## CS 360 Introduction to the Theory of Computing

# Midterm Practice Problems

1. Decide whether each the following statements is true or false, and defend your answer: give a short proof for each statement you believe is true, and give a counter-example to each statement you believe is false.

   (a) If $A$ is a finite language, then $A$ is context-free.

   (b) If $A$ and $B$ are regular languages, then there exists a context-free grammar for the language $\left(\overline{A} \cup \overline{B}\right)^*$.

   (c) For languages $A$ and $B$, if $A$ is regular and $B \subseteq A$, then $B$ is regular.

   (d) For every unambiguous context-free grammar $G$ over the alphabet $\{0,1\}$, there exists an ambiguous context-free grammar $H$ over the alphabet $\{0,1\}$ such that $\mathrm{L}(H) = \mathrm{L}(G)$.

2. Prove that the following two languages are regular.

   (a) The first language is

   $$A = \left\{ w \in \{0,1,2,3\}^* \;:\; \begin{array}{l} \text{consecutive symbols in } w \text{ never} \\ \text{differ in value by more than 1} \end{array} \right\}.$$

   For example, 0100123210, 1001210010, 321101, and $\varepsilon$ are strings in $A$, but 01200210, 132, and 30121210 are not.

   (b) The second language is

   $$B = \left\{ w \in \{0,1,2\}^* \;:\; \begin{array}{l} \text{every 2 in } w \text{ is immediately preceded and} \\ \text{immediately followed by two 0s} \end{array} \right\}.$$

   For example, 0100200200, 1002001100200, 1101, and $\varepsilon$ are strings in $B$, but 01200200, 2, and 002200 are not.

3. Define

   $$A = \left\{ u1v \;:\; u,v \in \{0,1\}^*,\ |u| = |v| \right\}.$$

   In words, $A$ is the language consisting of all strings over the alphabet $\{0,1\}$ that have odd length, and have a 1 in the middle position.

   (a) Prove that $A$ is a context-free language.

   (b) Prove that $\overline{A}$ is a context-free language.

   (c) Prove that $A$ is not a regular language.

4. Consider the language $A$ over the alphabet $\Sigma = \{0, 1\}$ defined as follows:

$$A = \{x \in \Sigma^* \; : \; x \text{ has exactly 100 times as many 0s as 1s}\}.$$

Prove that $A$ is context-free.

(If your proof includes the description of a CFG or PDA, you do not need to prove the correctness of the CFG or PDA.)

5. Recall that a context-free grammar $G$ over an alphabet $\Sigma$ is in *Chomsky Normal Form* if each of its rules has one of the following forms:

   (a) $S \to \varepsilon$ (where $S$ is the start variable)
   (b) $X \to YZ$ ($X$, $Y$ and $Z$ are variables, neither $Y$ nor $Z$ is the start variable)
   (c) $X \to \sigma$ (where $X$ is a variable and $\sigma \in \Sigma$)

   Let $\Sigma = \{0, 1\}$, suppose that $A \subseteq \Sigma^*$ is a context-free language, and assume $G$ is a context-free grammar for $A$ that is in Chomsky Normal Form.

   Describe how you could obtain from $G$ a new grammar $H$ (not necessarily in Chomsky Normal Form) for the language

$$B = \{x \in \Sigma^* \; : \; xy \in A \text{ for some } y \in \Sigma^*\}.$$

   You do not need to prove that the grammar $H$ generates $B$—just describe how to obtain $H$ from $G$.

6. Let $\Sigma = \{0, 1, 2\}$ and define a language $A \subseteq \{0, 1, 2\}^*$ as

$$A = \big\{r2s2t \; : \; r, s, t \in \{0, 1\}^*, \; |r| = |s| = |t|\big\}.$$

   Prove that $A$ is not context-free.

7. Prove that for every alphabet $\Sigma$ and every regular language $A \subseteq \Sigma^*$, the language

$$B = \Big\{ww^R \; : \; w \in A\Big\}$$

   is context-free.

8. Suppose $\Sigma = \{0, 1\}$ and $A \subseteq \Sigma^*$ is a regular language. Define

$$B = \big\{xy \; : \; x, y \in \Sigma^*, \; yx \in A\big\}.$$

   Prove that $B$ is regular.