϵ can be deformation retraction to X (in fact, is homotopy equivalent to X: if $r : X \to X_\epsilon$ is the retraction and $\iota : X_\epsilon \to X$ is the inclusion, $r\iota \simeq \mathrm{id}$ and $\iota r \simeq \mathrm{id}$). is homeomorphic to the filled in closed ball $D^3 \subseteq \mathbb{R}^3$. Then D^3 is certainly contractible to a point, say $\{*\}$. Hence:

$$X \simeq X_\epsilon \simeq D^3 \simeq \{*\}$$

Hence by transitivity X is contractible. It is certainly possible to show it is also retractible, however this is a bit harder to show.

1.2 Cell-Complex

As is often the case in mathematics, we understand well some simple examples, and would like to create more complicated examples using the simpler family of objects. There are many simple geometric shapes which we may choose, one of the most studied are disks. Define an n-cell to be a topological space homeomorphic to D^n. Note that a 1-cell is just an closed arc.

Definition 1.2.1: Cw Complex

A *CW complex* or a *cell complex* is a space constructed in the following form:

1. Start with a discrete set X^0 whose points are regarded as 0-cells
2. Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via gluing maps

$$\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$$

   where $\partial e^n_\alpha = S^{n-1}$. That is, X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with the collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. As a set, $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha / \sim$ where each e^n_α is an open n-disk.

3. The induction stops at a finite step, at which point we set $X = X^n$, or we continue this process indefinitely and set $X = \bigcup_n X^n$ in which case X is given the weak topology via inclusion ($A \subseteq X$ open if and only if $A \cap X^n$ for all n). If $X = X^n$, then X is said to be fintite dimensional with dimension n.

To break down the definition, it is perhaps more intuitively to see that we have

$$X^0 \hookrightarrow X^1 \hookrightarrow \cdots$$

where X_k is obtained by gluing copies of n-cells $(e^n_\alpha)_\alpha$ to X_{k-1} via gluing maps $g^k_\alpha : \partial e^k_\alpha \to X_{k-1}$. Then X is simply the colimit of this diagram with these maps

$$\varinjlim X_i = X$$

The term *CW* stands for "Closure-finite Weak topology".

For future reference, we define the following natural map associated to a cell-complex:

Definition 1.2.2: Characteristic Map

Let X be a cell complex and e^n_α associated to $\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$ be a cell forming this complex. Then there exists a map $\Phi_\alpha : e^n_\alpha \to X$ extending the map φ_α to be a homeomorphism from the interior of D^n_α to e^n_α, in particular $\Phi_\alpha : e^n_\alpha \hookrightarrow X^{n-1} \sqcup_\alpha D^n_\alpha \to X^n \hookrightarrow X$.

The name characteristic is due to its link to *Eulers characteristic* (see section 3.5.3).

Example 1.7: CW Complex

1. Let X^0 be some collection of points. Then we may form a graph by ataching a 1-cells (i.e. spaces homeomorphic to closed intervals) to points in X^0. If we so choose, we may form loops in our graphs.

2. $\mathbb{R}$ can be seen as having a CW-structure (i.e. homeomorhpic to a CW-complex) by taking the 0-skeleton $\mathbb{Z}$ and attaching the 1-cells $[n, n+1]$. This process can be generalized to give $\mathbb{R}^n$ a CW-structure.

3. The house with two rooms pictures earlier is a CW complex, can you see how? There should be 29 0-cells, 51 1-cells, and 23 2-cells.

4. One can get the sphere S^{n-1} by attching to a point the entire boundary $\partial D^n \to e_0$ (each X^k for $1 < k < n$ had no k-cells attaching, meaning $X^1 = X^2 = \cdots = X^{n-1}$). Alternatively, two lower dimensional disks can be attahed to the boundary of D^{n-2} (play with this for S^2)

5. Take $X^0 = \{*\}$ to be a single point. Then we may attach 2 1-cells. Then, take 1 2-cell, and identify two of it sides with one of the circles, and two of sides with the other circle (to do this, make the map a $2-1$ map onto the two 1-cells). If we straighten out the circle so that it looks like squares, we would get the following visual:

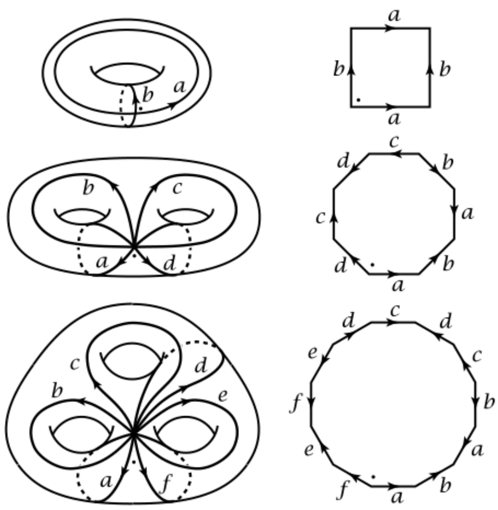


Figure 1.5: n-torus as CW complex, Hatcher

The other examples are when we are attaching a 1-cell to 4 or 6 circles respectively.

6. $\mathbb{R}\mathrm{P}^n$ is a CW-complex. We can see this by interpreting $\mathbb{R}\mathrm{P}^n$ as $S^n/(v \sim -v)$. In this case, $\mathbb{R}\mathrm{P}^n$ is $\mathbb{R}\mathrm{P}^{n-1}$ with the gluging $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ gluing antipodal points together. Think of the case of $\mathbb{R}\mathrm{P}^1 = S^1$ so that you are gluing the boundary of the disk D^2 to S^1; the strick is to glue the boundary at "twice the speed" around S^1. Hence $\mathbb{R}\mathrm{P}^n$ has one cell per dimension.

From this, we may define $\mathbb{R}\mathrm{P}^\infty = \bigcup_n \mathbb{R}\mathrm{P}^n$ to also be a cell-complex with one cell per dimension. The space $\mathbb{R}\mathrm{P}^\infty$ can be viewed as the space of liens through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

7. $\mathbb{C}\mathrm{P}^n$ is a CW-complex. We shall show how we can create $\mathbb{C}\mathrm{P}^n$ as the quotient of D^{2n} under the relation $v \sim \lambda v$ for $v \in \partial D^{2n}$. Take $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with last coordiantes real and non-negative. Then they are vectors of the form:

$$(w, \sqrt{1 - |w^2}) \in \mathbb{C}^n \times \mathbb{C} \qquad |w| \leq 1$$

Ceratinly, these vetctors are the graph of the functions $w \mapsto \sqrt{1 - |w|^2}$ and form the shape $D^{2n}_+ \subseteq S^{2n-1} \subseteq \S^{2n+1}$ and consists of vectors of the form $(w, 0) \subseteq \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Then putting the relation $v \sim \lambda v$ on S^{2n+1} can be identified with vectors in D^{2n}_+, and the idnetificaiotn is unique if the last coordiante is nonzero. If the last coordiant is zero, we jstu take $v \sim \lambda v$ for $v \in S^{2n-1}$.

Thus, w have $\mathbb{C}\mathrm{P}^n$ descrived as the quotient of D^{2n}_+ through $v \sim \lambda v$ for $v \in S^{2n-1}$. Finally, $\mathbb{C}\mathrm{P}^n$ is obtainedd from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching e^{2n} cells via $S^{2n-1} \mapsto \mathbb{C}\mathrm{P}^{n-1}$. Hence,

$$\mathbb{C}\mathrm{P}^n = e^0 \cup e^2 \cup \cdots e^{2n}$$

We may define $\mathbb{C}\mathrm{P}^\infty$ similarly to $\mathbb{R}\mathrm{P}^\infty$.

In many of our future constructions, we shall need to identify subcomplexes to use in various operations on complexes, we hence single out the following definitions:

Definition 1.2.3: Subcomplex

Let X be a cell complex. Then a *subcomplex* is a closed subspace $A \subseteq X$ that is a union of cells of X. A pair (X, A) consisting of a cell complex X and a subcomplex A is caleld a *CW pair*.

Since A is closed, the characteristic map of each cell in A has image contained in A, and the image of the attaching maps of A is conatiend in A, so A is itself a cell-complex!

Note that the closure of each cell of a cell-complex (or the colosure of a colleciton of cells) need not be a subcomplex (though they often will be). For example, take S^1 with the usual one 0-cell and one 1-cell. Attach a 2-cell to S^1 by gluing the boundary of e^2 to a *proper subset* (or subarc) of S^1. Then the closure ofhte 2-cell is not a subcomplex since it contains only a *part* of the 1-cell.

Example 1.8: Subcomplexes

1. Each skeleton X^n of a cell-complex X is a subcompelx.
2. Ecah $\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$ is a subcomplex for $k \leq n$. In fact, these are the only subcomplexes of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$
3. the natural inclusions $S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$ are cell complexes given the right CW structure of S^n, namely it would have to be the construction where we glue the two lower dimensional spheres instead of gluging the boundary of D^n.

1.2.1 Operations on Cell Complex

Cell complexes shall be a fundamental tool we use; many invariances will be defined on them. One of the ways we shall take advantage of these invariance is by defining new ways of constructing CW structures given already konwn ones. Most of these constructions work more generally in the category **Top**; any appropriate adjustments to the definitions can be made.

Definition 1.2.4: Product

Let X, Y be cell-complex. Then $X \times Y$ is a cell-complex with a cells being products $e^m_\alpha \times e^n_\beta$.

Note that if there are infinitely many cells, the the product topology on $X \times Y$ may have a finer topology than the CW topology on $X \times Y$, however we shall not conern ourselves with this.

Example 1.9: Product Of Cells

Let S^1 have the standard cell structure (attach I to a single point identifying the endpoints). Then $S^1 \times S^1 = T^1$ is a cell-structure. This is visualized in figure 1.5.

Definition 1.2.5: Quotient

Let (X, A) be a CW pair. Then X/A has a cell-structure respecting the universal property of quotients given as follows: The cells of X/A are the cells of $X - A$ plus a new 0-cell, the image of A in X/A.

Example 1.10: Quotient Of Cells

1. Take S^{n-1} with any cell structure, build D^n from S^{n-1} by attaching an n-cell in the natural way, and consider (D^n, S^{n-1}). Then D^n/S^{n-1} is S^n with the usual cell-structure.

Definition 1.2.6: Suspension

Let X be a cell complex. Then the *suspension* of X, denoted SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to a another point.

Visually, a suspension can be seen as so:

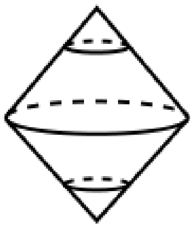


Figure 1.6: Suspension

The prototypical example of a suspension is $S(S^n) = S^{n+1}$. Since suspension are created via quotienting and products, we see that we can create a cell structure on the cone of X:

$$CX = (X \times I)/(X \times \{0\}) \cup (X \times I)/(X \times \{1\})$$

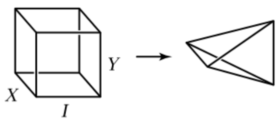
Though this might not seem useful, as we progress this construction will become increasingly more importnat. Of notable important is that maps can be suspended, if $f : X \to Y$, then we can suspend it to $Sf : SX \to SY$, which is given by the quoteint map of $f \times \mathrm{id} : X \times I \to Y \times Y$. Often, the suspension of a "non-simply connected" object shall be simply connected, and the suspension of a map shall make it nullhomotopic.

In a suspension, all the points of X are joined to a single point. We can generalize this by joining every point in X to the points to a space Y (and vice-versa):

Definition 1.2.7: Join

Let X, Y be topological space. Then a *join*, dentoed $X * Y$ is the quotient of the space $X \times Y \times I$ by $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Thus $X \times Y \times \{0\}$ collapses to X while $X \times Y \times \{1\}$ collapses to Y.

If X, Y are CW complexes, then there is a natural CW structure on $X * Y$ with the subspaces X and Y as subcomplexes, and the remaning cells being the product cells of $X \times Y \times (0, 1)$. This process of the join operation can be visualized like so:



In fact, if $X = Y = [0, 1]$, then a join would produce the above shape. The join $X * Y$ contains a copy of X and Y, and every point $(x, y, t) \in X * Y$ is on a unique line segment joining the points $x \in X \subseteq X * Y$ and $y \in Y \subseteq X * Y$. Hence, we may thinkg of every point in $X * Y$ as being written in the form

$$t_1 x + t_y y$$

where $0 \leq t_i \leq 1$, $t_1 + t_2 = 1$ and $0x + 1y = y$ and $1x + 0y = x$; that is all possible *convex combinations*. If we itterate and write $X_1 * \cdots * X_n$, we see that this is independent of bracketing and we can write every element in the form:

$$t_1 x_1 + \cdots + t_n x_n \qquad 0 \leq t_i \leq 1, t_1 + \cdots + t_n = 1$$

that is, we get every formal convex linear combinations of the elements. A special but important case is when each X_i is a singleton. In this case, the join of two points is a line segment, the join of 3 is a triangle, the join of 4 is a tetrahedron, and so forth. In general, the join of n points is a convex polyhydron of dimension $n - 1$ known as a *simplex*.

Example 1.11: Join

Let X_i be two points, and say they are placed in $\mathbb{R}^n$ on a coordinate axis at unit lengths. Then the join $X_1 * \cdots * X_n$ is the union of 2^n copies of $n - 1$ dimensional simplices. The radial projection from the origina gives a homeomorphism from $X_1 * \cdots * X_n$ to S^{n-1}.

Definition 1.2.8: Wedge Sum

Let X, Y be topological spaces. Then given points $x \in X$ and $y \in Y$, the *wedge sum* (with respect to x, y) is

$$X \vee Y = X \sqcup Y / x \sim y$$

If X, Y are cell complexes, $X \vee Y$ is a cell complex by taking the cell complex $X \sqcup Y$ and collapsing a subcomplex to a point. A simple example of a wedge sum would be for any cell complex X, the quotient X^n / X^{n-1} is a wedge sum of the n-spheres $\vee_\alpha S^n_\alpha$ with one sphere for each n-cell of X.

Definition 1.2.9: Smash Product

Let X, Y be topolocial spaces. Then given $x \in X$ and $y \in Y$, the *smash product* (with respect to x and y) is

$$X \wedge Y := X \times Y / X \vee Y$$

If X, Y are cell complex, then giving $X \times Y$ the cell-complex topology the smash product is a cell-complex. For an example, show that $S^1 \wedge S^1 = S^2$ and more generally $S^n \wedge S^m = S^{n+m}$.

Example 1.12: Linking Smash Product And Join

Show that

$$S(X \wedge V) \simeq X * Y$$

1.3 Homotopy Equivalence and Homotopy Extension

In this section, we shall prove some important theorems to find homotopy equivalent spaces. The first is way of collapsing some parts of our space to a point to simplify the problem, and the second varies the way we put subspaces together.

In general, if we take X and X/A, these two spaces may have diferent homotopy types, for example the torus T^2 and $T^2/(S^1 \vee S^1) \cong S^2$. The reason why the homotopy type changes is becasue $S^1 \vee S^1$ is not contractible, and hence change the homotopy; thus to insure an invariance in the homotopy type we must insure that the subspace we are contracting by is contractible. To see the usefulness of this criterion, we start with the following notion:

Definition 1.3.1: Homotopy Extension Property

Let $f_0 : X \to Y$ be a map and $f_t|_A : A \times I \to Y$ be a homotopy on $f_0|_A$. Then if we can extend this to a homotopy on X, $f_t : X \times I \to Y$, we say that (X, A) has the *homotopy extension property.*

Note how the pair (X, A) did not include information about the codomain. Thus, for any Y if $X \times \{0\} \to Y$ is continuous and $A \times I \to Y$ is a homotopy that agrees with $A \times \{0\}$, then it may be extended to $X \times I \to Y$. If this was only valid for certain domains, we would say that it had the homotopy extension proeprty with respect to Y.

Proposition 1.3.1: Condition For Homotopy Extension

A pair (X, A) has the homotopy extension property if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$

Try visualizing this condition before seeing the formal proof:

Proof **:**
let (X, A) have the homotopy extension property. Then the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ is homotopic to itself, and so extends to a map $X \times I \to X \times \{0\} \cup A \times I$, so $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Conversely, If $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ and X is Haudorff, it suffices to show it for when A must is be closed, since in fact this condition implies that A is closed. For if $r : X \times I \to X \times I$ is a retraction map onto $X \times \{0\} \cup A \times I$, then the image of r is the set of points $z \in X \times I$ with $r(z) = z$, i.e. a closed set if X is Haudorff, so $X \times \{0\} \cup A \times I$ is closed in $X \times I$ and hence A is closed in X.

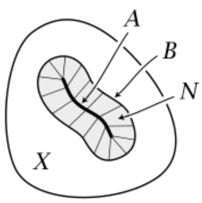
Now, given A is closed, then if we have two continuous maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$, we can combine them to get a map

$$X \times \{0\} \cup A \times I \to Y$$

which is continuous since it continuous on the closed sets $X \times \{0\}$ and $A \times I$. By assumption, we have a retraction map and so composing to get $X \times I \to X \times \{0\} \cup A \times I \to Y$, we get an extension $X \times I \to Y$, hence (X, A) has the homotopy extension property.

Example 1.13: Homotopy Extension Property

1. If A has mapping cylinder neighbourhood in X, then (X, A) has the homotopy extension property. A mapping cylinder neighbourhood is a closed neighbourhood N containing a subspace B (B is the "boundary" of N), with $N \setminus B$ an open neighbourhood of A such that there exists a map $f : B \to A$ and a homeomorphism $h : M_f \to N$ with $h|_{A \cup B} = \text{id}$.



To see that (X, A) has this property, first not that $I \times I$ retracts onto $I \times \{0\} \cup (\partial I) \times i$, and so $B \times I \times I$ retracts onto $B \times I \times \{0\} \cup B \times (\partial I) \times I$, and hence we induce a retraction of $M_f \times I$ onto $M_f \times \{0\} \cup (A \cup B) \times I$. Thus $(M_f, A \cup B)$ has the homotopy extension property. Thus, the homeomorphic pair $(N, A \cup B)$ does as well.

Given a continuous map $X \to Y$an a homotopoy restriction to A, we can take the constant homotopy on $X - (N - B)$ and then extend over N by applying the homotopy extension property for $(N, A \cup B)$, to the given homotopy on A and the constant homotopy on B

2. For an example of a pair without the homotopy extension property take (I, A) wehrer $A = \{0, 1, 1/2, 1/3, ...\}$. Then there is no continuous retractions $I \times I \to I \times \{0\} \cup A \times I$.

Proposition 1.3.2: Homotopy Extension For CW Pair

Let (X, A) be a CW pair. Then $X \times \{0\} \cup \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.

Proof **:**
We shall first directly work with the cells D^n, and then work with the space X. First, there is a retraction

$$r : D^n \times I \to (D^n \times \{0\}) \cup (\partial D^n \times I)$$

an easy such example would be the radial projection from the point $(0, 2) \in D^n \times \mathbb{R}$. From this retraction, we may construct a deformatino retraction from $D^n \times I$ onto $(D^n \times \{0\}) \cup (\partial D^n \times I)$ by setting

$$r_t = tr + (1 - t)\,\mathrm{id}$$

Since $X^n \times I$ is obatined from $(X^n \times \{0\}) \cup (X^{n-1} \cup \mathbb{A}^n) \times I$ by attaching copies of $D^n \times i$ along $(D^n \times \{0\}) \cup (\partial D^n \times I)$, we get a deformation retraction from $(X^n \times \{0\}) \cup (X^{n-1} \cup \mathbb{A}^n) \times I$. If we perform this deformation retraction during the t-interval $[1/2^{n+1}, 1/2^n]$, then the infinite concatenation of these homotopies is a deformation retraction of $X \times I$ onto $(X \times \{0\}) \cup (A \times I)$, since there is no problem with continuity of this deformation retraction at $t = 0$ since it's continuous at $X^n \times I$ (beign stationary in $[0, 1/2^{n+1}]$), and the CW complexes have the weak topology with respect to their skeleton, so the map is continuous if and only if its retractoin on each skeleton is continuous. But then we have the reatraction.

Theorem 1.3.1: Homotopy Equivalence Of Quotient

Let (X, A) satisfy the homotopy extension property and let A be contractible. Then the quotient map $\pi : X \to X/A$ is a homotopy equivalence.

Proof **:**
Let $f_t : X \to X$ be a homotopy extending the contraction of A where:

1. $f_0 = \mathrm{id}$
2. $f_t(A) \subseteq A$ for all $t \in I$

Then $\pi \circ f_t : X \to X/A$ sends A to a point, hence there is a well-defined map $\overline{f}_t : X/A \to X/A$, notably the following diagarm commutes for all t:

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ X/A \arrow[r, "\overline{f}_t"] & X/A \end{tikzcd}$$

In particular $\pi \circ f_t = \overline{f}_t \circ \pi$. When $t = 1$, we have that $f_1(A)$ is a point, hence f_1 induces the map

$g : X/A \to X$ in the following diagram:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\ X/A \arrow[r, "\overline{f}_1"] \arrow[ur, "g"] & X/A \end{tikzcd}$$

this diagram is certainly commutative, in particular $\pi \circ g = \overline{f}_1$, since

$$\pi \circ g(\overline{x}) = \pi \circ g \circ \pi(x) = \pi \circ f_1(x) = \overline{f}_1 \circ \pi(x) = \overline{f}_1(\overline{x})$$

Hence, the maps g and π are homotopy equivalence since

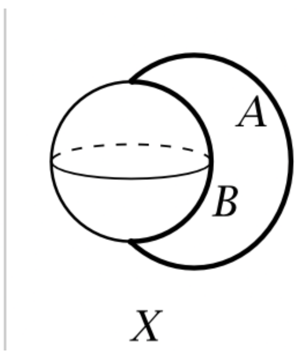
$$g \circ \pi = f_1 \simeq f_0 = \mathrm{id} \qquad \pi \circ g = \overline{f}_1 \simeq \overline{f}_0 = \mathrm{id}$$

via the homotopies f_t and $\overline{f}_t$, giving us the desired result.

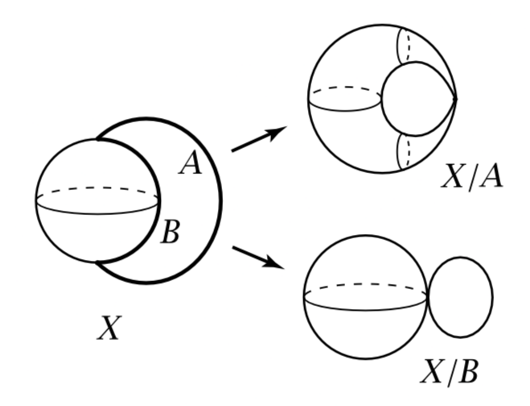
Example 1.14: Homotopy Equivalence Using Quotients

1. (Graphs) Let X be any graph with finitely many vertices and edges. If two endpoints of any edge of X are distinct, we can collapse the edge to a point, giving a homotopy equivalent graph with one fewer edge (namely since edges are contractible). This can be repeated until we get a graph X where each component has 1 vertex connected by loops (wedge sum of circles). We shall show in the next chapter that the wedge sum of circles are homotopy equivalent if they have the same number of circles.

2. Let X be the space S^2 along with an interval I attached at two distinct points. Let the interval I be labeled A, and let B be the arc on the sphere connecting the endpoints of A:



Give X a CW complex with the two endpoints of A and B being 0-cells, the interior of A and B beign 1-cells, and the rest of S^2 a 2-cell. Since A, and B are contractible, we get that X/A and X/B are homotopy equivalent to X:

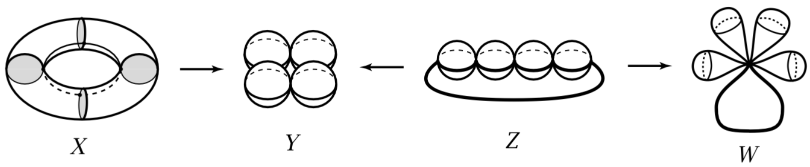


The space X/A is S^2/S^0, that is the sphere with two points identified, while X/B is $S^1 \vee S^2$. Thus,

$$S^2/S^0 \simeq S^1 \vee S^2$$

even though at first glance they may seem quite different

3. Let X be the union of a torus T with n meridional disks. To make a CW complex on X, take a longitudinal circle in the torus, hence intersecting each of the meridional disks at one point. These poitns of intersectio nare the 0-cells, the 1-cells are the rest of hte longitudinal circl an the boundary of the cmeridional disks, and the 2-cells are the remaining regions. We may collapse the meridional disks to a point to yeidl a homotopy equivalent space Y consisten of n 2-spheres, each tangent to it's neighors (we mah think of this as a necklace with n beads). From this intution, we may imagine a space Z which is a line of n 2-spheres each touching each other tangentially (a "strand of n-beads") with a line connecting the ends of the first and last 2-sphere. Then we can collapse the line to get your space Y, showing the two are homotopic. Finally, we may take the union of arcs on the 2-spheres to form a path starting form the endpoint of the first sphere and ending on the endpoint of hte last sphere, which we may then collapse to get a shape that resembles a "bouquet of flowers" with a handle:



4. (reduced suspension) Let X be a CW complex and $x \in X$ a 0-cell. Then in the suspension SX, there is a line segment $\{x\} \times I$ which we can colapse to produce the homotopy-equivalent space ΣX called hte *reduced suspension*. For example, if $X = S^1 \vee S^1$ with x the point of intersection, the SX is the union of two spherses intersection along the arc $\{x\} \times I$, so the reduced suspension is $S^2 \vee S^2$ (which can think of being a simpler space than the suspension). In general, it can be shown that:

$$\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$$

for arbitrary CW complex X, Y. The reduced suspension also has a simpler CW strucuture: in SX there are two 0-cells (the two suspensions points), and an $(n+1)$-cell $e^n \times (0,1)$ for each n-cell of X other than the 0-cell x. Then the reduced suspension ΣX is the same as the smash product $X \wedge S^1$, wince both spaces are the quotient of $X \times I$ with $(X \times \partial I) \cup \{x\} \times I$ collapsed to a point.

Another application of the extension property is another way of gluing shapes while keeping the homotopy class invariant:

Theorem 1.3.2: Homotopy Equivalence Via Gluing

Let(X_1, A) be a CW pair and elt $f, g : A \to X_0$ be attachings maps that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$$

relative to X_0

When we say that the two spaces are homotopy equivalent relative to X_0, we mean that there are maps between the two spaces that restrict to the idneity on X_0 such that they are homotopy equivalent via homotopies that restirct to the identity on Y.

Proof **:**
Let $F : A \times I \to X_0$ be a homotop yfrom f to g, and consider the space $X_0 \sqcup_f (X_1 \times I)$. Certainly this space containsn $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces. Then given a deformation retract of $X_1 \times I$ onto $(X_1 \times \{0\}) \cup A \times I$ as given in proposition 1.3.2, , we get an induced deformation retraction of $X_0 \sqcup_f (X_1 \times I)$ onto $X_0 \sqcup_f X_1$. Similarly, $X_0 \sqcup_f (X_1 \times I)$ deformations retracts onto $X_0 \sqcup_g X_1$. Both of these deformation retractions restrict to the iendity on X_0, so together they give a homotopy equivalence

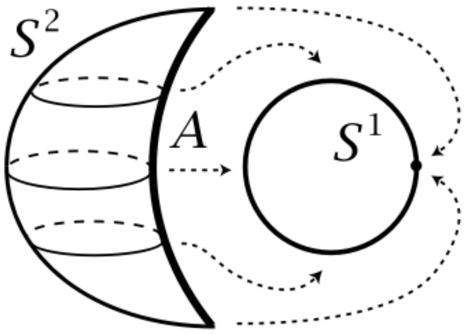
$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$$

relative to X_0, as we sought to show.

see this :`https://www.math3ma.com/blog/clever-homotopy-equivalences`

Example 1.15: Attaching Spaces

1. Let X be a sphere with two points identified. We shall rederive taht this is homotopy equivalent to $S^1 \vee S^2$. Take S^2 and S^1. Take an arc on S^2, and attach it to all of S^1. Since A is contractible, this map is homotopic to a constnant map attaching S^1 to S^1 via a constatn map, giving $S^1 \vee S^2$:



The result now follows since (S^2, A) is a C Wpair, S^2 being obatine from A by attaching a 2-cell

2. We may do the same with the necklace of beeds as seen in the previous example.a The necklace can be obtained from a circel by attaching n 2-sphers along arcs, so the necklace is homotopy-equivalent to the space obtained by attachign n 2-spheres to a circle at points, then sliding these attaching poitns around the circle until they all coincide, giving us the wedge-sum.

3. Let $CA = (A \times I)/(A \times \{0\})$ be the mapping cone of A. Then if (X, A) is a CW pair,

$$X/A \simeq X \cup CA$$

For, since CA is contractible subcomplex of $X \cup CA$, we see that:

$$X/A = (X \cup CA)/CA \simeq X \cup CA$$

4. Let (X, A) be a CW pair with A contractible in X (the inclusion map $A \hookrightarrow X$ is homotopic to a constant map). Then we shall show:

$$X/A \simeq X \vee SA$$

By the prevoius example, we have $X/A \simeq \cup CA$. Now, since A is contractible in X, the mapping cone of the inclusion $A \hookrightarrow X$ $(X \cup CA)$ is homotopy equivalent to the mapping cone of a constnat map, which is $X \vee SA$. As an example, $S^n/S^i \cong S^n \vee S^{i+1}$ for $i < n$¡ since S^i is contractibel in S^n if $i < n$. As a particular use-case, we get

$$S^2/S^0 \simeq S^2 \vee S^1$$

which we saw earlier, showing us we have two technics to find homotopy-equivalences.

Before moving on, we give a few important technical results when it comes to homotopies relative to a subspace.

Proposition 1.3.3: Homotopy Extension and Homotopy Equivalence

Let (X, A) and (Y, A) satisfy the homotopy extension property and $f : X \to Y$ be a homotopy equivalence with $f|_A = \text{id}$. Then f is a homotopy equivalence relative to A

Proof **:**
Hatcher p. 17 (prop 0.19)

Corollary 1.3.1: homotopy extension and deformation Retraction

Let (X, A) satisfy the homotopy extension proeprty and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retraction of X

Proof **:**
Apply the previous proposition to $A \hookrightarrow X$

Corollary 1.3.2: Homotopy Equivalence via Deformation Retracts

The map $f : X \to Y$ is a homotopy equivalence if and only if X is adeformation retraction of the mapping cylinder M_f. Hence, two spaces X and Y are homotopy equivalent if and only if there is a third space containing both X and Y as deformation retracts.

Proof **:**
Hatcher p.17

This last corrollary alludes to a more general notion known as *cobordism* which says that tow manifolds are *cobordic* if their union is the boundary of a manifold of one higher dimension. This shall not be further addressed in this book, but those interested can look at EYNTKA differential Geometry.

2

Fundamental Group

As we saw, homotopies allow us to study spaces up to continuous deformation along paths. It is thus natural try and study the paths on a space. In this section, we shall see that paths that are closed (that form a loop) shall naturally form a group structure, and this group structure shall be invariant under homotopy giving our first algebraic invariants on topological spaces. This group shall be called the *fundamental group* or *homotopy group*.

After having defined the fundamental group, we shall give many ways of computing it, how we can piece it together from the homotopy group of smaller spaces, and how homotopy groups of CW complexes can be calculated. Along the way, we shall see that most spaces X shall have a *universal cover*, say $U(X)$ that shall be simply connected (all paths are nullhomotopic) and will be a geoemtric representation of the fundamental group. We shall see that that there are many intermediary spaces between X and $U(X)$, and the corespondence between the intermediary spaces and the fundamental group of X shall mirror that of the famous *galois correspondence* between the intermediary fields and the subgroups of the galois group.

2.1 Path Homotopy and Fundamental Group

A lot of study is on homotopies, which will eventually bring you to homotopy groups, homology, exact sequences of homotopic groups and Hopf fibration of the sphere. An important class of homotopies worth singling out is homotopies which are “1-dimensional”. Working with one-dimensional one’s doesn’t mean we won’t be working with higher-dimensional objects, but the path in the space we use that will be homotopic will be one dimensional.

Definition 2.1.1: Path-Homotopic

Two paths $f, \tilde{f}$ in X are ***path-homotopic*** if

1. they have the same initial and final points: $f(0) = f'(0)$, $f(1) = f'(1)$
2. There exists a continuous function $F : I \times I \to X$ such that

   $$F(s,0) = f(s), F(s,1) = f'(s), F(0,t) = x_0, f(1,t) = x_1$$

   where $f(0) = f'(0) = x_0$ and $x_1 = f(1) = f'(1)$. We write

   $$f \simeq_p f'$$

   F is called a ***path-homotopy***

Example 2.1: Path Homotopy

1. If we take all function $I \to I$ such that $x \mapsto x^n$, then all of these are path-homotopic. In fact, any path such that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$ will be path-homotopic by the straight-line homotopy
2. More generally, if C is a convex set, then any two paths with the same initial and ending points are path-homotopic by the straight-line homotopy
3. "There must be more than the straight-line homotopy" you might be thinking, and you're right; think of previous examples we have worked with in the last chapter and see if you can find a path homotopy that is not a straight line
4. "Are there paths that are not homotopic?". That can be the case too. Take $X = \mathbb{R}^2 - \{0\}$ and

   $$f(s) = (\cos(\pi s), \sin(\pi s)) \qquad g(s) = (\cos(\pi s), 2\sin(\pi s))$$

   then these two functions are path homotopic using the straight-line homotopy, as you see visually:

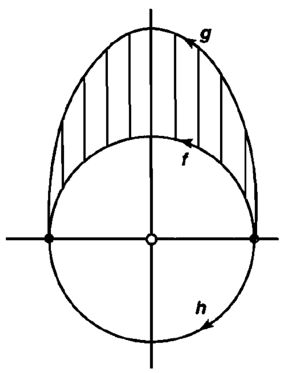


Figure 2.1: Same initial and end points, but one goes under point

In general, if the point 0 was not missing, we can have:

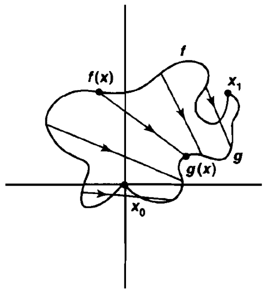


Figure 2.2: Convex space

However, the path $h(s) = (\cos(\pi s), -\sin(\pi s))$ is *not* path-homotopic. In the visual, this is the h path. It should perhaps seem intuitive that we we try to pass a continuous path *through* the point 0, we will introduce a discontinuity. The proof for this is a bit laborious at this point, so being satisfied with this "hole" intuition is useful.

5. Recall all polynomials $x \mapsto x^n$ from earlier? These are in the same equivalence class

$$[f] = [g]$$

And since $F'(x,t) = (x, (t-1)x + t0)$ is also a path-homotopy,

$$[id_0] = [f] = [g]$$

meaning it is also *nullhomotopic*

As may be expected from objects in a category, the morphisms form an object to itself froms a groupoid, and automorphisms form a group. This group is rather complicated and is the study of more recent mathematics[1], and so we shall construct a new much simplier algebraic invariant bassed off of the paths in a space.

Definition 2.1.2: identity path

If $x \in X$, denote e_x to be the constant path $e_x : I \to X$ such that $e_x(t) = x$ for $t \in I$

Therefore, the *id* in our previous example would be relabeled as e_0.

Next, we need to figure out a way defining an operation between paths. The natural candidate for this would be to extend one path to the next. However, we can extend *arbitrary* paths from one to the next, since that will make discontinuous paths (because the end point of one path might not be the initial point of another). Thus, we we try and define paths by through "combination", we need to impose this limit:

[1] See for example the study of Diff(M) for a manifold M

Definition 2.1.3: combining paths Operators: $*$

If f, g are paths in X and $f(1) = g(0)$, define $f * g$ by

$$(f * g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Example 2.2: Combining Paths

Try thinking of any two paths in $\mathbb{R}^2$ and combining them into 1.

Lemma 2.1.1: Well-defined: $*$ Preserves Equivalence Relation

if $f \simeq_p f'$, $g \simeq_p g'$, then

$$(f * g) \simeq_p (f' * g')$$

Proof **:**

Let F, G be homotopies between f and f', g and g' respectively. Basically, we'll just be doing just like in the equivalence relation, doubling the speed appropriately and using the pasting lemma to combine the two continuous functions. Let $H : I \times I \to Y$ such that

$$H(x, t) = \begin{cases} F(2x, t) & t \in [0, \frac{1}{2}] \\ G(2x-1, t) & t \in [\frac{1}{2}, 1] \end{cases}$$

then note that $H(x, 0) = f * g$ and $H(x, 1) = f' * g'$, and H is continuous by the pasting lemma, and so we're done

Because of this, can define $*$ on equivalence classes:

$$[f] * [g] = [f * g]$$

With the operation defined, we can look at the properties of it to see if it would follow all the axioms of a group. Remember that when we defined $*$, it only works on appropriate paths (with end and initial points matching). This will need to be considered in our up-coming examination of its properties:

Theorem 2.1.1: Properties Of $*$

1. (Associativity) Assume f, g, h are paths in X, $f(1) = g(0)$, $g(1) = h(0)$. Then
$$([f] * [g]) * [h] = [f] * ([g] * [h])$$
2. (identity) If f is a path in x, $f(0) = x_0$, $f(1) = x_1$, then
$$[f] * e_{x_1} = [f] = e_{x_0} * [f]$$
3. (invertible) Given a path f in X, define $\tilde{f}(t) = f(1-t)$ Then
$$[f] * [\tilde{f}] = [e_{f(0)}]$$
$$[\tilde{f}] * [f] = [e_{f(1)}]$$

Proof **:**
Some useful facts that will be used continuously

1. Let $k : X \to Y$ be a continuous map and F be a path homotopy between f and f'. Then $k \circ F$ is a path-homotopy between $k \circ f$ and $k \circ f'$ where
$$(k \circ F)(s, t) = k(F(s, t))$$
2. composition and $*$ are basically distributive: $k \circ (f * g) = (k \circ f) * (k \circ g)$.

As you probably noticed in the construction, the identity wasn't unique! This means that $*$ just falls short of actually being a binary operation. In fact, it is not actually a function, but a *partial function*; you can't take two arbitrary paths, but you need to select paths with the appropriate mid-points (as we noticed when defining $*$ originally!). If we restrict to paths that are always concatinatable, we will indeed have a binary function, but we usually want to think of $*$ being defined on our entire space.

We actually give a name to a "group" which uses a partial function (and hence, has multiple identities depending on how you must restrict the partial function to get a function):

Definition 2.1.4: Groupoid

A ***groupoid*** is a set (ex. G) with a ***partial function*** $* : G \times G \to G$ *and* a unary function $^{-1}: G \to G$ which then satisfy the group axioms (associatvitiy, inverse, identity). The slight difference between groups is that we now have:

1. (inverse) $a^{-1} * a$ and $a * a^{-1}$ is always defined
2. if $a * b$, is defined then $a * b * b^{-1} = a$ and $a^{-1} * a * b = b$ is also defined

How exactly the partial function $*$ is "partial" depends on the context – there is no one way of defining this partial function. Furthermore, we choose a unary function for the inverse because the "identity" is now no longer so "unique", as was shown in theorem 2.1.1.

Example 2.3: Groupoid Example

f we take $(\mathbb{R}, *, ^{-1})$ to be our groupoid with $*$ as we defined and $(f)^{-1} = \tilde{f}$ where $\tilde{f} = f(1-x)$ (i.e. reverse f), then we defined a groupoid!

With the algebraic structure defined and labeled, we can define the elements of our groupoid:

Definition 2.1.5: Path Space

Let $P(X) = \{\text{equivalence classes of paths } [f]\}$. We will call this the ***path space***

there is a natural map $P(X) \to X^2$, $[f] \mapsto (f(0), f(1))$. This map is surjective if and only if X is path connected. You can also think of the groupoid from earlier as

$$(P(X), *, ^{-1})$$

As mentioned, we still don't have a "unique" identity nor a total function, so we are yet to totally define a group. The limiting factor is that we can just combine any function with any function. However, if the initial- an end-points where the same, this would solve our problem! For this reason, we will once again restrict our attention, this time to paths that *loop*:

Definition 2.1.6: Loop

Given $x_0 \in X$ a ***loop based at*** x_0 is a pth which begins and ends at x_0. that is

$$f : I \to X, \qquad f(0) = f(1) = x_0$$

There is actually an advantage to adding this resctriction and thinking about loops: Imagine you have a plane with a point missing. Then having a loop around that point would mean you can't contract that loop to the constant map (as we've seen before, where we sneakily introduced a loop). This idea of not being able to contract loops will be our key concept! With this definition, we can consider the group structure on a subset of $P(X)$ such that the initial and end points match

Definition 2.1.7: Fundamental Group [at x_0]

Let

$$\pi_1(X, x_0) = \{[f] | f(0) = f(1) = x_0\}$$

be the path space with loops at x_0. Then $\pi_1(X, x_0)$ is a group under $*$, called *fundamental group* of X *based at* x_0, or the *first homotopy group*

Note: there *can* be more than one element in $\pi_1(X, x_0)$. We will soon explore spaces where this is the case, and also explore cases where there is only one element.

Sometimes $\pi(X)$ is used to represent the groupoid of the space X. One will naturally as if there is a 1st homotopy group, is there a 2nd? an nth? There is, and it's denoted $\pi_n(X, x_0)$. This higher value of n is where you can think of "dimensions" as I mentioned earlier. These, though, will not be studied here, and are the subject of homotopy theory.

Example 2.4: Fundamental Group

Let's consider $\mathbb{R}^n$ with the Euclidean topology and some $x_0 \in \mathbb{R}^n$ to define $\pi_1(\mathbb{R}^n, x_0)$. Then by the straight-line Homotopy, we can collapse every loop to the constant map, so there is only one element:

$$\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\}$$

More generally, if you have a convex space, then it's fundamental group will be trivial (since you can use the straight-line homotopy).

Note that this group is *not* necessarily abelian! We shall show that $S^1 \vee S^1$ has fundamental group as the free product of two copies of the integers, $\mathbb{Z} * \mathbb{Z}$.

So how many fundamental groups are there on a space? Does every point have a distinct type of group or are some of them isomorphic, or better yet, all of them isomorphic on the same space. The different answers to this will actually give us the topological property we're looking for in groups! That is, when thinking about a space, we usually think about the fundamental groups *up to isomorphism* and then the topologically invariant property are the *types* of fundamental groups. That means that the fundamental group does not rely on the point x_0 on which it was originally defined to get it's structure. To that matter, it will also not rely on the *entire* space X, just the component spaces, so the interesting thing to study are the path-connected components.

To get to that intuition, we first, specify how to define the isomorphism on the group

Proposition 2.1.1: Group Isomorphism Between Fundemental Groups

Let α be a path X $\alpha(0) = x_0$, $\alpha(1) = x_1$, then define the group isomorphism

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

where $\hat{\alpha}([f]) = [\alpha] * [f] * [\bar{\alpha}]$ and $\bar{\alpha}$ is the reverse of the funtion.

In words, what this functions does is go back, do the loop, then go back forward.

***Proof* :**

This is basically an Algebra proof. To show it's homomorphic:

$$\begin{aligned}
\hat{\alpha}([f] * [g]) = \hat{\alpha}([f * g]) &= [\alpha] * [f * g] * [\alpha] \\
&= [\alpha] * [f * e_{x_1} * g][\bar{\alpha}] \\
&= [\alpha] * [f * \alpha * \tilde{\alpha} * g][\bar{\alpha}] \\
&= ([\alpha] * [f] * [\bar{\alpha}]) * ([\alpha] * [g] * [\bar{\alpha}]) \\
&= \hat{\alpha}([f]) * h(g)
\end{aligned}$$

For bijectivity, we can define the two-sided inverse function $\hat{\alpha}^{-1}$ by reverseing $\tilde{\alpha}$, where it can be checked that

$$\hat{\alpha}(\hat{\alpha}^{-1}([f])) = \hat{\alpha}^{-1}(\hat{\alpha}([f])) = [f]$$

Making the homomorphism bijective, making it isomorphic

Corollary 2.1.1: Path-Connectedness And Isomorphism

If X is path-connected, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ for any $x_0, x_1 \in X$

This implies that for every path-connected component of X, it will have the *same* fundamental group at each point! Thus, when thinking of Fundamental groups, it is convenient of thinking of X as being a path-connected space (or at least, it will come down to breaking X up to it's path connected components). For the more Categorically inclined, there is also no *natural* isomorphism between points! This will come down to the fact that not every space is commutative[2]. The following corrollary may make this more clear:

Corollary 2.1.2: Fundamental Group Abelian

Let X be a path connectd space. Then $\pi_1(X)$ is abelian if and only if any transition map β_h (where $h(0) = b_0$ and $h(1) = b_1$) only depends on the endpoints of h.

Proof **:**
Let's say $\pi_1(X)$ is abelian. Consider:

$$\beta_{h_1}([f]) = [h_1 * f * \bar{h}_1]$$

Then:

$$\begin{aligned}\beta_{h_1}([f]) &= [h_1 * f * \bar{h}_1] \\ &= [h_1 * f * \bar{h}_2] * [h_2 * \bar{h}_1] \\ &= [h_2 * \bar{h}_1] * [h_1 * f * \bar{h}_2] \\ &= [h_2 * f * \bar{h}_2] \\ &= \beta_{h_2}([f])\end{aligned}$$

Conversely, if $\pi_1(X)$ were not abelian then there exists $[f], [g] \in \pi_1(X)$ such that $[f] * [g] \neq [g] * [f]$, that is $[\bar{g}] * [f] * [g] \neq [f]$. Now, pick some β_h. Since it is an isomorphism and $\beta_{\bar{g}h}$ is also a path-isomrophism:

$$\beta_{\bar{g}h}([f]) = \beta_h([\bar{g}] * [f] * [g]) = [h * \bar{g} * f * g * \bar{h}] \neq [h * g * \bar{h}] = \beta_h([f])$$

showing we have two different paths, completing the proof.

Something we shall not prove but is certainly worth noting is that the fundamental group of a topological group is *always*, hence the topological group of a lie group is always abelian!

Overall, if two path-connected spaces with different fundemental groups, they are topologically different. However, in general, different fundemental groups doesn't mean different spaces (ex. $\mathbb{R}$ and $\mathbb{R}^2$ have the same Fundemental group). The Fundamental group that is simplest to compute is the trivial one we have seen often. If a space has a trivial fundamental group, we give it a name:

[2]see ref:HERE for more details

Definition 2.1.8: Simply-Connected

X is *simply-connected* if X is path-connected, and $\pi_1(X, x_0)$ is the trivial group for some (and hence all) $x_0 \in X$

Note that $\pi_1(X, x_0) = 0$ but a space is *not* contracbie (take for example the warsaw circle, whih is $\sin 1/x$) but turned into a circle, which we shall show is not contractible once we show that $\pi_1(S^1) \cong \mathbb{Z}$). A theorem we shall not show but state right now for reference is Whitehead's theorem: a CW complex whose homotopy groups are all zero is contractible.

Lemma 2.1.2: simply connected simplification

If X is simply connected, $P(X) \cong X^2$ (i.e. a bijection),that is a in X is only determined by its endpoints.

Proof **:**
First, we'll show all paths are homotopic. Let α, β be paths from x_0 to x_1. Then

$$\begin{aligned}
[\alpha] * [\tilde{\beta}] &\in \pi_1(X, x_0) \\
[\alpha] * [\tilde{\beta}] &= [e_{x_0}] \\
[\alpha] * [\tilde{\beta}] * [\beta] &= [e_{x_0}] * [\beta] \\
[\alpha] &= [\beta]
\end{aligned}$$

With this, we can show injectivity. If $\pi([\alpha]) = \pi([\beta])$, then

$$\alpha(0) = \beta(0), \qquad \alpha(1) = \beta(1)$$

then $[\alpha] = [\beta]$ by proof above

And or surjectivity, given $(x_0, x_1) \in X^2$, there exists a path $\alpha(0) = x_0$, $\alpha(1) = x_1$ such that

$$\pi([\alpha]) = (x_0, x_1)$$

completeing the proof
I will have to look over that surjectivity claim again because it looks like it's breaking well-definedness

2.1.1 Functorial Properties

We've shown that we can define a fundamental group on a space, but what if we have another topological space Y, and a continuous function $f : X \to Y$, what of the group-structure we just build up? Well, we wouldn't of introduced fundamental groups if they weren't topological in nature, which means that continuous maps *preserve* them! This is also how the comment much earlier that in a sense topology is fundamentally the study of objects that don't change under continuation functions comes in.

So if we have a continuous function, can we define an appropriate *homomorphism* (the equivalent of continuous functions). This is what we'll do in this section.

Definition 2.1.9: induced homomorphic function

Suppose $h : X \to Y$ is continuous, with $h(x_0) = y_0$ (i.e. $h(X, x_0) \to (Y, y_0)$) . Then define

$$h_* : \pi_1(x, x_0) \to \pi_1(Y, y_0) \qquad h_*([\alpha]) = [h \circ \alpha]$$

Trivia Notice that it's a topology isomorphism which also specifies a point is a point. This makes our function a little more than continuous, it's continuous *and* preserves that point. Categorically, this means we're working in a slightly different category than Topology (Top): It's technically the Base-topology at x_0 ($\mathbf{Top}_{x_0}$) category.

Proposition 2.1.2: $(-)_*$ is a Functor

1. h_* is a a group-homomorphism
2. If $i : (X, x_0) \to (X, x_0)$ is the identity, then i_* is the identity
3. This map is functorial, that is, if $f : (X, x_0) \to (Y, y_0)$ and $(Y, y_0) \to (Z, z_0)$ then

$$(g \circ f)_* = g_* \circ f_*$$

Furthermore, if $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then f_* is an isomorphism.

Thus, there is a functor between the $\mathbf{Top}_{\{x_0\}}$ Category and the **Grp** Category

***Proof* :**
We'll prove each separately

1.

$$\begin{aligned} h_*([\alpha][\beta]) &= h_*([\alpha * \beta]) \\ &= [h \circ (\alpha * \beta)] \\ &= [(h \circ \alpha) * (h \circ \beta)] \\ &= [(h \circ \alpha)] * [(h \circ \beta)] \\ &= h_*([\alpha]) * h_*([\beta]) \end{aligned}$$

2. $i_*([\alpha]) = [i \circ \alpha] = [\alpha]$, hence it's the identity

3.

$$\begin{aligned} (g \circ f)_*[\alpha] &= [(g \circ f) \circ \alpha] \\ &= [g \circ (f \circ \alpha)] \\ &= g_*([f \circ \alpha]) \\ &= g_*(f_*([\alpha])) \end{aligned}$$

Thus, we've shown that the fundamental group is invariant to homeomorphisms! This is excellent, for now we can properly study them as a topological property of our space. The first indeavour we should seek out is how to reasonably calculate fundamental groups of spaces.

2.1.2 Covering Spaces and Lifting Property

The way we shall seek out fundamental groups is by associating with them a new universal object that shall geometrically represent the loops of a space (hence, geometrically represent the algebraic properties of the fundamental group). The main idea is that each space X can be "covered" by a large space E which will be easier to analyze (usually simply connected). Then we may ask for a "unviersal covering space" that will be a simply connected object that will somehow represent all possible "untangling" of loops, and will exist for almost every space we shall work with, the exception ususally being pathological spaces such as the Hawaian earings (which shall soon be defined)

Definition 2.1.10: Evenly Covered and Slices

Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be ***evenly covered*** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called the partition of $p^{-1}(U)$ into *slices*

if p does this to every neighbourhood in B, we call such a map p a covering map:

Definition 2.1.11: Covering Map

Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a ***covering map***, and E is said to be a *covering space* of B.

Example 2.5: Covering Maps

1. The identity map $i : X \to X$ is "trivially" a covering map

2. The next most trivial cover is if we have many copies of the object. If $X \times \{1, ..., n\}$ is n disjoint copies of X, then $p(x, i) = x$ for all i is again a covering map, since all the spaces are disjoint and when you limit to any space, it's homeomorphic to X. This function gives a nice visual for how you can think of covering spaces in general:

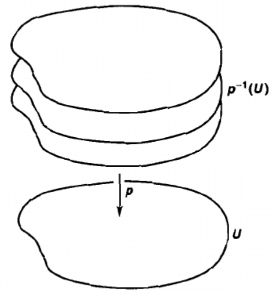


Figure 2.3: Covering space example

3. The map $p : \mathbb{R} \to S^1$ given by the equation

$$p(x) = (\cos(2\pi x), \sin(2\pi, x))$$

is a covering map. This comes down to geometric intuition about these trigonometric functions and circles. As an intuitive check that you get this function, if we take any interval $[n, n+1] \subseteq \mathbb{R}$, then that interval will wrap itself around the circle completely. Notice that $\mathbb{R}$ is simply connected, while we shall show that S^1 is not simply connected.

4. Another covering space for the circle is $p : S^1 \to S^1$ given by $z \mapsto z^n$ treating $S^1 \subseteq \mathbb{C}$.

5. It may be tempting to say that being a local homeomorphism implies p is a covering map, but this is not the case. Consider:

$$p : \mathbb{R}_{\geq 0} \to S^1 \qquad p(x) = (\cos(2\pi x), \sin(2\pi x))$$

This function is still surjective, and a local homeomorphism (any point $x \in \mathbb{R}_{\geq 0}$ has a neighorhood homeomorphic to it's image), and even is a covering map for *most* points. However, consider the point $b_0 = (1, 0)$ and some open neighborhood U around that point. Then the pre-image of U would be the same as in the third 3rd example, except for V_0 which would be $(0, \epsilon)$ for some ϵ depending on the open neighbourhood U. If we restrict p to $(0, \epsilon)$, then we do not get a homeomorphism, but an *embedding*, that is, a function for which if we restrict the co-domain to it's image, it becomes a homeomorphism. If we restrict to $(0, \epsilon)$ we'd get the open neighborhood starting at 0 and ending at ϵ (relative to where sin and cos sends ϵ.)

Anoher example would be:

$$E = [0, 1] \times \{0\} \cup (0, 1) \times \{1\}, B = [0, 1]$$

which is basically forcing surjectivity on the function $(0, 1) \to [0, 1]$. Then there isn't a neighborhood around $0 \in [0, 1]$, which is evenly covered. Again, this comes down to it being an embedding based on the domain rather than the co-domain.

6. the map $\varphi : \mathbb{C} \to \mathbb{C}^\times$ mapping $z \mapsto e^z$ is a covering map. It is in fact an ∞ to 1 covering map since $z + 2k\pi i \mapsto e^z$. Note that $\mathbb{C}$ is simply connected.

Proposition 2.1.3: Properties of Covering Maps

Let $p : E \to B$ be a covering map.

1. Every $y \in B$, the subspace $p^{-1}(y)$, has the discrete topology
2. The restriction of a covering map by an open $A \subseteq E$ set is a covering map: $p|_A : A \to B$[a]
3. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map
4. $p' : E' \to B'$ is also a covering maps, then
$$p \times p' : E \times E' \to B \times B'$$
is a covering map

[a] example 2.5 shows why this is not true for general subspaces

Proof **:**

1. This should be a straight forward test of understanding coverings. Take a neighborhood of y, $y \in U$. Since p is a covering map, then it's pre-image is partitioned into $\{V_\alpha\}$. Restricting our attention to any one slice V_α will lead to a homeomorphism (so for us, bijection) betweeen V_α and U, meaning one point of it maps to U. Since we can do this for every V_α, we get that $p^{-1}(y)$ is a collection of points intersected by open sets, meaning the subspace is discrete.
2. Let $A \subseteq E$. We'll show that $p(A)$ is open. If around every point $x \in p(A)$, there exists an open set V such that $V \subseteq p(A)$, then $p(A)$ is open (basis definition). Let's take a point $x \in p(A)$. Choose an open neighborhood V, $x \in V$. Then since p is an open cover, choose a neighbourhood that is evenly covered by p. Let $\{V_\alpha\}$ be the neighborhood. Since $x \in p(A)$, $p^{-1}(x) \in A$, so choose $y \in p^{-1}(x)$, and the appropraite V_β such that $y \in V_\beta$. Since A is open, $A \cap V_\beta$ is also open. Since p restricted to a V_β is a homeomorphism, $p(A \cap V_\beta)$ is *also* open. Note that $p(A \cap V_\beta) \subseteq p(A)$. Since we found an open neighborhood around x which is contained in $p(A)$, then $p(A)$ is open.
3. exercise
4. Let $b \in B$, $b' \in B$. Then we can take neighbourhoods U, U', $b \in U$, $b' \in U'$ which are evenly covered, so we have $\{V_\alpha\}$ and $\{V'_\alpha\}$. We can take the cross-product of the respective slices. They will still be disjiont, and restricting to slices still a homeomorphism, thus $p \times p'$ is a covering map

Example 2.6: Product of Covering Map

1. Take
$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$
where p is the same covering map of the circle we introduced earlier. Then this function is a covering map for the torus!
2. Notice that in the Torus we have a hoemomorphic copy of the figure 8 ($S^1 \vee S^1$). We thus

have a covering map for it given by proposition 2.1.3(4). There are multiple ways of covering this space. You can keep one circle and unwrap the other, or you can unwarp both. This space will also be a space that will have a universal cover.

As mentioned, the key property about covering spaces that is of interst to us is how it will represent fundamental groups. We now build up to "lifting" paths from our base space up to our covering space in a way that simplifies calculations:

Definition 2.1.12: Lifting

Let $p : E \to B$ be an [arbitrary] continuous function. Let $f : X \to B$ be a continuous map. A ***lifting*** of f is a continuous map $\tilde{f} : X \to E$ such that $f = p \circ \tilde{f}$. Similarly, we can say that this commutative diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

If you recall the theory of projective modules, you may recognize homomorphisms always be lifted. Key to us when dealing with homotopies is that lifts through covering maps preserve paths!

Lemma 2.1.3: Paths Lifted Through Covering Map

Let $p : E \to B$ be a covering map such that $p(e_0) = b_0$ and let $f : [0, 1] \to B$ be a continuous with inital point $f(0) = b_0$. Then it has a unique lifting to a path $\tilde{f}$ in E with initial point $\tilde{f}(0) = e_0$, that is the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "e_0 \mapsto b_0"] \\ {[0,1]} \arrow[ur, "0 \mapsto e_0"] \arrow[r, "0 \mapsto b_0"'] & B \end{tikzcd}$$

Proof **:**

(**Existence**) It is similar to how we showed connectedness. Let M be the largest element of $[0, 1]$ such that $f|_{[0,M]}$ has a lift $\tilde{f}$. Let $\tilde{f}(M) = e$, $f(M) = b$, then $p(e) = b$. Let U be an open set in B containing b such that U is evenly coevered $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$.

Let $\alpha_0 \in I$ be such that $e \in V_{\alpha_0}$. U is open, so $\exists \epsilon > 0$ such that $(M - \epsilon, m + \epsilon) \subseteq f^{-1}(U)$. Thus, we define $\tilde{f}_{(M- \epsilon, M+\epsilon)}$ to be $i \circ f|_{(M- \epsilon, M+\epsilon)}$ is the where $i : U \to V_{\alpha_0}$ is the canonical isomorphism, so $M = 1$.

(**Uniqueness**) Let's say g_1, g_2 are distinct lifts of f with $g_1(0) = g_2(0) = e_0$. Let $m = \inf \{t \in [0, 1] \mid g_1(t) \neq g_2(t)\}$. Then we may take $f(m) = b$. Taking U to be an open neighbourhood of b which is evenly covered so that $p^{-1}(U) = \bigsqcup_{\alpha \in I} V_\alpha$, we get

$$(m - \epsilon, m + \epsilon) \subseteq f^{-1}(U)$$

By continuity $g_1(m - \epsilon, m + \epsilon)$ is connected, so there exists α_1, α_2 st

$$g_1((m - \epsilon, m + \epsilon)) \subseteq V_{\alpha_1} \qquad g_2((m - \epsilon, m + \epsilon)) \subseteq V_{\alpha_2}$$

Since by definition of m $g_1(t) = g_2(t)$ for $t < m$, it must be that

$$V_{\alpha_1} \cap V_{\alpha_2} \neq \emptyset$$

But since these neighbourhood are either equal or disjoint, we have $\alpha_1 = \alpha_2$. From this, we may define the natural homeomorphism $i : U \to V_{\alpha_1}$. Since $f = p \circ g_1 = p \circ g_2$ we get

$$g_1|_{(m-\epsilon,m+\epsilon)} = \iota f = g_2|_{(m-\epsilon,m+\epsilon)}$$

But that contradictinos m being the least on which g_1 and g_2 agree, showing that it must be that $m = 1$.

Lemma 2.1.4: Homotopies can be lifted

Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Let $F : I \times I \to B$ be a continuous function such that $F(0, 0) = b_0$. Then there exists a unique lift $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0, 0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

***Proof* :**
Using lemma 2.1.3, we can lift every path of the homotopy. We need to then show its continuous. To do this, it is similar to the proof of compactness/connectedness and the arguments we just did with saying M is the largest value s such that $F(t, s)$ is continuous, and showed that it must be $M + \epsilon$, which lets us conclude that $M = 1$

Note that though paths are lifted, loops *need not be lifted*, in particular it is often the case that a loop is lifted to a path. However, if the path/loops are homotopic, the endpoints of two path-homotopic functions also agree:

Lemma 2.1.5: Path Homotopies can be lifted

Let f, g be paths starting and ending at b_0 (i.e. path-homotopic loops). If $f \simeq_p g$, then $\tilde{f}(1) = \tilde{g}(1)$

***Proof* :**

The path homotopy F between f, g lifts to a path homotopy between $\tilde{f}, \tilde{g}$. In particular, the end points of homotopic paths agree, therefore $\tilde{f}(1) = \tilde{g}(1)$.

Using this, we see that lifiing continuous maps is in fact directly tied with the homotopy group:

Theorem 2.1.2: Conditions for Maps being Lifted

Let $p : (\tilde{X}, e_0) \to (X, x_0)$ be a covering map and $f : (Y, y_0) \to (X, x_0)$ where Y is path conected and locally path connected[a]. Then the lift $\tilde{f}$ exists if and only if $f_*(\pi_1(Y_0)) \subseteq p_*(\pi_1(\tilde{X}, e_0))$

[a] for each $y \in Y$, there exists an open neighbourhood that is path connected, think of a comb-space for a connected, not locally path connected space

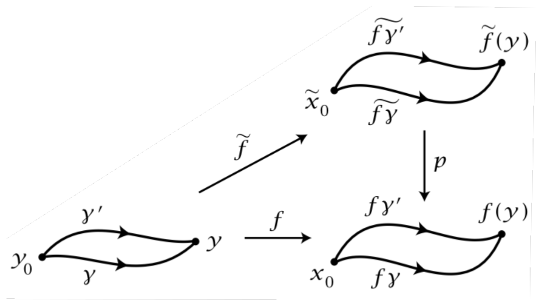
Proof **:**
If $\tilde{f}$ exists, then by functoriality $f_* = p_* \circ \tilde{f}_*$, hence $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, e_0))$.

Conversely, suppoe $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1, \tilde{X}, e_0)$. Choose $y \in Y$, and let γ be a path in Y from y_0 to y. Then $f\gamma$ is a path in X starting at x_0 and and going to $f(y_0)$. Then by lemma 2.1.5, $\tilde{f\gamma}$ lifts to a path in e_0.

Now, define

$$\tilde{f}(y) = \tilde{f\gamma}(1)$$

This map is well-defined: if γ' is antoher path from y_0 to y, then $h_0 = (f\gamma') * (\overline{f\gamma})$ is a loop at x_0. By assumtpion $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, e_0))$. We shall essentially be mimicking this visual:



Since $[h_0] \in p_*(\pi_1(\tilde{X}, e_0))$, there is a homotopy h_t of h_0 to a loop h_1 in Y that lifts to a loop $\tilde{h}_1$ in $\tilde{X}$ at basepoint e_0. The homotpoy h_t can be lifted to $\tilde{h}_t$ by the above lemma. Then since $\tilde{h}_1$ is a loop at e_0, so is $\tilde{h}_0$. By the uniqueness of the liftied paths, the first half of $\tilde{h}_0$ is $\tilde{f\gamma'}$ and the second half is $\tilde{f\gamma}$ traversed backwards, with comon midpoint $\tilde{f\gamma}(1) = \tilde{f\gamma'}(1)$. Hence, $\tilde{f}$ is well-defined.

What's left to show is that $\tilde{f}$ is continuous. Let $U \subseteq X$ be open neighbourhood of $f(y)$. Then choose $\tilde{U} \subseteq \tilde{X}$ containing $\tilde{f}(y)$ such that $p : \tilde{U} \to U$ is a homeomorphism. Choose a path-connected open enigh V of y such that $f(V)$
subseteqU. For paths from y_0 to poitns $y' \in V$, we can take a fixed path γ from y_0 to y followed by paths η in V from y to the poitns y'. Then the path $(f\gamma) * (f\eta)$ in X have lifts $(\tilde{f\gamma}) * (\tilde{f\eta})$ where $\tilde{f\eta} = p^{-1} f\eta$ and $p^{-1} : U \to \tilde{U}$ is the inverse of $p|_{\tilde{U}}$. Thus, $\tilde{f}(V) \subseteq \tilde{U}$, and $\tilde{f}|_V = p^{-1} f$, hence $\tilde{f}$ is continuous at any y, completing the proof.

Corollary 2.1.3: Unique Lifting Property

Let $p : \tilde{X} \to X$ be a covering map and $f : Y \to X$ be a continuous map. Then if $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ are two lifts of f that agree at one point Y and Y is connected, then $\tilde{f}_1$ an $\tilde{f}_2$ agree on all of Y, that is

$$\tilde{f}_1 = \tilde{f}_2$$

Proof **:**
Hatcher p. 62

As mentioned, the lifts are *not necessarily* loops. When projected using the covering space, they will become loops. The key is that all the paths in B at b_0 lift to paths that start at $e_0 \in p^{-1}(b_0)$ and end at $e_k \in p^{-1}(b_0)$. We may imagine the lifting correspondence as somehow "untangling" the paths. With path-homotopies being conserved with liftings and being homotopy-independent, we can now talk about fundamental groups when their lifted:

Definition 2.1.13: Lifting Correspondence

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be a lifting of f to E such that $\tilde{f}(0) = e_0$. Then we can define:

$$\varphi : \pi_1(B, b_0) \to p^{-1}(b_0) \qquad \varphi([f]) = \tilde{f}(1)$$

and φ is called the ***lifting correspondence*** derived from the corvering map p, e_0.

Note: it is important to choose a point e_0 since usually the covering space might have many locally homeomorphic neigbhourhoods, so you have to make a choice. The choice, however, doesn't really matter so much since there is a local neigbhourhood around e_0 will be homeomorphic to it's image.

Recall that the inverse of a point of a covering map is a discrete space, and hence we shall get a discrete group for this construction!

Proposition 2.1.4: Lifting Correspondence and Path-Connected Spaces

Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. If E is path-connected, then the lifting correspondence is surjective. If E is also simply-connected, then the lifting correspondence is bijective

Proof **:**

Assume E is path-connected and Let $e_1 \in p^{-1}(b_0)$. To show surjectivity, choose a path f_E in E such that $f_E(0) = e_0$, $f_E(1) = e_1$. Let $f = p \circ f_E$. Then f_E lifts f, and

$$f(0) = p(f_E(0)) = p(e_0) = b_0 \tag{2.1}$$
$$f(1) = p(f_E(1)) = p(e_1) = b_0 \tag{2.2}$$

Thus, $\varphi([f]) = f_E(1) = e_1$, and so φ is surjective, as we sougth to show.

Now assume E is simply connected and say $\varphi([f]) = \varphi([g])$ so that $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, the paths $\tilde{f} \simeq \tilde{g}$. Then we may take $p \circ \tilde{f}$ and $p \circ \tilde{g}$ which are certainy homotopic. Then certainly $p \circ \tilde{f} = f$ and $p \circ \tilde{g} = g$ are homotopic, but then $[f] = [g]$, showing injectivity.

What we've essentially proven is that we can deduce information of the fundamental group of a space given the covering map, and we can at least deduce all it's elements if the covering map is simply connected. At minimum, if we have a simply connected covering space, then the cardinality of $\pi_1(X, b_0)$ is equal to $p^{-1}(b_0)$!

Example 2.7: Lifts

1. Let $B = E = S^1$, and take $p : E \to B$ $z \mapsto z^2$. Notice that E is path-connected, hence the fundamental group surjects onto the pre-image of a point of the covering map. Let $b_0 = e_0 = (1, 0)$. Then $\varphi : \pi_1(S^1, b_0) \to p^{-1}(b_0) = \{(1, 0), (-1, 0)\}$.

   We'll exhibit this homomorphism explicitely. Take $e_{b_0} : [0, 1] \to S^1$, $e_{b_0}(t) = b_0 = (1, 0)$. Then the lifting correspondence of e_{b_0} is clearly $\tilde{e_{b_0}} = e_{e_0}$. Then:

   $$\varphi([e_{b_0}]) = e_{e_0}(1) = e_0 = (1, 0)$$

   now, let:

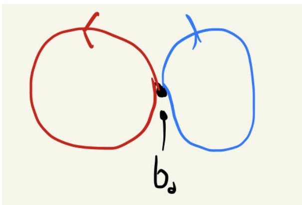
   $$f : I \to S^1, f(t) = (\cos(2\pi t)\sin(2\pi t))$$

   which is the funciton which will go around the circle once. Then the lifting correspondence is:

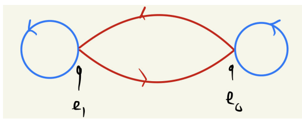
   $$\tilde{f} : I \to S^1, \tilde{f}(t) = (\cos(\pi t), \sin(\pi t))$$

   So $\tilde{f}(1) = (-1, 0)$ so $\varphi([f]) = (-1, 0)$ (which also verifies surjectivity). So for the first time, we showed that we have a *non-contractible loop* since the two homotopic classes are different! That is $[f] \neq [e_{b_0}]$

2. Let X be the figure 8 as we described in a pervious example (from the torus). Let the intersection of the circles be b_0. Take one loop going the left circle counter-clockwise, and one for the right circle, clockwise, so that $[f], [g] \in \pi_1(X, b_0)$. Let E represent the following lifting space:

   (a) Figure 8

   (b) Particular Lifting Space of Figure 8

   and take e_0 as in the image. Then

   $$\varphi([f]) = e_1$$
   $$\varphi([g]) = e_0$$
   $$\varphi([f] * [g]) = e_1$$
   $$\varphi([g] * [f]) = e_1$$

   This told us that $[f]$ and $[g]$ are different paths, but it seems that their product is the same. However, remember that φ is surjective, *not* injective, so φ might've only given us *partial* information on how the elements of this fundamental group interact. We can in fact work wtih a better covering space which will reveal more information.

3. Same X, same b_0, but this time, our lifting space E is different!

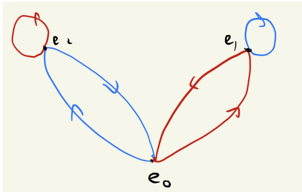


Figure 2.5: Different Lifting Space

Now,

$$\varphi([f] * [g]) = e_1 \tag{2.3}$$
$$\varphi([g] * [f]) = e_2 \tag{2.4}$$

Therefore, $[f] * [g] \neq [g] * [f]$, i.e. $\pi_1(X, b_0)$ is *not* abelian, as we claimed when $\pi_1(X, x_0)$ was introduced!!!

Proposition 2.1.5: Lifting Correspondence And Homomorphisms

Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. Then:

1. The induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective
2. Let $H = p_*(\pi_1(E, e_0))$. Then the lifting correspondence φ induces an injective map:
$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b)$$
where $\pi(B, b_0)/H$ is the collection of right cosets of H. The map is bijective if E is path connected
3. If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0

Proof **:**

1. Let $\tilde{h}$ be a loop in E at e_0 where $p_*([\tilde{h}])$ is the identity element. Since it's the identity, there exists a path homotopy F between $p \circ \tilde{h}$ and the constant loop. Then let $\tilde{F}$ be the lifting of F to E such that $\tilde{F}(0, 0) = e_0$. But then $\tilde{F}$ is the path homotopy between $\tilde{h}$ and the constant loop at e_0
2. Let f, g be loops in B based at b_0 with corresponding $\tilde{f}$ and $\tilde{g}$ in E that begin at e_0. Then $\varphi([f]) = \tilde{f}(1)$ and $\varphi([g]) = \tilde{g}(1)$. We will show that $\varphi([f]) = \varphi([g])$ if and only if $[f] \in H * [g]$. First, if $[f] \in H * [g]$, then $[f] = [h * g]$ where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Then the product $\tilde{h} * \tilde{g}$ is defined and is the lifting of $h * g$. Since $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$ which begin at e_0 must end at the same point in E. But then $\tilde{f}$ and $\tilde{g}$ end at the same point of E so $\varphi([f]) = \varphi([g])$.

   Conversely, suppose $\varphi([f]) = \varphi([g])$ so that $\tilde{f}$ and $\tilde{g}$ end at the same point in E. Then the product of $\tilde{f}$ and the reverse of $\tilde{g}$ is defiend, and it is a loop in E based at e_0. By directly computing, we get $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy in$E$ between $\tilde{H} * \tilde{g}$ and $\tilde{f}$, $p \circ \tilde{F}$ is a path homotopy in B between $h * g$ and f where $h = p \circ \tilde{h}$. But then $[f] \in H * [g]$, as we sought to show.

Furthermore, if E is path connected, then φ is surjective, meaning Φ is also surejctive.

3. Injectivity of Φ means $\varphi([f]) = \varphi([g])$ if and only if $[f] \in H * [g]$. Applying this to the constnt loops, we get that $\varphi([f]) = e_0$ if and only if $[f] \in H$. But $\varphi([f]) = e_0$ exactly whenthe lift of f that begins at e_0 ends at e_0.

Finally, we may present the fundamental group of the circle. For the long build-up of it, we'll make it a theorem

Theorem 2.1.3: Fundamental Groups Of The Circle

The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$

Proof **:**
Let $p : \mathbb{R} \to S^1$, and $p(t) = (\cos(2\pi t), \sin(2\pi t))$. Then p is a covering map such that $p(r) = p(s) \Leftrightarrow r - s \in \mathbb{Z}$. Let $b_0 = (1, 0) \in S^1$. Then $p^{-1}(b_0) = \mathbb{Z}$. Let our starting point in E be $e_0 = 0$. Then the lifting corresponding gives a map $\varphi : \pi_1(S^1, b_0) \to \mathbb{Z}$. Moreover, since $\mathbb{R}$ is simply connected, it follows that φ is a bijetion! So, it remains to show that φ is a homomorphism:

$$\varphi([f] * [g]) = \varphi([f]) + \varphi([g])$$

Let $\tilde{f}, \tilde{g}$ be lifts of f, g starting at 0, so $\varphi([f]) = \tilde{f}(1) \in \mathbb{Z} = p^{-1}(b_0)$. Define $\tilde{g}_2(t) = \tilde{g}(t) + \tilde{f}(1)$. Then

$$p(\tilde{g}_2(t)) = p(\tilde{g}(t) + \tilde{f}(1)) = p(\tilde{g}(t) + k) = p(\tilde{g}(t)) = g(t)$$

since $k \in \mathbb{Z}$, so the covering map projects this path onto g, meaning it is also a lift of g, just shifted over by $f(t)$ to where the path starts. We can see this by $\tilde{g}_2(0) = \tilde{f}(1)$. Thus, we can combine these two paths, $\tilde{f} * \tilde{g}_2$, and by what we've just shown, this is a lifts of $f * g$. Thus

$$\begin{aligned} \varphi([f] * [g]) &= (\tilde{f} * \tilde{g}_2))(1) \\ &= \tilde{g}_2(1) \\ &= \tilde{g}(1) + \tilde{f}(1) = \varphi([f]) + \varphi([g]) \end{aligned}$$

completeing the proof.

This also means that the fundamental group of the circle is abelian, which should makes sense if you try and draw loops on a circle. You will get back and forth and will be able to reverse the order no problem. Trying this on the figure 8, you might see how you can get stuck. We just formalized such intuition!

2.1.3 Fundamental Group Invariant Under Homotopy

If f is homeomorphic, then f_* is isomorphic. However, this is a rather rigid condition; it turns out that if $f \simeq g$, then $f_* = g_*$, showing that the fundamental group is invariant under the much looser condition of homotopy. If $f_t : (X, x_0) \to (Y, y_0)$ is a homotopy such that $f_t(x_0) = y_0$, we shall call it a basepoint-preserving homotopy.

Proposition 2.1.6: Invariance Of Induced Homomorphisms Under Homotopy

Let $f_t : X \to Y$ be a homotopy. Then $(f_0)_* = (f_1)_*$

Proof **:**
If $f_0 \simeq f_1$, then $[f_0] = [f_1]$, and $[f_0 \circ \varphi] = [f_1 \circ \varphi]$ for some loop φ (at any basepoint). Then $(f_0)_*([\varphi]) = [f_0 \circ \varphi] = [f_1 \circ \varphi] = (f_1)_*([\varphi])$

When working with homotopy equivalence, we may in fact drop the basepoint information. We first require the following 2 lemmas:

Lemma 2.1.6: Induced Maps, Retractions, and Deformation Retraction

Let A be a retract of X. Then for $a_0 \in A$, the induced map $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective, where $\iota : A \to X$ is the inclusion. If A is a deformation retraction of X, then ι_* is also surective, hence an isomorphism.

This means that X should have *at least* as many elements in it's fundamental group as A

Proof **:**
Let $r : X \to A$ be a retraction. Then $f \circ \iota = \mathrm{id}_A$. Thus, $(r \circ \iota)_* = (id_A)_* = id|_{\pi_1(A,a_0)}$. Thus

$$[f] = (r_* \circ \iota_*)([f]) = r_*(\iota_*([f]))$$

showing r_* is a left inverse, thus ι_* is injective.

If $r_t : X \to X$ is a deformation retraction of X onto A so that $r_0 = \mathrm{id}$ and $r_t|_A = \mathrm{id}$ and $r_1(X) \subseteq A$, then for any loop $f : I \to X$ based at $x_0 \in A$, the composition $r_t \circ f$ gives a homotopy of f to a loop in A, showing ι_* is surjective, and hence an isomorphism.

The group-theoretic equivalent statement is that a homomorpism $\varphi : G \to H$ restricts to the identity on H. If H is a normal subgroup, Then G is the direct product of H and $\ker(\varphi)$, and if H is not normal then we replace direct product with semi-direct product.

Lemma 2.1.7: Basepoint Path In Homotopy

Let $\varphi_t : X \to Y$ be a homotopy and h the path $\varphi_t(x_0)$ formed by the images of the basepoint $x_0 \in X$. Then there exists a β_h such that the following diagram commutes:

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

Proof **:**
Let $h_t(s) = h(ts)$ so that h_t is the restriction of h to the interval $[0, t]$ and reparamaterized so that the dmoain is still $[0, 1]$. Let $\overline{h_t}$ be the inverse path. Then if f is a loop pin X at the basepoint x_0, the product $h_t \circ (\varphi_t \circ f) \circ \overline{h_t}$ gives a homotopy of loops at $\varphi_0(x_0)$:

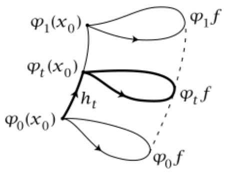


Figure 2.6: Visualization

Restricting this homotopy to $t = 0$ and $t = 1$, we get

$$\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$$

completing the proof

Theorem 2.1.4: homotopy Equivalence, Then Basepoint Invariant

Let $\varphi_t : X \to Y$ be a homotopy equivalent. Then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$

Proof **:**
We want to show that φ_* is an isomorphiam. Let $\Psi : Y \to X$ be the homotopy-inverse of φ so that $\varphi \circ \psi \simeq \text{id}$ an $\psi \circ \varphi \simeq \text{id}$. We shall force the necessary condition on φ_* via the following diagram:

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi \circ \varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

The composition of the first two maps is an isomorphism since $\psi \circ \varphi \simeq \text{id}$ implying $\psi_* \circ \varphi_* = \beta_h$ for some h. In particular, since $\psi_* \circ \varphi_*$ is an isomorphism, φ_* is injective. Similarly, the second and third maps shows that ψ_* is injective. Thus, the first two of the three maps are injection and their composition is an isomorphism, hecne φ_* must be surjective as well, completing the proof.

(A word leading for this example)

Example 2.8: Fundamental Group Does Not Characterize Spaces
Define the *pseudocircle* to be the set $X = \{a, b, c, d\}$ with the topology:

$$\{\{a, b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b\}, \{a\}, \{b\}, \emptyset\}$$

which can be thought of as the topology induced by the partial oarder $a < c$, $b < c$, $a < d$, $b < d$ where the open sets aer the donward-closed sets. Then we may define a continuous map

$f : S^1 \to X$ by

$$f(x,y) = \begin{cases} a & x < 0 \\ b & x > 0 \\ c & (x,y) = (0,1) \\ d & (x,y) = (0,-1) \end{cases}$$

Then we shall not show this now, but this is a *weak homotopy equivalence*, which we shall show later implies f induces an isomorphism on all homotopy groups! We shall later in this book introduce other concepts such as the singular homology and cohomology, and f will induce an isomorphism between these structures too! See this link for more details:

`https://en.wikipedia.org/wiki/Pseudocircle`

2.1.4 Interesting Consequences

Theorem 2.1.5: Can't make a whole

Let B^2 be the unit disc in $\mathbb{R}^2$. There is no retraction form B^2 to S^1. You can think of this as you cannot make a whole.

Proof **:**
The fundamental group of the circle is infinite, but the fundamental group of B^2 has 1 element (because its star-convex). Since the fundamental group of S_1 would need to injectively map to B^2, this implies there is *no* retraction!

Theorem 2.1.6: Brouwer Fixed Point Theorem

if $f : B^2 \to B^2$ is continuous, then there exists $x \in B^2$ such that $f(x) = x$

Proof **:**
Recall that $B^2 := \left\{(x,y) | x^2 + y^2 \le 1\right\}$ and thus is $\approx I^2$.

Assume $f : B^2 \to B^2$ is continuous. For the sake of contradiction, let's say that for every $x \in B^2$, $f(x) \neq x$. We will use f to construct a retraction of B^2 to its boundary S^1.

The key idea is that since $f(x) \neq x$, and both are in B^2, we can draw a straight line between x and $f(x)$. We can continue this line to the boundary. If we give a direction (let's say from $f(x)$ to x), we can then make a function $g(x)$ which will take our point x, and given this line, it will bring it to the boundary.

The key idea here is that if we had that $f(x) = x$, then we woouldn't know what ray to draw. for example, if $f(x) = \frac{x}{\|x\|}$ then every point will have a clear ray except $x = 0$. This unclearness of the ray is translated as f being discontinuous (in the case of the concrete f, we divided by 0).

More formally, we'll define $G : B^2 \times \mathbb{R}_{\ge 0} \to \mathbb{R}^2$.

$$G(x,r) = x + r(x - f(x))$$

since Everything is in $\mathbb{R}^2$, mutiplying, adding, subtracting is allowed. Note that

$$\|G(x,r)\|^2 = \|x\|^2 + r^2 \|x - f(x)\|^2 + 2rx \cdot (x - f(x))$$

so for $\|G(x,r)\| = 1$, we use the quadratic formula to solve r:

$$r = \frac{-2x \cdot (x - f(x)) + \sqrt{4(x \cdot (x - f(x)))^2 + 4(1 - \|x\|^2) + \|x - f(x)\|^2}}{2\|x - f(x)\|^2}$$

This formula is complicated, but it's okay: everything is positive, so everyhing is okay (since f has no fixed points). We can make r into a formula which will make sure that $G(x,r) = 1$ by calling the right hand side $r(x)$.

Thus, we can define

$$g(x) = G(x, r(x)) \in S^1$$

hence, $g(x)$ is continuous. Note that one the circle, $g(x) = x$, showing that it's a retraction. But we showed there can't be a retraction between the ball and the circle – a contradiction!

The following uses the fact that $\pi_1(S^1) = \mathbb{Z}$

Theorem 2.1.7: Fundamental Theorem Of Algebra

let $p(z)$ be a nonconstnat polynomial with coefficients in $\mathbb{C}$. Then $p(z)$ has a root in $\mathbb{C}$

***Proof* :**

Without loss of generailty let $p(z)$ be monic (we may scale the result at the end). Assume that $p(z)$ has no roots in $\mathbb{C}$. Then for each $r \in \mathbb{R}_{\geq 0}$, define

$$f_r(s) : S^1 \to S^1 \qquad f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

which is a loop based at 1. As we let r vary, we get loops f_r that are homotopic based at 1. Since f_0 is the trivial loop, we get that $[f_r] \in \pi_1(S^1)$ is the nullhomotopic element.

Now, fix r larger enough so that it's bigger than $\max |a_1| + \cdots |a_n|, 1$. Then for $|z| = r$, we get

$$|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1} > |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|$$

Thus, it follows that

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on $|z| = r$ and $0 \leq t \leq 1$. Now, replacing p by p_t in the formula of f_r and letting $t \to 0$ we get

$$f_{r,0}(s) = \omega_n(s) = e^{2\pi i n s}$$

Now, we know that $[w_n] \in \pi_1(S^1)$ maps to n in the isomorphism presented in the proof of the fundametnal group of S^1. Thus, we have:

$$0 = \varphi([f_r]) = \varphi([w_n]) = n$$

and hence $n = 0$, showing that only constant functions don't have roots.

Theorem 2.1.8: Burusk-Ulam Theorem In 2 Dimensions

For every continuous map $f : S^2 \to \mathbb{R}^2$, there exists a pair of antipodal points $x, -x$ such that $f(x) = f(-x)$

An interesting consequence of this is that S^2 is not homeomorphic to a subspace of $\mathbb{R}^2$, an intuitive but not obvious fact to prove directly

Proof **:**
For the sake of contradiction, let's say there is a continuous map $f : S^2 \to \mathbb{R}^2$ with no equal antipodal images. Then define the map $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

Now, imagining the sphere S^2 embeded in $\mathbb{R}^3$, $S^2 \subseteq \mathbb{R}^3$ centered at the origin, take η to be the loop given by the equator

$$\eta(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

Let $h = g \circ \eta$. Then, $g(-x) = -g(x)$, so

$$h(s + 1/2) = -h(s)$$

for all $s \in [0, 1/2]$. Now, $h : I \to S^1$ can be lifted to a path $\tilde{h} : I \to \mathbb{R}$. The equation $h(s + 1/2) = -h(s)$ then implies

$$\tilde{h}(s + 1/2) = \tilde{h}(s) + q/s \qquad q \in 2\mathbb{Z} + 1$$

This q is independed of s, since by solving the equation $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$, we see that q deepends continuosly on $s \in [0, 1/2]$, so q must be a constant since it is constrained to integer values. In particular

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q$$

Thus, h represents q times the generator of $\pi_1(S^1)$. Since q is odd, we conclude that h is *not* nullhomotopic. But h is the composition $g \circ \eta : I \to S^2 \to S^1$, and η is certainly nullhomotopic in S^2, so $g \circ \eta$ is nullhomotopic in S^1 by composing nullhomotopy of η with g - a contradiction. Thus, it cannot be that $f(x) \neq f(-x)$ forall x, as we sought to show.

Corollary 2.1.4: Union Of 3 Sets of Sphere contains Antipodal Points

Whenever S^2 is expressed as the union of three closed sets A_1, A_2, A_3, then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$.

Proof **:**
Let $d_i : S^2 \to \mathbb{R}$, $d_i(x) = \inf_{y \in A_i} |x - y|$ be the distance from x to A_i. This is certainly a continuous function, so we may apply the Burusk-Ulam theorem to

$$S^2 \to \mathbb{R}^2 x \mapsto (d_1(x), d_2(x))$$

namely, we can find a set of points such that $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$. If either of these two distances is zero, then $x, -x$ both lie in the set A_1 or A_2, since these are closed sets. On the other hand, if the distance from x and $-x$ to A_1 and A_2 are both strictly positive, then x and $-x$ must lie in neither of these sets, but hten it must lie in A_3.

It can be verified that this theorem fails when considering 4 sets (can you come up with a counter example?) Given the higher-dimensional analogue of Burusk-Ulamns theorem, the same theorem works for S^n beign covered by $n+1$ sets (while being able to cover it with $n+2$ sets non containing antipodal points). The above corollary even works for S^1: if it is covered by two closed sets, then one must contain antipodal points (closedenss is required, since S^1 can be covered by two half-open sets)

(Some interesting results I've ommited (mainly p. 349 Munkres): extending S^1 to B^2 map, vector field results, A 3×3 positive real matrix, then has positive real eigen value)

(A word on the complexity of coveing spaces of $\pi_n(S^m)$). The homotopy structure also is dependent on information beyond paths, for example $\pi_n(S^2)$ is nonzero for infinitely many values of n.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

Figure 2.7: known Higher order Homotopy groups of n-spheres

One last result that shall be stated, but certainly not proven due to it's complexity, is the characterization of all simply connected closed 3-manifolds. The simplest such manifold would be S^3. Up to homeoomorphism, are there any other? This was known as *poincare conjecture*, and it was proven that this was *not* the case; all closed simply connected 3-manifolds are homeomorphic to S^3.

2.2 Products, Fibred Products, and Van Kampen's Theorem

We have covered what happens if we have a continuous map, homeomorphism, and homotopy equivalence between two topological spaces. We shall now look at some usual constructions of topological spaces and see how it effects the fundamental group. Most of the products we have seen will require a bit more theory, namely the *van-kanmpen theorem*.

Proposition 2.2.1: Fundamental Group And Product Spaces

Let X, Y be path connected spaces. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

***Proof* :**
This follows immediately from the universal property of products: if $f : [0,1] \to X \times Y$ is a loop, then there exists continuous maps g, h such that $f(t) = (g(t), h(t))$. These are certainly loops in X and Y respectively, and similarly a homotopy between two paths f, f' induces a homotopy between g, g' and h, h'. Furthermore, any two paths g, h can be combined to form a path (g, h) in $X \times Y$. Hence, we have a bijection:

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Showing that this is a homomorphism is immediate.

Example 2.9: Fundamental Group Of Torus
Since $T^2 = S^1 \times S^1$, $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. In particular, any loop on T^2 is a loop around the minor axis combined with a loop from the major axis. Interestingly, these loops may knot:

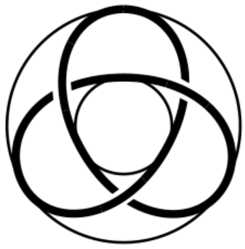


Figure 2.8: Torus Loop Knot

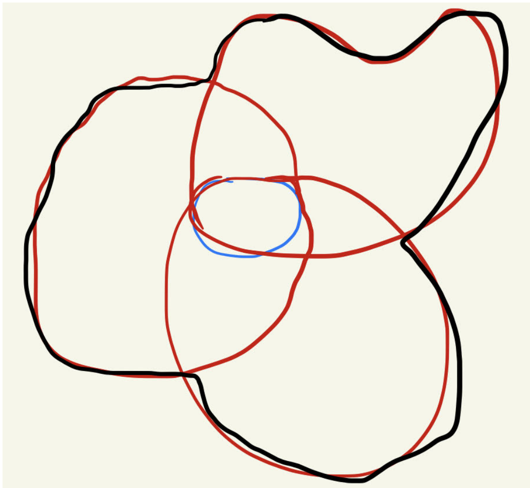
Giving us some interesting non-trivial paths.

A slight generalization of the above proposition is the following:

Proposition 2.2.2: Decomposing Spaces to find Homotopy

Let X be a topological such that it is the union of a collection of path-connected open sets A_α each containing the baspoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

To visualize:



and a loop f at x_0 is homotopic to $f \simeq (f_1 * f_2 * \cdots * f_n)$ where each $f_i \in A_i$, hence:

$$[f] = [f_1] * [f_2] * \cdots * [f_n]$$

breaking up the problem into one of finding the fundamental group each A_i.

Proof **:**
Let $f : I \to X$ be a loop with basepoint x_0. First, we may partition I so that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α, namely since f is continuous each $s \in I$ has an open neighbourhood $V_s \subseteq I$ that gets mapped to A_α by f; we may take these intervals to be sufficinetly small so that the closure is in A_α. Then by compactness finitely many V_s cover I, giving us our desried partition.

Let $f_i = f|_{[i-1,i]}$ and A_i be the A_α containing the iamge of f_i. Then $f = f_1 * f_2 * \cdots f_n$. Since $A_i \cap A_{i+1}$ is path connected for each i, there exists a g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i) \in A_i \cap A_{i+1}$. Then

$$(f_1 * \overline{g}_1) * (g_1 * f_1 * \overline{g}_2) * \cdots * (g_{n-1} * f_1)$$

which is certainly homotopic to f, and is a the desired composition, as we sought to show.

Note that we have not given how to find the homotopy group of this space (this shall be done using van-Kampens theorem). In the case where each component is simply connected, we get the following application:

Example 2.10: Fundamental Group of n-sphere
Write S^n as the union of A_1 and A_2 each homeomorphic to $\mathbb{R}^n$ and $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ (take A_1, A_2 to be the compliments of the two anitpodal points in S^n). Then choosing a basepoint $x_0 \in A_1 \cap A_2$, if $n \geq 2$ then $A_1 \cap A_2$ is path onnected and so by the lemma every loop in S^n based at x_0 is homotopic to a product of loops in A_1 or A_2. Both $\pi_1(A_1)$ and $\pi_1(A_2)$ are zero, and hence S^n is nullhomotopic for $n \geq 2$.

An important consequence that is usually rather difficult to prove classicaly becomes much easie

Corollary 2.2.1: Plane Not Homeomorphic To $\mathbb{R}^n$

$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$

***Proof* :**
The case of $n = 1$ is quickly disposted of since $\mathbb{R}^2 - \{0\}$ is path connectd, while $\mathbb{R} - \{0\}$ is not. When $n > 2$, then $\mathbb{R}^n - \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, hence by proposition 2.2.1

$$\pi_1(\mathbb{R}^n - \{x\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \cong \pi_1(S^{n-1})$$

hence, if $n = 2$ we get $\pi_1(\mathbb{R}^n - \{x\}) = 2$ and for $n > 2$ we get id. Then it is a simple application of proposition 2.2.2

For the $\mathbb{R}^m \ncong \mathbb{R}^n$ if $n \neq m$, we can use higher homotopy groups, but there will be an easier proof of this using homology in the next chapter.

Proposition 2.2.2 gave us the paths of a space that can be decomposed into spaces A_α whose intersection is path-connected. However, we did not present what type of group these path can produce, besides some trivial cases. This next theorem expands on the important result to gives us the group structure. It should be remarked that the result is entirely straightfroward, and is simply the application of the existence of pushouts and the functoriality discussed earlier:

Theorem 2.2.1: Van Kampen's Theorem

Let X be a topological space with is the union of two open and path connected subspaces U_1, U_2. Suppose that $U_1 \cap U_2$ is path connected and nonemtpy, and let $x_0 \in U_1 \cap U_2$. Then the push-out of

$$\pi_1(U, x_0) \leftarrow \pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$$

exists, namely the following diagram commutes:

$$\begin{tikzcd}
& \pi_1(U_1 \cap U_2) \arrow[dl, "\iota_{1*}"'] \arrow[dr, "\iota_{2*}"] & \\
\pi(U_1) \arrow[dr, dashed] \arrow[ddr, bend right, "j_{1*}"'] & & \pi(U_1) \arrow[dl, dashed] \arrow[ddl, bend left, "j_{1*}"] \\
& \pi(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \arrow[d, "k"] & \\
& \pi_1(X) &
\end{tikzcd}$$

Before proving, it is useful to formula the theorem more concretely for computational pruposes and over arbitrarily many components:

> Let X be the uion of path-connected open sets A_α each containing $x_0 \in X$ and each $A_\alpha \cap A_\beta$ is path-connected. Then the homomorphism $\varphi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective where $*_\alpha \pi_1(A_\alpha)$ is the free product of the groups. If furthermore $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then $\ker \varphi$ is generated by the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for

$\omega \in \pi_1(A_\alpha \cap \beta)$ and hence φ induces the isomorphism $*_\alpha \pi_1(A_\alpha)/N$.

In terms of presentations, if

$$\pi_1(U, x_0) = \langle u_1, u_2, ..., u_k \big| \alpha_1, \alpha_2, ..., \alpha_l \rangle$$
$$\pi_1(V x_0) = \langle v_1, v_2, ..., v_m \mid \beta_1, \beta_2, \cdots, \beta_n \rangle$$
$$\pi_1(U \cap V, x_0) = \langle w_1, w_2, ..., w_p \mid \gamma_1, \gamma_2, ..., \gamma_q \rangle$$

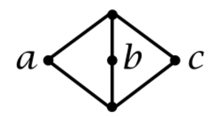
and

$$I : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$$
$$J : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$$

then if $A = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ and $B = \{\beta_1, \beta_2, ..., \beta_n\}$:

$$\boxed{\pi_1(X, x_0) = \langle u_1, u_2, ..., u_k, v_1, v_2, ..., v_m \mid A, B, I(w_1)J(w_1)^{-1}, ..., I(w_p)J(w_p)^{-1} \rangle}$$

where $I(w_1)J(w_1)^{-1}$ can be thought of as $I(w_1) = J(w_1)$. Note the intersection being path connected is a necesary condition: taking two arcs that cover S^1 shows that $\varphi : *_\alpha \pi_1(A_\alpha) \to \pi(S^1)$ is not surjective. Similarly for $A_\alpha \cap A_\beta \cap A_\alpha$ consider the suspension of 3 points which looks like



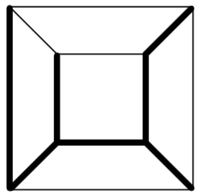
Then if $A_\alpha, A_\beta, A_\gamma$ is the compliment of the graph with respect to the points a, b, c, then A_α, A_β respects the conditions of Van-Kampens theorem and so $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$. However, the triple intersection is not path connected. Erroneously applying the theorem, we would get $\pi_1(X) \cong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}$, which is not isomorphic to $\mathbb{Z} * \mathbb{Z}$ (one has three disjoint infinite cyclic subgroups, the other does not).

***Proof* :**
Hatcher p. 44 (long, worth at some point adding but not critical)

Example 2.11: Use Of Van-Kampen

1. The simplest example would be a wedge-sum. Then if $X = \bigvee_\alpha X_\alpha$ all intersecting at a single point, then $*_\alpha \pi_1(X_\alpha) \cong \pi_1(X)$.

2. Let X be the graph:



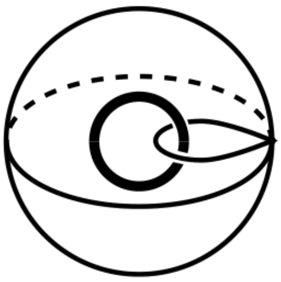
and T be the heavily shaded regions (T is a maximally spanning tree). Notice that T is contractible, and that there are 5 non-shaded edges. I claim that $\pi_1(X) = *_i^5\mathbb{Z}$. We may deduce this using the Van-Kampen Theorem. For each edge $e_\alpha \in X - T$, choose an open neighbourhood of $T \cup e_\alpha$ in X that retracts onto $T \cup e_\alpha$; call it A_α. Then the intersecton $A_\alpha \cap A_\beta$ is T, which is contractible. an path connected. Hence, we get

$$\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)$$

Since each A_α deformation retracts onto a circle, we see that we get that $\pi_1(X)$ is the free product $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

3. Let $X = \mathbb{R}^3 - S^1$. First, notice that this deformation retracts to the shape:

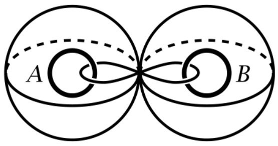
Then one can imagine that we modify this deformation retraction so that we instead have $S^1 \vee S^2$ like so:



Then since we have a deformation retraion, we have that

$$\pi_1(X) \cong \pi_1(S^1 \vee S^2) \cong \mathbb{Z}$$

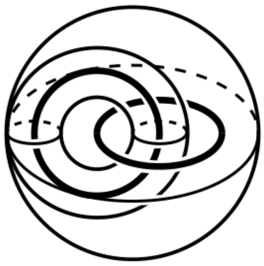
Since $\pi_1(S^2) \cong 0$. Similarly, if $X = \mathbb{R}^3 - (A \cup B)$ where A, B are two unliked circles, then we may deformation retrace to $S^1 \vee S^1 \vee S^2 \vee S^2$ to look like so:



We thus, get that

$$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$$

One the otherhand, if the two circles where linked, then we get a deformation retraction to $S^2 \vee (S^1 \times S^1)$:



which gives us:

$$\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$$

As we see, the compliment of a shape can give us some "knotting" information; this is the first steps into knot theory and knot invariances.

4. (torus knots) For each $m, n \in \mathbb{Z}$ where $\gcd(n, m) = 1$, define $K_{m,n} \subseteq \mathbb{R}^3$ to be the iamge of $f : S^1 \to S^1 \times S^1 \subseteq \mathbb{R}^3$ given by

$$f(z) = (z^m, z^n)$$

The knot $K_{m,n}$ winds around the torus m times in the longitudinal direction and n times in the meridional direction; the relatively prime criterion is to insure f is injective. We shall compute $\pi_1(\mathbb{R}^3 - K_{m,n})$, or equivanently (due to van-kampen) we may take the 1-point compactification and consider $\pi(S^3 - K_{m,n})$.

(quite interesting, Hatcher p. 47, the answer is $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$)

5. (Hawaian earings) Let $X \subseteq \mathbb{R}^2$ be the union of circles C_n of radius $1/n$ with center $(1/n, 0)$

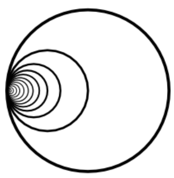


Figure 2.9: Hawaian Earings

We may think that the fundamental group of this space is an infinite free product of $\mathbb{Z}$, but it is in fact much more complicated. (Hatcher p.48)

An important application of Van-Kampens theorem is when used in tandem with cell-complexes, in particular 2 complexes for they "add relations" by adding homotopies. Let X be a 1-skeleton that we want to attach a collection of 2 cells e_α via the maps $\varphi_\alpha : S^1 \to X$; let the resulting space be labeled Y. These maps define a loop in X. If $s_0 \in S^1$ is a basepoint, then $\varphi_\alpha(s_0)$ is the baspoint of the image of φ_α. For varying α, the basepoints may differ, but we may at $\varphi_\alpha(s_\alpha) \in X$ take γ_α that makes each loop a loop at x_0: $\tilde{\gamma}_\alpha \varphi_\alpha \gamma_\alpha$. These loops may not be nullhomotopic, but after attaching the 2 cells, they will become nullhomotopic. This means that the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $X \hookrightarrow Y$ contains all $\tilde{\gamma}_\alpha \varphi_\alpha \gamma_\alpha$[3]. Since the kernel represents the added relation, we see that how we attach the cells represent new relations added to Y. The following proposition shows that the kernel consists entirely of paths as we just outlined them, and hence we exactly choose the group we get when working with 2-cells:

Proposition 2.2.3: Van-Kampen and Cells

1. If Y is obtained from X by attaching 2-cells as described above, then the inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N, hence
$$\pi_1(Y) = \pi_1(X)/N$$
2. If Y is obtained from X by attaching n-cells for fixed $n > 0$, then the inclusion $X \hookrightarrow Y$ induces an isomorphism
$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)$$
3. For a path-connected cell complex X, the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorpshim $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$

This essentially tells us that the fundamental group of a CW complex is determine by how we attach 2-cells!

***Proof* :**
Hatcher p.50

Example 2.12: Fundamental Group Of M_g

Recall in example 1.7(5) that we took the cell complex M_g constructed with one 0-cell, $2g$ 1-cells an one 2-cell. Note that $M_1 = T$ is the torus. We get that the 1-skeleton is the wedge sum of $2g$ circles, giving a fundamental group being the free-product of $2g$ copies of $\mathbb{Z}$. Then the 2-cell attached long the loops gives the commutators, namely $[a_1, b_1], ..., [a_g, b_g]$. Hence:

$$\pi_1(M_g) \cong \langle a_1, b_1, ..., a_g, b_g \bigm| [a_1, b_1], ..., [a_g, b_g] \rangle$$

Using this, we [finally] see that $M_g \not\simeq M_h$ if $g \neq h$ (hence $M_g \ncong M_h$). Since $\pi_1(M_g) \cong \pi_1(M_h)$ if and only if the number of copies aligned (finitely generated free $\mathbb{Z}$-modules have an invariant basis number).

[3] Note that it's path-independent: if η_α was a different path, then $\eta_\alpha \varphi_\alpha \tilde{\eta}_\alpha = (\eta_\alpha \tilde{\gamma}_\alpha)\gamma_\alpha \varphi_a \tilde{\gamma}_\alpha(\gamma_\alpha \tilde{\eta}_\alpha)$

Corollary 2.2.2: Every Group Appears As A Fundamental Group

Let G be a group. Then there exists a 2-dimensional cell complex X_G such that

$$\pi_1(X_G) \cong G$$

Proof **:**

Let $G = \langle g_\alpha \mid r_\beta \rangle$ where g_α are the generators and r_β are the relations. Then take skeleton to be $\bigvee_\alpha S^1$ and attach the 2-cells e^2_β specified by the words r_β.

Example 2.13: More Fundamental Groups

1. (non-orientable surfaces) Let N_g be the surface where we attach 2-cells to the wedge-sum of g circles by the words $a_1^2, ..., a_g^2$. Note that $N_1 = \mathbb{R}\mathrm{P}^2$ since it is D^2 with the antipodal points of ∂D^2 identified, and N_2 is the Klein bottle (the left image is the usual construciton of the Klein bottle):

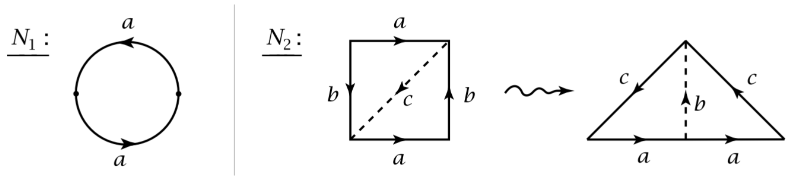


Figure 2.10: $\mathbb{R}\mathrm{P}^2$ and Klein Bottle

Then:

$$\pi_1(N_g) = \langle a_1, ..., a_g \mid a_1^2, ..., a_g^2 \rangle$$

Note that we also can slightly change how we glue the gth 2-cell to get an abelian group $(\mathbb{Z}/2\mathbb{Z})^n$ by choosing generators $a_1, a_2, ..., a_{g-1}, (a_1 + \cdots + a_g)$ with $2(a_1 + \cdots + a_g) = 0$.

Overalll, N_g is not homoopy equivalent to N_h if $g \neq h$, and is not homotopy equivalence to any M_g or M_h.

2. let $G = \langle a \mid a^n \rangle = \mathbb{Z}_n$. Then X_G is S^1 with the 2-cell e^2 attached via the map $z \mapsto z^n$. If $n = 2$, then $X_G = \mathbb{R}\mathrm{P}^2$. Note that when $n \geq 3$, then X_G is not a surface (a smooth 2-manifold). For example, if $n = 3$, then we can have a neighbourhood N of X_G that looks like the product of the tripod (inverse of a triangle, or a triangle in CAT(0)) with the interval I (the tripod can also be thought of as a vertex with three intervals attached to it). Similar results happen for $n \geq 4$. The resulting shape is like a torus with a $1/n$ twist at some point (for $n = 3$, it's a $1/3$ twist):

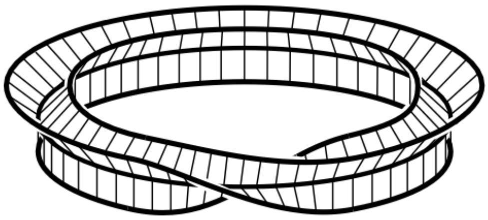


Figure 2.11: visual of X_3

2.3 Classification of Covering Spaces

We finish off this section by classifying covering spaces. let X be a path-connected, locally path-connected space. We shall show there is a similar theory between the varying cover-spaces and galois theory. In particular, every covering map shall be associated with a subgroup $\pi_1(X, x_0)$ (for any given basepoint $x_0 \in X$), where the partial order given by the covering maps is inverse to the ordering of the groups, that is we have a contra-variant functor between the partial orders (giving the galois correspondence).

Recall that if $p : E \to B$ is a covering space where E is simply connectd, then $\pi_1(B, x_0)$ is in bijective correspondence with $p^{-1}(x_0)$. Are there condition conditions on B for which there exist a covering map where E is simply connected? It turns out there is; this type of space a name:

Definition 2.3.1: Semilocally Simply-Connected

Let X be a topological space. Then if for each $x \in X$, there must exist a neighbourhood $U \subseteq X$ such that the map induced by the inclusion map $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial, X is called *semilocally simply-connected.*

To see why this is a necesary condition, suppose $p : \tilde{X} \to X$ is a covering space with $\tilde{X}$ simply connecd. Then every point $x \in X$ has a neighbourhood U having a lift $\tilde{U} \subseteq \tilde{X}$ projecting homeomrphically to U by p. Each loop in U lifts to a loop in $\tilde{U}$, and the lifted loop is nullhomotopic in $\tilde{X}$ since $\tilde{X}$ is simply connected. Hence, composing this nullhomotopy with p, the original loop in U is nullhomotopic in X.

All locally simply-conneced spaces are semilocally simply-connected. As an exmaple, CW complexes are locally contractible (appendix of hatcher), and hence shall all have universal covers.

Example 2.14: Not Semilocally Connected

Tkae $X \subseteq \mathbb{R}^2$ consisting of circles of radius $1/n$ centered at $(1/n, 0)$ (i.e. the hawaian earings). On the other hand, the cone $CX = (X \times I)/(X \times \{0\})$ on the shrinking wedge of circles is semilocally simply-connected since it is contractible, but is not locally simply-connected

Lemma 2.3.1: Universal Covering Map

let X be a path connected, locally path connectd, semilocally path connected space. Then there exists a simply connected space $\tilde{X}$ such that $p : \tilde{X} \to X$ is a covering map. Furthermore, this cover is unqiue up to isomorphism, and any other simply connected covering map commutes with $\tilde{X}$

***Proof* :**
Let X be a path connected, locally path connected, semilocally path connected space. Define the space

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

Since X is path-connected, the choice of x_0 is irrelevant. Define $p : \tilde{X} \to X$, $[g] \mapsto \gamma(1)$, which is well-defined since all paths in the homotopy end at the same point. Since X is path-connected, each point $x \in X$ is the endpoint of a path starting at x_0, and hence p is surjective.

We next define a topology on $\tilde{X}$. Let

$$\mathcal{U} = \{U \subseteq X \mid \pi_1(U) \to \pi_1(X) \text{ is trivial}\}$$

Note that is the above map is independent of base point since X is locally path connected (hence U is path-connected). Then since X is semilocally simply-connected, $\mathcal{U}$ forms a basis for X. Given $U \in \mathcal{U}$, and γ a path in X from x_0 to a point in U, elt

$$U[\gamma] = \{[\gamma * \eta] \mid \eta \text{ is a loop in } U\}$$

Clearly, $U_{[\gamma]}$ depends only on the homotopy clas of γ, justifying the notation. The map $p : U_{[\gamma]} \to U$ is surjective since U is path-connectd, and injective since different chocies of η joining $\gamma(1)$ to a fixed $x \in U$ are all homotopic in X since $\pi_1(U) \to \pi_1(X)$ is trivial.

Next, if $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\gamma']}$. For, if $\gamma' = \gamma * \eta$, then elements of $U_{[\gamma']}$ have the form $[\gamma * \eta * \mu]$, and hecne are in $U_{[\gamma]}$, while conversely elements of $U_{\gamma]}$ ahve hte form:'

$$[\gamma * \mu] = [\gamma * \eta * \overline{\eta} * \mu] = [\gamma' * \overline{\eta} * \mu]$$

and hence aer in $U_{[\gamma']}$. Thus, if $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we get $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. So if $W \subseteq U \cap V$, and contains $\gamma''(1)$, then $[\gamma''] \in W_{[\gamma'']}$ and

$$W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$$

showing that the collection of $U_{[\gamma]}$ forms a basis for a topology on $\tilde{X}$. From this, the bijection $p : U_{[\gamma]} \to U$ is a homeomorphism since it gives a bijection between the subsets $V_{[\gamma']} \subseteq U_{[\gamma]}$, and the sets $V \in \mathcal{U}$ contained in U. This then shows that $p : \tilde{X} \to X$ is continuous. It is also clear that $p : \tilde{X} \to X$ is a covering map. What remains to show si that $\tilde{X}$ is path-connected. For any point $[\gamma] \in \tilde{X}$, let γ_t be the path in X that equals γ on $[0, t]$ and is constant at $\gamma(t)$ on $[t, 1]$. Then the function $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ lifting γ that starts at $[x_0]$ (the homotopy class of constants paths at x_0) and ends at $[\gamma]$. Since $[\gamma]$ was an arbitrary point in $\tilde{X}$, we see $\tilde{X}$ is path-connected.

Finally, to show $\pi_1(\tilde{X}, [x_0]) = 0$, it suffices to show that hte iamge of htis group under π_* is triial since p_* is injective. Any element in the iamge of p_* are represented by loops γ at $x_8$that lift to loops in $\tilde{X}$ at $[x_0]$. We has seen that hte path $t \mapsto [\gamma_t]$ lifts γ starting at $[x_]$, and for this lifted path to be a loop it means that $[\gamma_1] = [x_0]$. Since $\gamma_1 = \gamma$, w have that $[\gamma] = [x_0]$, and so γ is nullhomotopic, hence the iamge of p_* is trivial, and so $\tilde{X}$ is also simply connected, completing the proof.

(showing uniquness?)

This construction shows that a space with these conditions always has a simple covering space, however most the time there shall be simpler means of constructing covering spaces. For example, we know a simple covering space of S^1, namely $\mathbb{R}$, and any product T^n has covering space $\mathbb{R}^n$ since commute with functors (as they are limits, but we also proved it explicitely).

Example 2.15: Simply Connected Covering Spaces
hatcher p. 65

Theorem 2.3.1: Galois Correspondence Of Fundamental Groups

Let X be a path-connectd, locally path-connected, semilocally simply-connected space. Then there is a bijection between the set of basepoint-preserving isomorphsism classes of path-connected covering space $p : (\tilde{X}, e_0) \to (X, b_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}), e_0)$.

Proof **:**
Let $\tilde{X}$ be the space we defined earlier and let $\sim$ be the equivalence relation given by:

$$[\gamma] \sim [\gamma'] \qquad \iff \gamma(1) = \gamma'(1),\ [\gamma * \gamma'] \in H$$

This is indeed a equivalence relations since H is a group, namely it is reflective since H cotnaisn th identity, symmetric since H Is closder under inverses, and transitive since H is closed under multiplication. Let $X_H = \tilde{X}/\sim$. Note that if $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma']$ if and only if $[\gamma * \eta] \sim [\gamma' * \eta]$. Hence, if two points in a neighbourhood $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_H¡ then the whole neighbourhoods are identified. Hence ,the projection map $X_H \to X$ givne by $[\gamma] \mapsto \gamma(1)$ is a covering map.

We know show that $\operatorname{im} p_* = H$. For any loop $\gamma in X$ at basepoint x_0¡ it lifts to $\tilde{X}$ starting at $[x_0]$ and endign at $[\gamma]$, so the iamge of this lifted apth in X_H is a loop if and only if $[\gamma] \sim [x_0]$, or equivalently $[\gamma] \in H$, as we sought to show.

Theorem 2.3.2: Uniqueness Of Covering Space

Let X be a path-conneced and locally path-connected. Then two path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are base-point isomorphic if and only if $\operatorname{im} {p_1}_* = \operatorname{im} {p_1}_*$.

An isomorphism here would be that the diagram commutes.

Proof **:**
If there is an isomorphism f betweeen $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$, then we get $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$, giving us the groups are equal, Conversely, if hte groups are equal, by the lifting criterion, we may lift p_1 to a map $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1$. We can do the same the other way to get

$p_1\tilde{p}_2 = p_2$. Then by uniqueness of lifting, $\tilde{p}_1\tilde{p}_2 = \mathrm{id}$ and $\tilde{p}_2\tilde{p}_1 = \mathrm{id}$ since the composition fixes the basepoint. These maps are certainly homomorphissm, and they are each other inverses, and hence, each are isomorphissm, as we sought to show.

Proposition 2.3.1: Base-Point Independence

If ignoring the basepoint, the correspondence gives a bijection between isomorphism class of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$

Proof **:**
Hatcher p. 68

Hence, a simply-connected coverign space X is a *universal cover* since it is unique *up to isomorphism*.

(Representing Covering Spaces by Permutations, Hatcher p. 68)

2.3.1 Deck Transformations

You may have noticed in the earlier theorem that the isomorphism between covering spaces that commuted in the diagram

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

has the same properties as fiber bundles. This is no coincidence; there is a relation between fiber bundles, covering maps, and fundamental groups. Since the study of this fiber-bundle morphisms on covering spaces pre-dates categorical nomenclature, we have the following widely-accepted name:

Definition 2.3.2: Deck Transformation

Let $p : E \to B$ be a covering map. Then an isomorphism:

$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

is called a *Deck transformation*. The collection of deck transformations is denoted $\mathrm{Deck}(p)$ or $\mathrm{Aut}(p)$. If we want to specify a particular group for a covering space, we will write $\mathrm{Aut}(p, E)$.

Since the spaces we are working with have unique universal covering maps up to isomorphism (ex. CW complexes), the notation $\mathrm{Deck}(p)$ is well-defined given we are always taking universal covers.

Example 2.16: Deck Transformation

1. Let $p : \mathbb{R} \to S^1$. Then a deck transformation willl be vertical translations, so
$$\mathrm{Deck}(p) = \mathbb{Z}$$
2. Let $p : S^1 \to S^1$ be given by $z \mapsto z^n$. Then $\mathrm{Aut}(p, S^1) = \mathbb{Z}/n\mathbb{Z}$.

By the unique lifing property, a deck transformation is *completely* determined by where it sends a single point (assuming E is path connected). Thus, the only map that fixes points of E is the identity.

Definition 2.3.3: Normal Covering

A covering space $p : E \to B$ is called *normal* if for each $x \in X$, and each pair of lifts $\tilde{x}, \tilde{x}'$ of x, there exists a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

For example, the covering space $\mathbb{R} \to S^1$ and the n-sheeted covering space $S^1 \to S^1$ are normal. Intuitively, a normal covering space is one with "maximal symmetry".

Proposition 2.3.2: Properties Of Normal Covering Spaces

Let $p : (E, e_0) \to (B, b_0)$ be a path-connected covering space of a path-connectd, locally path-connected space X, and let H be the subgroup $H = p_*(\pi_1(E, e_0)) \subseteq \pi - 1(X, x_0)$. THen:

1. The covering map is normal if and only if $H \trianglelefteq \pi_1(X, x_0)$
2. $\mathrm{Aut}(p, E)$ is isomorphic to the quotient $N(H)/H$ ($N(H)$ the normal closure of H).

In particular, $\mathrm{Aut}(p, E)$ is isomorphic to $\pi_1(X, x_0)/H$ if E is a normal covering. hence, the universal cover $p : E \to B$ has $\mathrm{Aut}(p) \cong \pi_1(X)$

Proof **:**
Hatcher p. 71

(more here on some interesting properties)

2.4 *Bonus Material

1. Bott Periodicity

3

Homology

When working with $\pi_n(X)$, we must keep track of the basepoint, for the composition of maps $f : S^n \to X$ must depend on the location of the basepoint. This makes computations much harder, in particular due to the nature of paths it renders many groups $\pi_n(X)$ non-abelian [1]. In homology theory, the objects will be independent of point of origin. This has a few consequences: for one this shall "abelianize" all our results, opening us to the theory of abelian groups and abelian categories more generally, a richer settings for many types of computations. To see how this abelienization takes place, take two vertices x, y with two edges a, b from x to y:

$$\begin{tikzcd} x \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & y \end{tikzcd}$$

Then we have ab^{-1} and $b^{-1}a$ as two loops. If, however, we conider the path ab^{-1} to be a loop starting at x and $b^{-1}a$ to be a loop starting at y, we see that these two are the "same". In particular, if we have no information on base-point, they are indistinguishable, namely $ab^{-1} = b^{-1}a$. Hence, we are no longer dealing with "loops" (since we don't have a basepoint), but *cycles*, which we shall shortly define more rigorously. Another consequence is that we shall need a new way of creating a group since we are no longer working with functions since we have no basepoint to anchor the cycles. The way we shall change our theory is by focusing on *boundaries* of shapes (these shall be our cycles), and seeing when:

1. a cycle is the boundary of a higher-dimensional shape (or a cyle of one higher dimension)
2. given a notion of orientation of a cycle, when does the sum of cycles cancel out?

Due to the generality of the above questions, homology starts out simpler with the introduction of a new type of strucure that can be imposed on a topological space X (similar to a CW complex)

[1] As we saw, using cell complexes we may craft any finitely generated group G as $\pi_1(X)$

that shall make it much easier to giv a notion of oriented cycle and boundary. Similarly to CW complexes, these structures will also have a noation of n-skeletons and gluing e_{n+1} cells to n-skeletons to "trivialize" relations. This shall lead to trying to describe "simple" simplexes known as "simplicial" and 'singular" simplexes from which we shall measure to what degree cycles fail to be boundaries of higher dimensional cycles.

After having established this structure, we move onto the more general notion of homology (known as singular homology), and show prove many nice theoretical results. We shall finish by showing that the simlicial homology group is isomorphic ot the singular homology group, combining our nice theoretical tools with computational tools.

(Hatcher gives a fun little hisotircal footnote that the homology used to be numbers, the betti number and the torsion coefficients, and when Poincare wrote about homology in the beginning of the 20th century that is what it was. the transition to the modern viewpoint with groups happened graudally for ~ 40 years, until essentially solidifed in 1952 by Eilenberg and Steenrod, p.131 of Hatcher)

(also see this link here)

3.1 Δ-Complex

(note somewhere why the inducd orientation on the boundary matches that of Stokes orientation given on manifolds)

In this section, we define the structre we shall impose on our space to define their homology, starting with the "building block":

Definition 3.1.1: n-Simplex

A *n-simplex* is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, v_1, ..., v_n$ that do not lie in a hyperlane of dimeension less than n. Equivaelntly, $v_1, v_0 ..., v_n, v_0$ are linearly independent. The points v_i are the vertices of the simplex an the simplex itself is denoted $(v_0, ..., v_n)$ (called the *barycentric coordiantes*)

Note that the ordering is part of the definition, which is why the vertices are in an $(n+1)$-tuple notation; this is to orient the edges. This is important for when we glue different edges together: the square, seen as two triangles, can be glued to a torus, T^2, real proejctive plane, $\mathbb{R}P^2$, or Klein bottle, K, depending on the orientation of the edges and the choice of points:

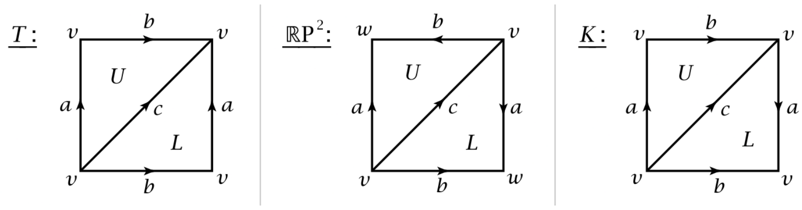


Figure 3.1: Different ways to glue a square

If we delete a vertex from an n-simplex, we get an $(n-1)$-simplex called the *face* of the simplex. if

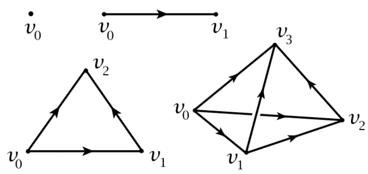
$(v_0, v_1, ..., v_n)$ is an n-complex, we shall represent a face via

$$(v_0, v_1, ..., \hat{v}_i, ..., v_n)$$

where $\hat{v}_i$ indicates the vertex is removed. By convention, the vertices of a face (or any subsimplex) will always be ordered according to their ordering of the larger simplex.

Example 3.1: n-simpliex

1. Here is the visual of a $0, 1, 2, 3$ simplex:



2. The standard n-simplex is defined as:

$$\Delta^n = \left\{ (t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i t_i = 1,\ t_i \geq 0,\ 0 \leq i \leq n \right\}$$

whose vertices must be, by definition, unit vectors along the coordiante axes. For example, Δ^n looks like:

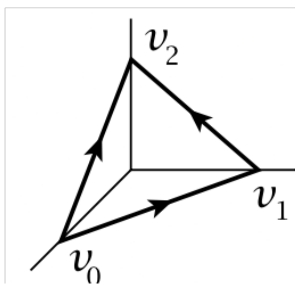


Figure 3.2: Δ^3

Definition 3.1.2: Boundary

Let Δ^n be the standard simplex. Then the union of all the faces of Δ^n is called the *boundary* of Δ^n and is written $\partial\Delta^n$. The open simplex $\operatorname{Int}\Delta^n$ is the interior of Δ^n

Definition 3.1.3: Δ-Complex

Let X be a topological space. Then a Δ-complex structure on X is a collection of continuous maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ with n_α depending on the index α such that

1. The restriction $\sigma_\alpha|_{\text{Int }\Delta^{n_\alpha}}$ is injective, and each point $x \in X$ is in the image of exactly one restriction $\sigma_\alpha|_{\text{Int }\Delta^{n_\alpha}}$. We label $\sigma_\alpha|_{\text{Int }\Delta^{n_\alpha}}$ as e_α^n.
2. Each restriction σ_α to a face Δ^n is one of the maps $\Sigma_\beta : \Delta^{n_\alpha - 1} \to X$.
3. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α (the topology of X the initial topology given by the maps σ_α)

figure 3.1 of (a covering space of) a torus, $\mathbb{R}\mathrm{P}^1$, and a Klein bottle are examples of Δ-complexes; the torus and the klein bottle each have six maps (one 0-simplex, three 1-simplices, two 2-simplices, each labeled on the diagram), while $\mathbb{R}\mathrm{P}^2$ has two 0-simplices (v, w in the covering space). We see here the importance of the orientation of the edges. We see that X can be represented of disjoint simpices $\Delta_\alpha^{n_\alpha}$ with $\Delta_\beta^{n_\alpha - 1}$ corresponding to the restriction of σ_β of σ_α to the particular face. Hence, any Δ-complexes can be described "combinatorially" as collections of n-simplices together with functions associtaing each face of an n-simplex to an $(n-1)$-simplex (similar to a cell-complex). In fact, though we will not verify this, the 1st condition insures that Δ-complexes can be thought of as CW complexes (note that a Δ-complex is Haudsorff).

Now, let X be a Δ-complex. We lead towards defining its' *simplicial homology group*. This group can be thought of as the generalization of a notion of a "path" or of the functions $[0,1]^n \to X$:

Definition 3.1.4: Group Of n-Chains

Let X be a Δ-complex. Then $\Delta_n(X)$ is the free abelian group generated by the open n-simplices e_α^n of X. Elements of $\Delta_n(X)$ are called *n-chains*, and can be written as :

$$\sum_\alpha n_\alpha e_\alpha^n$$

or similarly $\Sigma_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e_α^n.

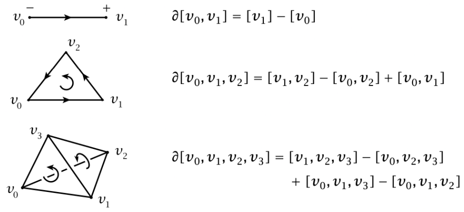
Examples of these groups shall soon be given. These groups generlize the notion of a path (or surface in higher dimension) without a basepoint. How would we get a loop (or closed surface)? This will come down to the boundary "cancelling itself out". To see this better, take $X = \Delta^n$ as a Δ-complex so that it can be written as $[v_0, v_1, ..., v_n]$. If $n = 2$, then it's boundary is given by:

$$\partial[v_0, v_1] = [v_1] - [v_0]$$

since orientation is preserved. Similarly, if $n = 3$, then

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

the following visual may aid in visualizing:



More generally, we see the following immerging pattern:

$$\partial(v_0, v_1, ..., v_n) = \sum_i (-1)^i (v_0, ..., \hat{v}_i, ..., v_n)$$

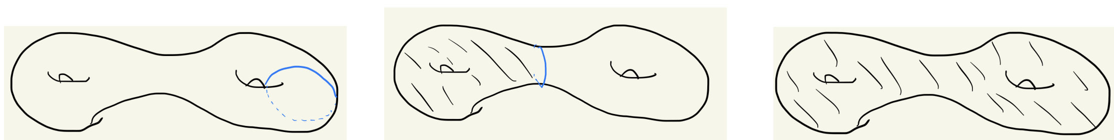
Taking the Δ-complex structure of a space X, we see that if the boundary is zero (for ex. if we have a closed loop), then we have a *cycle*. It can be easily verified that ∂ is a homomorphism, leading to the following important definition:

Definition 3.1.5: Boundary Homomorphism

Let X be a Δ-complex. then the *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined by:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \; \sigma_\alpha|_{(v_0,...,\hat{v}_i,...,v_n)}$$

The boundary map may or may not be zero, not all cycles of dimension lower are the boundary of a higher diensional cycle. Here is a visual that works for CW complexes (we shall see the homology on Δ-complexes is equivalent to CW complexes, it is simply easier to picture CW complex borders):



(a) Cannot Be Image of ∂_2 (b) Image of ∂_2 (c) Image of ∂_2 is Zero

Figure 3.3: Cycles and their Relations

Those familar with some algebra may start thinking that some sort of exactness is almost at play, and this is exactly the case. The key algebraic property of the boundry map that links it to exactness is the following:

Lemma 3.1.1: Complex Condition

The composition

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

produces the zero map, $\partial_{n-1} \circ \partial_n = 0$

Proof **:**
Plugging in, we get:

$$\partial_{n-1}(\partial_n(\sigma)) = \sum_{j<i}(-1)^i(-1)^j\ \sigma|_{(v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n)} + \sum_{j>i}(-1)^i(-1)^j\ \sigma|_{(v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n)}$$

Then the two summation on the right hand side cancel after switching i, j since the sum becomes the negative of the first.

This algebraic property can certainly be studied independently of the geometric intuitions from which it started and leads to a plethora of intersting results in algebraic geometry and algebraic topology. It is thus fruitful to give a purely algebraic definition of the phenomena happening in the above lemma:

Definition 3.1.6: Chain Complex and Exact

Given a chain of homomorhpisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

if $\partial_n\partial_{n+1} = 0$ for all n, then the above is called a *chain complex*, in other words:

$$\operatorname{im}\partial_{n+1} \subseteq \ker\partial_n$$

If equality holds, then the sequence is called an *exact sequence*.

As we see, not all cycles are boundary of a chain of one dimension higher. The following measure to what degree cycles “fail” to be boundaries of higher chains:

Definition 3.1.7: nth [Simplicial] Homology Group

Given a chian complex, the *nth homology group* is defined to be

$$H_n = \frac{\ker(\partial_n)}{\operatorname{im}\partial_{n+1}}$$

Elements of $\ker\partial_n$ are called *cycles*, elements of $\operatorname{im}\partial_{n+1}$ are called *boundaries*, elements of H_n are called *homology classes*, and if two elements are in the same cosets, they are called *homologous* (meaning their difference is a boundary)
In the case where $C_n = \Delta_n(X)$, the homology group will be denoted $H_n^{\Delta}(X)$ and is called hte *nth simplicial homology group* of X

As we see the homology group measures to what degree are cycles *not* the boundary of a higher dimensional chain. If all cycles are boundaries, then the homology is trivial, while if a cycles is not, “captures” this in some way. Since most homology groups we shall work with will be finitely generated, we shall interpret the torsion-free and torsion part of an abelian group to understsand what this “failure” means.

Example 3.2: Simplicial Homology Group

For reference, here are Δ-complexes for the torus, $\mathbb{RP}^2$, and klein bottle:

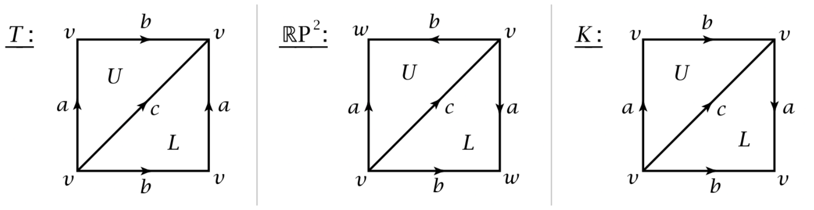


Figure 3.4: Reference Figure for Torus, Projective space, and Klein Bottle

1. Let $X = S^1$, with one vertex v an one edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$. The map $\partial_1 : \Delta_1(S^1) \to \Delta_0(S^1)$ is $\partial_1 \equiv 0$ since

$$\partial e = v - v$$

Hence $\operatorname{im}\partial_1 = 0$. On the other hand, we always have $\ker\partial_0 = \mathbb{Z}$, and so $H_0^{\Delta}(S^1) = \mathbb{Z}$. Since $\operatorname{im}\partial_1 = 0$, $\ker\partial_1 = \mathbb{Z}$. Certainly for $n \geq 0$, $\Delta_n(S^1) = 0$ since there is no simplices in these dimensions. Hence:

$$H_n^{\Delta}(S^1) = \begin{cases} \mathbb{Z} & n \in \{0,1\} \\ 0 & n \geq 2 \end{cases}$$

Notably, if the boundary maps are all zero, the homology groups of the complex are isomorphic to the chain groups $\Delta_n(X)$.

2. Let $X = T$ be the torus given the Δ-complex in figure 3.4, namely it has one vertex, three edges a, b, c, and two 2-simplices U, L, giving us $\Delta_0(T) = \mathbb{Z}$, $\Delta_1(T) = \mathbb{Z}^3$, and $\Delta_2(T) = \mathbb{Z}^2$. Like before, $\operatorname{im}\partial_1 = 0$, so we immediately have $H_0^{\Delta}(T) = \mathbb{Z}$. Next, $\ker\partial_1 = \mathbb{Z}^3$, and since $\partial_2 U = \partial_2 L = a + b - c$, writing $\Delta_1(T) = \ker\partial_1$ with basis $\{a, b, a+b-c\}$ we have:

$$H_1^{\Delta}(T) = \mathbb{Z} \oplus \mathbb{Z}$$

with basis $\{[a], [b]\}$. Since there is no 3-simplices, we have that $H_2\Delta(T) = \ker\partial_2$ which is given by $U - L$ since

$$\partial(pU + qL) = (p+q)(a+b-c) = 0 \qquad \iff \qquad p = -q$$

Hence, we overall have:

$$H_n^{\Delta}(T) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

3. Let $X = \mathbb{RP}^2$ with the Δ-complex strucutre given by figure 3.4, with two vertices v, w, three edges a, b, c and two 2-simpcles U, L. Then $\operatorname{im}\partial_1$ is generated by $w - v$, so

$$H_0^{\Delta}(\mathbb{RP}^2) \cong \mathbb{Z}$$

Since $\partial_2 U = -a + b + c$ and $\partial_2 L = a - b + c$, we get that ∂_2 is injective meaning it's kernel is zero and so:

$$H_2^\Delta(\mathbb{R}\mathrm{P}^2) = 0$$

Next, we see by staring at the image that $\ker \partial_1 = \mathbb{Z} \oplus \mathbb{Z}$ (find all paths that, under the coving map, are closed). By inspecting, we see it's basis is $a - b$ and c. The quotient $\ker \partial_1 / \operatorname{im} \partial_2$ is certainly a finite group since we are quotient two groups with the same rank. To find the quotient, take as bassi for $\ker \partial_1$ $(a - b + c, c)$. Then since $(a - b + c) + (-a + b + c) = 2c$, we see that the quotient must have order 2, thus:

$$H_1^\Delta(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$$

Giving us:

$$H_n(\mathbb{R}\mathrm{P}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

The torsion element can be thought of as somehow representing some "non-orientability" or that we have found some "twists" in our complex. We shall calculate $H_k(\mathbb{R}\mathrm{P}^n)$ using Cellular homologies in section 3.5.

4. For a moment assume homeomorphisms preserve homology groups. Let $X = S^n$ with the Δ-complex structure on S^n given by taking two copies of Δ^n and identifying their boundaries via identity maps, giving us a space homeomorphic to S^n. Then to compute $H_n^\Delta(S^n)$, we seee that $\ker \Delta_n$ is an infinite cyclic group generated by $U - L$ where U, L are the two n-simplices. Since the image is certainly trivial, we have:

$$H_n^\Delta(S^n) \cong \mathbb{Z}$$

Calculating explicitely the other dimension is rather difficult wihtout more advanced tools like the Snake lemma, however the fact that $H_n^\Delta(S^n) \cong \mathbb{Z}$ will be an important lemma. For now, the case of $n = 2$ can be studied. In particular, we have that any cycles on S^2 is clearly the boundary of of a chain one dimension higher. This should be made rigorous to check that:

$$H_n(S^2) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

5. (Klein Bottle here)

There are some natural questions that arise: is the simplicial homology group independent of homeomorphism? Of homotopy equivalence? These questions are hard to answer with the given Δ-complexes. What we shall do is gernalize to a new homology theory known as *singular homology groups*, which is easier to work theoretically, though harder to visuzlize, but can be defined on more general spaces, and in the end we sahll show they produce the same groups as simplicial homology groups.

(a quick word on how we used to use simplicial complexes instead of Δ-complex, which is why the homology is called *simplicial*. The disadvantage of this is that the structure is generally more cumbersome to work with, requiring many more faces, edges, and vertices, see p. 107 Hatcher)

3.2 Singular Homology

Definition 3.2.1: Singular n-Simplex

Let X be a space. Then a *singular n-simplex* in a space is a continuous map $\sigma : \Delta^n \to X$.

The term singular expresses how the map σ need only be continuous, there may be "singularities" where the image does not look like a simplex. This generality also means that each space X has many more singular n-simplexes, and that the notion of "boundary" is more obscure, relying on the domain of σ rather than a geometric intuition in the codomain. Similarly to before, let $C_n(X)$ be the free abliean group with basis being the set of all singular n-simplices in X, that is the elements of $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$. Elements of $C_n(X)$ are called *[singular] n-chains*, and are finite formal sums $\sum_i n_i\sigma_i$. Just as before, we define a boundary map:

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma|_{(v_0,\ldots,\hat{v}_i,\ldots,v_n)}$$

Naturally, there is implicitly an identification of $(v_0, ..., \hat{v}_i, ..., v_0)$ with ∂_{n-1} preserving odering of vertice so that we can think of the map $\sigma|_{(v_0,\ldots,\hat{v}_i,\ldots,v_n)}$ as being from $\Delta^{n-1} \to X$ (i.e. of a singular $(n-1)$-simplex). When not ambiguous, we shall write the border map $\partial_n : C_n(X) \to C_{n-1}(X)$ simply as ∂. Then we may consicely write the condition $\partial_n\partial_{n+1} = 0$ as $\partial^2 = 0$, and again the image of ∂ is called the boundary, while everything within the kernel is called the cycle. Thus, we again have the following definition:

Definition 3.2.2: [Singular] Homology Group

Let X be a topological spaces with singular n-simplices. Then

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

Unlike simplicial Homology groups, singular homology groups are evidently preserved under homeomorphisms, for there is a natural bijection $C_n(X) \to C_n(Y)$ that works well with the boundary condition (namely it shall map the shapes of the right dimensions to themselves[2]). Note that $C_n(X)$ is usually uncountable and may seem hard to work with, but as we shall see this flexibility gives us many more easly provable results. In fact, the singular homology group can be identified with the simplicial homology group by a special construction on a space X (labeled $S(X)$), by taking for each singular n-simplex σ a n-simplex Δ^n_σ with Δ^n_σ attached in the obvious way to the $(n-1)$-simplices of $S(X)$ that are restrictions of σ to the approrpiate faces. Then $H^\Delta_n(S(X)) = H_n(X)$ for all x, though $S(X)$ is usually huge.

(Hatcher p. 107 gives a word on how to interpret cycles in singular homology geometrically given maps from finite Δ-complexes).

[2]This result shall be proven later, but it can also be seen through DeRham Theory if you have done some Differential Geometry

(the key take-away is that elemetns in $H_1(X)$ are represented by oriented loops, elements in $H_2(X)$ are represented by maps of closed oriented surface into X, and so forth. With some work, it can be shown that an oriented 1-cycle $\bigsqcup_\alpha S^1\alpha \to X$ is zero in $H_1(X)$ if and only if it extends to a map of a compact oriented surface with boundary $\bigsqcup_\alpha S^1_\alpha$ into X, and similarly for 2-cycles. It was hopped in the early days of homology theory that the close connection with manifolds would have continued to higher dimensiosn, but that was in fact not the case! These notions would further be explored in a theory known as *bordism*)

With these intuition in mind, we may look at some useful properties of singular homologies:

Proposition 3.2.1: Homology group and path-connected components

Corresponding to the decomposition of a space X into its path-components X_α,

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

Proof **:**
This comes down to properties of continuous maps. Since singular simplex is always path-connected, the imge is too, so $C_n(X)$ splits as the direct sum of subgroups $C_n(X_\alpha)$. Then the boundary maps ∂_n preserves direct sums of decompositions taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, hence $\ker\partial_n$ and $\operatorname{im}\partial_{n+1}$ split similarly as direct sums, hence the homology group splits, that is:

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$

as we sought to show.

We shall later see that $H_n(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha H_n(X_\alpha)$. We may wish that $H_n(X \times Y)$ also has an easy structure, however this becomes a good bit more complicated, and shall be addressed in section 4.2.1.

Proposition 3.2.2: 0th Singular Homology Group for Path-Connected Space

If X is nonempty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence, for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$ for each path component.

Proof **:**
By definition, $H_0(X) = C_0(X)/\operatorname{im}\partial_1$ (since $\partial_0 = 0$). What we shall do is define a the surjective homomorphism:

$$\epsilon : C_0(X) \to \mathbb{Z} \qquad \sum_i n_i\sigma_i \mapsto \sum_i n_i$$

and show that $\ker\epsilon = \operatorname{im}\partial_1$, giving us the structure of $H_0(X)$.

($\operatorname{im}\partial_1 \subseteq \ker\epsilon$) Note that each singular 1-simplex $\sigma : \Delta^1 \to X$ has

$$\epsilon(\partial_1(\sigma)) = \epsilon(\sigma|_{(v_1)} - \sigma|_{(v_0)}) = 1 - 1 = 0$$

hence, they are conained in the kernel.

($\operatorname{im}\partial_1 \supseteq \ker\epsilon$) Suppose $\epsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i = 0$. If we can find an element in $C_1(X)$ for which $\sum_i n_i\sigma_i$ is the boundary, then we're done. Since the σ_i' are singular 0-simplices, their image is simply a point in X. For each i, choose a path $\tau_i : I \to X$ where $\tau_i(0) = x_0$ and $\tau_i(1) = \sigma_i(v_0)$, and let σ_0 be the singular 0-simplex with iamge x_0. Then τ_i can be seen as a singular 1-simplex, namely $\tau_1 : [v_0, v_1] \to X$, and

$$\partial \tau_i = \sigma_i - \sigma_0$$

Thus:

$$\partial\left(\sum_i n_i\tau_i\right) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$$

But then we see that $\sum_i n_i\sigma_i$ is a boundary, giving $\ker\epsilon \subseteq \operatorname{im}\partial_1$, as we sought to show.

Proposition 3.2.3: Homology Group of a Point

If X is a point, then

$$H_0(X) \cong \mathbb{Z} \qquad H_n(X) = 0,\ \forall n > 0$$

Proof **:**

There is a unique singular n-simplex σ_n for each n, and $\partial(\sigma_n) = \sum_i (-1)^i\sigma_{n-1}$ is a sum of $n+1$ terms, and thus is 0 when n is odd and σ_{n-1} when n is even ($n \neq 0$). Thus, we get

$$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with the boundary maps alternating between being an isomorphism and the trivial map except at the last $\mathbb{Z}$. Then it can be read off that the homology groups of this complex are all trivial except for $H_0 \cong \mathbb{Z}$.

For future reference, it is sometimes useful to defined a modified version of homology for which the homology is trivial in all dimensions (including 0).

Definition 3.2.3: Reduced Homology Group

Define the *reduced homology groups* $\tilde{H}_n(X)$ to be the homology groups of the augmented chain complex:

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where

$$\epsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i$$

Certainly, we should require X to be nonempty, or else we would have a nontrivial homology group in dimension -1. Since $\epsilon \circ \partial_1 = 0$, ϵ vanishes on $\operatorname{im}(\partial_1)$, an hence induces the map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(X)$. Thus:

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z} \qquad H_n(X) \cong \tilde{H}_n(X),\ n > 0$$

We can think of the extra $\mathbb{Z}$ in the agumented chain as representing the unique map $(\emptyset) \to X$ where

$\emptyset$ is the empty simplex with no verices. Then the augmetnation map ϵ is the boundary map since:

$$\partial(v_0) = (\hat{v}_0) = (\emptyset)$$

(it is worth noting that $H_1(X)$ is the abelianization of $\pi_1(X)$ whenever X is path connected. This shall not be proven, but see Hatcher 2.A for reference)

3.3 Homotopy Invariance

In this section, we shall show that homotopy equivalent spaces have isomorphic homology groups, in particular we shall show there is a functor so that the continuous map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for each n and that f_* is an isomorphism if f is a homotopy equivalence.

Definition 3.3.1: Pushforward

Let $f : X \to Y$ be a continuous map. Then f induces a homomorphsim $f_\sharp : C_n(X) \to C_n(Y)$ by mapping

$$\sigma \mapsto f \circ \sigma$$

which then is extended linearly:

$$f_\sharp \left(\sum_i n_i \sigma_i \right) = \sum_i n_i f_\sharp(\sigma_i) = \sum_i n_i f \circ \sigma_i$$

Since $f_\sharp \partial = \partial f_\sharp$, namely:

$$\begin{aligned} f_\sharp \left(\partial(\sigma)\right) &= f_\sharp \left(\sum_i (-1)^i \; \sigma_i|_{(v_0, \dots, \hat{v}_i, \dots, v_n)} \right) \\ &= \sum_i (-1)^i f(\sigma_i|_{(v_0, \dots, \hat{v}_i, \dots, v_n)}) \\ &= \partial(f_\sharp(\sigma)) \end{aligned}$$

the following diagram commute:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n+1}(X) \arrow[r] \arrow[d, "f_\sharp"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r] & \cdots \end{tikzcd}$$

In terms of categorie, f induces a homomorphism between complexes. In particular, it takes cycles to cycles since if $\partial(\alpha) = 0$ then $\partial(f_\sharp(\alpha)) = f_\sharp(\partial(\alpha)) = 0$, and $f_\sharp$ takes boundary to boundaries since $f_\sharp(\partial\beta) = \partial(f_\sharp\beta)$. Hence $f_\sharp$ induces a homomorphism:

$$f_* : H_n(X) \to X_n(Y)$$

Proposition 3.3.1: Functor between Top and Homology

let f, g be continuous maps. Then:

1. $(fg)_* = f_* g_*$
2. If f is a homeomorphism, f_* is an isomorhpism.
3. id_* is the identity map

Hence $(-)_* : \mathbf{Top} \to C(\mathbf{Ab})$ is a functor where $C(\mathbf{Ab})$ is the category of complexes with objects with complexes with objects $\mathbf{Ab}$ and morphisms being commutative diagrams as seen above.

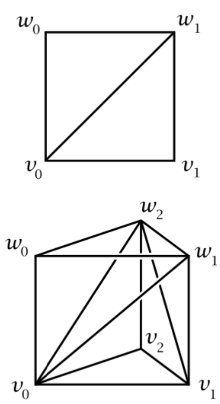
Proof **:**
exercise, note that f being a homeomorphism inducing f_* to be an isomorphism was already mentioned under the definition of singular homology.

Theorem 3.3.1: Homotopy, Then Isomorphism

Let $f, g : X \to Y$ be homotopic maps. Then they induce the same homomorpism:

$$f_* = g_*$$

Proof **:**
The key is to subdivide $\Delta^n \times I$ into simplices. In $\Delta^n \times I$, let $\Delta^n \times \{0\} = [v_0, ..., v_n]$, and $\Delta^n \times \{1\} = [w_0, ..., w_n]$, where v_i, w_i have the same image under the projection from $\Delta^n \times I \twoheadrightarrow \Delta^n$. To construct the $(n+1)$-simplex, we move a vertex v_i up to w_i, where we start at v_n and work backwards to v_0. In the $n \in \{1, 2\}$ case, this looks like so:



Concretely, we start by moving

$$[v_0, ..., v_n] \text{ to } [v_0, ..., v_{n-1}, w_n]$$

then move it to $[v_0, ..., v_{n-2}, w_{n-2}, w_n]$ and so on where we move $[v_0, ..., v_i, w_{i+1}, ..., w_n]$ up to $[v_0, ..., v_{i-1}, w_i, ..., w_n]$. Now, the region between the $(n+1)$-simplex $[v_0, ..., v_i, w_i, ..., w_n]$ has as lower face $[v_0, ..., v_i, w_{i+1}, ..., w_n]$ and as upper face $[v_0, ..., v_{i-1}, w_i, ..., w_n]$. Overall, $\Delta^n \times I$ is the union of these $(n+1)$-simplices, each intersecting the next in an n-simplex face.

Now, let $F : X \times I \to Y$ be a homotopy from f to g, and let $\sigma : \Delta^n \to X$ be a singular simplex. Define:

$$F \circ (\Sigma \times \mathrm{id})\Delta^n \times I \to X \times I \to Y$$

From this, we can define the following operator, called the *prism operator* as follows:

$$P : C_n(X) \to C_{n+1}(Y)$$
$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0,...,v_i,w_i,...,w_n]}$$

We shall show taht the prism operator satisfies

$$\partial P = g_\# - f_\# - P\partial$$

where the left hand side represnts the boundary of the prism, and the right hand side represents the top $\Delta^n \times \{1\}$ face, the bottom $\Delta^n \times \{0\}$ face, and the $\partial\Delta^n \times I$ sides of the prism. For this, we simply calculate:

$$\begin{aligned}\partial P(\sigma) = &\sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,...,\hat{v_j},...,v_i,w_i,...,w_n]} \\ &+ \sum_{j \leq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|_{[v_0,...,v_i,w_i,...,\hat{w}_j,...,w_n]}\end{aligned}$$

The terms where $i = j$ in the sum cancel out, with the exception of $F \circ (\sigma \times \mathrm{id})|_{\hat{v}_0,w_0,...,w_n]}$, which gives $g \circ \sigma = g_\#(\sigma)$, and the term $-F \circ (\sigma \circ \mathrm{id})|_{[v_0,...,v_n,\hat{w_n}]}$ which gives $-f \circ \sigma = -f_\$(\sigma)$. When $i \neq j$, then we get $-P\partial(\sigma)$ since

$$\begin{aligned}\partial P(\sigma) = &\sum_{j > i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,...,\hat{v_j},...,v_i,w_i,...,w_n]} \\ &+ \sum_{j > i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|_{[v_0,...,v_i,w_i,...,\hat{w}_j,...,w_n]}\end{aligned}$$

But now, if $\alpha \in C_n(X)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$$

since $\partial\alpha = 0$. Hence, $g_\#(\alpha) - f_\#(\alpha)$ is a boundary, hence $g_\#(\alpha), f_\#(\alpha)$ determines the homology class, but then g_* and f_* are equal on the homology class of α, completing the proof.

Corollary 3.3.1: Homotopy Equivalence, Then Isomorphism

Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for all $n \in \mathbb{N}$

Corollary 3.3.2: Homology Group Of Simply Connected Spaces

Let X be simply connected path connected space. Then:

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof **:**
Since $X \sim \{x_0\}$, $H_n(X) = H_n(\{x_0\})$, this follows from proposition 3.2.3

The key property that allowed us to derive the conclusion is the following

Definition 3.3.2: Chain Homotopy

The relationship

$$\partial P + P\partial = g_\# - f_\#$$

says that P is a *chain homotopy* between the chains $f_\#$ and $g_\#$

This property is more general than that of homotopy, and is the key fact in the proof to induce the homomorphism:

Proposition 3.3.2: chain homotopic chain maps

Chain homotopic chain maps induce the same homomorphism on homology

This shall be important when we generalize homology to more algebraic structures.

Proof **:**
The relation $\partial P + P\partial = g_\sharp - f_\sharp$ from the above theorem shows exactly this.

Since $f_\# \epsilon = \epsilon f_\#$ (where $f_\#$ is the identity map on the added groups $\mathbb{Z}$) gives us the induced homomorphisms $f_* : \tilde{H}_n(X) \to \tilde{H}_n(Y)$ for the reduced homology groups, and the properties above hold equally well for reduced homologies.

3.4 Relative Homology: Quotient Complexes

In geometry, we very often construct more complicated shapes given simpler ones. Often, we may decompose a information about a shape X into:

1. information on $A \subseteq X$ and X/A
2. information on $A, B \subseteq X$ where $A \cup B = X$

We shall show how we can in the right cases find on $A \subseteq X$ and X/A where we can find such information, or the right $A, B \subseteq X$ where we can compose all we need to know about X via A, B. This builds up to two importan theorems, the 2nd a direct application of the first:

1. Snakes Lemma Sequences
2. Mayer-Vietoris Sequences

In this section, we shall proof snakes lemma (Both the algebraic and topological), and prove some important results along the way.

An important simplification in homology comes when taking a subsapce $A \subseteq X$, and then we want to consider $C_n(X)/C_n(A)$, which essentially tells us to consider all homology information on the space A as trivial. We give this group a name:

Definition 3.4.1: Relative Chain Groups

Let X be a topological space and $A \subseteq X$ a subspace. Then

$$C_n(X, A) := C_n(X)/C_n(A)$$

is called the *relative homology groups* of X with respect to A.

Notice that the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, hence we have an induced map $\partial_n(X, A) \to C_{n-1}(X, A)$, and so $\partial^2 = 0$. Thus, we have the following:

Definition 3.4.2: Relative Homology Groups

Define

$$H_n(X, A) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

to be the *relative homology group* where ∂_n and ∂_{n+1} are defined as above. Elements of $H_n(X, A)$ are represented by *relative cycles* (n-chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$), and relative cycles α is trivial in $H_n(X, A)$ if and only it is a *relative boundary* $\alpha = \partial\beta + \gamma$, where $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$

Note that the quoteint $C_n(X)/C_n(A)$ *can* be viewed as a subgroup of $C_n(X)$ by considering it as having basis the singular n-simplcies $\sigma : \Delta^n \to X$ whose image is *not* contained in A, however the boundary map does not take this subgroup of $C_n(X)$ to the coresponding subgroup of $C_{n-1}(X)$, hence this is not something we shall be considering for it does not work well with the algebra we have built up.

We would like to link $H_n(X, A)$ with $H_n(X)$. We shall do so in the following long exact sequence:

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X,A) \to \cdots \to H_0(X,A) \to 0$$

This should naturally bring into mind the notion of a short exact sequence of complexes:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+2} \arrow[r] \arrow[d, "\iota"] & A_{n+1} \arrow[r] \arrow[d, "\iota"] & A_n \arrow[r] \arrow[d, "\iota"] & A_{n-1} \arrow[r] \arrow[d, "\iota"] & \cdots \\
\cdots \arrow[r] & B_{n+2} \arrow[r] \arrow[d, "\pi"] & B_{n+1} \arrow[r] \arrow[d, "\pi"] & B_n \arrow[r] \arrow[d, "\pi"] & B_{n-1} \arrow[r] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r] & C_{n+2} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{3.1}
$$

where when working with chain complexes we get each long exact sequence has maps $\partial_k : X_{k+1} \to X$. If $A_k \subseteq B_k$ where B_k is a topological space and $C_k = B_k/A_k$, then the above diagram commutes.

We shall now show that this short exact sequence of chain complexes stretches into a long exact sequence of homology groups:

Theorem 3.4.1: Algebraic Snake Lemma

Given a short exact sequence of chain complex as in equation (3.1):

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{\pi_*} H_{n-1}(C) \to \cdots$$

Visually, we can see the theorem tells us that we make chain together equation 3.1 into:

$$
\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & 0 \arrow[d, dotted] & \\
\cdots \arrow[r, dotted] & H_{n+2}(A) \arrow[r, dotted] \arrow[d, "\iota"] & H_{n+1}(A) \arrow[r, dotted] \arrow[d, "\iota"] & H_n(A) \arrow[r, dotted] \arrow[d, "\iota"] & H_{n-1}(A) \arrow[r, dotted] \arrow[d, "\iota"] & \cdots \\
\cdots \arrow[r, dotted] & H_{n+2}(B) \arrow[r, dotted] \arrow[d, "\pi"] & H_{n+1}(A) \arrow[r, dotted] \arrow[d, "\pi"] & H_n(B) \arrow[r, dotted] \arrow[d, "\pi"] & H_{n-1}(B) \arrow[r, dotted] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r, dotted] & H_{n+2}(C) \arrow[r, dotted] \arrow[d, dotted] \arrow[uur] & H_{n+1}(C) \arrow[r, dotted] \arrow[d, dotted] \arrow[uur] & H_n(C) \arrow[r, dotted] \arrow[d, dotted] \arrow[uur] & H_{n-1}(C) \arrow[r, dotted] \arrow[d, dotted] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

Note that this theorems work in the general context of homological algebra, for the notion of exact sequence of complexes is independent of any geometric considerations.

Proof :
We first define the map $\partial : H_n(C) \to H_{n-1}(A)$. We should keep in mind the following diagram:

$$
\begin{tikzcd}
 & A_{n-1} \arrow[d, "\iota"] \\
B_n \arrow[r, "\partial"] \arrow[d, "\pi"] & B_{n-1} \\
C_n &
\end{tikzcd}
$$

Let $c \in C_n$ be a cycle so that $c \in \ker \partial_n$. Since π is surjective, $c = \pi(b)$ for some $b \in B_n$. Then by definition $\partial : B_n \to B_{n-1}$ so $\partial(b) \in B_{n-1}$. Then by commutativity $j(\partial(b)) = \partial(j(b)) = \partial(c) = 0$ (since c is a cycle), thus $\partial b \in \ker(\pi)$. Since $\ker j = \operatorname{im} i$, $\partial b = i(a)$ for some $a \in A_{n-1}$. Since $i(\partial a) = \partial i(a) = \partial\partial b = 0$ since i is injective, $\partial a = 0$. Thus, define $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$. This is well defined since:

1. The element a is uniquely determined by ∂b since i is injective
2. for differnet choice b' for b, we get $j(b') = j(b)$, os $b' - b \in \ker(j)$ which is equal to $\operatorname{im}(i)$. Thus, $b' - b = i(a')$ for some a', ehcen $b' = b + i(a')$. But then this means we just chanage a to another representative (a homologous element) $a + \partial a'$, namely since
$$i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b = i(a'))$$
3. A difference choice of c within its homology class would have hte form $c = \partial c'$. To see this, since $c' = j(b')$ for some b',
$$c + \partial c' = c\partial j(b') = c + j(\partial b') = j(b + \partial b')$$
thus b is replaced by $b = \partial b'$, which gives ∂b and thus a is unchanged.

Finally, $\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism, for if $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via elements b_1, b_2, then
$$j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 = c_2$$
and
$$i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$$
hence
$$\partial([c_1] + [c_2]) = [a_1] + [a_2]$$
giving us the map ∂ is indeed a homomorphism. What's left to show is the following inclusions:

1. $\operatorname{im} i_* \subseteq \ker j_*$. Since $ji = 0$, $j_* i_* = 0$
2. $\operatorname{im} j_* \subseteq \ker \partial$. By definition of ∂, $\partial b = 0$ so $\partial j_* = 0$.
3. $\ker j_* \subseteq \operatorname{im} i_*$. Let $b \in B_n$ with boundary represent a homology class in $\ker j_*$, so $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjectiv,e $c' = j(b')$ for some $b' \in B_{n+1}$. Then since $\partial j(b') = \partial c' = j(b)$,
$$j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$$

Thus, $b - \partial b' = i(a)$ for some $a \in A - n$. Since $i(\partial a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ and i is injective, a is a cycle. Thus:

$$\iota_*[a] = [b - \partial b'] = [b]$$

showing that i_* maps onto $\ker j_*$

4. $\ker \partial \subseteq \operatorname{im} j_*$. By definition of ∂, if c represents a homology class in $\ker \partial$, then $a = \partial a'$ for some $a' \in A_n$. The element $b - i(a')$ is a cyle since $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$. Since
$$j(b - i(a')) = j(b) - ji(a') = j(b) = c$$
we get that $j_*[b - i(a')] = [c]$

5. $\ker i_* \subseteq \operatorname{im} \partial$. Given a cycle $a \in A_{n-1}$ such that $i(a) = \partial b$ for seom $b \in B_n$¡ then $j(b)$ is a cylce since $\partial j(b) = j(\partial b) = ji(a) = 0$, hence $\partial[j(b)] = [a]$

Thus, as an immediate consequence in the current topological settings we have:

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X,A) \to \cdots \to H_0(X,A) \to 0$$

Concretely, the map $\partial : H_n(X, A) \to H_{n-1}(A)$ take a class $[\alpha]$, represented by a relative cycle α, and maps it to the class represented by the cycle $\partial\alpha$ in $H_{n-1}(A)$. Essentially, the long exact seuqences measures the difference between $H_n(X)$ and $H_n(A)$. In particular, exactness at all n would imply $H_n(X, A) = 0$, then the inclusion $A \hookrightarrow X$ indcues an isomorphism $H_n(A) \cong H_n(X)$ for all n (the converse is also true, left as an exercise).

Note that we may augment these results to reduced homology groups when $A \neq \emptyset$ by applying the above to the short exact sequence $0 \to C_n(A) \to C_n(X) \to C_n(X, A) - 0$ to non-negative diemnsions, and add $0 \to \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \to 0 \to 0$ for the -1 dimension. Thus, $\tilde{H}_n(X, A)$ is the same as $H_n(X, A)$ for all n (given $A \neq \emptyset$)

Example 3.3: Application of Snake Lemma

Look at lecture notes for CW complexes and their faces.

1. Consider $(D^n, \partial D^n)$ and a long exact sequence of reduced homology groups. Then the ∂ map
$$\cdots \to H_i(D^n) \to H_i(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \to H_{i-1}(D^n) \to \cdots$$
are isomorphisms for all $i > 0$ since the remaining terms $\tilde{H}_i(D^n)$ are all zero for all i. Thus assuming the homology of the sphere is $H_n(S^n) = \mathbb{Z}$ and $H_k(S^n) = 0$ for $k \neq n$:
$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

2. Consider (X, x_0) for $x_0 \in X$ and a long exact sequence of reduced homology groups. Then
$$H_n(X, x_0) \cong \tilde{H}_n(X)$$
for all n since $\tilde{H}_n(x_0) = 0$ for all $n \in \mathbb{N}$.

A map $f : X \to Y$ with $f(A) \subseteq B$ (we may write $f : (X, A) \to X(B))$ induces homomorphisms

$f_\# : C_n(X, A) \to C_n(Y, B)$. The relation $f_\#\partial = \partial f_\#$ holds for relative chains (since it holds for chains), hence we have the induced homomorhpisms $f_* : H_n(X, A) \to H_n(Y, B)$

Proposition 3.4.1: Homotopy Between Relative Maps

Let $f, g : (X, A) \to (Y, B)$ be two maps where $f(A) \subseteq B$. If these maps are homotopic through maps of pairs $f_t : (X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$

***Proof* :**
This is just about checking that the previous work is still valid. Since the prism oeprator P form the above prove take $C_n(A)$ to $C_{n+1}(B)$, w have an induced prism operator $P : C_n(X, A) \to C_{n+1}(Y, B)$. When passing through the quotient, we still have

$$\partial P - P\partial = g_\# - f_\#$$

Hence, $f_\#$, $g_\#$ on relative chain groups are chain homotopic, hence they induce the smae homomorphisms on relative homology groups.

For future reference, we may generlize the long exact sequence given by the pair (X, A) to one by the triple (X, A, B), $B \subseteq A \subseteq X$ via

$$\cdots H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots$$

associated to the chian complexes from by short exact sequences:

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$$

An easy example would be if B is a point, then the long exact sequence formed by the triple (X, A, B) is just a long exact sequence of reduced homology for the pair (X, A).

We know build up to relating X with X/A. The key result we built up to is when we can delete, or excise, a part of the space and keep the same homology group:

Theorem 3.4.2: Excision Theorem

Let $Z \subseteq A \subseteq Z$ such that the closure of Z is contained in the interior of A ($\overline{Z} \subseteq \operatorname{Int} A$). Then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(X - Z, A - Z) \cong H_n(X, A) \qquad \forall n \in \mathbb{N}$$

Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X $\operatorname{Int} A \cup \operatorname{Int} B = X$, then inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B) \cong H_n(X, A) = H_n(A \cup B, A) \qquad \forall n \in \mathbb{N}$$

The equivalence is obtained by setting $B = X - Z$ and $Z = X - B$ so that $A \cap B = A - Z$ and $\bar{Z} \subseteq \operatorname{Int} A$ is equivalent to $X = \operatorname{Int} A \cup \operatorname{Int} B$ since $X - \operatorname{Int} B = \bar{Z}$.

To prove this theorem, we require a few lemmas. First Let X be a topological space and U a collection of subsest of X whose interior covers X. Let $C_n^U(X) \leqslant C_n(X)$ be chains $\sum_i n_i\sigma_i$ such that $\operatorname{im} \sigma_i \subseteq U_j \in U$ for some element in U. Then the boundary map takes $C_n^U(X)$ to $C_{n-1}^U(X)$, hence $C_n^U(X)$ forms a chain complex; let the respective homology gropu be denoted $H_n^U(X)$

Lemma 3.4.1: Homology Of Covered Space

The inclusion map $\iota : C_n^U(X) \hookrightarrow C_n(X)$ is a chian homotopy equivalence, in particular there exists a ma p$\rho : C_n(X) \to C_n^U(X)$ such that $\iota \circ \rho$ and $\rho \circ \iota$ are chain homotopic to the idnetity. THus, ι induces an isomorphishm:

$$H_n^U(X) \cong H_n(X) \qquad \forall n \in \mathbb{N}$$

Proof **:**
Hatcher p. 119, long!

Proof **:**
Of Excision Theorem p.124

Finally to finish the snakes lemma, we need to replace the relative homology groups with absolute homology groups:

Corollary 3.4.1: Snakes Lemma With Absolute Homology Groups

Let (X, A) be a pair where A is a nonempty closed subspace that is a deformation retract of some neighbourhood of X, $\pi : (X, A) \to (X/A, A/A)$ the natural quotient map, and $\pi_* : H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ the induced map. Then π_* is an isomorphism:

$$H_n(X, A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$$

Proof **:**
hatcher p. 124

This motivates the following definition

Definition 3.4.3: Good Pair

Let $A \subseteq X$ be a closed subspce that is a deformation retract of some neighbourhood of X. Then (X, A) is called a good pair

With this, we may upgrade the snake lemma to the following:

Theorem 3.4.3: Topological Snake Lemma

Let (X, A) be a good pair. Then there is an exact sequence:

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{\pi_*} H_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{\pi_*} \tilde{H}_{n-1}(X/A) \to \cdots$$

where i is the inclusion and j the quotient.

***Proof* :**
Immedaite consequence of above results.

Corollary 3.4.2: Homology Groups of Spheres

$$\tilde{H}_n(S^n) \cong \mathbb{Z} \qquad \tilde{H}_i(S^n) = 0 \quad (i \neq n)$$

***Proof* :**
For $n > 0$, take $(X, A) = (D^n, S^{n-1})$ so that $X/A = S^n$. Then the terms $\tilde{H}_i(D^n)$ in the long exact sequence for this pair are zero since D^n is contractible. Exactness of the sequence implies the maps

$$\tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1})$$

are isomorphisms for $i > 0$. Since $\tilde{H}_0(S^{n-i}) = 0$ for $n \neq i$ and $\tilde{H}_0(S^0) \cong \mathbb{Z}$ since S^0 has two connected components, we get the homology groups of S_n to be:

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

as we sought to show.

Corollary 3.4.3: No Retract For Border Of Spheres

The border of D^n does not retract, that is ∂D^n is not a retract of D^n. Hence, every map $f : D^n \to D^n$ has a fixed point

***Proof* :**
If $r : D^n \to \partial D^n$ is a retraction, then $r \circ \iota = \text{id}$, ι the inclusion map. Then the composition

$$\tilde{H}_{n-1}(\partial D^n) \xrightarrow{\iota_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$$

is the identity map on $\tilde{H}_{n-1}(\partial D^n) \cong \mathbb{Z}$. But ι_* and r_* are zero since $\tilde{H}_{n-1}(D^n) = 0$, a contradiction. Then the fixed point follow from Brouwers fixed point theorem (theorem 2.1.6).

(Hatcher elaboartes on how to generalize from (X, A) to arbitrary pairs on p.124-125)

Example 3.4: Finding Explicit Generators of $H_n(S^n)$
Hatcher p.125; it shall be the inclusion map Consider the homotopic equivalent $(\Delta^n, \partial\Delta^n)$.

Corollary 3.4.4: Excisions For CW Complex

Let X be a CW complex that is the union of subcomplexes A and B. Then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ indcues an isomorphism

$$H_n(B, A \cap B) \cong H_n(X, A) \qquad \forall n \in \mathbb{N}$$

Proof **:**
Let $A \cap B \neq \emptyset$. Since CW pairs are good, by corollary 3.4.1 we may pass to the quotient space $B/(A \cap B)$ and X/A, which are homeomrophic.

Corollary 3.4.5: Wedge Sum Homology Group

Let X_α be topological spaces and $\bigvee \alpha X_\alpha$ their wedge sum. Then the inclusion $\iota_\alpha : X_\alpha \hookrightarrow \bigvee X_\alpha$ indcues an isomorphism:

$$\bigoplus_\alpha : (\iota_\alpha)_* : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

given the wedge sum is formed at the same basepoint $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good.

Proof **:**
This follows from the proposition, since reducd homology is the same as the homology relative to a basepoint, giving us:

$$(X, A) = \left(\cup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}\right)$$

The following conseuqence is of great importance:

Theorem 3.4.4: Invariance Of Dimension

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be nonempty open nsubsets. Then if $U \cong V$, then $m = n$.

Proof **:**
Take $x \in U$, and let $A = \mathbb{R}^n - \{x\}$, $B = U$, so that $A \cup B = \mathbb{R}^n$ and $A \cap B = U - \{x\}$. Since $(B, A \cap B)$ is a good pair of a CW complex, by excision:

$$H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\})$$

By the long exact seuqence for the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, we get

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \tilde{H}_{k-1}(\mathbb{R}^n - \{x\})$$

Since $\mathbb{R}^n - \{x\}$ deformation retracts onto S^{n-1}, we get that

$$H_k(U, U - \{x\}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Thus, by the same reasoning:

$$H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq m \end{cases}$$

Since a homeomorphism $\varphi : U \to V$ indcuces an isomorphism $\varphi_* : H_k(U, U - \{x\}) \to H_k(V, V - \{h(x)\})$, we get that it must be that $n = m$, as we sought to show.

For a statement that may at fisrt seem rather complicated to prove explicitely using classical tools of topology ,the proof turns out to be rather simple. To me, this is a good indicator, since it shows that the notion of dimension is not in this context a very "subtle" idea, but instead quite a natural one.

Note that the converse is not true for general open sets (take a punctured $\operatorname{Int} D^n$ and $\operatorname{Int} D^n$, they have the same dimension but different homology groups). Limiting to simply connected sets, the case of $n = 1$ is easily checkable as an exercise, the case of $n = 2$ is true by the Riemann mapping theorem (which states the stronger condition that simply connected sets are in fact biholomorphic, to D^2). For $n \geq 3$, it starts getting more complicated; If U is contractible and still simply connected when adding the point at infinity. The extra condition is necessary, beign contractible does not imply being homeomorphic to D^3 (whiteheads manifold is a counter-example). These explorations continue in a class on Geometry.

The idea of the above theorem can be generalized to find when two spaces are not locally homeomorphic:

Definition 3.4.4: Local Homology Groups

Let X be a topological space and $x \in X$ be a [closed] point. Then

$$H_n(X, X - \{x\})$$

is called the *local homology group* of the space X at x.

If U is an open neighbourhood of X, then by excision we get

$$H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$$

meaning the local homology groups only depends on the local topology of X near x. Then a homeomorhpism induces:

$$H_n(X, X - \{x\}) \cong H_n(Y, Y - \{f(x)\})$$

for all x nad n. Hence, we see that the local homology group can be used to study spaces locally, and determine whether a sapce is locally homeomorphic at certain points.

3.4.1 Naturality

(Hatcher finishes off by checking the naturality of of short exact sequences being assocaited to chain complexes)

3.4.2 Mayer-Vietoris Sequences

As we saw, the snake lemma provided a useful way of linking information between homology groups and relative homology groups. There is another long exact sequence that gives us more information, and is in many cases easier to computer.

Theorem 3.4.5: Mayer-Vietoris Sequence

Let $A, B \subseteq X$ such that X is the union of the interior of A and B. Then we have the following exact sequence

$$\cdots \to H_n(A\cap B) \xrightarrow{\varphi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \xrightarrow{\varphi} H_{n-1}(A) \oplus H_{n-1}(B) \to \cdots \to H_0(X) \to 0$$

called the *Mayer-Vietoris sequence*.

The idea behind this sequence is that we often use it for induction proofs, where we know information about A, B and $A \cap B$, and build up to show information about the homology of $A \cup B$.

(look at commented link for another proof)

Proof **:**
Hatcher p.149

Let $C_n(A+B) \leqslant C_n(X)$ consitsing of chains that are sums of chains in A and B. Then $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A+B)$ to $C_{n-1}(A+B)$, so the collection of $C_n(A+B)$'s form a chain complex. Then by proposition (Hatcher 2.21), the inclusion $C_n(A+B) \hookrightarrow C_n(X)$ induces isomorphisms on homology groups. Then the Mayer-Vietoris squence will be the long exact sequence of homology groups asocitaed to the sort exact sequence of chain complexes given by:

$$0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

where $\varphi(x) = (x, -x)$ and $\varphi(x, y) = x + y$. (For exactness, see Hatcher p.150, for explicit ∂ see Hatcher too)

Naturally, we also get a formally identical Mayer-Vietoris sequence for redcued homology groups, namely by augmenting the previosu short exact sequence of complexes in the natural way:

$$\begin{tikzcd}
0 \arrow[r] & C_0(A\cap B) \arrow[r] \arrow[d, "\epsilon"] & C_0(A) \oplus C_0(B)) \arrow[r] \arrow[d, "\epsilon\oplus\epsilon"] & C_0(A+B) \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

If $A \cap B$ is path-connected, then we may think of Mayer-Vietoris sequences as an analog of van-Kapmen's theorem. In particular, the reduced Mayer-Vietoris sequence gives an isomorphism:

$$H_1(X) \cong \frac{H_1(A) \oplus H_1(B)}{\operatorname{im}\varphi}$$

which is exactly the abelianization of the van-Kapempen theorem, where H_1 is the abelianization of π_1 for path connected spaces (shown in 2.A of Hatcher)

(a bit on if A, B, and $A \cap B$ deformation retract onto U, V, and $U \cap V$ respectively, and how that works with CW complex, Hatcher p.150)

Example 3.5: Mayer-Vietoris Sequence

1. Let $X = S^n$, A, B the northern and southern hemispheres so that $A \cap B = S^{n-1}$. Then the reduced Mayer-Vietoris sequence of

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) = 0$$

   hence:

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

   This gives another way of calculating the homology of S^n using induction.

2. Let K be the Klein bottle, and see it as the union of two Möbius bans A and B glued at thier boundary by a homeomorphism. Then A, B, and $A \cap B$ are homotopy equivalence to circles. Then the important part of the Mayer-Vietoris sequence for the decomposition $K = A \cup B$ is

$$0 \to K_2(K) \to H_1(A \cap B) \xrightarrow{\varphi} H_1(A) \oplus H_1(B) \to H_1(K) \to 0$$

   The map $\varphi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $1 \mapsto (2, -2)$, since the boundary circle of a Möbius band wraps twice around the core circle. Since φ is injective, we get

$$H_2(K) = 0$$

   Furthermore, $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$, since we can choose $(1, 0)$ and $(1, -1)$ as a basis for $\mathbb{Z} \oplus \mathbb{Z}$. The higher homologys are all trivial as we saw earlier. Hence:

$$\tilde{H}_n(K) = \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

3. (A generalization of Mayer-Vietoris in some cases, p. 151 Hatcher)

Finally, the relative version of Mayer-Vietoris is sometimes useful. Let

$$(X, Y) = (A \cup B, C \cup D) \qquad C \subseteq A, \quad D \subseteq B$$

Then we would get the realtive Mayer-Vietoris sequence to be:

$$\cdots \to H_n(A \cap B, C \cap D) \xrightarrow{\varphi} H_n(A, C) \oplus H_n(B, D) \xrightarrow{\psi} H_n(X, Y) \xrightarrow{\partial}\to \cdots$$

which is a good exercise to check.

Example 3.6: Product With Sphere

Let X be a path-connected topological space. We shall show that

$$H_k(X \times S^n) \cong H_k(X) \oplus H_{k-n}(X)$$

3.4.3 Equivalence of Simplicial and Singular Homology

We have now developed the theory of homology through singular homologies and have developed computational technics via simplicial homologies. We now aim to show the two homology groups are isomorphic. We shall also want to draw parallel between relative homology groups, so to that end if X Is a Δ-complex with $A \subseteq X$ a subcomplex, we may define the realtive groups $H_n^{\Delta}(X, A)$ give $\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A)$ (Since A is the Δ complex fomrd by any union of simplices of X). Next, there is a cannonical homomorphsim $H_n^{\Delta}(X, A) \to H_n(X, A)$ induced by the chain map $\Delta_n(X, A) \to C_n(X, A)$ by sending each n-simplex of$X$ to its characteristic map $\sigma : \Delta^n \to X$. If $A = \emptyset$, the relative homology is the abolute homology.

For it, we shall require an important lemma

Lemma 3.4.2: The Five Lemma

Given the followign commutatative diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[r, "d"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E'
\end{tikzcd}$$

if the two rows are exact, and α, β, δ, and ϵ are isomorphisms, then so is γ.

Proof **:**
Hatcher p. 129, but is also a very common proof.

Theorem 3.4.6: Singular And Simplicial Homology

The homomorhpism $H_n^{\Delta}(X, A) \to H_n(X, A)$ is an isomorphism for all $n \in \mathbb{N}$ and all Δ-complex pairs (X, A)

Proof **:**
Hatcher p.128

Some interesting consequences can immediately be deduced. A finitely generated abelian group must have finitely generated subgroups[3], hence if $\Delta_n(X)$ is finitely gnerated, so must $H_n^{\Delta}(X)$ be, hence $H_n(X)$ would be too. Now that we have a good way of generating $H_n(X)$, it is worth mentioning that the torsion and non-torsion part of the group is given a name:

Definition 3.4.5: Betti Number And Toersion Coefficeints

The number of $\mathbb{Z}$ summand sis called the nth betti number of X¡and the integesr specifying the orders of the finite cyclic summands are called the *torsion coefficients*.

[3]This need not be the case for non-abelian groups

3.4.4 Retractions

Next, we look at how the induced map form retraction behaves. It turns to induce really simple group structures, as perahps can be expected given its a section map. We first require a lemma.

Lemma 3.4.3: Splitting Lemma

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abliean groups. Then the following are equivalent:

1. There is a splitting homomorphism $p : B \to A$ (i.e. a section map)
2. there is a splitting homomorphism $s : C \to B$ (i.e. a section map)
3. There is an isomorphism $B \cong A \oplus C$

If any of these are satisfeid, the exact sequence is said to split

Proof **:**
See any book on projeective modules, for ex. Dummit and Foote, or EYTNKA Algebra.

Proposition 3.4.2: Retraction And Split Sequence

Let $r : X \to A$ be a retraciton. Then:

$$H_n(X) \cong H_n(A) \oplus H_n(X, A)$$

Proof **:**

3.5 Cellular Homology and Euler Characteristic

We shall now take some time to give some applications and computations of important spaces, in particular we shall take the time to develop new methods of calculating homologies of CW complexes. For these computations, we shall explore the notion of *degree* which generalizes the notion of winding number and brings the idea of Euler Characteristic. Next define Cellular homology for CW complex and show it's isomorphic to singular (and hence simplicial) homology groups. We then briefly mention the induced homology of a retraction, and prove the imporant theorem of *invariance of domain.*

3.5.1 Degree

In complex analysis, the winding number of a funciton gives a lot of information about the number of roots as well as the values of path integrals. The following is a generalization of this concept:

Definition 3.5.1: Degree

A continuous map $f : S^n \to S^n$ induces a map $f_* : H_n(S^n) \to H_n(S^n)$ which must be of the form $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ depending on f. Then d is called the *degree* of f, and is denoted $\deg f$.

Proposition 3.5.1: Properties Of Degree

1. $\deg \mathrm{id} = 1$
2. If s is not surjective, then $\deg f = 0$. Note the converse is not true
3. $f \simeq g$, then $\deg f = \deg g$ (the converse is more difficult to prove, see ref:HERE[a])
4. $\deg fg = \deg f \deg g$. Hence $\deg f = \pm 1$, f is a homotopy equivalence since $fg \simeq \mathrm{id}$ implies $\deg f \deg g = \deg \mathrm{id} = 1$
5. $\deg f = -1$ if f is a reflect onto S^n, fixing the points in a subsphere S^{n-1}, and inerchanging the two complementary hemispheres
6. The antipodal points $-\mathrm{id} : S^n \to S^n$ given by $x \mapsto -x$ has degree $(-1)^{n+1}$
7. If $f : S^n \to S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$, in particualr it is antipodal.

[a]Hopf from 1925 showed it, corollary 4.25 in Hatcher

Proof **:**

1. Immediate since $\mathrm{id}_* = \mathrm{id}$
2. If $x_0 \in S^n - f(S^n)$, then f can be factored as
$$S^n \to (S^n - \{x_0\}) \hookrightarrow S^n$$
and $H_n(S^n - \{x_0\}) = 0$ since $S^n - \{x_0\}$ is contractible, hence $f_* = 0$
3. $f \simeq g$ then $f_* = g_*$ so $\deg f = \deg g$
4. Consequence of $(fg)_* = f_* g_*$
5. We can give S^n a Δ-compelx structure with these two hemisphers as its two n-simplcies Δ_1^n and D_2^n and n-chains $\Delta_1^n - \Delta_2^n$ represetns generators of $H_n(S^n)$, as seen in a preivou example. THus, the relfection interchangign Δ_1^n and Δ_2^n send this generator to its negative
6. this maps is a composition of $n+1$ reflections, each chagnign the sign of one coordiante in $\mathbb{R}^{n+1}$
7. If $f(x) \neq x$, then the line segment from $f(x)$ to $-x$ defined by $t \mapsto (1-t)(f(x) - tx)$ does not pass through the origin. Hence if f has no fixed poitns, the formula
$$f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

defines a homotopy from f to the antipodal points. The antipodal maps have no fixed points, so the fact that maps without fixed oints are homotopic to the antipodal points are a converse of that

For (2), can you think of a counter-example for the converse? Degree's immediately give a few interesting and famous results:

Theorem 3.5.1: Hairy ball theorem

Let $X = S^n$. Then X has a contiuous field of nonzero tanngent vectors if and only if n is odd

***Proof* :**
Let $V : S^n \to TS^n$ where $x \mapsto v(x)$ be the vector-field map. Thinking of $v(x)$ as a vector at the origin and as x as a vector, we see that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$.

Now, if $v(x) \neq 0$ for all x, then

$$\frac{v(x)}{|v(x)|}$$

is well-defined. Replacing $v(x)$ by this normalization, the vectors

$$\cos(t)x + \sin(t)v(x)$$

are in teh unit circle in the plane spanned by x and $v(x)$. Taking t from 0 to π, we get a homotopy:

$$f_t(x) = \cos(t)x + \sin(t)v(x)$$

from S^n to the antipodal map $-\operatorname{id}$. But then:

$$(-1)^{n+1} = \deg(-\operatorname{id}) = \deg(\operatorname{id})1$$

implying n is odd. Conversely if n is odd so that $n = 2k - 1$, we can take

$$v(\vec{x}) = (-x_2, x_1, ..., -x_{2k}, x_{2k-1})$$

Then $v(x)$ is orthogonal to x, so v is a tangent vector field on S^n and $|v(x)| = 1$ for all $x \in S^n$.

Proposition 3.5.2: Free Action on Sphere

$\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group that can act freely on S^n if n is even

***Proof* :**
Let $G \curvearrowright S^n$ so that there is a homomorphism $\rho : G \to \operatorname{Aut}_{\mathbf{Top}}(S^n)$. Since homeomorphisms have degree ± 1, an action by a group G on S^n induces a "degree" function $d : G \to \{\pm 1\}$, which is a homomorphism since $\deg fg = \deg f \deg g$. If the action is free, d sends nontrivial elements of G to $(-1)^{n+1}$ by property 7 above. But then, when n is even, d has trivial kernel, hence $G \subseteq \mathbb{Z}_2$.

We now proceed how to compute the degree of a map in the most common case, that is when for some $y \in S^n$, $f^{-1}(y)$ is finite. Say $x_1, x_2, ..., x_n$ are the points with disjoint neighbourhoods $U_1, U_2, ..., U_n$.

Then $f(U_i - x_i) \subseteq V - y$ for each i, and we have:

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V-y) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "\pi_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - x_i) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"] \arrow[ul, "\cong"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd} \tag{3.2}$$

where k_i and π_i are indcued by inclusion, and all triangles and square commute. The two isomorphisms in the upper half are givne by excision, the lower two isomorphissm from exact sequences of pairs. Given these four isomorphisms, the top two gropus in the diagram being identified as $H_n(S^n) \cong \mathbb{Z}$ and the top homomorphism f_* becomes multiplication by an integer, and is given a name:

Definition 3.5.2: Local Degree

Given the above setup, the *local degree* at x_i is the degree ofthe map f_* and is written as:

$$\deg f|_{x_i}$$

For example, if f is a homeomorphism, then $f^{-1}(y) = x$ is a single point, and so $\deg f|_{x_i} = \deg f = \pm 1$. More generally, if f maps to each U_i homeomorphically onto V (if f is a local homeomorphism), then we have

$$\deg f|_{x_i} = \pm 1 \qquad 1 \leq i \leq n$$

where the sign is usually rather easy to determine. From this, we can get the general degree:

Proposition 3.5.3: Computing Degree

Given the setup as above, we have:

$$\deg f = \sum_i (\deg f|_{x_i})$$

Proof **:**

In the diagram at equation (3.2), the term $H_n(S^n, S^n - f^{-1}(y))$ is the direct sum of $H_n(U_i, U_i - x_i) \cong \mathbb{Z}$ and $k - i$, the inclusion of the ith smmand. The π_i is the projection onto the ith summand since the upper triangle commutes and

$$\pi_i k_i = 0 \qquad j \neq i$$

since $\pi_i k_i$ facotrs through $H_n(U_j, U_j) = 0$. Identifying the outer group in teh diagram with $\mathbb{Z}$, commutativity of the lower triangles gives us that

$$\pi_i j(1) = 1 \implies j(1) = (1, ..., 1) = \sum_i k_i(1)$$

Commutativity of the upper square gives that the middle f_* takes $k_i(1)$ to $\deg f|_{x_i}$, thus

$$\sum_i k_i(1) = j(1) \mapsto \sum_i \deg f|_{x_i}$$

Commutativity of the lower square finally gives us:

$$\deg f = \sum_i \deg f|_{x_i}$$

as we sought to show.

Proposition 3.5.4: Suspension And Degree

Let $f : S^n \to S^n$ be a continuous map, $Sf : S^{n+1} \to S^{n+1}$ its suspension. Then:

$$\deg Sf = \deg f$$

***Proof* :**
Hatcher p.137. The key is htat

$$H_n(X) = H_{n+1}(SX)$$

Example 3.7: Computing Degree

1. For a map $f : S^n \to S^n$, let $q : S^n \to \bigvee_k S^n$ be the quotient map given by collapsing teh complement of k disjoint open balls B_i in S^n to a point, and let $\pi : \bigvee_k S^n \to S^n$ identify all the summands to a single sphere. Then take $f = \pi \circ q$. Then teh local degree of f at x_i is ± 1 since f i a homeomorphism near x_i. By precomposing p iwth reflection of the summands of $\bigvee_k S^n$ if necessary, we can make each loca ldegree either 1 or-1 giving us a map with degree $\pm k$

2. Let $n = 1$ so that we have S^1. Show that the map $f(z) = z^k$ has degree k for $k \in \mathbb{Z}$. Since suspension preserve degrees, another way of getting a degree k map from S^n to S^n is by taking repeated suspensions of the map $z \mapsto z^k$.

Note that for a map $f : S^n \to S^n$, the suspension Sf maps only 1 point to each of the poles of S^{n+1}, hence the local degree of Sf at each point must equal the global degree of Sf. hence, the local degree of a map can be any intger if $n \geq 2$, just like for the global degree

3.5.2 Cellular Homology

We now move onto computing the homology groups of cellular complex. These shall take advantage of the above degree calcualtions. Before computing, we estalish some important preliminary results:

Lemma 3.5.1: Cellular Homology Lemma

Let X be a CW complex. Then:

1. $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$ and a free abelian group for $k = n$, with basis in injective correspondence with the n-cells of X

$$H_n(X^n, X^{n-1}) = \mathbb{Z}^m$$

2. $H_k(X^n) = 0$ for all $k > n$. In particular, If X is fininite-dimensional, then $H_k(X) = 0$ for $k > \dim X$

3. The map $H_k(X^n) \to H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$, and surjects for $k = n$

$$\begin{aligned} &H_k(X^n) \cong H_k(X) && k < n \\ &H_n(X^n) \twoheadrightarrow H_n(X) && k = n \end{aligned}$$

Note that $\dim(X) \in \{m, \infty\}$.

Proof **:**

1. Since (X^n, X^{n-1}) is a good pair and X^n/X^{n-1} is the wedge sume of n-spheres, the result follows.

2. Take the long exact sequence:

$$\cdots \to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots$$

If $k \neq n$, the last term is zero by part (a), hence the middle map is surjective, and if $k \neq n-1$, the first term is zero hence the middle map is also injective, giving an isomorphism:

$$H_k(X^{n-1}) \cong H_k(X^n) \qquad k \notin \{n, n-1\}$$

Now, looking at the inclusion-induced homomorphism:

$$H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1}) \to \cdots$$

we get that they are all isomorphism excepet for the map to $H_k(X^k)$,

$$H_k(X^{k-1}) \to H_k(X^k)$$

which may not be surjective and the map from $H_k(X^k)$

$$H_k(X^k) \to H_k(X^{k+1})$$

which my not be injective. Hence, the non-trivial homomorpihsm are:

$$\cdots \cong H_k(X^{k-2}) \cong H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1}) \cong H_k(X^{k+1}) \cong \cdots$$

The first part of the sequence now gives (2) since $H_k(X^0) = 0$ for $k > 0$, giving us (a sequence, not exact!):

$$\cdots \to 0 \to H_k(X^k) \to H_k(X^{k+1}) \cong H_k(X^{k+1}) \cong \cdots$$

Now, if $\dim(X) = n$, then for any $k > n$ it is clear that any element of $H_k(X^n)$ is zero the kernel being zero, hence:

$$\cdots \to 0 \to H_k(X^k) \to 0 \to \cdots$$

giving us that the only interesting homology information is going on when $k = n$

3. For the finite dimensional case, the last part of the above sequence gives (3). For the infinite dimensional case, long (Hatcher p.138)

By using the above lemma, we get the following commutative diagram:

$$\begin{tikzcd}
 & & & & & 0 & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \arrow[urr] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n-1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 9 \arrow[ur] & & &
\end{tikzcd}$$

where $d_{n+1} = j_n\partial_{n+1}$ and $d_n = j_{n-1}\partial_n$. These are the "relative boundary maps" of ∂_{n+1} and ∂_n; notice that $d_n d_{n+1} = 0$. We thus give a name to the horizontal row:

Definition 3.5.3: Cellular Chain Complex

In the above diagram, the chain complex is called the *cellular chain complex*.

Hence, it has a homology group, let's label it as $H_n^{CW}(X)$. The key fact is the following:

Theorem 3.5.2: Cellular Homology

$$H_n^{CW}(X) \cong H_n(X)$$

***Proof* :**
Hatcher p.140

Corollary 3.5.1: Consequence Of CW Homology

1. $H_n(X) = 0$ if X is a CW complex with no $n$cells
2. If X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements.
3. If X is a CW complex having no two of its in adjacent dimensions, then $H_n(X)$ is a free abelian group with basis in one-to-one correspondence with the n-cells of X.

***Proof* :**

1. immediate

2. Since $H_n(X^n, X^{n-1})$ is free abelian on k-generators, the subgroup $\ker d_n$ must be generated by at most k elements, hence similarly for the quotient $\ker d_n / \operatorname{im} d_{n+1}$

3. The cellular boundary maps d_n are automatically zero, giving the result.

Example 3.8: CW Homology

Take $X = \mathbb{C}\mathrm{P}^n$, which has a CW structure with one cell each even dimension as we saw way back. Thus:

$$H_k(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k \in \{0, 2, 4, ..., 2n\} \\ 0 & \text{otherwise} \end{cases}$$

In general, our CW complex can be more complicated, and hence we need a way to compute d_n:

Proposition 3.5.5: Cellular Boundary Formula

Let X be a CW complex, and consider the celuular homology with boundry map $d_n : C_n(X^n, X^{n-1}) \to C_{n-1}(X^{n-1}, X^{n-1})$. For e^n_α is a cell with attaching map:

$$\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$$

and

$$d_{\alpha\beta} : S^{n-1} \xrightarrow{\cong} \partial e^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{\pi} X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta) \xrightarrow{\cong} S^{n-1}$$

representing the attaching map φ_α with respect to an $(n-1)$-cell e^{n-1}_β, then d_n is determined by how it acts on each e_α and

$$d_n(e^n_\alpha) = \sum_\beta \deg(d_{\alpha\beta}) e^{n-1}_\beta$$

***Proof* :**

It's a big comutative diagram chase, see Hatcher p.140

The above proposition now gives us all we need to calculate the homology of CW structures, it is fully based on the gluing maps:

Example 3.9: Computing Cellular Boundary Map

1. Let $X = M_g$ be a closed orientable surface of genus g with the usual CW structure of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of the commutative $[a_1, b_1], ..., [a_g, b_g]$. Then the associated celular chain is:

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

   Then d_1 must be zero since there is only one 0-cell. d_2 is also zero since each a_i or b_i appears with its inverse in $[a_1, b_1], ..., [a_g, b_g]$, hence the map $\Delta_{\alpha\beta}$ are homotopic to constants maps. Since d_1, d_2 ar eboth zero, teh homology groups of M_g are the same as the cellular chain groups, hence:

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{2g} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

2. Let $X = N_g$ be closed nonorientable surface N_g of genus g which has one 0-cell, g 1-cells ,and one 2-cell attached by the word $a_1^2 \cdots a_g^2$. We again ahve $d_1 = 0$ and $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ is given by $d_2(1) = (2, ..., 2)$ since ach a_i appeas in teh attaching words of the 2-cell with total exponent 2, which means $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$ of degree 2. Since $d_2(1) = (2, ..., 2)$, we get that d_2 is injective, hence $H_2(N_g) = 0$. If we change the basis for $\mathbb{Z}^g$ by replacing the last standard basis element $(0, ..., 0, 1)$ by $(1, ..., 1)$ we see that

$$H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$$

Hence:

$$H_n(N_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Notice that we have detected orientability of a closed conencted n-manifold by looking at $H_n(M)$, which was $\mathbb{Z}$ when M is orientable and 0 otherwise, this pattern generlizes, see re:HERE (theorem 3.26 in Hatcher)

3. (3-dimensional Torus, Hatcher p.142)

4. (Moore space) Let G be an abelian group and take $n \in \mathbb{N}$. Then we can construct a CW complex such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq 0$. The construction we will be doing is called the *Moore space*, and is often denoted $M(G, n)$

   By some group theory, we know that $G = F \oplus T$ for some torsion part T and free part F. The simplest case is $G = \mathbb{Z}$. Then for any n we have $X = S^n$. If $G + \mathbb{Z}/m\mathbb{Z}$, then for any n take S^n and attach e^{n+1} via $\varphi : S^n \to S^n$ of degree m (you can inductively take the suspension on $z \mapsto z^m$). For the general case (not necessarily finitely generated), take $F \to G$ to be the free map from the free abelian group F to G. The kernel of this map is free since F is free. Then if $\{x_\alpha\}$ is a basis for F, $\{y_\beta\}$ is a basis for the kernel K, and $y_\beta = \sum_\alpha d_{\beta\alpha} x_\alpha$, we define

$$X^n = \bigvee_\alpha S^n_\alpha \quad \implies \quad H_n(X^n) \cong F$$

   Now, to construct X from ^{n}¡ we shall attach cells e^{n+1}_β via $f_\beta : S^N \to X^n$ such that f_b projects onto the summand S^n_α with degree $d_{\beta\alpha}$. The cellular boundary map d_{n+1} will be the inclusion $K \hookrightarrow F$, hence X has the disred homology.

   Now, by taking teh wedge sum of Moore spaces for varying n, we can get a connected CW complex with arbitrary abelian groups G_n at every dimension.

5. Let $X = \mathbb{R}\mathrm{P}^n$. Then we saw that $\mathbb{R}\mathrm{P}^n$ has a CW structure of one cell, e^k for each $k \leq n$, and the attaching map e^k is the 2-sheeted covering $\varphi : S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$. Th comptue d_k, we

compute the degree os

$$S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \to q\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$$

The map $q \circ \varphi$ restricts to a homeomorphism from each $S^{k-1} - S^{k-1}$ onto $\mathbb{RP}^{k-1} - \mathbb{RP}^{k-2}$, and these two homeomrophsism are given by precomposing with antipodal maps of S^{k-1}, which as degree $(-1)^k$. THus:

$$\deg q \circ \varphi = \deg \mathrm{id} + \deg(-\mathrm{id}) + 1 + (-1)^k$$

Hence, d_k is either 0 or multplication by 2, depending on whether k is even or odd. Thus, we get:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \qquad n \text{ is even}$$
$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \qquad n \text{ is odd}$$

Thus, if n is *odd*:

$$\mathbb{H}_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0,\ k = n \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd}, 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

and if n is *even*:

$$\mathbb{H}_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd}, 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

6. (Lens space p.144)

As we noted under definition 2.1.8, having a trivial fundamental group does not imply the sapce is contractible. The same is true for when a space has only trivial homology groups, this does not imply it's contractible. We first give a name to such spaces:

Definition 3.5.4: Acyclic Space

Let X be a space. Then if $\tilde{H}_i(X) = 0$ for all i, we call such a space *acyclic*.

Theorem 3.5.3: Excised Sphere Is Acyclic

$S^n - B$ is acyclic where B is a k-cell

***Proof* :**
We do induction on k. Suppose it is true for $k-1$ cells. We ahve shown taht under at least one homomorhpism, i_* orj_* induced by $\iota : (S^n - B) \to (S^n - B_1)$ or $j : (S^N - B) \to (S^n - B_2)$, the iamge of a non-trivial 2 is not trivial.

Suppoe there exists a non-trivial $d \in \tilde{H}_i(S^n - B)$. Without loss of generailty suppose $\iota_*(\alpha) \neq 0$ in

$H_i(S^n - B_i)$. Let $B_1 = B_{1,1} \cup B_{1,2}$. Think:

$$[0,1] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$$

Then $\cap_i [a_i, b_i] = \{e\}$, Let $D_i I^{k-1} \times [a_i, b_i]$. Then:

$$E = h(I^{k-1} \times \{e\}) \qquad E = \cap D_i$$

The iamge of α under the indcued hom is trivial, so

$$\tilde{H}_i(S^n - E) = \{0\}$$

Now $S^n - E = \cup_i (S^n - D_i)$. Now there exitss a compset subset A such that the iamge of α vanishes, but $A \subseteq S^n - D_i$ for some i, a contradiction

Another example given by Hatcher is the space X obtained from $S^1 \vee S^1$ by attaching two 2-cells through $a^5 b^{-3}$ and $b^4 (ab)^{-2}$, see Hatcher p.142

Theorem 3.5.4: More Homology calculations

Let $0 \leq k < n$ and let $h : S^k \to S^n$ an embedding. Then

$$\tilde{H}_i(S^n - h(S^k)) \cong \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise} \end{cases}$$

***Proof* :**
We do induction on K, if $k = 0$, then $S^k \{p, q\}$. If we remove two points from S^n, then

$$S^n - h(S^k) \cong \mathbb{R}^n - \{0\}$$

hence, it deformation retracts to S^{n-1} Hence

$$\tilde{H}_i(S^n - S^-) \cong \begin{cases} \mathbb{Z} & i = n - 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, by induction suppose the theorem holds true in dim $= k - 1$. Consider $h : S^k \to S^n$. Take:

$$X = S^n - h(S^{k-1}) = X_1 \cup X_2$$

Let $X = S^n - h(E^k_+)$ and $X_2 = S^n - h(E^k_-)$ so that

$$X = X_1 \cup X_2 = S^n - h(S^{k-1}) \qquad X_1 \cap X_2 = A = S^n - h(S^k)$$

Then by Mayer=viatoris:

$$\tilde{H}_{i+1}(X_1) \oplus \tilde{H}_{i+1}(X_2) \to \tilde{H}_{i+1}(X) \to \tilde{H}_i(A) \to \tilde{H}_i(X_1) \oplus H_i(X_2)$$

First, we know the term on the left and right are zero, an have:

$$0 \to \tilde{H}_{i+1}(X) \to \tilde{H}_i(A) \to \tilde{0}$$

This gives

$$\tilde{H}_{i+1}(X) = \tilde{H}_i(A)$$

then $i + 1 = n - (k - 1) - 1$, gives the non-trivial result, and 0 otherwise.

skipped for now, since it required information from appendix 1.B, not covered curently in class.

Theorem 3.5.5: Generalized Jordan Curve Theorem

Let $n > 0$, Let C be a subset of S^n homeomorphic to the $n-1$ sphere. Then $S^n - C$ has precisely two components, of which C is a common topological boundary.

Proof **:**
As we know, we want to calculate

$$\tilde{H}_0(S^n - C) = \mathbb{Z}$$

(uhh not done?)

Theorem 3.5.6: Invriance Of Domain

Let $U \subseteq \mathbb{R}^n$ open, and $f : U \to \mathbb{R}^n$ injective continuous map. Then $f(U)$ is open in $\mathbb{R}^n$, and f is an embeddings.

Proof **:**
let $f : U \to S^n$. be continuous. want to show $f(U)$ is open in the usual topological way, for every point there is an open neighbourhood. Choose an open ball in U for which the closure is contained in U (which we may do since $U \subseteq \mathbb{R}^n$[a]). Then

$$S_\epsilon = \overline{B_\epsilon(x)} - B_\epsilon(x)$$

is a sphere of dim $n-1$¿ Then

$$f(S_\epsilon) \cong S^{n-1}$$

Thus, it separates S^n into two connected component: W_1, W_2. each W_i is open in S^n. Sicne $f(B_\epsilon)$ is connected, then $f(B_\epsilon) \cap f(S_\epsilon) = \emptyset$. So $f(B_\epsilon)$ is either in W_1 or W_2. If it sits in W_1, it must be equal to W_1, so $f(B_\epsilon)$ must be open, and is the neighbourhood we were looking for, completing the proof.

[a]since $\mathbb{R}^n$ is normal, or by direct metric manipulation

3.5.3 Euler-Characteristic

Recall the famous equation for polygons in $\mathbb{R}^2$:

$$V - E + F = 2$$

We can define a similar constnat for any CW complex X, namely its *Euler Characteristic.* defiend to e

$$\chi(X) = \sum_n (-1)^n c_n$$

where c_n is the number of n-cells of X. We sahll see that this quantity can in fact be defined purely homologically, showing it wil be independent of CW structure:

Theorem 3.5.7: Euler's Characteristic

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$$

Proof **:**
Hathcer p.146 (A purely algebraic proof!)

Recall that rank $H_n(X)$ counts the number of n-cells. Hence, this formula is a sum on the number of n-cell. Using this, prove that $\chi(X \times Y) = \chi(X)\chi(Y)$

3.5.4 Lefschetz Fixed Point Theorem

Next, let G be a free-abelian group. Take the group homomorphism $\varphi : G \to G$. Then we can rerpesent φ in a matrix form, say A. Then if we have a finite simplicial complex K[4] and we have a chain map $\varphi : C_p(K) \to C_p(K)$ (same G). Then we can write $\operatorname{tr}(\varphi, C_p(K))$. Then $H_p(K)$ is not necessarily free, but we can factor the torsion part out and make it free: $H_p(K)/T_p(K)$. These ranks are called *Betti numbers*. If φ is as above. it induces a map

$$\varphi_* : H_p(X)/T_p(K) \to H_p(K)/T_p(K)$$

Theorem 3.5.8: Hopf Trace Theorem

Let K be a finite (simplicial or CW) complex. Let $\varphi : C_n(K) \to C_n(K)$ be a chain map. Then

$$\sum_{n=0} (-1)^n \operatorname{tr}(\varphi, C_n(K)) = \sum_n (-1)^n \operatorname{tr}\left(\varphi_*, \frac{H_n(K)}{T_p(K)}\right)$$

Proof **:**
see Munkres algebraic topology or Hatcher

From this, we can define the Euler characteristic as:

$$\chi(K) = \sum (-1)^p \operatorname{rank}(C_p(K))$$

For a triangle with a line down the middle we get

$$\chi(X) = 4 - 5 + 2 = 1$$

Theorem 3.5.9: Euler Characteristic from Class

Let K be a simplicial complex. Let β_p be the rank of $H_p(K)$. Then:

$$\chi(K) = \sum_p (-1)^p \beta_p$$

[4] the theorem does generalize to CW complexes

Proof **:**
We use hopf trace formula. Let $\mathrm{id} : K \to K$. Then we get $i_\# : C_p(K) \to C_p(K)$ and a homomorphsim $i_* : H_P(K) \to H_p(K)$. Then the matrix representation give a trace of rank of C_p, and the trace of ι_* is the rank of H_p, but now we are done

Definition 3.5.5: Lefschetz Number

Let K be a finite simplicial complex, $h : K \to K$ a continuous map. Then

$$\Lambda(h) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(h_*(H_i(K)/T_i(K))$$

is called the *lefschetz* number of h

Theorem 3.5.10: Lefschetz Fixed Point Theorem

Let K be a finite complex, $h : K \to K$ continuous. If $\Lambda(h) \neq 0$, then h has a fixed point

Proof **:**
Say $h : K \to K$ where K is connected. Then $h_* : H_0(K) \to H_0(K)$ is an isomorphism. (conclude?)

here is a consequence: Let H be a finite acyclic complex. If $h : K \to K$ is continuous, then h has a fixed point

Proof. $\Lambda(h) = 1$, so it has a fixed point □

The antipodal map $a : S^n \to S^n$ has degree $(-1)^{n+1}$. and

$$\Lambda(a) = 1 + (-1)^n \operatorname{tr}(a, \mathbb{Z}) = 0$$

so $\operatorname{tr}(a, \mathbb{Z}) = (-1)^{n+1}$, but tha that is the degree of the map. Now consider $h : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$, so

$$H_0(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) = \mathbb{Z}, \qquad H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \qquad H_2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) = 0$$

so the Lefschetz number is not zero, so there must be a fixed point.

3.5.5 Homology of Groups

Definition 3.5.6: Chain Over Groups

Consider chains $\sigma_i n_i \sigma_i$ where each σ_i is a singular n-simplex, andthe coeffiients n_i are taken in some fixed group G (rather than just $\mathbb{Z}$.)Thne the collectio nis called

$$C_n(X; G)$$

and the relative version is:

$$C_n(X, A; G) = C_n(X; G)/C_n(A; G)$$

The smae boundary map works as before, that is

$$\sum_i n_i \sigma_i \overset{\partial}{\mapsto} \sum_{i,j} (-1)^j \sigma|_i$$

The reducd homology is now ending with:

$$\cdots \to C_0(X;G) \xrightarrow{\epsilon} G \to 0$$

and as before, is X is a point, then:

$$H_n(X;G) \cong \begin{cases} G & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore:

$$H_n^{CW}(X;G) \cong H_n(X;G) \qquad d_n\left(\sum_\alpha n_\alpha e_\alpha^n\right) = \sum_\alpha (\cdots)$$

Example 3.10: Homology With Coefficients
Take $\mathbb{R}\mathrm{P}^n$ and coefficients in $\mathbb{Z}/2\mathbb{Z}$. Recall it's cellular strucutre is given by $\mathbb{R}\mathrm{P}^n = e_1 \cup \cdots \cup e_n$. Now recall:

$$H_k(\mathbb{R}\mathrm{P}^k, \mathbb{R}\mathrm{P}^{k-1}) \xrightarrow{d_k} H_{k-1}(\mathbb{R}\mathrm{P}^{k-1}, \mathbb{R}\mathrm{P}^{k-1})$$

now we have $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Then $d_k = 1 + (-1)^k$, so if k is even we get 2, and if k is odd we get 0, but in $\mathbb{Z}/2\mathbb{Z}$, that means we always bet 0, hence d_k is a zero map. Hence, we get

$$H_k(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}/2\mathbb{Z} \qquad 0 \le k \le n$$

Lemma 3.5.2: Degree For Abelian Groups

let $f : S^k \to S^k$ be of degree m. Then $f_* : H_k(S^k;G) \to H_k(S^k;G)$ is multiplication by m.

Proof **:**
Hatcher p.154. Keep this commutative diagram in mind:

$$\begin{array}{ccccccc}
\mathbb{Z} \approx & \widetilde{H}_k(S^k;\mathbb{Z}) & \xrightarrow{f_*} & \widetilde{H}_k(S^k;\mathbb{Z}) & \approx \mathbb{Z} \\
\downarrow \varphi & \downarrow \varphi_* & & \downarrow \varphi_* & \downarrow \varphi \\
G \approx & \widetilde{H}_k(S^k;G) & \xrightarrow{f_*} & \widetilde{H}_k(S^k;G) & \approx G
\end{array}$$

3.6 Axioms of Homology

(Give a bit of homology theory, this is developped in EYTNKA Algebra)

Definition 3.6.1: Eilenberg–Steenrod Axioms

Let C be a category of pairs (X, A) of topologial spaces closed under products with intervals I and contains one point spaces $(P, \emptyset) =: P$. Then a sequence of functors $H_n(-) : C \to \mathbf{Ab}$ along with a natural transforamtion $\partial : H_i(X, A) \to H_{i-1}(A, \emptyset)$[a] (called the *boundary map*) that satisfy the following axioms form a *homology theory*:

1. (**Homotopy**) Homotopic maps induce the same homology, that is $f \simeq g : (X, A) \to (Y, B)$ is homotopic, then $f = g : H_n(X, A) \to H_n(Y, B)$
2. (**Excision**) IF (X, A) is a pari and U is a subset of A such that the clsoure of U is contained in the interior of A ,then the inclusino map
$$i(X \setminus U, A \setminus U) \to (X, A)$$
cinduces na isomorphism in homology
3. (**Dimension**) Let P be the one-point spce. Then $H - n(P) = 0$ for all $n \neq 0$
4. (**Additivity**) If $X = \bigsqcup_\alpha X_\alpha$, then
$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$$
5. (**Exactness**) Each pair (X, A) indcues a long exact sequence in homology via the inclusions $\iota : A \to X$ and $j : (X, \emptyset) \to (X, A)$:
$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots$$

If P is the one point space, $H_0(P)$ is called the *coefficient group.*

[a] usually short-handed to $H_{i-1}(A)$

Without the dimension aximo, this is usually called an *extraordinary homology theory*; examples include K-theory and cobordism theory. Simplicial, singular, and cellular homology are all examples of homology theory

4

Cohomology

We shall now develop a dual theory to homological algebra for it's favourable algebraic advantages. A homology was defined given a complex $C_\bullet$. We always have access to the contravariant $\mathrm{Hom}(C_\bullet, G)$ functor that we may wish to apply to study some properties of $C_\bullet$. Often the dual category doesn't bring us much benefit. However, in the case of homology, the dual category for homologies, where the arrows go the other direction, offers some nice algebraic simplifications.

The main thing that cohomology gives us is a new product called the *cup product*. This product is technically present in homology, but is not natural and is conputationally much harder to find. This shall introduce a new algebraic structure, a graded anti-commutative R-algebra structure on $H^\bullet(X, R)$ (given R is a commutative ring) which shall give us more insights on our spaces.

4.1 Cohomology of Spaces

To intuitively understand how to think of $\mathrm{Hom}(C^\bullet, G)$, let X be a 1-dimensional Δ-complex, i.e. a graph. Given a fixed abelian group G (or really any abelian object), the set $\mathrm{Hom}_{\mathbf{Set}}(X, G)$ forms an abelian group, which we denote $\Delta^0(X; G)$ (if $G = \mathbb{Z}$, in the finite dimensional case, this group is [not necessarily naturally] isomorphic to $\Delta_0(X)$). We may do the same for edges defining $\Delta^1(X; G)$ and more generally $\Delta^n(X; G)$. Given these function groups, we shall define a function δ between them. Starting with the simplest case:

$$\delta : \Delta^0(X; G) \to \Delta^1(X; G) \tag{4.1}$$

¡++¿ where φ maps to $\delta(\varphi)$ where if $e = [v_0, v_1]$ is andge:

$$\delta(\varphi)(e) = \varphi(v_1) - \varphi(v_0)$$

If we take equation (4.1) as the simplest of chain complexes:

$$\cdots \to 0 \to \Delta^0(X; G) \xrightarrow{\delta} \Delta^1(X; G) \to 0 \to \cdots$$

Then

$$H^0(X;G) = \ker\delta \qquad H^1(X;G) = \Delta^1(X;G)/\operatorname{im}\delta$$

Now, $H^0(X;G)$ describes maps that have the same value on the two vertices of an edge, that is

$$\delta(\varphi) = 0 \qquad \text{iff} \qquad \varphi(v_i) = \varphi(v_j),\ e = [v_i, v_j]$$

that is φ is *constant* on each component of G (so φ is "locally constant" with respect to edges). Hence, $H^0(X;G)$ is all functions from the sets of component of X to G (similarly to how $H_0(X)$ counts the number of path components).

Next, take $\psi \in H^1(X;G)$ where $\psi : X \to G$ is a function where $\psi(e) \in G$ for each edge. If $\psi = \delta(\varphi)$ for some $\varphi \in \Delta^0(X;G)$, then ψ is purely determined by it's values at it's "extrema", in this case the value of ψ is detrmine by the change of value between each each corresponding to a function φ. For future reference, notice how this is similar to finding a function F given a change-in-value function f (usually the derivative). This connection is made explicit with *deRham cohomology* which we do in EYTNKA Differential Geometry. This parrall goes even further; if there exists a φ such that $\delta\varphi = \psi$, then it is unique up to adding an element of the kernel of δ.

When X is a tree, this equation is always solvable. If X is not a tree (hence it has cycles), then the equation may fail. To measure to what degree this fails, we first pick a maximal tree. Then a ψ can be determined by a φ on this maximal tree, however the value of ψ on eac edge not in the maximal tree must equal the different of already-determined values of φ at two ends of edges, which may not work since ψ can have, in general, arbitrary values on edges. Thus, $H^1(X;G)$ can be thought of as the direct product of G's, one for each edge of X not in the chosen maximal tree. This is simlar to $H_1(X;G)$ which ocnsists of a direct sum of copies of G for each edge X not in the maximal tree as seen in example 2.11 (abelianized).

Now, moving up a dimension, let X be a 2-dimensional Δ-complex and define

$$\delta : \Delta^1(X;G) \to \Delta^2(X;G)$$

$$\delta\psi([v_0, v_1, v_2]) = \psi([v_0, v_1]) + \psi([v_1, v_2]) - \psi([v_0, v_2])$$

i.e. a signed sum of the values of ψ on the edges. Now, combining to form a chain:

$$\Delta^-(X;G) \xrightarrow{\delta} \Delta^1(X;G) \xrightarrow{\delta} \Delta^2(X;G) \tag{4.2}$$

notice that

$$\delta\delta(\varphi) = (\varphi(v_1) - \varphi(v_0)) = (\varphi(v_2) - \varphi(v_1)) - (\varphi(v_2) - \varphi(v_0)) = 0$$

Showing that $\operatorname{im}\delta_1 \subseteq \ker\delta_2$, Again making a chain complex from equation (4.2) we may define $H^2(X;G)$. There are many ways of thinking of elements of $H^2(X;G)$, each giving a useful insight. First, notice htat is $\delta\psi = 0$, then:

$$\psi([v_0, v_2]) = \psi([v_0, v_1]) + \psi([v_1, v_2])$$

showing that the cohomology groups measures the failure of ψ being additive. Another is the usual interpretaion, that it meausre the degree of failure of findign a $\varphi \in \Delta^0(X;G)$ such that $\psi = \delta\varphi$, since if $\psi = \delta\varphi$, then $\delta\psi = 0$ (since $\delta\delta\varphi = 0$). In particular, since the value of the equation $\psi = \delta\varphi$ only depends on values defined on elements, we may think of it as measuring the local obstruction. But since $H^1(X;G)$ is zero if and only if $\psi = \delta\varphi$, we can think of $H^1(X;G)$ as somehow measuring global obstruction.

Now, instead of working with $\Delta^n(X;G) = \mathrm{Hom}_{\mathbf{Set}}(X^n, G)$, we shall work with $C^1(X;G) = \mathrm{Hom}_{\mathbf{Grp}}(\Delta_n(X), G)$. This is equivalent, since $\Delta_i(X)$ i a fre abelian group with basis the n-simplices of X, and a homomorphism from a free object is uniquely determined by where it maps its basis (which, naturally, can be mapped arbitrarily). The group $C^n(X;G)$ is given a name:

Definition 4.1.1: Dual Group

Let X be a topological space and $C_\bullet(X)$ the singular chain groups. Then:

$$C^\bullet(X;G) := \mathrm{Hom}(C_\bullet(X), G)$$

is the *singular cochain groups* with coefficients in G.

The boundary homomorphism ∂_n generalizes to:

$$\delta\varphi([v_0, ..., v_{n+1}]) = \sum_i (-1)^i \varphi([v_0, ..., \hat{v_i}, ..., v_n + 1]) = \varphi(\partial[v_0, ..., v_{n+1}])$$

Hence:

$$\delta(\varphi) = \varphi \circ \partial$$

meaning δ is a proper dual map, since the $\mathrm{Hom}(-, G)$ functor maps $C_n \mapsto \mathrm{Hom}(C_n, G)$ and $\partial \mapsto (\varphi \mapsto \varphi \circ \partial)$. Now, to form a complex, let X be a topological. We may then form a chian:

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

whether singular, simplicila, or cellular. If we take the homology, we get $H_n(X)$. On the otherhand, if we apply the [contravariant] hom functor:

$$\cdots \leftarrow \mathrm{Hom}(C_n, G) \xleftarrow{\delta_n} \mathrm{Hom}(C_{n-1}, X) \leftarrow \cdots$$

In particular, each $\varphi \in C^n(X;G)$ assigned to $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$. Then $\delta = \partial^* : C^n(X;G) \to C^{n+1}(X;G)$ is given by composition:

$$\delta(\varphi) = \partial^*(\varphi) = \varphi \circ \partial$$

In particular:

$$\delta(\varphi)(\sigma) = \sum_i (-1)^i \varphi(\sigma|_{v_0, \ldots, \hat{v_i}, \ldots, v_{n+1}})$$

hence $\delta^2 = 0$, hence the cohomology groups are well-defined.

Definition 4.1.2: Cocycles And Coboundaries

Let δ be defined as above, then element of $\ker \delta$ are called *cocycles* nad element of $\mathrm{im}\, \delta$ are called *coboundaries*. A cochain φ is a cocycle if

$$\delta(\varphi) = \varphi \circ \partial = 0$$

i.e. $\varphi \in \mathrm{Hom}(X, G)$ vanishes on the boundary.

We give the image of $\mathrm{Hom}(-, G)$ a name:

Definition 4.1.3: Cochain Complex

Let X be a topological and $C_\bullet$ a complex. Then $C^*_\bullet = \operatorname{Hom}(C_\bullet, G)$ is a *cochain complex* with respect to G.

From this we get the comology group:

Definition 4.1.4: Cohomology Group

Let $C_\bullet$ be a chain complex. Then for some abelian group G, we may form a chain and dualize with $\operatorname{Hom}(-, G)$ to form the cochain $C^\bullet$. Then:

$$H^n(C; G) = \frac{\ker(\delta)}{\operatorname{im}\delta}$$

is the nth *cohomology group* of X.

We will show that the cohomology group is algebraically determined by the homology group and G. A first hope may be that $\operatorname{Hom}(-, G)$ commutes with taking the quotient:

$$H^n(C; G) \cong \operatorname{Hom}(H_n(C), G)$$

However, this need not be the case:

Example 4.1: Hom Functor Not Commuting With Quotienting

Take the chain:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

where the 2 map sends $x \mapsto 2x$. Then dualizing with $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ we get the (isomorphic) cochain:

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

Then $H_2(C) = 0$, but $H^2(C) = \mathbb{Z}/2\mathbb{Z}$, while $\operatorname{Hom}_G(H_2(C), \mathbb{Z}/2\mathbb{Z}) = 0$.

Though this is not the csae, this is not a bad first guess, and there is at minimum a homomorphism $h : H^n(C; G) \to \operatorname{Hom}_G(H_n(C), G)$. In fact, we can be more precise. First, we require a lemma:

Lemma 4.1.1: Free Resolution Homomorphism

Let F, F' be free resolutions of abelian groups H, H'. Then every homomorphism $\alpha : H \to H'$ can be extended a homomorphism of chain maps F to F':

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d, "\alpha_2"] & F_1 \arrow[r] \arrow[d, "\alpha_1"] & F_0 \arrow[r] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0
\end{tikzcd}$$

Proof **:**

Hatcher p.194, or Aluffi

Recall every abelian group has a free resolution of rank 2 (since every abelian group is a $\mathbb{Z}$-module, and $\mathbb{Z}$ is a PID).

Theorem 4.1.1: Universal Cofficient Theorem For Cohomology

Let C be a chain complex of free abelian groups with homology $H_\bullet(C)$. Then the cohomology groups $H^n(C;G)$ of the cochian complex $\mathrm{Hom}_G(C_\bullet, G)$ is given by:

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}_G(H-n(C), G) \to 0$$

Proof **:**
Denote the set of cycles and boundaries by:

$$Z_n = \ker \partial \subseteq C_n \qquad B_n = \operatorname{im} \partial \subseteq C_n$$

Take $[\varphi] \in H^n(C;G)$ so that $\varphi : C_n \to G$ where $\delta \circ \varphi = 0$, i.e. $\varphi \circ \partial = 0$. In other words, φ vanisehs on B_n. The restriction $\varphi_0 = \varphi|_{Z_n}$ then induces a quoitent homomorphism:

$$\overline{\varphi_0} : Z_n/B_N \to G \in \mathrm{Hom}_G(H_n(C), G)$$

If $\varphi \in \operatorname{im} \delta$ so taht $\varphi = \delta \circ \psi = \psi \circ \partial$, then φ is zero on Z_n, so $\varphi_0 = 0$, thus $\overline{\varphi_0} = 0$. Hence, we have a well-defined quoitnet map:

$$h : H^n(C;G) \to \mathrm{Hom}_G(H_n(C), G) \qquad \varphi \mapsto \overline{\varphi_0}$$

which is naturally a homomorphism. Next, this homomorphism is surjective. First, the short exact sequence:

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1}0 \to 0$$

splits since B_{n-1} is free, being a subgroup of a free abelian group C_{n-1}[a]. Hence, there is a projection homomorphism $\pi : C_n \to Z_n$ that restricts to the idnetity on Z_n. Composing π with φ_0, we get:

$$\varphi = \varphi_0 \circ \pi : C_n \to G$$

Then this "extended" homomrophism vanishes on B_n, i.e, it extends homomorphism $H_n(C0) \to G$ to elements of $\ker \delta$. Composing with the quotient map $\ker \delta \to H^n(C;G)$, we get a homomorphism from $k : \mathrm{Hom}(H_n(C), G) \to H^n(C;G)$. Then $k \circ h = \mathrm{id}$, showing h is surjective, and we get:

$$0 \to \ker h \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

It remains to analyze $\ker h$.

(finish here; hatcher p.191)

, we get to

$$0 \to \operatorname{coker} i^*_{n-1} \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}_G(H_n(C), G) \to\to 0$$

Next, we analyze $\operatorname{coker} i^*_{n-1}$ and show it depends on $H_{n-1}(C)$.

[a] a sub-module of a free $\mathbb{Z}$-module is free

To comput Ext, we have a couple of nice shorhands:

Proposition 4.1.1: Computing Exterior Group

Let H be finitely generated Then

1. $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$
2. $\text{Ext}(H, G) = 0$ if H is free
3. $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$

In particular, if we have homology groups H_n, H_{n-1} with torsion subgroups $T_n \subseteq H_n$ and $T_{n-1} \subseteq H_{n-1}$, then:

$$H^n(C; \mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$$

Proof **:**
hatcher p.196

Naturally, the shrot exact sequences in the universal coefficient theorme are natuarl, that is a chain map $\alpha : C \to C'$ of free abelian groups induces a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \text{Ext}(H_{n-1}(C), G) \arrow[r] & H^n(C; G) \arrow[r] & \text{Hom}(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \text{Ext}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*"] & H^n(C'; G) \arrow[r] \arrow[u, "\alpha^*"] & \text{Hom}(H_n(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*"] & 0
\end{tikzcd}$$

On the otherhand, the splitting is not natural, since it depends on teh projction $\pi : C_n \to Z_n$.

Together with the short-five lemma we get:

Corollary 4.1.1: Homology Isomorphism, then Cohomology Isomrophism

Let α be a map between chains. Then if α induces an isomorphism on homology groups, it induces an isomorphsim on cohomology gruops with any coefficeint group G.

Proof **:**
immediate

Note that instead of using abelian groups (i.e. $\mathbb{Z}$-modules), we could use general R-modules. If R is a PID, then then the universal coefficient theorem is exactly the same, while if R is not then we may have longer free resolutions. If $R = F$ is a field, then any F-module isalways free, hence we always have length 0 free resolutions.

Some remarks are in order: when $n = 0$, then $\text{Ext} = 0$, and so we simply get

$$H^0(X; G) \cong \text{Hom}(H_0(X), G)$$

Expicitely, we can see this since singular 0-simplies are just points, and a cochain in $C^0(X; G)$ is an arbitrary function $\varphi : X \to G$, not necessarily continuous. For this to be a cocycle, we have that each 1-simplex $\sigma[v_0, v_1] \to X$ w ehave

$$\delta(\varphi)(\sigma) = \varphi(\partial\sigma) = \varphi(\sigma(v_1)) - \varphi(\sigma(v_1)) = 0$$

Hence, it is equivalent ot saying that φ is constant on path components of X. THus, $H^0(X;G)$ is all the function from path-components of X ot G, which is the same as $\mathrm{Hom}(H_0(X),G)$.

Similarly, since $H_0(X)$ is free, $\mathrm{Ext}(H_0(X),G)0$, so

$$H^1(X;G) \cong \mathrm{Hom}(H_1(X),G)$$

Finally, when $G = F$ is a field, then Ext_F vanishes, and so

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F),F)$$

Example 4.2: Cohomology Groups

For reference, we recall the following Δ-complexes from example 3.2

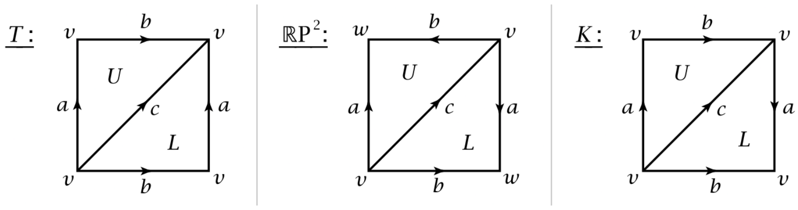


Figure 4.1: Reference Figure for Torus, Projective space, and Klein Bottle

In the proceeding calculations, we may either use the Δ-comlex or the cellular complex (we shall later go over how they are all equivalent, which is a rather routine algebraic argument)

1. Let $X = T = S^1 \times S^1$ and

$$\Delta_0(T) = \mathbb{Z} \qquad \Delta_1(T) = \mathbb{Z}^3 \qquad \Delta_2(T) = \mathbb{Z}^2$$

with the chain:

$$0 \to \Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial 1} \Delta_1 \to 0$$

taking the $\mathrm{Hom}(-,\mathbb{Z})$ functor, we get:

$$0 \to \mathrm{Hom}(\Delta_0,\mathbb{Z}) \xrightarrow{\partial^1} \mathrm{Hom}(\Delta_1,\mathbb{Z}) \xrightarrow{\partial^2} \mathrm{Hom}(\Delta_2,\mathbb{Z}) \to 0$$

By some basic algebra, if the generators of Δ_0 is v, the generators ofΔ_1 are a, b, c and the generators of Δ_2 are U, L, then the generators of the hom-dual are v^*, a^*, b^*, c^*, and U^*, L^*. which are all the usual δ function (for example $v^*(x) = 1$ if $x = v$ and 0 otherwise). Hence, we have:

$$0 \to \mathbb{Z} \xrightarrow{\partial^1} \mathbb{Z}^3 \xrightarrow{\partial^2} \mathbb{Z}^2 \to 0$$

Now, to find these mapping first notice that:

$$\partial^1(v^*)(e) = v^*(v) - v^*(v) = 0$$

hence $\partial^1 = 0$ is trivial. Next

$$\partial^2(a^*)(U) = a^*(a) + a^*(b) - a^*(c) = 1$$
$$\partial^2(a^*)(L) = a^*(c) - a^*(a) - a^*(b) = -1$$

Thus

$$\partial^2(a^*) = U^* - L^*$$

Similarly, we get:

$$\partial^2(b^*) = U^* - L^*$$
$$\partial^2(c^*) = L^* - U^*$$

We now have all the values needed to compute the cohomology groups. This is the same type of linear algebra we have been already doing, the result of which is:

$$H^n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

Now, if $G = \mathbb{Z}/2\mathbb{Z}$, then the dual is isomorphic to:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Z}/2\mathbb{Z}^2 \to 0$$

then we still have $\partial^1 = 0$ (the coefficents do not effect the maps), however we now have that $= -1$, hence $\partial^2(a^*) = \partial^2(a^*) = \partial^2(a^*) = U^* + L^*$. (Put down your computations)

$$H^n(X) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 0 \\ (\mathbb{Z}/2\mathbb{Z})^2 & n = 1 \\ (\mathbb{Z}/2\mathbb{Z}) & n = 2 \\ 0 & n \geq 3 \end{cases}$$

as we sought to show.

2. Let $X = K$ be the klein bottle. Then the cellular complex:

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

With homology group:

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & n = 1 \\ 0 & n \geq 2 \end{cases}$$

When dualizing the chain respect to $\mathbb{Z}$ we get:

$$0 \leftarrow \mathbb{Z} \overset{\delta}{\leftarrow} \mathbb{Z}^3 \overset{\delta}{\leftarrow} \leftarrow 0$$

since $(\mathbb{Z}^n)^* \cong \mathbb{Z}^n$. Let $H^n = H^n(K;\mathbb{Z})$. Now:

$$\delta^0(f)(a,b) = f(\delta_1(a,b)) = 0$$

Hence $\delta^0 = 0$. Nxt:

$$\delta^1(g)(n) = g(\partial_2(n)) = g(2n, 0)$$

Choosing the right basis for C^1, this map becomes

$$(a,b) \mapsto 2a$$

Hence, the cohomology groups are:

$$H^n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \\ 0 & n\geq 3 \end{cases}$$

3. Assuming cellular complexes are isomorphic to singular complexes, we have that:

$$\tilde{H}^k(S^n;G) = \begin{cases} G & n=k \\ 0 & \text{otherwise} \end{cases}$$

Also thought this link was cool: many ways of computing the homology of a torus here

Finally, we quickly go through how many of the concepts from homology, such as relative homology or reduced homology and others, naturally translate to cohomology

(hatcher p.199; reduced is naturally defined, it is homotopy invariant ecsision still works, and the axioms of cohomology translate. Simplical cohomology is naturally isomorphic, cellular cohomology follows too, snake lemma and Mayer-Vietoris transalte)

For cellular homology, in particular their skeletons

Proposition 4.1.2: Cellular Cohomology

$H^n(X;G) \cong \ker d^n / \operatorname{im} d^{n-1}$. Furthermore

$$(H^\bullet(X^n, X^{n-1};G), \delta_\bullet) \cong (\operatorname{Hom}(H^\bullet(X^n, X^{n-1};G), G), d_\bullet \circ h)$$

i.e. the cellular complex is isomorphic to the dual of the cellular chain complex after applying the $\operatorname{Hom}(-,G)$ functor.

Proof **:**
hatcher p.203

4.2 Cup Product

Let $G = R$ now be a ring. We shall show that we can put a graded ring structure on $C^\bullet(X;R)$. If you have done DeRham cohomology, the product we shall define can be thought of as the generalization of the wedge product and shall be behave very similarly:

Definition 4.2.1: Cup Product [on chains]

Let $f \in C^k(X;R)$ and $g \in C^\ell(X;R)$. Then the *cup product*, denoted $f \smile g \in C^{k+\ell}(X;R)$ is the cochain that evaluates on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by:

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0,\dots,v_k]})g(\sigma|_{[v_k,\dots,v_{k+\ell}]}) \in R$$

To see that the cup product works on cohomology classes, we require the following

Lemma 4.2.1: Derivation Property Of Cup Product

$$\delta(f \smile g) = (\delta f) \smile g + (-1)^k f \smile (\delta g)$$

Proof **:**
Let $\sigma : \Delta^{k+\ell+1} \to X$. Then:

$$(\delta f) \smile g(\sigma) = \sum_{i}^{k+1} (-1)^i f(\sigma|_{v_0,\dots,\hat{v}_i,\dots,v_{k+1}})g(\sigma|_{v_{k+1},\dots,v_{k+\ell+1}})$$

$$(-1)^k(f \smile \delta g)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i f(\sigma|_{v_0,\dots,\hat{v}_i,\dots,v_k})g(\sigma|_{v_k,\dots,v_{k+\ell+1}})$$

Adding these together, the last term of the first sume cancels with the first term of the second sum, the remaining terms are then exactly

$$\delta(f \smile g)(\sigma) = (f \smile g)(\partial\sigma)$$

since

$$\partial\sigma = \sum_{i}^{k+\ell+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]}$$

completing the proof.

From this, we see that the cup product of two cocyles is a cocyles, and the product of a cocycles and a coboundary (or a coboundary and a cocycle) is a coboundary since

$$f \smile (\delta g) = \pm\delta(f \smile g)$$

if $\delta f = 0$ and

$$(\delta f) \smile g = \delta(f \smile g)$$

if $\delta g = 0$. Thus, the cup-product is well-defined on cohomology groups:

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$$

Since on the level of chains the cup product is associative and distruvitive, so too it is on cohomologies. Furthermore, since R has an identity, so too does the cup product with the class $1 \in H^0(X;R)$ which is the 0-cocycle taking the value 1 and each singular 0-simplex.

Since the cup product for simplicial cohomology can be defined by the exact same formula for singular cohomologies, the canonical isomrophism between simplicial and singular cohomologies respect cup products.

Proposition 4.2.1: Cup Product Commutes With Continuous Map

Let $f : X \to Y$ be a continuous map, and let $f^* : H^n(Y; R) \to H^n(X; R)$ the induced cohomology maps. Then

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

and similarly for the relative case.

***Proof* :**
This is a simple computation with $f^\#$ (see hatcher p.210 if you need help)

Theorem 4.2.1: Graded-Commutative Structure

When R is commutative,

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

In paritcular, this means that $2(\alpha \smile \alpha) = 0$, which can more conisely be written as $2\alpha^2 = 0$.

***Proof* :**
Hatcher p.210

Definition 4.2.2: Cohomology Ring and Algebra

Let X be a topolocal space. Then $H^\bullet(X; R) = \bigoplus_k H^k(X; R)$ is a ring with the $+$ and $\smile$ operations, and is called the *cohomology ring* with respect to R and X. Multiplying by the scalars R, we get the *cohomology algebra* with respect to R and X.

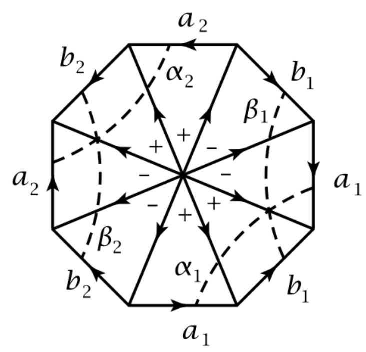
Naturally, these rings have a graded tructure. If A is a graded ring and $a \in A$, then we usually write $|a| = k$ to say that $a \in A_k$, and call it the *dimension* or *degree* of a. Note that by proposition 4.2.1,

$$f^* : H^\bullet(X; R) \to H^\bullet(f(X); R)$$

is a ring homomorphism.

Example 4.3: Cup Product and Cohomology Ring

1. Let M^g be a closed orientable surfaec of genus $g \geq 1$ with the usual Δ-complex structure. To inforce orientability and for reference, here is the Δ-complex when $g = 2$:



Naturally, grading of interest is:

$$H^1(M^g) \times H^1(M^g) \to H^2(M)$$

Let $\mathbb{Z}$ be the coefficients. By the universal coefficeint thoerem, $H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z})$; given any basis for $H_1(M)$, let $\alpha_1, \alpha_2, ..., \alpha_g, \beta_1, \beta_2, ..., \beta_g$ be the basis for $H^1(M)$. for each α_i or β_j, we can choose a representative cocycle, namely a $\varphi_i \in H^1(M)$ such that $\delta\varphi_i = 0$. Let α_i be the loop in M meeting a_i in one pont and disjoint form all other basis eleements a_j and b_j. Then define φ_i to have 1 on edges metting the arc α_i and the value 0 on all toher edges. Then φ_i counts the numer of intersections fo each edges with the arc α_i. We may similarly find representatives ψ_i for each arc β_i. We thus have a list of representatives $\varphi_1, \varphi_2, ..., \varphi_g$ and $\psi_1, \psi_2, ..., \psi_g$.

Now, applying we see that $\varphi_1 \smile \psi_1(\sigma)$ is zero when σ does not take have the outer edge b_1, as seen in the lower-right of the above figure. Now, if c is the 2-chain formed by the sum of all the 2-simplices with the signs indicated in teh center of the figure, we see that

$$\varphi_1 \smile \psi_1(c) = 1$$

Since M is 2-dimensional, $\partial c = 0$ (since c is not a boundary), $0 \neq c \in H_2(M)$. Since $(\varphi_1 \smile \psi_1)(c)$ is a generators of $\mathbb{Z}$, c represents a generator of $H_2(M) \cong \mathbb{Z}$, hence $\varphi_1 \smile \psi_1$ sends (c) to the dual $\gamma \in H^2(M) \cong \mathrm{Hom}(H_2(M), \mathbb{Z}) \cong \mathbb{Z}$ which generates it. Hence:

$$\alpha_1 \smile \alpha_2 = \gamma$$

Doing a similar computation, we getthat :

$$-(\beta_i \smile \alpha_j) = \alpha_i \smile \beta_j = \begin{cases} \gamma & i = j \\ 0 & i \neq j \end{cases}, \qquad \alpha_i\alpha_j = 0, \qquad \beta_i\beta_j = 0$$

Which gives us all the relations we need to determine the cupt product since it it distributive. This cup product is *not* commutative since $\alpha_i \smile \beta_i = -(\beta_i \smile \alpha_i)$.

The cohomology ring we have described is called the *exterior algebra*, if $\Lambda_{\mathbb{Z}}[\alpha_1, \alpha_2, ..., \alpha_n]$ is the exterior algebra on n generator respecting the relatoin

$$\alpha_i\alpha_j = -\alpha_j\alpha_i, \; i \neq j \qquad \alpha_i^2 = 0$$

Then

$$H^\bullet(T; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha, \beta] \qquad |\alpha| = |\beta| = 1$$

More generally, we can see that

$$H^\bullet(T^n; R) \cong \Lambda_R[\alpha_1, \alpha_2, ..., \alpha_n]$$

The same is true for any product of odd-dimensional spheres where $\alpha_i|$ is the dimension of the ith sphere.

2. Computation for N^g, in particular $N^1 = \mathbb{R}\mathrm{P}^2$ (Hatcher p.208)

   From this, we see that
$$H^\bullet(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$$

3. Let X be a 2-dimensional CW complex by attaching the boundary of the 2-cell S^1 bie the map $z \mapsto z^m$. We see using cellular cohomology (or cellular homology and the universal coefficient thoorem) that
$$H^n(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z}_m & n = 2 \\ 0 & n \geq 3 \end{cases}$$
   which might suggest the cup product must be trivial. However, if we take $\mathbb{Z}_m$ coefficients, we get:
$$H^n(X; \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & n \in \{0, 1, 2\} \\ 0 & n \geq 3 \end{cases}$$
   suggesting in this case that the cup product may not be trivial. (Hatcher p.209)

Naturally, the cup product works for relative cohomologies (Hatcher elaborates on all the different combinations, which I shall for now ommit)

Naturally, it may be asked what graded commutative R-algebras occurs as cup product aglebras on $H^\bullet(X; R)$ for some space X. This is called the *Realization problem*, and is in general quite difficult. If $R = \mathbb{Q}$, the probelm is almost solved, if $R = \mathbb{Z}/p\mathbb{Z}$, there is some progress, and for $R = \mathbb{Z}$, very little is known.

An interesting application of cohomology rings is showing a space cannot be split into a wedge-sum up to homotopy equivalent. For example:

Example 4.4: Cohomology Ring And Products

There is a natural isomorphism

$$H^\bullet(\bigsqcup_\alpha X_\alpha; R) \cong \prod_\alpha H^\bullet(X_\alpha; R)$$

and:

$$\tilde{H}^\bullet\left(\bigvee_\alpha X_\alpha; R\right) \cong \prod_\alpha \tilde{H}^\bullet(X_\alpha; R)$$

where we take the reduced cohomology in the 2nd isormopshim to be relative to a basepoint and take the relative cup product[a]

Let's now show that $\mathbb{C}\mathrm{P}^2$ is not homotopy equivalent to $S^2 \vee S^4$. In particular, with $\mathbb{Z}$ coefficients, we have:

$$H^\bullet(S^2;\mathbb{Z}) \cong H^\bullet(S^4;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$$

while

$$H^\bullet(\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z}[\alpha]/(\alpha^3)$$

But certinaly:

$$H^\bullet(S^2 \vee S^4;\mathbb{Z}) \cong \left(\mathbb{Z}[\alpha]/(\alpha)^2 \oplus \mathbb{Z}[\alpha]/(\alpha)^2\right) \ncong \mathbb{Z}[\alpha]/(\alpha)^3 \cong H^\bullet(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$$

Giving us the result. Can you use this fact to show that no map $f : S^4 \to S^2$ is nullhomotopic? (hint, reduce to showing that if f is null-homotopic, then $C_f \simeq SX \vee Y$ where $C_f = X \cup_f Y$) (see comment for details)

Note that taking the cohomology ring over different coefficients may yield different results; sometimes two spaces will have the same cohomology ring over one ring but not over another. For example, consider if $\mathbb{R}\mathrm{P}^5$ is homotopic to $\mathbb{R}\mathrm{P}^4 \vee S^5$. Notice that these have the same cohomology groups (exercise), and hence can't be distinguished that way. They even have the same cohomology ring over $\mathbb{Z}$. However, they do not have isomorphic cohomology rings over $\mathbb{Z}/2\mathbb{Z}$.

[a] Naturally, we also assume that the base-point are deformation retracts of some neighbourhoods in order for the isomorphism to hold

4.2.1 Küneth's Formula

here

4.3 Poincaré Duality

We now take a moment to look at the relation of homology and cohomology in the special case of [topological]*manifolds*. Compact manifolds have some incredible symmetries between their homology and cohomology groups that we shall explore, which lead to *Poincaré duality*.

First, we need to define a few terms in algebreo-topological manner

Definition 4.3.1: Orientation

Let M be a closed connected manifold of dimension n. Then if the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ we say that M is *orientable*, and if it is $\mathbb{Z}_2$ it is said to be *non-orientable*.

(prof. speed rolled the theory for now. Defined many important propertie of orientation and cap product)

Theorem 4.3.1: Poincaré Duality

Let M be a closed R-orientable n-manifold with fundamental call $[M] \in H_n(M;R)$. Then the map $D : H^k(M;R) \to H_{n-k}(M;R)$ given by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k, that is:

$$M_k(M;R) \cong H_{n-k}(M;R)$$

Proof **:**
See Hatcher p.237-240

5

Homotopy Theory

(Chapter 4 of Hatcher, very interesting!)

6

Spectral Sequence

(Vakil's covering of Spectral sequences is fascinating! Place holder to add here)

A

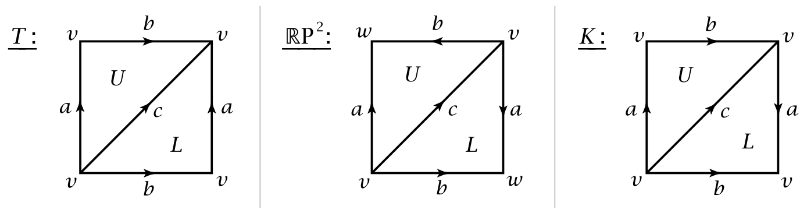
Common Structures and Related Algebraic Structure

I want to put a table with a topological space and it's corresponding fundamental group and (co)homology group.

`http://mathonline.wikidot.com/list-of-fundamental-groups-of-common-spaces`

A.1 Δ-Complex and Cell Complexes

The Δ-complexes for Torus, Klein Bottle, and $\mathbb{R}\mathrm{P}^2$:



A.2 Fundamental Groups and Covering Spaces

here

A.3 Homologies and Cohomologies

Note that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. Hence, $\tilde{H}_0(X)$ shall alway be ommited, since we are only working with path-connected spaces. To calculate the cohomology groups, if $T_n \subseteq H_n$ is the torison part:

$$H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$$

i.e., the torsion part "moves up" while the torsion-free part stays put.

Shape	Homology	Cohomology with $\mathbb{Z}$
$S^1 \simeq M^1$	$\tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(S^1;G) = \begin{cases} G & n=1 \\ 0 & \text{otherwise} \end{cases}$
S^k	$\tilde{H}_n(S^k) = \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(S^k;G) = \begin{cases} G & n=k \\ 0 & \text{otherwise} \end{cases}$
$S^1 \times S^1 = T^2$	$\tilde{H}_n(T^2) = \begin{cases} \mathbb{Z}^2 & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(T^2;G) = \begin{cases} \mathbb{Z}^2 & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$
M_g	$\tilde{H}_n(M_g) = \begin{cases} \mathbb{Z}^{2g} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(M_g;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$
K	$\tilde{H}_n(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(K;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=1 \\ \mathbb{Z}_2 & n=2 \\ 0 & \text{otherwise} \end{cases}$
$\mathbb{R}\mathrm{P}^2$	$\tilde{H}_n(\mathbb{R}\mathrm{P}^2) = \begin{cases} \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \begin{cases} 0 & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$
$\mathbb{R}\mathrm{P}^k$, k odd	$\tilde{H}_n(\mathbb{R}\mathrm{P}^k) = \begin{cases} \mathbb{Z}_2 & n \in \{1,3,...,2k-1\} \\ \mathbb{Z} & n=k \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & 2n<k \\ \mathbb{Z} & n=k \\ 0 & \text{otherwise} \end{cases}$
$\mathbb{R}\mathrm{P}^k$, k even	$\tilde{H}_n(\mathbb{R}\mathrm{P}^k) = \begin{cases} \mathbb{Z}_2 & n \in \{1,3,...,2k-1\} \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(\mathbb{R}\mathrm{P}^k;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & 2n \leq k \\ 0 & \text{otherwise} \end{cases}$
$\mathbb{C}\mathrm{P}^2$	$\tilde{H}_n(\mathbb{C}\mathrm{P}^2) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z} & n=4 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z} & n=4 \\ 0 & \text{otherwise} \end{cases}$
$\mathbb{C}\mathrm{P}^k$	$\tilde{H}_n(\mathbb{C}\mathrm{P}^k) = \begin{cases} \mathbb{Z} & n \in 2n \leq 2k \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) = \begin{cases} \mathbb{Z} & 2n < 2k \\ 0 & \text{otherwise} \end{cases}$
N^g	$\tilde{H}_n(N^g) = \begin{cases} \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}$	$\tilde{H}^n(N^g;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{g-1} & n=1 \\ \mathbb{Z}_2 & n=2 \\ 0 & \text{otherwise} \end{cases}$

TBD (Homology and Cohomology in $\mathbb{Z}/2\mathbb{Z}$ coefficients)

[1] Möbius Band

A.3.1 Cohomology Rings

Note that

$$\tilde{H}^{\bullet}\left(\bigvee_{\alpha} X_{\alpha}; R\right) \cong \prod_{\alpha} \tilde{H}^{\bullet}\left(X_{\alpha}; R\right)$$

Giving us some flexibility for finding cohomology rings:

Shape	Ring
S^k	$H^{\bullet}(S^k;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2),\ \lvert\alpha\rvert = k$ $H^{\bullet}(S^k;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^2),\ \lvert\alpha\rvert = k$
T^2	$H^{\bullet}(T^2;R) \cong \Lambda_R[\alpha,\beta],\ \lvert\alpha\rvert = \lvert\beta\rvert = 1$
M_k	$H^{\bullet}(T^k;R) \cong \Lambda_R[\alpha_1,\alpha_2,...,\alpha_k],\ \lvert\alpha_i\rvert = 1$
$\mathbb{R}\mathrm{P}^2$	$H^{\bullet}(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$ $H^{\bullet}(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha,\alpha^2)$
$\mathbb{R}\mathrm{P}^k$	$H^{\bullet}(\mathbb{R}\mathrm{P}^k;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{k+1})$
$\mathbb{R}\mathrm{P}^{\infty}$	$H^{\bullet}(\mathbb{R}\mathrm{P}^{\infty};\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha],\ \lvert\alpha\rvert = 1$ $H^{\bullet}(\mathbb{R}\mathrm{P}^{\infty};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha),\ \lvert\alpha\rvert = 2$
$\mathbb{R}\mathrm{P}^{2k}$	$H^{\bullet}(\mathbb{R}\mathrm{P}^{2k};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha,\alpha^{k+1}),\ \lvert\alpha\rvert = 1$
$\mathbb{R}\mathrm{P}^{2k+1}$	$H^{\bullet}(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha,\alpha^{k+1},\beta^2,\alpha\beta),$ $\lvert\alpha\rvert = 2,\ \lvert\beta\rvert = 2k+1$
$\mathbb{C}\mathrm{P}^2$	$H^{\bullet}(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3)$
$\mathbb{C}\mathrm{P}^k$	$H^{\bullet}(\mathbb{C}\mathrm{P}^k;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{k+1})$
$\mathbb{C}\mathrm{P}^{\infty}$	$H^{\bullet}(\mathbb{C}\mathrm{P}^{\infty};\mathbb{Z}) \cong \mathbb{Z}[\alpha]\ \lvert\alpha\rvert = 1$