COMS10013 - Exam Questions

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Preface

NOTE(2025): This exam was done via Blackboard due to Covid, so the 'pool' system of questions is different to an in-person exam. The types of exam question are representative.

One question should be selected randomly from each of the following pools of questions.

Pool 1 — Hessians & Fixed Points **[3]**

(1) Consider the function $z(x, y) = ax^3y + by^2 - 3axy$, where a and b are real, positive constants. Which of the following statements is true.

A. The point $(x, y) = (1, \frac{a}{b})$ is a local maxima of z.
B. The point $(x, y) = (1, \frac{a}{b})$ is a local minima of z.
C. The point $(x, y) = (1, \frac{a}{b})$ is a saddle point of z.
D. None of the above.

(2) Consider the function $z(x, y) = ax^3y + by^2 - 3axy$, where a and b are real, positive constants. Which of the following statements is true.

A. The point $(x, y) = (-1, -\frac{a}{b})$ is a local maxima of z.
B. The point $(x, y) = (-1, -\frac{a}{b})$ is a local minima of z.
C. The point $(x, y) = (-1, -\frac{a}{b})$ is a saddle point of z.
D. None of the above.

(3) Consider the function $z(x, y) = ax^3y + by^2 - 3axy$, where a and b are real, positive constants. Which of the following statements is true.

A. The point $(x, y) = (\sqrt{3}, 0)$ is a local maxima of z.
B. The point $(x, y) = (\sqrt{3}, 0)$ is a local minima of z.
C. The point $(x, y) = (\sqrt{3}, 0)$ is a saddle point of z.
D. None of the above.

(4) Consider the function $z(x, y) = ax^3y + by^2 - 3axy$, where a and b are real, positive constants. Which of the following statements is true.

A. The point $(x, y) = (-\sqrt{3}, 0)$ is a local maxima of z.
B. The point $(x, y) = (-\sqrt{3}, 0)$ is a local minima of z.
C. The point $(x, y) = (-\sqrt{3}, 0)$ is a saddle point of z.
D. None of the above.

Solution: The gradient of z can be computed as $\nabla z = (3)\, ay(x^2 - 1)$
$ax(x^2 - 3) + 2by$
which gives the zero vector for all four possible points. The Hessian of z is $H =$

(6) $axy3a(x^2-1)$
$3a(x^2-1)2b$
. Note $H(x,y) = H(-x,-y)$, so answers for (1) and (2) (resp. (3) and (4)) are the same. For $(x,y) = (1, \frac{a}{b})$, we have $\det(H) = 12a^2 > 0$ and $H_{11} = \frac{6a^2}{b} > 0$, so we have a local minima. For $y = 0$, we have $\det(H) = -9a^2(x^2-1)^2 < 0$ irrespective of x, so we have a saddle point (note a is non-zero).

Pool 2 — 2nd Order Diff Eq **[3]**

(1) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0, y(0) = 1 and y(\frac{\pi}{6}) = e^{\frac{\pi}{3}}$.
Given that $(y(\frac{\pi}{12}))^2 = 2e^{\frac{c\pi}{6}}$, find the value c. Please write your answer as a number, with no full stop at the end.

(2) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 26y = 0, y(0) = 1 and y(\frac{\pi}{10}) = e^{-\frac{\pi}{10}}$.
Given that $(y(\frac{\pi}{20}))^2 = 2e^{\frac{c\pi}{10}}$, find the value c. Please write your answer as a number, with no full stop at the end.

(3) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 17y = 0, y(0) = 1 and y(\frac{\pi}{2}) = e^{2\pi}$.
Given that $(y(\frac{\pi}{4}))^2 = 2e^{\frac{c\pi}{2}}$, find the value c. Please write your answer as a number, with no full stop at the end.

(4) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0, y(0) = 1 and y(\frac{\pi}{4}) = e^{-\frac{3\pi}{4}}$.
Given that $(y(\frac{\pi}{8}))^2 = 2e^{\frac{c\pi}{4}}$, find the value c. Please write your answer as a number, with no full stop at the end.

Solution: Solving the characteristic polynomial gives $\lambda = \alpha \pm \beta i$, giving the general solution $y(t) = e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$. The first IC gives $A = 1$. The second IC is of the form $y(\frac{\pi}{2\beta}) = e^{\frac{\alpha\pi}{2\beta}}$ which gives $B = 1$. The relation is of the form $(y(\frac{\pi}{4\beta}))^2 = 2e^{\frac{c\pi}{2\beta}}$. Evaluating the LHS gives $(\sqrt{2}e^{\frac{\alpha\pi}{4\beta}})^2 = 2e^{\frac{\alpha\pi}{2\beta}}$, so $c = \alpha$.

(1) 2

(2) -1

(3) 4

(4) -3

Pool 3 — Polar Form **[3]**

(1) Let $z = (3+i)^3 + (3-i)^3$. By considering the polar form of $3+i$ or otherwise, compute the modulus of z. Please write your answer as an integer, with no full stop at the end.

(2) Let $z = (-1+2i)^3 + (-1-2i)^3$. By considering the polar form of $-1+2i$ or otherwise, compute the modulus of z. Please write your answer as an integer, with no full stop at the end.

(3) Let $z = (2+i)^3 + (2-i)^3$. By considering the polar form of $2+i$ or otherwise, compute the modulus of z. Please write your answer as an integer, with no full stop at the end.

(4) Let $z = (1+3i)^3 + (1-3i)^3$. By considering the polar form of $1+3i$ or otherwise, compute the modulus of z. Please write your answer as an integer, with no full stop at the end.

Solution: By construction the expression is real, (and an integer). In general we have $z = (a+bi)^3 + (a-bi)^3 = 2a^3 - 6ab^2$ which can be derived in multiple ways algebraically (eg. by the polar form substitution suggested). The modulus is then just the absolute value of this expression. Alternatively, a numerical approach will suffice coupled with the stipulation that the result is an integer.

(1) 36

(2) 22

(3) 4

(4) 52

Pool 4 — Taylor Series [3]

(1) You will need a calculator for this question. Let $f(x) = e^{\frac{x^2}{2}+\frac{3}{4}}$, and let $T_n(x)$ denote the n'th Taylor polynomial approximation to f around the point $x_0 = 0$. Find the *minimum* value n such that the approximation $T_n(1)$ is within 0.1 of $f(1)$.

(2) You will need a calculator for this question. Let $f(x) = e^{\frac{x^2}{2}+\frac{4}{5}}$, and let $T_n(x)$ denote the n'th Taylor polynomial approximation to f around the point $x_0 = 0$. Find the *minimum* value n such that the approximation $T_n(1)$ is within 0.1 of $f(1)$.

(3) You will need a calculator for this question. Let $f(x) = e^{\frac{x^2}{2}+\frac{5}{6}}$, and let $T_n(x)$ denote the n'th Taylor polynomial approximation to f around the point $x_0 = 0$. Find the *minimum* value n such that the approximation $T_n(1)$ is within 0.1 of $f(1)$.

(4) You will need a calculator for this question. Let $f(x) = e^{\frac{x^2}{2}+\frac{6}{7}}$, and let $T_n(x)$ denote the n'th Taylor polynomial approximation to f around the point $x_0 = 0$. Find the *minimum* value n such that the approximation $T_n(1)$ is within 0.1 of $f(1)$.

Solution: Write $f(x) = Ae^{\frac{x^2}{2}}$ for convenience. By computing $f^{(n)}(0)$ for succssive n, we find $f^{(0)}(0) = f^{(2)}(0) = A, f^{(1)}(0) = f^{(3)}(0) = 0$, and $f^{(4)}(0) = 3A$ etc. and so $T_0(x) = T_1(x) = A$, $T_2(x) = T_3(x) = A(1+\frac{x^2}{2})$, and $T_4(x) = A(1+\frac{x^2}{2}+\frac{x^4}{8})$. Evaluating each T_n at $x = 1$, we find that $T_4(1)$ is within the required range.

Pool 5 — Abstract Differential Equation [3]

(1) Suppose the function $y(t)$ satisfies the differential equation $y'(t)+a(t)y(t) = b(t)$ where the functions $a(t)$ and $b(t)$ are not constant. Define the function $z(t) = y(2t)$. Which of the following differential equations is $z(t)$ a solution to?

A. $z'(t) + 2a(t)z(t) = 2b(t)$

B. $z'(t) + 2a(2t)z(t) = 2b(2t)$

C. $z'(t) + a(t)z(t) = b(t)$

D. $z'(t) + a(2t)z(t) = b(2t)$

(2) Suppose the function $y(t)$ satisfies the differential equation $y'(t)+a(t)y(t) = b(t)$ where the functions $a(t)$ and $b(t)$ are not constant. Define the function $z(t) = y(3t)$. Which of the following differential equations is $z(t)$ a solution to?

A. $z'(t) + 3a(t)z(t) = 3b(t)$

B. $z'(t) + 3a(3t)z(t) = 3b(3t)$

C. $z'(t) + a(t)z(t) = b(t)$

D. $z'(t) + a(3t)z(t) = b(3t)$

(3) Suppose the function $y(t)$ satisfies the differential equation $y'(t)+a(t)y(t) = b(t)$ where the functions $a(t)$ and $b(t)$ are not constant. Define the function $z(t) = y(4t)$. Which of the following differential equations is $z(t)$ a solution to?

A. $z'(t) + 4a(t)z(t) = 4b(t)$

B. $z'(t) + 4a(4t)z(t) = 4b(4t)$

C. $z'(t) + a(t)z(t) = b(t)$

D. $z'(t) + a(4t)z(t) = b(4t)$

(4) Suppose the function $y(t)$ satisfies the differential equation $y'(t)+a(t)y(t) = b(t)$ where the functions $a(t)$ and $b(t)$ are not constant. Define the function $z(t) = y(5t)$. Which of the following differential equations is $z(t)$ a solution to?

A. $z'(t) + 5a(t)z(t) = 5b(t)$

B. $z'(t) + 5a(5t)z(t) = 5b(5t)$

C. $z'(t) + a(t)z(t) = b(t)$

D. $z'(t) + a(5t)z(t) = b(5t)$

Solution: By making the substitution $s = ct$, where c is the relevant constant, and using the chain rule, we obtain $z'(t) = cy'(s)$. Substituting this into each candidate answer, **B** returns our original equation, so this is the correct answer.