COMS10013 - Analysis - WS3

This worksheet is taken from worksheets prepared by Chloe Martindale and Conor Houghton.

Questions

These are the questions you should make sure you work on in the workshop.

1. **A linear accelerated motion question**. A train is travelling from Bristol to London Paddington at the maximum speed of 55.9 m/s, 125 mph, when the driver activates the emergency brake. This causes the train to decelerate uniformly at 1.2 m/s^2. How far will the train travel until it stops and how long will this take, in seconds. Do this using differential equations, for example:

$$\frac{dv}{dt} = -1.2 \tag{1}$$

not by looking up formulas.
Solution:
Let

- $y(t)$ be the position of the train at time t,
- $v(t) = \frac{dy}{dt}$ be the velocity of the train at time t.
- $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ be the acceleration of the train at time t.

It is given that $y(0) = 0$, $v(0) = 55.9$, and that for $t \geq 0$ the acceleration $a(t) = -1.2$ is constant. Then

$$v(t) = \int a(t)dt = \int -1.2\, dt = -1.2t + v(0) = -1.2t + 55.9.$$

So the train reaches velocity $v = 0$ at time $t = 55.9/1.2 \approx 46.6$, that is, after 46.6 seconds. Also the position

$$y(t) = \int v(t)dt = \int -1.2t + 55.9\, dt = -0.6t^2 + 55.9t + y(0) = -0.6t^2 + 55.9t.$$

So the position of y at the moment the velocity v reaches zero, which we just computed occurs at time $t = 55.9/1.2$, is

$$y(55.9/1.2) = -0.6 \cdot (55.9/1.2)^2 + 55.9 \cdot (55.9/1.2) \approx 1302.0,$$

i.e., 1302 metres further on from the point when the brake was activated.

2. **Types of differential equations** Write down (but don't solve) an example of a differential equation that is:
   (a) First-order, linear but not homogeneous, with constant coefficients.
   (b) First-order, linear, homogeneous but without constant coefficients.
   (c) Second-order, linear, homogeneous, with constant coefficients.
   (d) Second-order, linear, not homogeneous, without constant coefficients.

Solution:

(a) Anything of the form

$$ay'(x) + by(x) = c,$$

where a, b, c are independent of x and y, and a and c are non-zero.

(b) Anything of the form

$$a(x)y'(x) + b(x)y(x) = 0,$$

where at least one of $a(x)$ and $b(x)$ is non-constant, and a is non-zero.

(c) Anything of the form

$$ay''(x) + by'(x) + cy(x) = 0,$$

where a, b, c are independent of x and y and a is non-zero.

(d) Anything of the form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = d(x),$$

where at least one of $a(x), b(x), c(x), d(x)$ is non-constant, and $a(x)$ and $d(x)$ are non-zero.

3. **Differential equations** Solve the following, linear, homogeneous, first-order, constant coefficients, differential equations once using direct integration and once with the *ansatz*.

(a) $y'(t) - y(t) = 0$ with initial condition $y(0) = 2$: this is the classic: $y = Ae^t$ and the initial condition gives $A = 2$.

(b) $y'(t) + 3y(t) = 0$ with initial condition $y(3) = 3$: this one isn't much different, $y = Ae^{-3t}$ and the initial condition isn't an initial condition, but a *boundary condition*, which is sneaky, but $Ae^{-9} = 3$ so $A = 3e^9$.

(C) $y'(t) - 6t^2y(t) = 0$ with initial condition $y(0) = 3$: now trying an Ansatz of the form Ae^{rt} won't work, because of the t^2 involved. So we're looking for an Ansatz that, when differentiated, somehow invokes the chain rule to make a t^2 term appear. We'll try $y = Ae^{rt^3}$ so that

$$y'(t) - 6t^2y(t) = Ae^{rt^3}3rt^2 - 6t^2Ae^{rt^3} = At^2e^{rt^3}(3r - 6)$$

and setting this to zero gives that $r = 2$. So our solution is $y(t) = Ae^{2t^3}$. We'll now invoke the initial condition to get $A = 3$.

4. **First order inhomogeneous equations**.

(a) $f'(t) + 5f(t) = 1$ with initial condition $f(0) = 2$: well staring it at a bit $f(t) = 1/5$ is a particular solution. We still need to find the complementary function, that comes from solving $f'(t) + 5f(t) = 0$. Luckily, Q3 has us well-equipped to solve this one, so we could either try direct integration or an Ansatz (of Ae^{rt}) to find the complementary function is Ae^{-5t}. So

$$y(t) = Ae^{-5t} + \frac{1}{5}$$

is the general solution. Putting $f(0) = 2$ gives $A = 9/5$.

(b) $f'(t) = t - f(t)$ with initial condition $f(1) = 3e^{-1}$: we'll start out with the complementary function, solving $f'(t) + f(t) = 0$; an Ansatz of Ae^{rt} yields A^{-t}.

Now we need to find a particular solution. Luckily, it's easy enough to just stare it and 'guess' a solution, but that's not particularly helpful if you can't see it. We want a function that has something to do with t, and when we differentiate it, it's not becoming more complicated. This makes $f(t) = at + b$ a good guess: let's try it. $f'(t) = a$ and hence $a = t - at - b$ so $a = 1$ and $b = -a$ or $f(t) = t - 1$ is a particular solution. The general solution is

$$f(t) = Ae^{-t} + t - 1$$

The so called initial condition gives

$$3e^{-1} = Ae^{-1}$$

so $A = 3$.

(c) $f'(t) + 2f(t) = \sin(t)$ with initial condition $f(0) = 9/5$: let's start with finding the complementary function, by looking at $f'(t) + 2f(t)$. With an Ansatz of Ae^{rt}, we'll swiftly find that $r = -2$.

Finding the particular solution here is less straightforward due to the confounding $\sin(t)$ term. We're looking for something that involves a $\sin(t)$ term, and when we differentiate it, other terms magically disappear. We'll try

$$f(t) = a \sin t + b \cos t\,.$$

Plugging this in gives

$$a \cos t - b \sin t + 2a \sin t + 2b \cos t = \sin t$$

so the cosine coefficients give $a + 2b = 0$ and the sine coefficients give $-b + 2a = 1$, or $-b - 4b = 1$ so $b = -1/5$ and $a = 2/5$. Our particular solution is:

$$f(t) = \frac{2}{5} \sin t - \frac{1}{5} \cos t$$

and general solution

$$f(t) = \frac{2}{5} \sin t - \frac{1}{5} \cos t + Ae^{-2t}$$

If we put in $t = 0$ we get

$$\frac{9}{5} = -\frac{1}{5} + A$$

or $A = 2$.

(d) $f'(t) - 2f(t) + t^2 = 0$ with initial condition $f(2) = 13/4 + 6e^4$. Let's find the complementary function first, from solving $f'(t) - 2f(t) = 0$. Hopefully this is familiar now, so we'll use an Ansatz of Ae^{rt} and find that $r = 2$.

Now we'll look for the particular solution: this is quite similar to (b), but this time, we're looking for a function that has something to do with t^2 (and that doesn't become more complicated when differentiating). Let's use an ansatz $f = at^2 + bt + c$ to get

$$2at + b - 2at^2 - 2bt - 2c + t^2 = 0$$

or $a = 1/2$, $b = a$ and $b - 2c = 0$ so $b = 1/2$ and $c = 1/4$. The general solution is therefore

$$f(t) = Ae^{2t} + \frac{t^2}{2} + \frac{t}{2} + \frac{1}{4}$$

and substituting gives

$$\frac{13}{4} + 6e^4 = Ae^4 + \frac{13}{4}$$

and hence $A = 6$.

5. **Second order equations** Solve the following equations for the given initial conditions:
   (a) $y''(t) = 4y(t)$ with initial conditions $y(0) = 1$ and $y'(0) = 0$: we'll try our standard Ansatz of Ae^{rt}; this gives $Ae^{rt}(r^2 - 4) = 0$ which gives two solutions $r = \pm 2$. So our solution is $Ae^{2t} + Be^{-rt}$. Now let's use the initial conditions to find $A = B = \frac{1}{2}$.

   (b) $y''(t) + 4y'(t) + 3y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = -2$: now $y = e^{rt}$ gives

   $$r^2 + 4r + 3 = 0$$

   or $(r + 1)(r + 3) = 0$ hence

   $$y = Ae^{-t} + Be^{-3t}\,.$$

   Let's now apply the initial conditions: $y(0) = 0$ tells us that $A + B = 0$. The second initial condition tells us that $-A - 3B = -2$, so $A = -B$ or $B = 1$ and $A = -1$.

   (c) $y''(t) + 2y'(t) + y(t) = 0$ with initial conditions $y(0) = 2$ and $y(1) = 3/e$: we'll try our standard Ansatz and find that $r^2 + 2r + 1 = 0$, or $(r + 1)^2 = 0$ so $r = -1$. Before, our Ansatz gave us two linearly independent solutions, which is what we need for a second order differential equation. However, because of the repeated root, we only get one. So somehow we need to 'wiggle' with the solution to get two. $Ae^{-t} + Bte^{-t}$ gets us there – by multiplying the second guess by t, we've created enough independence in our solution to be able to satisfy the initial conditions. Solving these conditions gives

   $$2e^{-t} + te^{-t}\,,$$

   and we can check that this works.

Extra questions

1. **Solutions as a vector space** The aim of this exercise is to prove most of the following theorem: the solutions to the second-order linear homogeneous differential equation $a\ddot{y}(t) + b\dot{y}(t) + cy(t) = 0$ form a vector space. Note that this also follows from a theorem stated in the lecture notes. Prove:
   (a) If $f(t)$ and $g(t)$ are two solutions to this differential equation, then $h(t) = f(t) + g(t)$ is also a solution to this differential equation: this is easy since $d(f + g)/dt = \dot{f} + \dot{g}$ and so on.

(b) If $f(t)$ is a solution to this differential equation, and s is any integer, then $k(t) = s \cdot f(t)$ is also a solution to this differential equation: again, follow from linearity of differentiation $d(sf)/dt = s\dot{f}$.

(c) The function $f_0(t) = 0$ (the function that is zero for all t) is a solution to the differential equation: easy to see from substituting zero into the equation.

Technically we would also have to show that addition of solutions is commutative and associative, but this is tedious and doesn't have anything to do with differential equations - the way to prove this would just be to show it for all functions. So you don't have to prove that here.

Note that this theorem holds even if a, b, c are functions of t, same proof, you never had to differentiate these constants, and for any order of linear differential equation, not just second order. The theorem doesn't hold for inhomogeneous equations however: for a challenge, can you see which parts of the proof don't work for $a\ddot{y}(t) + b\dot{y}(t) + cy(t) = d$ and for which d? **Solution**: so if f and g are solutions and you substitute in $\lambda f + \mu g$ you end up with $(\lambda + \mu)d = d$ so, basically, the linearity holds for $\mu = 1 - \lambda$; the third property, zero is a solution, doesn't hold unless d is zero.