3. Differential equations

LECTURE 3

2024/25

SOPHIE.STEVENS@BRISTOL.AC.UK

Workshop 2 summary:

3. Let $f(x,y)$ be a continuous function and let $\mathbf{w} = (w_1, w_2)$. In this question we'll show that

$$\nabla_{\mathbf{w}} f(x,y) = w_1 \frac{\partial f}{\partial x}(x,y) + w_2 \frac{\partial f}{\partial y}(x,y)$$

Start by considering the single-variable function $g(h) = f(x_0 + hw_1, y_0 + hw_2)$ where x_0, y_0 are fixed, arbitrary values for x and y

(i) Find the derivative of g at 0. What relation does this have to $\nabla_{\mathbf{w}} f$?

(ii) Let $a = x_0 + hw_1$ and $b = y_0 + hw_2$. Use the chain rule to find the derivative of g in terms of f_a and f_b.

(iii) Conclude to find the desired expression for $\nabla_{\mathbf{w}} f$

What we'll cover today:

- Solving differential equations
- Classification of ODEs
- First order linear homogeneous ODEs
- First order linear inhomogeneous ODEs
- Second order linear homogeneous ODEs
- (Second order linear inhomogeneous ODEs)

Solving a differential equation by **direct integration**

Example 1:

$$\frac{df}{dt} = t^2 + 2 \text{ and } \mathrm{f}(0) = 5$$

Example 2:

$$\frac{df}{dt} = 2f$$

[on board]

Ordinary Differential Equations (ODEs)

One unknown function and one unknown variable

E.g. $\frac{df}{dt} = e^{f} + 10 + \sin(t)$

The ODE is **linear** if the equation is linear in the unknown function :

$$a_n \frac{d^n}{dx^n} f(x) + a_{n-1} \frac{d^{n-1}}{dx^n} f(x) + \ \ldots + a_1 \frac{d}{dx} f(x) + a_0 = 0$$

The **order** of the ODE is the highest derivative that we take (i.e. n)

We (humans/machines) are good at solving linear equations

Classification Examples

- $f'(t) = e^f + 10 + \sin(t)$
- $f_x(x,t) = xe^{f(x,t)} + 10x^3 + \sin(t)$
- $f'(t) = 20f''(t) + f$
- $f''(t) = f(t)f'(t)$

- ODE; non-linear, 1st order
- Not an ODE
- ODE; linear; 2nd order
- ODE; non-linear, 2nd order

Types of linear ODE

HOMOGENEOUS

All terms contain exactly one power of the dependent variable and its derivatives:

- $\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

If $f(x)$ and $g(x)$ are solutions, so too is:

$$\alpha f(x) + \beta g(x)$$

- An order n homogeneous linear ODE will always have n independent solutions
- The solutions form a vector space

INHOMOGENEOUS

The right hand is not 0:

E.g. $\frac{dy}{dx}$ + 5 y = 6t

Solving first order homogeneous equations

$$y'(x) + A(x)\, y(x) = 0$$

1. Direct integration
2. Using an *Ansatz* : typically a good guess is ce^{rx}

Example: [board]

Solving first order inhomogeneous equations

$$y'(x) + A(x)\,y(x) = \mathrm{B}(x)$$

- Find a **particular solution**: any solution to the differential equation
- Find the solution to the corresponding homogeneous equation (**complementary function**): solve $y'(x) + A(x)\,y(x) = 0$
- **General solution = particular solution + complementary function**

Example: [board]

Solving second order homogeneous equations

$$y''(x) + A(x)y'(x) + B(x)y(x) = 0$$

- Let's assume $A(x)$ and $\mathrm{B}(x)$ are constants
- Ansatz 1: try $\mathrm{y} = ae^{rx}$
- Ansatz 2: try $\mathrm{y} = a_1e^{r_1x} + a_2e^{r_2x}$

Example: [board]

Summary

- Direct integration sometimes works
- Classification of ODEs (order, linear, homogeneous, inhomogeneous)
- First order linear homogeneous ODEs (via Direct Integration or Ansatz)
- First order linear inhomogeneous ODEs (solution = particular solution + complementary function)
- Second order linear homogeneous ODEs