STAT 231

Tutorial.

October 26, 2016

Roadmap

- The Chi-Squared distribution and its properties

- Practice Midterm

The Chi-Squared

* If $W \sim \chi^2_n$, $n = df$ = degrees of freedom,

$$W = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

Z_i's independent. $Z_i \sim N(0,1)$

* W takes values $(0, \infty)$

* $E(W) =$ degrees of freedom $= k$

  $V(W) = 2 \times$ degrees of freedom $= 2k$

* Moment generating function of a χ^2 distribution with $df = k$

$$\Rightarrow \boxed{M(t) = (1-2t)^{-k/2}}$$

Some Special Cases

* $df = 1 \implies W = Z^2$
* $df = 2 \implies W \sim \underline{\underline{Exp}}(2)$
* $df = \text{large} \implies W \sim G(k, \sqrt{2k})$ approximately
* $df \in (2, \text{large}) \implies$ Use the table

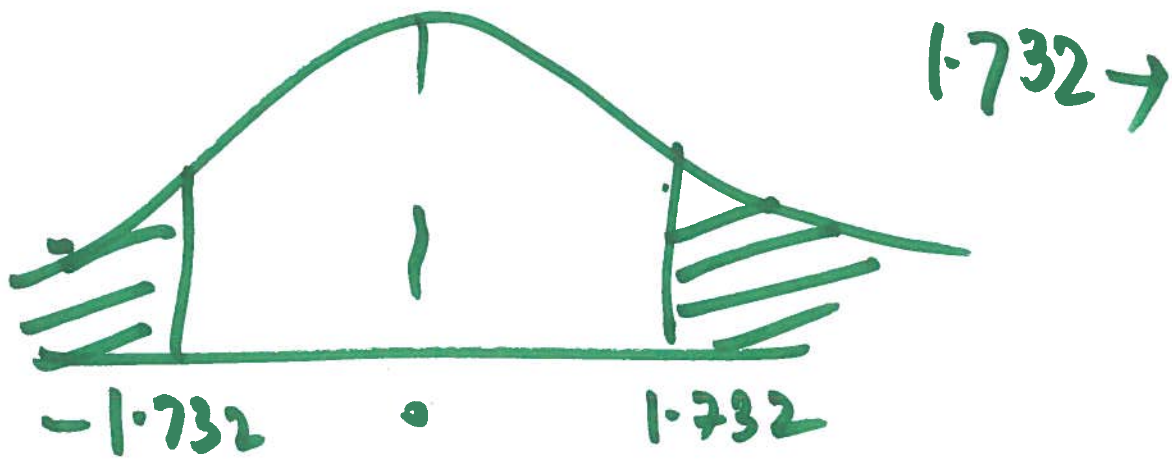
(a)(i)

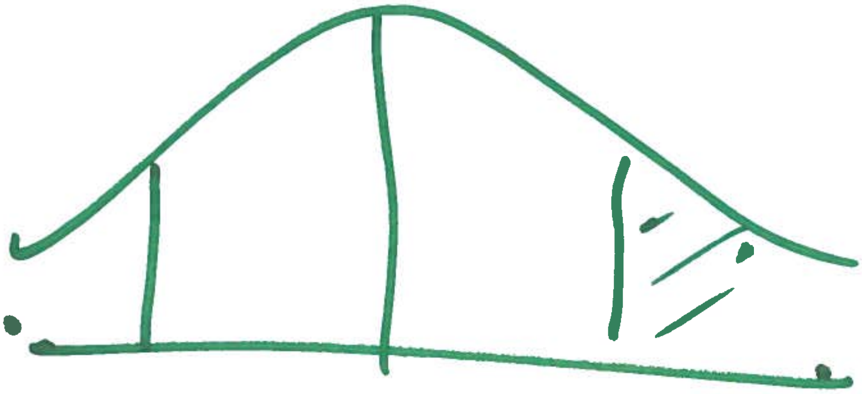
$X \sim \chi^2_1$, find $P(X > 3)$

$P(X > 3)$

$= P(Z^2 > 3)$ 1.732

$\Rightarrow P(Z > \sqrt{3}) + P(Z < -\sqrt{3})$





$1 - 0.95811 = 0.04189$

Prob $= 2 \times 0.04189 =$

If $X \sim \chi^2_2$, find $\underline{\underline{P(X>3)}} = ?$

$$\frac{1}{\mu} e^{-x/\mu}$$

Density function $= \dfrac{1}{2} e^{-\frac{x}{2}}$

$$\int_3^\infty \frac{1}{2} e^{-x/2}\, dx = e^{-3/2}$$

$\parallel$

Check!

If $X \sim \chi^2_{79}$, find $P(X>100)$.

If n is large,

$$X \underset{app}{\sim} G\Big(\underbrace{79}_{\mu}, \underbrace{\overset{?}{\sqrt{158}}}_{s.d.}\Big)$$

$$P(X > 100)$$

$$= P\left(\frac{X-79}{\sqrt{158}} > \overset{a}{\overset{\parallel}{\frac{100-79}{\sqrt{158}}}}\right)$$

$$= P(Z > \text{''})$$

Use the χ^2 table.

$X \sim \chi^2_9$; $P(X \le 5.4)$

Row = df

Entries → Quantiles

Row 40 Column = 0.7

Value = 38.859

$P(W \le 38.859) = 0.7$

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Examination
Test 2
Winter 2014
STAT 231

Times: Monday 2014-02-24 at 16:30 to 17:20 (4:30 to 5:20PM)
Duration: 50 minutes
Exam ID: 2709029
Sections: STAT 231 LEC 001,002,003,004
Instructors: Cyntha Struthers, Ilham Akhundov, Peisong Han, Suryapratim Banerjee

Special Materials

Candidates may bring only the listed aids.

- Calculator - Pink Tie

Normal and Chi-squared tables provided separately

Do not write on tables.

ID Number: ______________________

NAME (Please Print): ______________________

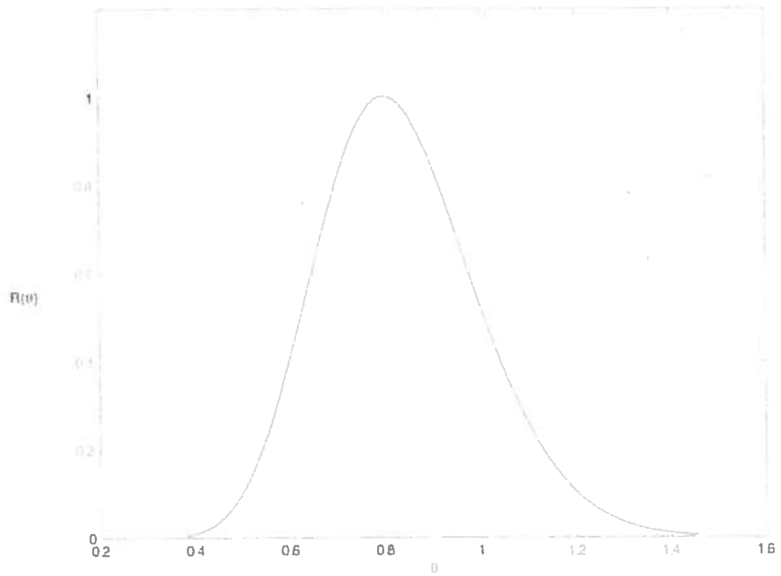
Signature: ______________________

Question	Mark	Maximum Mark	Marker Initials
1		5	
2		5	
3		3	
4		9	
5		3	
Total		25	

1. [5] Suppose $y_1, y_2, \ldots y_{30}$ are the observed values in a random sample from the Poisson distribution with probability function

$$P(Y = y; \theta) = \frac{\theta^y e^{-\theta}}{y!} \quad \text{for } y = 0, 1, \ldots \text{ and } \theta > 0$$

Suppose $\sum_{i=1}^{30} y_i = 24$. The graph of $R(\theta)$, the relative likelihood function of θ, for these data is given below:



Write your answer only in the space provided.

(a) The maximum likelihood estimate of θ is 0·8.

(b) An estimate of $p = P(Y \leq 1; \theta)$ is $\hat{p} =$ 0·809.

(c) The value $\theta = 0.9$ lies inside a 50% likelihood interval (True/False) True.

(d) The value $\theta = 1.2$ lies inside a 10% likelihood interval but outside a 15% likelihood interval

(True/False) TRUE.

(e) If $r(\theta_1) = \log R(\theta_1) = -3$ then $\theta = \theta_1$ is an implausible value of θ in light of the data

(True/False) TRUE.

4. [9] **Write your final answer only in the space provided.**

(*a*) **Without using Chi-squared tables** determine the following:

(*i*) If $X \backsim \chi^2(1)$ then $P(X > 3) =$ 0.084.

(*ii*) If $X \backsim \chi^2(2)$ then $P(X > 3) =$ 0.223.

(*iii*) If $X \backsim \chi^2(79)$ then the approximate value of $P(X > 100) =$ 0.0475

(*b*) **Using Chi-squared tables** determine the following:

(*i*) If $X \backsim \chi^2(9)$ then $P(X \leq 5.4) =$ 0.2.

(*ii*) If $X \backsim \chi^2(19)$ then $P(X > 30.1) =$ 0.05.

(*iii*) If $X \backsim \chi^2(7)$ then the value of a such that $P(X \leq a) = 0.025$ is $a =$ 1.69.

(*iv*) If $X \backsim \chi^2(7)$ then the value of b such that $P(X > b) = 0.025$ is $b =$ 16.01.

(*c*) For the following questions specify **the distribution and its parameters(s)**:

(*ii*) If $X \backsim G(2,3)$ and $Y \backsim \chi^2(5)$ independently then the distribution of $W = Y + \left(\frac{X-2}{3}\right)^2$ is χ^2_6.

(*ii*) If $X_i \backsim \chi^2(i)$, $i = 1, 2, \ldots, n$ independently then the distribution of $\sum_{i=1}^{n} X_i$ is $n(n+1)/2$.

5. [3] If the moment generating function of X is $M(t) = (1 - 2t)^{-15}$ for $t < 1/2$ then

(*a*) $E(X) =$ 30.

(*b*) $E(X^2) =$ 960.

(*c*) $Var(X) =$ 60

2. [5] In a large population a proportion θ of people have a certain characteristic. In a sample of $n = 180$ people chosen at random from this population there were 60 people with the characteristic.

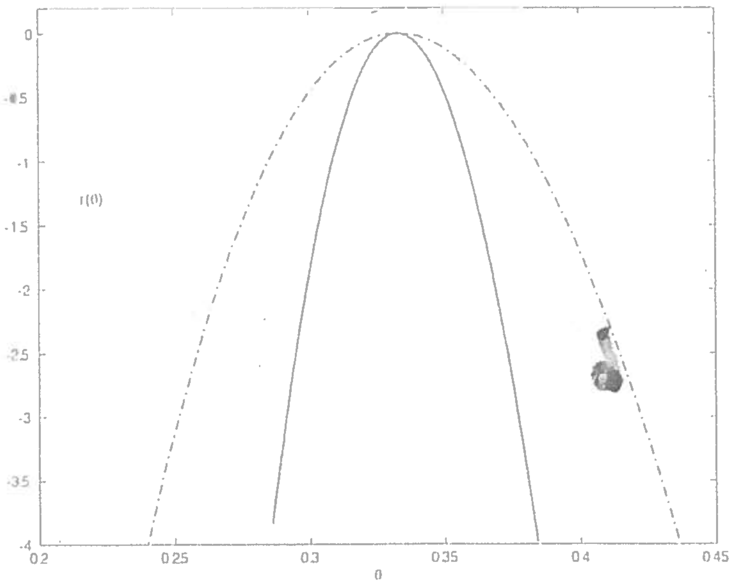
Write your answer only in the space provided.

(*a*) The maximum likelihood estimate of θ is $\hat{\theta} = 1/3$.

(*b*) If $R(\theta)$ is the relative likelihood function of θ for these data then $R(0.4) =$ 0·182.

(*c*) $\theta = 0.4$ is inside a 15% likelihood interval for θ (True/False) TRUE.

(*d*) Another sample of $n = 720$ people was taken and 240 people were observed to have the characteristic. The log relative likelihood functions $r(\theta) = \log R(\theta)$ for $n = 180$ and $n = 720$ are plotted on the graph below. On the graph clearly indicate which curve corresponds to $n = 180$ and which curve correponds to $n = 720$.



Dotted line
↓
n = 180

3. [3] Let

$$g(\theta) = \theta^a e^{-b/\theta} \quad \text{for } \theta > 0$$

The value of θ (in terms of a and b) which maximizes $g(\theta)$ is $\theta =$ $\boxed{\hat{\theta} = -b/a}$.

If $X \sim \chi^2_{19}$, then

$P(X > \underline{30.1}) = ?\ 1 - 0.95$

$= 0.05$

If $X \sim \chi^2_{7}$, find a such that

$P(X \leq a) = 0.025$

Row : 7 Column : 0.025

$a = 1.69$

If $X \sim N(2,3)$

$Y \sim \chi^2_5$ $\quad Z^2 = \chi^2_1$

$$W = \underset{\chi^2_5}{Y} + \left(\frac{X-2}{3}\right)^2 \sim \chi^2_6$$

What does W follow?

If $W_1 \sim \chi^2_{n_1}$, $W_2 \sim \chi^2_{n_2}$

and W_1 and W_2 are indep

$$W_1 + W_2 \sim \chi^2_{n_1+n_2}$$

$X \sim \chi^2_{7}$, find b

s.t $P(X > b) = 0.025$

Row = 7

Column = ~~Colum~~ 0.975

$$\boxed{b = 16.013}$$

If $X_i \sim \chi^2_{(i)}$ $i = 1, \ldots n.$

indep

$Y = \sum X_i \sim ?$

$Y \sim \chi^2_{1+2+3+\ldots+n}$

$= \chi^2_{\frac{n(n+1)}{2}}$

Moment generating function

$$M(t) = (1-2t)^{-15} \rightarrow \chi^2_{30}$$

$E(x) = 30$ (df)

$E(x^2)$

$V(x) = 60$ ($2 \times$ df)

$$V(x) = E(x^2) - (E(x))^2$$

$60 = " - 30^2$

$$P(Y=y) = \frac{\theta^{y} e^{-\theta}}{y!}$$

$n = 30$

$(y_1, \ldots, y_{30})$ $\sum y_i = 24$

SAMPLE

(a) $\hat{\theta} = \text{MLE} = 24/30 = 0.8$

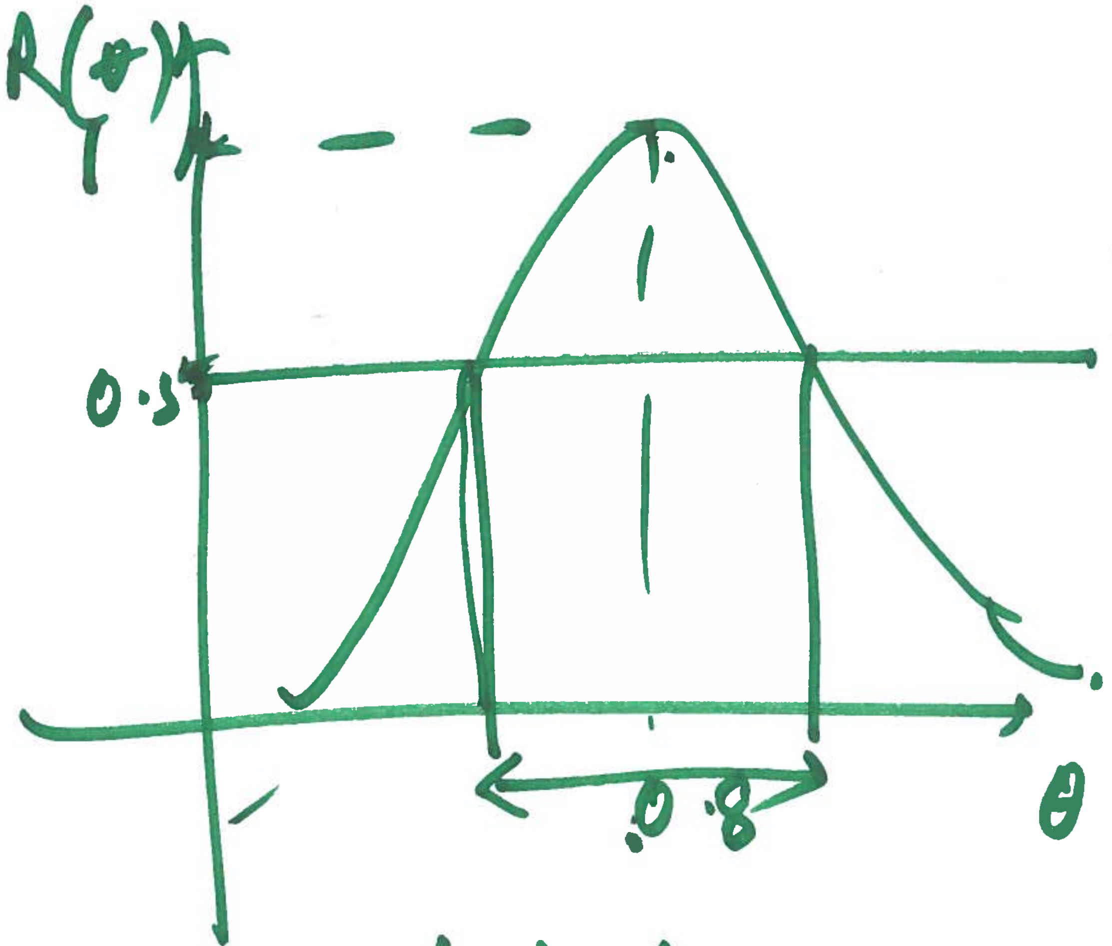
(b) MLE for $P(Y \leq 1) = ?$

$$P(Y \leq 1) = P(Y=1) + P(Y=0)$$

$$\boxed{= \frac{e^{-\theta}\theta^{1}}{1!} + \frac{e^{-\theta}\theta^{0}}{0!}}$$

MLE

$$\boxed{= e^{-0.8} \times 0.8 + e^{-0.8}}$$



$$R(\theta) = L(\theta) / L(\hat{\theta})$$

The value $\theta = 0.9$ lies on the 50% likelihood interval

TRUE OR FALSE

Draw the horizontal line 0·5 (on the y-axis) to get the 50% likelihood interval and check whether $\theta = 0{\cdot}9$ belongs to this interval.

If $r(\theta_1) = -3$, then θ_1 is IMPLAUSIBLE (TRUE OR FALSE?)

$$R(\theta) = L(\theta) / L(\hat{\theta})$$

$$r(\theta) = \log R(\theta).$$

To check whether some value of θ is PLAUSIBLE, we check whether

$$R(\theta) \geq 0.1$$

$$r(\theta_1) = -3 \Rightarrow R(\theta) = \underline{\underline{e^{-3}}}$$

2

Binomial (n, θ)

$n = 180$; $y = 60$.

(a) What is $\hat{\theta} = ?$ $60/180 = 0.\dot{3} = 1/3$

(b) $R(0.4)$.

(c) $\theta = 0.4$ fall in the 15%

likelihood interval ?

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})}$$

$$= \frac{\cancel{{}^{180}C_{60}}\ \theta^{60}(1-\theta)^{120}}{\cancel{{}^{180}C_{60}}\ \hat{\theta}^{60}(1-\hat{\theta})^{120}}$$

$R(0.4) =$ Plug $\theta = 0.4$

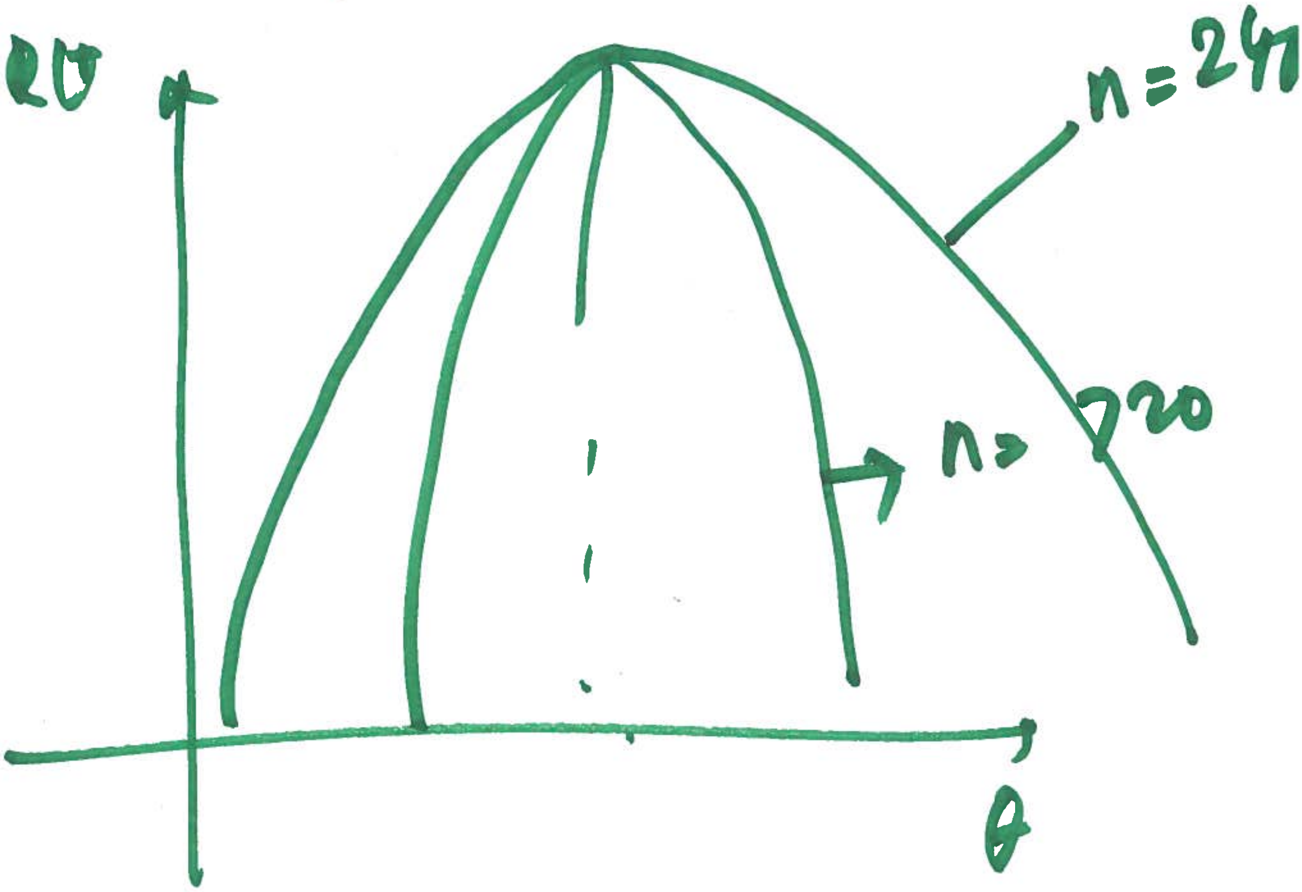
$\hat{\theta} = 1/3$ in the above equation

$= 0.182$

Since $R(0.4) > 0.15$, 0.4 must li...

in the 15% l.i

(a)



$n = 720$

$n = 240$

$$g(\theta) = \theta^{a} e^{-b/\theta}.$$

Find the value of θ that maximizes $g(\theta)$

$$\hat{\theta} = -b/a$$