COMS10013 - Analysis - WS5

Solutions

1. **Linear Approximation** The value of the function $\sin(x)$ is typically difficult to calculate. But let's suppose you urgently need to find the value of $\sin(0.1)$ but - catastrophe! - you've forgotten your calculator.
   Use the method of linear approximation of $\sin(x)$ at 0 to estimate $\sin(0.1)$.
   Using your calculator, how good is this approximation?
   Assume that you're seeking $\sin(0.1)$ in radians.

   Solution: We'll begin by using the linear approximation formula from the lectures, using $a = 0$:
   $$L(x) = \sin(0) + \cos(0)(x - 0)\,.$$
   Hopefully you remember $\sin(0) = 0$ and $\cos(0) = 1$ so that the linear approximation is just $L(x) = x$. Thus the approximation of $\sin(0.1)$ is 0.1.

   This is actually an instance of a special approximation phenomenon: when x is small, $\sin(x) \approx x$.

   How good is the approximation: $|\sin(0.1) - 0.1| = 0.000165....$ For most purposes, we can conclude that this approximation is pretty good.

2. **Root finding I:**

   (a) Pick an initial value and use the Newton root finding method to find the root of the equation
   $$f(x) = x^3 - 5\,,$$
   using say five iterations of the algorithm.
   Solution: The concrete solution to this problem obviously depends on your initial value: the technique is the same. Suppose g_0 is your initial guess (and hopefully you didn't cheat and use $g_0 = \sqrt[3]{5}$, otherwise this whole thing is moot).

   For the Newton root finding method, we'll put this into the iterative formula:
   $$g_{n+1} = g_n - \frac{g_n^3 - 5}{3g_n^2}\,.$$
   For example, if our initial guess is 2.5, then:
   $$g_1 = 2.5 - \frac{10.625}{18.75} = 1.933\ldots$$
   Continuing, we find that $g_2 = 1.735$, $g_3 = 1.710$, $g_4 = 1.710$ and $g_5 = 1.710$ (where I've rounded the guesses to 3 decimal places),

   (b) Compare your solution with $\sqrt[3]{5}$.
   Solution: For the guess of 2.5, we estimate the (rounded) root to be 1.710. If we use the unrounded value, we find that this differs from $\sqrt[3]{5}$ by about 3.3×10^{-15}.

3. **Root finding II:**

(a) Pick an initial value and use the Newton root finding method to find the root of the equation

$$f(x) = \sin(x)x^3 + \cos(x)\,,$$

using say five iterations of the algorithm.

(b) Evaluate f at your root guess. Was your initial value good?

Solution: This is essentially the same as the previous question, but now the differentiation is a bit harder and it's not obvious what the root actually is.

We'll use the iterative formula

$$g_{n+1} = g_n - \frac{\sin(g_n)g_n^3 + \cos(g_n)}{\cos(g_n)g_n^3 + 3g_n^2\sin(g_n) - \sin(g_n)}\,.$$

So for instance, trying the same guess of $g_0 = 2.5$ as in the previous question, we get:

$$\begin{aligned} g_1 &= 7.012 \\ g_2 &= 6.363 \\ g_3 &= 6.282 \\ g_4 &= 6.279 \\ g_5 &= 6.279 \end{aligned}$$

Already we can see that after five iterations, we're converging on a root (at least, if we round liberally to 3 decimal places).
We find that $f(g_5) = 2.14 \times 10^{-9}$, which helpfully is, as desired, pretty close to zero.

4. **Taylor series**

(a) Compute the Taylor series of $1/(1-x)^2$ at $x = 0$.
Solution: So this one is cool,

$$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3} \tag{1}$$

and

$$\frac{d^2}{dx^2}\frac{1}{(1-x)^2} = \frac{6}{(1-x)^4} \tag{2}$$

if you do a few more you might guess

$$\frac{d^n}{dx^n}\frac{1}{(1-x)^2} = \frac{(n+1)!}{(1-x)^{n+2}} \tag{3}$$

and you can prove this is true by induction. Now

$$\left.\frac{d^n}{dx^n}\frac{1}{(1-x)^2}\right|_{x=0} = (n+1)! \tag{4}$$

so

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n. \tag{5}$$

(b) Compute the Taylor series of $1/x$ at $x = 2$.
Solution: Again if you differentiate a few times you'll see

$$\frac{d^n}{dx^n}\frac{1}{x} = \frac{(-1)^n n!}{x^{n+1}} \tag{6}$$

and substituting that in to the Taylor series gives

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{2^{n+1}}\,. \tag{7}$$

5. **Computing with Taylor series**.
This exercise is to approximate $\sin(\pi/4)$ without using any trigonometric functions on your calculator.
Compute, without a calculator, the Taylor series of $\sin(x)$ at $x = 0$.
Compute the approximations $T_1(x)$, $T_3(x)$, $T_5(x)$, $T_7(x)$ from your series at $x = \pi/4$ to eight decimal places (you can use a calculator).
Solution: The main goal of this is to compute the Taylor series at $\sin(x)$: it's

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

so that the first few terms look like:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

i.e. the sign of the terms oscillates between ± 1, and we're only seeing odd powers of x.
Then

$$\begin{aligned}
T_1(\pi/4) &= 0.78539816\\
T_3(\pi/4) &= 0.70465265\\
T_5(\pi/4) &= 0.707143046\\
T_7(\pi/4) &= 0.707106470
\end{aligned}$$

and we're aiming for the true value: $\sin(\pi/4) = 0.7071067811865476\dots$.