COMS10013 - Resit Exam Questions

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Preface

NOTE(2025): This exam was done via Blackboard due to Covid, so the 'pool' system of questions is different to an in-person exam. The types of exam question are representative.

One question should be selected randomly from each of the following pools of questions.

1. (a) Consider the function $z(x, y) = \sin(x)\cos(e^{-y^2})$. Which of the following statements is true?
   - A. z has infinitely many local maxima.
   - B. z has infinitely many local minima.
   - **C. z has infinitely many saddle points.**
   - D. All of the above.

   (b) Consider the function $z(x, y) = \cos(x)\cos(e^{-y^2})$. Which of the following statements is true?
   - A. z has infinitely many local maxima.
   - B. z has infinitely many local minima.
   - **C. z has infinitely many saddle points.**
   - D. All of the above.

   (c) Consider the function $z(x, y) = \sin(y)\cos(e^{-x^2})$. Which of the following statements is true?
   - A. z has infinitely many local maxima.
   - B. z has infinitely many local minima.
   - **C. z has infinitely many saddle points.**
   - D. All of the above.

   (d) Consider the function $z(x, y) = \cos(y)\cos(e^{-x^2})$. Which of the following statements is true?
   - A. z has infinitely many local maxima.
   - B. z has infinitely many local minima.
   - **C. z has infinitely many saddle points.**
   - D. All of the above.

Solution: Let $z(x, y) = \cos(x)\cos(e^{-y^2})$, the analysis is virtually the same for all variants. Solving $\nabla z(x, y) = 0$ gives $(x, y) = (n\pi, 0)$, so we have infinitely many stationary points. The Hessian at $y = 0$ gives $H = \begin{pmatrix} -\cos(x) & 0 \\ 0 & 2\cos(x)\sin(1) \end{pmatrix}$ which has negative determinant irrespective of x, hence all stationary points are saddle points.

2. (a) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$, $y(0) = 0$, $y(\ln(2)) = -2$.
Compute $y(\ln(3))$. Please write your answer as an integer, with no full stop at the end.

(b) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = 0$, $y(0) = 0$, $y(\ln(2)) = -12$.
Compute $y(\ln(3))$. Please write your answer as an integer, with no full stop at the end.

(c) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 9\frac{dy}{dt} + 18y = 0$, $y(0) = 0$, $y(\ln(2)) = -56$.
Compute $y(\ln(3))$. Please write your answer as an integer, with no full stop at the end.

(d) The function $y(t)$ satisfies $\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 32y = 0$, $y(0) = 0$, $y(\ln(2)) = -240$.
Compute $y(\ln(3))$. Please write your answer as an integer, with no full stop at the end.

Solution: The characteristic polynomial has the form $\lambda^2 - 3\lambda\alpha + 2\alpha^2 = (\lambda - \alpha)(\lambda - 2\alpha)$, so the general solution has the form $y(t) = e^{\alpha t}(A + Be^{\alpha t})$. The first IC gives $A = -B$. The second IC gives $A = -B = 1$, so $y(t) = e^{\alpha t}(1 + e^{\alpha t})$. The answer is then of the form $3^\alpha(1 - 3^\alpha)$.

(a) -6

(b) -72

(c) -702

(d) -6480

3. (a) How many solutions are there to the equation $\bar{z} = z^2$ in the complex numbers? Please write your answer as an integer, with no full stop at the end.

   (b) How many solutions are there to the equation $\bar{z} = z^3$ in the complex numbers? Please write your answer as an integer, with no full stop at the end.

   (c) How many solutions are there to the equation $\bar{z} = z^4$ in the complex numbers? Please write your answer as an integer, with no full stop at the end.

   (d) How many solutions are there to the equation $\bar{z} = z^5$ in the complex numbers? Please write your answer as an integer, with no full stop at the end.

Solution: We can take the modulus to obtain $|z| = z^n$, so $|z| = 1$ or $z = 0$. Multiplying the original equation by z, we then obtain $z^{n+1} = 1$, which has $n+1$ solutions. Hence, we have $n+2$ total.

(a) 4

(b) 5

(c) 6

(d) 7

4. (a) Let $f(x) = \frac{2}{(1-x)^2}$. The Taylor series for $f(x)$ around $x_0 = 0$ is of the form $\sum_{n=0}^{\infty} a_n x^n$ for some sequence a_n. By computing a formula for a_n in terms of n, or otherwise, compute a_{10}. Please write your answer as an integer, with no full stop at the end.

   (b) Let $f(x) = \frac{3}{(1-x)^2}$. The Taylor series for $f(x)$ around $x_0 = 0$ is of the form $\sum_{n=0}^{\infty} a_n x^n$ for some sequence a_n. Compute a formula for a_n, in terms of n and hence compute a_{10}. Please write your answer as an integer, with no full stop at the end.

   (c) Let $f(x) = \frac{4}{(1-x)^2}$. The Taylor series for $f(x)$ around $x_0 = 0$ is of the form $\sum_{n=0}^{\infty} a_n x^n$ for some sequence a_n. Compute a formula for a_n, in terms of n and hence compute a_{10}. Please write your answer as an integer, with no full stop at the end.

   (d) Let $f(x) = \frac{5}{(1-x)^2}$. The Taylor series for $f(x)$ around $x_0 = 0$ is of the form $\sum_{n=0}^{\infty} a_n x^n$ for some sequence a_n. Compute a formula for a_n, in terms of n and hence compute a_{10}. Please write your answer as an integer, with no full stop at the end.

Solution: In the case $f(x) = \frac{c}{(1-x)^2}$, we have $a_n = c(n+1)$.

(a) 22

(b) 33

(c) 44

(d) 55

5. (a) Consider the differential equation $\frac{dy}{dx}+10y^2=e^x$. Which of the following correctly describes this differential equation?

   A. First-order, linear, inhomogeneous, with constant coefficients

   B. First-order, non-linear, inhomogeneous, with constant coefficients

   C. First-order, non-linear, homogeneous, with constant coefficients

   D. First-order, non-linear, inhomogeneous, with non-constant coefficients

(b) Consider the differential equation $\left(\frac{dy}{dx}\right)^2+10y=e^x$. Which of the following correctly describes this differential equation?

   A. First-order, linear, inhomogeneous, with constant coefficients

   B. First-order, non-linear, inhomogeneous, with constant coefficients

   C. First-order, non-linear, homogeneous, with constant coefficients

   D. First-order, non-linear, inhomogeneous, with non-constant coefficients

(c) Consider the differential equation $\frac{dy}{dx}+10xy=e^x$. Which of the following correctly describes this differential equation?

   A. First-order, non-linear, inhomogeneous, with non-constant coefficients

   B. First-order, linear, inhomogeneous, with non-constant coefficients

   C. First-order, linear, homogeneous, with non-constant coefficients

   D. First-order, linear, inhomogeneous, with constant coefficients

(d) Consider the differential equation $x\frac{dy}{dx}+10y=e^x$. Which of the following correctly describes this differential equation?

   A. First-order, non-linear, inhomogeneous, with non-constant coefficients

   B. First-order, linear, inhomogeneous, with non-constant coefficients

   C. First-order, linear, homogeneous, with non-constant coefficients

   D. First-order, linear, inhomogeneous, with constant coefficients