5. Approximating functions and the Taylor series

LECTURE 5

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Workshop 4 summary:

BOTH ALGORITHMS…

- Aim to find the minimum of a function
- Are iterative and (hopefully) converse to a minimum
- Require a stopping condition
- Are inexact
- Can get stuck in local minima
- Can equally well find the maximum

IN REAL LIFE…

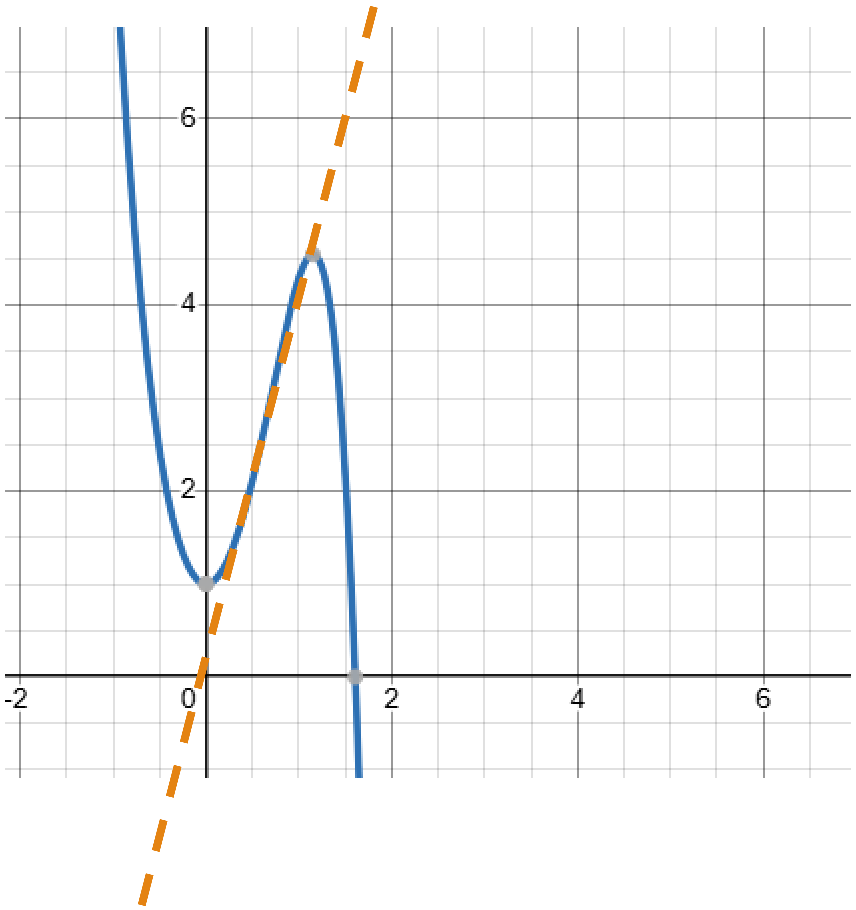
- Used in machine learning and neural networks
- “Cost function” optimised by Gradient Descent
- Nelder-Mead then Gradient Descent is often a good strategy

What we'll cover today:

- Linear Approximations of functions
- Approximating roots (Newton's method)
- Approximating functions (Taylor series)

Linear approximation

Linear Approximation of a function



Linear approximation of a function

Linear approximation of f at $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

- This is related to the gradient/tangent:

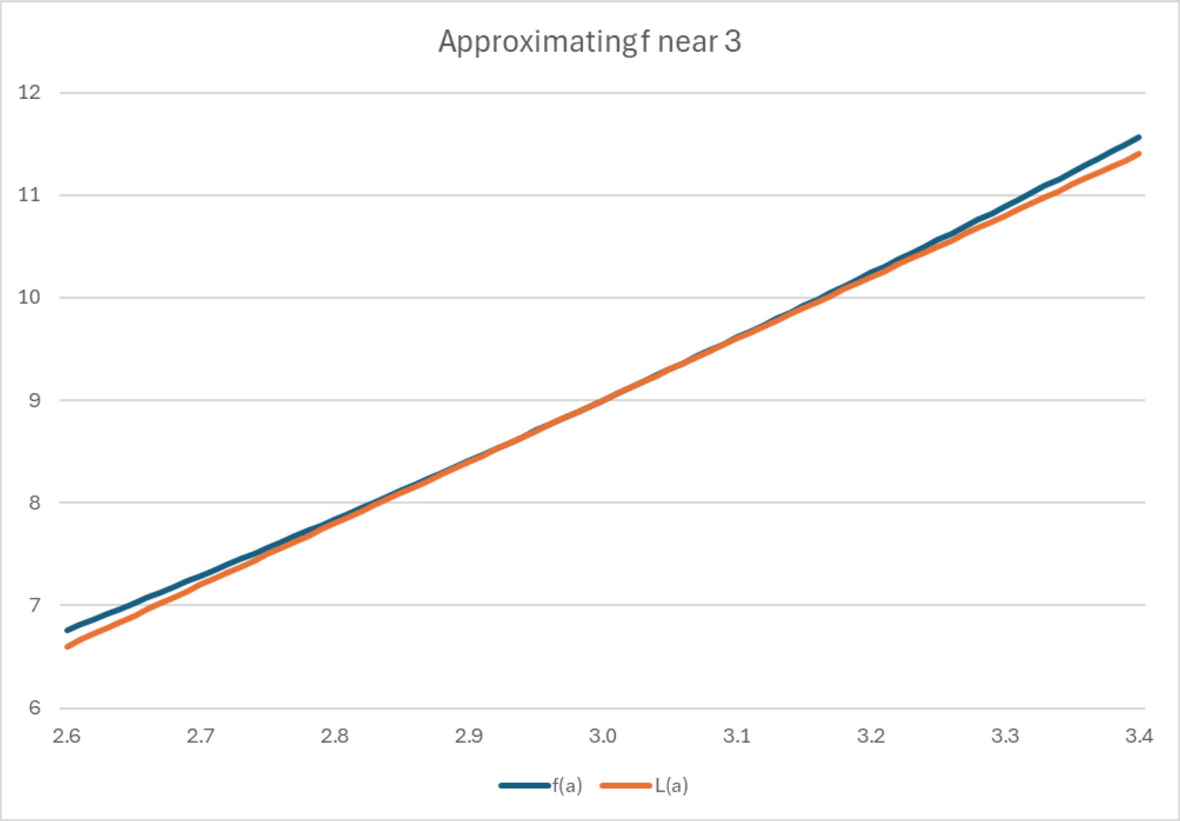
$$f'(a) = \frac{f(x) - f(a)}{x - a}$$

Example

Approximate $f(x) = x^2$ at $a = 3$

1. $L(x) = f(3) + f'(3)(x - 3)$
2. $L(x) = 9 + 6(x - 3) = 6x - 9$
3. Conclusion: at $\mathrm{a} = 3$, we can approximate x^2 by the line $y = 6x - 9$
4. Is it a good approximation?
   - $f(3) = 9;$
   - $L(3) = 6 \times 3 - 9 = 9$
5. What about 'near' 3?
   - $f(2.75) = 7.5625$
   - $L(2.75) = 7.5$

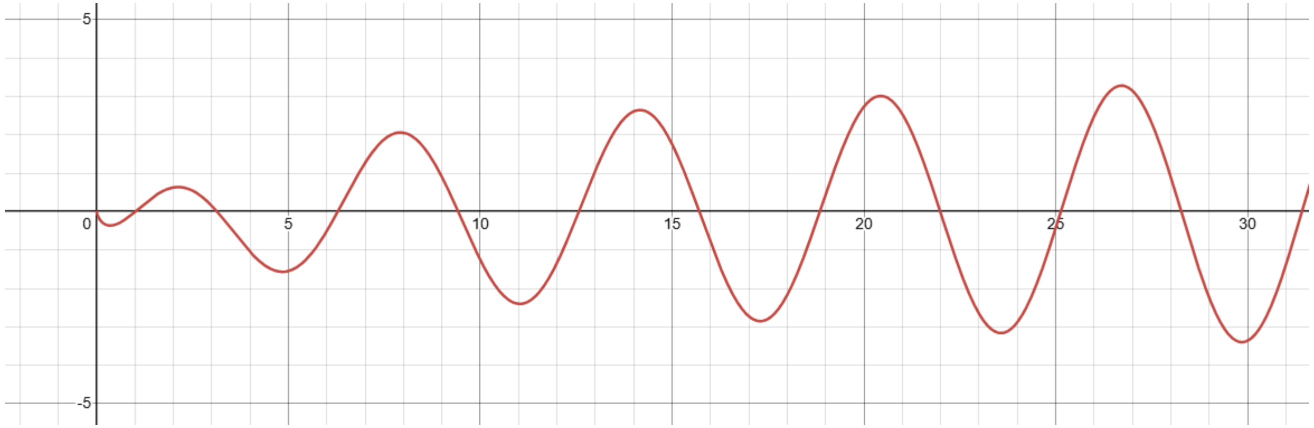
Example cont.



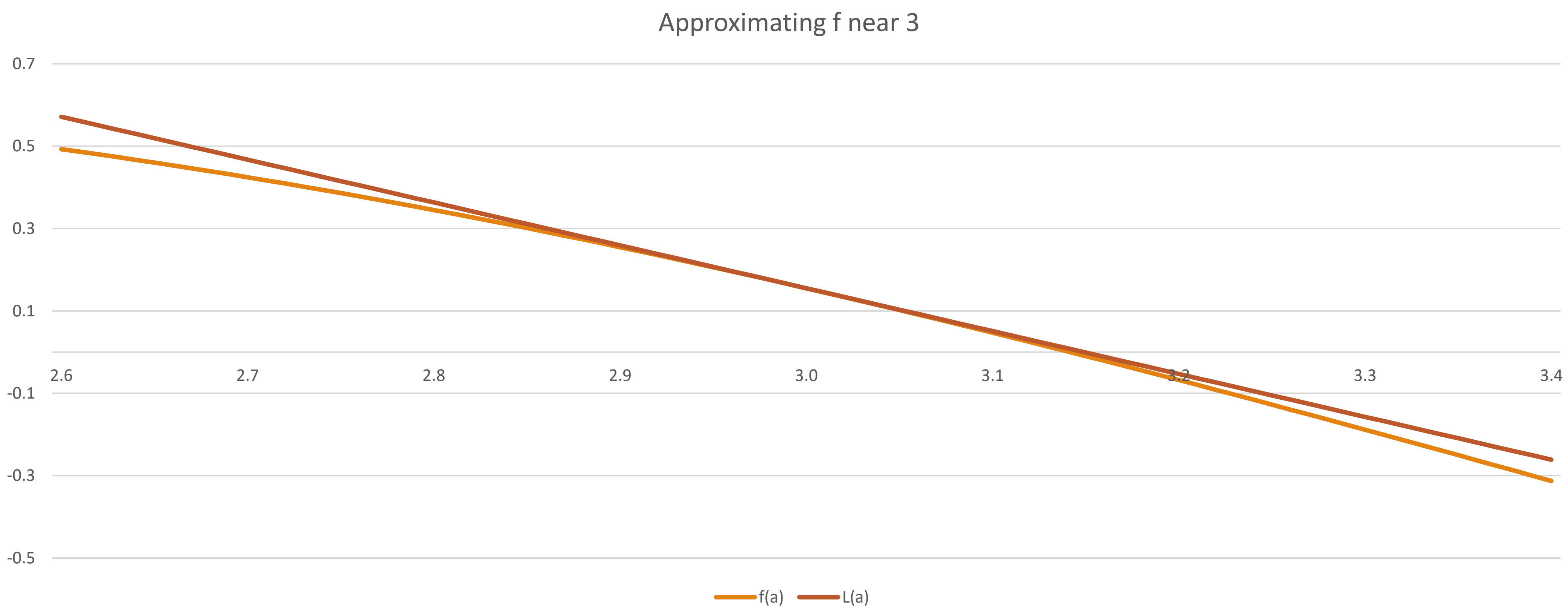
A more complicated example

Approximate $f(x) = \sin(x)\ln(x)$ at $a = 3$

1. $f'(x) = \cos(x)\ln(x) + \dfrac{\sin(x)}{x}$
2. $L(x) = f(3) + f'(3)(x-3) = 0.155 - 1.04(x-3)$
3. $L(x) = 3.28 - 1.04x$



Example cont.



Newton’s method

NUMERICAL ROOT FINDING

Can we find a root of this function?

Approximating roots of $f(x)$ (Newton's method)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example: $\mathrm{f}(x) = x^2 - 1$

- $x_0 = 5$
- $x_1 = 5 - \frac{25-1}{2\times 5}$ = 2.6
- Etc.

CODE:

```
x = 5
for _ in range(10):
    x = x-(x**2-1)/(2*x)
    print(x)
```

Why does Newton's method work?

We'd like to find x_1. We have x_0.

The linear approximation of f at x_0 is:

$$L(x_1) = f(x_0) + f'(x_0)(x_1 - x_0)$$

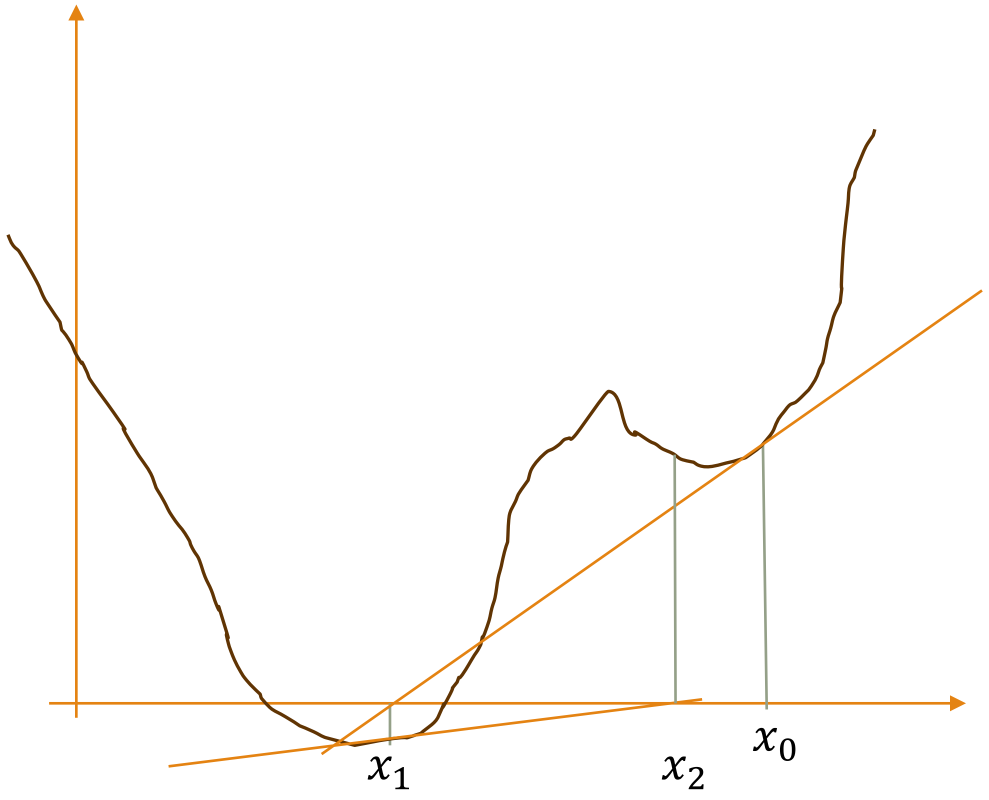
We want a root of f, so we set LHS to 0:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

Now we rearrange to get:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton’s method geometrically



Taylor series

HIGHER ORDER APPROXIMATIONS OF FUNCTIONS

A dilemma

Pic: https://santecares.com/2019/08/30/infographic-understanding-the-stages-of-memory-loss/

I really want to know what e^7 is, but I've forgotten the value of e

Our dilemma generalised:
We'd like to understand how a function behaves, but the function difficult to calculate/analyse

A solution!

- We'll approximate our function by a polynomial!

- Remember a polynomial (of degree t) looks like:

$$a_0 + a_1 x + a_2 x^2 + \ \dots + a_t x^t$$

  - Polynomials are quick to compute and easy to analyse

  - Our goal: make our function look like a polynomial

Taylor series expansion

Goal: Approximate $\boldsymbol{f(x)}$ at the point $\boldsymbol{x = a}$ by a polynomial $\boldsymbol{g(x)}$

$$\boldsymbol{g(x) = c_0 + (x-a)c_1 + (x-a)^2 c_2 + \ \ldots + (x-a)^M c_M}$$

We know that $f(x)$ is differentiable at $x = a$.

We need to find the coefficients c_i so that $f(a) \approx g(a)$

Taylor series expansion

Goal: Approximate $\boldsymbol{f(x)}$ at the point $\boldsymbol{x = a}$ by a polynomial $\boldsymbol{g(x)}$

$$\boldsymbol{g(x) = c_0 + (x-a)c_1 + (x-a)^2 c_2 + \ \ldots + (x-a)^M c_M}$$

- Let's differentiate on both sides:

$$f'(x) = c_1 + 2(x-a)c_2 + 3(x-a)^2 c_3 + \cdots + M(x-a)^{M-1} c_M$$
$$\Rightarrow \ f'(a) = c_1$$

- Let's continue:

$$f''(x) = 2c_2 + 3 \cdot 2(x-a)c_3 + \cdots + M \cdot (M-1)(x-a)^{M-2} c_M$$
$$\Rightarrow \ f''(a) = 2c_2$$

- Our general formula:

$$f^{(n)}(a) = \mathrm{n}! \ c_n \Rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor series expansion

We've found that a polynomial approximation of $f(x)$ at $x = a$ is given by

$$g(x) = f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2!} + \ \ldots + (x-a)^M \frac{f^{(M)}(a)}{M!}$$

More succinctly:

$$g(x) = \sum_{i=0}^{M} \frac{(x-a)^i}{i!} f^{(i)}(a)$$

This is the **Taylor series expansion of $\boldsymbol{f(x)}$ at $\boldsymbol{x = a}$**

Examples: [on board]

- e^x
- $\ln(1 + x)$
- $\sqrt{x}$

Why do we use Taylor series expansion?

- Gives an nth order approximation of a function
- The approximation is better as n increases

Summary

- The derivative can be used to give a linear approximation of a function
- We can use this to find roots of a function numerically, using Newton’s method
- We can use higher order derivatives to give us a better approximation of a function using Taylor series expansion