6. Complex numbers

LECTURE 6

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Workshop 5 summary:

What we’ll cover today:

- Complex numbers
- More about solving ODEs now we know about complex numbers
- What I expect you to know for the exam

Complex Numbers

What is an imaginary number?

- Philosophically, what is $\sqrt{2}$*?*
  - *A number that, when multiplied by itself, gives 2*

- $i = \sqrt{-1}$
- What is $\sqrt{-1}$?
  - *A number that, when multiplied by itself, gives -1*

What is a complex number?

- A complex number has a **real** part and an **imaginary** part:

$$z = a + ib \quad : a, b \in \mathbb{R}$$

- We can apply arithmetic rules to complex numbers (as long as we remember that $i^2 = -1$)
  - Addition: $(a_1 + i\, b_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$
  - Multiplication: $(a_1 + i\, b_1) \times (a_2 + ib_2) = a_1 a_2 + i a_1 b_2 + i b_1 a_2 + i^2 b_1 b_2 = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2)$

Examples with arithmetic

ADDITION

$$(5+3i)+(2+2i)$$
$$=7+5i$$

$$(5+3i)-(2+2i)$$
$$=3+i$$

MULTIPLICATION

$$(5+3i)\times(2+2i)$$

$$=5\times 2+5\times 2i+3i\times 2+3i\times 2i$$

$$=10+10i+6i+6i^2$$

$$=10\ -6+16i$$

$$=4+16i$$

Important attributes

- Let

$$z = a + ib \quad : a, b \in \mathbb{R}$$

- The **complex conjugate** of z is:

$$z^* = \bar{z} = a - ib$$

- The **absolute value** of z is:

$$|z| = \sqrt{zz^*} = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}$$

Example

$z = 5 + 3i$

1. Complex conjugate:

$$z^* = 5 - 3i$$

2. Absolute value:

$$|z| = \sqrt{(5 + 3i)(5 - 3i)}$$

- $(5 + 3i)(5 - 3i) = 5^2 - 15i + 15i\ - 3i^2 = 25 + 9 = 34$

$$\Rightarrow\ |z| = \sqrt{34}$$

What about division?

- Let $u = a + ib$ and $\mathrm{v} = c + id$
- Division: $\frac{u}{v} = \frac{a+ib}{c+id}$
- This doesn't look like $x + iy$
- Trick: multiply by one!
- $\frac{u}{v} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{ac+bd+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$
- We multiplied by the complex conjugate of the denominator

Example

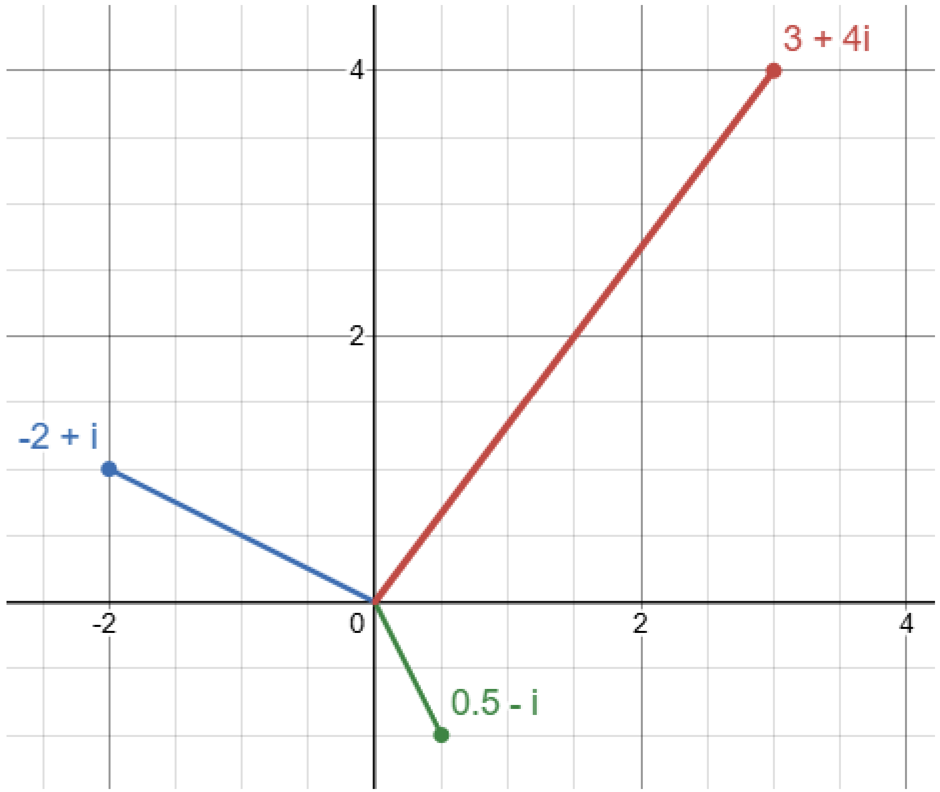
$$\frac{5+3i}{2+2i} = \frac{5+3i}{2+2i} \times \frac{2-2i}{2-2i}$$

Denominator: $(2+2i)(2-2i) = 2^2 - 2^2 i^2 = 8$

Numerator: $(5+3i)(2-2i) = 10 - 10i + 6i - 6i^2 = 16 - 4i$

$$\frac{5+3i}{2+2i} = \frac{16-4i}{8} = 2 - \frac{1}{2}i$$

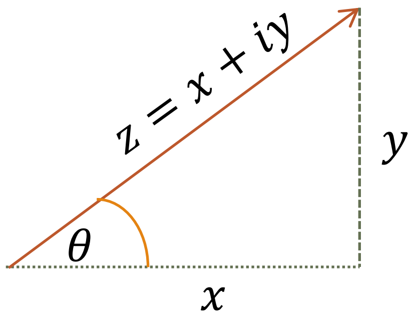
Visualising complex numbers



Polar representation

- In our visualisation, complex numbers had a **direction** and a **magnitude**
  - **Direction** = angle from horizontal
  - **Magnitude** = distance from origin = absolute value

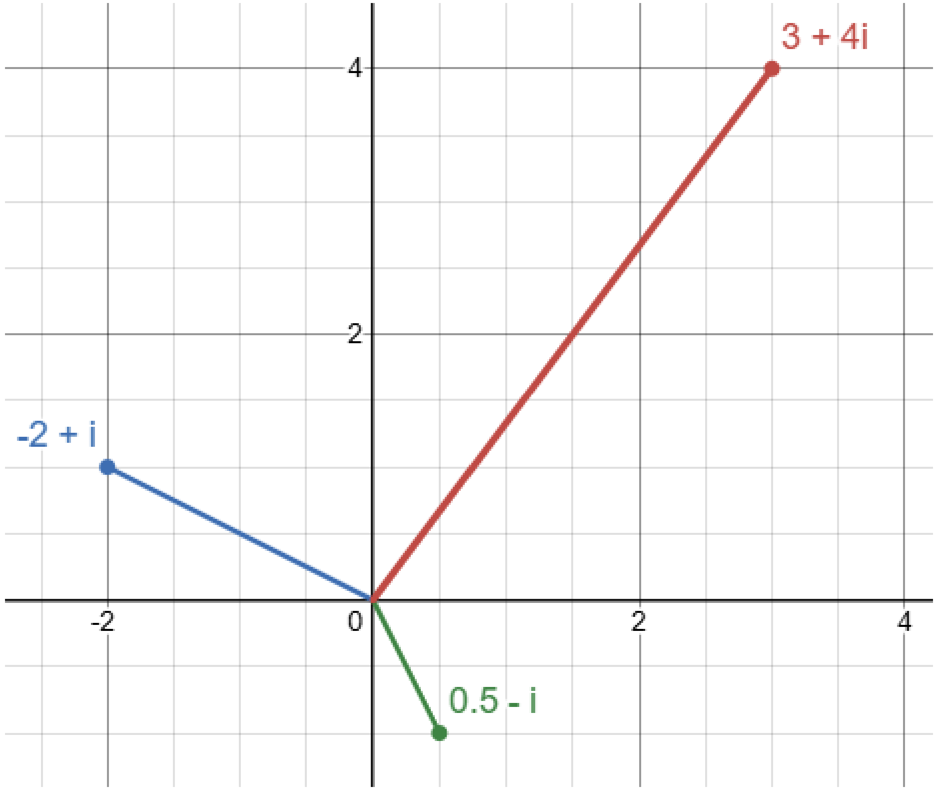
- An equivalent representation of a complex number is **polar representation**
- If $z = x + iy$:
  - Direction = angle = $\tan^{-1}\left(\frac{y}{x}\right) := \theta$
  - Magnitude = absolute value = $\sqrt{x^2 + y^2} := r$



- Polar representation: $z \to (r, \theta) \to re^{i\theta}$

$$\tan(\theta) = \frac{y}{x}$$

Example



$z = 3 + 4i$

- $r = |z| = 5$ (Pythagorean triple)
- $\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$
- Polar representation: $5e^{0.927i}$

$z = 0.5 - i$

- $r = |z| = \sqrt{0.25 + 1} = 1.118$
- $\theta = \tan^{-1}\left(\frac{-1}{0.5}\right) = -1.107$
- Polar representation: $1.118e^{-1.107i}$

Euler’s formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Arithmetic with polar representation

Suppose $z = x + iy = re^{i\theta}$

- $z^n = (x + iy)^n = ???$
- $z^n = \left(re^{i\theta}\right)^n = r^n\, e^{in\theta}$

Suppose $z' = x' + iy' = r'e^{i\theta'}$

- $z\, z' = (x + iy)(x' + iy') = ???$
- $z\, z' = \left(re^{i\theta}\right)\left(r'e^{i\theta'}\right) = r\, r'e^{i(\theta+\theta')}$

Differentiation with complex numbers?

$f: \mathbb{C} \to \mathbb{C}$

$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

Solving differential equations over $\mathbb{C}$

- We've got the ODE

$$f''(x) + f(x) = 0$$

- We'll try an Ansatz of Ae^{rx}
- Substituting this into our ODE:

$$Ar^2e^{rx} + Ae^{rx} = 0 \Rightarrow Ae^{rx}(r^2 + 1) = 0 \Rightarrow r = \pm i$$

- Solution is $Ae^{ix} + Be^{-ix}$
- Using Euler's formula:

$$\begin{aligned} &A\cos(x) + Ai\sin(x) + B\cos(-x) + Bi\sin(-x) \\ &= (A + B)\cos(x) + i\,(A - B)\sin(x) \\ &= C\cos(x) + iD\sin(x) \end{aligned}$$

Complex numbers summary

- $i = \sqrt{-1}$
- Complex numbers look like $\mathrm{a} + \mathrm{b}i$ and obey the laws of arithmetic
- We can define the absolute value and the complex conjugate
- We can write complex numbers in polar form $re^{i\theta}$…
- … and convert using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$
- Complex numbers can appear in the solution to ODEs

What have we seen in this section of Maths B?

1. Differentiation
2. Partial differentiation; classifying maxima and minima via Hessian; Gradient
3. ODEs (linear, homogeneous, inhomogeneous)
4. Optimisation (Gradient descent, Nelder-Mead)
5. Approximating functions (Linear Approximation, Newton root finding, Taylor series)
6. Complex numbers

Lecture 1: Differentiation

I EXPECT YOU TO…

- Understand the limit definition of a derivative
- Use properties about derivatives (e.g. product rule, chain rule)
- Know $\frac{d}{dx}x^n = nx^{n-1}$ and $\frac{d}{dx}e^x = e^x$

I DO NOT EXPECT YOU TO…

- Find the derivatives of functions from the limit definition
- Prove properties about derivatives (e.g. chain rule)
- Remember ‘obscure’ (e.g. trigonometric) derivatives

Lecture 2: Partial Differentiation

I EXPECT YOU TO…

- Calculate the partial derivative of a function
- Calculate the gradient ∇f
- Calculate the directional derivative
- Calculate the Hessian and use it to classify maxima/minima

I DO NOT EXPECT YOU TO…

- Apply the limit definition of a partial derivative
- Prove properties about partial derivatives
- Use the chain rule (as in Q3 on WS2)

Lecture 3: ODEs

I EXPECT YOU TO…

- Classify an ODE (e.g. linear, homogeneous)
- Solve an ODE by direct integration/separation of variables
- Solve first order linear homogeneous equations
- Solve first order linear inhomogeneous equations (using Complementary Function + Particular Solution)
- Solve second order linear homogeneous equations

I DO NOT EXPECT YOU TO…

- Find ‘obscure’ particular solutions
- Solve ODEs that are vastly different to what we’ve seen in the worksheets
- Apply anything beyond simple integration techniques

Lecture 4: Optimisation

I EXPECT YOU TO…

- Use calculus directly to optimise a function when possible
- Know the Gradient Descent algorithm
- Understand the structure of Gradient Descent/Nelder Mead (e.g. initial guess, iterations, stopping conditions)

I DO NOT EXPECT YOU TO…

- Know the details of the Nelder-Mead algorithm
- Apply Nelder-Mead Algorithm

Lecture 5: Taylor Series

I EXPECT YOU TO…

- Know about linear approximation
- Apply Newton's root-finding method
- Remember and apply the Taylor series to a differentiable function

I DO NOT EXPECT YOU TO…

- Memorise the Taylor series of functions

Lecture 6: Complex numbers

I EXPECT YOU TO…

- Know what a complex number is, perform arithmetic and calculate its absolute value and complex conjugate
- Switch between $x + iy$ representation and polar representation $re^{i\theta}$
- Know Euler's formula
- Recognise when a real ODE has complex solutions

I DO NOT EXPECT YOU TO…

- Suddenly start working with complex functions
- Know when a complex function is differentiable