Everything You Need To Know About MAT327

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September 25, 2022

Contents

(To incorporate. So we want to classify sufraces up to homeomorphism. HOwever, it is usually quite hard to find a homeomrophism (ex. a map to chagne a sphere inside out, or make a torus into a coffee cup). Instead, if we can find *topological invariatnts* on topological spaces (ex. a number, like dimension or number of holes, or more complicated objects like polynomials, groups, matrices ,vector-spaces etc), we can identify when two spaces are *not* homeomorphic, and even if each topological invariant only accompanies a unique object, though that is usually harder and falls under the label of classification theorem)

When topology was initially thought about, the goal was to be able to know how to classify different surfaces: What properties can we use to distinguish a sphere from a plane? The closed interval from the open interval? One manifold from the other? These questions then lead to further inverstigation in the concepts that were used to try and classify these shapes: How do we define dimension? Can we use the concept of distance (metric) to classify shapes? Can every space have a concept of distance defined on it? As these questions went on, we also asked how we can expand topologies by trying to take what we already know in set theory (subsets, equivalence relations, products) and using them to make topological spaces. All of these questions asking about the fundamental nature of objects grew into the domain of ***Topology*** [1]!

A good introduction to topology would mean understanding how these set-theoretic concepts translate to topology, some notion of why we settled on the definitions which have solidifed, and then analyse properties which are *intrinsic* a topology.

But what does it mean that a proeprty is *intrinsic* to a topology? The answer to this question compromises entire domains of study, but perhaps the best answer to this is found in *Category theory*. In section 1.2, we will elaborate on this very idea of intrinsicness using the philosophy of category theory (which will also be presented)

If you already have some algebraic background and are impatient to konw the answer, here is a summary:

> Given two topologies X, Y which are isomorphic (which we will define what it means for topologies to be isomorphic) then a property is "topological" if when one topology has it, the other must neccessarily have it, and if one doesn't have a properly the other neccessarily doens't have the properrty. For example, we earlier said that dimension is a topological property. So if $X \cong Y$, then if X is one-dimensional, Y must *also* be one-dimensional. However, we will show that a space being bounded (ex. $(0,1)$ is bounded,but $\mathbb{R}$ is not) will *not* be a topological property, that is, if $X \cong Y$, and X is bounded, it needn't be the case that Y is bounded! [2]

The formalization of topologies properties is a huge part of this doucment.

Another central question in topology (and actaully many fields of math) is what's the right way to compare/relate topologies? When this question was originally formulated by early topologists trying to classify shapes, topology was concerned with the properties of a geometric object that are preserved under stretching, twisting, crumpling and bending, but not tearing or gluing. This intuition will at first not at all be seen when we introduce topologies, but we will cover examples to show how it's the case. Nowadays, this intuition still stands, but topology is more than manipulating geometric objects. Two topologies will be related (which I'll write as $X \to Y$) if we can somehow "put" the topology of X into Y by "stretching, twisting, crumpling, or bending" the topology X. We will cover this intuition soon [3]

[1] In mathematics, topology (from the Greek words $\tau o\pi o\sigma$, 'place, location', and $\lambda o\gamma o\sigma$, 'study')

[2] boundedness is a *metric* property, and is preserved under *metric isomorphisms* (isometric functions)

[3] But unfortunately not in full rigour, since that would requier more category theory

Most of this document is dedicated to understanding different topological properties and how topologies relate, in the goal that you can show that if one topological space differs from another by that property, you know their different. In other words, your learning to classify topological spaces (which is the natural foundation you need to ask bigger questions later)!

I have another intuition on Topology after doing Real analysis (MAT457): In real analysis, we extensively work with *limits*. Essentially, real analysis feels like the study of construting things with limtis and manipulating object through limits. Because of this, the sequential definnition of continuity is in some may the most ueseful definition. Similarly, the definition of compactness through is more important than "any open cover has a finite subcover". However, There are spaces that are not well suited for limits (for example, inseperable Hilbert spaces, or $\mathbb{R}^\omega$, or the set of functions from $\mathbb{R}^\omega$ ot $\mathbb{R}^\omega$). Thus, topolgoical concepts where abstracted, refined, and distilled into the *key* properties of the concepts like compactness, connectedness, continuity, and so forth, so that they work in the fullest generality and so that they are still very simple to work with (in a sense, the fact that the definitions can be so simple is a beautiful work of genius). Thus, studying topology is really studying what it means to be a "space" in the most general possible setting. We have essentially agreed that what it means to be a sapce is if we are *closed* by taking as many "unions" of objects as we want (unions in quote because I feel like we can replace the word "union" with words like "gluding", "combining", "adding" in the write context), and that if we are working with the "dual" of obects (closed sets), then we still should have some flexibility in a "natural way", which means finite, but not more complicated than that; if we were to allowed more than finite intersections of closed sets, we will have to start having a structure on the closed sets instead of keeping track of the main thing that is open sets. In that way, letting closed sets be closed under finite intersection is in a sense making sure there is no "structure" in regared to clsoed sets (not making it closed under clsoed sets makes it... something?? I know allowing countable closure will make the euclidean topology too vague, but I should think about what happens if we don't care about finite closure of closed sets).

(a prof randomly mentioned that $\mathbb{R}/\mathbb{Q}$ with the usual quotient topology has no open sets beside the trivial ones!)

Chapter 1

Introduction to Topology

In this chapter, we will cover the definition of topologies, with some examples, how to compare topologies, and the concepts which will naturally arise given the definition of topology we will present (closure, interior, countability, etc.)

1.1 Basic definition and construction

Interestingly, with abstraction, the eventual definition of topology that is currently taken seems far away from any geometric notion. To my surmise, the direction in which the abstraction went was to think of topology as giving a notion of "localness" to an object. Be careful to not think of localness as distance: those two concepts will merge in nice topological spaces (as we'll eventually show with metric spaces in section 1.7), but it does not merge nicely in general (like the finite-compliment topology). The idea of "localness" linked to the explanation of topology about finding properties of the space is that it was figured out that the way to think about the setting in which to see these properties is in a setting were we look at smaller chunks of our space in intersting smaller components, and if we have the right "lens", we can conclude information about these smaller pieces to the entire space as well! This is a common trick in math: using local information to find global information.

We will elaborate on all of this after the definition and examples:

Definition 1.1.1: Topological Space

Given a set X, a *topological space* is a pair $(X, \mathcal{T}_X)$ where $\mathcal{T}_X \subseteq \mathcal{P}(X)$ is a set of subsets of X such that:

1. $\emptyset$ and X are in $\mathcal{T}_X$
2. $\mathcal{T}_X$ is closed under arbitrary unions
3. $\mathcal{T}_X$ is closed under finite intersection's

An element inside $\mathcal{T}_X$ is called *open*.

By convention, $Y \subseteq X$ is *open* if and only if $Y \in \mathcal{T}_X$.

Example:
Example

1. Given X being *any* set, then $\mathcal{T}_X = \{\emptyset, X\}$ is a topology. This is the *trivial topology*, or *indiscrete topology*. In terms of localness around points, these points are all considered to be in the one neigborhood. So in here, we can't really consider a *chunk* or *sub-part* of our geometric object, we only ever consider the whole: there are *no* local properties of points besides *whole thing*.

2. Given X to be any set, then $\mathcal{T}_X = P(X)$ is a topology. This is called the *discrete topology*. In terms of localness, we see every point as it's own thing: the local property is so percice the smallest local level are points themselves. Since the union of points is also in the topology, every possible subset of X is also in the topology. It's like we're looking through every possible level at once, like we can see the quantum level, the atomic level, the chemical level, the biological level, and the emperical level all at once.

3. Given X to be any set, then $\mathcal{T}_X = \{Y \subseteq X \mid Y = \emptyset \text{ or } |X \setminus Y| < \infty\}$. This set is called the *co-finite topology*. or *finite-compliment topology*.

   Note: why not just say that U is open if it's finite? It's because the arbitrary union of U's would no longer be finite. In this compliment world, unions will *shrink* $(X \setminus (U_1 \cup U_2))$, making sure it's always finite, and intersection are *finite*, insuring our topology.

4. (If you have done some Analysis) Take $n \geq 1$, $X = \mathbb{R}^n$. Then

   $$\mathcal{T}_X = \{\text{euclidean open sets in } \mathbb{R}^n\}$$

   Notationally written as $(\mathbb{R}^n)_{std}$, $(\mathbb{R}^n)_{euc}$, or just $\mathbb{R}^n$ if one abuses notation

   Note: If you took analysis, when you first saw this topology, you probably saw it defined it with open balls. However, there is no metric here to produce the definition of an open ball. We *can* define a metric on $\mathbb{R}^n$, and it can define this very topology; that will be addressed once we reach metric spaces (section 1.7). For now, you only need to think about these as a collection of sets

5. The set $\mathbb{C}$ with all "open circles", similarly to $\mathbb{R}^2$. These two will actually be topologically equivalent, hence mathematically showing how these two are geometrically the same (where thinking geometrically is *part* of what topology is [a])! My current intuition on differentiating these two is *algebaic*: $\mathbb{R}^2$ and $\mathbb{C}$ are different because we put a different multiplication structure on them. We will make this intuition rigorous soon (definition 1.2.3).

6. Let X be any set, if $x_0 \in X$, then $\mathcal{T}_X = \{Y \subseteq X \mid Y = \emptyset \text{ or } x_i \in Y\}$. This topology is came *particular point topology*

7. Let X be any set, with $<$ as be a [total relation] relation on X (so $\forall a, b \in X, a \neq b$, either $a < b$ or $b < a$ and if $a < b$, $b < c$, then $a < c$). With this, define

(a) $(a, b) := \{c \in x : a < c,\, c < b\}$

(b) $(-\infty, a) := \{c \in X : c < a\}$

(c) $(a, \infty) := \{c \in X : a < c\}$

Then $\mathcal{T}_X$ consists of all *unions* of the above sets for all choices of $a, b \in X$, together with X itself. $\mathcal{T}_X$ is called the "*order topology*"

8. here is an image of all topologies on 3 points

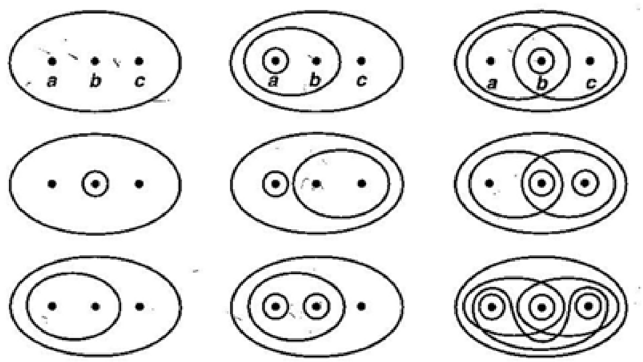

Figure 1.1: from Munkres p.76

[a]Just be careful with this intuition! It's useful in the beginning of topology, and even looking at topology historically (it started with the study of geometric objects), but as time goes on the "locallness" intuition becomes more important

Note: As an important technically, it *is* possible that an arbitrary of intersection of open sets are open, it just need not be the case! For example $\bigcap_{i\in\mathbb{N}}(-i, 1) = (-1, 1)$, which is clearly open in the topology of example 6. This is a cheeky example, but there do exist more interesting examples when we get to *product spaces* (section 2.2).

To make this notion of local property more succinct, let's take $\mathbb{R}$ with the standard topology and $x \in \mathbb{R}$. Then we know that $(x-1, x+1)$ contains x, that is x is contained in the open set $(x-1, x+1)$. Topologists would usually say that $(x-1, x+1)$ *is the neighborhood of* x. However, there doesn't need to be a unique neighbourhood for a point. The point x actually has uncountably many neighbourhoods, since $(x-\epsilon, x+\epsilon)$ for all $\epsilon \in \mathbb{R}$ is a valid neighborhood. So when thinking about localness, about *open neighbourhoods* in topology, think of it as the fundamental starting point for what a topology does. This doesn't mean one should ignore points, for many times the underlying space will affect the topology too, but what distinguishes topologies are the open sets, the neighbourhoods.

Example: Open neighbourhoods on a Sphere
To understand this localness even more, let's take a sphere for a moment and consider some topologies on it. Taking the trivial topology on the sphere will make you *only* look at the entire sphere as a thing (you don't see points or subsets, you see the thing which results from all of them). However, taking the discrete topology is like looking at *all* possible subsets as local properties. So, you can think of this as the two extremes of "level of precision" with which we want to look at the local properties of the sphere. Any other topology on the sphere is looking at different *levels* of topologies, and different *types* of topologies will be different "types" of local ways of looking at the sphere. topologies

Also notice that open sets can be arbitrary union and remain open, while closed sets must at most be in finite unions to remain closed. This comes down to if you let "both" infinities exist (that is, both can be arbitrary, or even intersection of infinitely countable sets) then a lot of flexibility will pop up adding more sets than is useful. For example $(\frac{-1}{n}, \frac{1}{n})$ where $n \in \mathbb{N}$ is a collection of open sets in euclidean $\mathbb{R}$. The intersection of all of them is $\{0\}$. We can do this around any point, and have every point be open sets . But then the arbitrary union of open sets is open, and since every one-point set is open, *every* possible combination of point is open. Thus, the topology of Euclidean $\mathbb{R}$ would just be $\mathcal{P}(R)$, that is the power set of $\mathbb{R}$. Because of this, a lot of topologies which would be distinguishable with different local properties and interesting would become identical. So, by restricting to "finiteness" for intersection, we essentially get rid of some wiggle room and force constraints that are happen throughy the use of finiteness. This is actually a very good example to show how bringing in infinities (of different types of infinities even, since we can have an *arbitrary* union, not just countable) will add flexibility to what you're dealing with − sometimes too much like what we just showed. This course will also make you gain an intuition of how infinities add certain flexibilities in the right circumstances.

If you were curious, there is a domain in mathematics in which there are infinite union and intersection of sets. The domain is called *measure theory*, and the space with this property is called a σ-algebra. However, only *countable* unions and intersections are permitted. These spaces are the foundation for much of real analysis and defining important concepts like the [Lebegue] integral.

With some notion of topologies, we can quickly go over the most basic relation between topologies: whether one contains the other:

Definition 1.1.2: Finer And Coarser

Let X be a set, and $(X, \mathcal{T}_1), (X, \mathcal{T}_2)$ be two topologies. THen

1. $(X, \mathcal{T}_1)$ is *finer* then $(X, \mathcal{T}_2)$ if $T_1 \supseteq T_2$
2. $(X, \mathcal{T}_1)$ is *coarser* then $(X, \mathcal{T}_2)$ if $T_1 \subseteq T_2$

Example:

1. The trivial topology is coarser than all other topologies on it's respective X
2. The discrete topology is finner than all other topologies on it's respective X
3. Take $\mathbb{R}$ generated by open balls and $\mathbb{R}$ generated by open squares. Then both of these

topologies are finer *and* coarser than each other. This means that they contain each other, and so define the same "localness".

Note that two topologies need not be *comparable* with another, i.e., neither has to be finer or coarser than the other.

Example:
If we have $X = \{a, b, c\}$, and $\mathcal{T}_1 = \{\{a\}, \{b, c\}, X, \emptyset\}$, as well as $\mathcal{T}_2 = \{\{a, b\}, \{c\}\}$. Then these two are both topologies, However, $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_1 \not\supseteq \mathcal{T}_2$

This next example is a bit of a hard, but worth taking the time to pour over. The topologies introduced in this example will come back often for counter-examples:

Example: of Incomparable Topologies
Let $\mathbb{R}$ be the underlying set, andt take the set generated by intervals $[a, b)$ to be the topology. What it means to be generated I'll come back to, but for now just thinki about taking all possible sets of the form $[a, b)$ and taking all possible unions of them. This topology is called the **lower-limit topology**, and is usually labeled $\mathbb{R}_l$. This topology is *finer* than the standard topology. This is because every open interval can be written as a countably finite union of half-open intervals. Furthermore, the standard topology is *coarser* than $\mathbb{R}_l$, since it would require an arbitrary intersection of elements of the standard topology to create an element $[a, b)$

Now consider the following topology: Take all open intervals $A = \{(a, b) \mid a < b\}$, but additionally, take $K = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and take all sets of the form $B = (a, b) \setminus K$. Then $A \cup B$ generates a topology. This topology is called the **K-topology**, and is usually denoted $\mathbb{R}_k$. By construction, this topology is *strictly finer* than he standard topology. Since A is the standard topology, A we have all of $\mathcal{T}_{\mathbb{R}}$. We also have sets of the form $(0, 1) \setminus K$, however, the fact that $0 \notin K$ will make the set $(-1, 1) \setminus K$ not constructable from open sets of $\mathcal{T}_{\mathbb{R}}$; around the point 0, there does not exist an open interval! So, $\mathbb{R}_K$ is strictly finer than $\mathbb{R}_{\text{euc}}$

Now, though the $\mathcal{T}_l$ and $\mathcal{T}_K$ are both finer than $\mathcal{T}_{euc}$, they are in fact *incomparable*. That is, one is not finer or coarser than the other! the set $[0, 1) \in \mathcal{T}_l$ is not in the K-topology: $[0, 1) \notin \mathcal{T}_K$, since again, it would require an infinite union of elements, and the sets of the form $(a, b) \setminus K$ have even less elements from the sets of the form (a, b) meaning they won't contribute in a meaningful way. Converserly, $(-1, 1) \setminus K \in \mathcal{T}_K$, but $(-1, 1) \setminus K \notin \mathcal{T}_l$. Since the closed part is on the *left*, it will follow the same reason as it did for the euclidean topology! Note that if the closed part was on the *right*, the problem would been avoided!

Note that this what was also meant earlier when I said there are topologies with different local properties! Not every topology defines the same idea of localness, of open neighborhoods, and you can even have topologies that define incomparible idea of openness.

1.2 Continuity

Continuous maps used to be that you can't lift your pen off your paper. Then it became the epsilon-delta definition, which evolved to the pre-image of an open set is open. We will now gain a rigorous understanding of where these ideas come form, and answer questions like "why the pre-image of open sets?" and "what does continuity really try to capture?"

As a start, you can think of continuous functions as the homomorphisms of topology. Homomorphisms tell you that the space you are mapping to preserves some structure of the space you mapped from. This same criterion applies in topology, a continuous map between two set X, Y with different topologies $\mathcal{T}_X, \mathcal{T}_Y$ mean that $\mathcal{T}_Y$ has some similar "localness" properties as $\mathcal{T}_X$ that the continuous map brings over.

Definition 1.2.1: Continuous Functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then $f : X \to Y$ is said to be *continuous* if

$$\forall U \in \mathcal{T}_Y,\ f^{-1}(U) \in \mathcal{T}_X$$

Note: Don't forget that functions map *points* from the underlying set, not open sets in the topology. The definition of continuous functions is dealing with sets, but don't let it confuse you when you're solving problems that there are points still involved.

Example:

1. You can think of most functions in 1st year calculus: for example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, or $f(x) = \sin(x)$, and so forth. Noe that $f(x) = \log(x)$ is only continuous if the domain is positive [a]
2. Take the domain and co-domain to be $R \times R$. In the domain, the domain, the open sets are $(a, b) \times \mathbb{R}$, that is, the first position is the euclidean domain, while the second is the entire line. The co-domain is the inverse: $\mathbb{R} \times (a, b)$. These two topologies are *incomparible*, however
$$f(x, y) = (0, x)$$
Is an open set. If we take any open set in the co-domain $\mathbb{R} \times (a, b)$, then
$$f^{-1}(\mathbb{R}, (a, b)) = (a, b) \times \mathbb{R}$$
which we know is open since that is a set in the topology of the domain. Since the open set in the co-domain was arbitrary, that means every open sets has an open pre-image, meaning f is continuous.
3. A constant map $f(x) = y$ for some fixed y is *always* continuous function. if V is an open set in Y such that $y \in U$, then $f^{-1}(V) = X$. If $y \notin V$ then $f^{-1}(V) = \emptyset$. Both those sets are open sets, and thus, f is continuous.

[a] or, if you already took complex analysis $f : \mathbb{C} - \{0\} \to \mathbb{C}$

The question might be asked "why not the image of open sets is open"? This comes down to being okay with not preserve the entire topology of the *domain*, but miss some elements in the co-domain, and instead preserve the entire topology of the *co-domain* and miss some elements in the domain. This way, when working with the image of a cotninuous function, we know we can think of any open set there having an analogue in the domain (that is, I can think of $V \in \mathcal{T}_Y$ and know there is a $U \in \mathcal{T}_X$ such that $f^{-1}(V) = U$), instead of needing to be careful that there is an open set $U \subseteq X$ such that $f(U) = V$. The main reason prioritising the co-domain is that we are usually trying to figure some information about the co-domain by mapping to it. Since if we have a function, $f : X \to Y$ it must be that $f^{-1}(Y) = X$, but it's not necessarily the case that $f(X) = Y$ (continuous functions need not be surjective), it's more natural to try an study topologies by putting them in the co-domain.

We might still want to give a name to maps whose image of open sets are open:

Definition 1.2.2: Open Maps

Let $f : X \to Y$ be a map such that if $U \subseteq X$ is an open set, then $f(U) \subseteq Y$ is an open sets. We call f an *open map*.

At first glance, one might not think of an example on open map which is not continuous, so here is one to keep in mind

Example: : Continuous but not open

Take

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Since $(a, b) \in \mathbb{R}$, and if $a > 0$ then $f^{-1}((a, b)) = (a, b)$. If $a \leq 0$, then $f^{-1}((a, b)) = (-\infty, b)$. Since both pre-images ar are open (the set $(-\infty, b) = \bigcup_{n \in \mathbb{N}} (-n, b)$) the map is continuous. However, The set $(-1, 1)$ maps to $[0, 1)$, which is *not* open in $\mathbb{R}$.

The other way is also not necessarily true:

Example: : Open, but not Continuous

Let $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x$, with the domain having the euclidean metric, and co-domain having the discrete metric. Then it's clear that the image of every open set is open, but the pre-image of every set need not be open (like $\{0\}$).

Open maps are useful in terms of the continuity of f^{-1}, in particular, if f is an open map, then f^{-1} is continuous.

Proposition 1.2.1: Swapping The Domain And Co-Domain

Let $f : X \to Y$ be a continuous function. Then X can have a finer topology, or Y can have a coarser topology

Proof **:**
This is a simple carrying the definition question. The domain can be replaced with a *finer* topology (since the pre-images of open sets will remain there), while the co-domain can be replaced wit a *coarser* topology (since there are less open sets to take to the pre-image).

However, this does not imply that the co-domain or the domain are finer, coarser, or even comparable to each other:

Example: of Continuous Function Between *Incomparable* Topologies

Take

$$T_1 = \{\emptyset, [a,b], [a,b]^c, \mathbb{R}\}, \qquad T_2 = \{\text{euclidean topology}\}$$

And the function which takes $f([a,b]) = 0$, and $f([a,b]^c) = 1$. This function is in fact continuous. Take any open set $U \in T_2$. Then there are 4 cases:

1. if $0, 1$ are not in U, then $f^{-1}(U) = \emptyset$
2. if $0 \in U$ but $1 \notin U$, then $f^{-1}(U) = [a,b]$
3. if $1 \in U$ but $0 \notin U$, then $f^{-1}(U) = [a,b]^c$
4. if $1 \in U$ *and* $0 \in U$, then $f^{-1}(U) = \mathbb{R}$

This is true for any arbitrary open set $U \subseteq T_2$, and so f is continuous! Even though T_1 and T_2 are *not* comparable, this funciton is *still* continuous. This means that he "cruxs" of what makes a function continuous is the function itself.

Thinking about preserving types of localness might feel strange here. This comes down to the map f being very particular. It maps entire parts of the domain to either 0 or 1, and 0 or 1 are about of uncountably many open sets. This shows how much the continuous map is important in defining the relation between the two spaces (in a sense similar to how algebraic homomorphisms are important in defining relation betweeen seemingly "incomparable" algebraic structures)

Making a non-trivial continuous function between two bizzare space might make us want to know the limits of what can be done between topologies. This answer really comes down to the topologies in question, but here is an example:

Example: : Limits of Continuous Functions Between Spaces

Take

$$\mathcal{T}_2 = \{\emptyset, (-\infty, 2) \cup (2, \infty), (-1, 1), (-\infty, 2) \cup (2, \infty) \cup (-1, 1), \mathbb{R}\}$$

Take $f : \mathbb{R} \to \mathbb{R}$ where the domain has the $\mathcal{T}_2$ topology, while the co-domain has the eucilidean topology, and f is *not* the constant function (for that will always define the trivial topology for the image). Since f is not the constant function, it must map to at least two values. Open sets containing those values must have an open pre-image. Let's say $f((-1,1)) = 0$. So far, an open set in the co-domain containing 0 would be open in the pre-image. Since f is not a constant map, ew must map the other points to another place in the co-domain. Let's say $f(B) = 1$. We want to figure out what B works so that f is continuous.

A set V in the co-domain containing 1 would need to hae an open preimage. If has has 0 and 1, it's still open because the union of open sets is open. If we had some points $x \in (-1, 1)$ mapping to something else, say -1, then the pre-image of 0 wouldn't be open, but the pre-iamge of any open set must be open. Simiarly for b. However, now we must ask ourselves where do we map, say, 1.5? If we map it to 0 or 1, then it would be part of the pre-image of U of V, and that would make it not-open. Conversely, if we make it, say, -2 (i.e. some arbitrary un-picked number), then $(-3, 3)$ which is open in $\mathbb{R}$ would not be open in $\mathcal{T}_2$, since the pre-image would be $\{-2\} \cup (-1, 1)$, which is not open. Therefore, there is nowhere for 1.5 to map to wihtout breaking continuity. THerefore, in this instance, the "type" of topology we have actually limited us what type of topology we have!!

This mapping problem can be fixed if we have the open sets of our topology cover all of $\mathbb{R}$ (without considering $\mathbb{R}$ to make the covering work, because then every topology "trivially" covers the underlying set [a]). In our previous example, let's replace $(-1, 1)$ with $[-2, 2]$. Map that to 0, the other set to 1. This is now a continuous function.

[a] I'm not using the idea of open covering, or of a covering map here!

Though there is still a limitation to continuous functions. A continuous function in that scenario is still limited to having a finite image. Take any domain that has a finite topology. Since the pre-image has to be open, if one of those open sets in the domain "split" and some map to one point and some map to another, then taking an open set containing one of those points will have a non-open pre-image. Therefore, continuous maps rely heavily on what topology from it' underlying set it takes. It can always take the trivial topology and so be a constant function, but it might not be able to take other topologies depending on the domain and co-domain.

We concentrated a lot on the domain having a non-standard topology rather than the co-domain. However, both play their own role in a function being continuous. The co-domain plays the role of providing the sets whose pre-image have to be open, so it can determine the density of sets in the domain. However, that is holy dependent on the function itself, which as we've seen can have a finite topology on as the domain, and so the function will only map to finitely many elements, which in our case is still possible even if the topologies are incomparable.

Another example of the relation between domain and co-domain with continuous functions is if we take the same underlying set X, but have the discrete topology in the domain. Then for whatever topology we have in the co-domain, the pre-image of that open set is a set, which is open in the domain since the topology of the domain compromises the power-set. However, if the co-domain has the discrete topology, then there are *too many* open sets in the co-domain for most functions to be continuous. Since every arbitrary set in the co-domain is open, every pre-image also has to be open.

It is in fact interesting to take a quick look at topologies with discrete co-domains, cause in a sense they show continuous functions which work "universally", since if it's continuous in the discrete topology, then you can take coarser topologies in the co-domain, and it will still be continuous, and since every topology is coarser than the discrete topology , then what we're finding are the function that are "universally" continuous given any topology on the co-domain (given the same underlying set X for the domain and co-domain)

For example, the constant function on any domain is continuous. If the domain has a finite topology which covers X without using X, then given n open sets in the domain, map them each to single points in X, and that will also be a continuous function. If I can think of more, I'll add them here.

With these ideas, we can ask if there are easy ways to manipulate continuous functions. We will use these time an time again to construct functions:

Proposition 1.2.2: Rules for constructing continuous functions

These 6 things:

1. **Constant function:** if $f : X \to Y$ maps all of X int othe single poitn $y_0 \in Y$, then f is continuous
2. **inclusion:** If a subsapce A is a subsapce of X, the inclusio n$i : A \to X$ is continuous
3. **composition:** if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous
4. **Restricting the domain:** if $f : X \to Y$ if continuous, and if A is a subspace of X, then $f|_A : A \to Y$ is continuous
5. **Restricting or expanding the range:** Let $f : X \to Y$ be continuous. if Z is a subsapce of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subsapce then the function $h : X \to Z$ obtaine by expanding the range of f is continuous
6. **local formulation of continuity**: The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each U_α.

Solution p.108 ▼

Note: taking a subspace is continuous, but that doesn't mean that continuous functions from the original space *to* another space will also be continuous from the subspace *to* that other space

example

The way I like to think about that is through this diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "f"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

In the diagram, f is continuous between X and Y, and i is the inclusion map. With the example we showed earlier, we see how f need not be continuous on the subspace.

Here is one more rule that is so widely used that it's given it's own name:

Lemma 1.2.1: Pasting Lemma

Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to B$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h :\to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$

Proof **:**
p.109

With this in mind, let's keep building on our notion of continuity by also adding that every open set maps to an open set. With this added condition, we almost have two identical topological spaces. Such a function is sometimes called *bi-continuous*. If a function is by-continuous, the topologies are basically comparable [1]. If the function is also bijective, then all open sets of the domain map to the co-domain, and vice-versa, meaning the two topologies are basically the same. When a functions is bi-continuous and bijective,we say it's is a *homeomorphism*

Definition 1.2.3: homeomorphism

If $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces, a map $f : X \to Y$ is a *homeomorphism* if

1. f is bijective
2. $f^{-1}(\mathcal{T}_Y) = \mathcal{T}_X$ (i.e. continuous both ways)

$(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are *homeomorphic* if there exists a homeomorphism between them

[1] If they are *actually* comparable I'll have to check later

Note: if $f : X \to Y$ is a *homeomorphism*, then $f^{-1} : Y \to Y$ is also a homeeomorphism

It can intuitively be said that $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are "homeomorphic" if there is a bijection between both (X, Y) and $\mathcal{T}_X, \mathcal{T}_Y)$. However, This bijection must respect the property in the above definition.

A homeomorphism is the strongest condition you can put between two topological spaces. It is the isomorphism of topologies [2]. You will encounter many other topological properties a space can have, and if a space is homeomorphic with another space, that space will have all the same properties, since we can exactly map one set of open sets onto another.

Example:
$\mathbb{R}^2$ and $\mathbb{C}$ are topologically the same. The map

$$\mathbb{R}^2 \to \mathbb{C} \qquad (x, y) \to x + iy$$

Will define a heomeomorphism.

Corollary 1.2.1: Identical Topologies

$f : X \to Y$ are homeomorphic, then the two topologies are idnetical $(\mathcal{T}_X = \mathcal{T}_Y)$

***Proof* :**
exercice

There is a slightly weaker condition than homeomorphism that is also used a lot. If a function is homeomorphic *onto it's image*, we call that an embedding

Definition 1.2.4: imbedding

Let X, Y be topological spaces. then if $f : X \to f(X)$ is a homeomorphism, then f is called an *imbedding*

1.2.1 Continuity on the real Numbers

Will get back to this. Wanted to put this here:

`https://math.stackexchange.com/questions/42308/continuous-bijection-from-0-1-to-0-1/42318#42318`

[2] which is actually the vocabulary used in Category theory: homeomorphisms *are* the isomorphisms of the Top Category

1.3 Generating topologies (basis)

When you think about the topology on $\mathbb{R}$, you usually don't think about, say, the element $(1, 2) \cup (10, 11)$. This is because what you're is here is actually one for the *basis* of a topology. The idea of a basis is that you can take a smaller subset of open sets which will *generate* the rest of your topology.

Definition 1.3.1: Basis Of Topology

Let X be a set. A set $\beta \subseteq P(X)$ is a basis if

1. $\forall x \in X$, $\exists B_0 \in \beta$ such that $x \in B_0$
2. $\forall B_1, B_2 \in \beta$, $x \in B_1 \cap B_2$, then $\exists B_3 \in B$ such that $x \in B_3 \subset B_1 \cap B_2$

Note: Don't confuse the idea of basis from linear algebra where you also need the basis elements to be the *unique* elements that some to any other element. Here, uniqueness is not a thing.

With the property a basis would need, we can ask what topology would be defined with a basis. This is *how* we will define topologies with a basis:

Lemma 1.3.1: Basis Generating A Topology

Let β be a basis. Then

$$\mathcal{T}_\beta := \{U \subseteq X \mid \forall x \in U, \exists B_0 \in \beta \text{ such that } x \in B_0 \subseteq U\}$$

generates a topology

Proof **:**
Take $\mathcal{T}_\beta$ as described in the lemma. Then $\emptyset \in \mathcal{T}_\beta$ vacuously. Next, $X \in \mathcal{T}_\beta$ since for every $x \in X$, there exists a basis element β_0 such that $x \in \beta_0 \subseteq X$. In fact, every basis element containing x will satify this condition.

Next, Let's take a collection of elements $U = \{U_\alpha\}_\alpha$. Is their union in $\mathcal{T}_\beta$? Take any $x \in U$. Then $x \in U_\alpha$ for some α. Since $U_\alpha \in \mathcal{T}_\beta$, then we know by definition that there exists a basis element β_x such that $x \in \beta_x \subseteq U_\alpha$. Since this was true for arbitray $x \in U$, then $U \in \mathcal{T}_\beta$, meaning $\mathcal{T}_\beta$ is closed under arbirary unions

Finally, Let's consider a finite number of open sets $\{U_1, ..., U_n\}$. Let's start with when there are two elemetns U_1, U_2, and take come $x \in U_1 \cap U_2$. Then by the definition of this topology, there exists a basis element such that $x \in \beta_1 \subseteq U_1$ and $x \in \beta_2 \subseteq U_2$. So then:

$$x \in \beta_1 \cap \beta_2$$

but then, by the 2nd condition of a basis, there exists a 3rd element such that $x \in \beta_3 \subseteq \beta_1 \cap \beta_2$. Since x was arbitrary, then $U_1 \cap U_2 \in \mathcal{T}_\beta$.

For the case with n elements, we can show this inductively. The case where $n = 1$ is trivial, so let's assume it works for $n-1$ elements: $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}_\beta$. Now add U_n. However, this is *exactly* the same case as we had with two basis elements (since $U_1 \cap \cdots \cap U_{n-1}$ is just another basis elements now!). Thus, we apply the same logic and get $U_1 \cap \cdots \cap U_n \in \mathcal{T}_\beta$, showing that $\mathcal{T}_\beta$ is indee a topology

Since this is *the* way of constructing a topology given a basis, we give this type of topology a name:

Definition 1.3.2: Topology Generated With Respect To β

Let β be a basis. Then we say that

$$\mathcal{T}_\beta := \{U \subseteq X \mid \forall x \in U, \exists B_0 \in \beta \text{ such that } x \in B_0 \subseteq U\}$$

is the topology generated by β

Note: It is important to remember that *every* topology has a basis! If $\mathcal{T}_X$ is a topology, then Let $\beta = \mathcal{T}_X$. It is an exercise worth checking to see that β is a basis!

The way I like thinking about a basis is that you need to find a set in which all *unions* will be the topology you want.

Example:

1. $\beta = \{\{x\} \mid x \in X\}$ generates the discrete topology
2. $\beta = \{X\}$ generates the trivial topology
3. $\beta = \left\{\text{open rectangles in } \mathbb{R}^2\right\}$ generates a topology homeomorphic to the standard topology, and so it *is* the standard topology.

   Proof **:**
   We'll show this after introducing lemma 1.3.3

   This shows how a space (like $\mathbb{R}^n$) can have *different* bases. In general, there is *no* minimal or smallest basis generating a given topology.

You might ask yourself "why not have a set in which all unions and *intersection* create a topology"? We can define that too. There is no real advantage of using one over the other, it comes down to the amount of structure that is present in one verus the other. In the basis definition, since the intersection of elements is already inside it, then if we are using a basis, we have less to show when proving theorems with sub-basis. However, constructing topologies can be easier with sub-basis.

Definition 1.3.3: Subbasis

p.82

1.3.1 Simplification With Basis

Lemma 1.3.2: smallest possible topology containing β

Let β be a basis. Then $\mathcal{T}_\beta$ is the *smallest* topology on X for which elements of β are open

Proof **:**
Simple proof; if smaller than contradiction.

If one finds the topology of a space, some proofs become easier to do:

Lemma 1.3.3: Simplification Due To Basis

Let X be a set ,β, β' be bases for X generating $\mathcal{T}, \mathcal{T}'$ respectively. then

$$\mathcal{T} \text{ is finer then } \mathcal{T}' \quad \Leftrightarrow \quad \forall B_0' \in \beta', x \in B_0', \exists B_o \in \beta \text{ st } x \in B_0 \subseteq B_0'$$

Proof. simple proof; exercise. □

With this, we can more easily prove that $\beta = \left\{\text{open rectangles in } \mathbb{R}^2\right\}$

Proof **:**
Good exercice

We can further use a basis to simplify continuity proofs

Lemma 1.3.4: Continuity With Basis

Let X, Y be topological spates. Let β be a basis for the topology on Y. Then $f : X \to Y$ is continuous if and only if $\forall B \in \beta,\ f^{-1}(B)$ is open in X

Proof **:**
expanding the pre-iamge of open sets is open using the basis generating deftinition of topology

Notice how this matches the definition of continuity given to you in MAT257:

Definition 1.3.4: Continuous Function in MAT257

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be continuous if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \epsilon$$

Or equivalently, when generalized, the pre-image of open sets are open.

Notice that it says “for all epsilon” which you can think of as ”for any open set in the image”. THen it says “there exists a delta” which can be thought of “there is an open set in the pre-image”. The next line is to show that you can keep the openness when you pass through the function, so "every pre-image of an open set is open"

If you're learnt what it means to be open from MAT257, you might recall that U is open if $\forall x \in U, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. As you must see, this comes from a basis generated by epsilon balls. More on this basis when covering metric spaces.

> **Important Note**: This definition of continuity using basis will become your dominant way of concretely proving continuity (someitmes abstractly, you require the pre-image of open sets, and sometimes other properties given circumstances). So many times, you'll take a basis elemetn $\beta \subseteq Y$, takes it's preimage, which you want to show it's open, which then you'll sow that for any point in $x \in \beta$, then $x \in U \subseteq \beta$ where U is a open (since every topology is a basis for itself).

Also, we might ask ourselves if we have a topology, can we find a basis? The trivial is that yes, every topology is its own basis, but we want a better criterion for finding it coarser basis:

Lemma 1.3.5: Finding Basis When Given A Topology

Let X be a topological space. Supposer that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X, and each $x \in U$, there is an elemetn $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X

Proof **:**
p.80

Overall, you will use basis when thinking about topologies quite often to simpilfy the technicalities of proving things like continuous funcitons.

1.4 Countability Axioms

(This was introducd much later in the book, but I feel it's better introduced here since it really relates to generating a space: If a space is first-countable, then sequences let us detect closure, Hausdorff, and Compactness! So Perhaps this can be added with a further discussion of basis!!)

Definition 1.4.1: Second-Countability

A space X is called ***secound-countable*** if it has a countable basis.

Example:
$\mathbb{R}$ is second-countable, because all open intervals with rational end-points will form a basis

We will be using this concept when we talk about seperation axioms. Unfortunately in this course, we won't treat it seperatly and see consequences of it. However, I do intend to return to this at some point since it's a chapter in munkres.

Real quick, since there's a second-countable, there's a first countable:

Definition 1.4.2: First-Countability

look at wiki (he skimmed over it)

And there isn't a 3rd-countable.

1.5 Closed sets

We have defined topology with *arbitrary* open sets, but having *finite* intersection. We can actually reverse the two, with *finite* union but *arbitrary* intersection. It turns out that what's key is that *one* of these two operations is finite (as we explained earlier), but it doesn't matter which!

What we'll explore in this section is how these two concepts are actually linked! That is, if you define it one way, you can get the other, and how can we use the properties of one to understand the other (like borders, connectedness, closure, interior, etc.). This will come down to the fact that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} (A_i)^c$$

This set-theoretic property will mean that we can be very interested in the compliments of open sets. We will give them a name:

Definition 1.5.1: Closed Sets

Let X be a set and $\mathcal{T}_X$ be a topology on X. A set $A \subseteq X$ is closed if it's compliment $A^c = X \setminus A$ is open

Note: There is no good reason closed sets were picked over open sets, but they give a different flavour when examining topologies. So it just shows that we can look at topologies with two different sets and get the same thing.

Example:

a,b is closed in $\mathbb{R}$ (it is the compliment of $(-\infty, a) \cup (b, \infty)$)

1. $\emptyset$ an $\mathbb{R}$ are always closed
   (a) From this, can conclude that in the trivial topology, everything is closed
   (b) similarly, in the discrete topology, everything is open and closed
2. arbitrary intersection are always closed
3. finite unions are always closed

The fact that finite unions and arbitrary intersections are closed should give an idea how closed is like being "co-open". You can basically define a topology as follows

Theorem 1.5.1: Thinking Of Topology In Terms Of Closed Sets

Let X be a topological space. Then the following conditions hold:

1. $\emptyset$ and X are closed
2. arbitrary intersections of closed set are closed
3. finite unions of closed sets are closed

There turns out to be no real advantage in using one set of axioms over the other. We could've had finite union and arbitrary intersection. However, one of those must be chosen (or else you can have $\bigcap_n (\frac{-1}{n}, \frac{1}{n}) = \{0\}$ which will start breaking how we think of basis, and a lot of other things) so by convention we choose arbitrary union and finite intersection.

As an example of how closed sets can be used basically interchangeably with open sets, here continuity with closed sets:

Theorem 1.5.2: Continuity And Closed

The function $f : X \to Y$ is continuous if and only if $\forall V \subseteq Y$ closed, $f^{-1}(V)$ is closed

Proof. Literally one line:

$$f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$$

because of set-theoretic reasons! □

Proposition 1.5.1: basis formulation with closed sets

If $\mathcal{T}$ has basis β, and U is a closed set, then there is an equivalent notion of for every $x \in U$, there is a V such that $x \in V \subseteq U$

***Proof* :**
see:

`https://en.wikipedia.org/wiki/Base_(topology)#Base_for_the_closed_sets`

Sometimes, this is worth working with, but since arbitrary union of closed sets need not be closed, this will not always be the idea tool to use.

We can also define the analogous idea of open maps with closed maps:

Definition 1.5.2: closed Maps

Given topological spaces X, Y. Then $f : X \to Y$ is an closed map if the image of closed sets are closed

Note that a map can be closed, but not continuous

Example: Continuous but not closed
Let $f : \mathbb{R}^2 \to \mathbb{R} \qquad (x, y) \mapsto x$, where $\mathbb{R}^2$ has the topology:

$$\{S \times T | S \text{ open and } \subseteq \mathbb{R}\,, T \text{ open and } \subseteq \mathbb{R}\}$$

and $\mathbb{R}$ has the usual topology.

`https://www.mathcounterexamples.net/continuous-maps-that-are-not-closed-or-not-open/`.

1.5.1 Clopen

You might ask yourself if being closed implies not-open. That's however not the case, since $\emptyset$ is both open and closed in any topology. There's a word for sets which are both open and closed:

Definition 1.5.3: Clopen

In a topological space X, $A \subseteq X$ is *clopen* if A is open *and* closed

We will get back to these types of sets extensively when dealing with connectedness – it will become a defining feature.

1.5.2 Closure

With some basic idea of closed sets, we can take advantage of the property of *arbitrary intersection* to formulate the next definition:

Definition 1.5.4: Closure

Let X be a topological space. If $A \subseteq X$, the *closure* $\overline{A}$ is the smallest closed subset of X containing X:

$$\overline{A} = \bigcap_{A \subseteq V,\ V \text{ is closed}} V$$

such a set is non-emtpy because X is closed. Also note that for any closed set, trivially, $A = \overline{A}$

Note that

1. A is an arbitrary set, it need not be open
2. there is not necessarily a smallest open ball that contains that set. That's because openness is not closed under arbitrary intersection. However, because open sets have arbitrary union, we can define a dual notion for open sets

Definition 1.5.5: Interior

The *interior* of a open set A, labled $\text{int}(A)$ is the lartest oep nset contained in A:

$$\text{int}(A) = \bigcup_{A \supset V,\ V \text{ is open}}$$

Example:

1. With this new way of looking at closure, try proving $\overline{(0,1)} = [0,1]$.
2. Similarly, try proving $\text{int}([0,1]) = (0,1)$.

With the idea of closure, we can bring up some logical consequences of the definition:

Theorem 1.5.3: "Border"-Like Property

$x \in \overline{A} \Leftrightarrow$ every open set U containing x satisfies $U \cap A \neq \emptyset$

Proof. Very good exercise! for the inverse, contradiction is much easier □

I say border-like property, because this idea gets us to the heart of what closure does. Notice that $\overline{A}$ is closed, and yet the theorem is centered around the *open* set A. So to say that for any *open* set U intersecting the *open* set A is non-empty gives information about the *closed* set $\overline{A}$ is actually quite interesting, especially because open sets are supposed to be some more granular way of looking at a geometric object than the entire object (like the trivial topology) but more coarse then looking at every point (like the discrete topology). This talk of open sets relating to this type of closed set will lead us to talking about the tiny difference between $\overline{A}$ and A in certain topologies with appropriate A. What I mean by this is that this condition is in a sense "trivial" for elements which are inside A since if $x \in A$ and $x \in U$ then there intersection is clearly non-empty. However, $\overline{A}$ is contains A, and so can be bigger. But it saying that if there is such a point, for any open set that contains this point, it *must* intersect with the potentially smaller A, meaning it must be, in a sense, arbitrarily close to A, but $\overline{A}$ is still possibly bigger than A. I keep saying things like "possibly" and "can be" because it is possible that A is open and $A = \overline{A}$ in certain topologies or certain sets in topologies. The interesting things happen when this is not the case, and in that case, $\overline{A}$ will be strictly bigger, *and* closed, so the compliment of an open set.

However, there are limits to thinking about that property as a border in the intuitive sense.

For example, $\overline{\mathbb{R}} = \mathbb{R}$ (since $\mathbb{R}$ is closed) and so $\mathbb{R}$, which doesn't have a border as is intuitive, is still it's own closure. Another example would be

the following topology

Example: Huge Closure

Let

$$T_{\text{large closure}} = \{\emptyset, (-1,1), (-\infty,2) \cup (2,\infty), (-\infty,2) \cup (-1,1) \cup (2,\infty), \mathbb{R}\}$$

In this topology, the closed sets are $[-2, 2]$, $(-\infty, 1] \cup [1, \infty)$, and $\mathbb{R}$. Therefore,

$$\overline{(0,1)} = [-2, 2]$$

that is, the border is quite substantial! So a more general intuition is that we look at open sets near the *edge* of A. In our $T_{\text{large closure}}$ example, there isn't much locally going on locally around the borders, for the very construction of our set is particular in only caring for open sets that fit very particular properties so that the closure of $(-1, 1)$ is purposely large.

Note that whatever "extra" we add to A when taking $\overline{A}$ is closed if A is open. Since $\overline{A}$ is closed then $X \setminus \overline{A}$ is open, and $(X \setminus \overline{A}) \cup A$ is open. Therefore $\left((X \setminus \overline{A}) \cup A\right)^c$, which is equal to $\overline{A} \setminus A$, is closed.

To get more of an idea of what is meant by how open sets "look" like on the "border", notice that the "border" did not have any open set. This is in fact a result of the definition. Take for example:

$$T_{\text{weird example}} = \langle \{(0,1), (2,3), (-\infty, 4) \cup (4, \infty), \mathbb{R}\} \rangle$$

which would be:

$$T_{\text{weird example}} = \{\emptyset, (0,1), (2,3), (-\infty,4) \cup (4,\infty), (0,1) \cup (2,3), (0,1) \cup (2,3) \cup (-\infty,4) \cup (4,\infty), \tag{1.1}$$

$$(0,1) \cup (-\infty,4) \cup (4,\infty), (2,3) \cup (-\infty,4) \cup (4,\infty), \mathbb{R}\} \tag{1.2}$$

In this example In this case, the two closed sets of interets are

$$[-4, 4]\,, [-4, 2] \cup [3, 4]$$

And so

$$\overline{(0,1)} = [-4, 2]$$

When I was trying to construct these examples, you might've noticed that I had to make sure that *no* open sets can be inside the border. This is another way of thinking about the theroem from above:

Corollary 1.5.1: Closure's border and open sets

Borders cannot have open sets inside them

***Proof* :**
Notice that the open set is not inside that border. Let's say as a hypothetical example that is was in there. then:

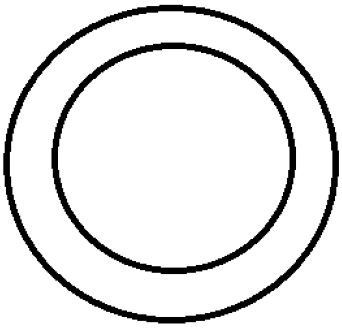

Figure 1.2:

if there was an open inside it then visually we'd get

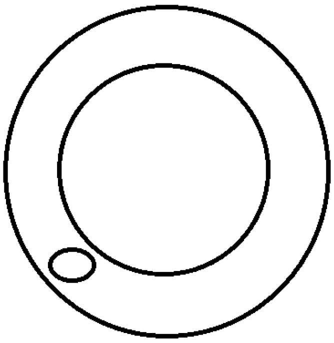

Figure 1.3:

However, then we'd have all that's in black here:

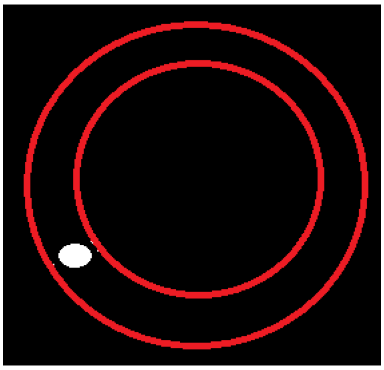

Figure 1.4:

to be the closed, but then the new closure would be all that is black here:

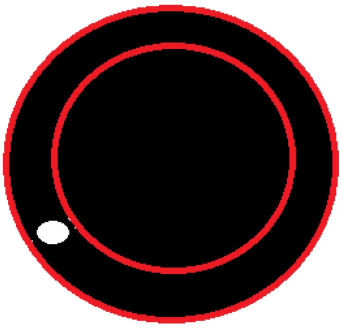

Figure 1.5:

Therefore, another way of thinking about the relation between border and open sets is that the closure of a set is the smallest amount of room added so that an open set cannot fit in the border.

You do much smaller closures if your open sets (excluding X) cover all of the underlying set. I don't have time to think through such scenarios, but it is worth doing.

You might be used to thinking about borders being "infinitely thin". Border with this property will happen on *at least* Hausdorff spaces. It might work on spaces with a weaker condition, but it will work on Hausdorff spaces. We'll get back to that later when such spaces are introduced.

One more interesting thing: We talked about how closedness is in a sense co-openness in that it does a lot of the same function, but we only switched the convention of which operation is finite and which is arbitrary many times. This here also let's us related geometry closed and open sets.

Theorem 1.5.4: Continuity And closure

The function $f : X \to Y$ is if and only if $\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

Proof. simple, in lec on 24th □

1.5.3 Limit Points

Limit points are a closely related concept:

Definition 1.5.6: Limit Points

Let X be a topological space, and $A \subseteq X$. We say that $x \in X$ is a *limit point* of A if and only if $\forall U \subseteq X$ open such that $x \in U$, then $U \cap (A \setminus \{x\}) \neq \emptyset$. We'll denote $\lim(A)$ to be the set of limit points of A.

Note that it is $x \in X$, not $x \in A$. The limit point does not need to be in A, that is, there is the notion of a set being able to be *arbitrarly close* to it's edge, but not have it.

Limit points are similar to closure, in that they both capture some idea of edgeness, as we've discussed the intuition behind the definition of closure. However, now the intersection does not have $\{x\}$. The difference this makes I think can be first illustrated by looking at examples which contrast it to closure of sets:

Example:
let $X = \mathbb{R}$

1. $A_1 = [0, 1]$, then the limit points are $[0, 1]$
2. $A_2 = (0, 1)$, then the limit points are $[0, 1]$
3. $A_3 = \{0\}$, then there are no limit points. This differs from closure since the closure of A_3 here would be A_3.
4. $A_4 = \left\{\frac{1}{n}, n \in \mathbb{Z}_+\right\}$, then the limit point is only 0. If this were the closure, then this would instead be $A_4 \cup \{0\}$

That last one should tell you what a limit point really is trying to focus on. It's trying to focus on the points which if you take some sequence with points within your set A, then that sequence will *converge*. That's why 1 in the last set does not converge in A_4. It should also be noted that we're more interested when a point is *not* part of A, so that something like a constant sequence doesn't count as interesting, because we're interesting in seeing if there are points *around our point x* which will get closer and closer to x, so in a sense limit points capture some nature of the neighborhood around our set A. This idea of capturing what's going on in the neighborhood of a point is key to topology, which is why this definition is worth intuitively understanding

We can see how closure relates to this. From the point of view of limit points, closure also lets in constant sequences. From the point of view of closure, limit points focus on edges which are in a sense "surrounded" by other points.

Note that closeness here for limit points is completely contingent on the topology, which should make sense if you think of topology as the study of closeness of stuff. Take for example $\mathbb{R}$ with the trivial topology and $\{0, 1\} \subseteq \mathbb{R}$. then $\lim(A) = X$.

Proof. Since $\mathbb{R}$ is our only non-empty open set, we'll only need to look at $\mathbb{R}$. Take $0 \in \mathbb{R}$. Then $\mathbb{R} \cap \{1\} \neq \emptyset$. Similarly for 1. Naturally this is true for any point in $\mathbb{R}$ too. Therefore, the limit points of A are in fact all of $\mathbb{R}$. □

This is true because there is no real sense of closeness in $\mathbb{R}$ as we understand it. If we were to take the standard topology on $\mathbb{R}$, and consider if 1 is a limit point in A, then: $(0.5, 2) \cap \{0\} = \emptyset$, meaning that we found an open set containing 1, which when intersected with A is empty, which shows us that in this topology, the points in A are not "close" enough to be limit points. We'll get back to this other condition which fixes this "pathological" idea of closeness/localness when dealing with Hausdorff spaces.

As mentioned, the closure of A and the limit points of A are very similar. We can in fact directly relate them:

Theorem 1.5.5: Relating $\overline{A}$ With Lim(A)

$$\overline{A} = A \cup \lim(A)$$

As we've already seen, $\overline{A}$ is a little more general, because from the point of view of limit points, it accepts constant functions. Though when proving, thinking about it the other way is easier.

If you are familiar with algebra, there is one more advantage to looking at closure this way. In algebra, there is an idea of being *closed*, meaning that if $a, b \in G$, then $a * b \in G$. Then in a sense, closure here has a similar idea, where sequences is like the binary operation (analogously), and a closed set contains all it's limits.

Proof **:**
exercice (double-subset inequality simple proof)

As I mentioned, closure brings up some idea of what a edge, or "border" is, that is, some points on the very "limits" of what it means to be in the set. That is, if you "move" *any* further in the wrong direction, you're out of the set and in the complement of the set. What if we only wanna concentrate on those points are on the edge? In a sense, that means if we take an "edge" points.

Definition 1.5.7: Border Of Set

$\partial(A) = \overline{A} \setminus \text{int}(A) = X \setminus (\text{int}(A) \cup (\text{int}(X \setminus A))) = \overline{A} \cap \overline{X \setminus A}$

This construction might also make you ask yourself what other definitions can we make using the idea of closure and closed sets! The answer to this is interestingly concicse: Given a set $A \subseteq X$, and X has some topology, then there are maximum of *14* intersting construcitons!

Theorem 1.5.6: Kuratowski

Consider the collectio nof all subsets A of the topological space X. The operation of closure $A \to \overline{A}$ and complementation $A \to X \setminus A$ are functions from this colleciton to itself. Then starting from a given set A, one can form no more than 14 distinct sets by applying these two operations

Proof **:**
This was an exercice in the book! p.102 #21

Example:
Let $A = \{0\} \cup (1, 2) \cup (2, 3) \cup (\mathbb{Q} \cap (4, 5))$. Show this set has all 14 charactersitics.

1.6 Hausdorff

We already mentioned Hausdorff spaces a couple of times. The idea of Hausdorff is that around any two arbitrary points, there exists a neighbourhood which contains each point respectively and doesn't intersect. This gives some forced "localness" properties to our space, especially if there are uncountably many points. In order to properly understand them, introducing them with sequences might make it intuitive on why we will defined it the way it is. How sequences act in Hausdroff spaces are in fact the motivation for the creator of the axiom which creates Hausdorff spaces.

We can define sequences converging to x in terms of the eventually being inside every open set containing x, as we've shown in MAT257. However, in more general framework, this does *not* make x unique. If we take the trivial topology, then sequences as outlined in the previous sentence converge not to a single point, but to *every* point, since there is only one non-empty open set. The Hausdorff condition ensures that we *only* converge to one point!

Definition 1.6.1: Hausdorff Space

A topological space X is called *Hausdorff* if for all pairs of distinct points $x_1, x_2 \in X$, there exists open sets U_1, U_2 such that $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Figure 1.6: visualization

This would mean that we can think of spaces which are fine enough to have some "distance" between points.

Example: Hausdorff Spaces

These are the basic examples

1. $\mathbb{R}$ is clearly a Hausdorff space
2. the discrete topology is Hausdorff
3. the trivial topology is *not* Hausdroff for $|X| \geq 2$

4. $\mathbb{R}^n$ is Hausdorff
5. $S^1 \subset \mathbb{R}^2$ is Hausdorff (the unit circle)
6. If $\mathcal{T}$ is a Hausdorff space, and $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\mathcal{T}'$ is also a Hausdorff space.

Example: Not Hausdorff

An example of how a space is not Hausdorff, how not all points have "distance" between them, is

$$\mathcal{T}_{\mathbb{N}} = \{(-n, n) \mid n \in \mathbb{N}\}$$

In this topology, the points -0.5 and 0.5 is not separated by an open set. Even if we do:

$$\mathcal{T}_{\mathbb{R}} = \{(-n, n) \mid n \in \mathbb{R} - \{0\}\}$$

It still doesn't work for, say $2, 4 \in \mathbb{R}$.

Note: As an exercise to show how connected closed and open sets are, we can do the following:

Definition 1.6.2: Hausdorff In Terms Of Closed Sets

$\forall x_1 \neq x_2, \exists M_1, M_2 \subseteq X$ closed such that $(X \setminus M_1) \cap (X \setminus M - 2) = \emptyset$ then

1. $(X \setminus M_1) \cap (X \setminus M - 2) = \emptyset$
2. $x_1 \in X \setminus M_1$
3. $x_2 \in X \setminus M_1$

Which is equivalently

1. $M_1 \cup M_2 = X$
2. $x_1 \notin M_1$
3. $x_2 \notin M_2$

Lemma 1.6.1: Finite Sets In Hausdroff Spaces

If X is Hausdorff, and $A \subset X$ is finite, then A is closed

Proof **:**

It is enough to show singletons are closed, since closed sets are closed under finite unions. Let $x \in X$. Take all $g \in X$ such that $g \neq x$. then there exists $U_g \subset X$ which is open such that $g \in U_g$

and $x \notin U_g$ by Hausdorff property. Then

$$\bigcup_{g \neq x} U_g = X \setminus \{x\}$$

which is open, and hence is complement is closed, and hence $\{x\}$ is closed, and hence any finite set is closed

Note: If a topology has finite sets which are closed, this does not mean that the topology is Hausdorff. The finite-compliment topology has closed sets, but it is not Hausdorff (take $\mathbb{R}$ with this topology, then any convergent sequence must converge to an open set which has all but finitely many elements). The axiom which says that finite sets are closed is called the T_1 axiom.

Recall how we were talking about being convergent? We can quickly state that formally here now:

Definition 1.6.3: Convergent sequence

Let X be a topological space, and $a_1, ..., a_n, ...$ be a sequence of element of X. We say that this sequence *converges* to $a \in X$ if and only if for all open sets U such that $a \in U$, $\exists N$ such that $\forall n \geq N, a_n \in U$.

The point of this definition is to give another definition of continuity:

Proposition 1.6.1: Continuity With Respect To Sequences

Let $f : X \to Y$ be a continuous function and $(x_n)_{n=1}^{\infty}$ a convergent sequence. Then $(f(x_n))_{n=1}^{\infty}$ is a convergent sequence, i.e. continuity preserve's convergent sequences

***Proof* :**
exercise

Lemma 1.6.2: Hausdorff Spaces Means At Most 1 Element When Converges

If X is Hausdorff, every sequence converges to *at most* 1 element.

***Proof* :**
simple proof by contradiction. (also $\Rightarrow\Leftarrow$ is a symbol for contradiction)

1.6.1 Relation With Previous Concepts

Proposition 1.6.2: Product Of Hausdroff Spaces

X, Y are Hausdorff if and only if $X \times Y$ is Hausdroff.

Proof **:**
easy proof

Proposition 1.6.3: closure of open set in Hausdroff space

Let $A \subseteq X$ be a *open* set. Then $\overline{A}$ add's a "trivial" amount to A, that is

$$A \subseteq B \Rightarrow \overline{A} \subseteq B$$

Solution If it were not the case, then the border would have an open set that doesn't intersect A, a contradiction. ▼

Corollary 1.6.1: closed Hausdorff set's interior

Let $A \subseteq X$ be a *closed* set. Then int$A$ removes a "trivial" amount to A, that is

$$B \subsetneq A \Rightarrow B \subseteq \text{int}(A)$$

Proposition 1.6.4: Subspace Of Hausdroff Space Is Hausdroff

here

Proof **:**
here

The quotient of a Hausdorff space need not be Hausdorff:

Example: Counter-Example
here

1.7 Metric Spaces

(Mention how there are structure that were invented that all generlize to being topological structures. Ex. metric structures or differentiable structure. Measurable structures is not, kind of an exception maybe? I think in logic topology also comes back)

Metric spaces are different from topological spaces. While in topology, we have an underlying set X with a set of subsets of it $\mathcal{T}_X$, defining $(X, \mathcal{T}_X)$, a metric spaces a set X with a *metric* d_X defined on it (X, d_X) (which we'll define in a moment). This makes these two spaces categorically different [3]. However, we introduce these spaces since every metric space also creates a topological space! Hence, a very natural way of studying metric spaces is through topology. For example, we can show that every metric spaces (with the induced topology by the metric) is Hausdorff. However, continuity will not preserve all the properties of a metric space!! For example, a metric can define whether a sapce is bounded. But, we will show examples of a continuous function betweeen $f : X \to Y$, where X is

[3]As in, metric spaces are objects in the **Met** Category, while topologies are objects in the **Top** Category

bounded, while Y is not! Another such example is completeness, where we can make a continuous map between $(-1,1) \to \mathbb{R}$, but $(-1,1)$ is clearly not complete, while $\mathbb{R}$ is.

If we take a further step back for a moment and see the bigger picture in Analysis, metric spaces are perhaps the most common topological space mathematicians use. There are some even more specific spaces, outlined in the bellow image:

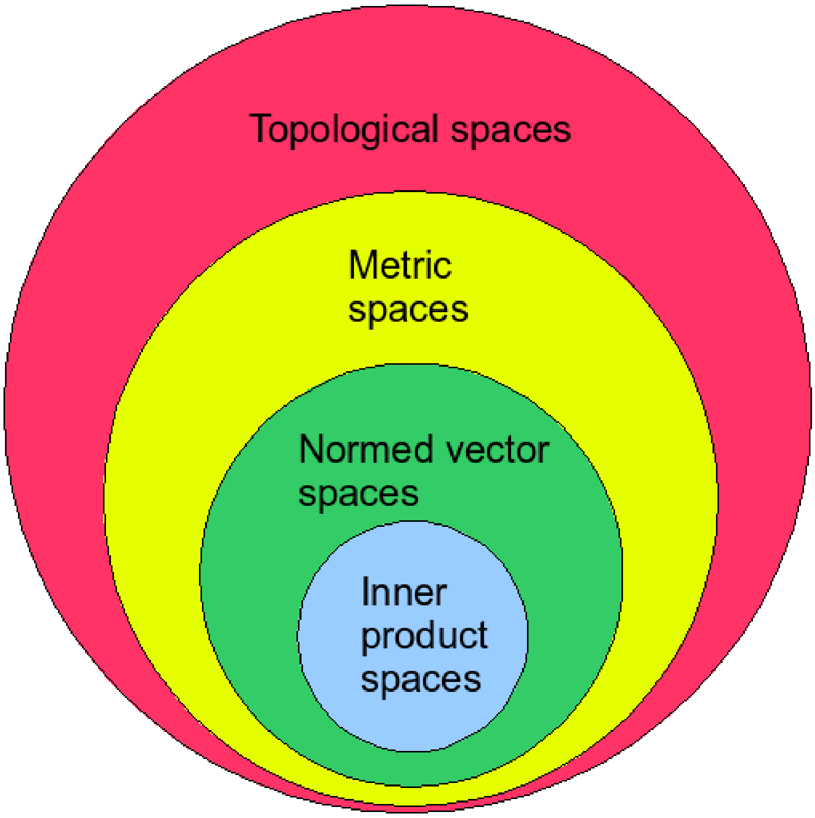

Figure 1.7: Hiarchy of spaces

and covering metric space's can also be seen as the first step into analysis.

This and more is covered in this chapter.

1.7.1 Basic Definition And Examples

We'll put the definition of metric here as a reminder

Definition 1.7.1: Metric

Let X be a set. A *metric* on X is the function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

such that

1. $d(x, y) = d(y, x)$
2. $d(x, y) + d(y, z) \geq d(x, z)$
3. $d(x, y) = 0$ if and only if $x = y$

Note: the fact that it is an *if and only if* is in fact very important. Without that you can say

$$d(x, y) = 0 \qquad \forall x, y \in X$$

and that would be a metric, and you can go on to say it would define the indiscreet topology, but this was purposely left out because it would then not make all metric spaces Hausdorff (as we'll show soon)

From this, we can define the basis. Take $\epsilon > 0$. Then

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

Definition 1.7.2: Metric Topology

If d is a metric on X, we call the *metric topology* on X, to e the topology generated by the basis

$$\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$$

One might ask themselves why is this the definition of a metric space? I personally like to think of it Categorically

> If X has a metric d, then the coarsest topology which makes $d : X \times X \to \mathbb{R}$ continuous is the one we just described

To me, this is what it means to describe a metric topology, since we have reaosn to augment the notation of metric to a topological one in a way that ins't merely convention, but with purpose. If one were to continue categorically, there might even be a universal property hidden here.

Lemma 1.7.1: Metric topology is Hausdorff

The metric topology is Hausdorff

Proof. Let X be a metric topology. Take any two distinct points x_1, x_2. Since they're distinct, let $r = d(x_1, x_2)$ where r is non-zero. Consider $\epsilon = \frac{r}{10}$. Then $B_\epsilon(x_1) = \{y \in X \mid d(x_1, y) < \epsilon\}$ and

$B_\epsilon(x_2) = \{y \in X \mid d(x_2, y) < \epsilon\}$ Therefore:

$$B_\epsilon(x_1) \cap B_\epsilon(x_2) = \emptyset$$

by definition of open balls. Hence, metrizable spaces are metric topologies. □

Now, you can think that we might need to start carrying around the metric when defining a topological space which has a definable metric (ex. $(\mathcal{T}_X, X, d)$). However, many metrics will sometimes define the same space. And sometimes, we want to think of spaces which have a metric not as topological spaces but as their own space. So we want to make the distinction of a space with a particular metric, and the idea of a topology which *can* be defined by a particular metric. The following terms distinguish between these two concepts:

Definition 1.7.3: Metrizable Topologies

If X is a topological space, we say that X is metrizable if there exists a metric on X which induces the topology of X.

Definition 1.7.4: Metric Space

A *metric space* is a metrizable topological space together with a **specific** metric d inducing the topology of X. We usually write

"Let (X, d) be a metric space"

Note: Sometimes, a metric space in other places of math is not defined in terms of metrizable space, but *just* in terms of a metric being definable on the space. It will always turn out that if you have a metric spaces, it is also a metrizable topology, but that can be considered a subtly.

Note: Note also that d is a continuous function! So some topological property of $\mathbb{R}$ is important when thinking of metric spaces!

Example:
The typical MAT257 metric spaces. Take $\mathbb{R}^2$. $\mathbb{R}^2$ is metrizable. For example

1. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
2. $|x_1 - x_2| + |y_1 - y_2|$
3. $\max(|x_1 - x_2|, |y_1 - y_2|)$
4. $\min(1, \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})$
5. 0 if $x = y$ and 1 if $x \neq y$ (the discrete metric, which creates the discrete topology)
6. Given a metric d, $\overline{d} = \min(d, 1)$. A cool analogy the prof used for this metric is as follows: Imagine you have a teleporter which can get you anywhere. It takes an hour to boot. If it takes less than an our to get somewhere, you'll walk. Otherwise you'll use the teleporter.

A quick observation on the metric on $\mathbb{R}$. say you have the euclidean metric. Then take

$$\overline{d}_\epsilon = \begin{cases} d(x,y) + \epsilon & x \neq y \\ 0 & x = y \end{cases}$$

Then this metric is in fact equivalent to the discrete metric! If $B_{\overline{d}_\epsilon}(x, \frac{\epsilon}{2})$, then B only contains x, making it equivalent to the discrete metric.

In general, for finite domains or domains which are discrete subsets of $\mathbb{R}$ (relative to the euclidean topology), something like this will hold.

We can also ask if some notion of the coarsest possible metric topology. However, The thing is that the moment you have the non-discrete metric, since we have that the co-domain is $\mathbb{R}$, and d is a constant, continuous

$$d(x,y) \leq d(x,z) + d(z,y)$$

meaning we just shrunk we we considered our open sets. So this does not seem to be a *natural* question to ask. Indeed, it might even be that there *isn't* a coarsest space

Lemma 1.7.2: Subspace Of Metric Space Is Metric Space

If (X, d) is a metric space, and $Y \subseteq X$, then $(Y, d|_Y)$ is also a metric space, and the subspace topology on Y coincides with the metric topology induced by $d|_Y$

Proof **:**
the proof have 2 sets:

1. Can check that $d|_Y$ is a metric
2. $\forall g \in Y, \epsilon > 0, B_d(g, \epsilon) \cap Y = B_{d|_Y}(g, \epsilon)$

which would finish the proof

Definition 1.7.5: Continuity For Metric Spaces

Let (X, d_X), $Y, d_Y)$ be metric spaces. Then $f : X \to Y$ is continuous if and only if

$$\forall, x \in X, \exists \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \epsilon$$

which is the same definition in MAT257

We can also define some property which strongly rely on the metric:

Definition 1.7.6: Diamater

For $A \subseteq X$, define the diamater of A to be

$$\operatorname{diam}(A) = \sup(d(a_1, a_2) \mid a_1, a_2 \in A)$$

Definition 1.7.7: Bounded

We say that (X, d) is *bounded* if $\operatorname{diam}(X) < \infty$

Theorem 1.7.1: Bounding Any Metric Spaces

Given $\overline{d}$, then the metric topology for $\overline{d}$ is the same as for d.

Proof. □

We can also return to our earlier discussion about border of sets in metric spaces. Let say a set A is open in a metric topology. Consider $x \in \overline{A} \setminus A$. When x is not Hausdorff, we know that it is possible that the border of a set be weird. Hausdroff fixes that for us

Theorem 1.7.2: Border Of Set In Metric Space

Take A to be a non-empty open in the metric space X. Then $x \in \overline{A} \setminus A$ if and only if

$$\forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset \wedge B_\epsilon(x) \cap (X \setminus A) \neq \emptyset$$

intuition

Proof. recall that

$$x \in \overline{A} \setminus A \Leftrightarrow \text{every open set } U \text{ containing } x \text{ satisfies } U \cap A \neq \emptyset$$

We will be using that fact for this proof:

($\Rightarrow$) Let's start with $x \in \overline{A} \setminus A$. For the sake of contradiction, let's say

$$\exists \epsilon > 0 \qquad B_\epsilon(x) \cap A = \emptyset \vee B_\epsilon(x) \cap (X \setminus A) = \emptyset$$

That is, either the epsilon ball is completely seperate from A, completely seperate from the compliment of A. If it's completely seperate from A then it contradicts the equivalent definition of closure "every open set U containing x satisfies $U \cap A \neq \emptyset$", since we have a an open set $B_\epsilon(x)$ containing x but not intersecting A. Therefore, it would have to be that $B_\epsilon(x) \cap (X \setminus A) = \emptyset$, that is, it is completely seperate from the compliment of A. However, $x \in \overline{A} \setminus A$, meaning that x is in some subset of X which is not part of A, that is some comliment of A, and so $B_\epsilon(x) \cap (X \setminus A) \neq \emptyset$, a contradiciton again. Therefore, there does not exist an epsilon for which an epsilon ball around x is either completely inside A or outside of A, and therefore any epsilon ball containing A but intersect both A

($\Leftarrow$) Let's take any point x that satisfies that condition. If we consider any open set U that contains x and non-trivially intersects with A, then since X is a metric space generated by all epsilon balls, $\exists \epsilon_0$ such that $B_{\epsilon_0} \subseteq U$, and we know that by assumption, that any ball around x will intersect A, therefore, it must hold.

Therefore, it must always be the case in metric topologies that the border has this "infinitely thin" property ☐

What made me originally think of this is imaginning a thick border, thick enough so that $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq \overline{A} \setminus A$. That is, you can think an epsilon ball around the border. However, that would then contradict that for any point in the border, $U \cap A \neq \emptyset$, therefore, the border must be really thin.

1.7.2 Product Spaces and Metrics

Given two metric spaces, if we take the product of them, we can ask ourselves what would be the metric on $X \times Y$ which would define the natural product topology? It turns out to be the following metric:

Definition 1.7.8: product metric

Let (X, d_X), (Y, d_Y) be metric spaces. Then

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max(d_x(x_1, x_2), d_y(y_1, y_2))$$

Lemma 1.7.3: Metric Product Space

1. $(X \times Y, d_{X\times Y})$ is a metric space
2. The metric space coincides with the product topology

Proof. Oct 1st lecture ☐

Theorem 1.7.3: Metrizablility Of Product Sapces

Let $(X_n)_{n\in\mathbb{Z}_+}$. Then every X_n is metrizable if and only if $X = \prod_{n\in\mathbb{Z}_+} X_n$ is metrizable (if X has the product topology)

***Proof* :**
Oct 1st lecture

Chapter 2

Constructing Topologies

In this chapter, we'll mainly be covering how to take the product of topologies and still have a topology, and how to define the notion of a sub-topology – in other words, we're integrating two more fundamental ideas when analyzing objects.

2.1 subspace (or Induced) topologies

We'll start with seeing how to create a topology if we have a subset $A \subseteq X$. The idea here is that sometimes we want to look at smaller parts of our space and see the topological properties it has:

Definition 2.1.1: Subspace Topology

Let X be a topological space, an Y be a subset of X. We definite the *subspace topology* on Y as:

$$\mathcal{T}_Y = \{U \cap Y \mid U \text{ is open in } X\}$$

Note: Y need not be open. if it is open, then the topologies are comparable.

Example:

1. basic examples
2. Give a word of warning on subspace topologies of product topologies (refer to homework)

Theorem 2.1.1: Basis For Y Relative To X

1. if β is a basis for the topology on X, then $\{B \cap Y \mid B \in \beta\}$ is a basis for the subspace topology for Y
2. Let $X \supseteq Y \supseteq Z$. Let X have a topology $\mathcal{T}_X$, and let $\mathcal{T}_Y$ be the induced topology on Y. Then the topologies induced on Z form X and Y are the same

Combining with product topologies:

Proposition 2.1.1: Product Of Subspace Topologies

Let X, Y be topological spaces, and $A \subseteq X$, $B \subseteq Y$. The subspace topology on $A \times B$ is the same as the product topology of the subspaces topologies on A, B.

Solution ▼

using the bases, it should be intuitive.

Note: CONVENTION: If you see some arbitrary set (ex. $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$), then this set will be a subspace topology of $\mathbb{R}$.

And like before, we can also define the idea of subspaces by using closed sets:

Lemma 2.1.1: Subspace Topology And Closed Sets

Let $Y \subseteq X$ [equipped with the subspace topology]. Then $A \subseteq Y$ is closed in Y if and only if $\exists V \subseteq X$ closed in X such that $A = V \cap Y$

Proof. good exercise

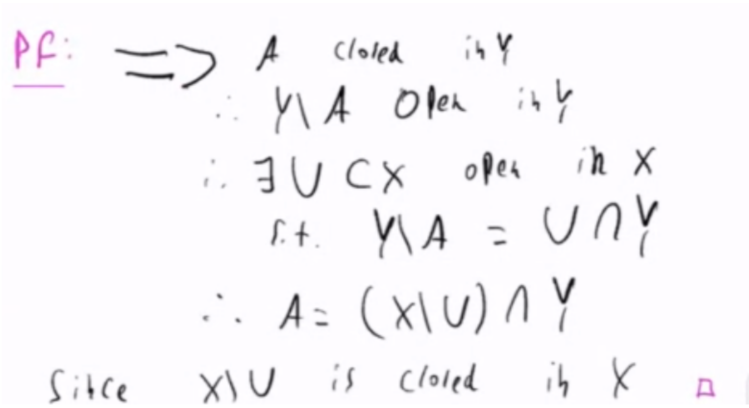

Figure 2.1: from lecture

□

Corollary 2.1.1: "transitivity" of closed sets between topology and subspace topology

Let X be a topological space. Suppose $Y \subseteq X$ is closed and $A \subseteq Y$ is closed in Y. The A is closed in X

Proof. Again, good exercice □

2.2 Product topology

This is a very common technique to generate new topologies, taking together smaller ones and producing a larger one

Definition 2.2.1: Product Topology basis

The *product topology* is generated by the basis

$$\beta = \{S \times T \mid S \subseteq X \text{ is open }, T \subseteq Y \text{ is open } \}$$

Corollary 2.2.1: Product Topology basis

The basis β define in 2.2.1 is indeed a topology

Proof **:**
Recall $\cap$ and $\cup$ operation being "point-wise".

We can also generalise the top definition to larger products:

Definition 2.2.2: Product Topology on n Topologies

The *product topology* is generated by the basis

$$\beta = \{S_1 \times \cdots \times S_n | S_i \subseteq X_i \text{ is open } \}$$

Note: Really make sure you have your intuition down on how to think about subsets of $X \times Y$. For example, the point $(1, 1)$ is not covered by $(1, 0) \cup (0, 1)$ visually, we we get something like

here

Another thing to watch out for is that $A \subseteq X \times Y$, this doesn't mean that $A = U \times V$, $U \subseteq X$, $V \subseteq Y$. For example: $A \subseteq X \times X$, $A = \{x \times x | x \in X\}$. By no means is this set of the form $U \times V$. Be very wary of this! Especially when thinking about subspaces of $X \times Y$!

Example:

1. You are already familiar with $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. You might've seen this as open balls. However, this topology looks more like squares:

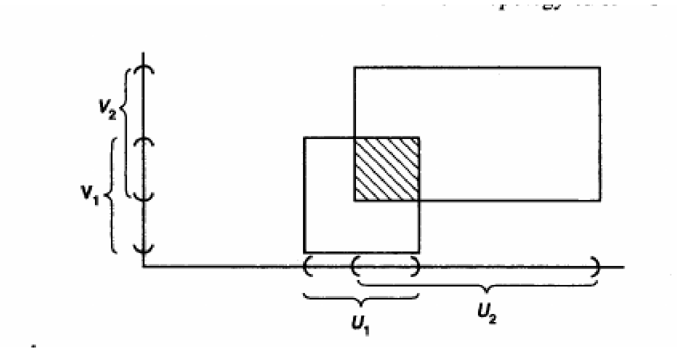

Figure 2.2: Notice that everything is a square

We will introduce open-balls once we define *metric-spaces*, and show how these two topologies are actually *homeomorphic*, and so can consider both interchangeably

2. Take the topology $\mathcal{T} = \{\emptyset, \{a,b\}, \{a\}\}$ on $X = \{a,b\}$. Then the product topology on $X \times X$ is $\{\emptyset, X \times X, \{(a,a)\}, \{(a,a),(a,b)\}, \{(a,a),(b,a)\}, \{(a,a),(a,b),(b,a)\}\}$. Notice that the last open set in the list is not in the basis.

With this construction, what we've done is start with two topologies, and combine them together to form a product. However, we often wonder if we can have an arbitrary topology Z and ask whether there exists two topologies X, Y such that $Z = X \times Y$. You can imagine that the analysis of a space can be greatly simplified if we can *project* down. This brings us to the following idea:

Theorem 2.2.1: Projection And Product Topologies

The product topology on $X \times Y$ is the coarsest topology for which π_X and π_Y are continuous

***Proof* :**
Let $\pi_X : X \times Y \to X$ such that $\pi_X(u,v) = u$. We'll first show that it's continuous. If we let $U \subseteq X$ be open, then $pi_X^{-1}(U) = (U, Y)$, and since U is open in X and Y is open in Y, then $pi_X^{-1}(U)$ is open. Since U was arbitrary, that means that π_X is continuous. Similarly for π_Y.

To shwo that $X \times Y$ is the corsest topology for which *both* π_X and π_Y are continuous (at the same time). Let's say we pick $\mathcal{T}_{X'} \subseteq \mathcal{T}_X$. That means there is an open set $U \in \mathcal{T}_X$ such that $U \notin \mathcal{T}_{X'}$. THis will now become a problem, since if we pick $U \in X$, and consider $\pi_X^{-1}(U) = (U, Y)$, this will no longer be open, since $U \notin \mathcal{T}_{X'}$. Similarly for Y. Thus $X \times Y$ *must* be the coarsest topology for which *both* π_X and π_Y are continuous.

If you are more Categorically inclined, then what's going on here is that in the **Set**-Category, we can define product sets $X \times Y$ (as sets) by the following universal property

$$\begin{tikzcd} & X \arrow[d, "\pi_\alpha"] \\ Y \arrow[ur, "f"] \arrow[r, "f_\alpha"] & X_\alpha \end{tikzcd}$$

When working with product topologies, we want this same diagram to commute, except now we have continuous funcitons instead of regular function. And it turns out that the (categorical)*limit* which makes us find this universal property is the projection functions. The reason to formulate in this manner is to show that the following universal property is satisfied:

Lemma 2.2.1: π_I Is An Open Map

here

Proof **:**
Simple proof

Note: THis example requires Hausdorffness
It need not be that π_X or π_Y be closed maps! It might seem tempting, since $id : X \to X$ is open and closed, so $\pi_X : X \times Y \to X$ would be seem to be closed. However take positive portion of the graph $1/x$ (label this set Γ). It can be shown that in Hausdorff spaces, graphs are closed. However, this image of this set is $(0, \infty)$, which is open! This will hold if Y is compact.

We showed that if $f : Z \to X \times Y$ is continuous, there must exist two "coordinate" functions $f_1 : Z \to X$, $f_2 : Z \to Y$ which are continuous, but what if we have the converse: $f : X \times Y \to Z$; does the "coordinate" function here need to be continuous? The answer is no, we can find a counter-example

Example: not continuous function for $X \times Y \to Z$
p.112 exercice 12

Also, here are some easy mistakes to make the first time you work with the product topology:

1. $(1, 0) \cup (0, 1)$ does not cover $(1, 1)$. The product topology defines a whole new grid of points.
2. If $U \subseteq \mathbb{R}^2$ is open, this doesn't mean that $U = V \times W$, V, W open in $\mathbb{R}$! It's true that $V \times W$ is an elemetn of $\mathbb{R}^2$, but if U is open in $\mathbb{R}^2$, then the union of elemetns of the form $V \times W$ (i.e., the basis for $\mathbb{R}^2$) are elements in $\mathbb{R}^2$. That is, not all open sets of $\mathbb{R}^2$ can e written as $V \times W$!

Example:
Take

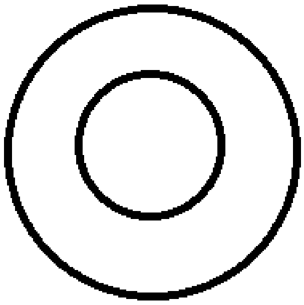

Figure 2.3: counter-example

To be some open set somewphere in $\mathbb{R}^2$. This set is the *arbitrary* union of open sets of the form $V \times W$! It can get even worse. In $\mathbb{R}^2$. take $V = \{(x, x) | x \in \mathbb{R}\}$. This set is closed (as it can be shown to be the compliment of an open set), but it is in no way represented by closed sets of the form $U \times V$. where U, V have more than 1 element. It is true that $x \times x$ is closed, and the *arbitrary* union of these closed sets are indeed closed, but as at this point know, the arbitrary union of closed sets need not be closed.

2.2.1 The Infinite Case

When we defined the product topology for more than 2 topologies, we were sneaky in insuring the definition limited to *fintie* products (notice the n in the definition). This was purposeful! We'll Now explore what happens in the infinite case.

The goal is the show what the flexibility of letting there be an infinite number of topologies bring. We'll give this topology a new name, and also figure out a definition of the product topology on infinitely many topological spaces.

Here is the first generalization of the product topology:

Definition 2.2.3: Cross product of infinite topologies (Box Topology)

Let $\mathcal{I}$ be an indexing set for topologies X_i, $i \in \mathcal{I}$. Then let us take as basis for a topology:

$$\prod_{i \in \mathcal{I}} U_i$$

where U_i is open in X_i. This topology is called the ***box topology***.

And here is the second keeping the finiteness of product topologies:

Definition 2.2.4: Cross product of infinite topologies (Product Topology)

Let $\mathcal{I}$ be an indexing set for topologies X_i, $i \in \mathcal{I}$. Then $\prod_{i\in\mathcal{I}} X_i$ has as basis all sets of the form $\prod U_i$, where U_i is open in X_i, for each i, and U_i is *equal* to X_i for all but finitely many values of i.

Note: Notice that definition 2.2.4 is named *product topology*. This definition is actually the *same* as product topology! Notice in both cases we have a finite amount of open sets in the cross product which (which are non trivial). Since in any element, most spaces are X_i (the entire space) while finitely can be U_i, these two spaces are *topologically the same*: **all** properties that apply to one apply to the other, they are indistinguishable topologically.

Note also that the box topology is thus generally *finer* than the product topology.

So why would one be preferable to the other? What does this restriction to only finitely many positions in any element being U_i? This comes down to the universal property of products *failing* on the box topology, that is, you can't split a continuous function $f : A \to \prod_{i\in\mathcal{I}} X_i$ to component functions $f_i : A \to X_i$! That is, The product topology is more useful than the box topology since continuous maps are continuous if and only if the component maps are continuous. We will generalize theorem 2.2.1 to the infinite case, then show how it fails for the box topology:

Theorem 2.2.2: Continuous Maps On Product Topologies

Endow X with the product topology. Let $f : Y \to X_I$, where $f(y) = (f_i(y))_{i\in I}$, f splits, that is:

$$\begin{tikzcd} & X \arrow[d, "\pi_i"] \\ Y \arrow[ur, "f"] \arrow[r, "f_i"'] & X \end{tikzcd}$$

where π_i is the projection map. Then f is continuous if and only if $\forall i \in I$ f_i is continuous

Proof **:**
here

Example: Failing on the Box-Topology
Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be the function st

$$f(t) = (t, t, t..., t, ...)$$

This functions splits into coordinate functio $f_n(t) = t$. As we've shown in 2.2.2, if $\mathbb{R}^\omega$ has the product topology, every f_n is continuous. However, if $\mathbb{R}^\omega$ has the box topology, f is no longer continuous, even if every component function is continuous. Take the open set:

$$V = \prod_n \left(\frac{-1}{n}, \frac{1}{n}\right) = (-1, 1) \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \cdots \in \mathbb{R}^\omega$$

and consider $f^{-1}(V)$. If $f^{-1}(V)$ were open ,in $\mathbb{R}$, it would containg some interval $(-\delta, \delta)$ around the point 0. This would mean that $f((-\delta, \delta)) \subseteq B$, so that, applying π_n to both sides of the inclusion:

$$f_n((-\delta, \delta)) = (-\delta, \delta) \subseteq \left(\frac{-1}{n}, \frac{1}{n}\right)$$

for all n. However, for n larger enough, δ will be too big ($n > \frac{1}{\delta}$). Thus, the inverse image will be will be $\{0\}$, which is not open. Thus f is not open!

This here is an example of what we mentieond when discusing why you can only have either unions or intersections be arbitrary applied: Adding the 2nd infinity makes it too flexible! Here's adding that many open sets will sneakily introduce "infinite intersections", which as we mentioned before is not a property we want when dealing with topologies.

Though the box-topology fails in this important criterion, there are a couple of theorems which work on Box topologies, and so on product topologies:

subspace works well, Hausdorff works well.

Proposition 2.2.1: Closure Of A Product Is Product Of Closure

Let X be the product topology of the sets $\{X_j \mid j \in J\}$ is family of topological spaces and $A_j \subseteq X_j$. Then $\overline{\prod_{j \in J} A_j} = \prod_{j \in J} \overline{A_j}$

Solution Proof here ▼

2.3 Quotient Topology

As usual as in most categories we work in, we can also define the dual notion of sub-object with quotient-object. In our case, that would be the quotient topology. What's interesting about the **Top** category is that the quotient objects don't really behave well. You can't generally take the product of quotient topologies, you can't take the subspace of a quotient topology and still have a topology. However, unlike the **Grp** Category, you do not need a *congruence relation* to define a quotient object, only an equivalence relation. To refresh your memory, a *congruence relation* is an equivalence relation which *preserves* the algebraic structure (so $[x \cdot y] = [x] \cdot [y]$). in the **Top** Category, we don't need an equivalent notion of *congruence relatoin*; defining quotients in term of equivalence relations *only* will suffice! In this way, quotient objects in the **Top** category are more flexible, but at the cost that many related objects we like to work with (ex. sub-objects, Cartesian product objects) will not be well-defined without extra conditions.

2.3.1 Basic Properties And Definition

The intuitive first example given with quotient topologies is by "glueing" surfaces together. so we can take an interval, and glue together the two end points, and get a circle. Or we can get a rectangle, and glue opposite edges together in the right way, and get a torus. Or we can glue all the points together and get a sphere.

To understand how to combine these points and treat them as *one* points, we start by defining a quotient map, which will let us define the quotient topology:

Definition 2.3.1: Quotient Map

Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective map. The map p is say to be a ***quotient map*** if

$$\text{A subset } U \text{ is open in } Y \text{ if and only if } p^{-1}(U) \text{ is open in } X$$

This condition is also sometimes called *strong continuity*. For a continuous function, if $p^{-1}(U)$ is open, then it it need not be that U is open. In this way, the function must preserve *more* open sets from the domain.

It's kind of a weird extra condition at first. The intuition given by the prof is that if you have space to walk around once you've glued stuff toegether, then once you pull that back, you should still have space to walk around. Also can be thought of as the smallest set which the equivalence class of the quotient space is preserved.

Another intuition for quotient maps is that its *almost* a homeomorphism. If it is injective, then it is a homeomorphism

We can also formulate the equivalent definition with closed sets with the usual strategy $(f^{-1}(Y \setminus B) = X \setminus f^{-1}(B))$

There are a couple of other ways of thinking about quotient maps that might be intuitive. Let $f : X \to Y$ be a function, and take $A \subseteq X$. we say that A is ***saturated*** (with respect to f) if A contains every set $p^{-1}(y)$ that it intersects – or as the professor said it $f^{-1}(f(A)) = A$. That is, if $y \in Y$ such that a point $x \in p^{-1}(y)$ is also in A, then A must contain all of $p^{-1}(y)$, for every y. With this formulation, we can equivalently say that p is a quotient map if it a continuous and p maps *saturated* open sets of X to sets of Y (or saturated closed sets of X to closed sets of Y)

There are also two special kind of quotient maps. If p is a surjective continuous function that is also *open* or *closed*, then it is also a quotient map. However, the condition of open/closed is too strong of a condition too: there are quotient maps which are neither open or closed. This is because the $p^{-1}(U)$ being open condition is weaker than the image of open sets are open condition, and so the converse of open map is not the converse of $p^{-1}(U)$ being open since if a map is not an open map, it might still satisfy the $p^{-1}(U)$ being open condition.

The idea of a quotient map is similar to quotient groups. Since the function is continuous, we already got the "homomorphic" property. The extra condition of $f^{-1}(V)$ open then V open ensures us that the function is more than just projecting down to a smaller space, but it's also making sure that if the fiber of a set $f^{-1}(U)$ is open, then it had to come from an open set U.

Example:

1. This examples brings in this “gluing” idea that will come up often when looking at quotient topology.Let Let $p : [0, 1] \cup [2, 3] \to [0, 2]$, both subspaces of $\mathbb{R}$. We'll define p so that it “glues” these two intervals together:

$$p(x) = \begin{cases} x & \text{for } x \in [0,1] \\ x-1 & \text{for } x \in [2,3] \end{cases}$$

   Notice that p is surjective, showing the gluing property. It is also closed. if we take a closed basis element $[a, b]$ in the domain, then $p([a, b])$ will either the a closed interval. Finally, this map is continuous. If we take a closed interval $[a, b]$ in the co-domain [a], then the pre-image is clearly either a closed interval or the union of two closed intervals

   As an aside, this map is *not* an open map. $[0, 1]$ is open in the subspace topology in the domain, but $[0, 1]$ is not open the subspace topology of the co-domain. Furthermore, if we got rid of the point where we were about to glue (say $[0, 1) \cup [2, 3]$), then we no longer have a quotient map. It will still be surjective and continuous. It would no longer be closed ($p([0, 1) = [0, 1)$, where $[0, 1)$ is closed in the co-domain), but that as we showed doesn't disprove enough. To show it's not a quotient map, notice that $[2, 3]$ is open in the domain, and $p(p^{-1}([2, 3])) = [2, 3]$, meaning that it's saturated, but $p([2, 3]) = [1, 2]$ is *not* open in $[0, 2]$.

2. Projection maps are quotient maps. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This map is surjective, continuous, and open (as we've shown before).

   As an aside, note that π_1 is *not* a closed map! The subset $C = \{(x, y)|xy = 1\}$ is closed in $\mathbb{R} \times \mathbb{R}$, but $\pi_1(C) = \mathbb{R} - \{0\}$ is *not* closed in $\mathbb{R}$. Furthermore, if we were to take $p : C \cup \{0\} \to \mathbb{R}$, then this map is continuous and surjective, but the one point set $\{0\}$ is open in A and saturated with respect to p, but its image is not open in $\mathbb{R}$

3. Here is a more general example. Let X be a topological space and $A \subseteq X$. if $r : X \to A$ is a continuous map such that $r|_A(a) = a$ (r is the identity on A), then r is a quotient map

   Solution We'll first show that if $r : X \to Y$ and $f : Y \to X$ are continuous maps such that $f \circ r \cong id$ (that is, f is a continuous right inverse),then r is a quotient map.

   the map r is already continuous and surjective by the right inverse; it's only left to show that if $r^{-1}(U)$ is open, then os is U. since f is continuous, it's pre-iamge is also continuous, that is

$$f^{-1}(p^{-1}(U))$$

   is open. But now it's just a matter of going through the definition:

$$f^{-1}(p^1(U)) = (p \circ f)^{-1}(U) = id(U) = U$$

   thus U is open. To show that r is a quotient map if we find the appropraite f. and indeed,

$$f : A \to X \qquad f(a) = a$$

   is a right-inverse, thus r is a quotient map! Retraction makes will come back big-time when we talk about the Fundametnal group of the circle (definition 7.3.1) ▼

4. Let $\pi_1 : R \times \mathbb{R} \to \mathbb{R}$ be the projection othe first coordiante. Let $A \subseteq R \times R$ such that $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R})$ be obtained by restrictin π_1. Then q is a quotient map, but it is neither open or closed.

   Solution FIrst show not open or closed, then take from hw. ▼

[a] I said closed interval instead of closed set because I'm using the *basis* definition of continuity, combined with the fact that we can alternate between open and closed sets

With the idea of a quotient map, we can define the topology which is formed from it:

Definition 2.3.2: Quotient Topology

If X is a sapce and A is a *set* and if $p : X \to A$ is a surjective map, then there exits exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the ***quotient topology*** induced by p.

It should be quite natural to see how a topology would come from p. You would let $\mathcal{T}_A$ be be defined as consisting of subsets $U \subseteq A$ such that $p^{-1}(U)$ is open in X. [1].

Example:
Munkre gave this example. Take $p : \mathbb{R} \to \{a, b, c\}$ by

$$p(x) = \begin{cases} a & \text{if } x > 0 \\ b & \text{if } x < 0 \\ c & \text{if } x = 0 \end{cases}$$

Then you can check that p is a quotient map, and so induces the quotient topology on $\{a, b, c\}$. So, by pushing open sets through p, we can see that the inducd topology is:

$$\{\{a, b, c\}, \{a\}, \{b\}, \{a, b\}\}$$

]

Remember the equivalence intuition presented earlier? We can generalize this with the idea of partitioning our space, and letting the partitions collapse to a single point in the quotient topology (like he borders being part of the same partition)

Definition 2.3.3: Quotient Space

Let X be a topological space, and let X^* be a partition of X int odisjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point X to the elemetns o X^* containing it. In the quotient topology induced by p, the sapce X^* is called the ***quotient space*** of X.

[1] which kind of makes me think of Algebra, that you're kind of pushing the structure forward onto the quotient group

Personally, I like thinking about this in the categorical way. If there is a partition, then there is a surjective map from $X \to X*$. If we now map X into a topological space and $\to$ into a continuous function, what topology can I put on $X*$ in order for function to remain continuous? [2]. The answer is *yes*: Take the indiscrete topology on $X*$. Then $p^{-1}(X*) = X$ and $p^{-1}(\emptyset) = \emptyset$. The natural question then becomes, can we go finer? Even better, is there a finnet topology for which p is continuous? If we go too fine, the discrete topology then it's possible that p is no longer continuous. Fortunately we can answer the question: U is open in $X*$ if and only if $p^{-1}(U)$ is open in X. If we went any finer, then we would have a set that's *not* open in X! Thus, this is the finnest set. Notice how this matches the definition of a quotient topology! In fact, this is the *reason* for the definition of a quotient map when looking at math from a categorical perspective! The advantage of this is that we can relate our concept of "quotient" from other fields. For example, Thinking about quotient spaces is like thinking about G/N, the quotient group. One can show that $f : X^* \to \{\text{quotient sapce}\}$ is homeomorphic.

The overall takaway is that if we have a surjective map between X and $X*$, we will define the topology on $X*$ based on p.

Example:

1. Let X be the closed unit ball
$$\{x \times y \mid x^2 + y^2 \leq 1\}$$
in $\mathbb{R}^2$, and let X^* be the partition of X consisting of all the one-point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{x \times y \mid x^2 + y^2 = 1\}$. Then we can induce a topology on X^* which is homeomorphic with the subspace of $\mathbb{R}^3$ called the ***unit 2-sphere*** defined by
$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$
Let $\phi : X* \to S^2$,
$$\phi(x, y) = \{$$
it's quite a complicated map, I'll do it later.
2. Let X be the rectangle $[0, 1] \times [0, 1]$. Define the partition X^* to be the singletons if it's an interio point, the piar $[x \times 0, x \times 1]$ and $[0 \times y, 1 \times y]$ for the edges, and the 2 corner points form their own set
$$\{0 \times 0, 1 \times 0, 1 \times 1, 0 \times 1\}$$
This quotient space is homeomoprhic to the torus.

[2] or, categorically, I'm trying to establish a functor between the **Set**-Category and the **Top**-Category

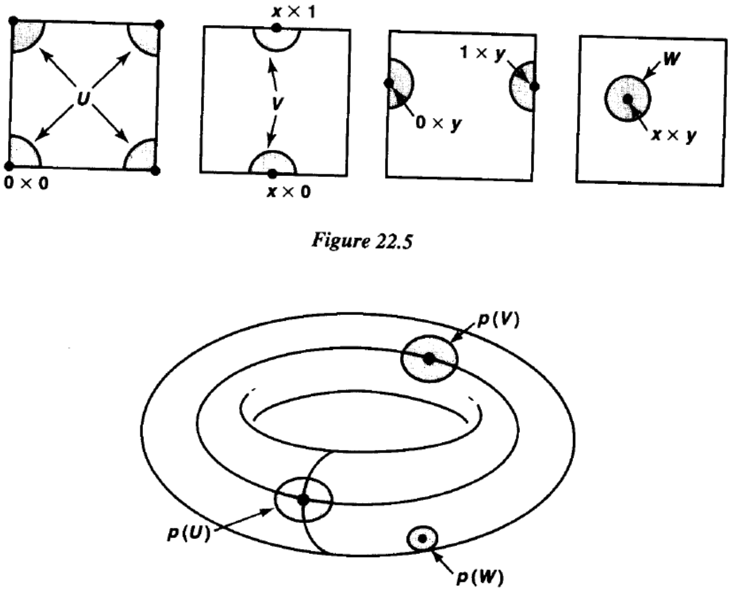

Figure 2.4: torusExampe

> **Note**: this partition construction relating to equivalent relations means we can define a quotient maps through equivalent relations!! Super useful and probably the way to think about it when construction quotient maps: Take f to be a surjective function and define the image through a relationhsip: $f(a) = f(b) \Leftrightarrow aRb$. Example:
>
> Let S be a set, and $f : \mathbb{R} \to S$ be a surjective map of sets such that $f(a) = f(b) \Leftrightarrow a - b \in \mathbb{Q}$. Prove that the induce qutoeint topology on S is the trivial topology.

When we studied product spces, we had a criteron for determining whether a map $f : Z \to \prod X_i$ *into* a product spcae was continuous. Its counterpart in the theory of quotient spaces is a criterion for determining when a map $f : X^* \to Z$ *out of* e a quotient sapec is continous.

Theorem 2.3.1: Universal Property Of Quotient Maps

Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map that is contant for each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to X$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is quotient map

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

This theorem is useful to find continuous maps on the quotient space.Here's a really easy intution. Imagine you have the interval I which can be mapped to S^1 by the quotient map, and let's say you have another space Z:

$$\begin{tikzcd} I \arrow[d, "p"'] \arrow[dr, "g"] & \\ S^1 \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

Then in order to find a continuous map form S^1 to Z, it suffices to find a continuous map form I to Z, and we can find a continuous map f as well! We'd know that $g = f \circ p$

Proof. □

2.3.2 relationship between previous concepts

We can ask ourselves what properties we've explored previously do quotient maps hold. It turns out that not a lot. Though a quotient map is almost homeomorphic, a lot can break due to lack of injectivity.

Subspaces do not behave well. if $p : X \to Y$ is a quotient space, then taking A as a subpsace of X and considering $p : A \to p(A)$, then p need not be a quotient map. The following theorem fixes this as well as possible:

Theorem 2.3.2: Subspaces And Quotient Maps

Let $p : X \to Y$ be a quotient map; let A be a subsapce of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting to p.

1. If A is either ope nor closed in X, then q i a quotien map.
2. if p is either an open map or closed map, then q is a quotient map.

Proof. Munkre p.140 □

Composition of maps still work well, which follow form the equation $p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U)$

The product of two quotient maps need not be quotient. Example at p.143

The Hausdorff condition does not behave well. X can be Hausdorff, but X^* need not be Hausdorff.

One more fact that I will mention: Every topological space can be obtained by quotienting a Hausdorff space!

link here

2.4 Weak Topology

We have given many constructions of topologies including the subspace, product, and quotient topology. All of these are special cases of one of two types of topologies: *initial topologies* and *final topologies*. These two types of topologies will generlize the results we've presented, and bring a new way of defining topologies that will give some minimal condition (for the initial topology) or maximal condition (for the final topology) for certain maps to be continuous:

Definition 2.4.1: Weak [initial] Topology

Let $F = \{f : X \to Y_\alpha\}_{\alpha \in A}$ be a collection of maps from a set A to topological spaces Y_α indexed by $\alpha \in A$. Then the coarsest topology on X which makes each f continuous is called the *weak topology*. In particular, a basis for the weak topology generated by F consists of $\{f_\alpha^{-1}(U)\}$ for all open sets $U \subseteq Y_\alpha$ for all $\alpha \in A$.

Note that it is not interesting to ask what is the finnest topology on X that makes all the maps continuous; that would simply be the discrete topology on X. Note too that since the weak topology $\mathcal{T}$ is the coarest topology on X that makes all the maps continuous, if $\mathcal{T}'$ is another topology on X that makes each map in F continuous, then $\mathcal{T}' \subseteq \mathcal{T}$. Note too that we showed in the definition how to actually create the weak topology, however for the skeptical reader it should be verified that the weak topology does indeed exist

Example:
Weak TopologyweakTopEx

1. Let $\{X_\alpha\}_{\alpha \in A}$ be some collection of topologies and $X = \prod_{\alpha \in A} X_\alpha$. Let $\Pi = \{\pi_\alpha : X \to X_\alpha\}$ be the collection of functions. Then the weak topology on X generated by F is *precisely* the product topology of X

2. Let X be any topological space and $A \subseteq X$ any subset. Let $\iota : A \to X$ be the inclusion map. Then the weak topology on A generated by ι is the *subspace* topology.

3. Let X be a normed space (over a topological field $\mathbb{F}$). Then X has a natural topology defined by the norm (the norm induces a metric, which induces the metric topology). let $X^* := \mathrm{Hom}(X, \mathbb{F})$ be the dual of X (the collection of all continuous linear maps from X to $\mathbb{F}$). Then we will define the weak topology on X to be the initial topology generated by

X^*. In this topology, a net $x_\alpha \to x$ if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$. We will sometimes write $x_\alpha \xrightarrow{w} x$ to emphaize weak convergence. If X is infinite dimensional, the weak topology of X is coarser than the norm topology on X.

(properties that are key to weak topologies can be found here)

One of the great advantages of of defining weak topologies is that we can make the collection of f_α and spaces X_α anything we want. This flexibility will the definition's of many interesting topologies on funcitonal spaces (ex. collection of smooth functions on compact spaces, collection of bounded continuous functions, or simply collection of bounded functions, and so forth). All such spaces are of key importance in funtional analysis. Weak topologies are also a special case of a much more general construction, which we'll explore at the end of this chatper.

Finally, there is also the dual of weak topologies where we take maps that map *into* X.

Definition 2.4.2: Final Topologies

Let Y be a set, $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces, and $F = \{f_\alpha : X_\alpha \to Y\}$ be a collection of maps into Y. Then the the finnest topology on Y that makes each f_α continuous is called the *final topology*. In particular, $U \subseteq X$ is open if and only if $f_\alpha(U)^{-1}$ is open for each $\alpha \in A$

Again, it should be verified that this is indeed a topology. While the weak (or initial) topology seeks for a domain with the coarsest topology, final topologies seek a codomain with the finnest topology. This is because the coarsest topology on the co-domain will always create continuous maps. Unlike the initial topology, the final topology is not given the name "strong" to contrast the name "weak" for initial topologies. There is in fact a topology known as the strong topolgoy on a space X: If $\|\cdot\|$ is a norm on X[3], the strong topology on X is the topology generated by all the balls $B_r(x)$ for all $x \in X$ and $r > 0$. The contrast between strong and weak topologies will make more sense when studying functional analysis, where there we would much rather a space have a strong topology, but it will sometimes be unfeasible (ex. the space of all smooth functions, or infinite dimensional vector spaces won't have a good norm defined on them), and so we will "weaken" the topology that will create topology that will not be induced by something as strong as a norm (usually, it the weak topology will be induced by a countable collection of seminorms, see Folland chapter 5 for more information).

Example: Final Topologies

Let X be an arbitrary topological space an $A \subseteq X$ a subspace. Let $\pi : X \to A$ be the projection map. Then the final topology on A is the *quotient topology*.

SSE

[3] See definition ref:HERE

Chapter 3

Topological Invariances and Metric spaces

We know begin the study of topological properties which are *invariant* between topologies. What invariance means is that if we have a homeomoprhism between two topologies, the properites will be present in both. We will make one exception in this chapter from this idea to deviate to metric spaces after we've studied Hausdorff spaces, since they present a very interesting example.

3.1 Connectedness

Connectedness is another property which is between points and the whole thing which talks about the nature of the geometry. This property gives us some idea of whether an open set can be "broken down further" to open sets. If a set is connected, then relative to that open set, you cannot break it down to the disjoint union of two (or more [1]) other open sets. An example of something that is clearly disconnected is two open balls $B_1(-10)$ and $B_1(10)$ in the euclidean topology. Similarly, each respective balls are connected in the euclidean topology.

It might be tempting to then say that it gives some idea of contiguity between points. However, that is not necessarily the case: In a topology which I'll outline later, we'll see how $B_r(x)$ will *always* be disconnected for any r and x with *any* underlying set with the exception of singletons. The topologies in which this holds is not even necessarily topologies which are intuitively disconnected. For example, in the euclidean topology, we'll show how balls are connected, but if you also add the closure of balls to your open sets, then you will actually get that every set is disconnected.

3.1.1 Basic Properties And Definitions

Recall clopen sets. We are bringing them back here to talk when sets *are* clopen, and what does it imply when they are:

[1] there's no real point in saying *or more* since then you can union all but one of the sets, and you're back to having 2 open sets

Lemma 3.1.1: separation of sets regarding clopen sets

Suppose $\mathcal{T}_X$ be a topological space such that $X = A \cup B$ and $A \cap B = \emptyset$, where A and B are also arbitrary sets. Then the following are Equivalent:

1. A is clopen
2. B is clopen
3. A, B are closed
4. A, B are open

Proof. □

Recall earlier when we introduced clopen sets, with the example of $\emptyset$ and X as the "trivial" clopen sets, and then we introduced what I called the separation lemma. There is something intuition in saying that space X is completely disjoint (meaning $\overline{A} \cap B = A \cap \overline{B} = \emptyset$) if X is as described in that lemma. This will turn out to be the basis for our thinking about connectedness.

Definition 3.1.1: Connected

Let X be a topology, and take the set $A \subseteq X$. A is **connected** if and only if the only clopen subsets of A are $\emptyset$ and A.

I like to think of open sets of an idea of "mandatory" overlap between all open sets in A – that is, the intuition for connectedness should really come from the topology rather than the underlying set [2].

Example:

1. The interval $I = (0, 1)$ is connected in the euclidean topology. Let's say, for the sake of contradiction, that $I = A \cup B$, $A \cap B = \emptyset$, A, B are open (equivalent to them being clopen by 3.1.1). Then A and B must be the union of open intervals. One of them must contain the points (0.., so let that be in A. Since A is the union of open intervals, there is a sub-interval in A such that $(0, \epsilon) \in A$, and some points $(\epsilon, ..$ are not in A (since B cannot be non-empty). Let's say $(\epsilon, \epsilon + \delta) \in B$. However, this would imply that $\epsilon \notin A$ and $\epsilon \notin B$. There are also no singleton intervals. Thus, $I \neq A \cup B$, $A \cap B \neq \emptyset$, A, B open, showing that I is connected in the euclidean topology. A more formal proof of this fact is presented in theorem 3.1.2.
2. $X = (0, 1) \cup (2, 3) \subset \mathbb{R}$. Then A is clopen in X, and so disconnected.
3. The trivial topology is connected.
4. the discrete topology is totally disconnected (only singletons are connected)
5. Let X be any infinite set with the finite-compliment topology. Then if $A \subseteq X$ is infinite, it is connected (try taking $A \cup B = A, A \cap B = \emptyset$, A, B open)

[2] naturally, sometimes the underlying set (like $\mathbb{R}$) has properties that are a great influence to a topology (like euclidean topology), but then we chose a topology which has our properties, so again it comes down to topology

6. A space is said to be *totally* disconnected if the only connected components are the open sets. Is the discrete topology totally disconnected? Are there others?

7. Is $\mathbb{R}_l$ connected? What sets are conected?

In this way, if we have a disconnected set X, so take $X = A \cup B, A \cap B = \emptyset$, A, B clopen. Then A and B are sort of independent from each other. Notice that $\forall c \subseteq X$, c is open if and only if $c \cap A$ is open and $c \cap B$ is open. So in a sense ,the openness of those sets don't have anything in common – they don't affect each other. Similarly for closed sets too.

Lemma 3.1.2: Connectedness and independence

See previosu paragraph

***Proof* :**
exercice

Connectedness might seem at first intuitive, but it is actually very much based on topology, and we *must* cover weird topological examples to get a better feel of connectivity:

Example: Super Clopen
Take

$$\mathcal{T}_{\text{super-clopen}} = \langle \{ \dots, [-3, 2], (2, 1), [1, 0], (0, 1)[1, 2], (2, 3), \dots \} \rangle$$

then the entire real line is covered in open sets. Now consider $[1, 2]$. This set is open. by definition. The compliment of it $(-\infty, 1) \cup (2, \infty)$ is the union of all other elements, so it is *also* open, and therefore $[1, 2]$ is also closed. Therefore it is clopen. In fact, every open set in this topology is clopen. Therefore, the union of any open sets will produce a set which has more than the trivial clopen subset in the subspace topology, and more generally, any set $A \subseteq X$ would have more than the trivial clopen subsets unless A is element of the basis of $\mathcal{T}_{\text{super-clopen}}$ topology. So this topology has a lot of connected components, but anything bigger than that is not connected.

There is an even more extreme example of this, where only single point sets are connected; such a topology is called *totally disconnected.* The discrete topolgoy is an obvious one, but much subtler ones exist! Will will show that a topology generated by intervals (a, b) are connected. BUt, $\mathbb{R}_l$ (the lower-limit topology, generated by $[0, 1)$ intervals) is totally disconnected! You can see that $(\infty, 0) \cup [0, \infty)$ is a disconnection, showing that $\mathbb{R}_l$ is disconneced, and you can then generalize this for arbitrary subset of $\mathbb{R}_l$! This shows that any geometric intuition about connectedness is depdent on the topology in which you aquired that intuition!

In the example above, you might think that this super disconnectedness occured because of a weirdly desinged topology. This need not be the case:

Example: Making the Euclidean Topology a Bit Finner
Take $\mathbb{R}$ as the underlying space. Take this as the basis for a topology:

$$\mathcal{T}_{\text{extremely disconnected}}\langle\{y \in X \mid d(x,y) < \epsilon, \forall\epsilon\}\rangle \cup \langle\{y \in X \mid d(x,y) \leq \epsilon, \forall\epsilon\}\rangle$$

That is, the set of all epsilon balls around every point, and the set of epsilon balls *with* their closure in the euclidean topology. This set is strictly finer than the euclidean topology, since every basis element of $\mathcal{T}_{eu}$, is in $\mathcal{T}_{\text{super disconnected}}$, but not the converse. Let's not take an arbitrary interval $I = (a,b)$, and let $r = b - a$. Intervals are connected in $\mathbb{R}$. However, consider

$$S = \left(a + \frac{r}{10}, b - \frac{r}{10}\right) \subseteq I$$

Take the compliment of it in the subspace topology

$$S^c = \left(a + \frac{r}{10}, b - \frac{r}{10}\right)^c = \left(a, a + \frac{r}{10}\right] \cup \left[b - \frac{r}{10}, b\right)$$

which is open in I since $I \cap \left[a - 1, a + \frac{r}{10}\right] \cup \left[b - \frac{r}{10}, b + 1\right] = S^c$, and therefore S is clopen, and so is a clopen subset of I, and so I is *not* connected!! You can do even weirder stuff too if you tried: you can split any ball in half. That could be a good exercice.

Fun fact! This space actually has a name!! spaces where the closure of open sets is open, it's called *extremely disconnected*!

We can even find examples of spaces with topologies we are really familiar with, but the underlying set forces the intersciton between the topology and the set to be disconnected:

Example: Ecudliean Topology on $\mathbb{Q}$
(from tutorial, put some place appropriate). Note that $(\mathbb{Q}, \mathcal{T}_{Eu})$ is totally disconnected (only singletons are connected) If we take any $A \subseteq \mathbb{Q}$, $|A| \geq 2$. Let's say for some two points $x < y$. Then we can find some $\epsilon \in \mathbb{R}$ such that $x < \epsilon < y$. Then $((-\infty, \epsilon) \cap \mathbb{Q}) \cup (\epsilon, \infty)$. Then for any A now, it will be the union of two open sets, meaning that it is there must be more than 1 trivial clopen set.

In all these cases, there was a very small gap between the disconnected spaces, which might lead you to believe all connected spaces have some idea os closeness in common. THIs nead not be the case:

Example: Connected space with large gaps
Let

$$I^2 = I \amalg I, A \subseteq I^2, \mathcal{T}_{I^2} = \langle\{\{(a,0),(0,a)\} \,|a \in \forall(c,d) \subseteq I\}\rangle$$

that is, I^2 is all elements of the form $(a,0)$ and $(0,b)$ for all $a, b \in I$, and the topology on it are all the intervals who "mirror" each other on both sides. Note that this topology strictly coarser than the subsapce topology on I^2 of the product topology. The point of this topology is to "glue" two open sets together so that what used to be two clopen sets in the relative subsapce will not be the only clopen set, making it connected.

In particular:

$$\{((a,b),0),(0,(a,b))\} \qquad a \neq 0$$

is connected. I'll prove this later, got algebra homework.

Note also that some of the spaces mentioned before are Hausdorff, while others are not. So "oddities" in connectedness will not be fixed if you choose a hausdorff space.

Also, here is an exerice I thought to be cool:

Example: not homeomorphic
Show that $(0,1)$, $(0,1]$ and $[0,1]$ are each not homeomorphic to each other

Solution The solution is ingeinous. Take two of them. Assume one is homeomorphic to the other. Then the iamge of a connected set is open. Rmove a single point from the co-domain (make the co-domain one of the closed ones). Then ▼

Suppose that there exists imbeddings $f : X \to Y$ and imbedding $g : Y \to X$. Give an example where that doesn't imply they are homeomorphic

Solution here ▼

Show $\mathbb{R}$ and $\mathbb{R}$ are not homeomorphic if $n > 1$

Solution Same principle as the first one! ▼

This last one shows how $\mathbb{R}^n$ and $\mathbb{R}$ are *not* topologically similar under the euclidean topology!! Better intuition of this will come with the Fundamental group (chapter 7).

3.1.2 Properties of Connected Spaces

Lemma 3.1.3: Subspace Topology And Disconectedness

If $Y \subseteq X$, and $Y = A \cup B$, $A \cap B = \emptyset$, A, B are non-empty then if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, then Y is disconnected. The closure of $\overline{A}$ and $\overline{A}$ are taken in X.

The purpose of this lemma is that it never refers to the subspace topology on Y.

***Proof* :**
Consider $\overline{A} \cap Y$. We know that $A \subseteq \overline{A} \cap Y$, and so

$$B \cap (\overline{A} \cap Y) \subseteq B \cap \overline{A} = \emptyset$$

Where the second subset inclusion because getting rid of Y will naturally make that true, and the last equal is by assumption. Therefore, $A = \overline{A} \cap Y$ is closed. Likewise for B. And so we're done.

One last lemma before we learn how to construct connected spaces. This one should be intuitive

Lemma 3.1.4: Connected Sets Within Disconnected Spaces

Let $X = A \cup B$, $A \cap B = \emptyset$ A, B clopen. Let $Y \subseteq X$. Assume Y is connected. Then either

$$Y \subseteq A \qquad \text{or} \qquad Y \subseteq B$$

Proof. if A is clopen, then $Y \cap A$ is clopen in Y. But Y is also connected, so $Y \cap A = \emptyset$ or $Y \cap A = Y$. If $Y \cap A = \emptyset$ then $Y \subseteq B$. If $Y \cap A = Y$ then $Y \subseteq A$. □

Corollary 3.1.1: Singleton's Are Always Connected

Given any topology and any set, singletons are connected

Proposition 3.1.1: Subspace Of Connected Sets

Let X be connected. Then $A \subseteq X$ need not be connected

Solution Take $\mathbb{R}$ with the euclidean topology and $[0, 1] \cup [4, 5] \subseteq \mathbb{R}$. Then $[0, 1]$ $\cup[4, 5]$ are both clopen, meaning there there is more than the trivial clopen set. ▼

However, there is one condition in which a subsapce is guaranteed to be a connected space:

Proposition 3.1.2: Connected Subspace Of Connceted Space

LEt A be a connected subspace of X. If $A \subseteq Bsse\overline{A}$, then B is also connected. In other words, if B is A and some limit points, it's such that's still connected.

Solution p.150 ▼

Proposition 3.1.3: Finite Product Of Connected Sets

Finite product of connected sets are connected

***Proof* :**
Let X, Y be connected spaces, and take $X \times Y$. Let's say there exists two open subsets of $A, B \subseteq X \times Y$ such that $A \cup B = X \times Y$, and $X \cap B = \emptyset$. Then

$$(A_1, A_2) \cup (B_1, B_2) = (A_1 \cup B_1, A_2 \cup B_2) = (X, Y)$$

implying that $A_1 \cup B_1 = X$ and $A_1 \cap B_1 = \emptyset$ for open set $A_1, B_2 \subseteq X$. But then that would imply X has a seperation, implying its disconnected – a contradiction. Similarly for Y. Thus, $X \times Y$ is connected.

Proposition 3.1.4: Arbitrary product of Connected Spaces

Arbitrary products of connected spaces need not be connected

Solution p.151 example 6. ▼

Proposition 3.1.5: quotient maps and connectedness

LEt $p : X \to y$ be a quotient map. If eahc set $p^{-1}(\{y\})$ is connceted, and Y is connected, then X is connected

Proof **:**
homework.

3.1.3 Constructing connected spaces

Proposition 3.1.6: points in common between connected sets

Let $\{A_i\}_{i\in I}$ be subsets of X which are connected. Assume $p \in \bigcap_{i\in I} A_i$. Then

$$Y = \bigcup_{i\in I} A_i$$

is connected

Solution ▼

For example $A = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0\right\}$ is connected. You can use that previous Lemma

Proposition 3.1.7: Adding Limit Points Keeps You Connected

Suppose $A, B \subseteq X$, A is connected, and

$$A \subset B \subset \overline{A}$$

then B is connected

Solution Tuesday Oct 5th lecture ▼

This next proposition is essentially what the Intermediate Value Theroem is in general topology (theorem 3.1.3). We will get back to this when we get to connectedness of the real line.

Proposition 3.1.8: Connectedness And Image Of Continuous Function

The image under a continuous map of a connected space is connected, that is

$$f : X \to Y$$

continuous, and X connected, then $f(X)$ connected.

Solution This one is not too bad. Start with some $A \subseteq Y$ which is assumed to be clopen. You'll use properties of continuity to and connectedness to eventually reach that $A = Y$ or $A = \emptyset$. ▼

Theorem 3.1.1: Finite Product Of Connected Spaces

A finite product of connected spaces is connected

intuition

Proof. Thursday lecture, beginning □

3.1.4 Connectedness for The Real lines

Let $X = \mathbb{R}$ with the euclidean topology. We'll now study some special properties of the connectedness on this space

Theorem 3.1.2: $[0, 1]$ Is Connected

The closed interval $I = [0, 1]$ is connected

Proof. This is Heine-Borel but in a different formulation.

As usual with connectedness proofs, we need to show there is only the trivial clopen sets. let $A \subseteq I$ be a clopen set by possibly replacing $A \leftrightarrow I \setminus A$. Without loss of generality, let $0 \in A$. Define

$$m := \sup(b \in I \mid [0, b] \subseteq A)$$

that is, m is the largest number such that $[0, m] \subseteq A$. We'll show that $m = 1$, showing that $A = I$, and thus I only have the trivial clopen sets, and so is connected

First, let's show that $m \in A$ (since the supremum doesn't *have* to be inside A). Assume then that $m \in I \setminus A$. Then $\exists \epsilon > 0$ such that $(m - \epsilon, m + \epsilon) \cap I \subseteq I \setminus A$. THe $\cap I$ is because of the subspace topology (recall that it's open in I if U is an open set, then $U \cap I$ is open in I). This would imply that $\forall b \in (m - \epsilon, m + \epsilon)$, $[0, b] \nsubseteq A$. However, that's a contradiction by the way we defined m, since that would imply that $b \leq m - \epsilon$.

Next, we need to show that $m = 1$. Let's assume $m < 1$. Since A is open, then there exists a $\epsilon > 0$ such that $[m, m + \epsilon) \subseteq A$, which means that $[0, m + \frac{\epsilon}{2}] = [0, m] \cup [m, m + \frac{\epsilon}{2}] \subseteq A$, again a conradiction since m must be the largest such number.

Thus, $A = [0, 1]$ $\square$

Corollary 3.1.2: any interval is connected

$[r_1, r_2]$ is connected, and so is (r_1, r_2), and so $[r_1, r_2)$ and $(r_1, r_2]$ is connected

Note that $\mathbb{R}$ is connected since $\bigcup_{n\in\mathbb{N}}(-n, n) = \mathbb{R}$. We can also show it by

$$f(x) = \frac{1}{1-x} - \frac{1}{x}$$

since it is a homeomorphism between $(0, 1)$ and $\mathbb{R}$.

With that, we can also show that $\mathbb{R}^n$ is connected.

Finally, there's also the generalization of the famous 1st year analysis theorem:

Theorem 3.1.3: Intermediate Value Theorem (Ivt)

Suppose X is connected, and $f : X \to [0, 1]$ is continuous. then $f(X)$ is an interval. That is, for all $x \in [0, 1]$ there exists a $c \in X$ such that $f(c) = x$.

Proof. We've already shown that the image of connected sets are connected, and that what is connected in subspace topologies of $\mathbb{R}^n$ are intervals, or products of, and so we're done. $\square$

Example:
Here are some examples that gave tricks I think are important to know:

1. Let $f : S^1 \to \mathbb{R}$ be continuous. Show that there exists a point $x, -x \in S^1$ such that $f(x) = f(-x)$

   Proof **:**
   Let's assume there does not exist a $x, -x \in S^1$ such that $f(x) = f(-x)$, that is, $f(x) \neq f(-x)$ for all $x \in \S^1$. Let A be the upper hemisphere $\cup(1, 0)$, and let B be the lower hemisphere $\cup(-1, 0)$. But, for every $x \in A$, $\exists y \in B$ such that $y = -x$. Then since S^1 is a subspace of a Hausdorff space, it is also Hausdorff, and so for every $x \in A$, we can find two open sets $U_{f(x)}, V_{f(-x)}$ such that $U_{f(x)} \cap V_{f(-x)} = \emptyset$. We can do this for every point, forming

   $$U = \bigcup_{a\in A} U_f(a) \qquad V = \bigcup_{b\in B} V_{f(b)}$$

   (Check that $U \cap V = \emptyset$, then take the pre-image, which would cover S^1, but still be disconnected)

2. Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) =$ x. The point is called a *fixed point* of f. What happens if $X = [0, 1)$ or $(0, 1)$?

Proof **:**
Let's say $x \neq f(x)$ for all $x \in [0,1]$. The trick is to define this function: $g : [0,1] \to [-1,1]\,, g(x) = x - f(x)$. Then since $[0,1]$ is connected, $g([0,1])$ is connected. Since $0 \notin g([0,1])$, then it cannot be that $g([0,1]) = [-a,0) \cup (0,b)$, since that would be a discontinuity,

so it must either be in the lower or upper region. Without loss of generality, we can say that $g([0,1]) \subseteq (0,1]$. Here is then were we would use the fact that the limit point exists to pull an interesting conclusion!

3.1.5 Path Connectedness

In weird topologies, connectedness can produce results that we don't usually think as "connected" when we think in a Euclidean way (like example (# here once I get referencing working)). We still want to preserve our notion of connectedness from euclidean space in some manner. If we stick to Euclidean intuitions, we can think something is connected if we never have to "leave" it to get to another point. This is actually how a general counter-example to this intuition for connectedness in general was constructed: by choosing a topology which in the euclidean intuition is disconnected, but the sets are open.

This particular concept of openness is very useful and usually what most think when they say connected. Thus, we will make our intuition of connected in a euclidean a name:

Definition 3.1.2: Path Connectedness

Let X be a topological space. A **path** in X is a continuous map $f : [0,1] \to X$. If every two points $a, b \in X$ can be connected by a path, we say that X is **Path-Connected**

Note: Munkres lets f be from any closed interval to X, which is functionally the same, but makes book-keeping a lot harder

Example:

1. Given $\mathbb{R}^n$ with the euclidean topology, then for every $x_0 \in \mathbb{R}^n$ $B(x_0)$ is a path-connected set.
2. The unit circle S^1 is path connected
3. Here's a more challenging one if you konw your Algebra. The space $GL_n(\mathbb{C})$ is path connected. (hint: link to the idnetity matrix. If you're still stuck: here)
   (a) If you have cracked the other one, then you can try to show that $GL_n(\mathbb{R})$ is *not* path connected (which will come down to the invertibility matricies in $\mathbb{R}$ are split between $+$ and $-$)

Lemma 3.1.5: Path-Connected $\Rightarrow$ Connected

If X is path-connected, then it's connected

Proof. We'll do the contra-positive: if it's not connected, it's not path connceted. Suppose $X = A \cup B$, $A \cap B = \emptyset$. Let $f : [0,1] \to X$. Then since $[0,1]$ is connected, $f([0,1])$ is connected, so it must be contained in either A or B. Thus, there is no path from $a \in A$ to $b \in B$, thus it's not path connected.

There is also a categorical proof the professor presented. $\square$

We have seen path-connectedness is stronger than connectedness becasue of this proof. The converse does not necessairly hold:

Example: : Topologist Sin-curve
The topologists sin curve union $\{0\} \times (-1,1)$ is the typical example of connected, but not path-connected set.

Let $S = \{(0,0)\} \cup \{(x, \sin(1/x)) | 0 < x < 1\}$ and $f = (f_1, f_2) : [0,1] \to S$ is path-connected. We'll show that if $f(0) = (0,0)$, then $f(t) = (0,0)$ for all t.

For the sake of contradiction, assume that $f(t)$ is not always $(0,0)$. For convinence, remove any constant initial part of the interval, so that we get $0 = \sup\{t | f([0,t]) = \{(0,0)\}\}$ (so that we consider the equiavlent function which immediately leaves 0). By theorem 2.2.2, we know that if f is continuous, then so are it's component functions, meaning f_1 and f_2 is continuous. Since f_2 is continuous, then $\forall \epsilon, \exists \delta$, s.t. $|t - 0| < \delta \Rightarrow |f_2(t) - 0| = |f_2(t)| < \epsilon$. Since the supremum of f_2 is 1, let $\epsilon = 1$ so that $|f_2(t)| < 1$ for all $t < \delta$. Since $\delta > 0$, we can pick a t_0 such that $0 < t_0 < \delta$, and where $f_1(t) > 0$ (since this function is oscillating in a undestandable manner for $t > 0$). Since f_1 is continuous *and* it's domain is an interval, then by the IVT (3.1.3) the image of an interval is an interval. Thus, $f_1([0,t_0]) = [0, f_1(t_0)]$. Since $f_2(t) = \sin(1/f_1(t))$ for all t with $f_1(t) \neq 0$ and $\sin(1/x)$ aps $(0,\epsilon)$ onto $[-1,1]$ for all $\epsilon > 0$, it follows that $[-1,1]$ is in the image of f_2 restricted to $[0,t_0]$ – a contradiction to the assumption that $t_0 < \delta$. Thus, there cannot exist a a non-constant path from $(0,0)$.

Keep the topologist sin-curve when thinking about path-connectedness! For example, note that if we just take $(-1,1)$, then it is path connected. This implies that it's not true that if A is path-connected, then $\overline{A}$ is path-connected, sinc the closure of the interval here will produce a non path-connected space. This is interesting, for it shows that closure might not play nicely with path-connectedness since we might use a nasty infinity when approaching a point! [a]

At first, this might seem an unintuitive of a proof but the "finiteness" of this will come from the fact that $[-1,1]$ is something called *compact* in the Euclidean topology. This is important, for that "finiteness" of paths because of the compactness of closed interval in $\mathbb{R}$ will have great consequences.

[a] and since we're limited to $[0,1]$ in the domain, a compact set, messing with infinites like this can break things!

To make the converse hold, we need that a space is connected and *locally* path-connected. This idea that if it holds *locally* plus some extra condition gives us a global concept harkens back to what I said at the begining of section 1.1, where I said that local properties can play a role in prove larger things. We will get back to local-path connectedness shortly (section 3.1.6)

Proposition 3.1.9: Product Of Path-Connected Spaces Is Path-Connected

If $\{X_i\}_{i\in I}$, are path connected spaces, then so is $\prod_{i\in I} X_i$

Proof **:**
Naturally, we give $\prod_{i\in I} X_i$ the product topology. This proof becomes very easy if we use theorem 2.2.2, where f is continuous to $\prod_{i\in I} X_i$ if and only if every component map is also continuous.

Proposition 3.1.10: image of path-connected spaces is path connected

If X is path connected, then $f(X)$ is path connected

Proof **:**
This is quite easy if you recall that the composition of continuous function is continuous (proposition 1.2.2).

3.1.6 Local Connectedness

As we mentioned in 3.1.2, if a set is connected, open sets on it "don't interact with others". We make this concept more rigorous, by defining a way of looking at a space by seperating it into it's connected components

Definition 3.1.3: Connected Components

Given X, define an equivalence relation on X by settting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called teh ***components*** (or the connected components) of X

Verifying it's an equivalence class is pretty simple

Theorem 3.1.4: Connected Components Of A Space

The components of X are connected disjoint subspace sof X whose union is X, such that each nonempty connected subspace of X interescts only one of them

Proof **:**
simple proof – exercise

Definition 3.1.4: Path-Connected Components

Given X, define an equivalence relation on X by settting $x \sim y$ if there is a path-connected subspace of X containing both x and y. The equivalence classes are called teh ***path-components*** of X

Verifying this equivalence relation requries a bit more work, but is worth going over.

We have the equivalent theorem for path-components:

Theorem 3.1.5: Path-Connected Components Of A Space

The path-components of X are connected disjoint subspace sof X whose union is X, such that each nonempty path-connected subspace of X interescts only one of them

Proof **:**
very similar to 3.1.4

Example:

1. Take $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Then since $\mathbb{Q}$ is totally disconnected, every component and path-component are just the singletons
2. Let $\overline{S}$ be the topologist sine-curve. Then $\overline{S}$ has 1 connected component, and 2 path-components: One is the curve S, while the other is eh vertical interval $V = 0 \times [-1, 1]$.

   Super interestingly, if you delete all poitns of V having rational second coordinate, then you get a space that has only one connected component, but *uncountably many* path-components!

Definition 3.1.5: Locally Connected

A space X is said to be ***locally connected at x*** if for every neighborhood U of x, there is a connected neigbhorhod V of x contained in U. If X is locally connected at each of its points, it is said simply to be *locally connected*. Similarly, a space X is said to be ***locally path-connected at x*** if for every neighorhood U of x, there is a path-connected neigborhood V of x contained in U. If X is locally path connected at each of its points, then it is siad to be *locally path connectd*

Example:
Every interaval in $\mathbb{R}$ is connected and path-connected, and hence locally connected/path-connacted. The subspace $[0, 1) \cup (0, 1]$ is not connected, but locally connected. The topologists's sine curve is connected but not locally connected. The rationals $\mathbb{Q}$ are neither connected or locally connected

Theorem 3.1.6: Locally Connected If And Only If Open Components

A space X is locally connected if and only if for eveyr open set U of X, each component of U is open in X

Proof **:**
simply carry out the definition (p.161 Munkre)

similarly

Theorem 3.1.7: Locally Path-Connected If And Only If Open Components

A space X is locally path-connected if and only if for eveyr open set U of X, each path-component of U is open in X

3.2 Compactness

Compactness really captures the idea that you only need a finite amount of open sets to represent a set. THis is a super-useful property, especially when you're trying to avoid oddities that come up with weirdness of uncountable spaces (ex. in a non-compact space, you can have a uncountable discrete subspace of X, which can be difficult to work with). My general way of thinking about compactness is that it gets rid of infinities, which can spoil some things.

3.2.1 Basic Properties And Definitions

Definition 3.2.1: Cover

A collection $\mathcal{A}$ of subess of a space X is said to be a ***cover***, or to be a ***covering*** of X, if the union of the elements of $\mathcal{A}$ is equal to X. it is called an ***open coering*** of X if its elements are open subset of X.

Definition 3.2.2: Compactness

A space X is said to be ***comapct*** if every open covering $\mathcal{A}$ of X contains a finite subcolletion that also a cover

Note: Notice that we're talking about the *space*, not a subset of X. We'll get to that soon.

Example:

1. The trivial topology is compact
2. finite spaces are compact

3. the real line is *not* compact ($\mathbb{R} = \bigcup_{n\in\mathbb{N}}(-n, n)$, so no finite subset of that union can cover $\mathbb{R}$)

4. $(0, 1]$ is *not* compact.

5. When we worked with connectedness, we saw that adding open sets to a topology might break connectedness. Same in compactness. Let $X = \mathbb{R}$. If $\mathcal{T} = \{\emptyset, X\}$, and $\mathcal{T}' = \mathcal{P}(X)$, then X is compact in $\mathcal{T}$ (since $\mathcal{T}$ is finite), but $\mathbb{R}$ is *not* compact in $\mathcal{T}'$ (since the set $\{x\}_{x\in\mathbb{R}}$ doesn't have a finite-subvcover!) This shows how Compactness is a very topological property! The topology on a set can change whether a set is compact or not. Note that the converse is true: if $\mathcal{T}'$ is a finer topology than $\mathcal{T}$ and X is comapct in $\mathcal{T}'$, then X is compact in $\mathcal{T}$. Let A be a cover for X from $\mathcal{T}$. Then A is also a cover for A in $\mathcal{T}'$. Then A admits a finite sub-cover. Then X is also compact in $\mathcal{T}$.

6. here is an example of a space which has an "infinity" which we'll be able to take advantage to show it's not compact. Take $\mathbb{R}_K$, the K-topology. Recall that this topology is *finer* than the euclidean topology since it has $\mathcal{T}_\mathbb{R}$ as well as all sets of the form $(a, b) - K$ where $K = \{1/n | n \in \mathbb{N}\}$. Take the set $[0, 1]$. We'll show it has no finite subcover. Take:

$$U = [0, 1) - K \qquad n_1 = (\frac{1}{2}, 1], U_n = (\frac{1}{n+1}, \frac{n-1}{)}, n \geq 2$$

   the key here is that U is open!! Take $\mathcal{C} = U \cup \{U_n\}_{n\in\mathbb{N}}$. Then it should be clear that *no* proper subcover (including finite ones)! This should make more intuitive the idea that compactness is about there being no weird infinites, cause if there are we can produce this.

(bonus) You can show that $\mathbb{R}_K$ is connected (Munkre's hint was that $(-\infty, 0), (0\infty)$ inheret their usual properties from $\mathbb{R}$), and show it's *not* path connected

<++>

Definition 3.2.3: Sub-Cover

If $Y \subseteq X$, S is a collection of subsets of X, we say S ***covers*** Y if $\bigcup_{B\in S} B \supseteq Y$

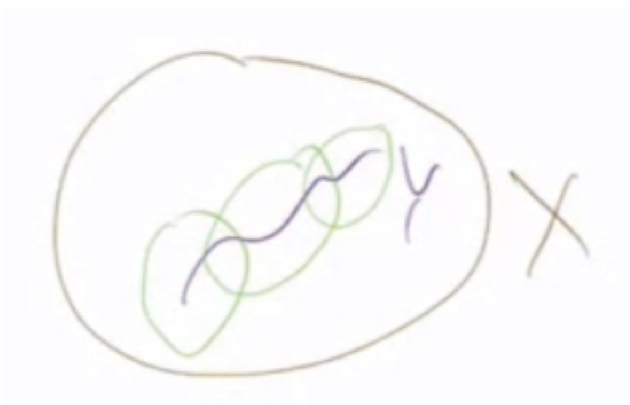

Figure 3.1: subcover

Lemma 3.2.1: Compactness With Cub-Cover

Let $Y \subseteq X$. THen Y is compact if and only if ***every*** S which open covers Y has a finite subset wihch also open covers Y.

Proof. proof both ways:

($\Rightarrow$) We know that $Y = \bigcup_{B \in S}(B \cap Y)$. Therefore, there exists $S' \subseteq S$ such that $Y = \bigcup_{B \in S'}(B \cap Y)$, which means that $Y \subseteq \bigcup_{B \in S'} B$

($\Leftarrow$) Let T be an open cover of Y, so $T = \{S_i \cap Y, i \in I\}$ (I indexing set) for S_i an open set in X. And so

$$Y = \bigcup_{i \in I} T_i \subset \bigcup_{i \in I} S_i$$

And so $S = \{S_i, i \in I\}$ covers Y, and so by assumption there exists a finite $J \subseteq I$ such that $S' = \{S_j \mid j \in J\}$ covers Y. so there's a T' which also covers Y.

□

3.2.2 Properties of Compact Spaces

Here are a couple of properties of compact spaces:

Theorem 3.2.1: closed subsets of compact spaces are compact

Let $Y \subseteq X$ be closed, and X be compact. Thne Y is compact

Proof. Let S be a collection of open sets in X which cover Y. So let $T = S \cup \{X \setminus Y\}$ be that cover union the compliment of Y. So T is also an open cover in X, so we can use the fact that X is compact. We know there exists a finite $T \subseteq T$ which is an open cover of X. Finally, now take $T' \cap S$, which is a finite subset of S which covers Y, and so is compact. □

This is actually incredibly powerful! If you are closed and a subspace of a compact space, you are *comapct*! This gives you an idea of the nature of compactness and it's relation to closed sets. So in a sense, we found another way of thinking about closed sets when our space is compact.

The converse is not necessarily true:

Example: compact but not closed

1. Let $X = \{1, 2, 3\}$ with the trivial topology. Then $Y = \{1\}$ is commpact, but *not* closed. You can also take $\mathcal{T} = \{\emptyset, \{1\}, X\}$, in which case, $\{1\}$ is compact, *open not closed.*
2. A little less trivial of an example. Let X be an infinite set with the finite-complement topology. Then every subset of X is compact (worth checking). [a]

[a] The cool thing about this example is that it's T_1 (recall what I said singletons are closed in the weaker T_1 condition?), so it's an example which is almost Hausdroff, which will be the proper condition for the converse.

We will talk about a condition for the converse when we add Hausdorffness.

Next, we will prove an incredibly powerfull theorem. This theorem will essentially be the proof of the Extreme Value Theorem (EVT) once we think of compactness in $\mathbb{R}^n$ (theorem 3.2.8)

Theorem 3.2.2: Continuity and Compactness

The image of a compact space under a continuous map is compact. That is, if $f : X \to Y$ is continuous, and $U \subseteq X$ is comapct, then $f(U)$ is compact.

Proof. Easy proof □

The converse is not necessarily ture:

Example: V compact, $f^{-1}(V)$ not Compact

Let

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 0$$

Then 0 is compact, however $f^{-1}(0) = \mathbb{R}$, which we shown earlier is not compact.

Theorem 3.2.3: Tychonoff's Theorem on finite spaces

If X, and Y are compact, then so is $X \times Y$.

Proof. This will also involve understanding the tube lemma. Worth going over for that lemma at least □

Example:

Earlier, we said that $\pi X \times Y \to X$ is *not* a closed map. However, if Y is compact, it *is* a closed map

Solution ▼

This final theorem is useful for characterising compactness using *closed* sets instead of open sets.

Definition 3.2.4: Finite Intersection Property

Let X be a set and $\mathcal{C}$ be a colleciton of subset of X. We say $\mathcal{C}$ has the ***finite intersection peroperty*** (FIP) if for every finite subcolleciton $\{C_1, ..., C_n\} \subseteq \mathcal{C}$,

$$\bigcap_i = 1^n C_i \neq \emptyset$$

Example:

1. if we pick some $x_0 \in X$, then we can let $\mathcal{C}$ be the colleciton of *all* subsets containing x_0, and so any finite subcolelction would also contain x_0.
2. Another would be $\mathcal{C} = \{(n, \infty) \mid n \in \mathbb{Z}\}$ since any finite subcollection of this set is non-empty because they're all rays.

Proposition 3.2.1: FIP And Compactness

Let X be a topological space. Then X is compact if and only if every collection $\mathcal{C}$ of *closed* subsets of X which has FIP satisfies $\bigcap_{c \in \mathcal{C}} c \neq \emptyset$

Solution p.169-70 ▼

3.2.3 Compact Hausdorff spaces

Compact Hausdorff spaces are used quite extensively, and so deserve some elaboration on their properties:

Theorem 3.2.4: compact subspaces of Hausdroff Spaces are closed

Suppose X is a Hausdorff space, and $Y \subseteq X$ is compact. Then Y is closed as a subset of X.

Proof **:**
Let $x_0 \in X \setminus Y$. We will find an open set containing x_0 that's disjoint from Y. Since X is Hausdorff, then $\forall y \in Y$ $\exists$ open U_y, V_y in X such that

1. $y \in Y$, $x_0 \in V_y$
2. $U_y \cap V_y = \emptyset$

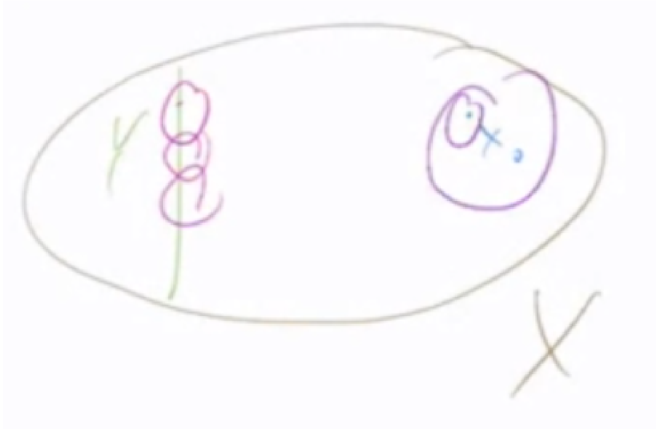

Figure 3.2: subspaceYcompactHausdorff

We can already see what's going on in Y: we're getting an open cover! That is $S = \{U_y \mid y \in Y\}$ covers Y. Therefore, $\exists S' \subseteq S$ such that S' is finite and S' still covers Y. Since $U_y \cap V_y = \emptyset$, and there is now only a *finite* number of U_y, then we can create a new open set which intersects every V_y:

$$\left(\bigcup_{y \in Y'} U_y \right) \cap \left(\bigcap_{y \in Y'} V_y \right) = \emptyset$$

which means that $V_{x_0} := \bigcap_{y \in Y'} V_y$ is disjoint from Y, so V is open and $x_0 \in V$.

From here, we simply repeated the process for every $x_0 \in X \setminus Y$. This will give us a collection set

$$\bigcup_{x \in X \setminus Y} V_x = X \setminus Y$$

meaning that $X \setminus Y$ is the union of open sets, and so is open. Thus, it's compliment Y is closed, as we sought to show.

This is actually a really powerful theorem if you think about it. If you have a compact Hausdorff space, *every* clsoed set is compact! So finiteness is a little bit everywhere.

Note that The converse is true by theorem 3.2.1. If we get rid of the Hausdorff condition, it is still possible to find spaces in which all clsoed sets are compact, and all compact sets are closed

Example:
The trivial topology on any set is clearly always compact (and closed). Thus

: All closed sets are compact $\not\Rightarrow$ Compact Hausdorff Space

Lemma 3.2.2: Disjointness of points

If $Y \subseteq X$, Y is compact, X is Hausdorff, $x_0 \in X \setminus Y$. Then there exists open sets U, W such that $x_0 \in U, Y \subseteq W$. (i.e., there's a distance between the two by an pen set)

Corollary 3.2.1: making continuous function homeomorphisms

Let $f : X \to Y$ be bijective and continuous. If X is compact, and Y is Hausdorff, then f is a homeomoprhism

Proof **:**

Let V be a closed set

So V is compact

So $f(V)$ is compact

So $f(V)$ is closed

so f is a closed map, meaning f^{-1} is continuous

So f is a homeomorphism.

We can use this fact to show the following result:

Corollary 3.2.2: Incomparability of Compact Hausdorff Spaces

Let $\mathcal{T}, \mathcal{T}' \subseteq P(X)$, where $\mathcal{T}$ is a Compact Hausdorff space

1. if $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, then it is compact, but not Hausdorff
2. if $\mathcal{T}'$ is strictly coarser than $\mathcal{T}$, then it is Hausdorff, but not compact

Therefore, if we have the same underlying set X for two topologies, each of which is compact and Hausdroff, then they cannot be comparible.

Proof **:**

($\mathcal{T}' \subset \mathcal{T}$) Let $\{U_i\} \subseteq T'$ such that $X = \cup_i U_i$. Since $\mathcal{T}'$ is finer than $\mathcal{T}$, then $\exists x_1, ..., x_n$ such that $X = \cup_{j=1}^{n} U_{x_j}$ still compact.

For the other way, let's take $id : (X, \mathcal{T}) \to (X, \mathcal{T}')$ Note that the identity is a continuous bijection, and so $\mathcal{T}'$ is *nto* Hausdorff, for it it were, then we'd have acontinuous bijjection from a compact space to a Hasudorff spaec, meaning the identity is a homeomorphism, meaning $\mathcal{T} = \mathcal{T}'$, a contradiction.

($\mathcal{T} \subset \mathcal{T}'$) means that $\mathcal{T}'$ is Hausdorff (for easy reasons), then use the identity map again.

We can also weaken compactness to local compactness, which will imply something called regularity. This will be covered once we introduce the concept of regular spaces.

Theorem 3.2.5: Compact In Metric $\Rightarrow$ Closed And Bounded

if (X, d) is a metric space and $Y \subseteq X$ is compact, then Y is clsoed and boudned

Proof. Clsoed because Hasudroff, bounded because compactness means take finite balls, then you can create a bigger ball with an epsilon bigger than all of of those. □

The converse is not necessarily true.

Example: Closed and Bounded, but not Compact
Let $X = \mathbb{R}$, and define

$$d(x, y) = \begin{cases} |x-y| & |x-y| < 1 \\ 1 & |x-y| \geq 1 \end{cases}$$

which makes $(\mathbb{R}, d)$ into a metric space (or the discrete metric too). $\mathbb{R}$ is not compact, but $\mathbb{R} \subseteq B_d(0, 2)$, so it is closed and bounded.

One last thing to note. Being a compact Hausdorff space is an insufficient condition to be a metric space. I couldn't find any example which I can put here without defining a new topology (the order topology), so maybe I'll do that in the introduction another time to explain this:

$$\omega_1 + 1 \text{ with the order topology is not metrizable}$$

3.2.4 Compactness of real line

The fact that the any non-finite part of the real line is compact might at first very strange. Surely for an interval $[0, 1]$, there is *some* infinite covering which will not a finite sub-cover. As it turns out, the fact that the "end-points" are in $[0, 1]$, that is, there isn't any point in $[0, 1]$ which we can *perpetually approach without reaching* is the key to making it compact.

Theorem 3.2.6: $[0, 1]$ Is Compact (Heine-Borel)

$[0, 1]$ Is Compact

Proof **:**
Let $X = [0, 1]$, and let S be an open cover for X. We want to show there exists a finite sub-cover for X. Let

$$T := \{t \in [0, 1] | \exists \text{ finite subset of } S \text{ which covers } [0, t]\}$$

We want to show that $1 \in T$. To do so, define $m := \sup(T)$. We'll show hat $m = 1 \in T$.

1. $m > 0$: Since S is a cover, then there exsits $U \in S$ such that $0 \in U$. Since U is open in $\mathbb{R} \cap [0, 1]$, then U is the union of open intervals intersected with $[0, 1]$, so there must exist a $a \in [0, 1]$ such that $[0, a) \subseteq U$, meaning $\left[0, \frac{a}{2}\right] \subseteq U$. Clearly, U covers$\left[0, \frac{a}{2}\right]$, so we must at least have $M \geq \frac{a}{2} > 0$. Thus $M > 0$.
2. $m = 1$. Let's say $m < 1$. Note that we don't know if $[0, m]$ is compact since m is the supremum, not the maximum. However, we know that $[0, m - \delta]$ *is* compact for appropriate

$\delta > 0$. We'll show that with the properties of the euclidean topology, we can find a compact closed interval $[0, m + \delta]$ for appropriate $\delta > 0$. Since we're in the euclidean topology, we know that there must exist some $U \in S$ such that $m \in U$. Thus, $\exists \epsilon > 0$ such that $(m - \epsilon, m + \epsilon) \subseteq U$. The fact that this exists is *key*: it is the core property of the euclidean topology we're using to show compactness. The fact that there always exists this tinnier epsilon band, and that inside it, we'll have $\left[m - \frac{\epsilon}{2}, m + \frac{\epsilon}{2}\right]$, will carry the proof. Since m is the supremum of value such that all closed intervals $[0, m]$ are compact, take the finite sub-cover which covers $\left[0, m - \frac{\epsilon}{2}\right]$. Let $S' \subseteq S$ be the finite sub-cover which covers $\left[0, m - \frac{\epsilon}{2}\right]$. Then define

$$S'' := S' \cup U$$

Then

$$\left[0, m - \frac{\epsilon}{2}\right] \cup U = \left[0, m - \frac{\epsilon}{2}\right] \cup (m - \epsilon, m + \epsilon) \subseteq \bigcup_{V \in S''} V$$
$$\Rightarrow \left[0, m + \frac{\epsilon}{2}\right] \subseteq \bigcup_{V \in S''} V$$

meaning $m + \frac{\epsilon}{2} \in T$, but then m wouldn't be the supermom, a contradiction. Thus, $m = 1$.

3. $m \in T$. Finally, we must show that the supermom is inside T. As before, $\exists U \in S$ such that $m \in U$ $(1 \in U)$, so again by the property of the subspace topology of $[0, 1]$, exists some $\epsilon > 0$ such that $(m - \epsilon, m] \subseteq U$. Since m is the supremum, we know that there exists a finite sub-cover which covers $[0, m - \frac{\epsilon}{2}]$. Let this finite cover be $S' \subseteq S$. Then:

$$S'' = S' \cup \{U\}$$

Then S'' covers $[0, m] = [0, 1]$, completing the proof.

If you didn't understand this proof in MAT257, this is probably the time to take a moment and ponder on this proof here. It is in a sense a novel proof. Out of necessity, Now that we've proved this, you can re-visit the Topologist sin-curve and check how this "finiteness" is important in the proof of it

Corollary 3.2.3: $[a, b]$ Is Compact

$[a, b]$ Is Compact

Proof. $[a, b]$ is either empty, finite, or homeomorphic to $[0, 1]$ □

Corollary 3.2.4: higher dimensions

$[0, 1]n \subseteq \mathbb{R}^n$ is compact

Proof. We showed the product of two compact spaces are compact, so it's easy enough to show that the product of finitely many compact spaces are compact □

Theorem 3.2.7: Closed And Bounded In $\mathbb{R}^n$ Imply Compact

$A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded (for the euclidean metric)

Proof **:**

($\Rightarrow$) That it's compact means that impies it's closed and bounded comes from the fact that A is a subspace of a hausdorffspace $\mathbb{R}^n$, thus by 3.2.5 it is closed and bounded

($\Leftarrow$) Now we need to show that in $\mathbb{R}^n$, closed and bounded are also *sufficient* conditions for compactness in $\mathbb{R}^n$. This is actually an easy consequence of what we've already shown. If you are bounded, then there isn't any "rays" when taking the product of $[a, b]$. Since we are closed, A will be the union of:

$$[a_i, b_i]^n$$

And since every $[a_i, b_i]^n$ is compact, and the union of *finitely* many of these will remains compact (notice we're dealing with the finite union of closed sets, so we don't need to worry abou the infinite case), thus it will be compact.

Note: You might've grown accustom to closed and bounded implies compact, and maybe you think in some nice spaces this is the case (like Hasudorff, or metric spaces). This really need not be the case! Take the euclidean metric on the rational numbers. Closed and bounded is *not* compact. More generally, any non-complete space (that is, spaces were Cauchy sequences are not convergent sequences) will not have this implication hold in general. If you know some analysis already, then you might be interested in konwing that completeness is *not* a topological property, meaning this implication is not a completely topological one!

Even in cases where there is a complete metric, it's not necessarily the case, though examples of those are more complicated [a].

The condition for the converse being true is unfortunately beyond this course [b]

[a] no infinite-dimensional Banach Spaces as metric spaces have the property that closed and bounded imply compact

[b] A metric space has the "Heine-Borel" property if and only if it is complete, σ-compact, and locally compact. We don't define σ-compactness in this document

Example:

The power of this theorem in $\mathbb{R}^n$ should no be understated. For example, S^n is compact. You can imbedd it in $\mathbb{R}^{n+1}$, and there is a closed and bounded set! This is thus a really powerfull tool in the right context.

These theorems make the following theroem a one-liner:

Theorem 3.2.8: Extreme Value Theorem (EVT)

If X is compact, and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ achieves it's max and min

Proof. $f(X)$ is also compact, and so in $\mathbb{R}$ is closed and bounded. □

3.2.5 Compactness and metric spaces

As with many things in topology, we like to find concepts which let's us detect properties of a space. Many times, this happens when two concepts merge given a nice enough space. This is exactly what happens in compact metric spaces, with the concepts we're about to introduce

Define uniform continuity. Why in metric space. Brief word on generalization.

next,

$$\text{uniform continuity} \Rightarrow \text{continuity}$$

example

when the two merge: compact metric spaces

$$\text{uniform continuity} \Leftrightarrow \text{continuity}$$

3.2.6 Locally Compact

This is not very important for the course, but it was introduced in tutorial and I personally find it intersting.

The coolest thing introduced in this section is the idea of taking a space, and making it a subspace of a *compact* space. One we get to Stone-Čech compactification.

Definition 3.2.5: Local Compactness

A space X is said to be *locally compact at x* if there is soe mcompact subsapce C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be ***locally compact***.

Local connectedness has that for *any* neigbhorhood around a point, there is a connected set that is contained in that neigbhorhood. This conditions is thus in a sense *weaker*. It will be satisfied once we add the condition that X is a Hausdorff space, but why weaken it in the first place? It will be to introduce the idea of *compactification*

Theorem 3.2.9: 1-Point Compactification

Let X be a space. Then X is locally compact Hausdroff if and only if there exists a space Y satisfying the following conditions

1. X is a subspace of Y
2. The set $Y - X$ consists of a singel point
3. Y is a compact Hasudroff space

If Y and Y' ar etwo spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X

Proof **:**
p. 183

Definition 3.2.6: One-Point Compactification

IF Y is a compact Hausdorff spaec and X is a proper subspace of Y whose closure equals Y, then Y is said to be the ***compactification*** of X. If $Y - X$ equals a single point, then Y is called the ***one-point compactification of*** X

Example:

1. THe one point compactification of the real line is homeomorphic with the circle!
2. Similarly, the one-point compactification of $\mathbb{R}$ is homeomorphic to S^2
3. One might ask if the one-point compactification of $\mathbb{C}$ is the same as $\mathbb{R}^2$, since both are homeomorphic. And indeed they are, but since their algebraic structure differs, we usually don't think of it in this way, and say $\mathbb{C} \cup \{\infty\}$ is the compactification of $\mathbb{C}$, and often call it the *Riemann sphere* or *extended complex plane*.

The rest of these results are trying to match the usual notion of "locality" when a definition of "locally X" is given

Theorem 3.2.10: Locally Compact in Hausdroff Space

Let X be a Hausdorff space. Then X is locally compact if and only if given x in X, and given neigbhorhood U of x, there is a neigbhorhood V of x such that $\overline{V}$ is comapct and $\overline{V} \subseteq U$

Proof **:**
p.185

Corollary 3.2.5: closed/open subsets of X also locally compact

Let X be a locally comapct Hausdrff space; let A be a subspace of X. If A is clsoed in X or open in X, then A is locally compact

Proof **:**
p.185

Corollary 3.2.6: Compact Hausdorff space and locally compact

A space X is homeomrophic to an open subspace of a comapct Hausdorff space if and only if X is locally compact Hausdorff

Proof **:**
This directly follows from theorem 3.2.9 and corollary 3.2.5

3.3 Nets

I'll eventually tranfer my notes on sequences from 257 here. Essentially,

1. If a space is Hausdorff then sequences converge to at most 1 point
2. a point is in $x \in \overline{A}$ if there exists a sequence $\{x_n\}$ in A such that $\{x_n\} \to x$
3. If $f : X \to Y$ is continuous, then f preserves sequences.

If we add the condition of first countability to each space, then these become if and only if's. However, we would want a construction that doesn't require us to add a condition to our space for these to be if and only if's. We Thus present this generalization of sequences

Definition 3.3.1: Net

A *net* is a function $\eta : S \to X$ whose domain is a directed set. A *directed set* is defined to be a pair $(S, \leq)$ where S is a set and $\leq$ is a relation on S satisfying

1. For all $s \in S$, $s \leq s$
2. for all $s, t, u \in S$, $s \leq t$, $t \leq u$ then $s \leq u$
3. For all $s, t \in S$ there exists $u \in S$ with $s \leq u$ and $t \leq u$

Definition 3.3.2: Filter

A *filter* on a set X is a collection $\mathcal{F} \subseteq 2^X$ that is

1. donward direct: $A, B \in \mathcal{F}$ implies there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$
2. noempty: $\mathcal{F} \neq \emptyset$
3. upward closed ;$A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$

a *proper filter* has the additional property that

4) there exists $A \subseteq X$ such that $A \notin \mathcal{F}$

Definition 3.3.3: Net Converging

We say that η *converges* to x if and only if its eventually a filter

$$\mathcal{E}_\eta := \{A \subseteq X | \exists t \in S \text{ such that } s \leq t \Rightarrow \eta s \in A\}$$

Look at *Topology: A Categorical Approach* chapter 3 for more details. Should add here.

Chapter 4

Expanding Hausdorff Property: Separations axioms

A lot of nice things started to happen when we introduce the Hausdorff condition on spaces. However, that still let's in some edge cases which do not confrom with the euclidean topology intuitions we've built up in analysis courses. $\mathcal{T}_{\mathbb{R}}$ will have a stronger condition, called *normal* (or T_4) which will let us use our intuition we've built up before. We'll now build up from the Hausdorff condition to the normal condition to show what other nice results happen while strengthening the Hausdorff condition. Once we reach the normal condition, we'll show some really intersting consequences of normal the existence of normal spaces!

4.1 Regularity and Normality

Definition 4.1.1: Regular

Suppose X is a topological space in which one-point sets are closed. X is ***regular*** if given $x \in X$, and a closed set B in X such that $x \notin B$, then there exists disjoint open sets U, W such that $x \in U$ $B \subseteq W$.

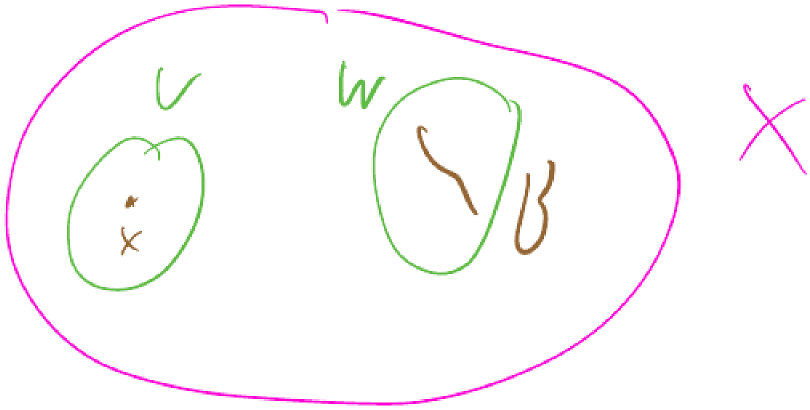

Figure 4.1: visual from lectures

> **Note**: Note that we are starting with the assumption that one point sets are closed. The converse of this, though it might seem intuitive enough, is not necessarily true (perhaps the trivial topology is a "pathological" example)

Example:
Here's a Hausdorff space that is not regular. One example is the space $\mathbb{R}_K$ constructed by Munkres. The underlying set is the reals, and the basis is chosen as the usual open intervals, along with all sets of the form $(a, b) \setminus K$ where $K = \{1/n \mid n \in \mathbb{N}\}$. This topology is clearly finer than the usual topology on the reals, and the reals form a Hausdorff space under the usual topology, so $\mathbb{R}_K$ is also Hausdorff (passing to a finer topology preserves Hausdorffness).

However, $\mathbb{R}_K$ is not a regular space. For instance, the closed subset K in this space (closed because its complement is open by construction in this topology) and the point 0 cannot be separated by disjoint open subsets.

Definition 4.1.2: Normal

X is ***normal*** if we give disjoint closed subspace $A, B \subseteq X$, then there exists disjoint open subsets u, W such that $A \subseteq U$ and $B \subseteq W$

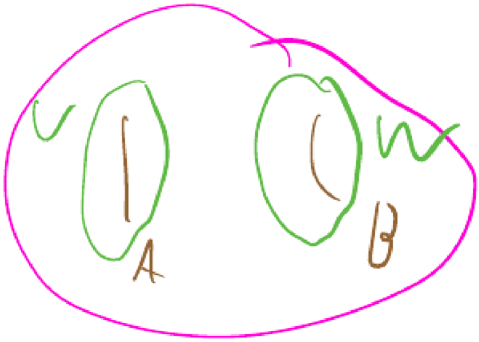

Figure 4.2: visual from lecture

Don't let image decieve you that we're going up "dimensions" of things (i.e. from point to line), but rather we're going from points to *closed set.* This might seem obvious if you just read it, but the visual is a bit miss-leading. There isn't something "higher" dimensionally than a open or closed set (perhaps the entire space, but you can't really define much there)

Normal is a stronger condition than regular. For example take

$$\mathcal{T} = \{\text{here}\}$$

Then $\mathcal{T}$ is regular, but not normal.

Chain of Implications: Given that points closed, then:

$$\text{Normal} \Rightarrow \text{Regular} \Rightarrow \text{Hausdorff}$$

Lemma 4.1.1: equivalent formulations ("zoom-in" axiom)

Let X be a topological space such that one-point sets are closed then

1. X is regular if and only if given $x \in X$, and open U containing x, $\exists$ open set V such that $x \in V$, and $\overline{V} \subseteq U$
2. X is normal if and only if given a closed set $A \subseteq X$ and open set U containing A, then $\exists$ open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$

Proof. **intuition**:

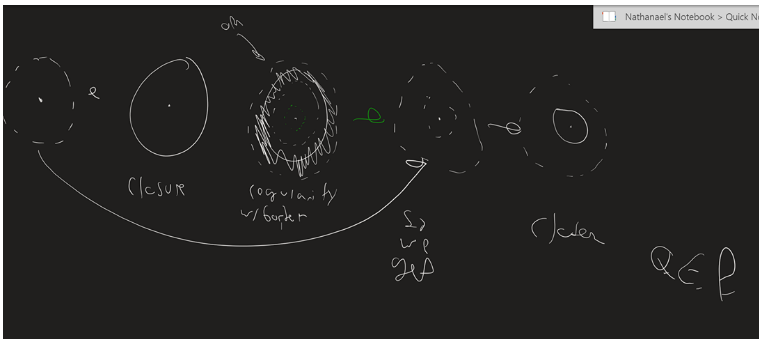

Figure 4.3: visual illustration of proof

Similarly for normality, except you can start with the closed set A instead of the center point $\square$

Note that regular spaces behave nicely under subspaces and product spaces

Theorem 4.1.1: subspaces of Regular Spaces

A subspace of a regular space is regular.

Theorem 4.1.2: Product of Regular Spaces

A Product of regualr spaces is regular.

Proof. Munkre p.197 $\square$

However, Though normal spaces have a stronger condition, they do not behave as nicely!

Example:
For example, take the Sorgenfrey plane:

$$\beta_{\mathbb{R}_l^2} = \{[a, b) \times [c, d) \mid a < b, c < d, a, b, c, d \in \mathbb{R}\}$$

This space is not normal. TO show this p.198 of Munrkes (long proof)

Note that Since $\mathbb{R}_l$ is regular, then $(\mathbb{R}_l)^2$ is also regular. Normal spaces are in a sense worse than regular spaces because they don't have this condition! I like to think of it as if one adds further restrictions on a space (or a concept) then there is less flexibility given because more conditions need to be satisfied in order for things to work.

However, one one must note that the converse is actually true, that is, if the product space is normal, then the spaces are normal

Proposition 4.1.1: $X \times Y$ Normal Implies X, Y Normal

title

Solution If A, A' are disjiont closed subsets of X, then $A \times Y$, $A' \times Y$ are disjiont closed of $X \times Y$. Therefore, we can use normality and find O, O' open disjoint neighbourhoods. Write $O = \bigcup_i U_i \times V_i$, and $O' = \bigcup_j U_j' \times V_j'$ where eac hU_i, U_j' is open in X and V_i, V_j' is open in Y. Pick $y \in Y$. Then

$$\bigcup \{U_i \mid y \in V_i\} \qquad \bigcup \{U_j' \mid y \in V_j'\}$$

are disjiont open neighbourhoods of A and A' ▼

There is one weird space I found that is said to be "vacuously normal" but not regular or Hausdorff, which confuses. me. Let $S = \{0, 1\}$ be the underlying set. Then Take the following is as the open sets:

$$\{\emptyset, \{1\}, \{0, 1\}\}$$

Note that the closed sets are:

$$\{\emptyset, \{0\}, \{0, 1\}\}$$

This space is not Hausdorff since $\{\}$ is not closed. However, normality looks like it applies.

4.2 What spaces are normal?

Though regular spaces act more nicely when being manipulated, normal spaces on themselves are strong enough to have lots of strong equivalent implications.

Lemma 4.2.1: Metrizable Spaces Are Normal

Metrizable spaces are normal

***Proof* :**
Let X be a metrizable space. Assign it the metric d. Let A, B be disjoint closed sets. We want to show that there exists two open sets (U, V), each containing their respective closed sets, each open. Take an $a \in A$. Then choose an $\epsilon_a > 0$ such that $B_{\epsilon_a}(a) \cap B = \emptyset$. This follows from $a \notin B$ and B being closed, so the limit poitns of A and B so they both have their border. If there was a point $a \in \partial A$ and an ball $B_\epsilon(a)$ that intersected B, then we can shrink the epsilon ball by the Hausdorff property of Metric spaces. So $d(a, B) > 0$. This is the key observation. From here the proof develops quite naturally. Take

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \qquad \text{and} \qquad V = \bigcup_{b \in B} B(b, \epsilon_b, 2)$$

what's left to show is that they do not intersect. Take $x \in U \cap V$. Then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$, $b \in B$. Then by the triangle inequality, and some manipulating of inequalities, we get

$$d(a,b) < \frac{\epsilon_a + \epsilon_b}{2}$$

If $\epsilon_a \leq \epsilon_b$, then $d(a,b) < \epsilon_b$, so the ball $B(b, \epsilon_b)$ contains the point a, a contradiction. Similarly if $\epsilon_b \leq \epsilon_a$. Thus, the two open sets are disjiont, proving normality observation

There is a second proof that I found here.

Lemma 4.2.2: Compact Hausdorff Spaces Are Normal

Compact Hausdorff Spaces Are Normal

Proof. longer proof. Oct 22nd. □

If we weaken the condition of compactness a bit to locally compact, we'll get that this implies it's regular:

Lemma 4.2.3: Locally Compact Hausdorff Spaces Are Regular

Locally compact Hausdorff spaces are Regular

Proof **:**
p.202

Theorem 4.2.1: Second-Countable Regular Implies Normal

Every second-countable regular space is normal

Note: the converse is not true: normal spaces are *not* necessarily second-countable. Take the example from Munkres.

Proof **:**

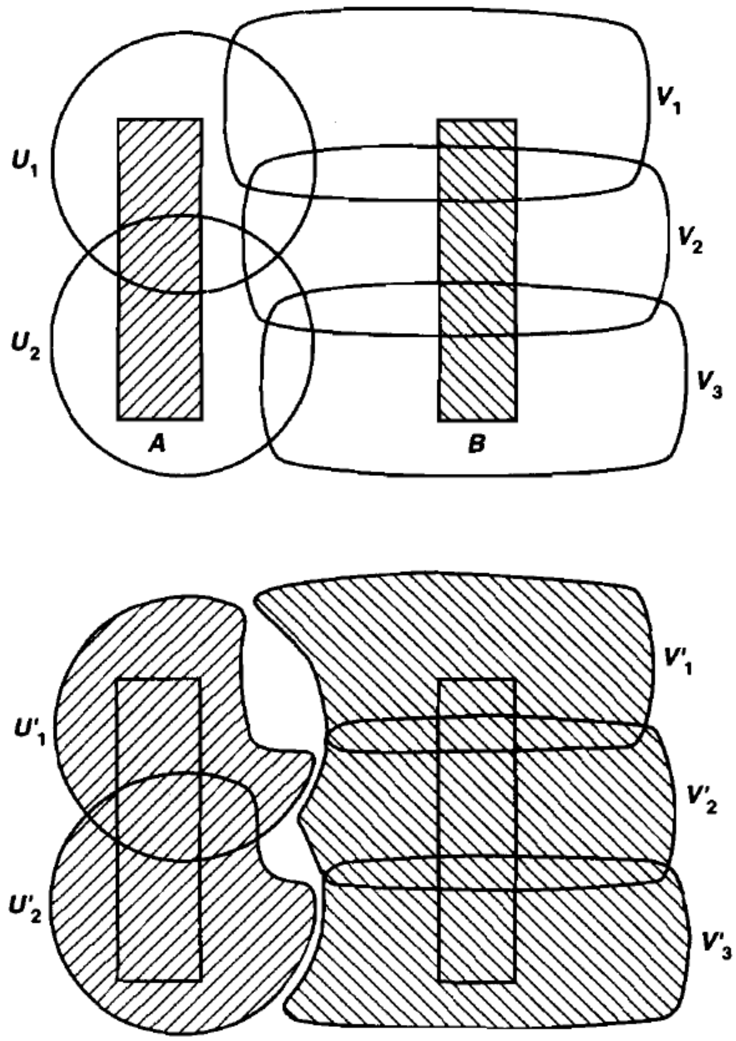

Figure 4.4: visual from Munkre

oct. 22nd

Proposition 4.2.1: Metrizable implies normal

Metrizable implies normal

***Proof* :**
here

However, normal does not imply metrizable:

Example: Of Normal, but not Metrizable, Spaces
`https://math.stackexchange.com/questions/157906/normal-non-metrizable-spaces`

4.3 Urysohn Lemma

Urysohn lemma is one of the larger topological insights, and has a great multitude of applicatoin in other very large and important thoerems (hence why it's called a lemma, it's like the key step for many important theorems).

Theorem 4.3.1: Urysohn

Let X be a normal space. Let A, B be disjoint closed sets. THere exists a continuous function $f : X \to [0, 1]$ st $A \subseteq f^{-1}(0)$, and $B \subseteq f^{-1}(1)$

Proof. p.207-210 □

Example:
Let X be a normal connected space. Then X must have uncountably many poitn

Solution Let X have more than 1 point. Then $x, y \in X$. Since X is Normal, then ▼

4.4 Metrization Theorem

As a consequence of Urysohn' Lemma, we have a couple of very neat results:

Theorem 4.4.1: Urysohn Metrization Theorem

If X is regualr and second-countable, then X is metrizable

Proof. here □

4.5 Tiezte Extension Theorem

Theorem 4.5.1: Tietze Extension Theorem

Let X be a normal space and let $A \subseteq X$ be a closed subset. let $f : A \to [0,1]$ be a continuous funciton. THen there exists a continuous function $F : X \to [0,1]$ such that $F|_A = f$.

Similarly if we replace $[0,1]$ by $\mathbb{R}$

Proof. here □

Chapter 5

Elements of Functional Analysis

See Munkres topology

Chapter 6

Dimension Theory

See Munkres Topology

Chapter 7

Fundamental Group

Recall how connectedness captured some idea of not being able to break down your set, and path-connectedness gave us the stronger notion that you must be able to "get" to any point within a "finite" amount of time (refer to topologist sin-curve for intuition). The key idea in this section is the generalization of the concept of path-connectedness from points to other topological spaces – in particular, we'll be interested in the loops we can form on spaces. This concept will turn out to be more useful because there will be many spaces which can be classified by seeing what loops can actually collapse down to a single points in a continuous fashion. For example, on a sphere, any loop can collapse to a single point, but on a torus, the loop going around the way you would grab it cannot. This turns out to be a naturally topological distinction. [And this generalization of path-connectedness is the right way to go because we are looking for some properties of a component, maybe? Will think about how this ties in later because I think the intuition tied to path-connectedness will be really cool].

Note that from now on, for notational simplicity, we'll take the convention that

$$I = [0, 1]$$

7.1 Long build-up

There is a lot of little pieces that need to fit together to see the use of the fundamental group. First, here's the basic ontology for this area:

Definition 7.1.1: Homotopy

Let X, Y be topological spaces, and suppose we have two maps f, f' which are continuous maps from X to Y. We say that f is ***homotopic*** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x), \qquad F(x, 1) = f'(x)$$

F is called a homotopy between f and f', and we write this as $f \simeq f'$

Basically, if we can continuously deform one shape into another, the two shapes are homotopic!

Note that the topology from which we define continuity ($X \times I$) has *both* the topology of X and the Euclidean topology to define it. I suspect the "continuous deformation" is more reliant of the euclidean portion, though I'll confirm this with further pondering. Also, it is worth pointing out that though I said its continuous deformation of the *shape*, technically, two *functions* are homotopic – homotopy is defined on the function space. Thinking about the graphs of functions is where the continuous deformation of shapes intuition comes in.

Example:

1. Let $f, g : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, x^3)$ and $g(x) = (x, e^x)$. Then the map $F : R \times I \to \mathbb{R}^2$,
$$F(x,t) = (x, (1-t)x^3 + te^x)$$
Is a homotopy. $F(x, 0) = (x, x^3)$, and $F(x, 1) = (x, e^x)$, the way is easily checked to be continuous. Thus
$$f \simeq g$$
2. Take $f : I \to I$ $x \mapsto x^2$. Then
$$F(x,t)((1-t)x^2 + tx)$$
is a homotopy. More generally, x^2 can be any polynomial, and this homotopy will work.
3. Take $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 0$. Then:
$$F : \{\mathbb{R}\} \times I \to \mathbb{R} \qquad F(x,t) = (1-t)f(x)$$
is a continuous function, with $F(x, 0) = f(x)$ and $F(x, 1) = 0$. Thus, every continuous function on $\mathbb{R}$ is homotopic to the constant function.

   If we think of the graphs of the functions, the point $\{0\}$ with the trivial topology any shape of $\mathbb{R}$ with the euclidean topology can be continuously deformed to on another.

When function's are homotopic to the constant function, we give function a special name:

Definition 7.1.2: nullhomotopic

If f is homotopic to a constant map, we say that f is ***nullhomotopic***

The idea of nullhomotopy is saying that if you deform your space in a continuous such that you can bring it to a constant point, then we can think of the two being, in some sense, topologically trivial [1].

You perhaps noticed I used the same idea for my homotopic examples of the $(1-t)a + tb$ amendment, that is, the usual function that creates *paths* between two things (recall in MAT257 how you used this function in the proof of MVT?). We give Homotopies that use this a name

[1] not necessarily trivial in the most general sense, because compactness, normality, connectedness, and all the other properties might still be present in this "trivial" case, but it's trivial when it comes to trying to form to bending it around

Definition 7.1.3: Straight-Line Homotopy

If F is a homotopy such that it's construction is:

$$F(x,t) = ((1-t)f(x) + tg(x))$$

then it is a Straight-line homotopy

You might've also noticed that the notation was basically an equivalence notation ($\sim$). That is actually precisely how you should think about what a two homotopic paths are: *equivalent up to continuous deformation*

Lemma 7.1.1: Equivalence Relation With Homotopy

For fixed X, Y, $\simeq$ is an equivalent relation

Proof **:**
We will prove the 3 properties for an equivalence relation:

(symetry) We need to show that $f \simeq f$. The identity map would do

(reflectivity) We need to show that $f \simeq f' \Rightarrow f' \simeq f$. the $1-t$ trick will do

(transitivity) We need to show that $f \simeq f', f' \simeq f'' \Rightarrow f \simeq f''$. Going twice as fast will do.

Since we have an equivalence relation, we an bring in our intuitions about them and form an equivalence class

Definition 7.1.4: Equivalence Class Of homotopies

We write $[f]$ for the equivalence class of f with respect to $\simeq$

A lot of study is on homotopies, which will eventually bring you to homotopy groups, homology, exact sequences of homotopic groups and Hopf fibration of the sphere. For now, we'll work with homotopies which are "1-dimensional" (which we can eventually generalize). Working with one-dimensional one's doesn't mean we won't be working with higher-dimensional objects, but the path in the space we use that will be homotopic will be one dimensional.

7.1.1 Path-Homotopy

The natural one-dimensional object to work with then is a path:

Definition 7.1.5: Path-Homotopic

Two paths $f, \tilde{f}$ in X are ***path-homotopic*** if

1. these have the same initial and final points: $f(0) = f'(0)$, $f(1) = f'(1)$
2. There exists a continuous function $F : I \times I \to X$ such that

$$F(s, 0) = f(s), F(s, 1) = f'(s), F(0, t) = x_0, f(1, t) = x_1$$

where $f(0) = f'(0) = x_0$ and $x_1 = f(1) = f'(1)$. We write

$$f \simeq_p f'$$

F is called a ***path-homotopy***

The professor also said something I thought interesting: That in a sense "we got one minute of time to change it, and we must do it in that time". I think there's some truth to that because you somehow can't go on "forever" perhaps.

Example:

1. If we take all function $I \to I$ such that $x \mapsto x^n$, then all of these are path-homotopic. In fact, any path such that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$ will be path-homotopic by the straight-line homotopy
2. More generally, if C is a convex set, then any two paths with the same initial and ending points are path-homotopic by the straight-line homotopy
3. "There must be more than the straight-line homotopy" you might be thinking, and you're right.
4. "Are there paths that are not homotopic?". That can be the case too. Take $X = \mathbb{R}^2 - \{0\}$ and

$$f(s) = (\cos(\pi s), \sin(\pi s)) \qquad g(s) = (\cos(\pi s), 2\sin(\pi s))$$

then these two functions are path homotopic using the straight-line homotopy, as you see visually:

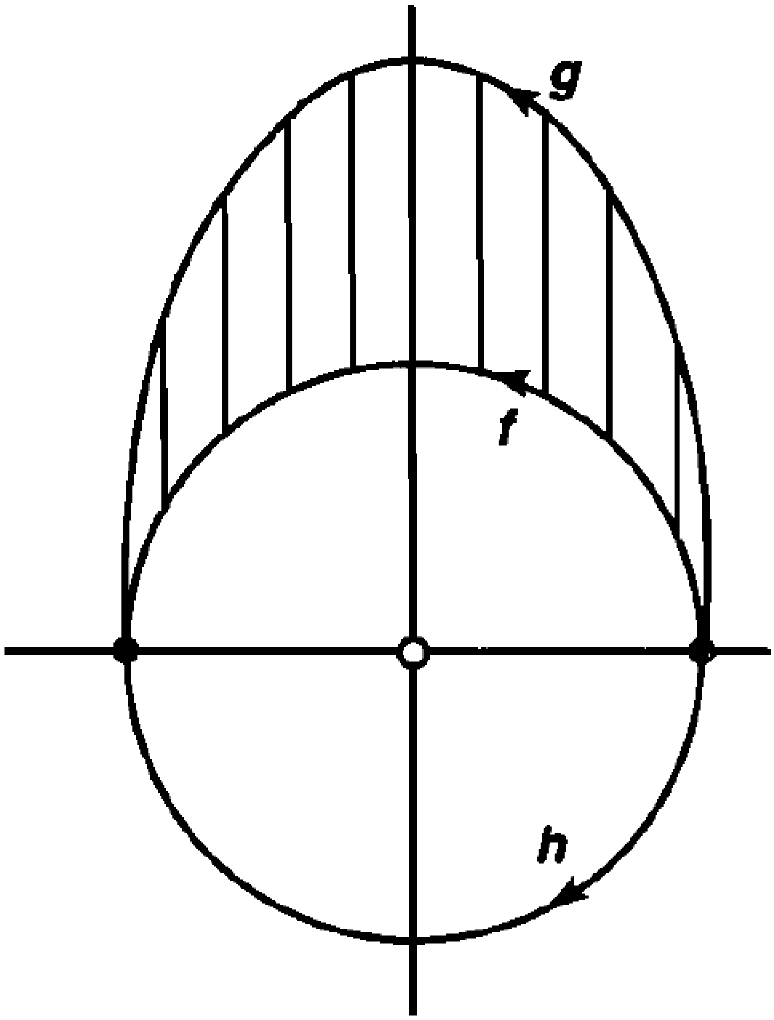

Figure 7.1: Same initial and end points, but one goes under point

In general, if the point 0 was not missing, we can have:

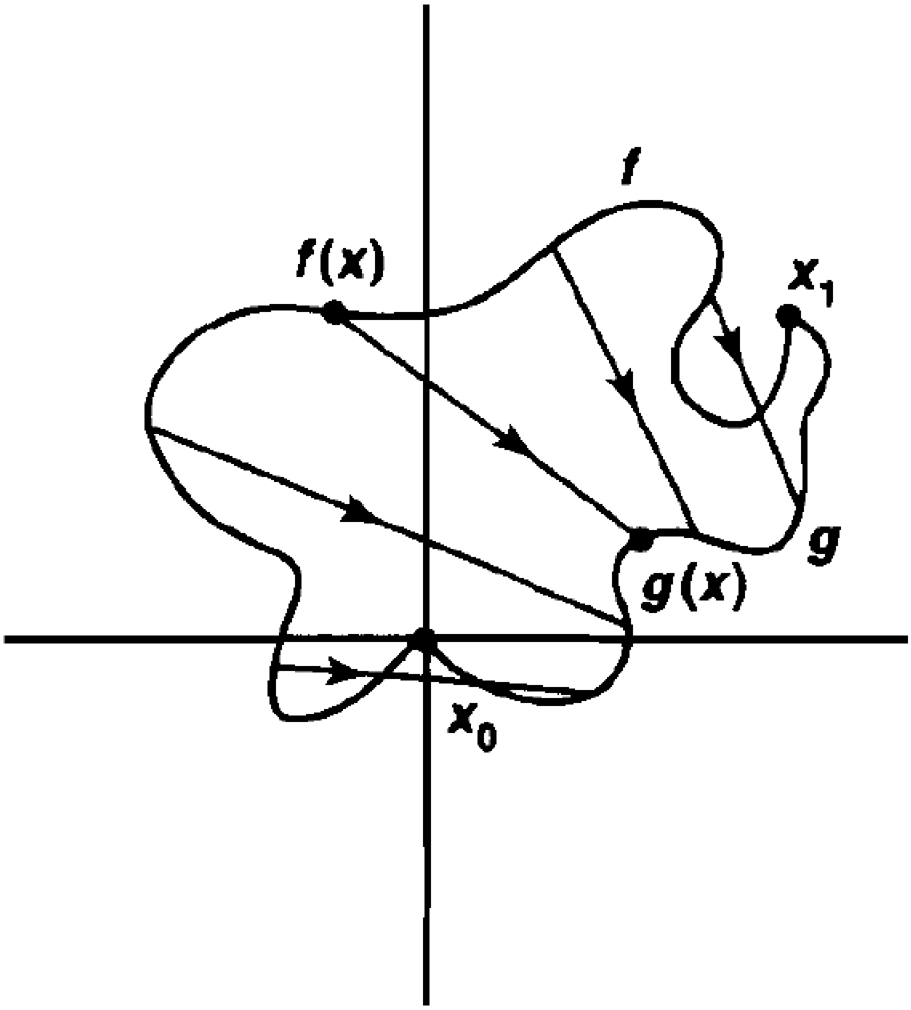

Figure 7.2: Convex space

However, the path $h(s) = (\cos(\pi s), -\sin(\pi s))$ is *not* path-homotopic. In the visual, this is the h path. It should perhaps seem intuitive that we we try to pass a continuous path *through* the point 0, we will introduce a discontinuity. The proof for this is a bit laborious at this point, so being satisfied with this "hole" intuition is useful.

Note that the equivalence relation constructions works just as well on path-homotopies, just more book-keeping. Because of the lower-dimension, the two will *not* define the same equivalence classes, so we define the equivalence

Definition 7.1.6: Equivalence Class Of Path

If f is a path in X, we write $[f]$ for the equivalence class of f with respect to $\simeq_p$

Example:
Recall all polynomials $x \mapsto x^n$? from earlier? These are in the same equivalence class

$$[f] = [g]$$

And since $F'(x,t) = (x, (t-1)x + t0)$ is also a path-homotopy,

$$[id_0] = [f] = [g]$$

meaning it is also *nullhomotopic*

Note: This is an equivalence class, but it will not necessarily form a quotient spce: We are not forming an equivalence class on the points, but on the functions on the space. You'd neeed to define a topology on the paths in order to define a quotient topology on that function space.

7.1.2 Algebraic build-up

As mentioned in the note, this new collection of equivalences classes will not necessarily be a topology. However, we *can* form Algebraic structures one them! To start, we'll need to define the basic's that are present in algebraic structures: identities and a binary operation:

Definition 7.1.7: identity path

If $x \in X$, denote e_x to be the constant path $e_x : I \to X$ such that $e_x(t) = x$ for $t \in I$

Therefore, the id in our previous example would be relabeled as e_0.

Next, we need to figure out a way defining a relation between paths. The natural candidate for this would be to extend one path to the next. However, we can extend *arbitrary* paths from one to the next, since that will make discontinuous paths (because the end point of one path might not be the initial point of another). Thus, we we try and define paths by through "combination", we need to impose this limit:

Definition 7.1.8: combining paths Operators: $*$

If f, g are paths in X and $f(1) = g(0)$, define $f * g$ by

$$(f * g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Example:
Try thinking of any two paths in $\mathbb{R}^2$ and combining them into 1.

Lemma 7.1.2: Well-defined: $*$ Preserves Equivalence Relation

if $f \simeq_p f'$, $g \simeq_p g'$, then
$$(f * g) \simeq_p (f' * g')$$

Proof. Let F, G be homotopies between f and f', g and g' respectively. Let $H : I \times I \to Y$. Basically, we'll just be doing just like in the equivalence relation, doubling the speed appropriately and using the pasting lemma to combine the two continuous functions:

$$H(x,t) = \begin{cases} F(2x,t) & t \in [0, \frac{1}{2}] \\ G(2x-1,t) & t \in [\frac{1}{2}, 1] \end{cases}$$

then note that $H(x,0) = f * g$ and $H(x,1) = f' * g'$, and H is continuous by the pasting lemma, and so we're done ☐

Because of this, we can say
$$[f] * [g] = [f * g]$$

With the operation defined, we can look at the properties of it to see if it would follow all the axioms of a group!! Remember that when we defined $*$, it only works on appropriate paths (with end and initial points matching). This will need to be considered in our up-coming examination of its properties:

Theorem 7.1.1: Properties Of $*$

1. (Associativity) Assume f, g, h are paths in X, $f(1) = g(0)$, $g(1) = h(0)$. Then
$$([f] * [g]) * [h] = [f] * ([g] * [h])$$
2. (identity) If f is a path in x, $f(0) = x_0$, $f(1) = x_1$, then
$$[f] * e_{x_1} = [f] = e_{x_0} * [f]$$
3. (invertible) Given a path f in X, define $\tilde{f}(t) = f(1-t)$ Then
$$[f] * [\tilde{f}] = [e_{f(0)}]$$
$$[\tilde{f}] * [f] = [e_{f(1)}]$$

Proof **:**
Some useful facts that will be used continuously

1. Let $k : X \to Y$ be a continuous map and F be a path homotopy between f and f'. Then

$k \circ F$ is a path-homotopy between $k \circ f$ and $k \circ f'$ where

$$(k \circ F)(s,t) = k(F(s,t))$$

2. composition and $*$ are basically distributive: $k \circ (f * g) = (k \circ f) * (k \circ g)$.

As you probably noticed in the construction, the identity wasn't unique! This means that $*$ just falls short of actually being a binary operation. In fact, it is not actually a function, but a *partial function*. That is you can't take two arbitrary paths, but you need to select paths with the appropriate mid-points (as we noticed when defining $*$ originally!). If we restrict to such paths, we will indeed have a binary function, but we usually want to think of $*$ being defined on our entire space.

We actually give a name to a "group" which uses a partial function (and hence, has multiple identities depending on how you must restrict the partial function to get a function):

Definition 7.1.9: Groupoid

A ***groupoid*** is a set (ex. G) with a ***partial function*** $* : G \times G \to G$ *and* a unary function $^{-1}$: $G \to G$ which then satisfy the group axioms (associatvitiy, inverse, identity). The slight difference between groups is that we now have:

1. (inverse) $a^{-1} * a$ and $a * a^{-1}$ is always defined
2. if $a * b$, is defined then $a * b * b^{-1} = a$ and $a^{-1} * a * b = b$ is also defined

How exactly the partial function $*$ is "partial" depends on the context – there is no one way of defining this partial function. Furthermore, we choose a unary function for the inverse because the "identity" because the identity is now no longer so "unique", as was shown in theorem 7.1.1.

Example:

If we take $(\mathbb{R}, *, ^{-1})$ to be our groupoid with $*$ as we defined and $^{-1}(f) = \tilde{f}$ where $\tilde{f} = f(1-x)$ (i.e. reverse f), then we defined a groupoid!

With the algebraic structure defined and labeled, we can define the elements of our groupoid:

Definition 7.1.10: Path Space

Let $P(X) = \{$equivalence classes of paths $[f]\}$. We will call this the ***path space***

there is a natural map $P(X) \to X^2$, $[f] \mapsto (f(0), f(1))$. This map is surjective if and only if X is path connected. So You can alos think of the groupoid from earlier as

$$(P(X), *, ^{-1})$$

7.1.3 Making it a group

As mentioned, we still don't have a "unique" identity nor a total function, so we are yet to totally define a group. The limiting factor is that we can just combine any function with any function. However, if the initial- an end-points where the same, this would solve our problem! For this reason, we will once again restrict our attention, this time to paths that *loop*:

Definition 7.1.11: Loop

Given $x_0 \in X$ a ***loop based at*** x_0 is a pth which begins and ends at x_0. that is

$$f : I \to X, \qquad f(0) = f(1) = x_0$$

There is actually an advantage to adding this resctriction and thinking about loops: Imagine you have a plane with a point missing. Then having a loop around that point would mean you can't contract that loop to the constant map (as we've seen before, where we sneakily introduced a loop). This idea of not being able to contract loops will be our key concept!

With this definition, we can consider the algebra on a subset of $P(X)$ such that the initial and end points match!!

Definition 7.1.12: Fundamental group at x_0

Let

$$\pi_1(X, x_0) = \{[f] | f(0) = f(1) = x_0\}$$

be the Path space with loops at x_0. Then $\pi_1(X, x_0)$ is a group under $*$! That is, if $[f], [g] \in \pi_1(X, x_0)$, then $[f] * [g] \in \pi_1(X, x_0)$, and $(\pi_1(X, x_0), *)$ is a group!!

Note that $\pi_1(X, x_0) \subseteq P(X)$. We call $\pi_1(X, x_0)$ the ***fundamental group*** of X *based at* x_0, or the *first homotopy group*

Note: there *can* be more than one equivalence relation in $\pi_1(X, x_0)$. We will soon explore spaces where this is the case, and also explore cases where there is only one element.

Sometimes $\pi(X)$ is used to represent the groupoid of the space X.

One will naturally as if there is a 1st homotopy group, is there a 2nd? an nth? There is, and it's denoed $\pi_n(X, x_0)$. This higher value of n is where you can think of "dimensions" as I mentioned earlier. These, though, will not be studied here, and are the subject of homotopy theory.

Example:
Let's consider $\mathbb{R}^n$ with the Euclidean topology and some $x_0 \in \mathbb{R}^n$ to define $\pi_1(\mathbb{R}^n, x_0)$. Then by the straight-line Homotopy, we can collapse every loop to the constant map, so there is only one element:

$$\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\}$$

More generally, if you have a convex space, then it's fundamental group will be trivial (since you can use the straight-line homotopy).

Note that this group is *not* necessarily abelian! This example

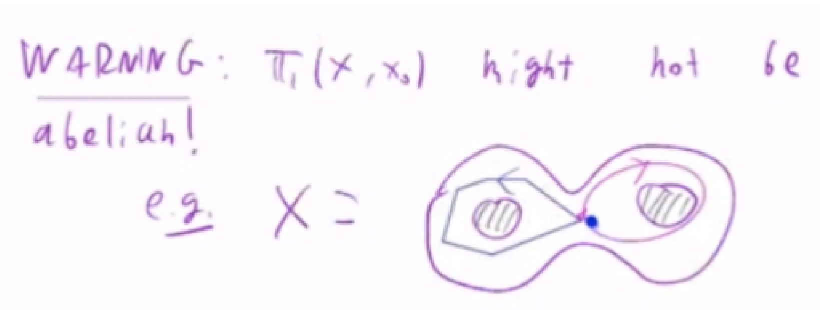

Figure 7.3: *not* abelian (i.e. homotopic) because of holes

Will eventually be used to illustrate why in section (HERE).

So how many fundamental groups are there on a space? Does every point have a distinct type of group or are some of them isomorphic, or better yet, all of them isomorphic on the same space. The different answers to this will actually give us the topological property we're looking for in groups! That is, when thinking about a space, we usually think about the fundamental groups *up to isomorphism* and then the topologically invariant property are the *types* of fundamental groups. That means that the fundamental group does not rely on the point x_0 on which it was originally defined to get it's structure. To that matter, it will also not rely on the *entire* space X, just the component spaces, so the interesting thing to study are the path-connected components.

To get to that intuition, we first, specify how to define the isomorphism on the group

Proposition 7.1.1: Group Isomorphism Between Fundemental Groups

Let α be a path X $\alpha(0) = x_1$, $\alpha(1) = x_1$, then define

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

where

$$\hat{\alpha}([f]) = [\tilde{\alpha}] * [f] * [\alpha]$$

where $\tilde{\alpha}$ is the reverse of the funtion.
In words, this means that you go back, to the loop, then go back forward.
This function is a group isomorphism.

The professor said this is a semi-direct product, and that it is an outer-automorphism. However, it looks like an inner-automorphism to me, so I'll have to check later.

***Proof* :**
This is basically an Algebra proof. To show it's homomorphic:

$$\begin{aligned}\hat{\alpha}([f]*[g]) = \hat{\alpha}([f*g]) &= [\alpha]*[f*g]*[\alpha] \\ &= [\tilde{\alpha}]*[f*e_{x_1}*g][\alpha] \\ &= [\tilde{\alpha}]*[f*\alpha*\tilde{\alpha}*g][\alpha] \\ &= ([\tilde{\alpha}]*[f]*[\alpha])*([\tilde{\alpha}]*[g]*[\alpha]) \\ &= \hat{\alpha}([f])*h(g)\end{aligned}$$

For bijectivity, we can define the two-sided inverse function $\hat{\alpha}^{-1}$ by reverseing $\tilde{\alpha}$, where it can be checked that

$$\hat{\alpha}(\hat{\alpha}^{-1}([f])) = \hat{\alpha}^{-1}(\hat{\alpha}([f])) = [f]$$

Making the homomorphism bijective, making it isomrophic

Note: This will imply that for every path-connected component of X, it will have the *same* fundamental group! Thus, when thinking of Fundamental groups, it is somewhat convention to thinkg about X as being a path-connected component (or at least, it will come down to breaking X up to it's path connected components)

For the more Categorically inclined, there is also no *natural* isomorphism between points! This will come down to the fact that not every space is commutative. I'll amend this once I actually know why it's true.

With this isomorphism at hand, we can now say what is meant when we say "*The* Fundemenatal Group" of a space X.

Corollary 7.1.1: Path-Connectedness And Isomorphism

If X is path-connected, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ for any $x_0, x_1 \in X$

This means that if you have two path-connected spaces with different fundemental groups, they are topologically different. However, in general, different fundemental groups doesn't mean different spaces (ex. $\mathbb{R}$ and $\mathbb{R}^2$ have the same Fundemental group)

Definition 7.1.13: Simply-Connected

X is ***simply-connected*** if X is path-connected, and $\pi_1(X, x_0)$ is the trivial group for some (and hence all) $x_0 \in X$

Lemma 7.1.3: simply connected simplification

If X is simply connected, $P(X) \simeq X^2$ (i.e a bijection), that is,

Proof **:**
First, we'll show all paths are homotopic. Let α, β be paths from x_0 to x_1. Then

$$[\alpha] * [\tilde{\beta}] \in \pi_1(X, x_0)$$
$$[\alpha] * [\tilde{\beta}] = [e_{x_0}]$$
$$[\alpha] * [\tilde{\beta}] * [\beta] = [e_{x_0}] * [\beta]$$
$$[\alpha] = [\beta]$$

With this, we can show injectivity. If $\pi([\alpha]) = \pi([\beta])$, then

$$\alpha(0) = \beta(0), \qquad \alpha(1) = \beta(1)$$

then $[\alpha] = [\beta]$ by proof above

And or surjectivity, given $(x_0, x_1) \in X^2$, there exists a path $\alpha(0) = x_0$, $\alpha(1) = x_1$ such that

$$\pi([\alpha]) = (x_0, x_1)$$

completeing the proof
I will have to look over that surjectivity claim again because it looks like it's breaking well-definedness

7.1.4 Is it an invariance?

We've shown that we can define a fundamental group on a space, but what if we have another topological space Y, and a continuous function $f : X \to Y$, what of the group-structure we just build up? Well, we wouldn't of introduced fundamental groups if they weren't topological in nature, which means that continuous maps *preserve* them! This is also how the comment much earlier that in a sense topology is fundamentally the study of objects that don't change under continuation functions comes in.

So if we have a continuous function, can we define an appropriate *homomorphism* (the equivalent of continuous functions). This is what we'll do in this section.

Definition 7.1.14: induced homomorphic function

Suppose $h : X \to Y$ is continuous, with $h(x_0) = y_0$ (i.e. $h(X, x_0) \to (Y, y_0)$) . Then define

$$h_* : \pi_1(x, x_0) \to \pi_1(Y, y_0) \qquad h_*([\alpha]) = [h \circ \alpha]$$

Trivia: Notice that it's a topology isomorphism which also specifies a point is a point. This makes our function a little more than continuous, it's continuous *and* preserves that point. Categorically, this means we're working in a slightly different category than Topology (Top): It's technically the Base-topology (bTop) category.

Proposition 7.1.2: Properties Of h_*

1. h_* is a a group-homomorphism
2. If $i : (X, x_0) \to (X, x_0)$ is the identity, then i_* is the idnetity
3. THis map is functorial, that is, if $f : (X, x_0) \to (Y, y_0)$ and $(Y, y_0) \to (Z, z_0)$ then

$$(g \circ f)_* = g_* \circ f_*$$

what this means is that it's a functor between the **Top** Category and the **Grp** Category

Solution We'll prove each separately

1.

$$\begin{aligned} h_*([\alpha][\beta]) &= h_*([\alpha * \beta]) \\ &= [h \circ (\alpha * \beta)] \\ &= [(h \circ \alpha) * (h \circ \beta)] \qquad = [(h \circ \alpha)] * [(h \circ \beta)] = h_*([\alpha]) * h_*([\beta]) \end{aligned}$$

2. $i_*([\alpha]) = [i \circ \alpha] = [\alpha]$, hence it's the identity
3.

$$\begin{aligned} (g \circ f)_*[\alpha] &= [(g \circ f) \circ \alpha] \\ &= [g \circ (f \circ \alpha)] \\ &= g_*([f \circ \alpha]) \\ &= g_*(f_*([\alpha])) \end{aligned}$$

▼

Thus, we've shown that the fundamental group is invariant to continuous functions! This is excellent, for now we can properly study them as a topological property of our space.

7.2 Use of Fundamental group

So far, we only defined a fundamental group that was trivial: the fundamental group of a convex space. What we'll do now is classify ways of finding different fundamental groups on spaces.

7.2.1 Covering spaces

This first construction will allow us to find the fundamental group of a circle. As further motivation to understand covering spaces, they will come back when we study Riemann surfaces and Complex Manifolds!! So having a good grasp now is solidly useful

Definition 7.2.1: Evenly Covered and Slices

Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be ***evenly covered*** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called the partition pf $p^{-1}(U)$ into *slices*

if p does this to every neighbourhood in B, we call such a map p a covering map:

Definition 7.2.2: Covering Map

Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a ***covering map***, and E is said to be a *covering space* of B.

Example:

1. The identity map $i : X \to X$ is "trivially" a covering map
2. A generalization of the previous example, if $X \times \{1, ..., n\}$ is n disjoint copies of X, then $p(x, i) = x$ for all i is again a covering map, since all the spaces are disjoint and when you limit to any space, it's homeomorphic to X. This function gives a nice visual for how you can think of covering spaces in general:

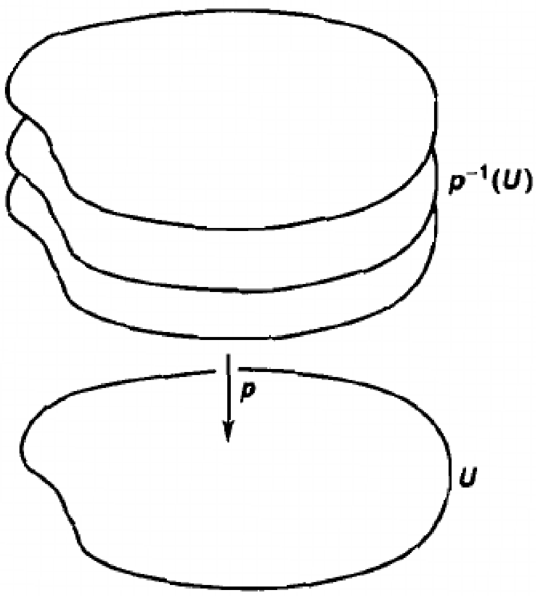

Figure 7.4: Covering space example

3. The function $\pi_x X \times Y \to X$ with Y given the discrete topology is a covering map. The

point of Y having the discrete topology is that that when taking the pre-image, and thinking about the homeomorphism, you need it to be *bijective*, and in particualr *injective*. This will be where using the discretenss of Y would be important.

More non-trivial examples can be found here

`https://en.wikipedia.org/wiki/Covering_space#Examples`

Corollary 7.2.1: properties of Covering Maps

1. Every $y \in B$, the subspace $p^{-1}(y)$, has the discrete topology
2. By the restrictions of a covering map, we will automatically get that it's an open map

***Proof* :**

1. This should be a straight forward test of understanding coverings. Take a neighborhood of y, $y \in U$. Since p is a covering map, then it's pre-image is partitioned into $\{V_\alpha\}$. Restricting our attention to any one slice V_α will lead to a homeomorphism (so for us, bijection) betweeen V_α and U, meaning one point of it maps to U. Since we can do this for every V_α, we get that $p^{-1}(U)$ is a collection of points intersected by open sets, meaning the subspace is discrete.
2. Let $A \subseteq E$. We'll show that $p(A)$ is open. If around every point $x \in p(A)$, there exists an open set V such that $V \subseteq p(A)$, then $p(A)$ is open (basis definition). Let's take a point $x \in p(A)$. Choose an open neighborhood V, $x \in V$. Then since p is an open cover, choose a neighbourhood that is evenly covered by p. Let $\{V_\alpha\}$ be the neighborhood. Since $x \in p(A)$, $p^{-1}(x) \in A$, so choose $y \in p^{-1}(x)$, and the appropraite V_β such that $y \in V_\beta$. Since A is open, $A \cap V_\beta$ is also open. Since p restricted to a V_β is a homeomorphism, $p(A \cap V_\beta)$ is *also* open. Note that $p(A \cap V_\beta) \subseteq p(A)$. Since we found an open neighborhood around x which is contained in $p(A)$, then $p(A)$ is open (by 1.3.2).

So with these in mind, define an interesting covering space? The examples were on disconnected spaces since that's kind of easy to construct: In general, we try to create covering spaces on path-connected spaces (as it's also what Fundamental Group is basically defined on). Here is our first non-trivial example:

Example:
The map $p : \mathbb{R} \to S^1$ given by the equation

$$p(x) = (\cos(2\pi x), \sin(2\pi, x))$$

is a covering map

Solution This comes down to geometric intuition about these trigonometric functions and circles. As an intuitive check that you get this function, if we take any interval $[n, n+1] \subseteq \mathbb{R}$, then that interval will wrap itself around the circle completely.

Let's start by showing that around any point for which the x values are positiev ($x > 0$) has a neighbourhood which is evenly covered. Let $U \subseteq S^1$ be that part of the circle. If you think about our function, that would mean that

$$p^{-1}(U) = \bigcup i \in \mathbb{Z}(n - \frac{1}{4}, n + \frac{1}{4})$$

or, visually:

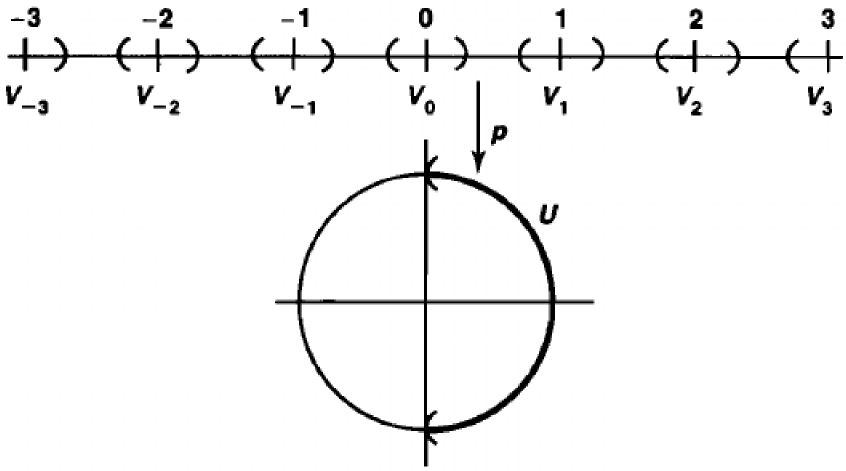

Figure 7.5: $x > 0$ U preimage

What we'll do now is take advantage of this functions properties and of compactness. Take some V_n from the preimage of $p^{-1}(U)$. Take it's closure $\overline{V_n}$. Then by 3.2.6 it is compact.Since our function is continuous and dealing with intervals, by the IVT (3.1.3), V_n maps surjectively onto U. It is already injective by construction. Then by 3.2.2 p_{V_n} is an open map from $\overline{V_n}$ to $\overline{U}$. Since it's Hausdorff, it means that we can get rid of the border in the domain and co-domain and it's still a homeomorphism (hw exercice). Since the function is already continuous, then it is a homeomorphism.

We restrice to $x > 0$, however we can do this for every half of the sphere $x < 0$, $y > 0$, $y < 0$, which will cover every point with an evenly covered neighbourhood, completing the proof. ▼

7.2.2 Properties of Covering Maps

As usual, we might wonder if we can restrict the domain and have a covering map, or take the product space and still have a covering map, or equivalent conditions for covering maps that are equivalent.

One of the two unique properties of covering maps is how p is a *local homeomorphism*: Around every point $e \in E$, there exists a neighbourhood $U \subseteq E$ such that when we restrict to that neighbourhood, our map becomes a homeomorphism to an open subset $V \subseteq B$. Is this condition itself enough to define a covering map? If you think about it, when doing the circle example, what was really key in the proof was proving that restricting will give us homeomorphisms: if two sheets overlap, we can consider them part of the same sheet, making it feel like we can still say it's evenly covered! However, this turns out not to be the case, for a subtle reason:

Example: of a Local Homeomorphism That's Not a Covering Map
Take p before but restrict the domain to $\mathbb{R}_+$:

$$p : \mathbb{R}_+ \to S^1 \qquad p(x) = (\cos(2\pi x), \sin(2\pi x))$$

This function is still surjective, and a local homeomorphism, and even is a covering map for *most* points. However, consider the point $b_0 = (1, 0)$ and some open neighborhood U around that point (not necessarily the same U as our example for the covering map of S^1). Like before, we'd have the same idea for the pre-images, except for V_0 which would be $(0, \epsilon)$ for some ϵ depending on the open neighbourhood U. If we restrict p to U, then we do not get a homeomorphism, but an *embedding*, that is, a function for which if we restrict the co-domain to it's image, it becomes a homeomorphism. If we restrict to $(0, \epsilon)$ we'd get the open neighborhood starting at 0 and ending at ϵ (relative to where sin and cos sends ϵ.)

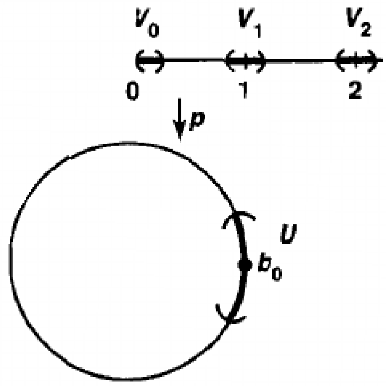

Figure 7.6: visual in Munrkes. Doesn't show embedding, but you can imagine the open set of the embedding starts at b_9

The professor also gave an easier example. If you take the projection map

$$E = [0, 1] \times \{0\} \cup (0, 1) \times \{1\}, B = [0, 1]$$

which is basically forcing surjectivity on the function $(0, 1) \to [0, 1]$. Then there isn't a neighborhood around $0 \in [0, 1]$, which is evenly covered. Again, this comes down to it being an embedding based on the domain rather than the co-domain.

This example shows how it's important to be weary when taking a subspace of the domain of the covering map. So how should we be careful when restricting the domain so that our function remains to be a covering map?

Proposition 7.2.1: Subspace Of Covering Space

Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map

Proof **:**
here

When taking the product of covering maps, we run into no problems

Proposition 7.2.2: Product Of Covering Maps

If $p : E \to B$ and $p' : E' \to B'$, are covering maps, then

$$p \times p' : E \times E' \to B \times B'$$

is a covering map

Proof **:**
Let $b \in B$, $b' \in B$. Then we can take neighbourhoods U, U', $b \in U$, $b' \in U'$ which are evenly covered, so we have $\{V_\alpha\}$ and $\{V'_\alpha\}$. We can take the cross-product of the respective slices. They will still be disjiont, and restricting to slices still a homeomorphism, thus $p \times p'$ is a covering map

Example:

1. Take
$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$
where p is the same covering map of the circle we introduced earlier. Then this function is a covering map for the torus! I'll put an image here soon.

2. Using lemma 7.2.1, and we restrict our attentiont to just the two circles we see in the image before, then we can think of covering the figure 8:

   figure here

   There are multiple ways of covering this surce. You can keep one circle and unwrap the other, or you can unwarp both, or you can have the universal covering space for it.

   figure here, really cool

Munkre present two more examples, and elaborates on the nature of the torus being represented in $\mathbb{R}^3$, but I'll get to that later

7.2.3 Fundamental group of a circle

We know have the tools to compute the fundamental group of a circle! The construction and the answer might at first seem unintuitive, but the prof gave this intuition for how to think about it:

> If you think about a circle, and a loop on the circle, you can think of the loop going around once to have had a 2π angle rotation. Similarly, the trivial loop had 0. You can't have something in between cause the you didn't have a loop. Similarly, you can have 4π, 6π, ..., $2k\pi$ angles of rotation. The fundamental group of a circle will capture the fact that each of these will be unique path homotopies!
>
> So in a sense, what the intuition is building up to for the fundamental group is asking the quesiton "how many ways can I wrap a piece of string around it, such that the start and end point are at some fixed point (like you put a tack to hold it down), and then you can move your string in the usual way you can move your string with a homotopy"

If you look at the examples at the end of last section, we see that the circle, the figure eight, the torus, where all covered by a "larger" space above it that got projected down. However, we didn't formally say how this space is found or achieved. We will start by that.

Definition 7.2.3: Lifting

Let $p : E \to B$ be a continuous function. Let $f : X \to B$ be a continuous map. A ***lifting*** of f is a continuous map $\tilde{f} : X \to E$ such that $f = p \circ \tilde{f}$. Similarly, we can say that this commutative diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

We can think of it as *lifting* in the following way: If we have a point $f(x) \in B$, then what the lift does is take our point x and puts it above one of the points in B which projects down to B – you can think of the corkscrew from the covering of S^1!

The reason we care about lifts and covering maps is because lifts with covering maps preserves paths!

Lemma 7.2.1: paths can be lifted

Let $p : E \to B$ be a covering map such that $p(e_0) = b_0$. Any path $f : [0, 1] \to B$ with inital point $f(0) = b_0$. THen it has a unique lifting to a path $\tilde{f}$ in E with initial point $\tilde{f}(0) = e_0$

***Proof* :**
(not the proof in Munkre since Munrke uses Lebegue number lemma which we didn't cover. Gonna use Profs proof)

(**Existence**) It is similar to how we showed connectedness. Let M be the largest elemetn of $[0, 1]$ such that $f|_{[0,M]}$ has a lift $\tilde{f}$. Let $\tilde{f}(M) = e$, $f(M) = b$, then $p(e) = b$. Let U be an open set in B containing b such that U is evenly coevered $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$.
Let $\alpha_0 \in I$ be such that $e \in V_{\alpha_0}$. U is open, so $\exists \epsilon > 0$ such that $(M - \epsilon, m + \epsilon) \subseteq f^{-1}(U)$.

Thus, we define $\tilde{f}_{(M-\epsilon,M+\epsilon)}$ to be $i \circ f|_{(M-\epsilon,M+\epsilon)}$ is the where $i : U \to V_{\alpha_0}$ is the canonical isomorphism, so $M = 1$.

(**uniqueness**) in the slides, it's also intuitive (The Nov19th file)

We will now expand this to lift produce spaces:

Lemma 7.2.2: F can be lifted

Let $p : E \to B$ be a covering map, and let $p(e_0) = b_0$. Let $F : I \times I \to B$ be a continuous function such that $F(0,0) = b_0$. Then there exists a unique lift $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof **:**
TUsing lemma 7.2.1, we can lift every path of the homotopy. We need to then show its continuous. To do this, the prof did an interesting proof that is similar to the proof of compactness/connectedness with saying M is the largest value s such that $F(t,s)$ is continuous, and showed that it must be $M + \epsilon$, which lets us conclude that $M = 1$

Nov24th video

And now we use this to be able to lift path homotopies!

Lemma 7.2.3: Path Homotopies can be lifted

If F is a path homotopy, then so is $\tilde{F}$

Proof **:**
IT follows since all lifts of constant paths are constant paths

Finally, we'll show that the endpoints of two path-homotopic functions also agree:

Lemma 7.2.4: Paths can be lifted

Let f, g be paths starting and ending at b_0. Assume $f \simeq_p g$. Then $\tilde{f}(1) = \tilde{g}(1)$

Proof **:**
The path homotopy F between f, g lifts to a path homotopy between $\tilde{f}, \tilde{g}$. Therefore, $\tilde{f}(1) = \tilde{g}(1)$

Note: Note that these are *paths* not loops. It will happen that when projected using the covering space, they will become loops!

With path-homotopies being conserved with liftings, we can now talk about fundamental groups when their lifted:

Definition 7.2.4: Lifting Correspondence

Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be a lifting of f to E such that $\tilde{f}(0) = e_0$.
Then we can define:

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0) \qquad \phi([f]) = \tilde{f}(1)$$

and ϕ is called the ***lifting correspondence*** derived from the corvering map p, e_0.

Note: it is important to choose and e_0 since usually the covering space might have many locally homeomorphic neigbhourhoods, so you have to make a choice. THe choice, however, doesn't really matter so much since there is a local neigbhourhood around e_0 will be homeomorphic to it's image.

Since we'll be basically working with path-connectd space:

Proposition 7.2.3: Lifting Correspondence and Path-Connected Spaces

Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. If E is path-connected, then the lifting correspondence is surjective. If E is also simply-connected, then the lifting correspondence is bijective

Solution Assume E is path-connected. Let $e_1 \in p^{-1}(b_0)$. Let f_E be a path in E such that $f_E(0) = e_0$, $f_E(1) = e_1$. Let $f = p \circ f_E$. Then f_E lifsts f, and

$$f(0) = p(f_E(0)) = p(e_0) = b_0 \tag{7.1}$$
$$f(1) = p(f_E(1)) = p(e_1) = b_0 \tag{7.2}$$

Thus, $\phi(p([f])) = f_E(1) = e_1$, and so ϕ is surjective, as we sougth to show.

Now assume E is simply connected. Nov26th. Proving injectivity. ▼

What we've essentially proven is that we can deduce information of the fundamental group of a space given the covering map!

Example:

1. Let $B = E = S^1$, and take $p : E \to B$ $z \mapsto z^2$, and take $b_0 = e_0 = (1, 0)$. Visually:

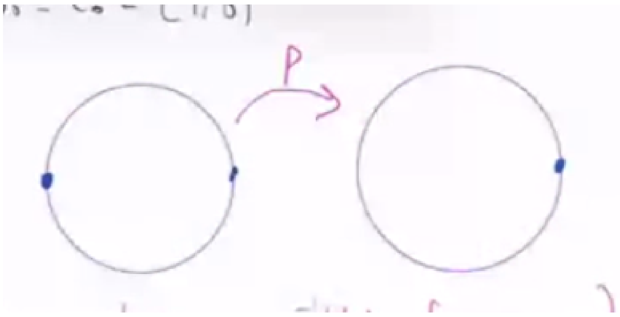

Figure 7.7: visual example of pre-iamge of b_0

Then $\phi : \pi_1(S^1, b_0) \to p^{-1}(b_0) = \{(1,0),(-1,0)\}$.
(Show that it's a homomorphism?)
We'll take two function. Take: $e_{b_0} : [0,1] \to S^1$, $e_{b_0}(t) = b_0 = (1,0)$. Then the lifting correspondence of e_{b_0} is clearly $\tilde{e_{b_0}} = e_{e_0}$. Then:

$$\phi([e_{b_0}]) = e_{e_0}(1) = e_0 = (1,0)$$

now, let:

$$f : I \to S^1, f(t) = (\cos(2\pi t)\sin(2\pi t))$$

which is the funciton which will go around the circle once. Then the lifting correspondence is:

$$\tilde{f} : I \to S^1, \tilde{f}(t) = (\cos(\pi t), \sin(\pi t))$$

So $\tilde{f}(1) = (-1,0)$ so $\phi([f]) = (-1,0)$ (which also verifies surjectivity). So for the first time, we showed that we have a *non-contractibel loop* since the two homotopic classes are different!!! That is $[f] \neq [e_{b_0}]$

2. This time, we'll be a little less formal. take for granted for a moment that we're dealing with a homomorphism. Let X be the figure 8 as we described in a pervious example (from the torus). Let the intersection of the circles be b_0. Take one loop going the left circle counter-clockwise, and oen for the right circle, clockwise, so that $[f], [g] \in \pi_1(X, b_0)$. Let E represent the following lifting space:

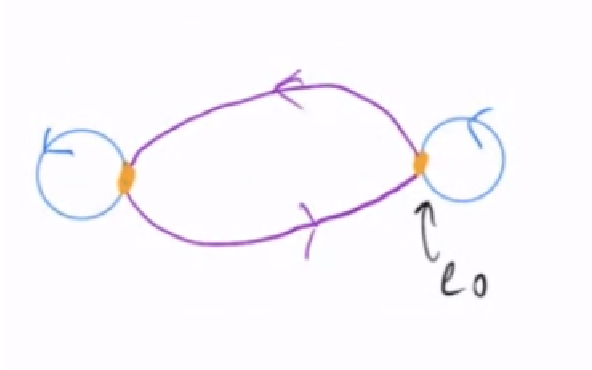

Figure 7.8: Lifting Space

and take e_0 as in the image. Then

$$\phi([f]) = e_1 \tag{7.3}$$
$$\phi([g]) = e_0 \tag{7.4}$$
$$\phi([f] * [g]) = e_1 \tag{7.5}$$
$$\phi([g] * [f]) = e_1 \tag{7.6}$$

okay, this told us htat $[f]$ and $[g]$ are different paths, but it seems that their product is the same. However, remember that ϕ is surjective, *not* injective, so ϕ might've only given us *partial* information on how the elements of this fundamental group interact. We can in fact work wtih a better covering space which will reveal more information:

3. Same X, same b_0, but this time, our lifting space E is different!

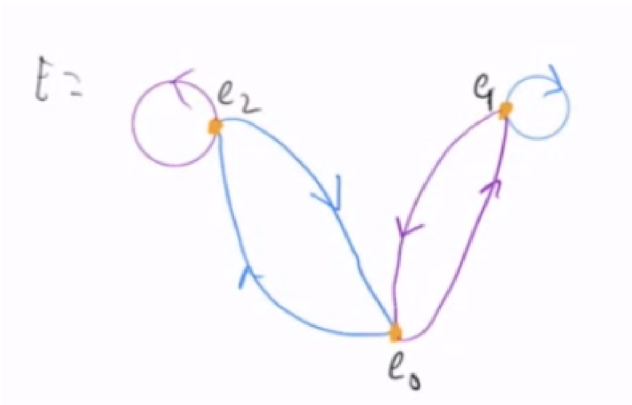

Figure 7.9: Different Lifting Space

Now,

$$\phi([f] * [g]) = e_1 \tag{7.7}$$
$$\phi([g] * [f]) = e_2 \tag{7.8}$$

Therefore, $[f] * [g] \neq [g] * [f]$, i.e. $\pi_1(X, b_0)$ is *not* abelian, as we claimed when $\pi_1(X, x_0)$ was introduced!!!

We'll finally present the fundamental group of the circle. For the long build-up of it, we'll make it a theorem

Theorem 7.2.1: Fundamental Groups Of The Circle

The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$

***Proof* :**
Let $p : \mathbb{R} \to S^1$, and $p(t) = (\cos(2\pi t), \sin(2\pi t))$. Then p is a covering map such that $p(r) = p(s) \Leftrightarrow r - s \in \mathbb{Z}$. Let $b_0 = (1, 0) \in S^1$. Then $p^{-1}(b_0) = \mathbb{Z}$. Let our starting point in E be $e_0 = 0$. Then the lifting corresponding gives a map $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$. Moreover, since $\mathbb{R}$ is simply connected, it follows that ϕ is a bijetion! So, it remains to show that ϕ is a homomrophism:

$$\phi([f] * [g]) = \phi([f]) + \phi([g])$$

Let $\tilde{f}, \tilde{g}$ be lifts of f, g starting at 0, so $\phi([f]) = \tilde{f}(1) \in \mathbb{Z} = p^{-1}(b_0)$. Define $\tilde{g}_2(t) = \tilde{g}(t) + \tilde{f}(1)$. Then

$$p(\tilde{g}_2(t)) = p(\tilde{g}(t) + \tilde{f}(1)) = p(\tilde{g}(t) + k) = p(\tilde{g}(t)) = g(t)$$

since $k \in \mathbb{Z}$, so the covering map projects this path onto g, meaning it is also a lift of g, just shifted over by $f(t)$ to where the path starts. We can see this by $\tilde{g}_2(0) = \tilde{f}(1)$. Thus, we can combine these two paths, $\tilde{f} * \tilde{g}_2$, and by what we've just shown, this is a lifts of $f * g$. Thus

$$\begin{aligned}
\phi([f] * [g]) &= (\tilde{f} * \tilde{g}_2))(1) \\
&= \tilde{g}_2(1) \\
&= \tilde{g}(1) = \tilde{f}(1) = \phi([f]) + \phi([g])
\end{aligned}$$

completeing the proof! Visually, this is what we just did with g_2:

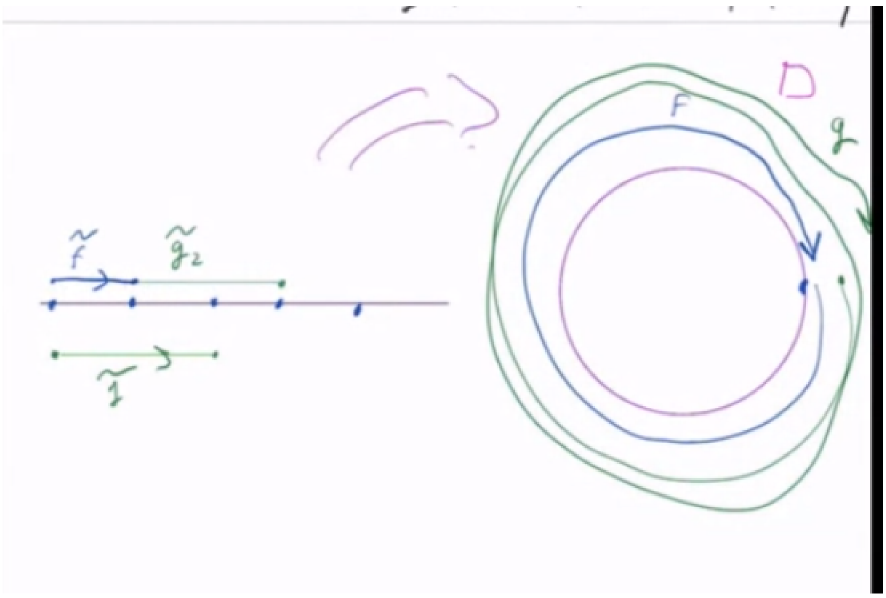

Figure 7.10: showing the logic for the lifting

This also means that the fundamental group of the circle is abelian, which should makes sense if you try and draw loops on a circle. You will get back and forth and will be able to reverse the order no problem. Trying this on the figure 8, you might see how you can get stuck. We just formalized such intuition!

There's one more theorem worth covering with cosets p.346.

7.2.4 Fundamental Group of finite space

I also found this interesting enough to create a section for it. Will fill in the details later:

`https://en.wikipedia.org/wiki/Pseudocircle`

7.2.5 Further explorations

`http://mathonline.wikidot.com/list-of-fundamental-groups-of-common-spaces`

7.3 Retractions and Contractions

We will not elaborate on important results that arrive as a consequence of the fundametnal group of a cricle. The eventuall conclusion of this is a theorem called *Browers Fixed point Theorem* which we will use to cool effect.

Retractions were introduced when we talked about quotient spaces as a good exampe of quotient maps. The relation between retraction maps and fundametnal groups of spaces their on is extremely intuitive, as we'll see, making them an excellent tool to analyze the fundamental group of spaces:

Definition 7.3.1: Retraction

If $A \subseteq X$, a ***retraction*** of X onto A is a continuous map $f : X \to A$ such that $r|_A$ is the identity. The supspace is the *retract* of X

Example:

1. A is a point
2. If $X = Y \times Z$, and $A = Y \times \{z_0\}$, then $f(y,z) = (y, z_0)$ is a retraction.
3. if A is an annulus, and we have a circle in the annulus, then pinching everything to A is a retraction.
4. $X = \mathbb{R}$, $A = \mathbb{R}_{\geq 0}$, then $\sigma(t) = |t|$ is a retraction

Lemma 7.3.1: induced maps and retractions

If A is a retract of X. for $a_0 \in A$, the induced map $j_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective, where $j : A \to X$ is the inclusion

This means that X should have *at least* as many elements in it's fundamental group as A

Proof **:**
Let $r : X \to A$ be a retraction. Then $f \circ j = id_A$. Thus, $(r \circ j)_* = (id_A)_* = id|_{\pi_1(A,a_0)}$. Thus

$$[f] = (r_* \circ j_*)([f]) = r_*(j_*([f]))$$

showing r_* is a left inverse, thus j_* is injective. Visually,

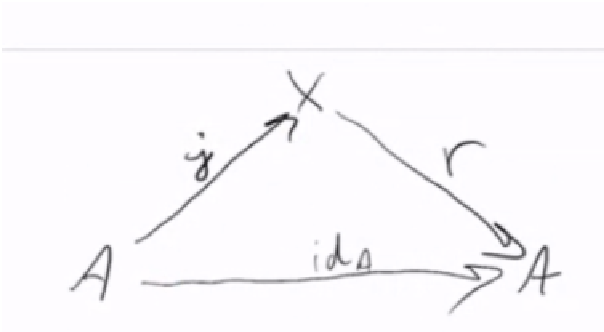

Figure 7.11: visual

We can finally prove a result you maybe heard jokes about all over the place:

Theorem 7.3.1: Can't make a whole

Let B^2 be the unit disc in $\mathbb{R}^2$. There is no retraction form B^2 to S^1. You can think of this as you cannot make a whole.

Proof **:**
The fundamental group of the circle is infinite, but the fundamental group of B^2 has 1 element (because its star-convex). Since the fundamental group of S_1 would need to injectively map to B^2, this implies there is *no* retraction!

7.4 Browser's fixed point theorem

Theorem 7.4.1: Brouwer Fixed Point Theorem

if $f : B^2 \to B^2$ is continuous, then there exists $x \in B^2$ such that $f(x) = x$

Proof **:**
Recall that $B^2 := \{(x, y)|x^2 + y^2 \leq 1\}$ and thus is $\approx I^2$.

Assume $f : B^2 \to B^2$ is continuous. For the sake of contradiction, let's say that for every $x \in B^2$, $f(x) \neq x$. We will use f to construct a retraction of B^2 to its boundary S^1.

The key idea is that since $f(x) \neq x$, and both are in B^2, we can draw a straiht line between x and $f(x)$. We can continue this line to the boundary. If we give a direction (let's say from $f(x)$ to x), we can then make a function $g(x)$ which will take our point x, and given this line, it will bring it to the boundary.

The key idea here is that if we had that $f(x) = x$, then we woouldn't know what ray to draw. for example, if $f(x) = \frac{x}{\|x\|}$ then every point will have a clear ray exept $x = 0$. This unclearness of the ray is translated as f being discontinuous (in the case of the concrete f, we divided by 0).

More formally, we'll define $G : B^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}^2$.

$$G(x, r) = x + r(x - f(x))$$

since Everything is in $\mathbb{R}^2$, mutiplying, adding, subtracting is allowed. Note that

$$\|G(x, r)\|^2 = \|x\|^2 + r^2 \|x - f(x)\|^2 + 2rx \cdot (x - f(x))$$

so for $\|G(x, r)\| = 1$, we use the quadratic formula to solve r:

$$r = \frac{-2x \cdot (x - f(x)) + \sqrt{4(x \cdot (x - f(x)))^2 + 4(1 - \|x\|^2) + \|x - f(x)\|^2}}{2 \|x - f(x)\|^2}$$

This formula is complicated, but it's okay: everything is positive, so everyhing is okay (since f has no fixed points). We can make r into a formula which will make sure that $G(x, r) = 1$ by calling the right hand side $r(x)$.

THus, we can define

$$g(x) = G(x, r(x)) \in S^1$$

and by 1.2.2, $g(x)$ is continuous. Note that one the circle, $g(x) = x$, showing that it's a retraction. But we showed there can't be a retraction between the ball and the circle – a contradiction!

Appendix A

Topological Groups

I'm not sure yet if I want this to be an entire chapter, probably if I find a lot of cool things about it

A.1 Basic Properties And Definitions

Definition A.1.1: definition name

A **topological group** G is a group that is also a topological space satisfying the T_1 axiom, such that the map $G \times G$ into G sends $x \times y$ into $x \cdot y$, and the map of G into G sending x intox^{-1} are continuous maps. Throughout the following exercices, let G denote the topological group

Example:

1. $(\mathbb{Z}, +)$ is a topological group with $\mathbb{Z}$ having the discrete topology.
2. $(\mathbb{R}, +)$ is a topological group with $\mathbb{R}$ having the euclidean topology. Let's check $+$ is continuous. We want to show that $+^{-1((a,b))}$ is open, that is, $+^{-1} = \{x, y \in \mathbb{R} | x + y \in (a, b)\}$. Clearly, the pre-image of this set is $\bigcup_{i \in (a,b)} \bigcup_{x \in \mathbb{R}} (x - i, x + i)$, which is the union of open sets, meaning the pre-iamge is open. furthermore, the inverse function is also open since if (a, b) is an open set in the co-domain, $(-b, -a)$ is the pre-image.

Interesting things with how to relate subgroups and subspaces to form topological subroups

I way of defining homogeniety

A way of relating G/H with the quotient topology