To Do

Read Sections 5.1 - 5.3 (Hypothesis Testing) and Sections 6.1 – 6.2.

Do 5.1 – 5.8 for Midterm Test 2.

See detailed information posted on Learn regarding material covered by Midterm Test 2 (4:40 - 6:10 on Tuesday November 15).

Today's Class

(1) **Likelihood Ratio Test for testing**

H_0: $\theta = \theta_0$.

(2) **Simple Linear Regression**

Likelihood Ratio Test Statistic

Recall that, an exact pivotal quantity does not always exist for constructing a confidence interval for a parameter *θ*.

In these cases we used an approximate Normal pivotal quantity or the approximate pivotal quantity based on the likelihood ratio statistic:

$$\Lambda = -2\log\left[\frac{L(\theta)}{L(\tilde{\theta})}\right] \sim \chi^2(1) \text{ approximately}$$

Likelihood Ratio Statistic

$$\Lambda = -2\log\left[\frac{L(\theta)}{L(\tilde{\theta})}\right] \sim \chi^2(1) \text{ approximately}$$

Recall that we used this result to justify the fact that a 15% likelihood interval is an approximate 95% confidence interval.

We can also use this result to construct an approximate hypothesis test of H_0: $\theta = \theta_0$.

Relative Likelihood and Plausible Values - Review

Suppose for a given data set *y* we have assumed a model depending on a single unknown parameter θ and we have constructed the relative likelihood function

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})}$$

If the value of $R(\theta_0)$ is large (close to one) then θ_0 is a very plausible value of θ based on the observed data.

If the value of $R(\theta_0)$ is small (close to zero) then θ_0 is a very implausible value of θ based on the observed data.

Likelihood Ratio Test for a Single Parameter

$R(\theta_0)$ is large (close to 1) if and only if

$$-2\log\left[\frac{L(\theta_0)}{L(\hat{\theta})}\right]$$ **is small (close to 0)**

$R(\theta_0)$ is small (close to 0) if and only if

$$-2\log\left[\frac{L(\theta_0)}{L(\hat{\theta})}\right]$$ **is large**

Likelihood Ratio Test for a Single Parameter

To test H_0: $\theta = \theta_0$ we use the likelihood ratio test statistic defined by

$$\Lambda(\theta_0) = -2\log\left[\frac{L(\theta_0)}{L(\tilde{\theta})}\right]$$

If H_0: $\theta = \theta_0$ is true we expect to observe small values of $\Lambda(\theta_0)$.

Large observed values of $\Lambda(\theta_0)$ provide evidence against H_0: $\theta = \theta_0$.

Likelihood Ratio Test for a Single Parameter

To calculate the approximate *p*-value we note that

$$\Lambda(\theta_0) = -2\log\left[\frac{L(\theta_0)}{L(\tilde{\theta})}\right] \sim \chi^2(1) \text{ approximately}$$

if H_0: $\theta = \theta_0$ is true.

Calculating the Approximate *p*-value

Let the observed value of the likelihood ratio statistic $\Lambda(\boldsymbol{\theta_0})$ (a random variable) be denoted by

$$\lambda(\theta_0) = -2\log\left[\frac{L(\theta_0)}{L(\hat{\theta})}\right] = -2\log R(\theta_0)$$

The approximate *p*-value for testing H_0: $\boldsymbol{\theta} = \boldsymbol{\theta_0}$ is:

$$\begin{aligned} & p - value \\ &= P[\Lambda(\theta_0) \geq \lambda(\theta_0); H_0] \\ &\approx P[U \geq \lambda(\theta_0)] \;\; if \; U \sim \chi^2(1) \\ &= 2\{1 - P[Z \leq \sqrt{\lambda(\theta_0)}]\} \quad \text{where } Z \sim G(0,1) \\ &= 2\{1 - P[Z \leq \sqrt{-2\log R(\theta_0)}]\} \end{aligned}$$

Likelihood Ratio Test for Binomial Model

Suppose *y* successes have been observed in a Binomial(*n*,*θ*) experiment.

The likelihood function is

$$L(\theta) = \theta^{y}(1-\theta)^{n-y} \quad \text{for } 0 < \theta < 1$$

and the relative likelihood function is

$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})} = \frac{\theta^{y}(1-\theta)^{n-y}}{\hat{\theta}^{y}(1-\hat{\theta})^{n-y}} \quad \text{for } 0 < \theta < 1$$

where $\hat{\theta} = \dfrac{y}{n}$

Likelihood Ratio Test for Binomial Model

The likelihood ratio test statistic for testing H_0: $\theta = \theta_0$ is

$$\Lambda(\theta_0) = -2\log\frac{L(\theta_0)}{L(\tilde{\theta})} = -2\log\left[\frac{\theta_0^Y(1-\theta_0)^{n-Y}}{\tilde{\theta}^Y(1-\tilde{\theta})^{n-Y}}\right], \qquad \tilde{\theta} = \frac{Y}{n}$$

The observed value is

$$\lambda(\theta_0) = -2\log\left[R(\theta_0)\right] = -2\log\left[\frac{\theta_0^y(1-\theta_0)^{n-y}}{\hat{\theta}^y(1-\hat{\theta})^{n-y}}\right], \qquad \hat{\theta} = \frac{y}{n}$$

and $$p-value \approx 2\{1-P[Z \le \sqrt{\lambda(\theta_0)}]\}$$

Previous Method for Hypothesis Test for Binomial Model

Previously (e.g. ESP experiment) we used the test statistic $D = |Y - n\theta_0|$ with observed value $d = |y - n\theta_0|$

If n is large then

$$p-value \approx P\left(|Z| \geq \frac{d}{\sqrt{n\theta_0(1-\theta_0)}}\right) = P\left(|Z| \geq \frac{|y-n\theta_0|}{\sqrt{n\theta_0(1-\theta_0)}}\right)$$

$$= 2\left[1-P\left(Z \leq \frac{|y-n\theta_0|}{\sqrt{n\theta_0(1-\theta_0)}}\right)\right] \quad \text{where } Z \sim G(0,1)$$

Two Methods for Hypothesis Testing for Binomial Model

We have two methods for testing H_0: $\theta = \theta_0$ for Binomial model.

Use the likelihood ratio test statistic with

$$p - value \approx 2\{1 - P[Z \le \sqrt{-2\log R(\theta_0)}]\} \quad \text{where } Z \sim G(0,1)$$

or use the asymptotic Normal test statistic with

$$p - value \approx 2\left[1 - P\left(Z \le \frac{|y - n\theta_0|}{\sqrt{n\theta_0(1-\theta_0)}}\right)\right] \quad \text{where } Z \sim G(0,1)$$

Binomial Example

Suppose n = 30, y = 10 and H_0: θ = 0.3.

Then θ = 10/30 = 1/3, the observed value of the likelihood ratio statistic is

$$\lambda(0.3) = -2\log\left[R(0.3)\right] = -2\log\left[\frac{(0.3)^{10}(1-0.3)^{20}}{(1/3)^{10}(1-1/3)^{20}}\right]$$

$$= -2\log(0.9251) = 0.156$$

and

$$p - value \approx 2[1 - P(Z \le \sqrt{0.156})] \text{ where } Z \sim G(0,1)$$

$$= 2[1 - P(Z \le 0.39)]$$

$$= 2(1 - 0.65173)$$

$$= 0.69654$$

Binomial Example

Alternatively

$$p-value \approx 2\left[1-P\left(Z \le \frac{|10-30(0.3)|}{\sqrt{30(0.3)(1-0.7)}}\right)\right] \quad \text{where } Z \sim G(0,1)$$

$$= 2[1-P(Z \le 0.40)]$$

$$= 2(1-0.65542) = 0.68961$$

The approximate *p*-values are nearly identical (0.69654 versus 0.68916) and the conclusion is the same.

There is no evidence based on the observed data to contradict H_0: $\theta = 0.3$.

Important Examples in Course Notes

Example 5.3.2 - likelihood ratio test for Exponential data

Example 5.3.3 - Gaussian distribution with known variance σ^2

Problems 5.8 and 5.9 – hypothesis tests for Poisson data

Example: STAT 230 and 231 Final Grades

No.	S230	S231		No.	S230	S231		No.	S230	S231
1	76	76		11	87	76		21	98	83
2	77	79		12	71	50		22	80	88
3	57	54		13	63	75		23	67	52
4	75	64		14	77	72		24	78	75
5	74	64		15	96	84		25	100	99
6	60	60		16	65	69		26	94	94
7	81	85		17	71	43		27	83	83
8	86	82		18	66	60		28	51	37
9	96	88		19	90	96		29	77	90
10	79	72		20	50	50		30	77	67

Example: STAT 230 and 231 Final Grades

Why might we be interested in collecting data such as these?

What might be a reasonable choice for the target and study population?

What are the variates? What type are they?

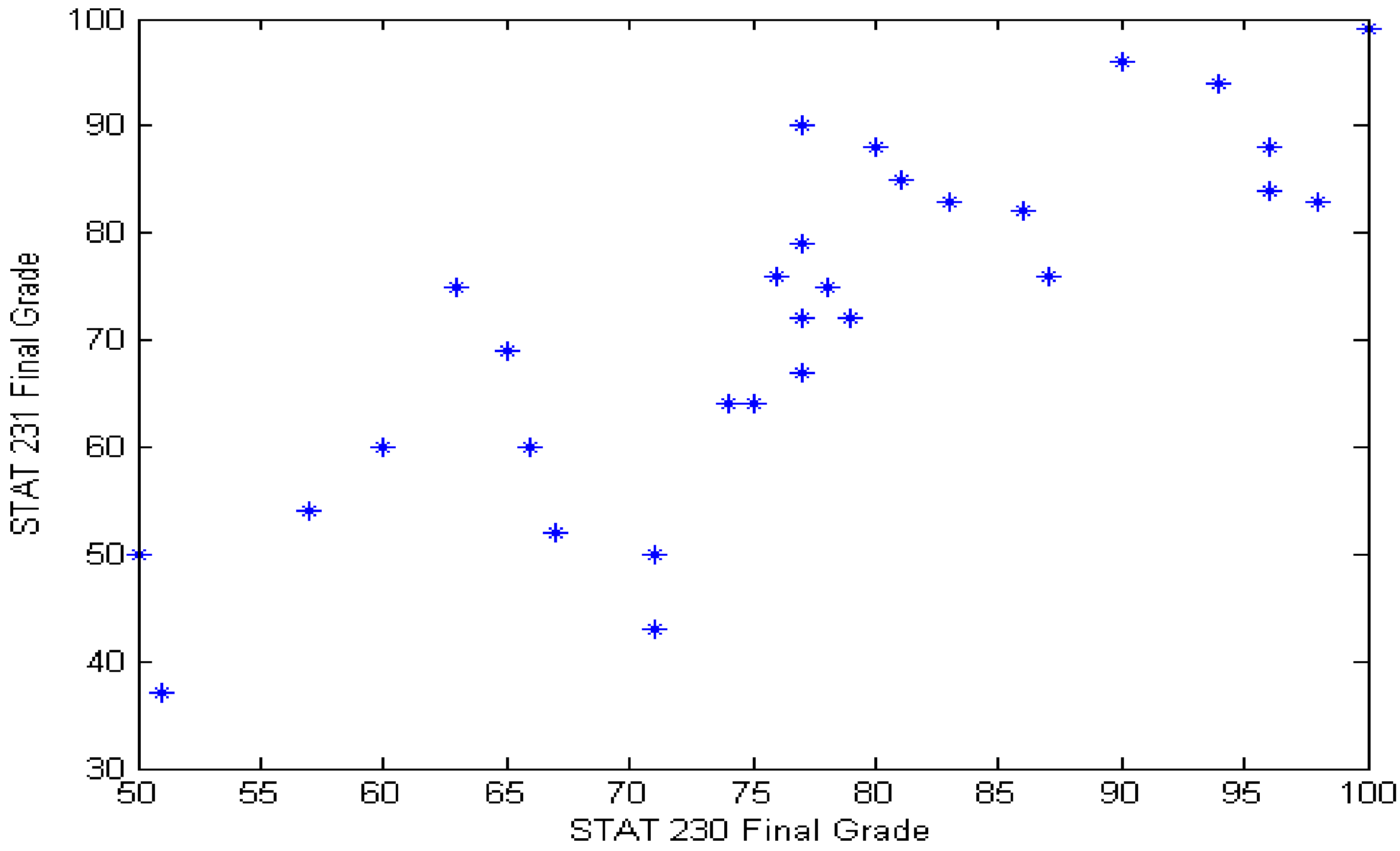
What is the explanatory variate?

What is the response variate?

How do we summarize these data numerically and graphically?

What model could we use to analyse these data?

Scatterplot of Data



Sample Correlation - Review

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

where $S_{xx} = \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2, \; S_{yy} = \sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2,$

and $S_{xy} = \sum_{i=1}^{n} \left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)$

Recall: -1 ≤ *r* ≤ 1

Sample Correlation for STAT 230/231 Final Grades

For these data

S_{XX} = 5135.8667, S_{XY} = 5106.8667,

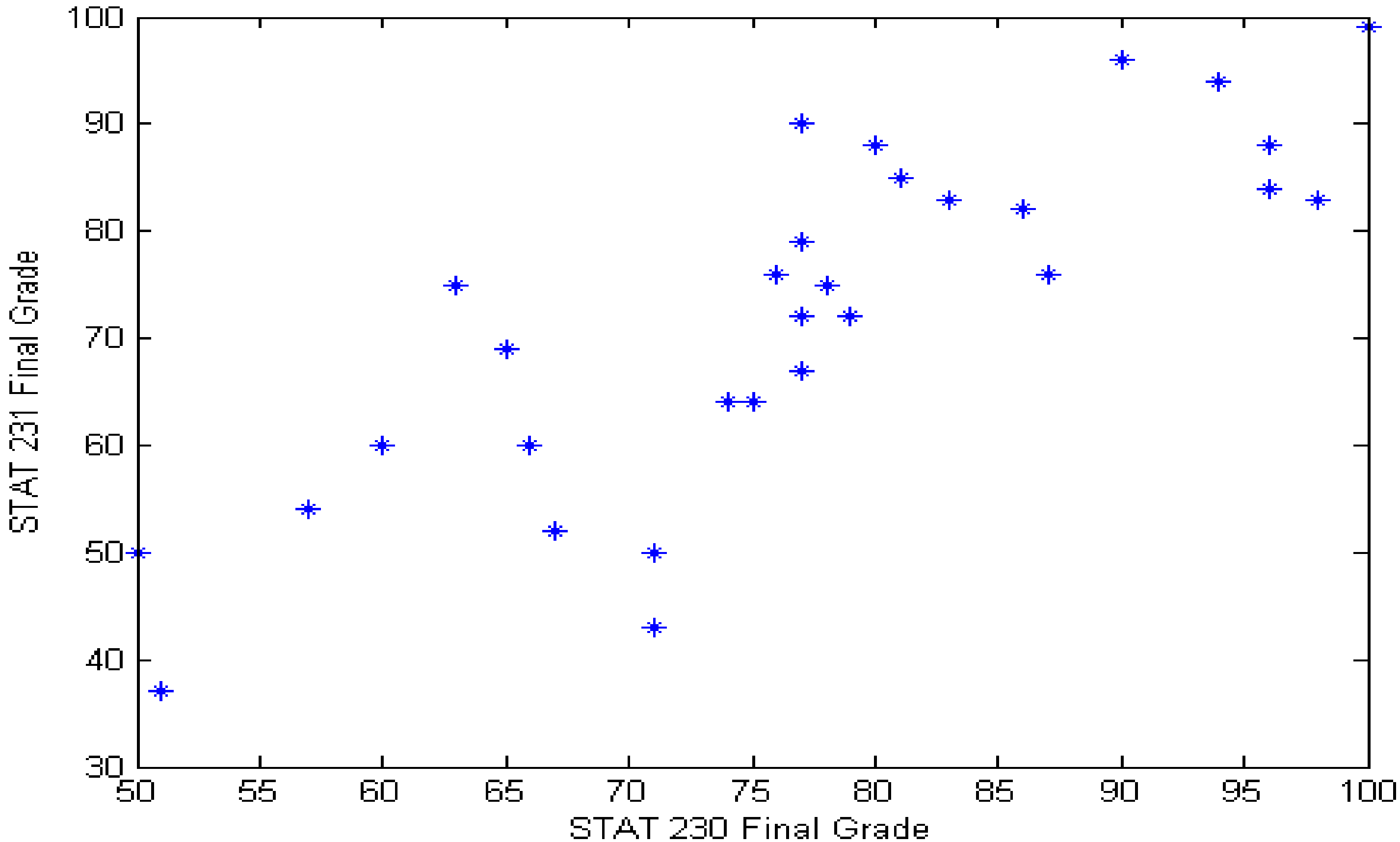
S_{YY} = 7585.3667

$$r = \frac{5106.8667}{\sqrt{(5135.8667)(7585.3667)}} = 0.82$$

Since *r* is close to 1 we would say that there is a strong positive linear relationship between STAT 230 final grades and STAT 231 final grades.

Question

How do we fit a straight line to these data?



Least Squares Estimates

To determine the fitted line $y = \alpha + \beta x$ which minimizes the sum of the squares of the distances between the observed points and the fitted line we need to find the values of α and β which minimize

$$g(\alpha, \beta) = \sum_{i=1}^{n} \left(y_i - \alpha - \beta x_i \right)^2$$

Least Squares Estimates

These values are determined by solving simultaneously the equations

$$\frac{\partial g}{\partial \alpha} = \frac{\partial}{\partial \alpha} \sum_{i=1}^{n} \left(y_i - \alpha - \beta x_i\right)^2 = \sum_{i=1}^{n} 2\left(y_i - \alpha - \beta x_i\right)\left(-1\right) = 0$$

$$\frac{\partial g}{\partial \beta} = \frac{\partial}{\partial \beta} \sum_{i=1}^{n} \left(y_i - \alpha - \beta x_i\right)^2 = \sum_{i=1}^{n} 2\left(y_i - \alpha - \beta x_i\right)\left(-x_i\right) = 0$$

Least Squares Estimates

These equations can be written as

$$(1) \quad \bar{y} - \alpha - \beta\,\bar{x} = 0$$

$$(2) \quad \sum_{i=1}^{n} \left(y_i - \alpha - \beta\, x_i \right) x_i = 0$$

Substituting (1) into (2) gives

$$\sum_{i=1}^{n} \left[y_i - \bar{y} - \alpha - \beta \left(x_i - \bar{x} \right) \right] = 0$$

Least Squares Estimates

Therefore

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i (y_i - \bar{y})}{\sum_{i=1}^{n} x_i (x_i - \bar{x})} = \frac{\sum_{i=1}^{n} y_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{S_{XY}}{S_{XX}}$$

and $\hat{\alpha} = \bar{y} - \hat{\beta}\,\bar{x}$

STAT 230 and 231 Final Grades

For these data

$$\bar{x} = 76.7333 \qquad \bar{y} = 72.2333$$

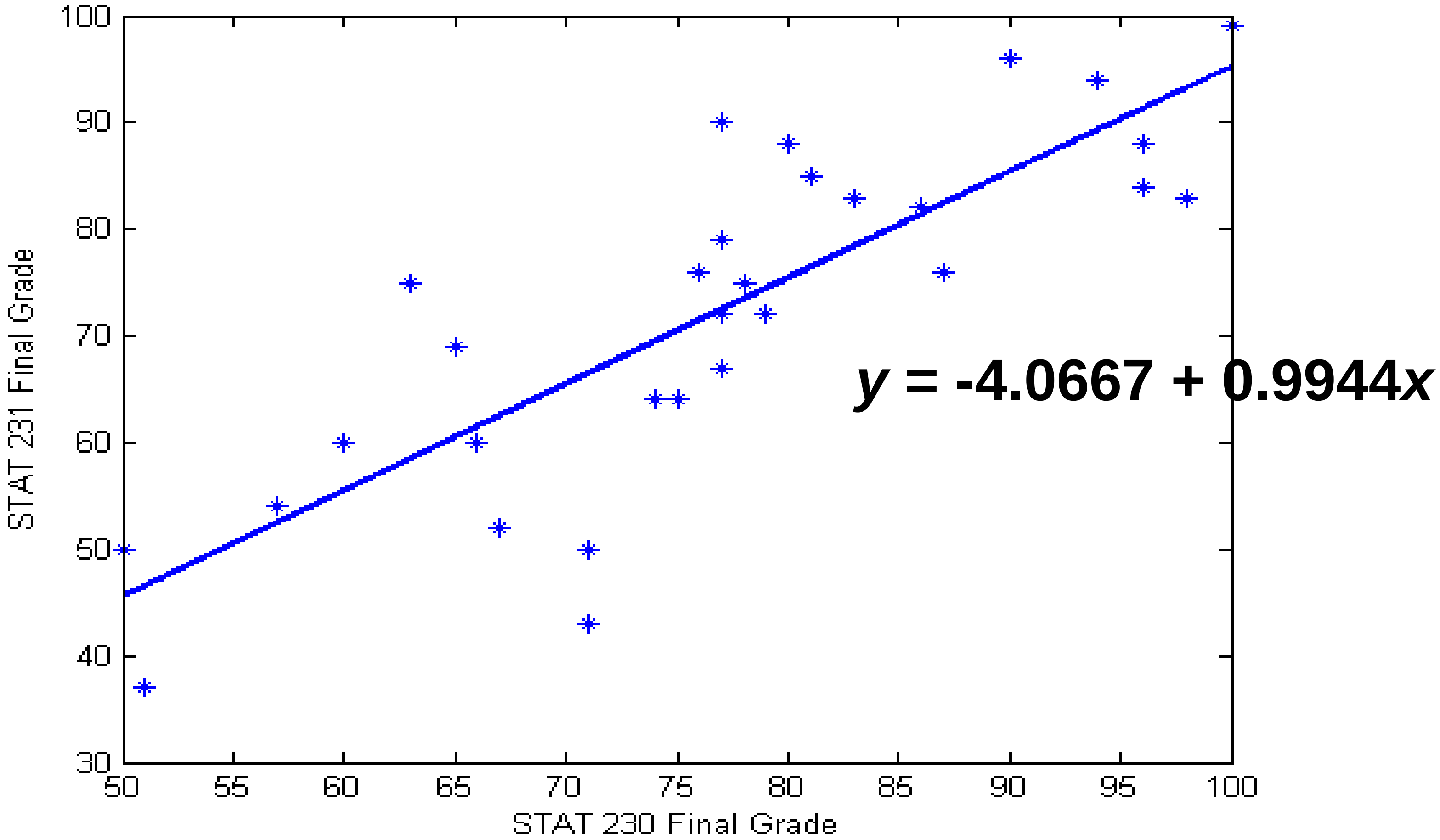
$$S_{XX} = 5135.8667 \qquad S_{XY} = 5106.8667 \qquad S_{XY} = 7585.3667$$

$$\hat{\beta} = \frac{S_{XY}}{S_{XX}} = \frac{5106.8667}{5135.8667} = 0.9944$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\,\bar{x} = 72.2333 - \left(\frac{5106.8667}{5135.8667}\right)(76.7333) = 4.0667$$

The fitted line is: *y* = -4.0667 + 0.9944*x*

Scatterplot with Fitted Line



STAT 231 versus 230 Final Grades

Based on these data, what is the best estimate of your STAT 231 final grade?

STAT 231 versus 230 Final Grades - Model?

If your final grade in STAT 230 was x = 75, then the least squares estimate of your STAT 231 final grade is

$y = -4.0667 + 0.9944(75) = 70.51$

What can we say about the uncertainty in this estimate?

STAT 231 versus 230 Final Grades - Model?

We need a statistical model in order to obtain an interval estimate of your final grade.

We need a model which captures the fact that not everyone with a final grade of $x = 75$ in STAT 230 gets a final grade of 70.51 in STAT 231.