University of Waterloo
Department of Electrical and Computer Engineering

ECE 380, Analog Control Systems
Final Examination
December 8, 2005

Professor J.G. Thistle

Instructions:

- Time allowed: 2.5 hours.
- No aids allowed other than a nonprogrammable calculator.
- The exam comprises 3 questions with a total value of 100 points; answer all of them.
- Two sheets of semilog graph paper are provided for Question 3; if you use them, label them with your name and student number and put them inside your answer booklet.

The following formulas may be of use:

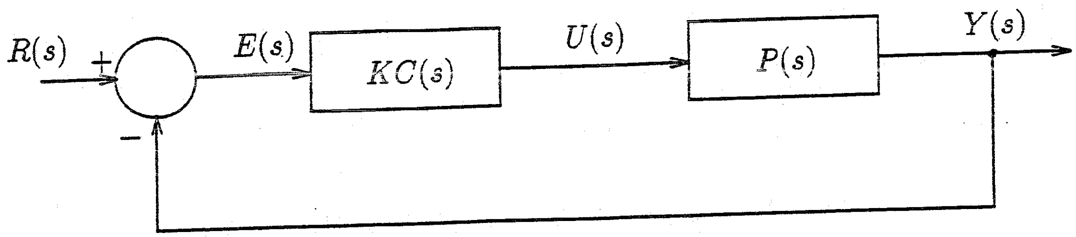
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta) \; ; \; \theta = \mathrm{Cos}^{-1}\zeta$$

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t)$$

$$\omega_{\max} = \frac{1}{\sqrt{\alpha}\,\tau}$$

$$\angle C(j\omega_{\max}) = \mathrm{Sin}^{-1}\frac{\alpha - 1}{\alpha + 1}$$

All questions relate to the following system:



Question No. 1 (35 points)

(a) **(15 points)** Sketch the root locus for the case where

$$C(s)P(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

Compute intersection points of branches, and show all of your calculations. For what values of K is the feedback system stable? Explain your answer.

(b) **(20 points)** Sketch the Nyquist plot for the case where

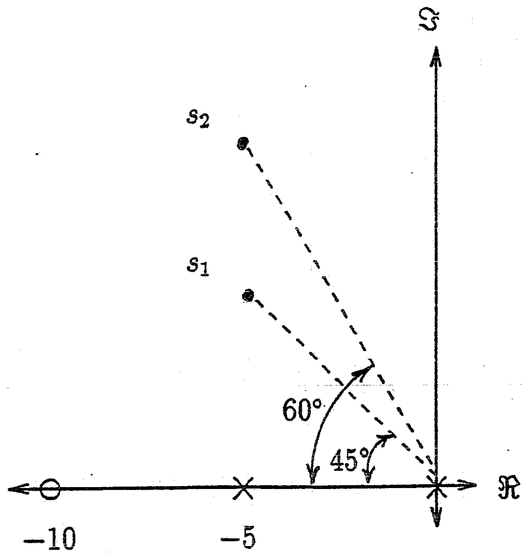
$$C(s)P(s) = \frac{s+1}{s^2(s+9)}$$

For what values of K does the system have a gain margin of 10 db or more? Explain your answer. For what value of K is the phase margin the largest? What is this maximum phase margin?

– See solutions to assgt. 6.

Question No. 2 (35 points)

~~Let $KC(s)$ be an ideal PD controller and let $P(s)$ model a second-order plant.~~ Let the two poles of $C(s)P(s)$ be 0 and -5, and let its only zero be -10:



(a) **(5 points)** Show that the point s_1 lies on the root locus.

(b) **(20 points)** Suppose that s_1 is a pole of the closed-loop system. Find the step response of the closed-loop system, giving numerical values of all constants appearing in your expression.

(c) **(5 points)** We wish to convert the controller into a lead compensator by adding a pole. Where should this pole be situated in order that s_2 lie on the root locus?

(d) **(5 points)** Suppose we add in series with the controller a system with transfer function

$$\frac{s+0.1}{s-p}$$

Where should we position the pole p in order that the velocity error constant be increased by a factor of 5? Explain.

2. (a)

$$\angle C(s_1)P(s_1) = \angle(s_1 - (-10)) - \angle(s_1 - (-5)) - \angle(s_1 - 0)$$

$$= 45^\circ - 90^\circ - 135^\circ$$

$$= -180^\circ$$

so, the angle relation is satisfied, and s_1 lies on the root locus.

(b) First, we can use the amplitude relation to find the appropriate gain:

$$K = \frac{1}{|G(s_1)P(s_1)|} = \frac{|s_1 - (-5)|\,|s_1 - 0|}{|s_1 - (-10)|}$$

$$= \frac{5 \cdot 5\sqrt{2}}{5\sqrt{2}}$$

$$= 5$$

$$\therefore KC(s)P(s) = 5\frac{s+10}{s(s+5)}$$

Hence

$$\frac{Y(s)}{R(s)} = \frac{KC(s)P(s)}{1+KC(s)P(s)} = \frac{5(s+10)}{(s(s+5)) + 5(s+10)}$$

$$= \frac{5(s+10)}{s^2 + 10s + 50}$$

i.e. $$\frac{Y(s)}{R(s)} = \frac{50}{s^2+10s+50}\left(1 + \frac{s}{10}\right)$$

$$\zeta\omega_n = 10, \quad \omega_n = \sqrt{50} \Rightarrow \zeta = \frac{1}{\sqrt{2}}$$

$\therefore$, its step response is

$$y(t) = 1 - \sqrt{2}\, e^{-10t} \sin\left(5t + \frac{\pi}{4}\right) + \frac{1}{10} \cdot 10\, e^{-10t} \sin 5t \qquad (t \geq 0$$

(c) $\angle C(s_2) P(s_2)$

$$= \angle(s_2 - (-10)) - \angle(s_2 - (-5)) - \angle(s_2 - 0) - \angle(s_2 - p),$$

where p is the compensator pole.

Hence, $\angle C(s_2) P(s_2) = 60° - 90° - 120° - \angle(s_2 - p)$

$$= -150° - \angle(s_2 - p)$$

We should therefore arrange for $\angle(s_2 - p)$ to be $30°$. Do some trigonometry —

e.g., law of sines:

$$\frac{\sqrt{3}/2\ \cancel{\sin 60°}}{-5 - p} = \frac{\cancel{\sin 30°}\ 1/2}{|s_2 - (-5)|}$$

& $$\frac{\sqrt{3}/2\ \cancel{\sin 60°}}{|s_2 - (-5)|} = \frac{\cancel{\sin 30°}\ \frac{1}{2}}{5}$$

So $-5 - p = 3.15$

$\Rightarrow \quad p = -20$

2(d) To increase K_v by a factor of 5, the ratio between the zero and pole of the lag compensator ~~should~~ must be ≥ 5. So position the pole at -0.02, so as not to shift the closed-loop poles more than necessary.

Question No. 3 (30 points)

Let

$$P(s) = \frac{10}{s(s0.100 + 1)}.$$

(a) (20 points) Design a controller $KC(s)$, consisting of a gain K and a lead compensator $C(s) = \frac{\alpha\tau s + 1}{\tau s + 1}$, such that the phase margin is 60° and the steady-state error resulting from a unit-ramp input $r(t) = tu_{-1}(t)$ is 0.10.

(b) (10 points) Repeat part (a), with a lag compensator in place of the lead compensator.

3 (a) For steady-state error spec., need

$$\frac{1}{K_v} \leq 0.10 \iff K_v \geq 10$$

w/o compensation, we have

$$\lim_{s \to 0} s\,P(s) = 10,$$ so we

can use a gain of 1 for K

→ Draw Bode plot of $K\,P(j\omega)$

– uncompensated PM = 45°

→ need to add 15°

→ try for 20° of additional PM

→ Set $\angle C(j\omega_{max}) = \sin^{-1}\frac{\alpha-1}{\alpha+1} = 20°$

$$\frac{\alpha-1}{\alpha+1} = \sin 20° = 0.342$$

$$\iff 0.658\,\alpha = 1.342$$

$$\iff \alpha = 2.0$$

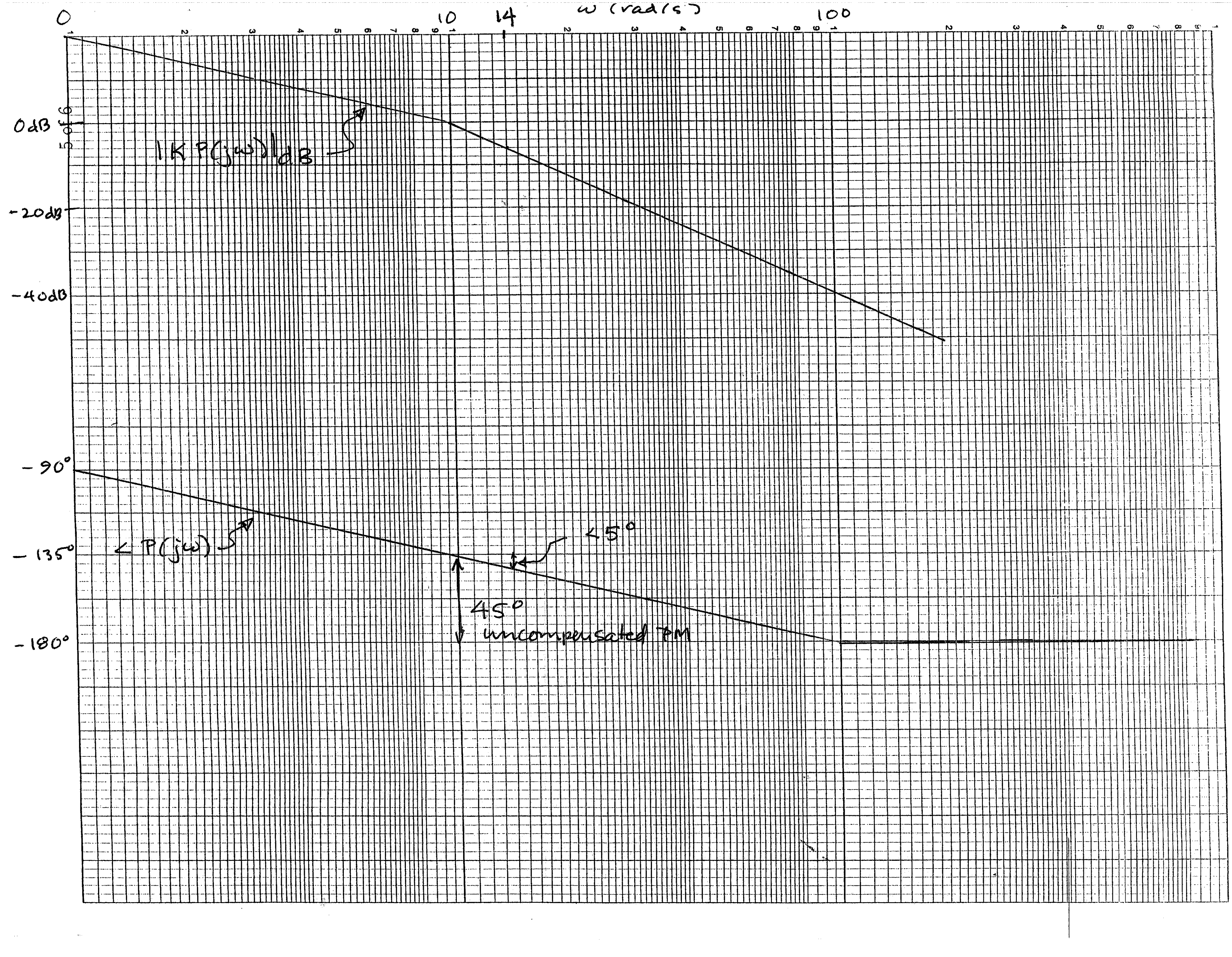
Now $10\log_{10}\alpha = 10\log_{10}2.0 = 6$ dB,

so, by the Bode plot, we should set

$\omega_{max} = 14\ s^{-1}$, so $\frac{1}{\sqrt{\alpha}\,\tau} = 14 s^{-1} \iff \tau = \frac{1}{\sqrt{2}\,14}$ s

At this frequency, the uncompensated phase is $> -140°$, so the resulting compensator,

$$1 \cdot \frac{\frac{\sqrt{2}}{14}s + 1}{\frac{1}{\sqrt{2}\,14}s + 1}$$ solves the control problem.



ω (rad/s)
0
10
14
100
0dB
-20dB
-40dB
|K P(jω)|dB
-90°
-135°
-180°
∠P(jω)
∠5°
45°
uncompensated PM

(Again, we can set $K = 1$.)

3 (b) The uncompensated phase is $-120°$ at about $4.3\ s^{-1}$.

At this frequency, the uncompensated gain is about 6.6 dB, so set

$$20 \log \alpha = -6.6 \text{ dB}$$

$$\rightarrow \quad \alpha \sim 1/2$$

Now, the upper corner frequency of the compensator phase plot should be (at most) $4.3\ s^{-1}$, so set

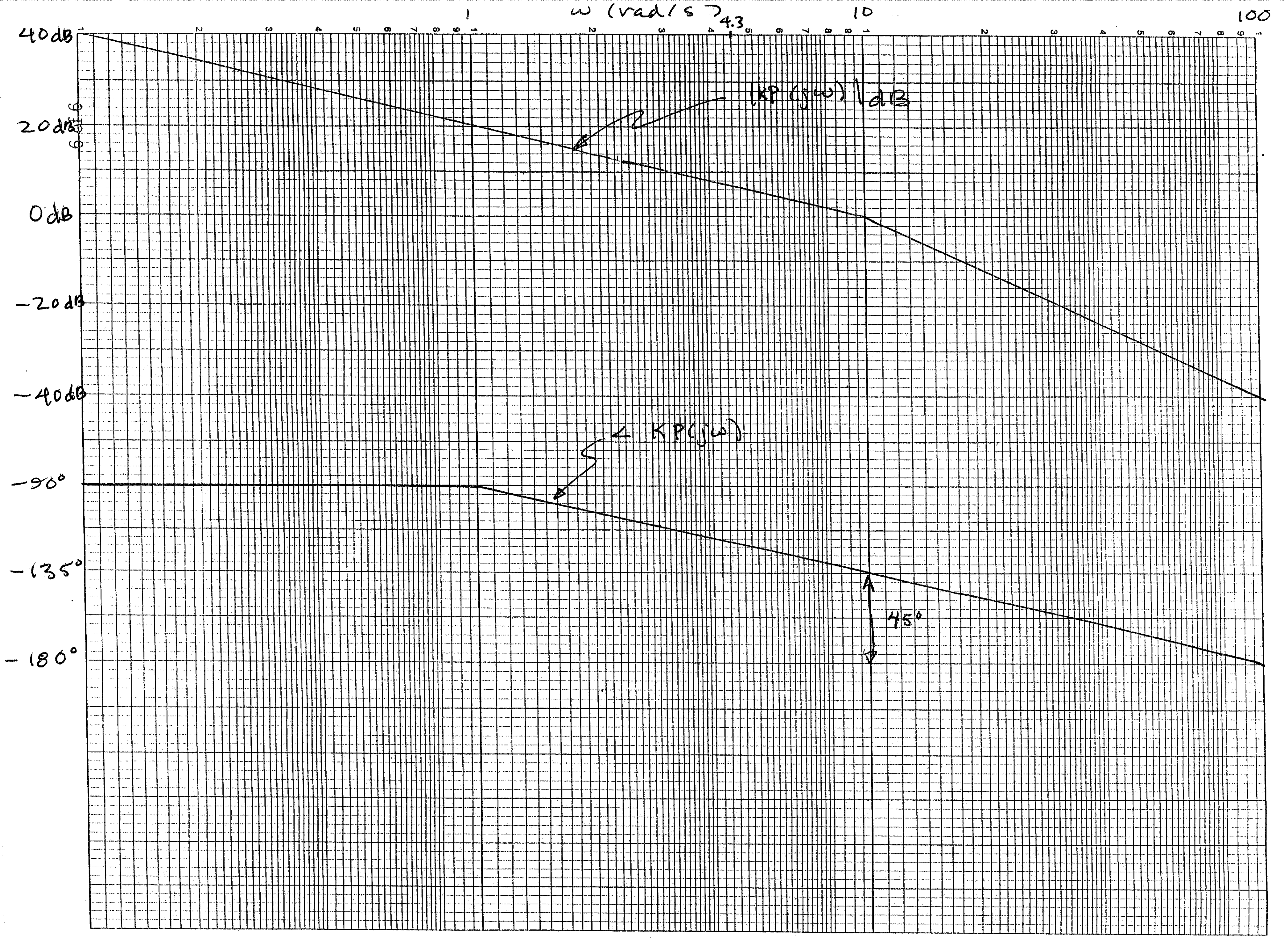
$$\frac{10}{\alpha \tau} = 4.3\ s^{-1}$$

$$\leftrightarrow \quad \tau = \frac{20}{4.3} \sim 4.7\ s$$

$\therefore$ a lag compensator of the form

$$\frac{2.35\, s + 1}{4.7\, s + 1}$$

solves the control problem.



ω (rad/s)
1
4.3
10
100
40dB
20dB
0dB
-20dB
-40dB
-90°
-135°
-180°
|KP(jω)|dB
∠ KP(jω)
45°