STAT 231

October 3, 2016

Course Notes: $\leq$ Chapter 3 for the midterm. (approx).

Roadmap

- Estimation techniques, and Likelihood function
- Relative Likelihood function $R(\theta)$

Log Relative - - - - .. $r(\theta)$

- Invariance property of the MLE.
- Model Selection
    - Graphical methods (Q-Q plot, Run Chart)
    - Numerical methods (Expected versus observed frequencies)

Model: $Y_i \sim f(y_i; \theta)$

$i = 1, \ldots n$

Y_i's independent.

Likelihood function

$$L(\theta; y_1, \ldots y_n) = \prod_{i=1}^{n} f(y_i; \theta)$$

$\parallel$

function of θ = unknown parameter.

f = probability function if Y is discrete.

= density function if Y is continuous

= Product of the distribution (density) function, evaluated at the sample point

$\hat{\theta}$ is called the MLE (Max. Likelihood Estimate) if $\hat{\theta}$ maximizes $L(\theta)$.

So. $\hat{\theta}$ maximizes $L(\theta)$

$\iff \hat{\theta}$ maximizes $\ell(\theta)$

because log is a monotonic function

We then take the log of the likelihood function and maximize $\ell(\theta)$

COMMON DISTRIBUTIONS

- **Binomial** : $Y \sim Bin(n, \pi)$

π = unknown parameter

y = observed # of successes in your sample.

$$\boxed{\hat{\pi} = y/n =}$$ sample proportion

- **Poisson** : $Y \sim Poi(\mu)$

$y_1, \ldots y_n \rightarrow$ Sample.

$$\boxed{\hat{\mu} = \bar{y}}$$

Exponential Distribution

$$Y_i \sim Exp(\mu) \quad i = 1, \ldots n.$$

$$\boxed{\hat{\mu} = \bar{y}}$$

Gaussian Distribution $G(\underbrace{\mu, \sigma}_{\theta})$

$$Y_i \sim G(\mu, \sigma) \quad i = 1, \ldots n$$

indep.

$$\boxed{\hat{\mu} = \bar{y}}$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum (y_i - \bar{y})^2}$$

$\neq$

not sample variance.

Example

Suppose Y_i is a r.v. with the following density function

$$f(y;\theta) = \frac{2y}{\theta} e^{-y^2/\theta}.$$

$$y > 0.$$

where θ is the unknown parameter.

Sample = $\{y_1, \ldots, y_n\}$ drawn from this dist^n.

Find $\hat{\theta}$ = MLE for θ

Step 1 Likelihood function ?

$$L(\theta; y_1, \ldots y_n) = \frac{2y_1}{\theta} e^{-y_1^2/\theta} \cdot \frac{2y_2}{\theta} e^{-y_2^2/\theta}$$

$$\ldots \frac{2y_n}{\theta} e^{-y_n^2/\theta}.$$

$$= \frac{2^n (y_1 \cdot y_2 \ldots y_n)}{\theta^n} e^{-\frac{1}{\theta}\sum y_i^2}.$$

Step 2 The Log- Likelihood function

$$l(\theta) = \log(2^n \cdot y_1 \ldots y_n) - n \ln \theta$$

$$- \frac{1}{\theta} \sum y_i^2.$$

Step 3 $\quad dl/d\theta = 0$ and solve.

$$-\frac{n}{\theta} + \frac{1}{\theta^2}\sum y_i^2 = 0$$

$$\boxed{\hat{\theta} = \sum y_i^2 / n}$$

A few notes:

$$L = \underline{P(Y_1 = y_1)} \ldots \quad P(\underline{Y_n = y_n})$$

As $n \uparrow$, the likelihood function values become smaller and smaller.

The Relative Likelihood function

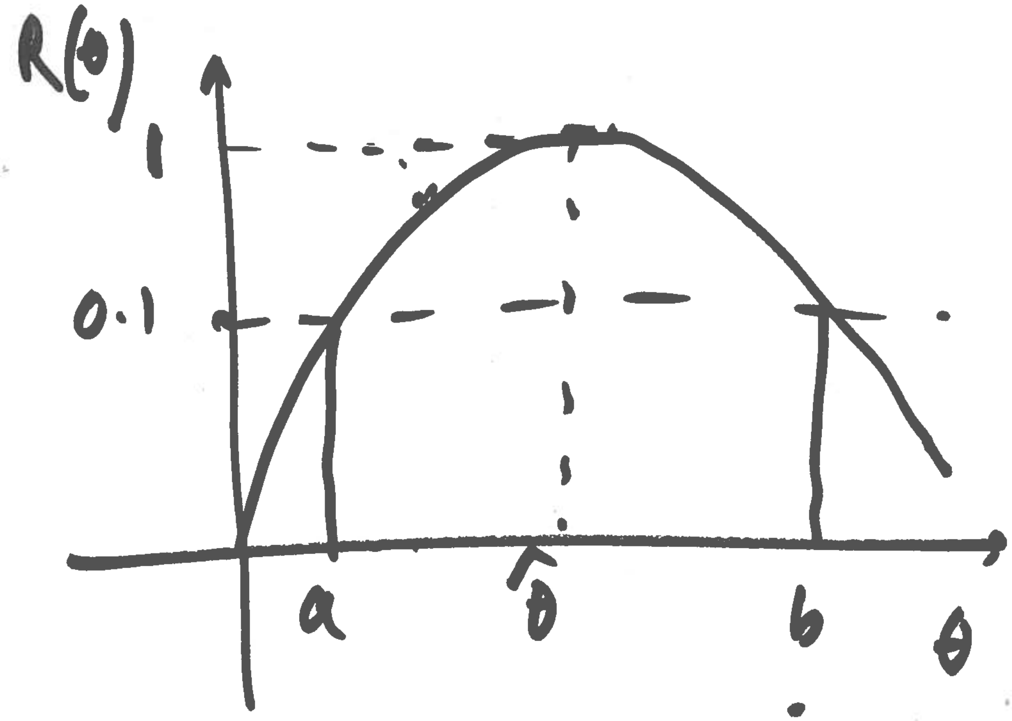
$$R(\theta) = \frac{L(\theta)}{L(\hat{\theta})}, \text{ where } \hat{\theta} = \text{MLE}.$$

The log-relative likelihood function $= r(\theta) = \log R(\theta)$

$$= \log \frac{L(\theta)}{L(\hat{\theta})}$$

$$= \log L(\theta) - \log L(\hat{\theta})$$

$$\boxed{r(\theta) = \ell(\theta) - \ell(\hat{\theta})}$$



$R(\theta) \geq 0$; $R(\theta) \leq 1$

R has its maximum at $\theta = \hat{\theta}$

$$\boxed{R(\hat{\theta}) = 1}$$

Question: From the graph above, find all θ; s.t $R(\theta) \geq 0.1$

Construct PLAUSIBLE intervals for θ.

<u>Example</u> Suppose $Y \sim Bin(200, \pi)$

$y = 80 = \#$ of successes

Calculate $R(0.5)$

$$L = {}^{200}C_{80}\, \pi^{80} (1-\pi)^{120}$$

$\hat{\pi} = 0.4$ ($80/200$ = Sample proportion)

$$R(0.5) = \frac{L(0.5)}{L(0.4)} = \frac{{}^{200}C_{80}(0.5)^{80}(0.5)^{120}}{{}^{200}C_{80}(0.4)^{80}(0.6)^{120}}$$

INVARIANCE PROPERTY OF THE MLE

<u>Theorem:</u> If $\hat{\theta}$ is the MLE for θ, then $g(\hat{\theta})$ is the MLE for $g(\theta)$

<u>Example:</u> Suppose $Y_1, \ldots Y_n \sim Poi(\mu)$

Find the MLE for $2\mu + 3$

MLE for $2\mu + 3 \rightarrow 2\hat{\mu} + 3$

$$= 2\bar{y} + 3$$

Example 2 $\quad Y_1, \ldots Y_n \sim Exp(\theta)$

Find the MLE for $Var(\bar{Y})$ $\quad$ indep.

$Var(Y_i) = \theta^2$ (for exponential)

$$Var\left(\frac{Y_1 + Y_2 + \ldots Y_n}{n}\right)$$

$$= \frac{1}{n^2} V(Y_1 + Y_2 + \ldots Y_n)$$

$$= \frac{1}{n^2}\left[V(Y_1) + V(Y_2) + \ldots V(Y_n)\right]$$

$$= \frac{1}{n^2} \cdot n\theta^2 = \theta^2/n$$

$\cancel{\bar{Y}^2/n}$

$$\hat{\theta} = \cancel{\theta}\ \bar{y}$$

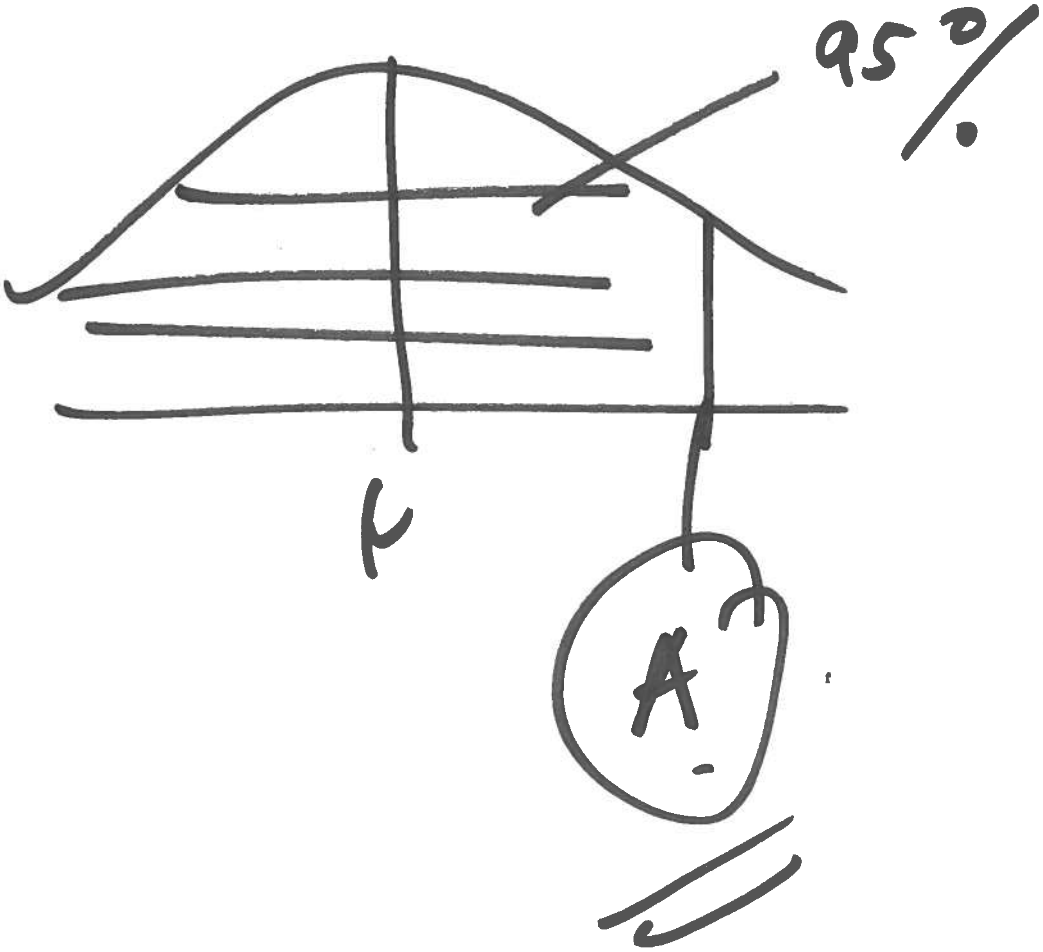
$\therefore$ the estimate for $V(\bar{Y}) = \hat{\theta}^2 / n$

$$= \bar{Y}^2 / n.$$

<u>Example</u>: Objective: To estimate the 95^{th} percentile of a Normal Distribution)

$Y_1, \ldots, Y_n \sim G(\mu, \sigma)$ indep

$\{y_1, \ldots y_n\}$



$A = \mu + 1.65\sigma.$

MLE for A

$$\hat{\mu} + 1.65\hat{\sigma}$$

$$\bar{y} + 1.65\sqrt{\frac{1}{n}\sum(y_i - \bar{y})^2}$$

If we want to find the MLE for $g(\theta)$, we find $\hat{\theta}$ and the MLE is then $g(\hat{\theta})$.

Left as exercise.

Find the MLE for the IQR of a Gaussian (μ, σ)

$\{y_1, \ldots y_n\}$