COMS10013 Analysis exam questions

2 questions should be randomly selected from the following by Blackboard (these are meant to be harder):

1. You will need a calculator for this question. Give, to two decimal places, the value of x_0 for which the coefficient of x in the Taylor expansion of $f(x) = (\ln x)^{\ln x}$ is equal to zero. Please write your answer as a number.
   Solution: $x_0 = e^{1/e} \approx 1.44$.
   Check: The coefficient of x is zero if and only if $f'(x_0) = 0$. Rewriting $\ln x^{\ln x}$ as $e^{\ln x \ln \ln x}$ and differentiating gives $f'(x) = \left(\frac{1}{x} \ln \ln x + \frac{1}{x}\right) \cdot \ln x^{\ln x}$ which is zero if and only if $x = e^{1/e}$.

2. Let $z(x, y) = x^2 y + y^2 + 2axy$, where a is a real constant. For which value of a does z have exactly one point with gradient zero? Please write your answer as a number.
   Solution: 0
   Check: $\nabla z = \begin{pmatrix} 2xy + 2ay \\ x^2 + 2y + 2ax \end{pmatrix}$, which is zero at $(0, 0)$, $(-a, -a^2/2)$, and $(-2a, 0)$. These points are all equal if and only if $a = 0$.

3. You will need a calculator for this question. Suppose that on 1st January 2020 the population of Bristol was 686,000 and that on 1st January 2021 the population of Bristol was 694,000. A common way to model population growth is via the differential equation $\frac{dp}{dt} = cp$, where $p(t)$ is the population at time t, measured in years, and c is a real constant. Suppose that this model applies to Bristol and compute the value of c to 4 decimal places. Please write your answer as a number.
   Solution: 0.0116
   Check: $p(t) = Ae^{ct}$. Setting 2020 to be time $t = 0$ gives $A = 686000$ and $p(1) = 694000$, so $c = \ln(694000/686000)$.(N.B. to 4 decimal places, first approximating 694000/686000 to 4 decimal plaves and then taking log of that gives the same answer as computing the log of 694000/686000 to four decimal places).

4. You will need a calculator for this question. Let $y(t)$ be the number of people resident in the UK infected with COVID-19 at time t (measured in weeks) and let $r(t) = \frac{dy}{dt}$ be the rate at which the number of infected people is changing at time t. Suppose that $\frac{dr}{dt} = \frac{1}{30} r$ and that on week $t = 0$ of 2021 $r(t) = r(0) = -1$ and $y(t) = y(0) = 89,971$. Compute the lowest integer value of t for which $y(t) < 1$.

Solution: 241
Check: $r(t) = Ae^{\frac{1}{30}t}$ and by I.C. $A = -1$. Integrating gives $y(t) = -30e^{\frac{1}{30}t} + c$ and by I.C. $c = 90001$. Setting $y(t) = 1$ gives $3000 = e^{\frac{1}{30}t}$ so $t = 30\ln(3000) \approx 240.19$.

4 questions should be randomly selected by Blackboard from the following (these are meant to be easier):

1. How would you rewrite $f(x) = x^{\sin x}$ in order to differentiate $f(x)$ using the chain rule and product rule?
   (a) $f(x) = e^{\sin x \ln x}$ (this is the correct answer)
   (b) $f(x) = x^{-ie^{ix}}$
   (c) $f(y) = (\arcsin y)^y$

2. Let $z(x, y) = e^x y + 4y^3 x$. Compute $\nabla_{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} z \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$. You should write your answer as a number.
   Solution: 13
   Check: $\nabla z = \begin{pmatrix} e^x y + 4y^3 \\ e^x + 12y^2 x \end{pmatrix}$ so
   $$\left(\nabla z \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 2 \cdot (e^0 \cdot 1 + 4 \cdot 1^3) + 3 \cdot (e^0 + 12 \cdot 1^2 \cdot 0) = 13.$$

3. The differential equation $\frac{dy}{dt} + ty = 0$ is
   (a) First-order, linear, homogeneous, with constant coefficients.
   (b) First-order, non-linear, homogeneous, with constant coefficients.
   (c) First-order, linear, non-homogeneous, with constant coefficients.
   (d) First-order, linear, homogenous, with non-constant coefficients. (this is the correct answer).

4. Given that $\frac{dy}{dx} + 2xy = \frac{1}{xe^{x^2}}$ and $y(1) = 0$, what is $y(e)$?
   (a) $\frac{1}{e^{e^2}}$ (this is the correct answer)
   (b) e^{e^2}
   (c) e^{2-e^2}

5. Given that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$, that $y(0) = 1$ and $y(\pi/2) = e^{-\pi/2}$, what is $\left(e^{\pi/4} y(\pi/4)\right)^2$? Please write your answer as a number, with no full stop at the end.
   Solution: 2
   Check: $y(x) = Ae^{-x}\sin(x) + Be^{-x}\cos(x)$. ICs give $A = B = 1$. So $y(\pi/4) = \frac{2}{e^{\pi/4}\sqrt{2}} = \frac{\sqrt{2}}{e^{\pi/4}}$.

6. Suppose that $y(x)$ satisfies $x\frac{dy}{dx} + y = 0$. Given that $y(1) > 0$, which of the following is the biggest number?

   (a) $y(3.1)$ (this is the correct answer)

   (b) $y(200)$

   (c) $y(-0.4)$

   (d) $y(88.3)$

   Solution check: $y = c/x$, where c is an integration constant, and since $y(1) > 0$ we know that c is positive. Therefore the max value of $y(x)$ occurs for the minimum positive x.

7. Given that $\frac{dy}{dx} + 2y = e^{2x}$ and that $y(1) = e^2/4$, which of the following values is $y(1/2)$?

   (a) $e^{1/4}$

   (b) $e/4$ (this is the correct answer)

   (c) $e/2$

   (d) $e^{1/2}$

   Solution check: $y = e^{2x}/4 + ce^{-2x}$, and by I.C. $c = 0$. So $y(1/2) = e/4$.

8. At which value of x does $y(x) = e^x/x$ have a global minimum? Please write your answer as a number, with no full stop.
   Solution: 1
   Check: $\frac{dy}{dx} = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$ has a zero only at $x = 1$. $\frac{d^2y}{dx^2} = e^x\left(\frac{1}{x} - \frac{2}{x^2} + \frac{2}{x^3}\right) = e > 0$ at $x = 1$, so the only extremal point is a minimum, hence is global. (But the students can get there with only dy/dx and the context of the question)