Part 2: Analysis

Question 1 (5 marks)
This question is about differentiation.

(i) In one sentence explain the infinitesimal approach to differentiation and use it to show that $\frac{d(x^3)}{dx} = 3x^2$

(ii) Using the chain rule, product rule and $d\sin x/dx = \cos x$, differentiate: $f(x) = x\sin x^2$

(5 marks)

Solution: In the infinitesimal approach you work out $[f(x+dx)-f(x)]/dx$ and, when it is safe to do so, you set dx to zero[2 points]. For x^3 you expand $(x+dx)^3$ using the binomial theorem[1 point]. Finally[2 points]

$$f'(x) = \sin x^2 + 2x^2\cos x^2$$

Question 2 (5 marks)
This question is about partial derivatives.

(i) Write down the definition of $\partial f(x,y)/\partial x$ and of ∇f.

(ii) If $f(x,y) = x\sin xy + y^2$, what is ∇f and what is the derivative along a unit vector in the $(1,1)$ direction?

(5 marks)

Solution: So[1 point]

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

and[1 point]

$$\nabla f = (f_x, f_y)$$

or in the particular example[1 point]

$$\nabla f = (\sin xy + xy\cos xy, x^2\cos xy + 2y)$$

and the directional derivative is given by the sum of both terms divided by $\sqrt{2}$[2 points] (half marks for correct answer without the normalization).

Question 3 (5 marks)
This question is about complex numbers.

(i) Write $z = \frac{1+2i}{2+i}$ in the form $x + iy$.

(ii) Write $z = 1 + \sqrt{3}i$ in the polar form.

(iii) The quaternions are a type of generalization of complex numbers. Instead of just i there are three imaginary numbers i, j and k, and these all square to minus one: $i^2 = j^2 = k^2 = -1$. In addition $ijk = -1$ and the numbers are *anti-commutative*: $ij = -ji$, $jk = -kj$ and so on. Lots of other relationships can be derived from these rules, for example if you multiply $ijk = -1$ you get $jk = i$, or if you switch it $jik = 1$ and multiply by j you get $ik = 1$. If $z = 1 + 3i + 2j$, what are zi and iz?

(5 marks)

Solution: (i) Division gives $z = (4 + 3i)/5$[1 point], (ii) the polar form has length two and angle $\pi/3$[2 points] and (iii) $zi = i - 3 + 2k$ [1 point]whereas $iz = i - 3 - 2k$[1 point].

Question 4 (5 marks)

This question is about differential equations.

(i) In one sentence explain the difference between a homogeneous and an inhomogeneous differential equation.

(ii) Solve $\frac{df}{dt} = 3f$ with $f(0) = 4$.

(iii) Solve $\frac{df}{dt} = 3f + t$ with $f(0) = 1$.

(5 marks)

Solution: So a homogenous equation only has f and its derivative[1 point]. As for the examples, the first one is $f = 4\exp(3t)$[2 points]; the second one $f(t) = 10/9\exp(3t) - t/3 - 1/9$[2 points]

Question 5 (5 marks)

This question is about optimization. Briefly describe gradient flow.

(5 marks)

Solution: In gradient flow you minimize a function by moving a small distance in the direction opposite the gradient.[5 points]

END OF PAPER