Department of Computer Science
University of Bristol

# COMS30030 - Image Processing and Computer Vision

Lecture 03

# Frequency Domain & Transforms

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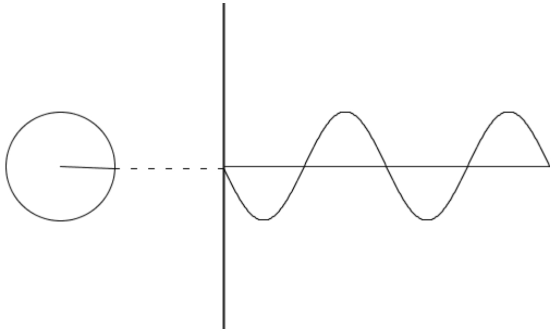
# Signals as Functions

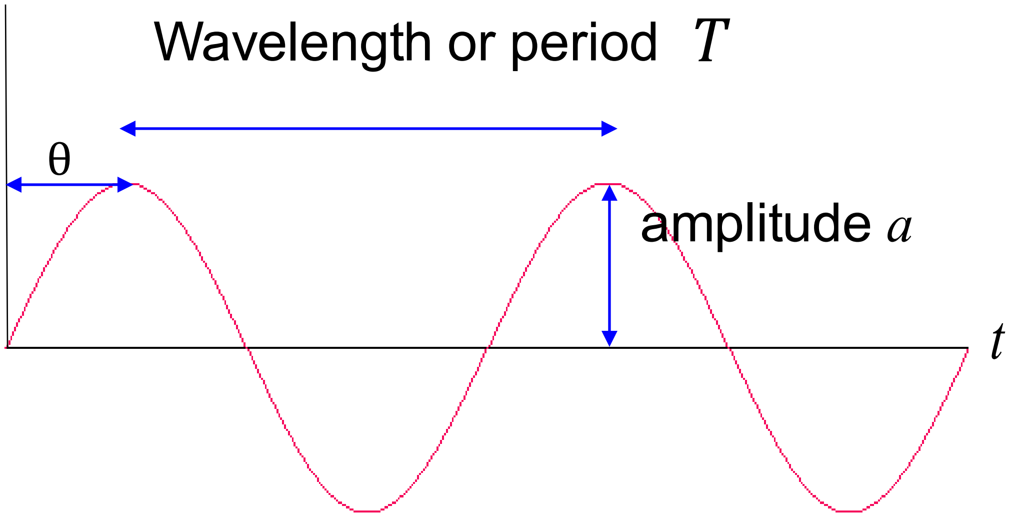
Frequency - allows us to characterise signals:

- Repeats over regular intervals with Frequency $u = \frac{1}{T}$ cycles/sec (Hz)
- Amplitude $a$ (peak value)
- the Phase θ (shift in degrees)

Example: sine function

$$f(t) = a\, sin 2\pi u t$$



# Fourier's Theorem

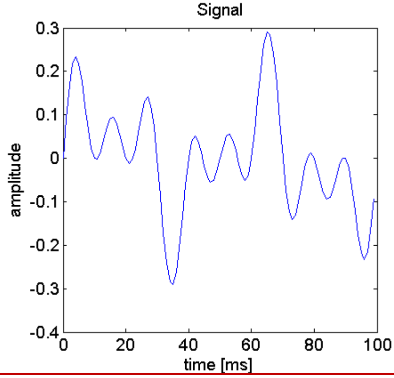
$$f(x) = \int a_n \cos(nx) + b_n \sin(nx) \;\; \delta n$$

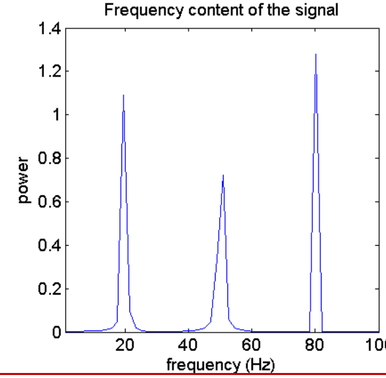
Jean-Baptiste Joseph Fourier

- The sines and cosines are the Basis Functions of this representation. $a_n$ and $b_n$ are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

# Intuition I: Simple 1D example

d = a + b + c

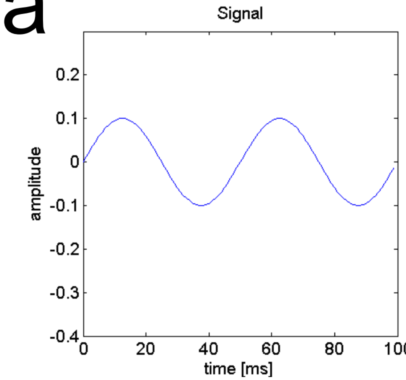


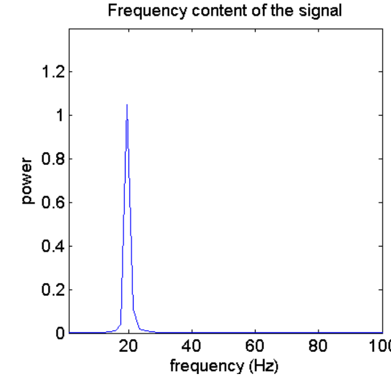




time domain

frequency domain

a

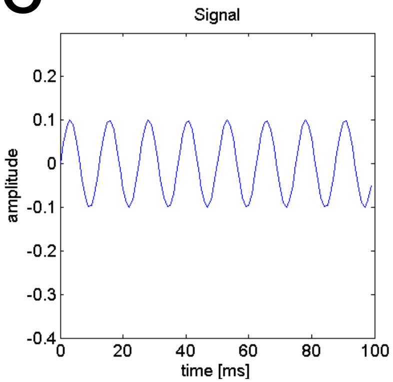


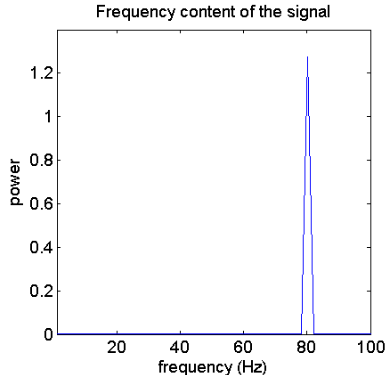




time domain

frequency domain

c

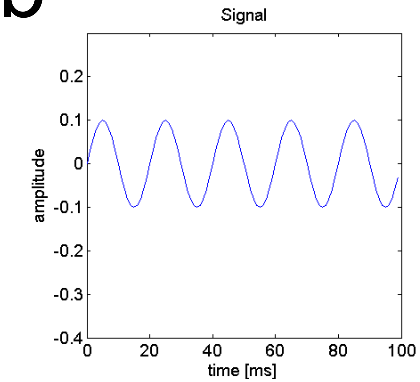


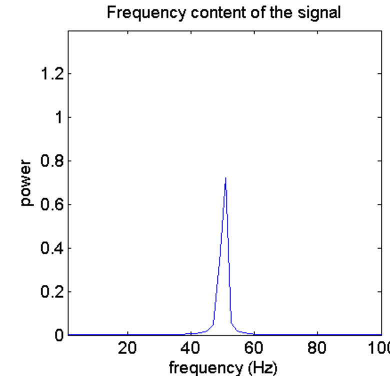




time domain

frequency domain

b




time domain

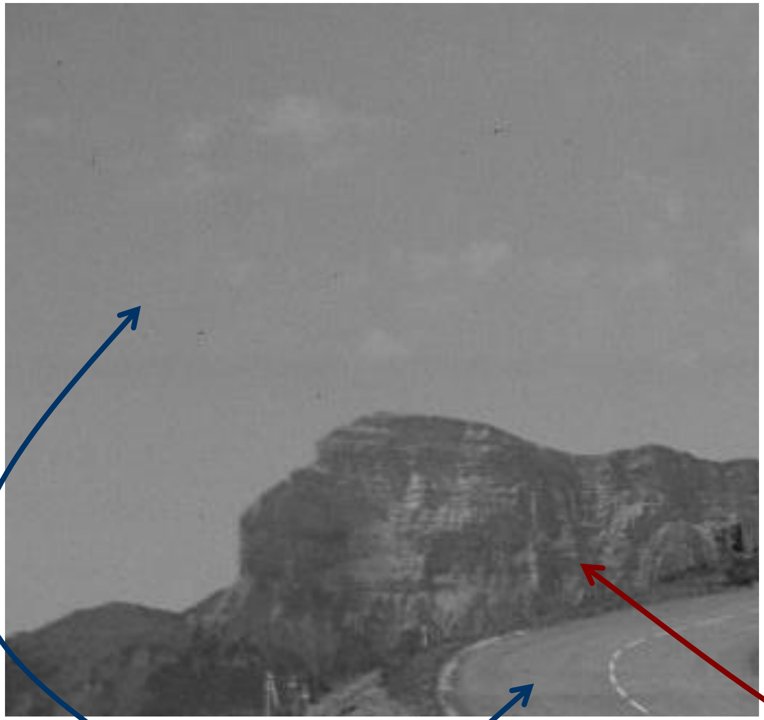
frequency domain

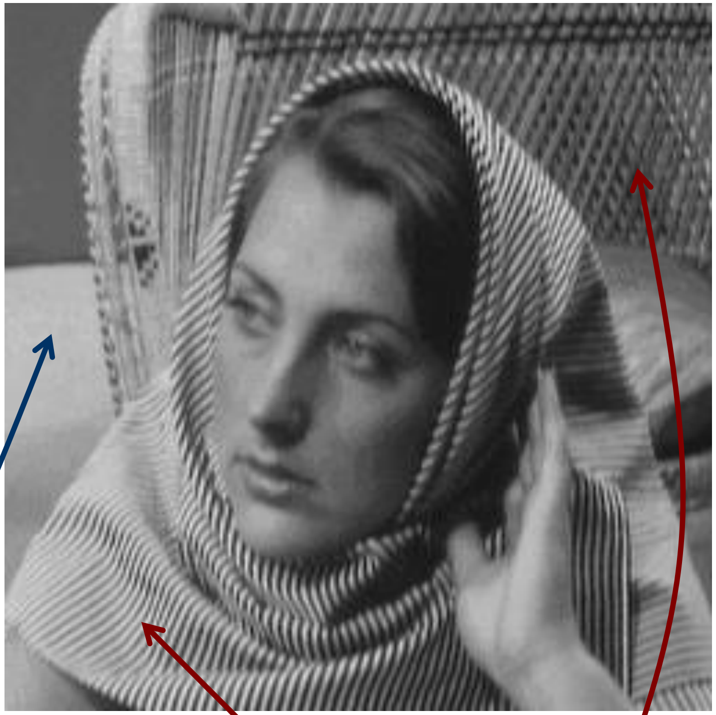
# Intuition II: Simple 1D example

Animation by Lucas V Barbosa

# Intuition III: Concept of Frequency in Images

*Rate of change of intensity along the two dimensions*



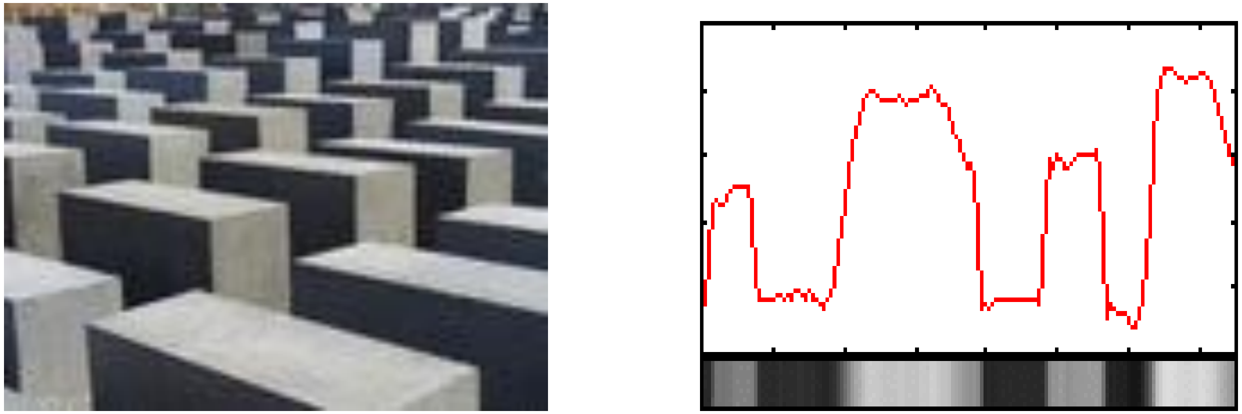


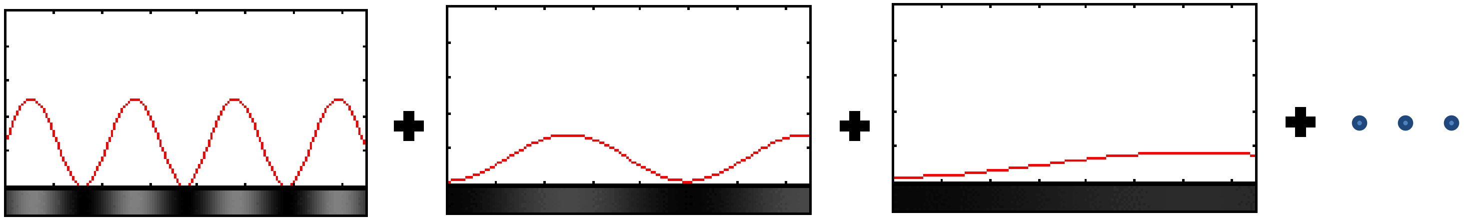
Slowly changing → low frequency

Rapidly changing →high frequency

# Intuition IV: Images as waves!?

Take a single row or column of pixels from an image → a 1D signal





From ImageNagik

# 2D Fourier Transform: Continuous Form

- The Fourier Transform of a continuous function of two variables $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\ e^{-i2\pi(ux+vy)}\ dxdy$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)\ e^{i2\pi(ux+vy)}\ dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

# 2D Fourier Transform: Discrete Form

- The FT of a discrete function of two variables, *f(x,y)*, is:

image

kernels (probing functions)

$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \boxed{f(x,y)} \boxed{e^{-i2\pi(\frac{ux+vy}{N})}} \quad \text{for } u,v = 0,1,2,\ldots,N-1 \,.$$

- Conversely, given $F(u,v)$, we can obtain *f(x,y)* by means of the *inverse FT:*

$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \; e^{i2\pi(\frac{ux+vy}{N})} \quad \text{for } x,y = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

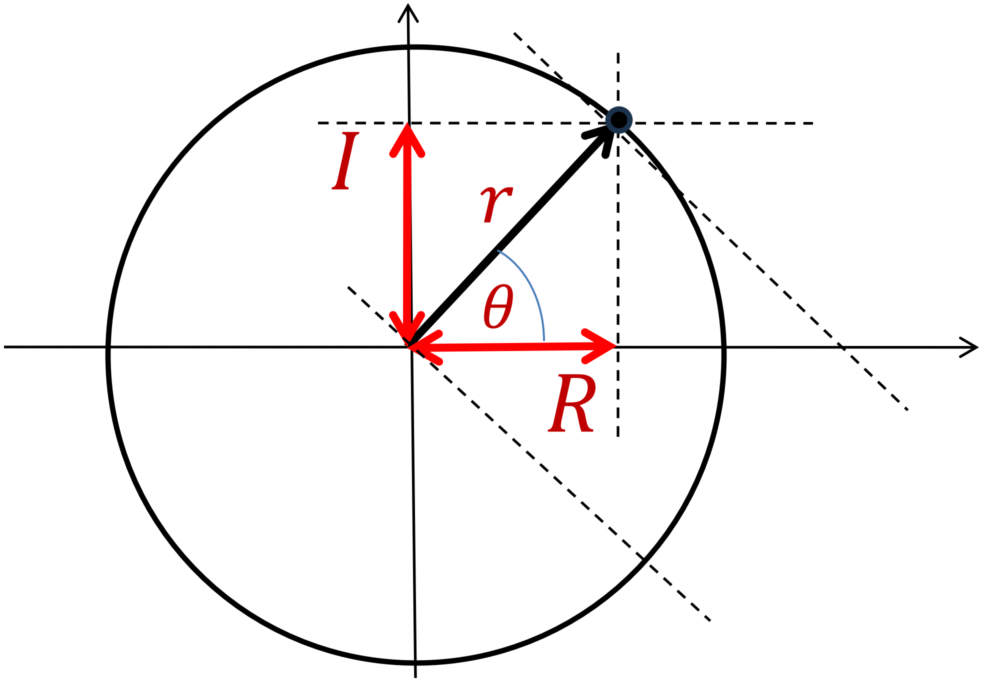
Note, they constitute a lossless representation of data.

# Euler's Formula

$e^{i2\pi(\frac{ux+vy}{N})}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The kernel is associated with a complex number $(r, \theta)$ in polar coordinates or $R(u,v), I(u,v)$ in standard complex notation.



# 2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function *f(x,y)* multiplied by a particular kernel at a particular frequency and orientation specified by *(u,v)*:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - i\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

$$\text{for } u,v = 0,1,2,\ldots,N-1\,.$$

All kernels together form a new orthogonal basis for our image.

# 2D Fourier Transforms

- Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- Thus, each term of the Fourier Transform is composed of the sum of all values of the image function $f(x,y)$ multiplied by a particular kernel at a particular frequency and orientation specified by $(u,v)$:

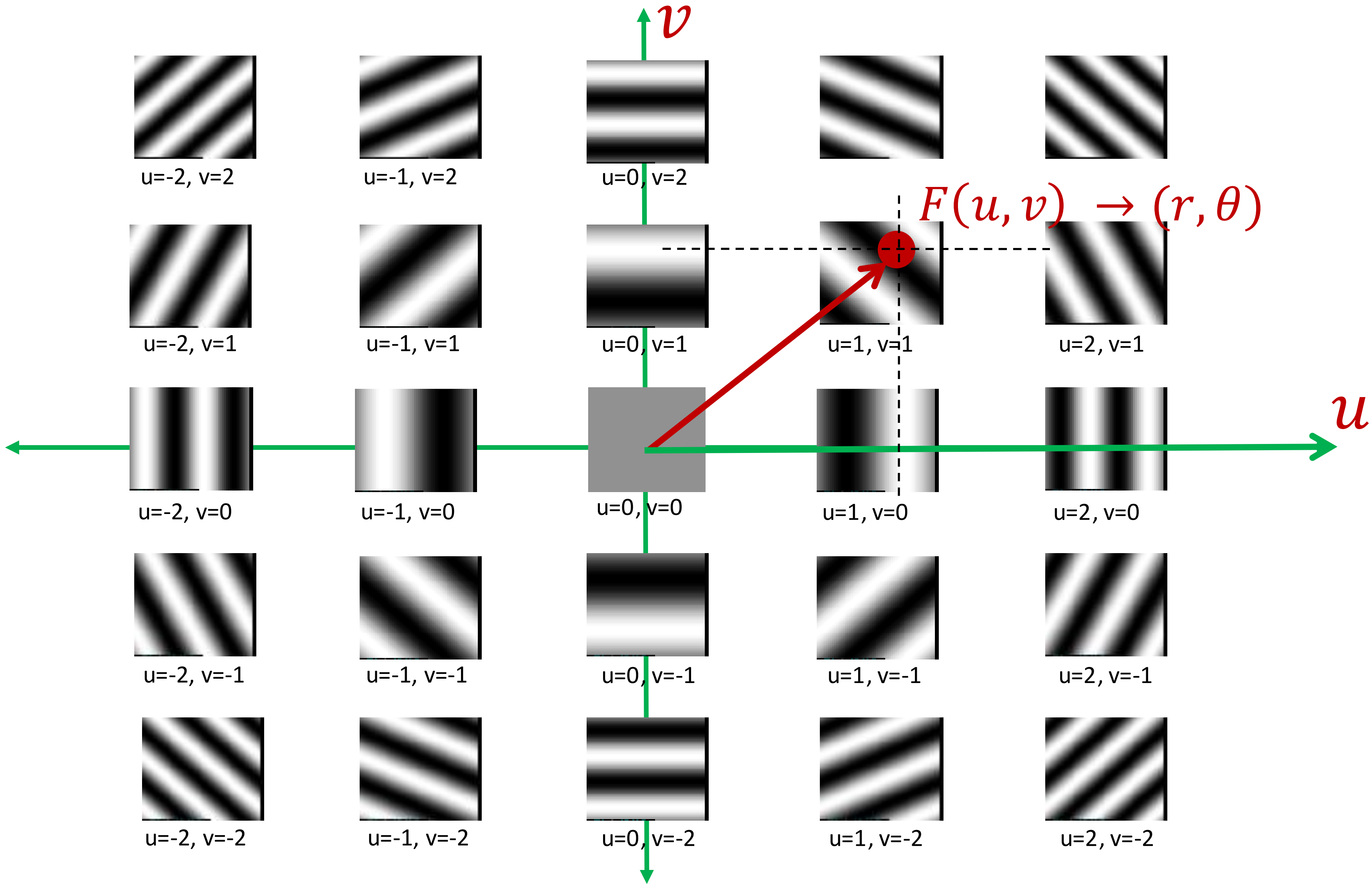
$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\underbrace{\cos\left(\frac{2\pi(ux+vy)}{N}\right)}_{1} - \underbrace{i\sin\left(\frac{2\pi(ux+vy)}{N}\right)}_{0}\right]$$

for $u,v = 0,1,2,\ldots,N-1$ .

The slowest varying frequency component, i.e. when *u=0,v=0* → average image graylevel

All kernels together form a new orthogonal basis for our image.

# ‘Fabric’ of the 2D Fourier Space (as kernels)



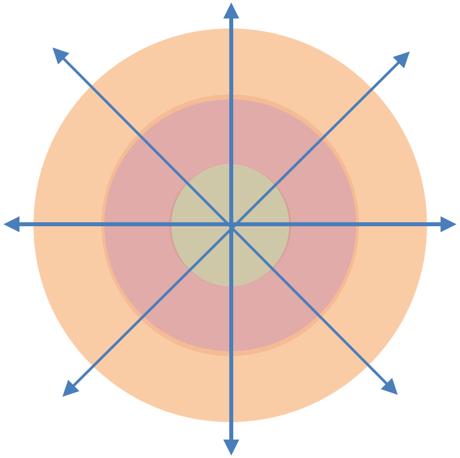
# Power Spectrum and Phase Spectrum

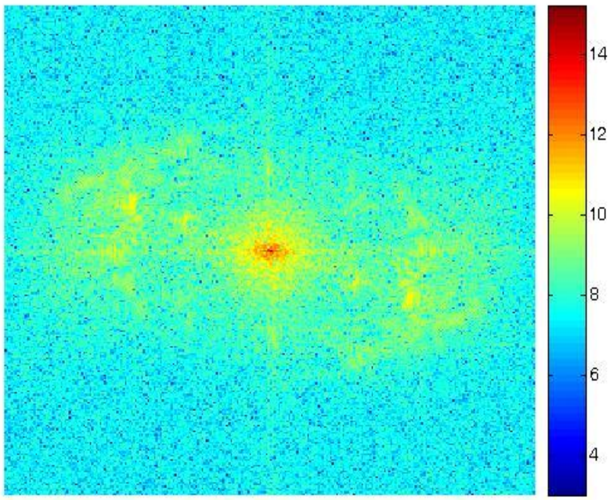
$f(x, y)$

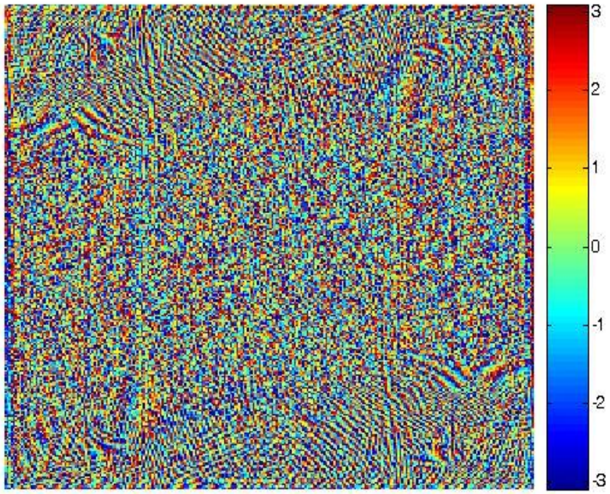
Low to Low-ish frequencies

Mid-range frequencies

High frequencies



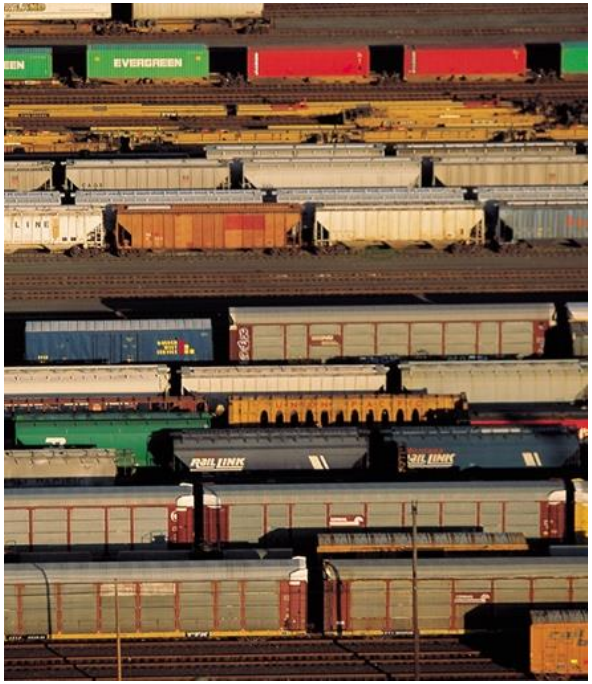


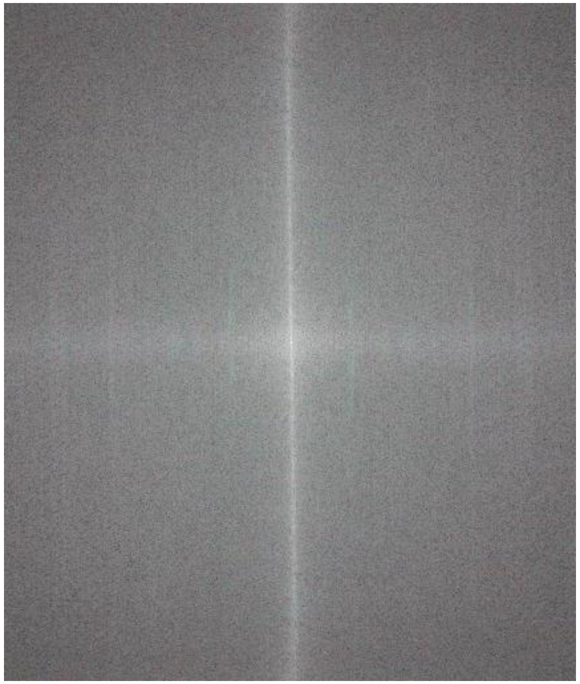

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

# Example: Relating Frequencies to Images

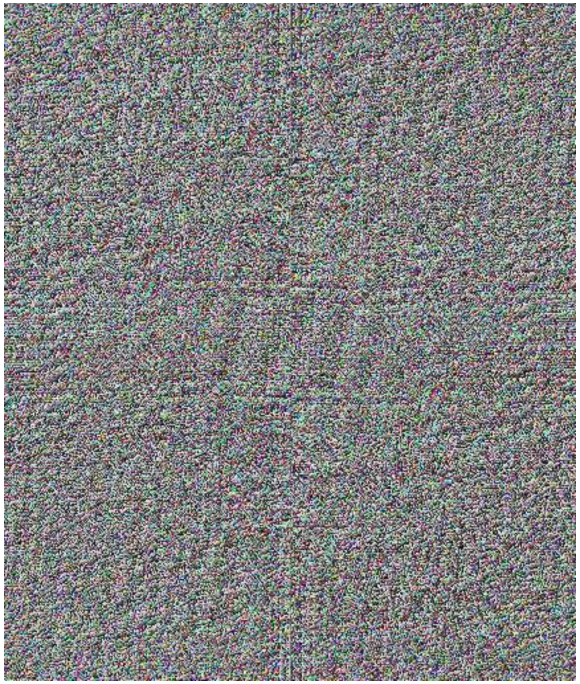



Magnitude



$|F(u,v)|$

Phase



$\angle F(I)$

# The Frequency Domain

- $F(u,v)$ is a complex number and has real and imaginary parts:

$$F(u,v) = R(u,v) + iI(u,v)$$

- Magnitudes (forming the Magnitude Spectrum):

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

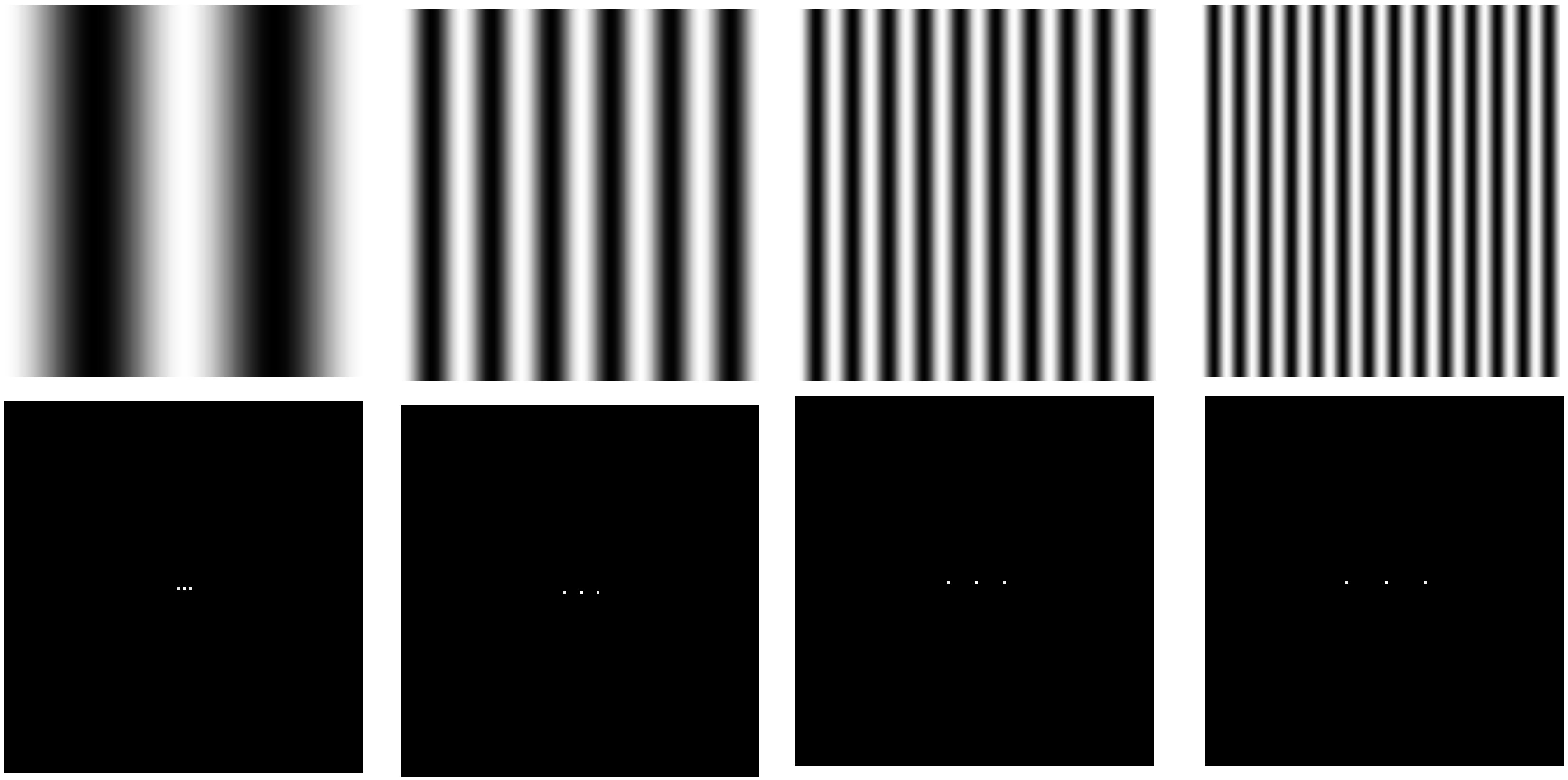
- Phase Angles (forming the Phase Spectrum):

$$\theta(u,v) = \tan^{-1}\left[I(u,v)/R(u,v)\right]$$

- Expressing $F(u,v)$ in polar coordinates $(r, \theta)$ :

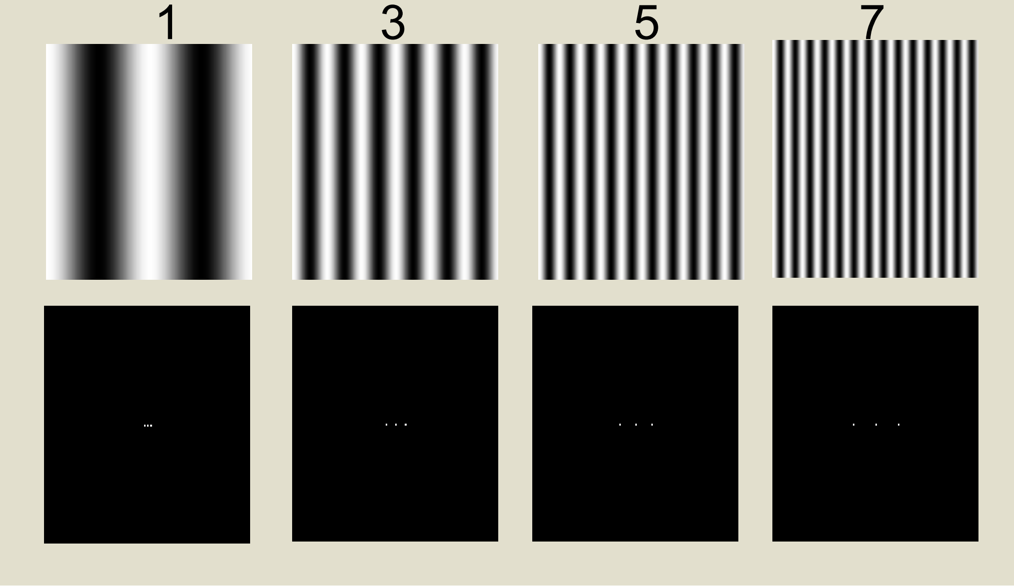
$$F(u,v) = |F(u,v)|e^{i\theta(u,v)} = re^{i\theta}$$

# Spatial Domain ⟷ Frequency Domain



# Perpendicular relationship

Ideal edge and line structures have a concentration along a line passing through the origin in the frequency domain and in a **direction perpendicular to their orientation** - they are 'constructed' by adding together all 2D sinusoidal waves that 'travel' perpendicular to the edge or line.



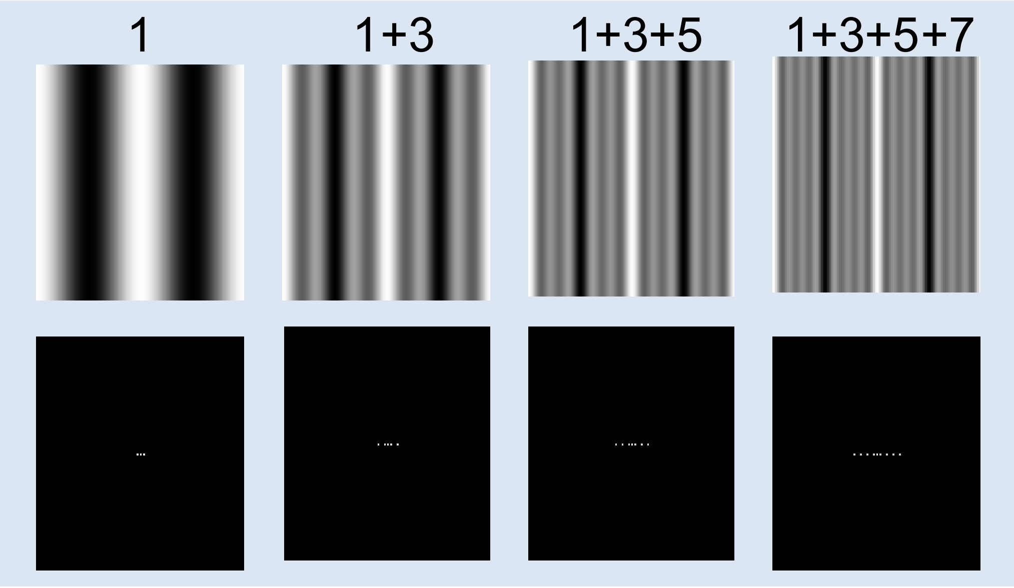
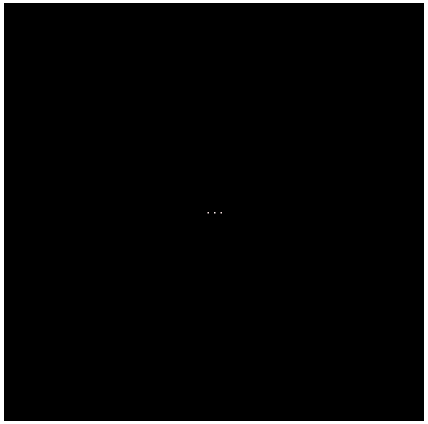
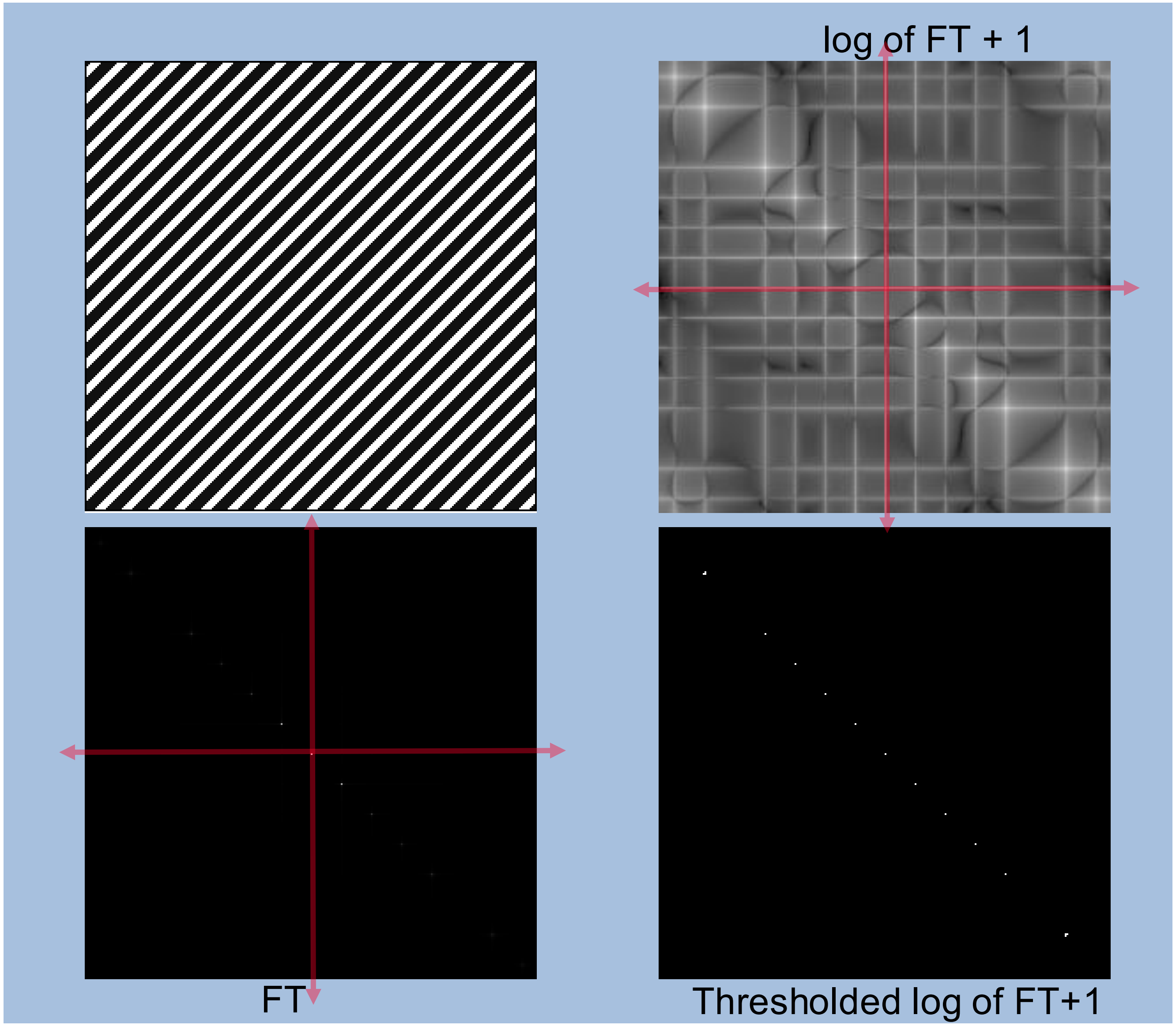



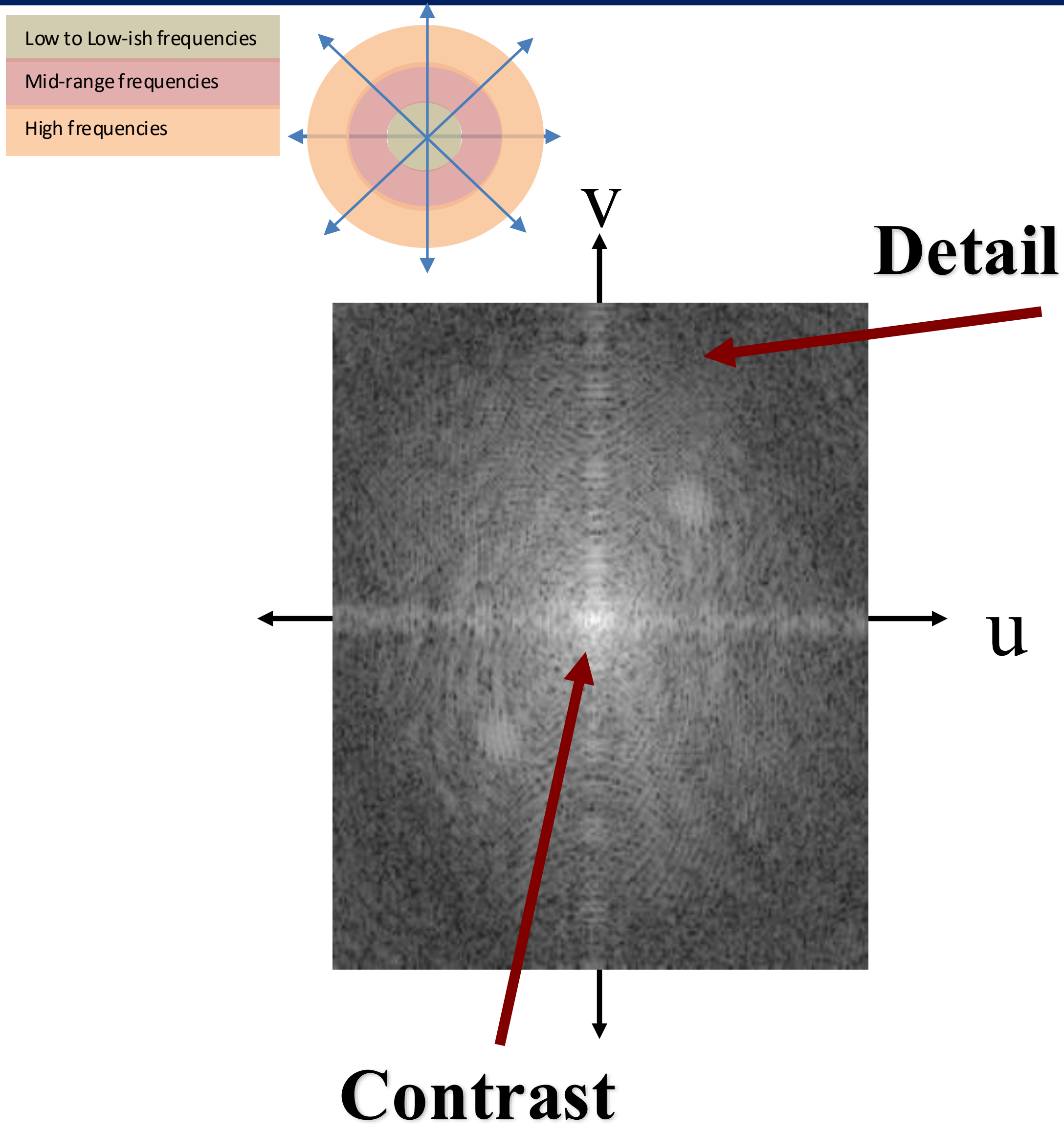

Image FT

# Spatial Domain ⟷ Frequency Domain

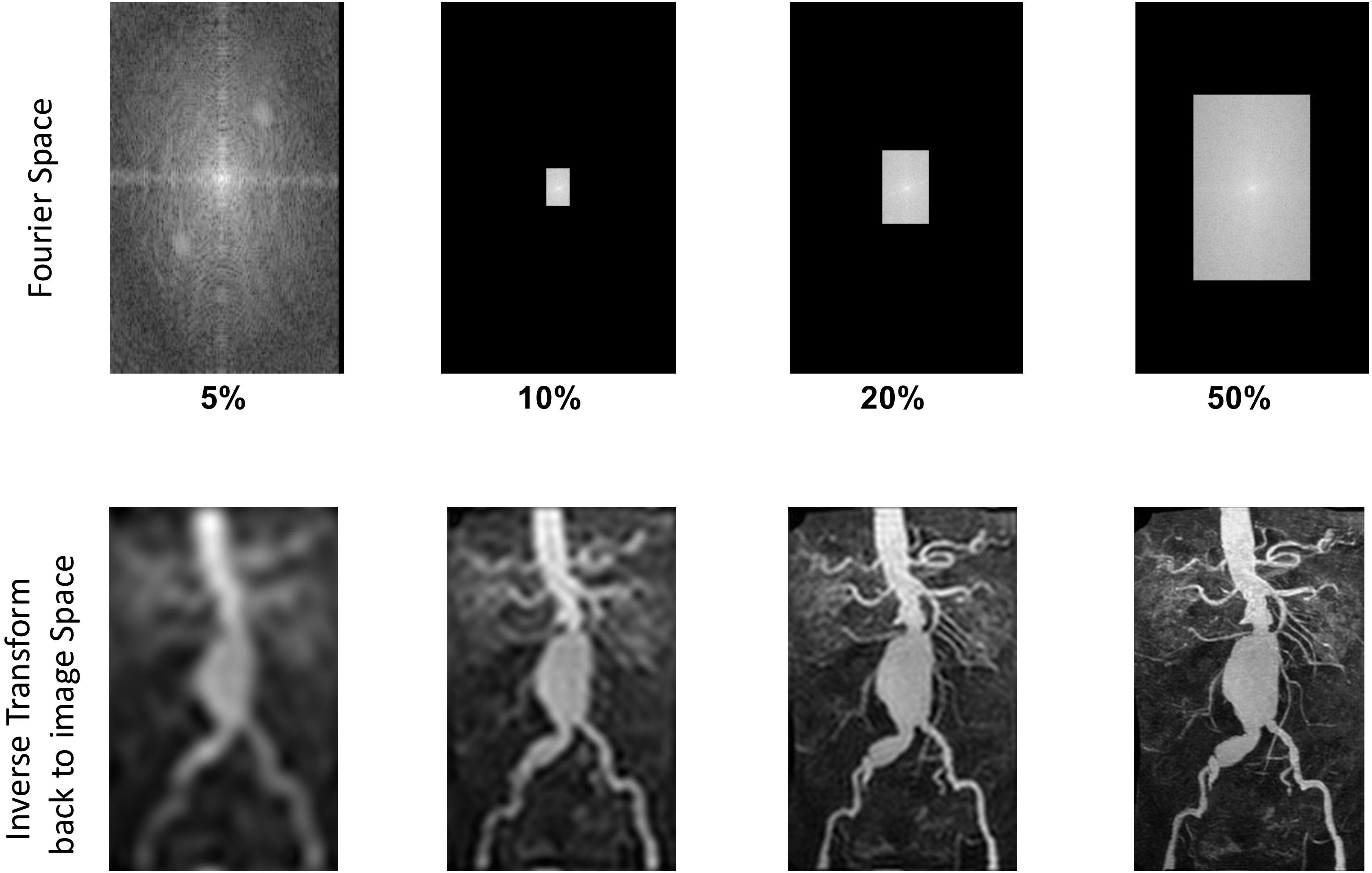




# Relating Frequencies to Images

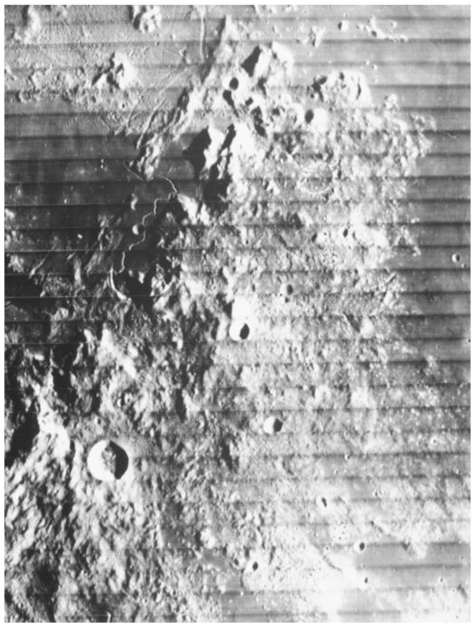


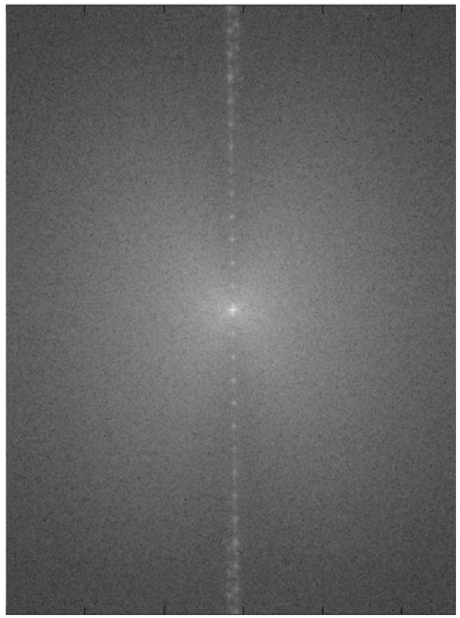
# Example: Relating Frequencies to Images



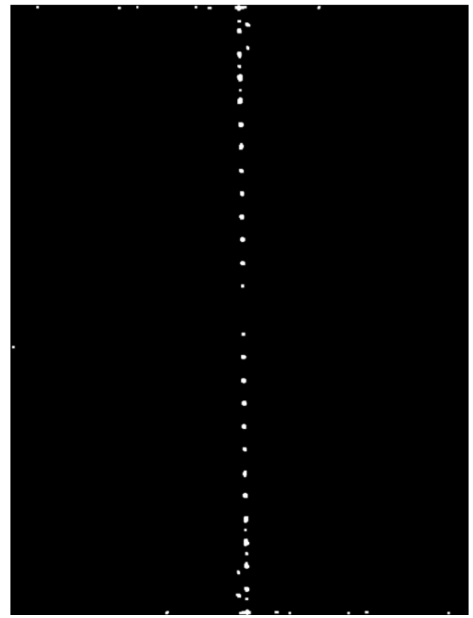
# Example: Manipulating the FS

## Lunar orbital image (1966)





$|F(u, v)|$



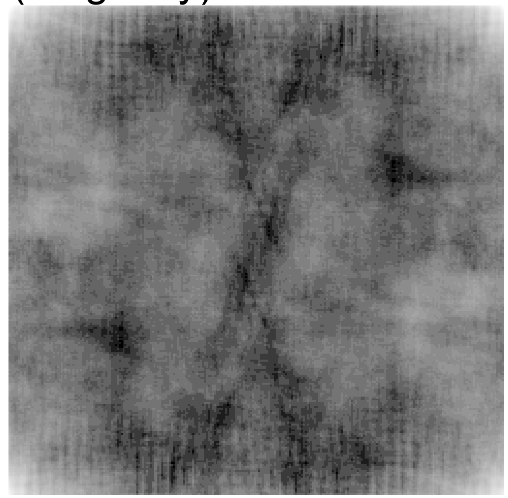
Mask used to remove peaks

$\mathrm{i}FFT(F(u, v))$

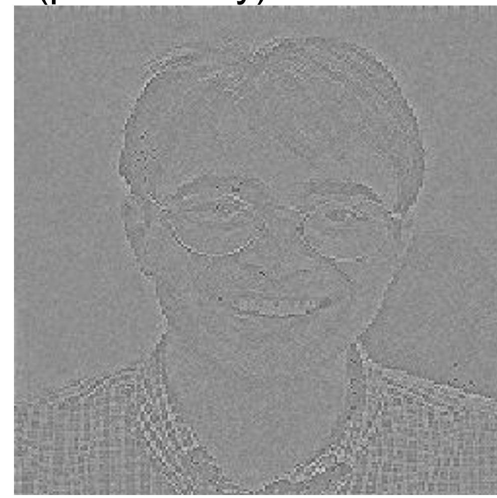
Slide by A. Zisserman

# Importance of Phase

ifft(mag only)



ifft(phase only)



ifft(mag(Peter) and Phase(Andrew))

ifft(mag(Andrew) and Phase(Peter))

# Separability

- Important property of the FT: *Separability*
- If a 2D transform is separable, the result can be found by successive application of two 1D transforms. This is a principle aspect of the Fast Fourier Transform (FFT).

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v) \quad e^{\frac{-j2\pi ux}{N}} \quad \text{where} \quad F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y) \quad e^{\frac{-j2\pi vy}{N}}$$

( , ) ( , ) ( , )

*1D row transforms*

*1D column transforms*

# Recall: 2D Discrete Convolution

- The discrete version of 2D convolution is defined as:

$$g(x,y)=\sum_{m}\sum_{n} f(x-m,y-n)h(m,n)$$

| | | $y-1$ | $y$ | $y+1$ | |
|---|---|---|---|---|---|
| | | | | | |
| $x-1$ | | 43 | 12 | 61 | |
| $x$ | | 44 | 45 | 60 | |
| $x+1$ | | 43 | 50 | 61 | |
| | | | | | |

| | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | -1 | 0 | 1 |
| $0$ | -2 | 0 | 2 |
| $1$ | -1 | 0 | 1 |

$$\underbrace{\phantom{f}}_{\boldsymbol{f}} * \underbrace{\phantom{h}}_{\boldsymbol{h}} = \mathbf{-68} = \begin{aligned} &f(x+1,y+1)h(-1,-1)\\ &+f(x+1,y)h(-1,0)\\ &+f(x+1,y-1)h(-1,1)\\ &+f(x,y+1)h(0,-1)\\ &+f(x,y)h(0,0)\\ &+f(x,y-1)h(0,1)\\ &+f(x-1,y+1)h(1,-1)\\ &+f(x-1,y)h(1,0)\\ &+f(x-1,y-1)h(1,1)\end{aligned}$$

# Convolution in the Spatial/Frequency Domain

Convolution Theorem:

Convolution in spatial domain
*is equivalent to*
multiplication in frequency domain
*(and vice versa)*

$$h = f * g \qquad \textbf{\textit{implies}} \qquad H = FG$$

$$h = fg \qquad \textbf{\textit{implies}} \qquad H = F * G$$

# Deriving the Convolution Theorem

NOT EXAMINABLE

$$h(x) = f(x) * g(x) = \sum_y f(x-y)g(y)$$

$$H(u) = \sum_x \left( \sum_y f(x-y)g(y) \right) e^{(-iux2\pi/N)}$$

$$H(u) = \sum_y g(y) \left( \sum_x f(x-y)\, e^{(-iux2\pi/N)} \right)$$

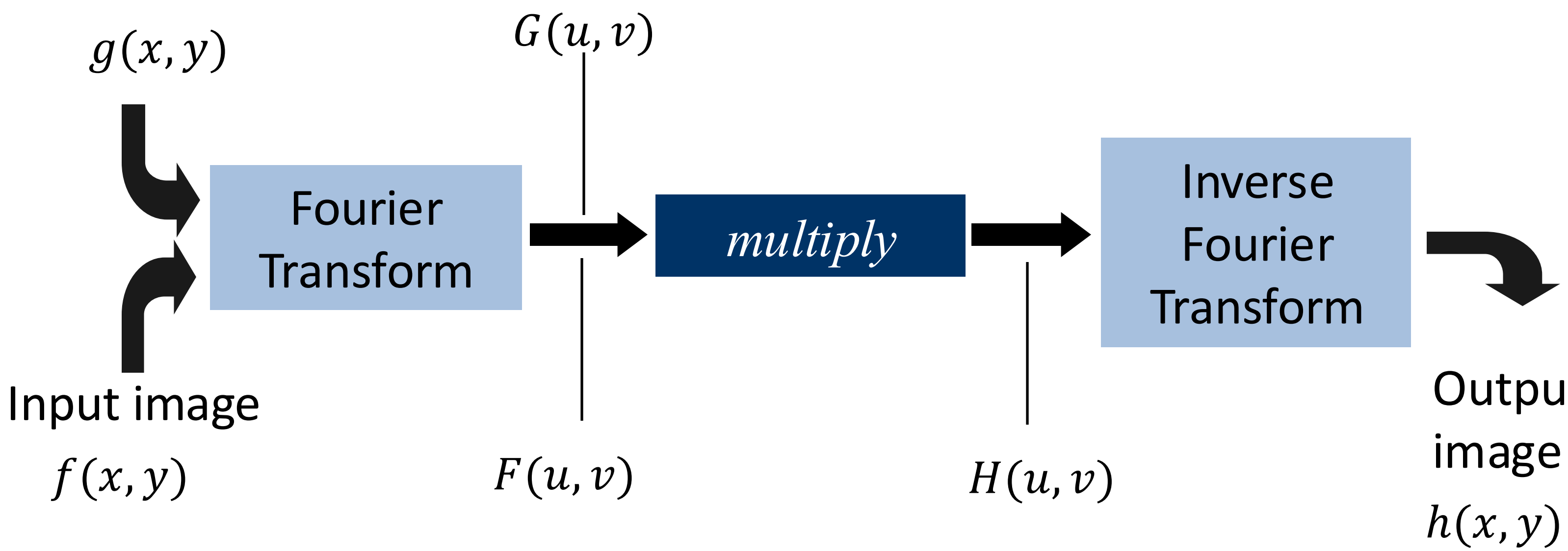
$$H(u) = \sum_y g(y) \left( F(u)\, e^{(-iuy2\pi/N)} \right)$$

$$H(u) = \sum_y g(y)\, e^{(-iuy2\pi/N)} F(u) = G(u) \cdot F(u) = F(u) \cdot G(u)$$

# Fast Filtering using the Convolution Theorem

1D: $H(u) = F(u)G(u)$

2D: $H(u,v) = F(u,v)G(u,v)$



# Recall: Spatial Low/High Pass Filtering

***f***

## Low Pass

***h=***

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 1 |
| 1 | 1 | 1 |

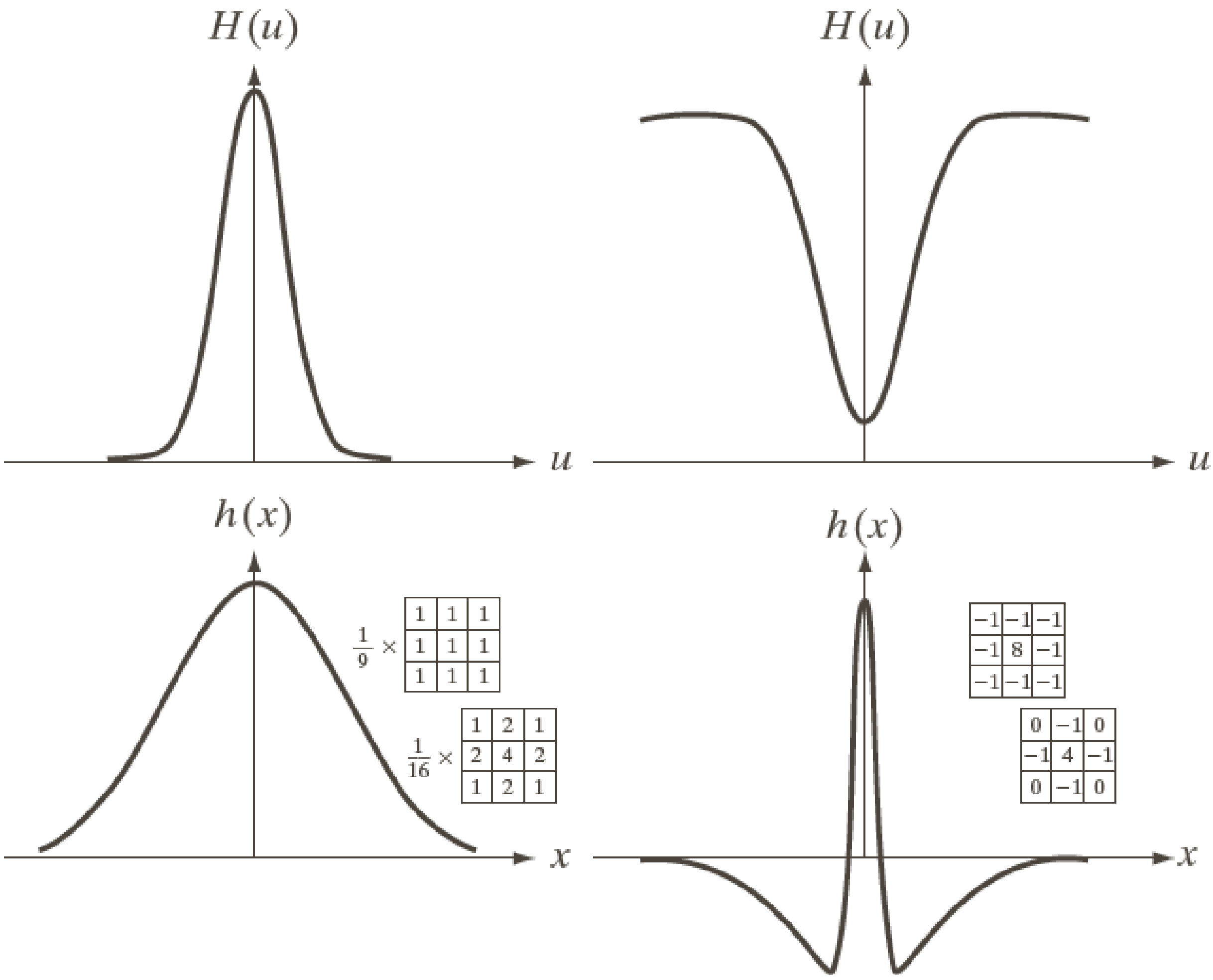
## High Pass

***h=***

| | | |
|---|---|---|
| -1 | -1 | -1 |
| -1 | 8 | -1 |
| -1 | -1 | -1 |

***f * h***

# Frequency Domain Low/High Pass Filtering

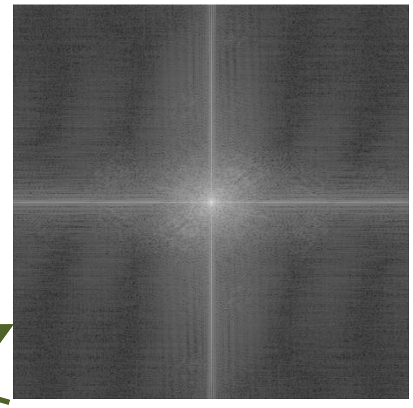



a c
b d

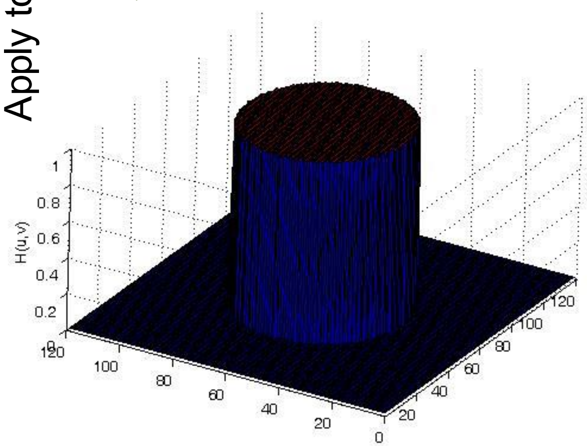
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter

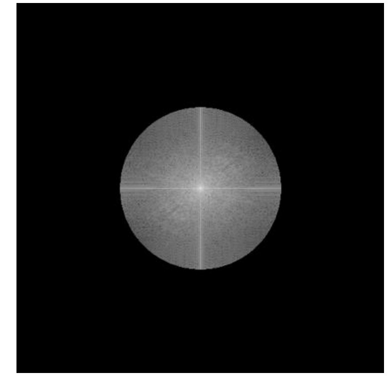
# Ideal Low Pass Filter

- 1D: turning the “treble” down on audio equipment!
- 2D: smooth image



Apply to freq. domain



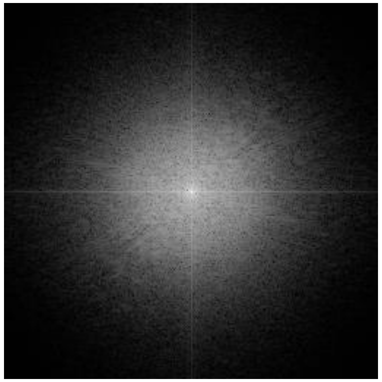



$$H(u,v) = \begin{cases} 1 & r(u,v) \leq r_0 \\ 0 & r(u,v) > r_0 \end{cases}$$

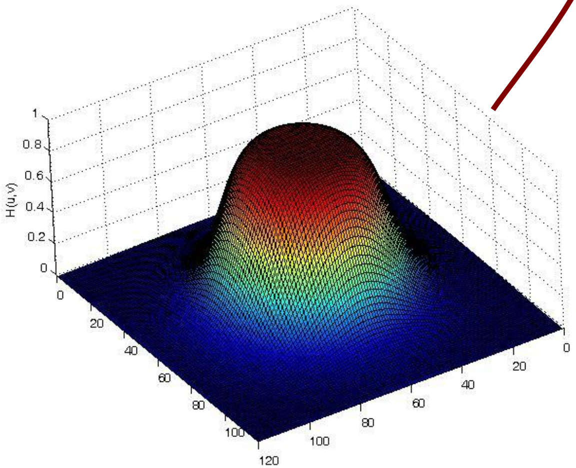
$r(u,v) = \sqrt{u^2 + v^2}$, $r_0$ is the filter radius

# Butterworth's Low Pass Filter

*Input image*



After applying to freq. domain

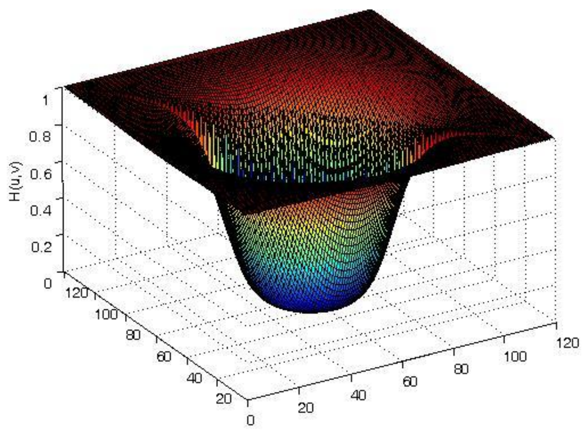


*After filtering*

$$H(u,v) = \frac{1}{1 + [r(u,v)/r_0]^{2n}} \quad \text{of order } n$$

# Butterworth's High Pass Filter

*Input image*

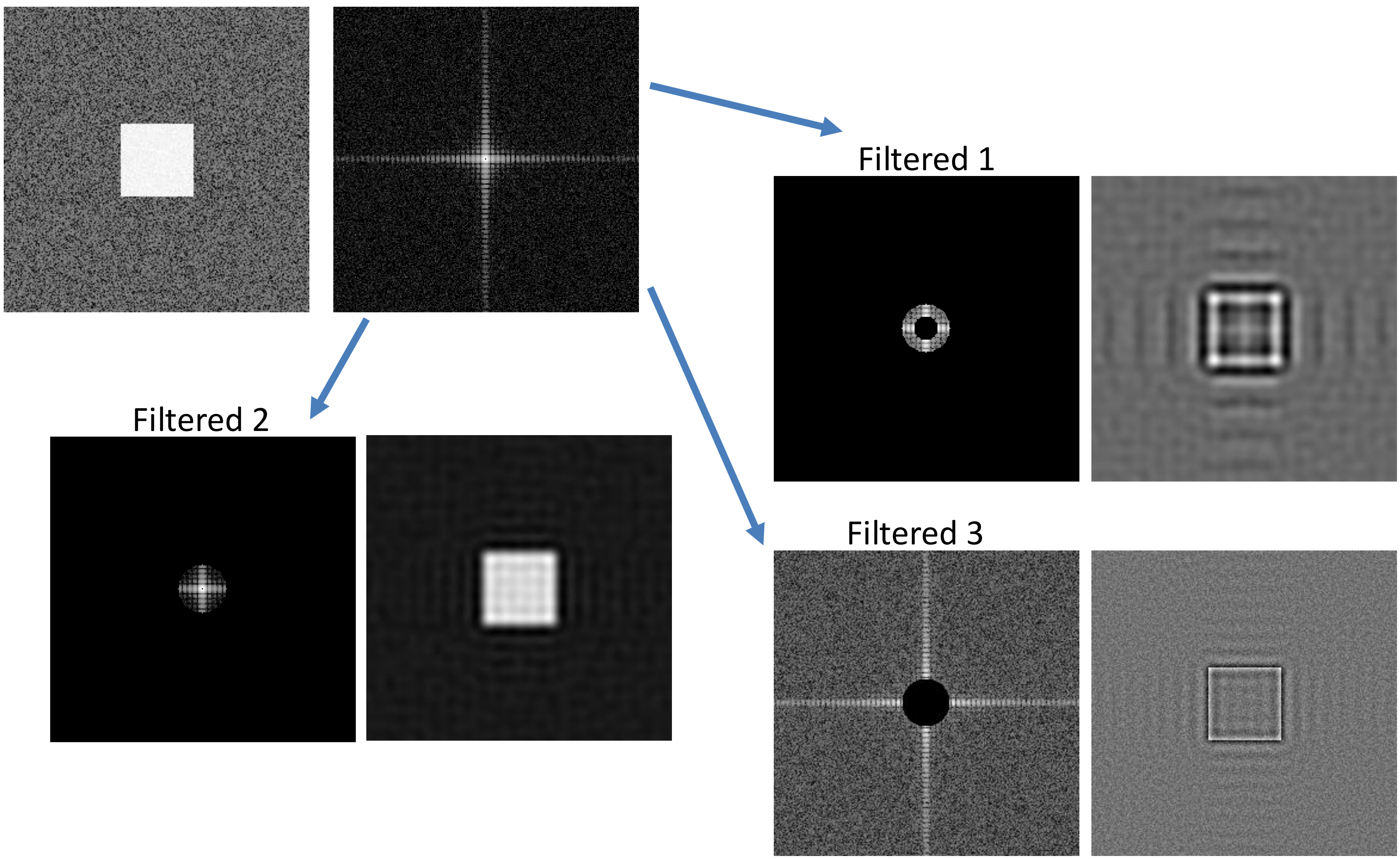



*After filtering*

Order of $n=3$

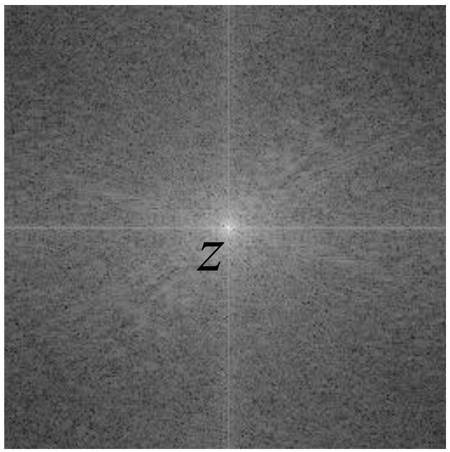
$$H(u,v) = \frac{1}{1 + [r_0 / r(u,v)]^{2n}} \quad \text{of order } n$$

# Other Custom/Example filters



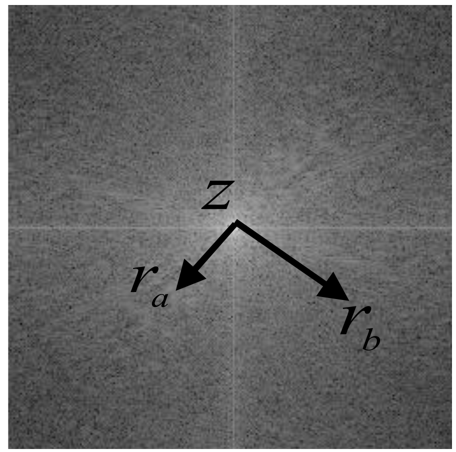
# Other Custom/Example filters

- Fourier space, with origin at $z=(u=0,v=0)$.



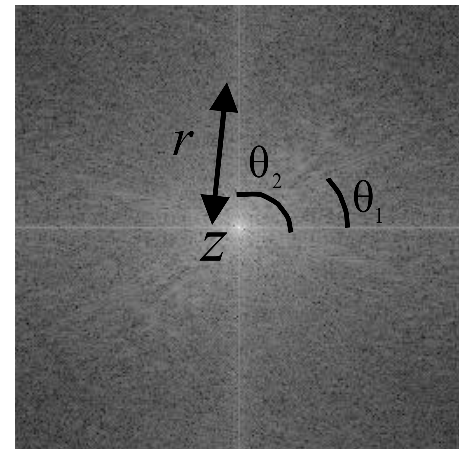

$$a \le u \le b$$
$$c \le v \le d$$

box




$$-r_b \le u \le r_b$$
$$\pm\sqrt{r_a^2 - u^2} \le v \le \pm\sqrt{r_b^2 - u^2}$$

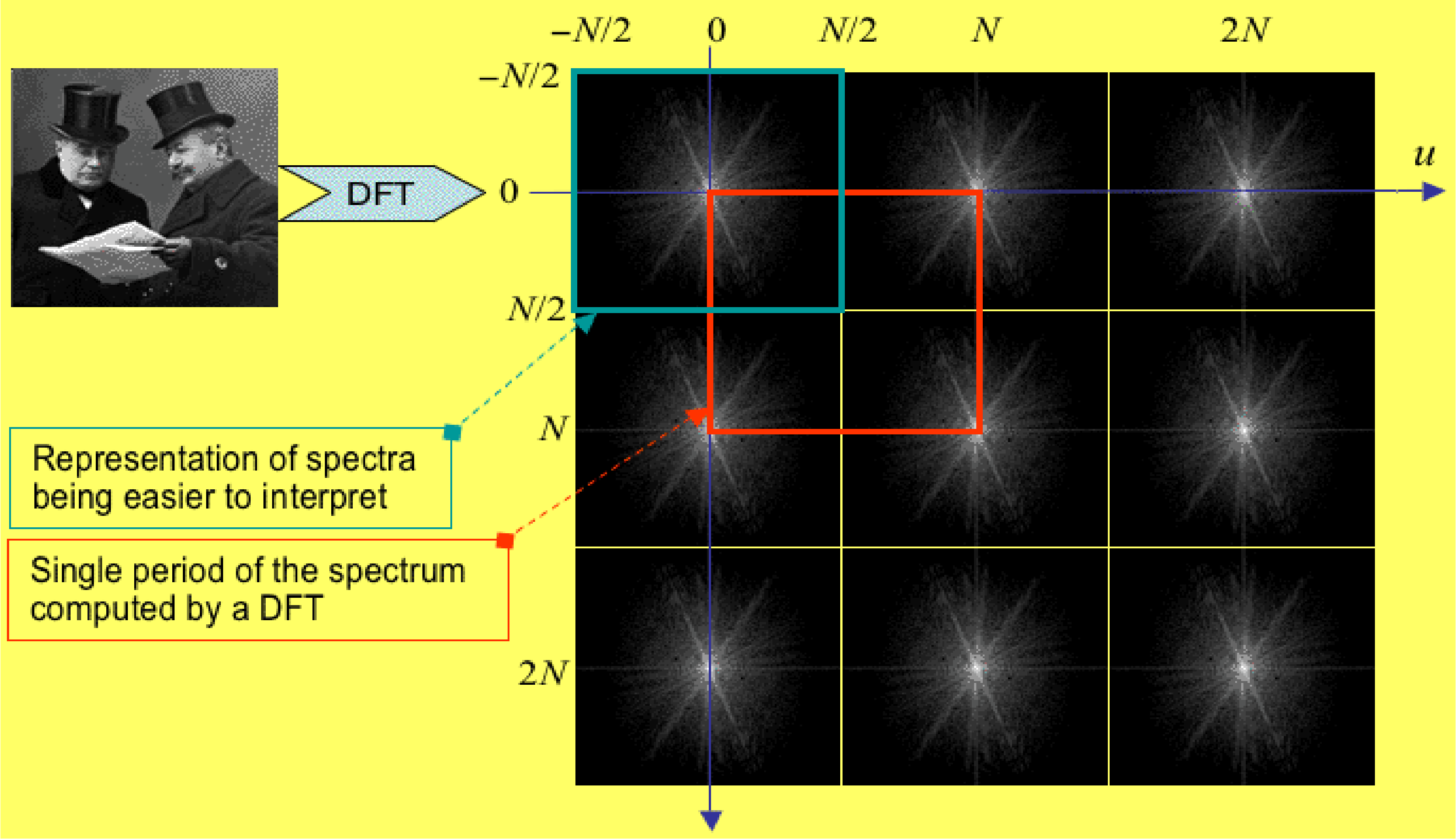
ring




$$u^2 + v^2 = r^2$$
$$\theta_1 \le \tan^{-1}\frac{v}{u} \le \theta_2$$

sector

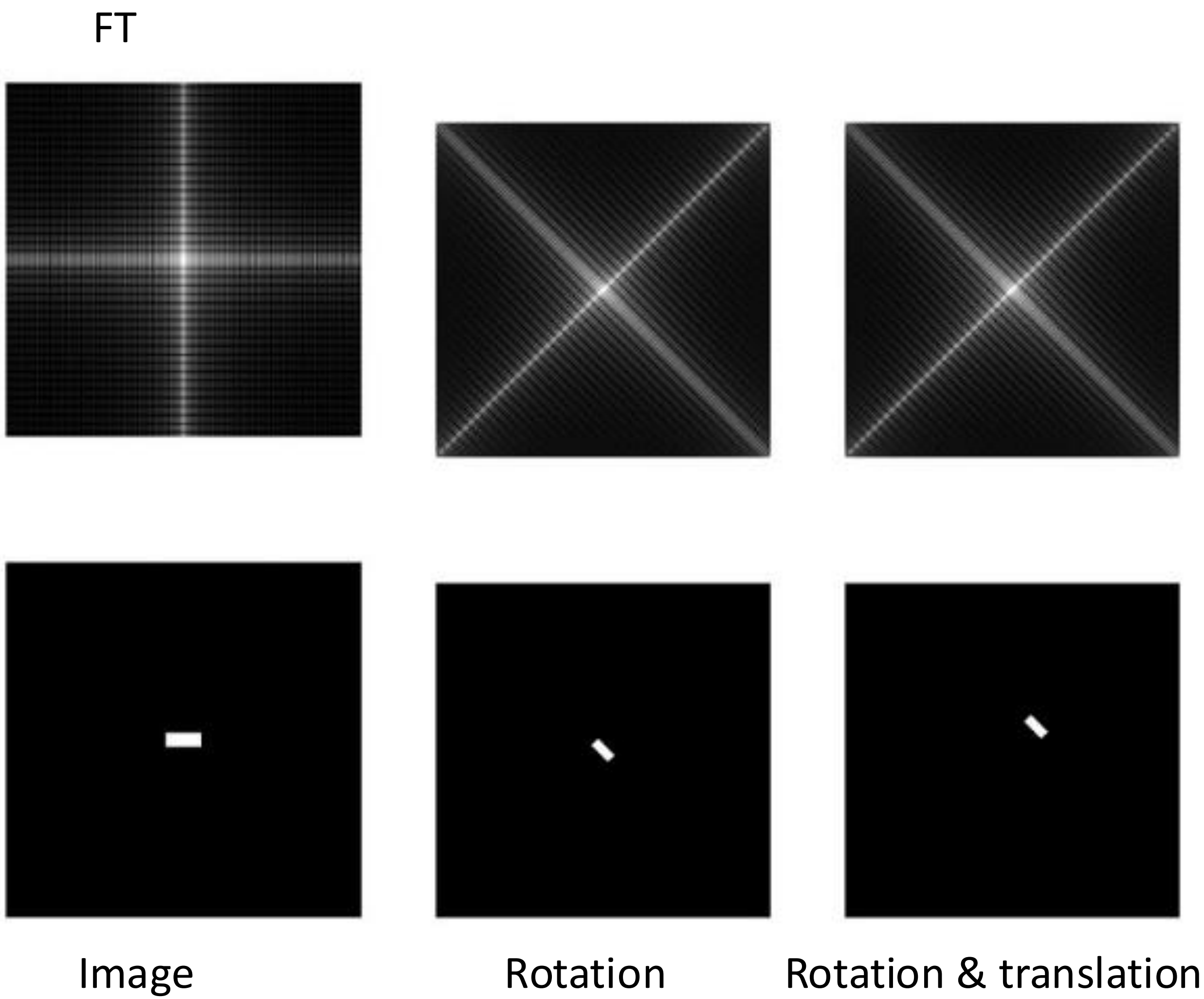
# Periodic Spectrum



# Self-Study: Effects of Rotation/Translation Illustrated

**Translation or shift in Spatial Domain**
Spatial shifts result in a linear phase change in the frequency domain, but no change in the magnitude spectrum. Hence, the magnitude spectrum of a line or dot, for example, looks the same wherever it is in the image.

**Rotation in Spatial Domain**
Rotation of an image in the spatial domain results in a corresponding rotation in the Fourier domain.



# Self-Study: Summary of Filter Definitions

**Notation notice! Using $D$ instead of $r$**

Lowpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

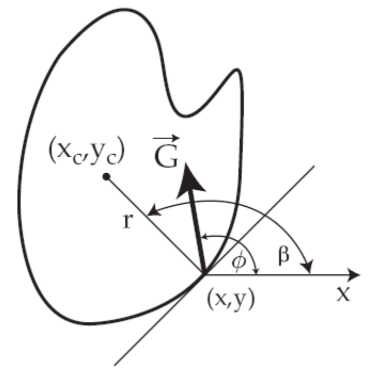
| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$ | $H(u,v) = e^{-D^2(u,v)/2D_0^2}$ |

Highpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$ | $H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$ |

# Next Lecture

Edge Detection




| | | | | |
|---|---|---|---|---|
| $\phi_1 = 0$ | $(r,\beta)_{1_1}$ | $(r,\beta)_{1_2}$ | $\cdots$ | $(r,\beta)_{1_{n_1}}$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\phi_j$ | $(r,\beta)_{j_1}$ | $(r,\beta)_{j_2}$ | $\cdots$ | $(r,\beta)_{j_{n_1}}$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $\phi_k = \pi$ | $(r,\beta)_{k_1}$ | $(r,\beta)_{k_2}$ | $\cdots$ | $(r,\beta)_{k_{n_1}}$ |

Hough Transform