COMS10013 - Analysis - WS1

This worksheet is partly taken from worksheets prepared by Chloe Martindale and Conor Houghton.

Questions

These are the questions you should make sure you work on in the workshop.

1. Apply the derivative definition to prove the sum rule for differentiation:

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)\,.$$

   Hint: Start by applying the derivative definition (with $\lim_{h\to 0}$) to the expression on the left.

2. (a) For what values of x does the graph

$$f(x)=2x^3-6x+3$$

   have a horizontal tangent?

   (b) Describe what a horizontal tangent represents in terms of the rate of change of the function.

   (c) How would you classify the x values that you found in part (a) into (local) maxima and minima? Aim your explanation at a high school student who hasn't seen second order derivatives.

3. In this question we're going to look at the quotient rule for differentiation. Let

$$F(x)=\frac{f(x)}{g(x)}\,.$$

   We'll show that

$$\frac{d}{dx}F(x)=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{g(x)^2}\,.$$

   (a) What properties of g do we need for F to be well-defined?

   (b) There's a nice trick in the proof of the quotient rule: consider the (equivalent) equation $f(x)=F(x)g(x)$. Use the product rule to calculate $\frac{d}{dx}f(x)$ and then use what you have found to extract the quotient rule.

4. Differentiate the following functions with respect to x, stating when you're using the sum/product/quotient/chain rules:

   a) $3x^2$

   b) $(x+2)^2$

   c) ae^{cx} where a and c are constants.

   d) $\exp x^2$

   e) $\exp 1/x$

f) $\frac{4x+5}{2-3x}$

5. In 1965, Moore's law came about: Moore[1] predicted that

   > The complexity for minimum component costs has increased at a rate of roughly a factor of two per year. Certainly over the short term this rate can be expected to continue, if not to increase.

   We're going write this as a differential equation.

   (a) Identify the variable that is changing.

   (b) In words, what is your function describing?

   (c) What differential equation captures Moore's law? You're looking to write something of the form $\frac{d}{dx}f(x) = g(x)$ (with your choice of variable names, and with $g(x)$ capturing Moore's law).

Extra questions

These are extra questions you might attempt in the workshop or at a later time.

1. Use your knowledge of the fact that $\frac{d}{dx}e^x = e^x$ to show that

$$\frac{d}{dx}\log(x) = \frac{1}{x}\,.$$

2. Differentiate x^x with respect to x.

3. Use Python (or your programming language of choice) to differentiate

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

[1] Here's a link to Moore's original paper: `http://cva.stanford.edu/classes/cs99s/papers/moore-crammingmorecomponents.pdf`. And here's an interesting distraction about the relevance of Moore's law today, by Intel's Ann Kelleher: `https://download.intel.com/newsroom/2022/new-technologies/ann-kelleher-iedm-2022.pdf`