Differential equations

A differential equation is an equation for an unknown function, or functions, where the equation involves derivatives of the function.

Differential equations are everywhere, as we typically observe and measure events by how they change (a derivative). For example Newton's law in dynamics, $F = ma$ is a differential equation since a, the acceleration, is the second derivative of position; in quantum mechanics, Schrödinger's equation, or even more complicated equations like the Dirac equation, are differential equations. In biology, the Hodgkin-Huxley equation links the rate of change of the voltage inside a neuron to its inputs, in finance, the Black-Scholes equation links the rate of change value of an option to the value of the underlying commodity, climate modelling is just a giant set of differential equations.

We can solve some equations exactly, others we can't solve, in fact, most non-linear equations cannot be solved exactly, but we can often solve them approximately using computers and knowing about the solutions to linear equations can give us a good understanding of how differential equations work.

Solving by direct integration

Before we discuss terminology and methods, we'll look at a couple of examples, here is a very simple example, say

$$\frac{df}{dt} = t^2 + 2 \tag{1}$$

and we know the initial value of $f(t)$, it is $f(0) = 5$. We can solve this by **direct integration**, we write

$$df = (t^2 + 2)dt \tag{2}$$

and integrate both sides

$$\int df = \int (t^2 + 2)dt \tag{3}$$

or

$$f = \frac{1}{3}t^3 + 2t + C \tag{4}$$

where C is the arbitrary constant of integration. Now substitute in $t = 0$ to get

$$5 = C \tag{5}$$

and the solution is

$$f = \frac{1}{3}t^3 + 2t + 5 \tag{6}$$

Now you could object that there is nothing in the proofs and definitions we looked at to say you can treat the df/dt like a fraction and multiply across by dt, in fact, you most explicitly cannot treat df/dt as a fraction in most circumstances, but there are theorems that say that it works in cases like the one we just looked at where we are going to integrate. In other words, don't think the fact df/dt looks like a fraction allows you to treat it like a fraction in all circumstances, but, in fact, there are theorems we won't look at that says that you can in this situation. This theorem is called **the fundamental theorem of calculus** and it basically says that the inverse of differentiation, the anti-derivative, is the indefinite integral. There is a short table of integrals at the end of these notes in case you need reminding.

Here is another simple example:

$$\frac{df}{dt} = 2f \tag{7}$$

This means

$$\int \frac{df}{f} = 2\int dt \tag{8}$$

and using $\int dx/x = \log x + C$ this gives

$$\log f = 2t + C \tag{9}$$

or

$$f = e^{2t+C} \tag{10}$$

or

$$f = Ae^{2t} \tag{11}$$

Here we have taken the $\exp C$ and, since C is an arbitrary constant, we can just rewrite it as another arbitrary constant $A = \exp C$. It might appear that this is cheating since $\exp C$ is always positive, but with complex numbers that isn't always the case and, in fact, there is a bit of messing going on with the sign when you do the integration, so it actually all works out. You can check the putative solution is the solution by differentiating:

$$\frac{d}{dt}Ae^{2t} = 2Ae^{2t} = 2f \tag{12}$$

Ordinary Differential Equations (ODEs) and their classification

We are going to look at differential equations with just one unknown function and one variable; this is an **ordinary** differential equation, for example

$$\frac{d\theta}{dt} = w + \sin\theta - t \tag{13}$$

for constant w is an ordinary differential equation, albeit a hard one, whereas

$$\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial x^3} - 6\phi\frac{\partial\phi}{\partial x} = 0 \tag{14}$$

for a function $\phi(x,t)$ that depends on x and t is a **partial** differential equation and not an ordinary differential equation. We won't be looking at partial differential equations here. If you are curious, this very difficult partial differential equation is called the Kortweig de Vries equation and it describes waves in a shallow channel; it was written down to describe a odd wave the engineer John Scott Russell spotted in a canal near Edinburgh.

Linear ODEs

An ordinary differential equation is **linear** if $f(t)$ is the unknown function and all the terms in the equation are of the form

$$A(x)\frac{d^nf(x)}{dx^n}$$

for $n \geq 0$. This means that terms are a product of a(n arbitrary and likely different) function involving only x and a derivative of f (of f itself). Importantly, this (potentially higher order) derivative is not squared or cubed etc – it appears only as a power one term. Thus

$$\frac{df}{dt} = tf + t^2$$

is a linear equation but
$$\frac{df}{dt} = f^2$$
is not, nor is
$$\frac{df}{dt} = \sin f$$
nor is
$$f\frac{df}{dt} = t\,.$$
Generally speaking, we are very good at solving linear equations but often struggle with nonlinear ones. Some nonlinear equations are linear equations in disguise, for example
$$f\frac{df}{dt} = f^2 \tag{15}$$
is actually the linear equation
$$\frac{1}{2}\frac{dg}{dt} = g \tag{16}$$
where $g = f^2$, but genuinely nonlinear equations can be hard, we will look at a few, fairly easy nonlinear equations in the worksheet.

Homogeneous and inhomogeneous linear ODEs

An linear ordinary differential equation can be **homogenous** or **inhomogeneous**; if f is the unknown function and t the variable then homogeneous equation is one where all the terms have an f or a derivative of f in, so
$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = \sin t \tag{17}$$
is inhomogeneous, whereas
$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = 0 \tag{18}$$
is homogeneous. The order of a ordinary differential equation is the order of the highest derivative, so the example above is second order because of the d^2f/dx^2 term whereas
$$\frac{df}{dt} = \sin f \tag{19}$$

is first order. Finally, the easiest examples have constant coefficients like

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f = 0 \tag{20}$$

whereas more difficult, but for linear cases, often still solvable examples have coefficients that depend on the variable, like

$$\frac{d^2f}{dt^2} + 2t\frac{df}{dt} + f = 0 \tag{21}$$

Solving first order linear homogeneous ODEs

Here we'll look at solving first order linear ordinary differential equations with constant coefficients. In this section we will look at homogenous equations.

The first method is direct integration, here is another example:

$$\frac{df}{dt} = \frac{1}{t} \tag{22}$$

with $f(2) = 3$ then

$$\int df = \int \frac{dt}{t} \tag{23}$$

so

$$f = \log t + C \tag{24}$$

where C is a constant of integration. A differential equation tells you how a function is changing in terms of its current value; this means that to know the solution completely, without an arbitrary constant, you usually need to know the function at at least one place: if I tell you the speed something is moving at, you can work out how far it will move but you can't work out where it will be unless I also tell you where it started. Often this extra information is an initial condition, $f(0)$ but in this problem we have been given the value at $t = 2$:

$$3 = f(2) = \log 2 + C \tag{25}$$

so $C = 3 - \log 2$ and the solution is

$$f = \log\frac{t}{2} + 3 \tag{26}$$

where we have used the law of logs that says $\log a - \log b = \log a/b$.

A second method of solving differential equation is the **ansatz**. The word ansatz is German for guess and the idea here is that you just guess the solution, a good guess often looks like

$$f(t) = Ae^{rt} \tag{27}$$

with constants r and A and then see if that works out, so basically you are guessing the solution has a particular form, in this case an exponential and you substitute in and check if it works out, if the solution doesn't have that form it won't work, if it does, you've solved the equation. The easiest example would be equations that look like

$$\frac{df}{dt} = 4f \tag{28}$$

Substitute in the ansatz and you get

$$Are^{rt} = 4Ae^{rt} \tag{29}$$

Cancel the A and the $\exp rt$ and you get

$$r = 4 \tag{30}$$

and so the ansatz works provided $r = 4$ and so the solution is

$$f = Ae4t \tag{31}$$

If you had used a bad guess, say $f = At^r$ for constant r it just won't work:

$$Art^{r-1} = 4At^r \tag{32}$$

and then dividing across by At^{r-1} we get

$$r = 4t \tag{33}$$

which isn't a constant.

Solutions as vector spaces

In your linear algebra lectures you have learned about vector spaces, a set of objects is a vector space is it has an appropriate linear structure, so if

$$V = \{v_1, v_2, \dots, v_n\} \tag{34}$$

is a set it is a vector space if for any v_i and v_j in the set

$$v = v_i + v_j \tag{35}$$

is also in the set and, if v is in the set, so is cv where c is a number and finally, the set contains a "zero vector", say e so that if v is in the set $v + e$ is also in the set. We haven't set what type of number c is and you can have vector spaces over the real numbers where c has to be a real number or, for example, vector spaces over the complex numbers, where c can be complex.

Vector spaces are very useful, in part because it is a very common structure where things that work on simple examples work on much harder ones, always useful in mathematics. In the case of vector spaces, we have simple examples like three-dimensional vectors where we can imagine the vectors by imagining vectors in space, but there are also complicated infinite-dimensional vector spaces where we can use the intuition we build up in three dimensions to prove theorems.

An example of an infinite dimensional vector space is the space of functions, if $f(x)$ is a function and $g(x)$ is a function and c_1 and c_2 are real numbers then

$$h(x) = c_1 f(x) + c_2 g(x) \tag{36}$$

is also a function. The map $e(x) = 0$ is a zero function. Now one thing we do in the case of finite-dimensional vector spaces is find basis vectors, **i**, **j** and **k** for example; the idea of basis vectors for the space of functions is a powerful one, at the heart, for example, of Fourier analysis; we won't really discuss that here, but bear it in mind when you see Fourier analysis in later years.

Another important idea for finite dimensional vector spaces is matrix operations, linear transformations of space are written as matrices. Now differentiation and integration are also linear:

$$\frac{d}{dx}[c_1 f(x) + c_2 g(x)] = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx} \tag{37}$$

and

$$\int [c_1 f(x) + c_2 g(x)] dx = c_1 \int f dx + c_2 \int g dx \tag{38}$$

and so, in a way, differentiation and integration can be thought of as infinite-dimensional matrices. In fact the function $\exp x$ has the property that

$$\frac{de^x}{dx} = e^x \tag{39}$$

and

$$\int e^x dx = e^x + C \tag{40}$$

so, with a little messing about the integration constant, we can think of the exponential as an eigenfunction of these operators. Again, these ideas go beyond this unit, but hopefully you can see that there is an important and powerful approach to differential equations which thinks about the vector space of functions.

We have seen another interesting point though, often if we have a solution to a differential equation, well a homogeneous first order linear differential equation, then if we have a solution we can multiply it by a constant and still have a solution; the constant is fixed by the initial condition, not the differential equation. In fact this is part of a larger theorem, which we state here without proving: the solutions to a homogeneous linear differential equation of order n form an n- dimensional vector space which is a subspace of the function vector space. In other words, the general solution to such an equation is a function

$$f(t) = A_1 f_1(t) + A_2 f_2(t) + \ldots + A_n f_n(t) \tag{41}$$

with n different linearly independent basis functions $f_i(t)$ and n integration constants A_i.

Solving first order linear inhomogeneous ODEs

First order inhomogeneous linear ordinary differential equations are, in principle, solvable but, in practice, we can't always do the integrals we need to do.

In this section, we'll study equations of the form

$$\frac{dy}{dx} + a(x)y(x) = b(x)$$

for some functions $a(x), b(x)$. We'll actually restrict to the easier case where $a(x)$ is a constant.

To solve this equation, we first find a *particular solution.* That is, any solution to the differential equation. There is a whole series of ad hoc approaches to this; here well look at the case where $f(t)$ is an exponential, so for example

$$\frac{dy}{dx} + 3y = 5e^{-x} \tag{42}$$

Now, we again use an ansatz of $y = A \exp \lambda x$:

$$A\lambda e^{\lambda x} + 3Ae^{\lambda x} = 5e^{-x} \tag{43}$$

and for this to work we need $\lambda = -1$, in which case we can cancel the exponentials and get

$$-A + 3A = 5 \tag{44}$$

or $A = 5/2$, hence a particular solution is

$$y = \frac{5}{2}e^{-x}\,. \tag{45}$$

Although our particular solution is a solution to our ODE, it's not the only solution: to find the family of solutions (the 'general solution') we must also consider the solution of the corresponding homogeneous aspect: i.e. solve

$$\frac{dy}{dx} + 3y = 0$$

In other words, we ignore the terms of the equation that don't involve our function f, and solve this. We'll solve this using our methods for solving first order linear homogeneous ODEs, and a good Ansatz of Ae^{rx} reveals that a solution is Ae^{-3x}. The solution to the homogeneous part of the equation is called the *complementary function.*

Our general solution is the particular solution plus the complementary solution:

$$y = \frac{5}{2}e^{-x} + Ae^{-3x}.$$

Solving second order homogeneous linear ODEs

A second order homogeneous linear ODE is an equation of the form

$$f''(x) + A(x)f'(x) + B(x)f(x) = 0\,.$$

We won't say too much about these, except to say that in general, an Ansatz is a good approach:

First, lets solve a second order differential equation:

$$y''(t) - y(t) = 0 \tag{46}$$

and, since we don't have a method, lets try an ansatz, guessing $y = A \exp rt$:

$$Ar^2 \exp rt - A \exp rt = 0 \tag{47}$$

Cancel what can be cancelled and you get

$$r^2 = 1 \tag{48}$$

or $r = \pm 1$ so the solutions are

$$y = A_1 e^t + A_2 e^{-t} \tag{49}$$

It is easy to check this works:

$$y''(t) = A_1 e^t + A_2 e^{-t} = y \tag{50}$$

The first thing to notice is how easy this was; here's another example

$$y''(t) + y'(t) - 6y = 0 \tag{51}$$

Substitute for $y = A \exp rt$ and

$$r^2 + r - 6 = 0 \tag{52}$$

which factorizes to $(r + 3)(r - 2) = 0$ and the solution is

$$y = A_1 e^{-3t} + A_2 e^{2t} \tag{53}$$

Of course, it won't always be that easy, these two examples have equations for r that are easy to factorize into real numbers, we will look at a more complicated example soon. There are other even more complicated examples where the equation for r has a repeated root, these will still have two solutions even if there is only one value of r, this involves a different ansatz; it isn't difficult but for reasons of time we won't look at those cases here.

One thing to notice is that we have a two-dimensional space of solutions, with two arbitrary constants, A_1 and A_2. This makes sense, imagine throwing a ball into the air, to map the position of the ball through time you would need not only to know where you threw it from, but how fast. In the same way that the arbitrary constant for first-order differential equations is often fixed by initial conditions, for second order the two conditions are often fixed by an initial condition for $y(0)$ and $y'(0)$

As an example, say

$$y'' + 3y' + 2y = 0 \tag{54}$$

with $y(0) = 0$ and $\dot{y}(0) = 1$ then

$$y = A_1 e^{-t} + A_2 e^{-2t} \tag{55}$$

and $y(0) = A_1 + A_2 = 0$ and $\dot{y}(0) = -A_1 - 2A_2 = 1$, substituting $A_1 = -A_2$ from the $y(0)$ equation gives $1 = -A_2$ and so

$$y = e^{-t} - e^{-2t} \tag{56}$$

Recalling integration

There are two important ideas related to integration, the first is **the area under a curve**, the area between $f(x)$ and the x-axis from a to b is

$$A = \int_a^b f(x)dx \tag{57}$$

There is a whole "infinitessimals" discussion of how this can be calculated, basically you think about adding little bits

$$A(\delta) = \sum_{n=0}^{N} f(a + n\delta)\delta \tag{58}$$

where N is chosen so that $a + N\delta = b$; the area under the curve, and hence the integral, is th limit of $A(\delta)$ as $\delta \to 0$. It turns out there are lots of technical details to deal with to make this work, but, in any case it isn't the thing we are interested in here, we are interested in the **antiderivative**, for a function $f(x)$ we want to find $F(x)$ such that

$$\frac{d}{dx}F(x) = f(x) \tag{59}$$

It turns out, due to something called the **fundamental theorem of calculus** that this is the same calculation, more or less, and that the antiderivative is the indefinate integral

$$F(x) = \int f(x)dx \tag{60}$$

Thus integration is the opposite of differentiation, so, for example

$$\frac{dt^n}{dt} = nt^{n-1} \tag{61}$$

so

$$\int t^{n-1}dt = \frac{1}{n}t^n + C \tag{62}$$

C is the integration constant, an arbitrary number. The idea is that

$$\frac{d}{dt}t^n = nt^{n-1} \tag{63}$$

but so is

$$\frac{d}{dt}\left(t^n + C\right) = nt^{n-1} \tag{64}$$

for any constant C so if you think of the indefinite integral as being the backwards of differentiation what you get isn't fully defined because you can add an arbitrary constant without changing anything.

This leads us to a list of indefinite integrals

- $\int t^n dt = t^{n+1}/(n+1) + C$.
- $\int \exp rt dt = \exp rt/r + C$ where r is constant.
- $\int 1/t dt = \log t + C$
- $\int \sin t dt = -\cos t$
- $\int \cos t dt = \sin t$

In addition you can do a **substitution**, for example if you wanted

$$\int xe^{x^2}dx \tag{65}$$

you might want to do a change of variables to $u = x^2$ since you know how to integrate $\exp u$ when you dn't know how to integrate $\exp x^2$. However, you also need to change the dx, there is a rule for this

$$dx = \frac{1}{du/dx} du \tag{66}$$

This du/dx is called the Jacobian factor and there are some difficult points about its sign we are ignoring here. In the example this would tell you that

$$dx = \frac{du}{2x} \tag{67}$$

or $du = 2xdx$ so

$$\int xe^{x^2} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C \tag{68}$$

Now you'll see this only worked because there was an x knocking around to allow us to change the variable and this suggests some things are hard, even impossible, to integrate. We haven't looked at all the tricks, indeed we haven't looked at **integrating by parts**, one of the most important tricks, but it is indeed nonetheless true that some integrals are hard and many are impossible.

Summary

Differential equations are important, they are equations which include derivatives of the unknown function; a useful and common class of differential equations are the first order linear homogeneous ordinary differential equations. We can solve these using direct integration or using an ansatz: this leaves an arbitrary constant which is often fixed by an initial condition. Integration, in turn, is often hard, but we can do it for some functions, this might involve a change of variable, if you do change variables watch out for the Jacobian factor.