Section 2: Analysis

In this section on Analysis, use the Section 2 Answer Sheet for your answers. There are 8 questions. For multiple choice questions there is only one correct answer.

Question 1 (3 marks)

Determine which of the following is the derivative of $\frac{\sin x + x}{2+\cos x}$. Enter your answer by crossing exactly one of the boxes on your answer sheet.

a. $\frac{\cos x+1}{(2-\sin x)^2}$

b. $\frac{(\cos x+1)(2+\cos x)-(\sin x+x)\sin x}{(2-\sin x)^2}$

c. $\frac{\cos x+1}{-\sin x}$

d. $\frac{\sin x+x}{-\sin x}$

e. $\frac{(\cos x+1)(2+\cos x)+(\sin x+x)\sin x}{(2+\cos x)^2}$

(3 marks)

Solution: Using the product rule and the chain rule

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x + x}{2+\cos x}\right) &= \left(\frac{d}{dx}(\sin x + x)\right) * (2+\cos x)^{-1} \\ &+ \left(\frac{d}{dx}(2+\cos x)^{-1}\right) * (\sin x + x) \\ &= (\cos x + 1) * (2+\cos x)^{-1} \\ &+ \sin x (2+\cos x)^{-2}(\sin x + x)\end{aligned}$$

Hence E is correct.

Question 2 (2 marks)

If $f'(2) = 5$, $g(4) = 2$, $g(2) = 1$, $f(2) = -1$ and $g'(4) = 3$, determine the value of $\frac{d}{dx}(f(g(x)))$ at the point $x = 4$. The answer is an integer between -99 and 99. Enter your answer according to the instructions.

(2 marks)

Solution: $\frac{d}{dx}(f(g(x))) = g'f'(g(x)) = g'(4)f'(2) = 15$

Question 3 (2 marks)

If $f(x,y) = cos\left(\frac{x}{y}\right)$ determine which of the following is ∇f. Enter your answer by crossing exactly one of the boxes on your answer sheet.

a. $\left(-\frac{1}{y}\sin\frac{x}{y}, \frac{x}{y^2}\sin\frac{x}{y}\right)$

b. $\left(-\sin\frac{x}{y}, \frac{1}{y^2}\sin\frac{x}{y}\right)$

c. $\left(\sin\frac{x}{y}, -\frac{1}{y^2}\sin\frac{x}{y}\right)$

d. $\left(-\frac{x}{y}\sin\frac{x}{y}, \frac{x}{y^2}\sin\frac{x}{y}\right)$

e. $\left(\frac{1}{y}\sin\frac{x}{y}, -\frac{x}{y^2}\sin\frac{x}{y}\right)$

(2 marks)

Solution: a is correct

Question 4 (3 marks)

Determine the derivative of $f(x, y)$ in the $(3, -4)$ direction at the point $(\pi, 2)$, rounded to three decimal places. The answer is a real number between -1 and 1. Enter your answer according to the instructions.

(3 marks)

Solution:

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{3^2 + (-4)^2} = 5 \\
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{3}{5}, -\frac{4}{5}\right) \\
\nabla f \cdot \vec{u} &= -\frac{1}{5}\left(\frac{3}{y} + \frac{4x}{y^2}\right)\sin\frac{x}{y} \\
\nabla f \cdot \vec{u}\,(\pi, 2) &= -\frac{1}{5}\left(\frac{3}{2} + \frac{4\pi}{4}\right)\sin\frac{\pi}{2} \\
&= -0.92831853071 = 0.928 \text{ to three dp}
\end{aligned}$$

Question 5 (5 marks)

Consider the pair of complex numbers $z_1 = 10 + 8i$ and $z_2 = \frac{3-(62/3)i}{1-(4/3)i}$. Determine the answers to the following questions. Enter your answer for each question by crossing exactly one box, labelled z_1 or z_2, on the answer sheet. The allocation of marks per question is indicated in brackets.

A. Which is closest to the origin? (1 mark)

B. Which has rotated by the least amount in absolute terms? (2 marks)

C. Which is closest to $z_3 = 1 + i$? (2 marks)

(5 marks)

Solution: Answers: a.z_2, b.z_1, c.z_1

$$\frac{3-(62/3)i}{1-(4/3)i} * \frac{1+(4/3)i}{1+(4/3)i} = 11-6i$$
$$|z_1| = \sqrt{10^2+8^2} = 12.81$$
$$|z_2| = \sqrt{11^2+6^2} = 12.53$$
Hence z_2 is closer to the origin.
$$arg(z_1) = \tan^{-1}\left(\frac{8}{10}\right) = 0.675$$
$$arg(z_2) = \tan^{-1}\left(\frac{11}{-6}\right) = -1.071$$
Hence z_1 has rotated the least.
$$|z_1 - z_3| = \sqrt{9^2+7^2} = \sqrt{130}$$
$$|z_2 - z_3| = \sqrt{10^2+(-6)^2} = \sqrt{136}$$
Hence z_1 is closest to z_3.

Question 6 (2 marks)

For the 1st order ordinary differential equations (ODE) 1-5 below, classify them according to the following terms:

A. Order, a natural number.

B. Linear or non-linear.

C. Homogeneous or non-homogeneous.

D. Has constant coefficients or does not have constant coefficients.

1. $y' - e^x y + 7 = 0$
2. $y' - e^x y = 0$
3. $3y'' - 2y' = 3$
4. $2\frac{d^3y}{dx^3} + \cos x \frac{dy}{dx} = 0$
5. $2\frac{d^5y}{dx^5} + \cos y \frac{d^3y}{dx^3} = 0$

Enter your answers as follows. For each of the 5 equations, cross exactly one of the boxes in the first table, in the column representing the order of the equation. Then, cross all that apply in the second table, where L stands for linear, H stands for homogeneous, and C stands for constant coefficients.

(2 marks)

Solution: Answers:

1. 1,1,2,2
2. 1,1,1,2
3. 2,1,2,1
4. 3,1,1,2
5. 3,2,1,2

Question 7 (5 marks)

The function $y(t)$ satisfies the 2nd order ODE $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2 = 0$ with $y(0) = 1$ and $y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}$. Given that $\left(y\left(\frac{\pi}{3}\right)\right)^3 = \frac{1}{A}e^{B\pi}$, determine the values of A and B. The answers are integers between -99 and 99. Enter your answers according to the instructions.

(5 marks)

Solution: Answer: $A = 8$, $B = 1$
Solution:

$$
\begin{aligned}
r^2 - 2r + 2 &= 0, \\
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= 1 \pm i, \\
y(t) &= e^t \left(M \cos t + N \sin t\right), \\
y(0) &= 1 = M, \\
y(\pi/4) &= e^{\pi/4}\left(\frac{1}{\sqrt{2}} + N\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}} \implies N = 0 \\
y(t) &= e^t \cos t \\
y(\pi/3) &= e^{\pi/3}/2 \\
(y(\pi/3))^3 &= e^{\pi}/8
\end{aligned}
$$

END OF QUESTIONS

END OF PAPER