COMS10013 - Analysis - WS6

Solutions

1. **Complex numbers**: calculate the following complex numbers in the form $(a + bi)$:
   (a) $(2 + 3i) + (5 - 2i) = 7 + i$
   (b) $(-1 + i)(-1 - i) = 2$
   (c) $(1 - i)^3 = -2 - 2i$
   (d) $(1 + i)/(1 - i) = i$; to see this, multiply by $(1 + i)/(1 + i)$

2. **More complex numbers**: Compute the real part, imaginary part, norm (i.e. absolute value), and complex conjugate of the following numbers:
   (a) i: the real part is 0, the imaginary part is 1, the norm is 1, the complex conjugate is $-i$.
   (b) $3 - 2i$: the real part is 3, the imaginary part is -2, the norm is $\sqrt{13}$, the complex conjugate is $3 + 2i$.

3. **Polar form**. Convert between rectangular $(a + ib)$ and polar $re^{i\theta}$ form:
   (a) i: gives $e^{i\pi/2}$.
   (b) $2 - i$: the norm is $\sqrt{5}$ and the angle is some annoying angle whose tan is 1/2.
   (c) $3e^{i\pi/2}$ is $3i$.
   (d) e^{1+2i}, this is also annoying, we have

   $$e^{1+2i} = e \times e^{2i} = e[\cos 2 + i \sin 2]$$

   which I guess you could work out with a calculator.

4. **More on Polar form**.
   (a) The complex conjugate of $re^{i\theta}$: we'll first convert $z = re^{i\theta}$ to $z = r(\cos(\theta) + i \sin(\theta))$. Then the complex conjugate is $z^* = r(\cos(\theta) - i \sin(\theta))$. Using the identities $\sin(-\theta) = i \sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$, we get that

   $$z^* = r(\cos(-\theta) + i \sin(-\theta)) = re^{-i\theta}\,.$$

   (b) What is the formula for the inverse of a complex number in polar form (e.g. $1/re^{i\theta}$, give the solution in polar form again) and what does this mean geometrically?
   We're looking for a number $z = r'e^{i\theta'}$ so that

   $$(re^{i\theta})(r'e^{i\theta'}) = 1.$$

   Multiplying this out, we're looking for r', θ' so that

   $$rr'e^{i(\theta+\theta')} = 1\,.$$

   Looking at the magnitude (absolute value) of both sides, we see that $rr' = 1$, so that $r' = 1/r$. We can write the right-hand side as e^{0i}, which makes it clear that $\theta' = -\theta$. So the inverse is $\frac{1}{r}e^{-i\theta}$.
   Geometrically, we've scaled the complex number (from having distance to the origin of r to now having distance to the origin of $1/r$) and we've reflected in the real axis.

5. **Second order equations** $y''(t) = -y(t)$ with initial conditions $y(0) = 1$ and $y'(0) = 0$. Let's start with our ansatz $y(t) = Ae^{rt}$ and see what happens. With this ansatz, we get

$$Ae^{rt}(r^2 + 1) = 0$$

which, now that we know about complex numbers, we can solve to give us $r = \pm i$. So our solution is a linear combination of these r values, namely

$$Ae^{it} + Be^{-it}\,.$$

Using the initial conditions, we get from $y(0) = 1$ that

$$A + B = 1$$

and from $y'(0) = 0$, that

$$Ai - Bi = 0 \Rightarrow (A - B) = 0$$

so that $A = B = \frac{1}{2}$ and

$$y(t) = \frac{e^{it}}{2} + \frac{e^{-it}}{2}\,.$$

We can write this in rectangular form using Euler's formula:

$$y(t) = \frac{1}{2}\left(\cos(t) + i\sin(t) + \cos(-t) + i\sin(-t)\right)\,.$$

We'll use the fact that $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$ to get

$$y(t) = \cos(t)\,.$$

So even though the process of finding our solution used complex numbers, the solution itself didn't!

Extra questions

1. **Equations with complex solutions**. Solve the following equations over the complex numbers
   (a) $x^2 - 2x + 5 = 0$: For this we'll use the quadratic formula to get roots

   $$\frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

   (b) $x^2 - 2x + 8 = 0$: This question is pretty much the same. The quadratic formula gives roots

   $$\frac{2 \pm \sqrt{4 - 32}}{2} = 1 \pm i\sqrt{7}$$

   (c) $x^2 - ix - 1 = 0$: this is less clear cut. Let's see what the quadratic formula gives:

   $$\frac{-i \pm \sqrt{(-i)^2 + 4}}{2} = \frac{-i \pm \sqrt{3}}{2} = \pm\frac{\sqrt{3}}{2} - \frac{i}{2}$$