Department of Computer Science
University of Bristol

COMS30030 - Image Processing and Computer Vision

Lecture 03

Frequency Domain & Transforms

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Signals as Functions

Frequency - allows us to characterise signals:

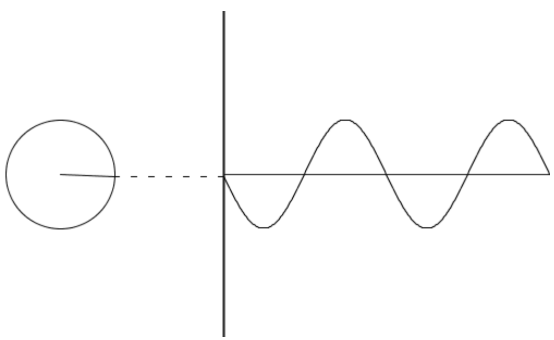
– Repeats over regular intervals with Frequency $u = \dfrac{1}{T}$ cycles/sec (Hz)

– Amplitude a (peak value)

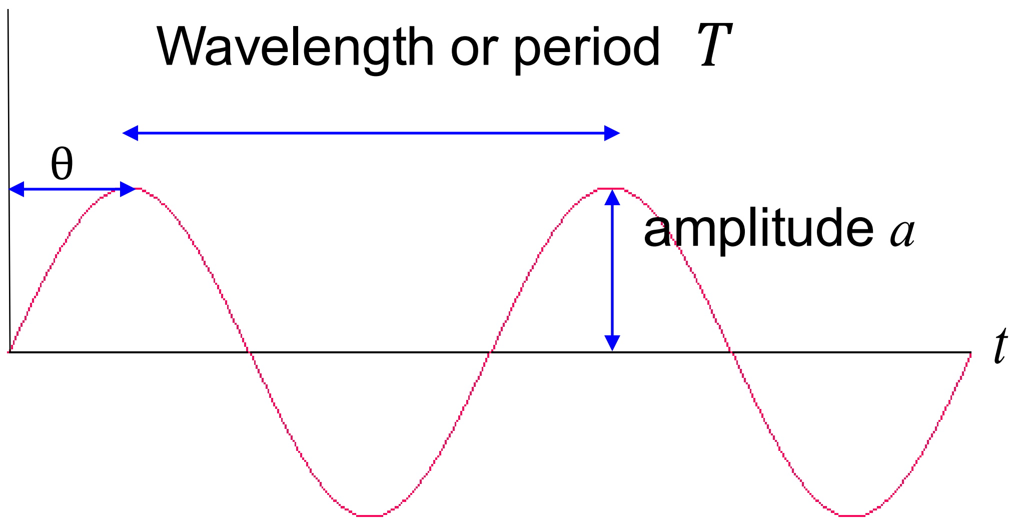
– the Phase θ (shift in degrees)

Example: sine function

$$f(t) = a\, sin 2\pi u t$$



From giphy.com



Fourier's Theorem

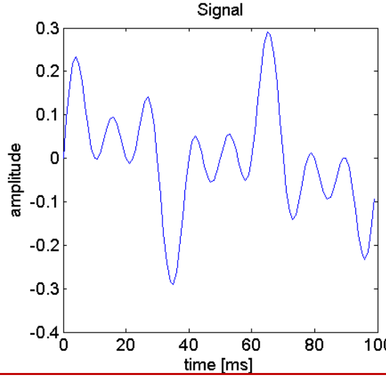
$$f(x) = \int a_n \cos(nx) + b_n \sin(nx) \;\; \delta n$$

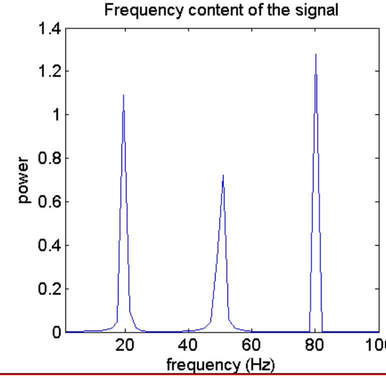
Jean-Baptiste Joseph Fourier

- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

Intuition I: Simple 1D example

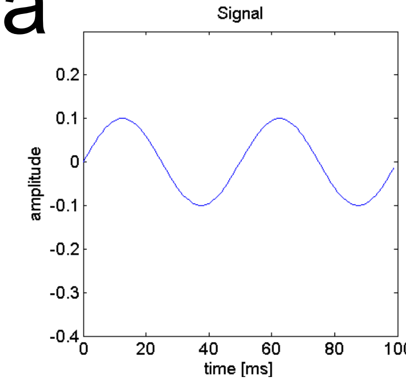
d = a + b + c

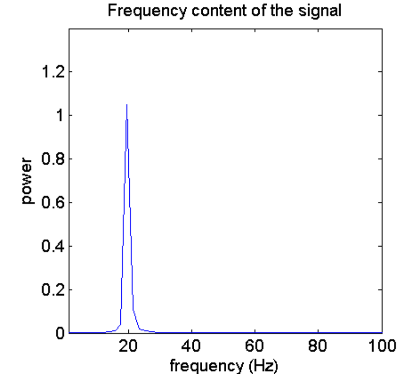




time domain frequency domain

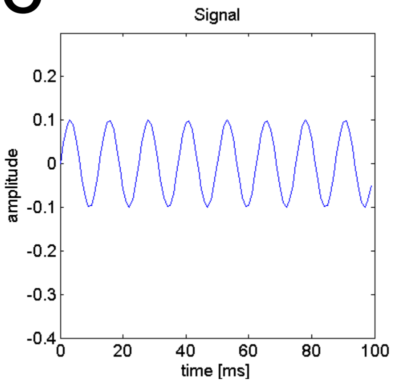
a

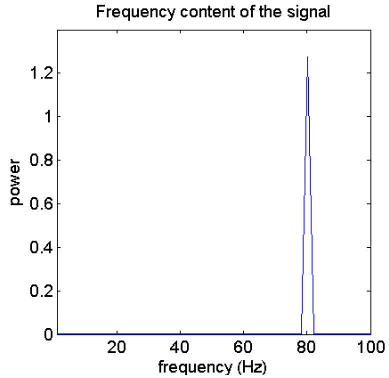




time domain frequency domain

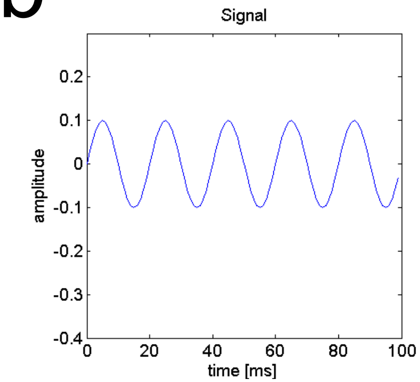
c

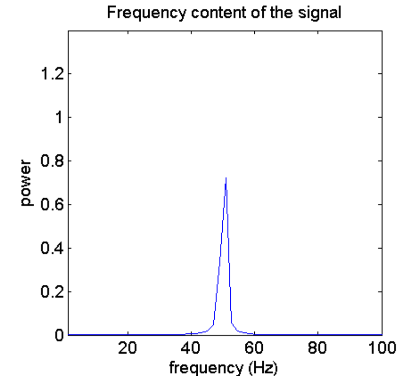




time domain frequency domain

b





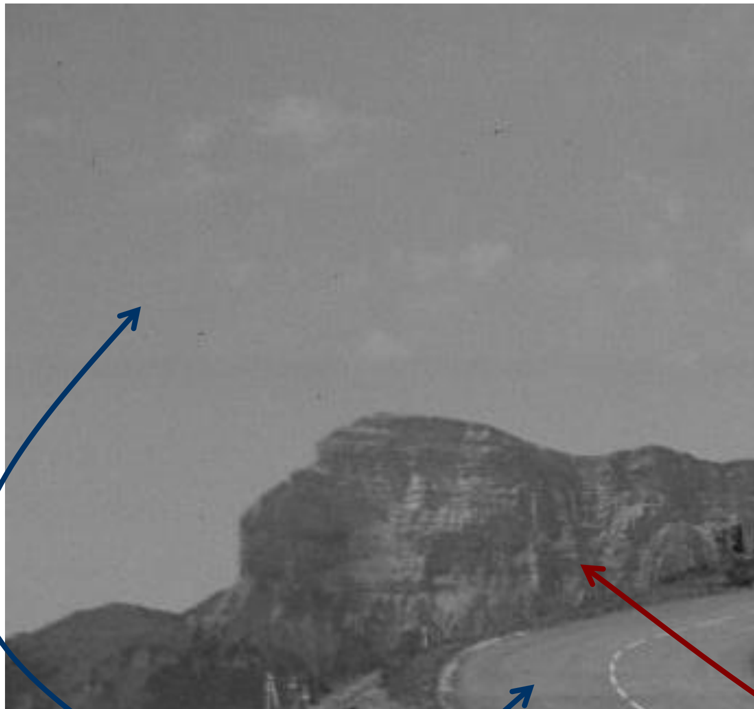
time domain frequency domain

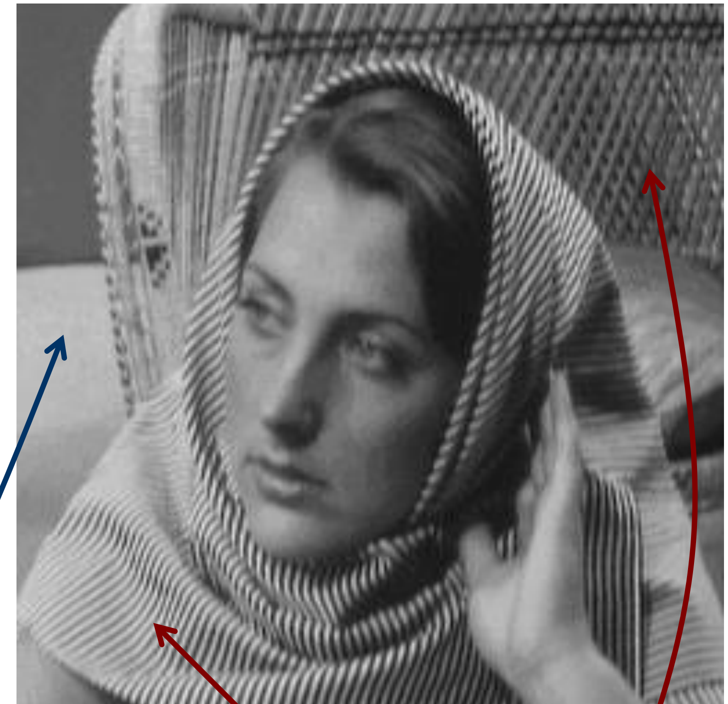
Intuition II: Simple 1D example

Animation by Lucas V Barbosa

Intuition III: Concept of Frequency in Images

Rate of change of intensity along the two dimensions



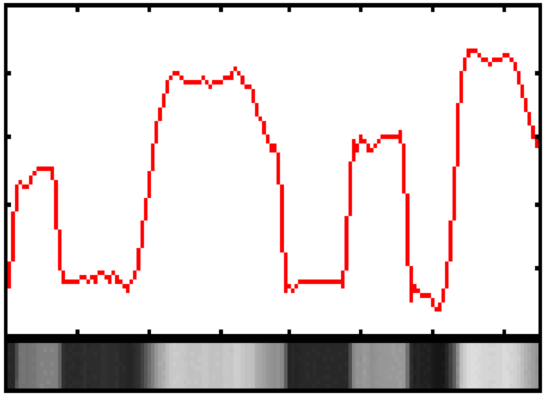


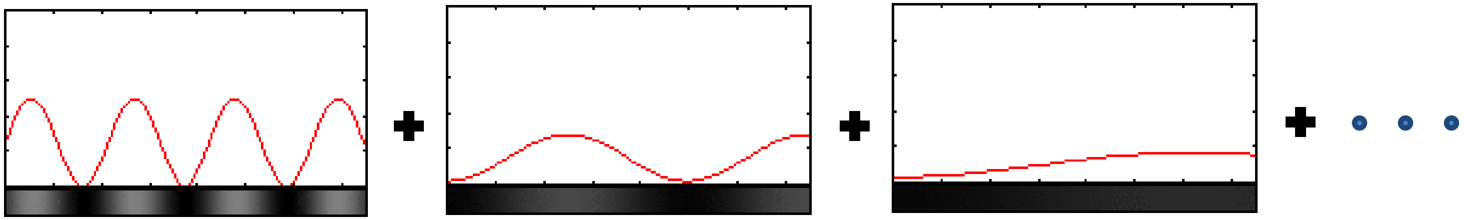
Slowly changing → low frequency

Rapidly changing →high frequency

Intuition IV: Images as waves!?

Take a single row or column of pixels from an image → a 1D signal





From ImageNagik

2D Fourier Transform: Continuous Form

- The Fourier Transform of a continuous function of two variables $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\; e^{-i2\pi(ux+vy)}\; dxdy$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\; e^{i2\pi(ux+vy)}\; dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform: Discrete Form

- The FT of a discrete function of two variables, $f(x,y)$, is:

image

kernels (probing functions)

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \boxed{f(x,y)} \boxed{e^{-i2\pi(\frac{ux+vy}{N})}} \quad \text{for } u,v = 0,1,2,\ldots,N-1\,.$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse FT:*

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v)\; e^{i2\pi(\frac{ux+vy}{N})} \quad \text{for } x,y = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

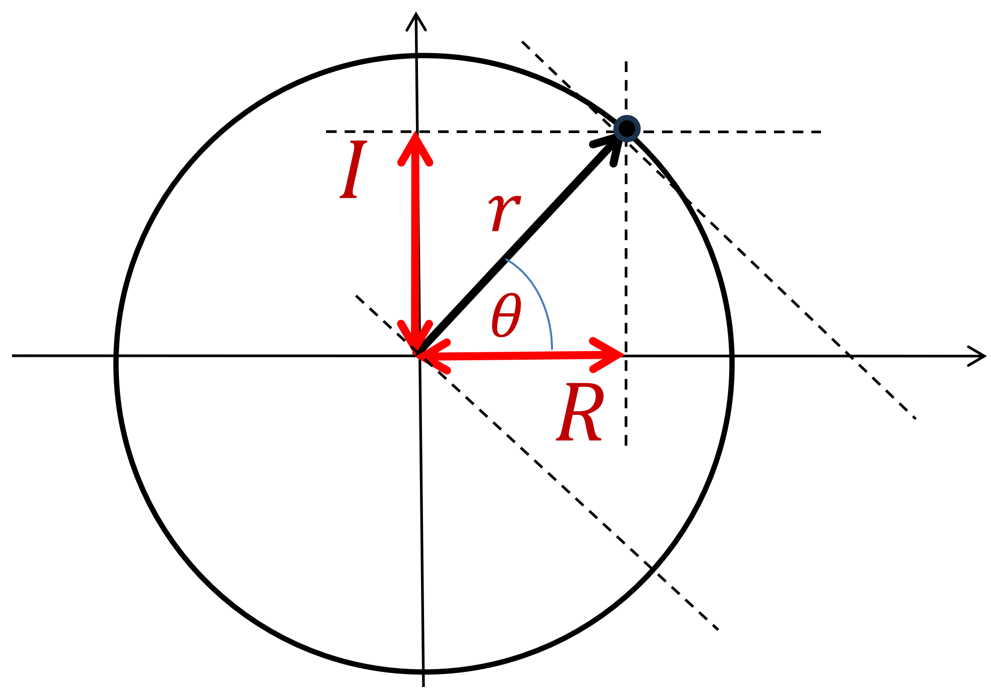
Note, they constitute a lossless representation of data.

Euler's Formula

$e^{i2\pi(\frac{ux+vy}{N})}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The kernel is associated with a complex number (r, θ) in polar coordinates or $R(u, v), I(u, v)$ in standard complex notation.



2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function *f(x,y)* multiplied by a particular kernel at a particular frequency and orientation specified by *(u,v)*:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - i\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

$$\text{for } u,v = 0,1,2,\ldots,N-1\,.$$

All kernels together form a new orthogonal basis for our image.

2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function *f(x,y)* multiplied by a particular kernel at a particular frequency and orientation specified by *(u,v)*:

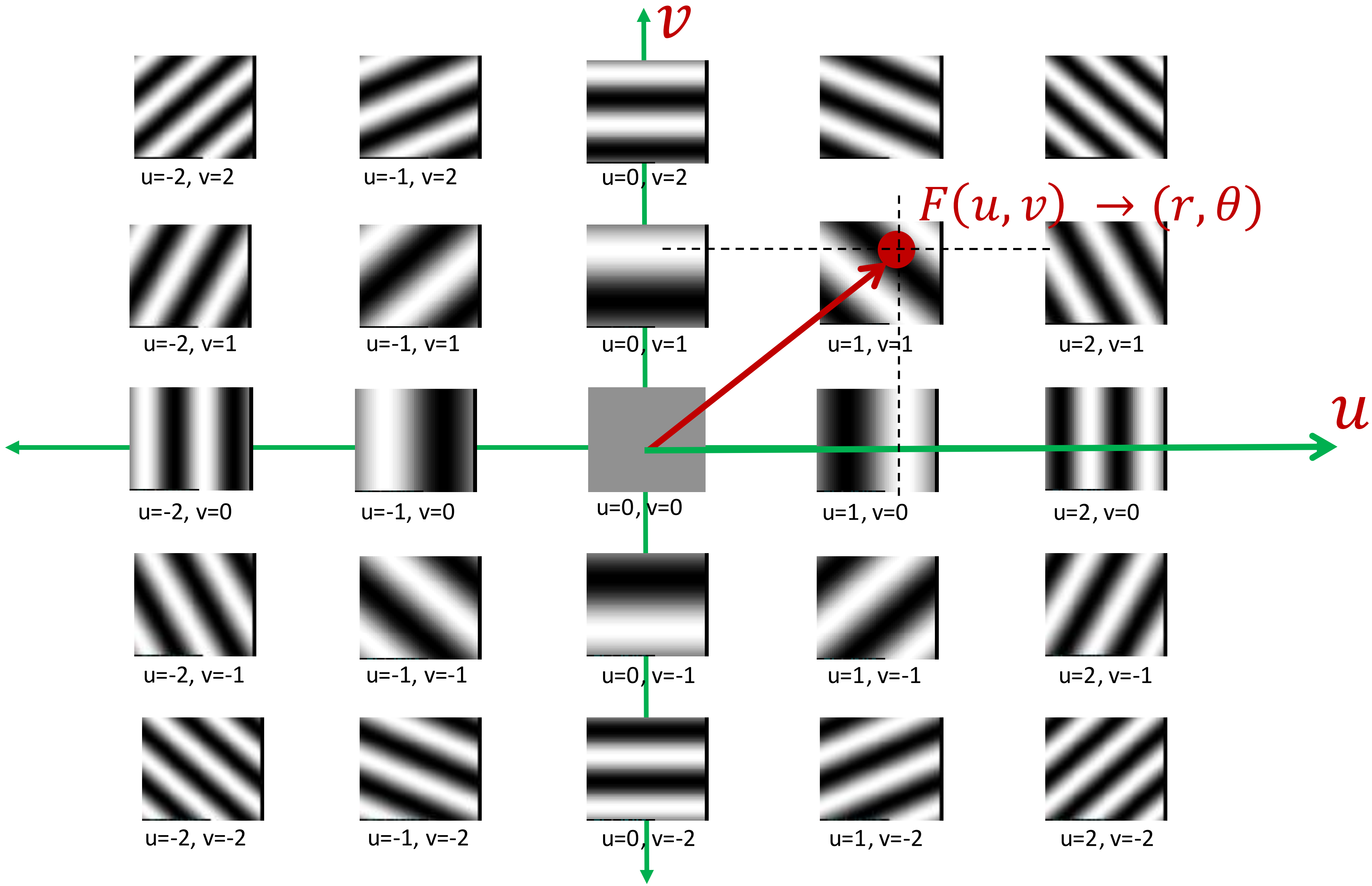
$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\overset{1}{\cos\left(\frac{2\pi(ux+vy)}{N}\right)} - i\overset{0}{\sin\left(\frac{2\pi(ux+vy)}{N}\right)}\right]$$

for $u, v = 0,1,2,\ldots,N-1$.

The slowest varying frequency component, i.e. when *u=0,v=0* → average image graylevel

All kernels together form a new orthogonal basis for our image.

'Fabric' of the 2D Fourier Space (as kernels)



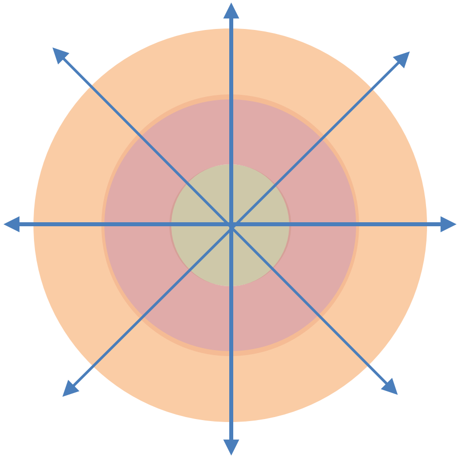
Power Spectrum and Phase Spectrum

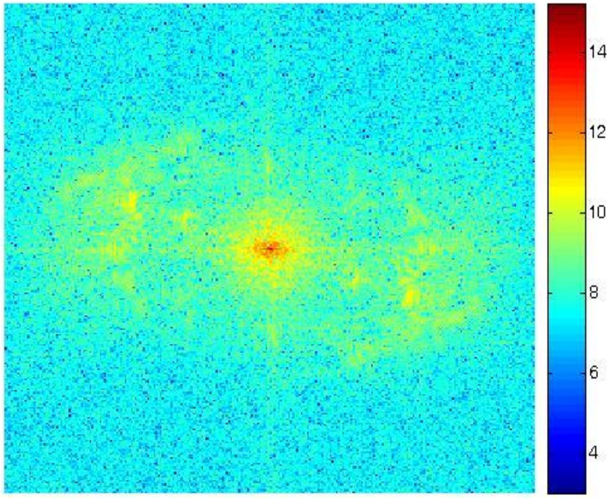
$f(x,y)$

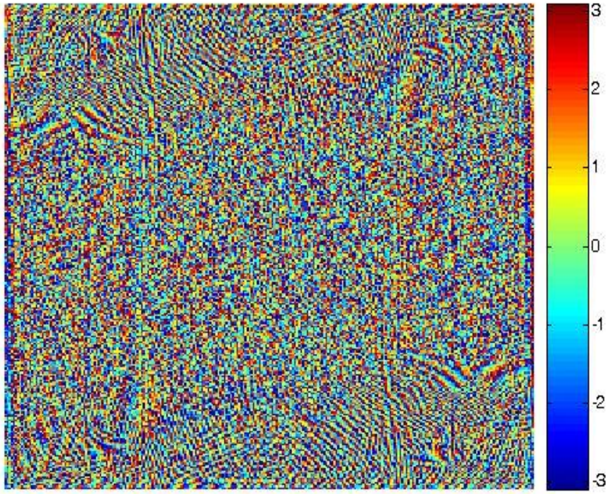
Low to Low-ish frequencies

Mid-range frequencies

High frequencies



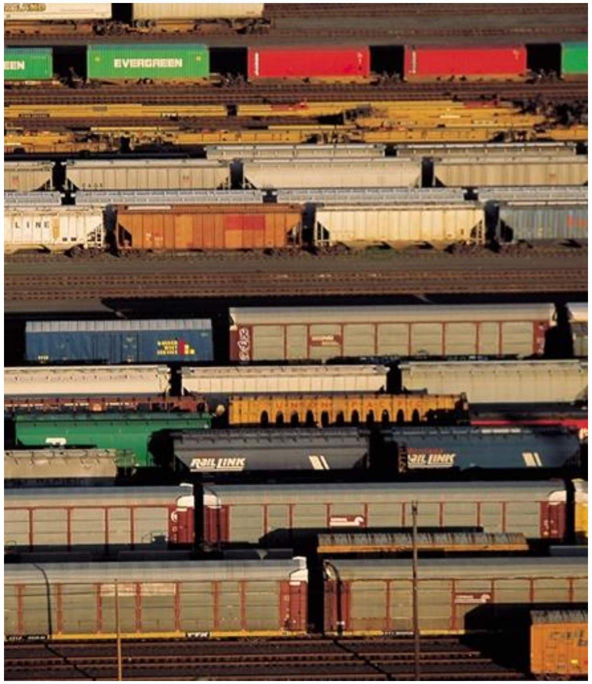




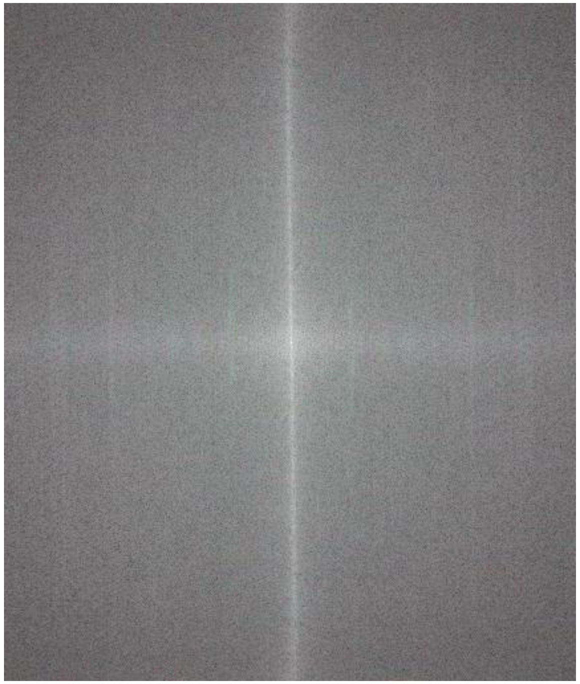
$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

Example: Relating Frequencies to Images

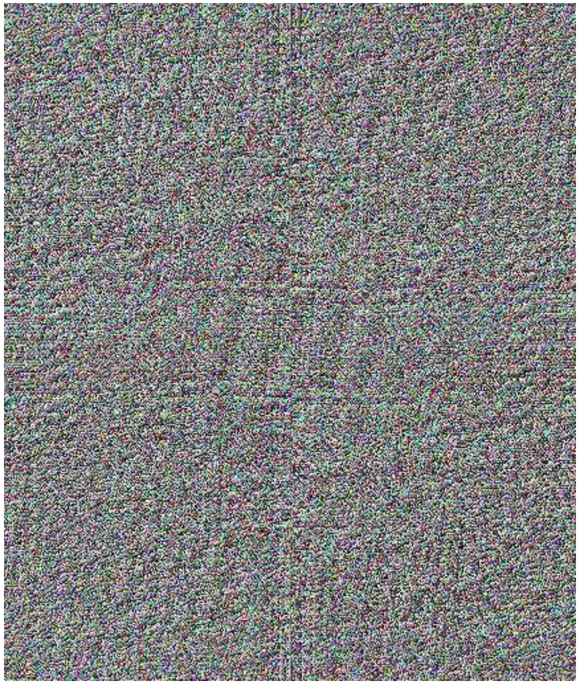


Magnitude



$|F(u,v)|$

Phase



$\angle F(I)$

The Frequency Domain

- *$F(u,v)$* is a complex number and has real and imaginary parts:

$$F(u,v) = R(u,v) + iI(u,v)$$

- Magnitudes
  (forming the Magnitude Spectrum):

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

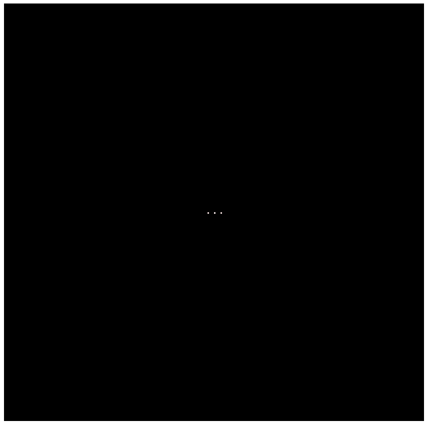
- Phase Angles
  (forming the Phase Spectrum):

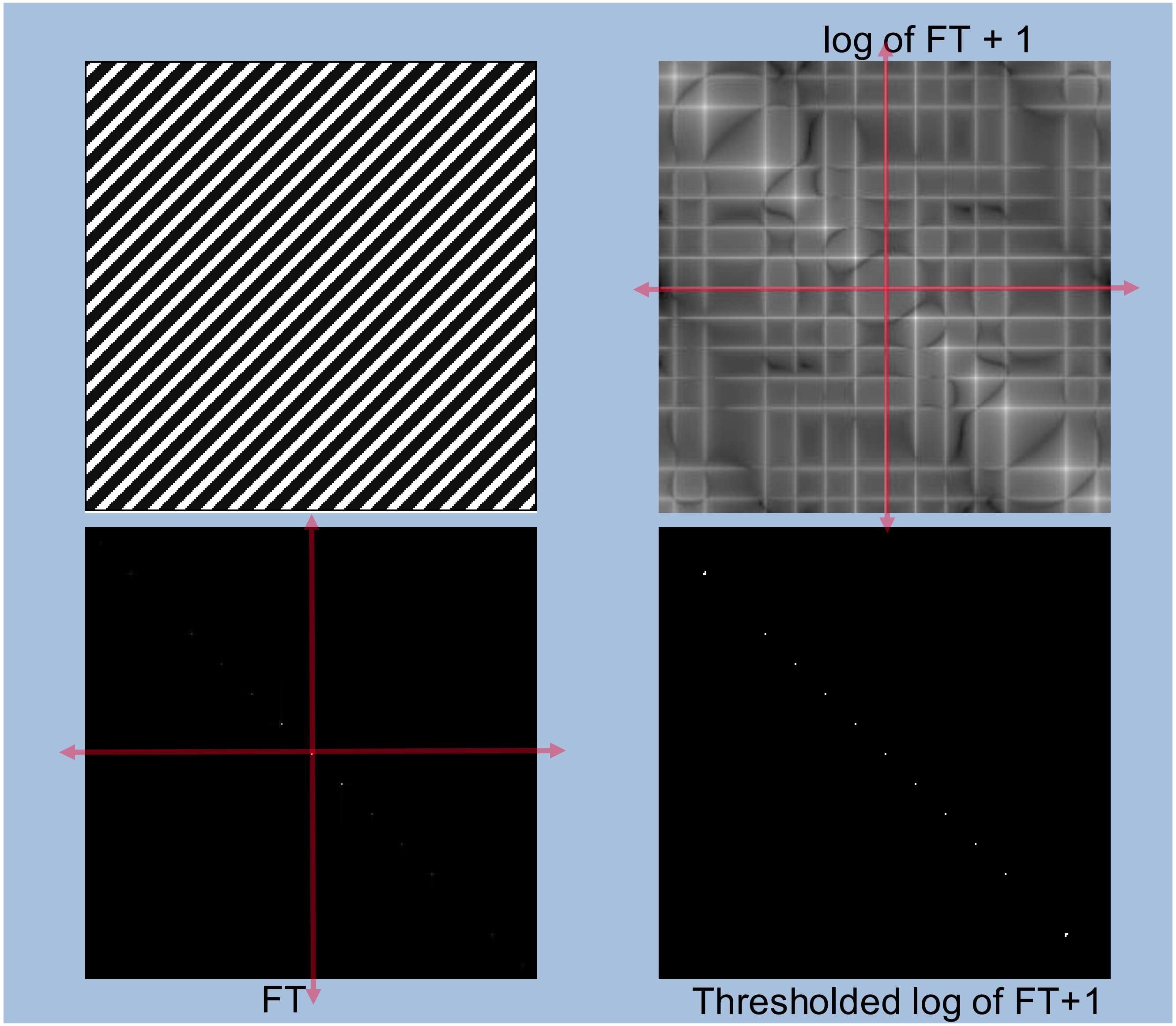
$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

- Expressing *$F(u,v)$* in polar coordinates *(r, θ)* :

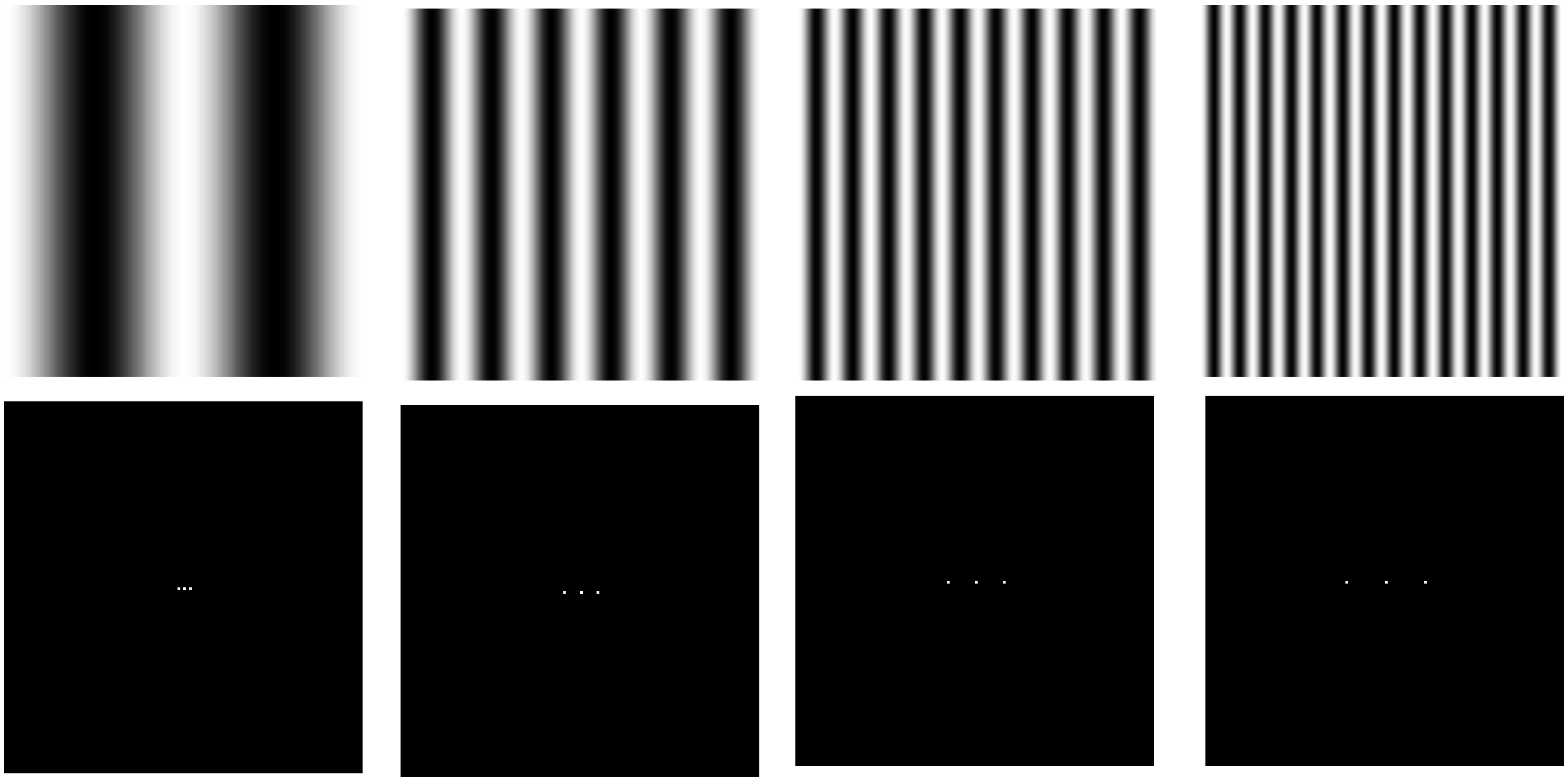
$$F(u,v) = |F(u,v)|e^{i\theta(u,v)} = re^{i\theta}$$

Spatial Domain ⟷ Frequency Domain



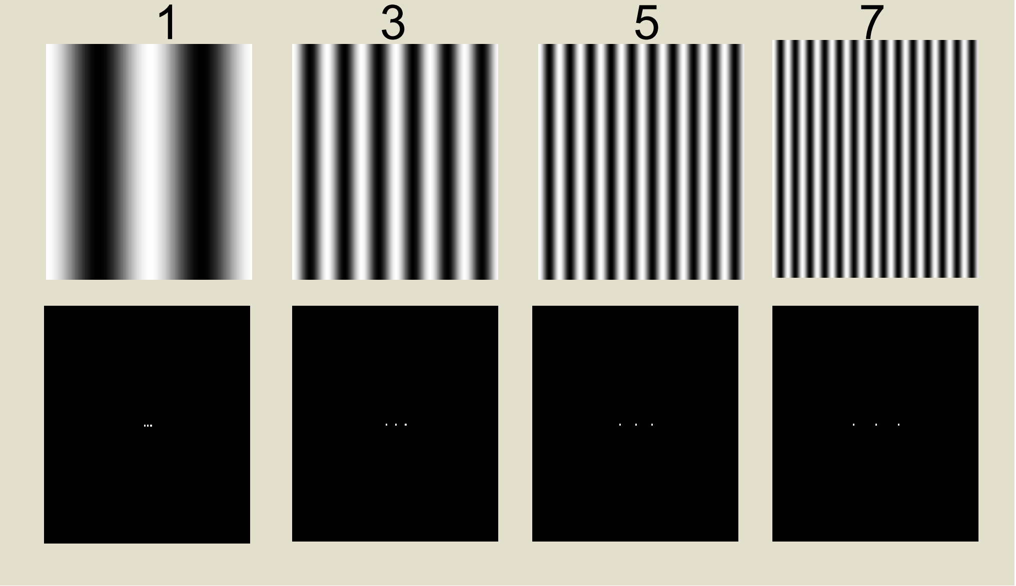


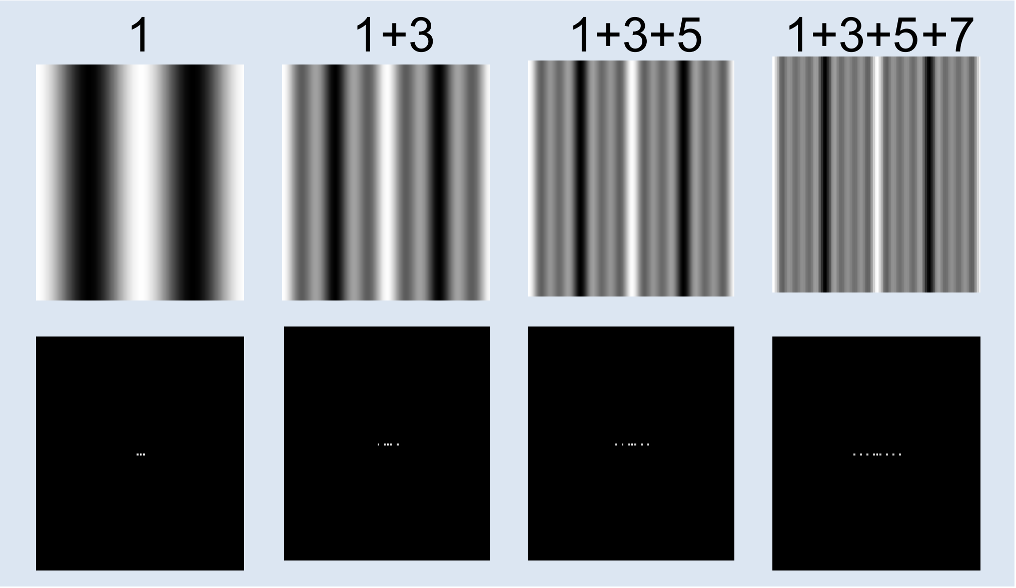
Spatial Domain ⟷ Frequency Domain

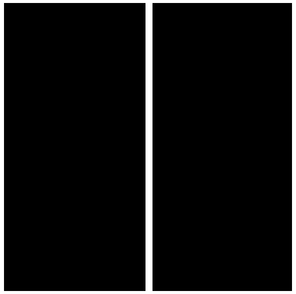


Perpendicular relationship

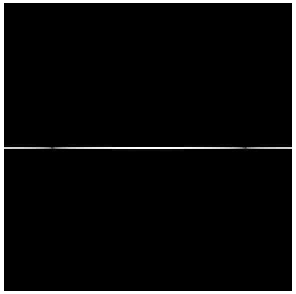
Ideal edge and line structures have a concentration along a line passing through the origin in the frequency domain and in a **direction perpendicular to their orientation** - they are 'constructed' by adding together all 2D sinusoidal waves that 'travel' perpendicular to the edge or line.





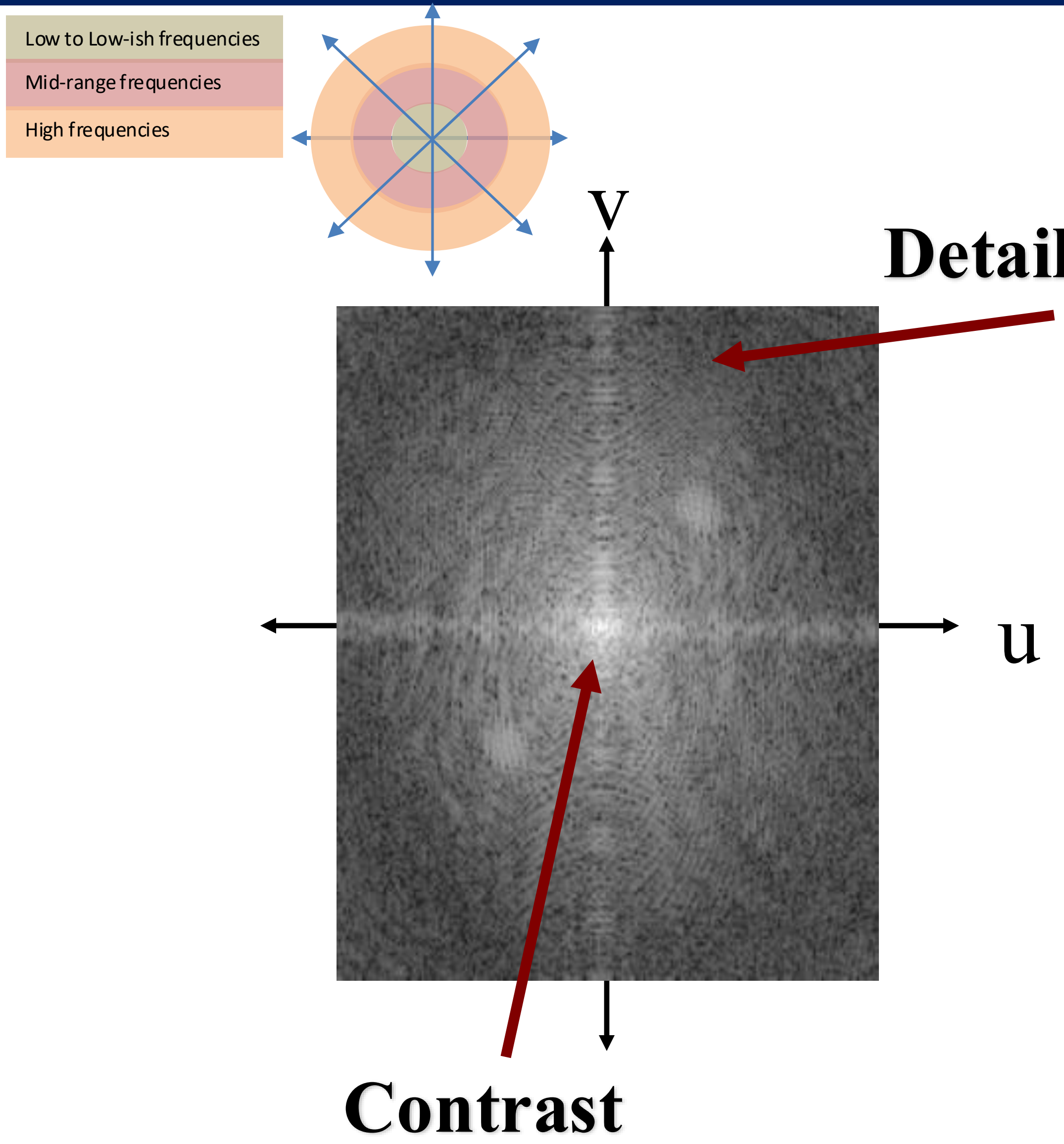


Image

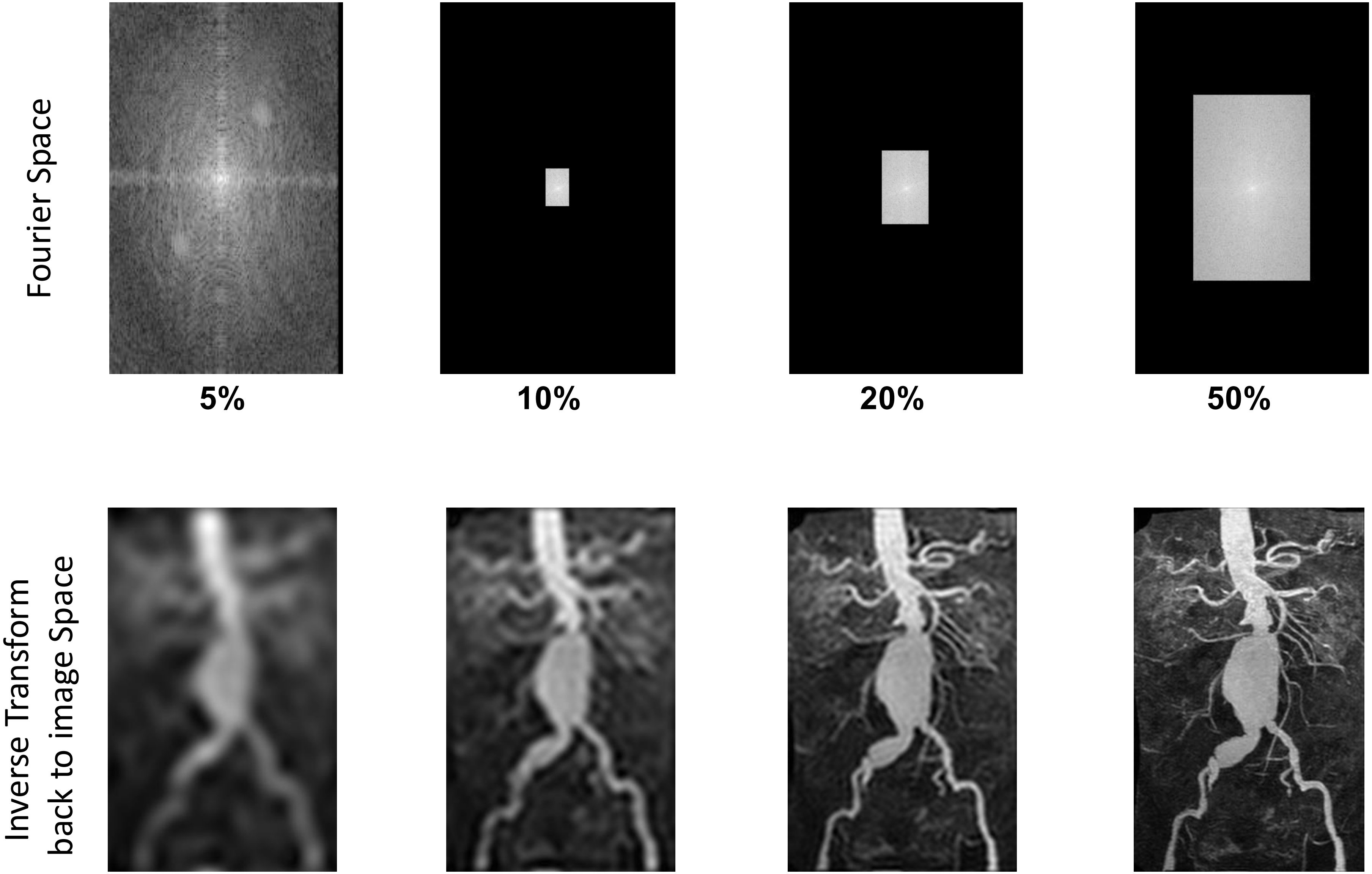


FT

Relating Frequencies to Images

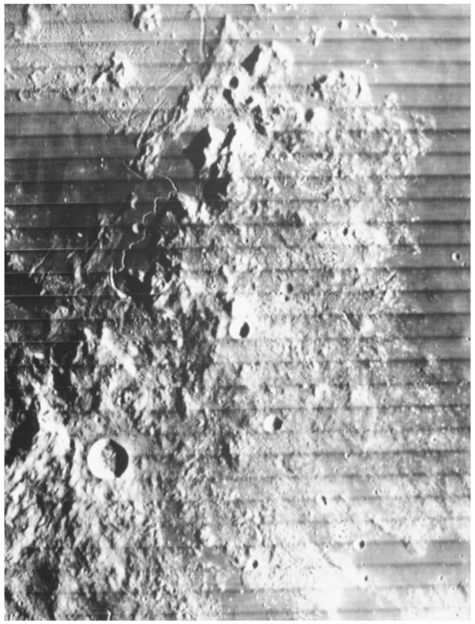


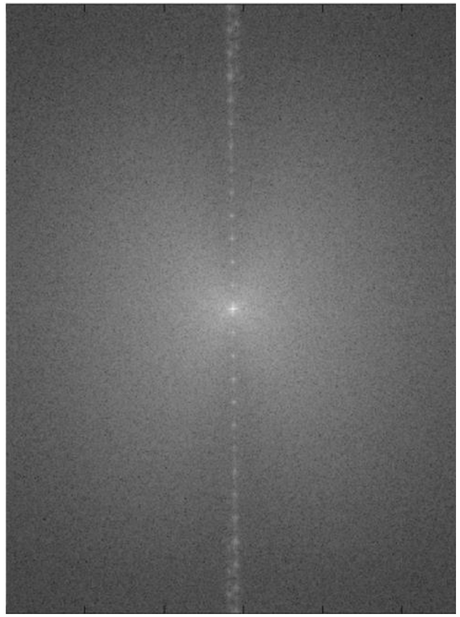
Example: Relating Frequencies to Images



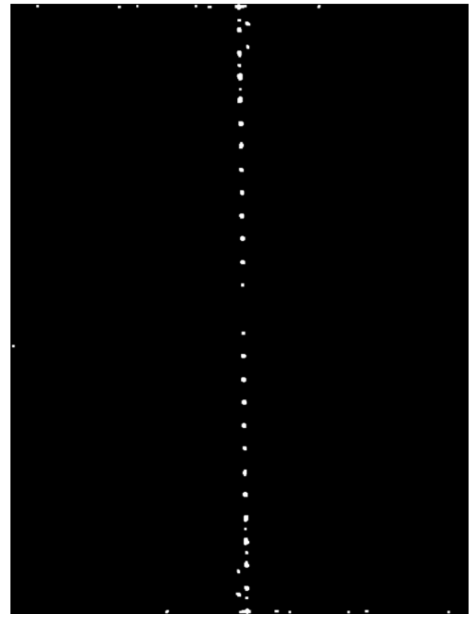
Example: Manipulating the FS

Lunar orbital image (1966)





$|F(u,v)|$



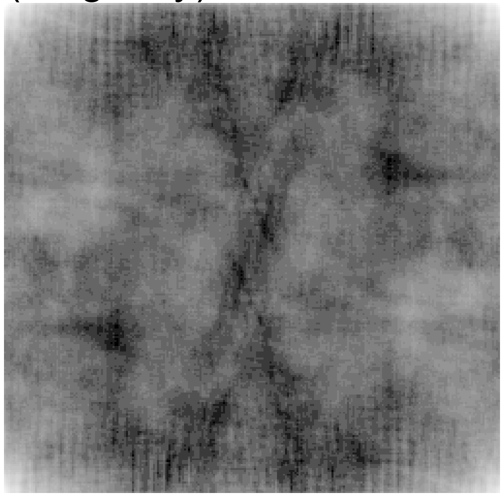
Mask used to remove peaks

$\mathrm{i}FFT(F(u,v))$

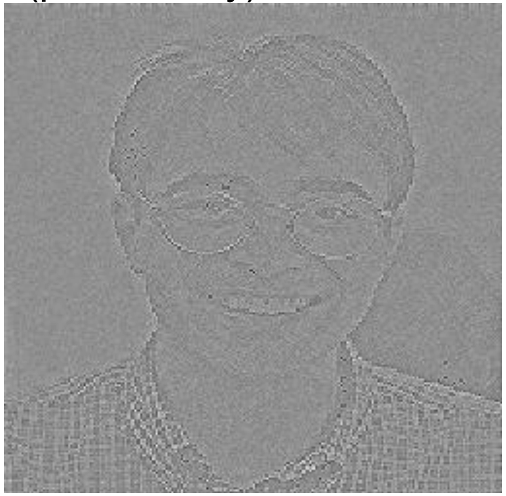
Slide by A. Zisserman

Importance of Phase

ifft(mag only)



ifft(phase only)



ifft(mag(Peter) and Phase(Andrew))

ifft(mag(Andrew) and Phase(Peter))

Separability

- Important property of the FT: *Separability*
- If a 2D transform is separable, the result can be found by successive application of two 1D transforms. This is a principle aspect of the Fast Fourier Transform (FFT).

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v) \quad e^{\frac{-j2\pi ux}{N}} \quad \text{where} \quad F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y) \quad e^{\frac{-j2\pi vy}{N}}$$

(,) (,) (,)

1D row transforms

1D column transforms

Recall: 2D Discrete Convolution

- The discrete version of 2D convolution is defined as:

$$g(x,y) = \sum_{m}\sum_{n} f(x-m, y-n)h(m,n)$$

$$\underbrace{\begin{array}{c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{y-1} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{y+1} & \multicolumn{1}{c}{} \\ \cline{2-6} & & & & & \\ \cline{2-6} x-1 & & 43 & 12 & 61 & \\ \cline{2-6} x & & 44 & 45 & 60 & \\ \cline{2-6} x+1 & & 43 & 50 & 61 & \\ \cline{2-6} & & & & & \\ \cline{2-6} \end{array}}_{f} * \underbrace{\begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{-1} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{1} \\ \cline{2-4} -1 & -1 & 0 & 1 \\ \cline{2-4} 0 & -2 & 0 & 2 \\ \cline{2-4} 1 & -1 & 0 & 1 \\ \cline{2-4} \end{array}}_{h} = -68 = \begin{array}{l} f(x+1,y+1)h(-1,-1) \\ + f(x+1,y)h(-1,0) \\ + f(x+1,y-1)h(-1,1) \\ + f(x,y+1)h(0,-1) \\ + f(x,y)h(0,0) \\ + f(x,y-1)h(0,1) \\ + f(x-1,y+1)h(1,-1) \\ + f(x-1,y)h(1,0) \\ + f(x-1,y-1)h(1,1) \end{array}$$

Convolution in the Spatial/Frequency Domain

Convolution Theorem:

Convolution in spatial domain
is equivalent to
multiplication in frequency domain
(and vice versa)

$h = f * g$ ***implies*** $H = FG$

$h = fg$ ***implies*** $H = F * G$

Deriving the Convolution Theorem

NOT EXAMINABLE

$$h(x) = f(x) * g(x) = \sum_y f(x-y)g(y)$$

$$H(u) = \sum_x \left(\sum_y f(x-y)g(y) \right) e^{(-iux2\pi/N)}$$

$$H(u) = \sum_y g(y) \left(\sum_x f(x-y)\, e^{(-iux2\pi/N)} \right)$$

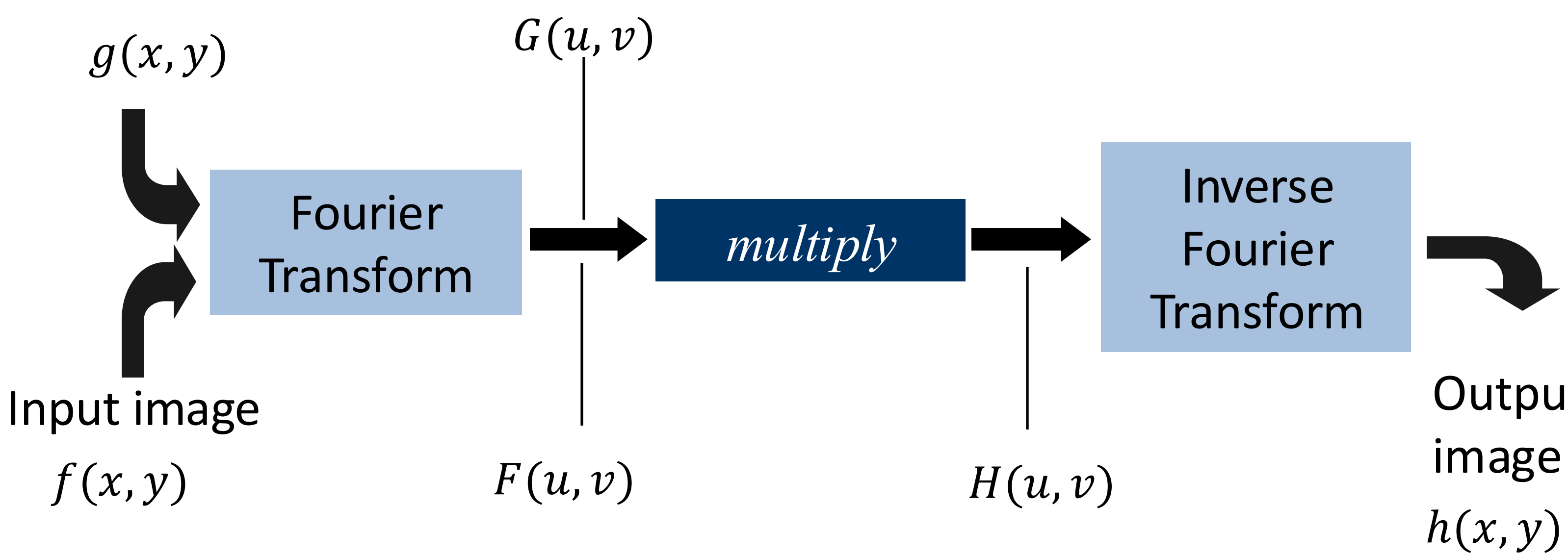
$$H(u) = \sum_y g(y) \left(F(u)\, e^{(-iuy2\pi/N)} \right)$$

$$H(u) = \sum_y g(y)\, e^{(-iuy2\pi/N)}\, F(u) = G(u) \cdot F(u) = F(u) \cdot G(u)$$

Fast Filtering using the Convolution Theorem

1D: $H(u) = F(u)G(u)$

2D: $H(u,v) = F(u,v)G(u,v)$



Recall: Spatial Low/High Pass Filtering

f

Low Pass

h=

1	1	1
1	1	1
1	1	1

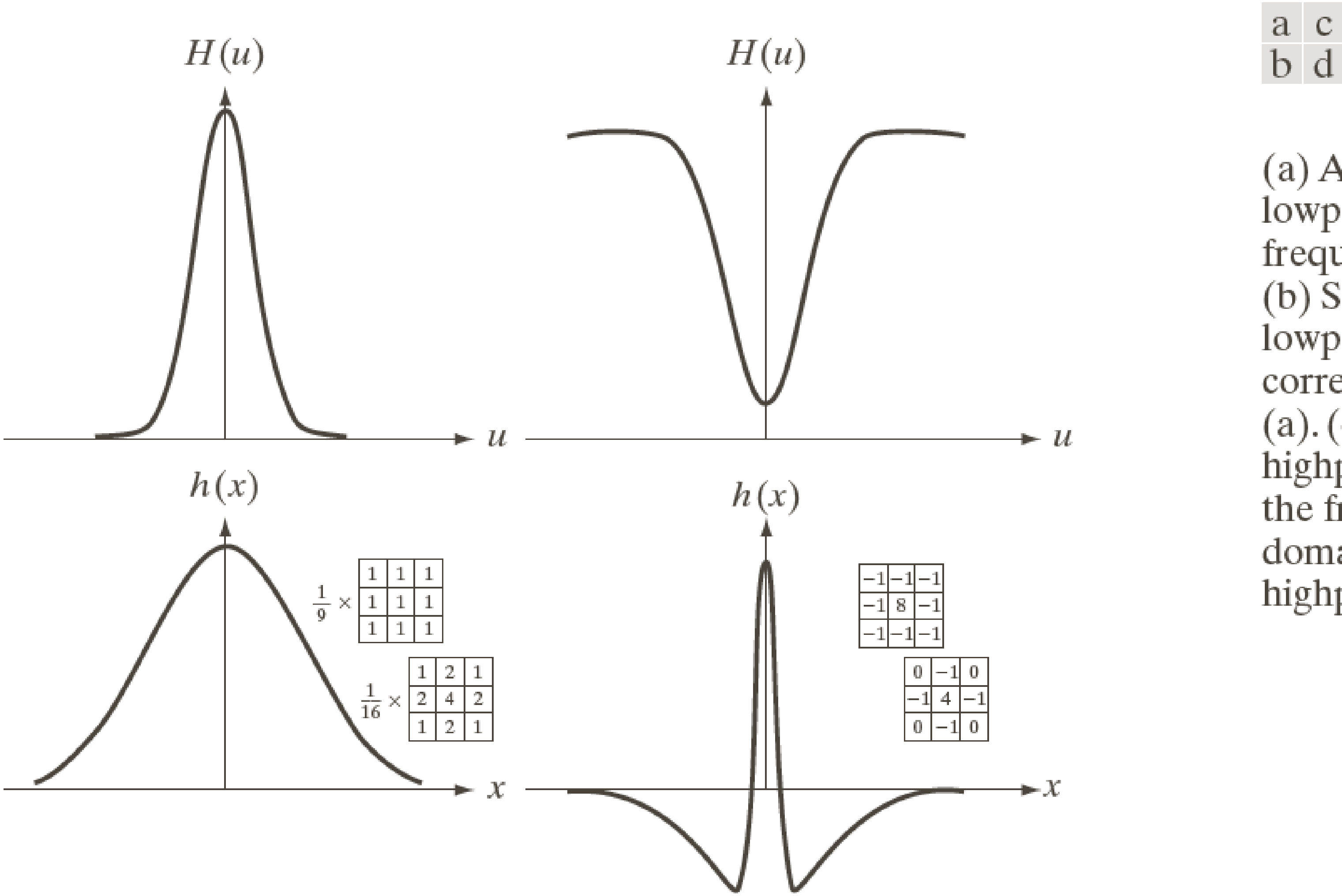
High Pass

h=

-1	-1	-1
-1	8	-1
-1	-1	-1

f * h

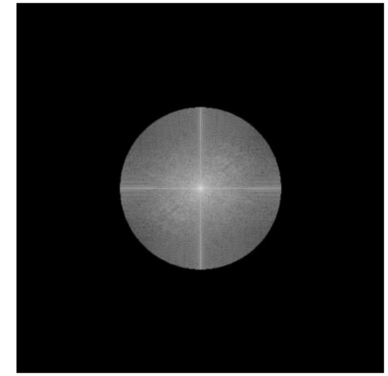
Frequency Domain Low/High Pass Filtering

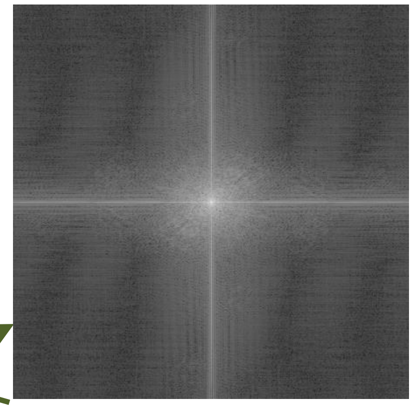


(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter

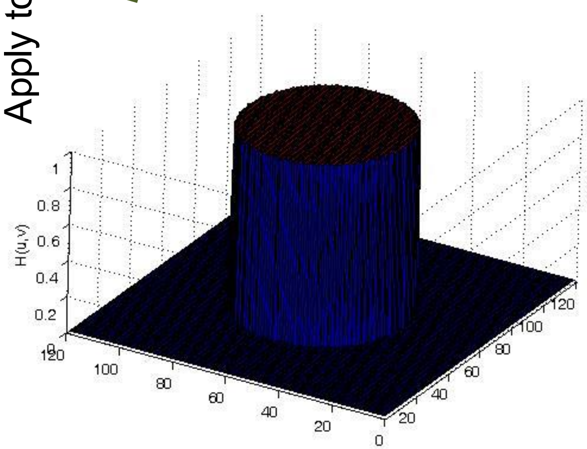
Ideal Low Pass Filter

- 1D: turning the “treble” down on audio equipment!
- 2D: smooth image





Apply to freq. domain

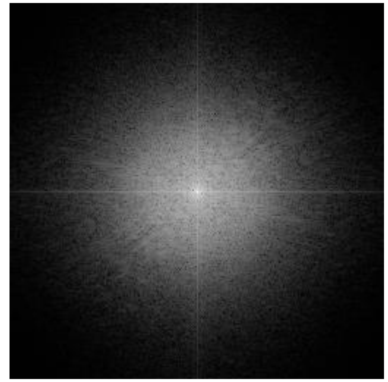


$$H(u,v) = \begin{cases} 1 & r(u,v) \le r_0 \\ 0 & r(u,v) > r_0 \end{cases}$$

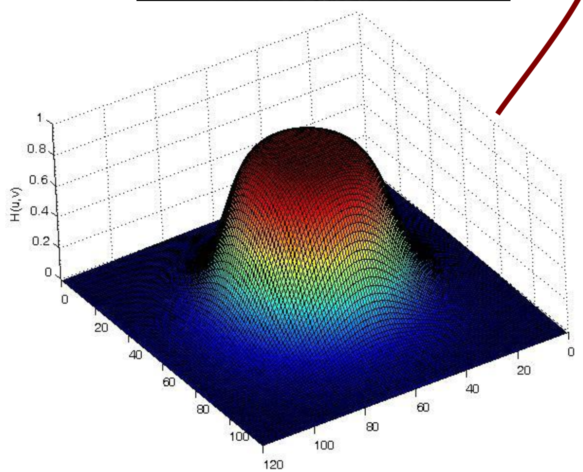
$r(u,v) = \sqrt{u^2 + v^2}$, r_0 is the filter radius

Butterworth's Low Pass Filter

Input image



After applying to freq. domain

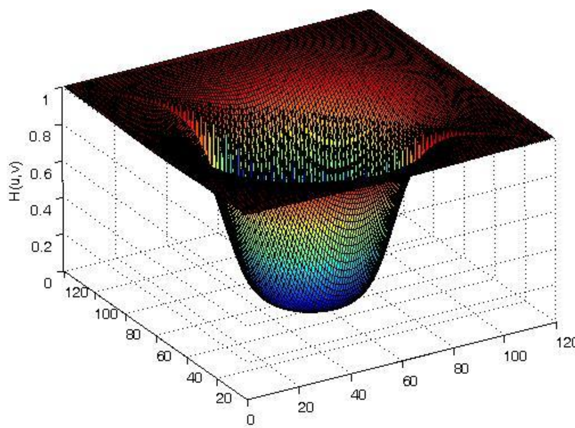


After filtering

$$H(u,v) = \frac{1}{1 + [r(u,v)/r_0]^{2n}} \quad \text{of order } n$$

Butterworth's High Pass Filter

Input image

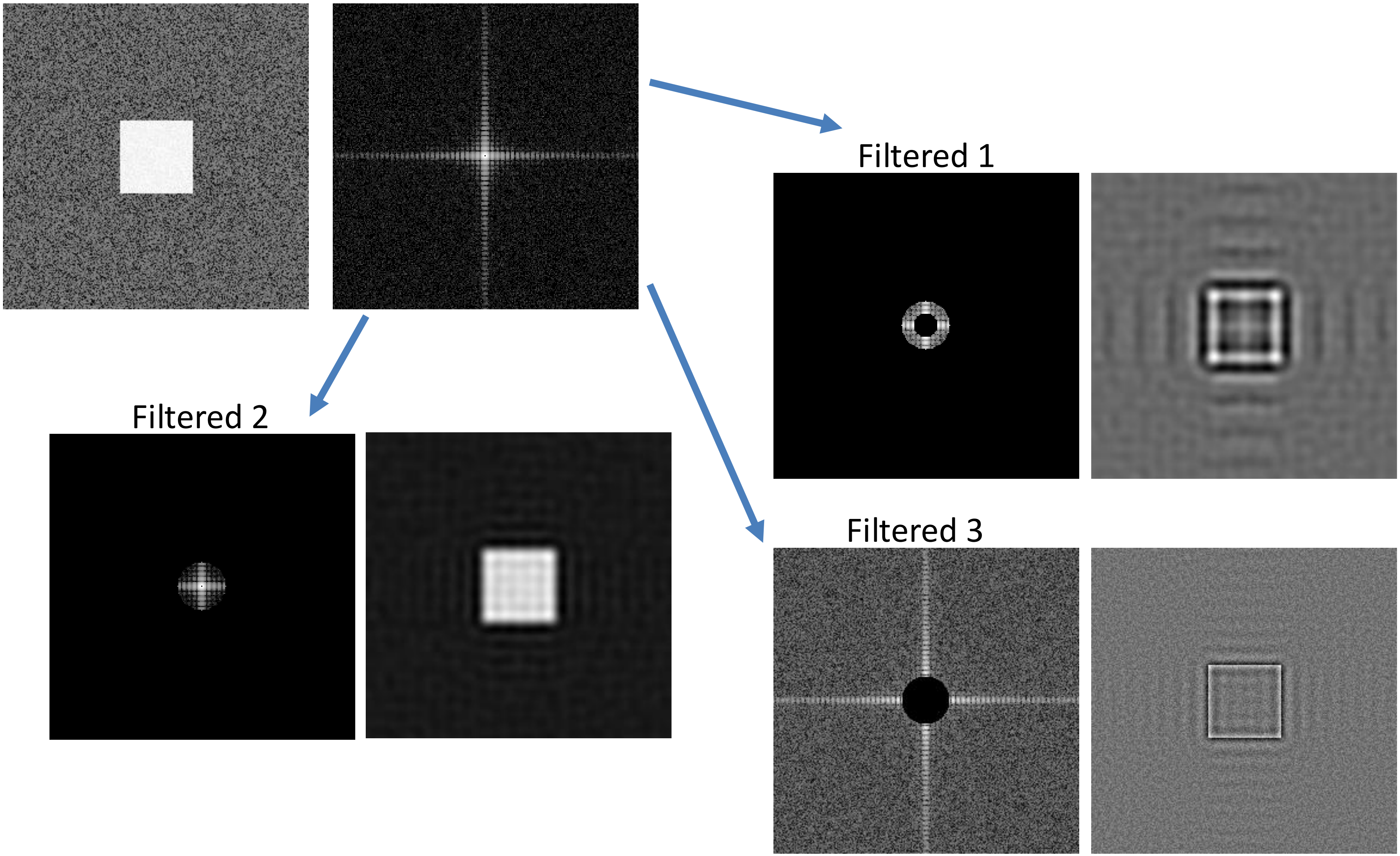


After filtering

Order of $n=3$

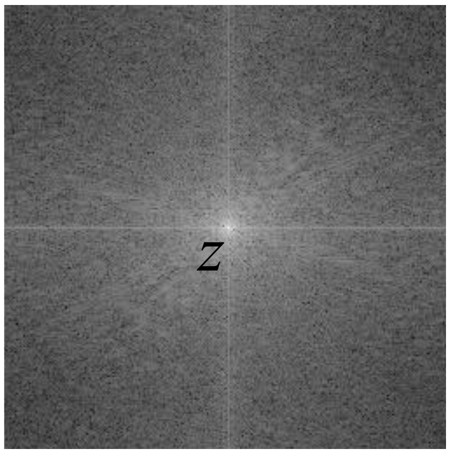
$$H(u,v) = \frac{1}{1 + [r_0 / r(u,v)]^{2n}} \quad \text{of order } n$$

Other Custom/Example filters



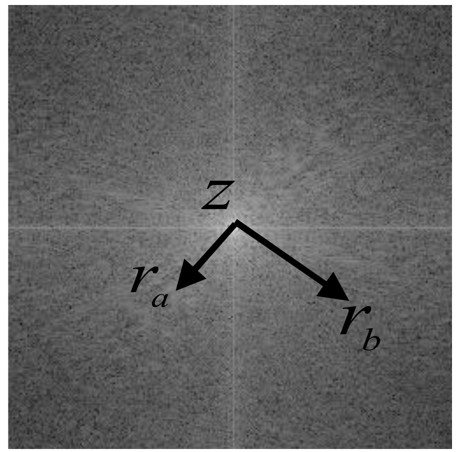
Other Custom/Example filters

- Fourier space, with origin at $z=(u=0,v=0)$.



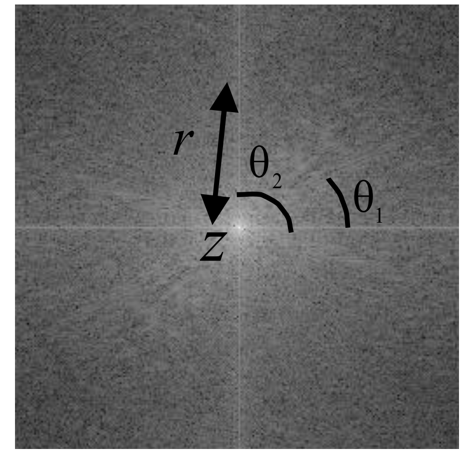
$$a \le u \le b$$
$$c \le v \le d$$

box



$$-r_b \le u \le r_b$$
$$\pm\sqrt{r_a^2 - u^2} \le v \le \pm\sqrt{r_b^2 - u^2}$$

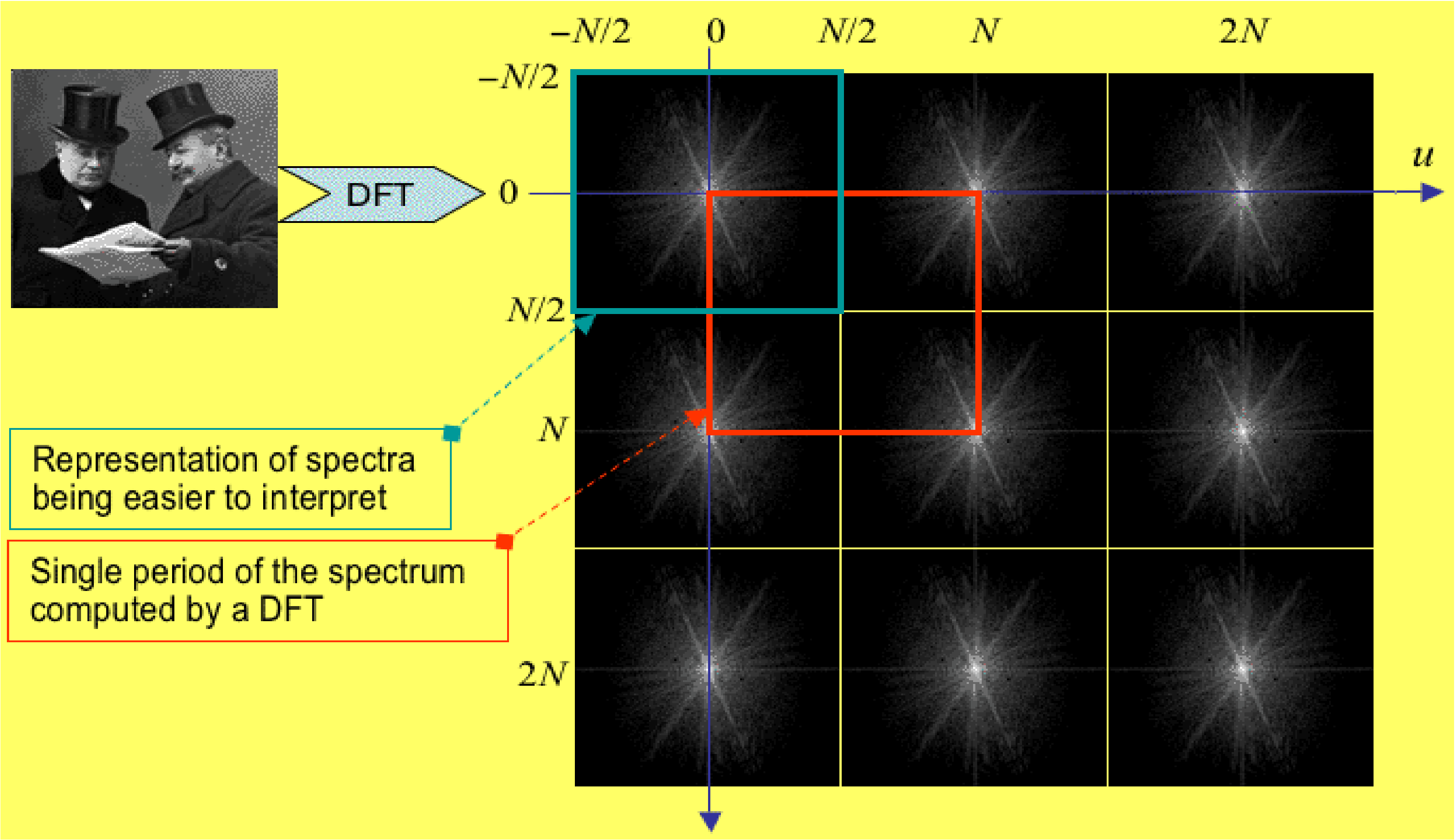
ring



$$u^2 + v^2 = r^2$$
$$\theta_1 \le \tan^{-1}\frac{v}{u} \le \theta_2$$

sector

Periodic Spectrum



Self-Study: Effects of Rotation/Translation Illustrated

Translation or shift in Spatial Domain
Spatial shifts result in a linear phase change in the frequency domain, but no change in the magnitude spectrum. Hence, the magnitude spectrum of a line or dot, for example, looks the same wherever it is in the image.

Rotation in Spatial Domain
Rotation of an image in the spatial domain results in a corresponding rotation in the Fourier domain.

FT

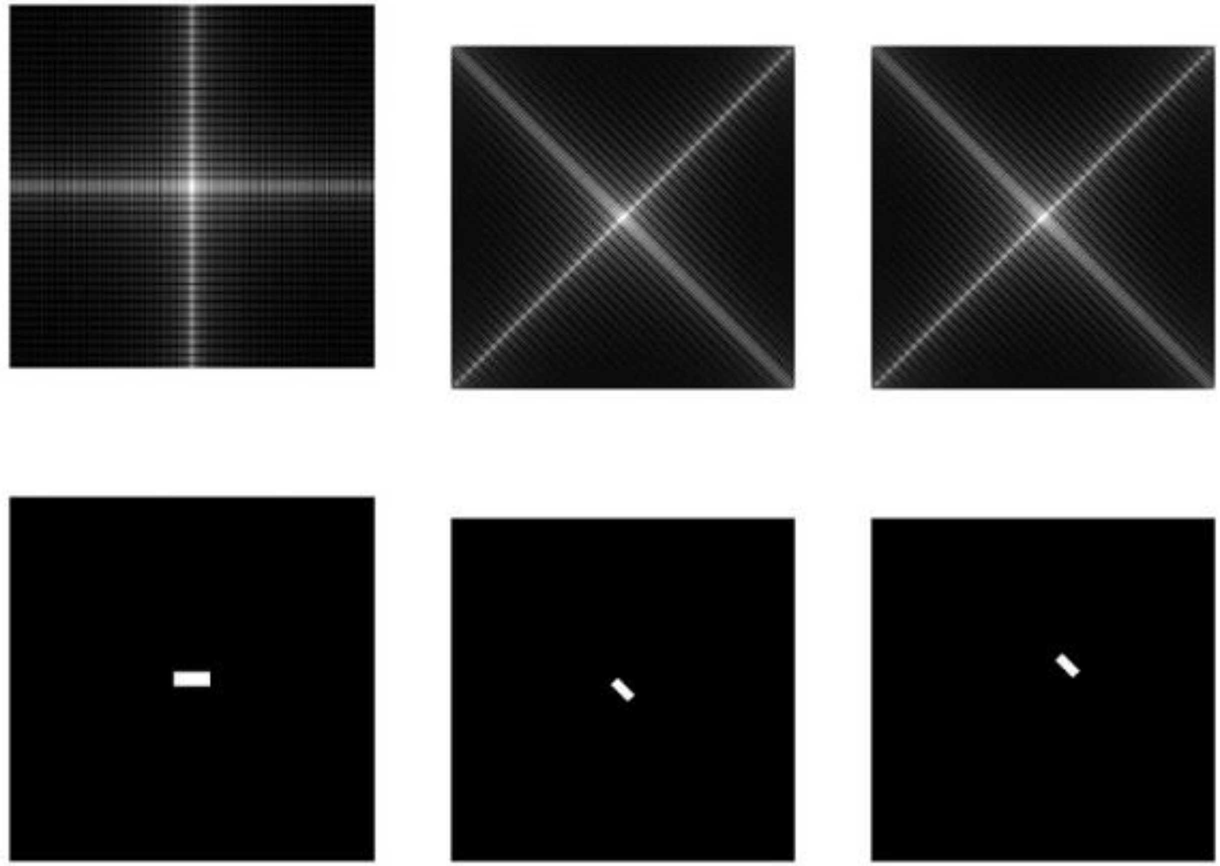


Image | Rotation | Rotation & translation

Self-Study: Summary of Filter Definitions

Notation notice! Using D instead of r

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

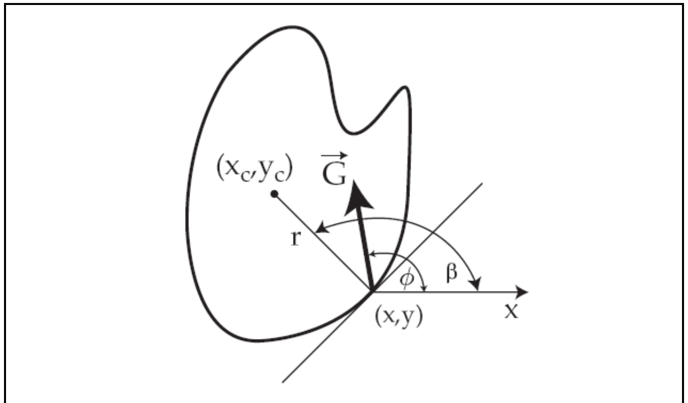
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

Next Lecture

Edge Detection



$\phi_1 = 0$	$(r,\beta)_{1_1}$	$(r,\beta)_{1_2}$	$\cdots$	$(r,\beta)_{1_{n_1}}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
ϕ_j	$(r,\beta)_{j_1}$	$(r,\beta)_{j_2}$	$\cdots$	$(r,\beta)_{j_{n_1}}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\phi_k = \pi$	$(r,\beta)_{k_1}$	$(r,\beta)_{k_2}$	$\cdots$	$(r,\beta)_{k_{n_1}}$

Hough Transform