CS 33

Data Representation (Part 2)

Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2nd Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.

Why Should I Use Unsigned?

- *Don't* use just because number nonnegative
  - easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- *Do* use when using bits to represent sets
  - logical right shift, no sign extension

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Note that "sizeof" returns an unsigned value. (Recall that, when mixing signed and unsigned items in an expression, the result will be unsigned.)

Word Size

- (Mostly) obsolete term
  - old computers had items of one size: the word size
- Now used to express the number of bits necessary to hold an address
  - 16 bits (really old computers)
  - 32 bits (old computers)
  - 64 bits (most current computers)

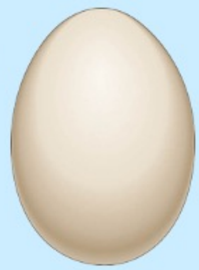
Byte Ordering

- Four-byte integer
  - 0x76543210
- Stored at location 0x100
  - which byte is at 0x100?
  - which byte is at 0x103?

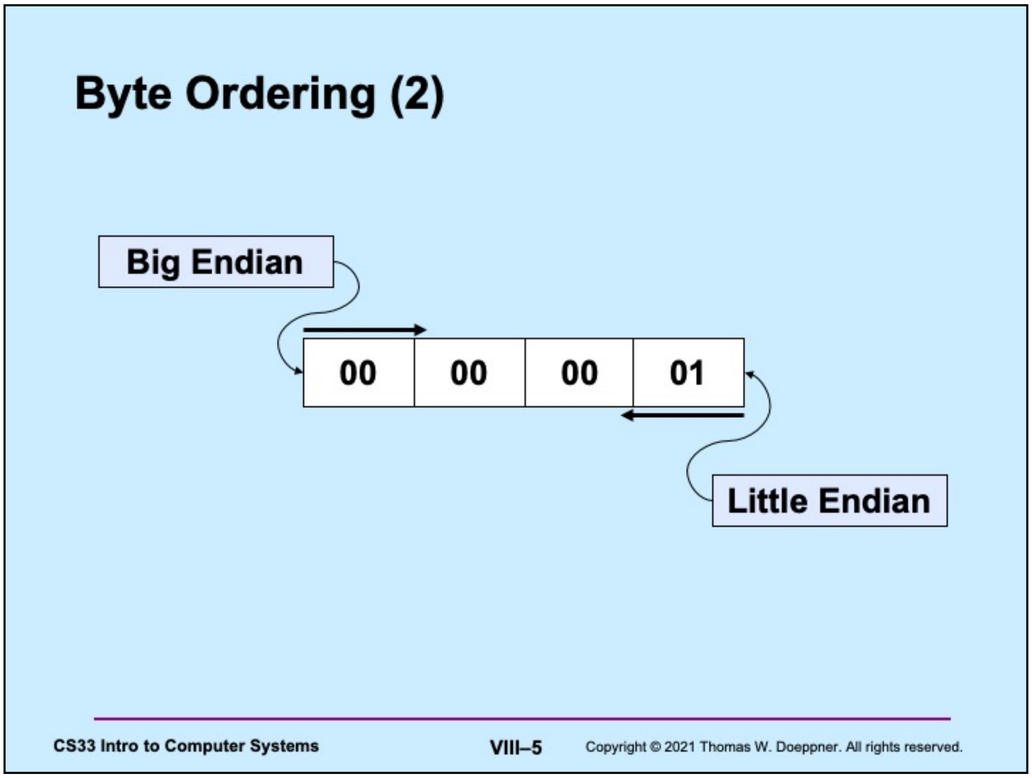
?

10	32	54	76	Little-endian
0x100	0x101	0x102	0x103	

76	54	32	10	Big-endian
0x100	0x101	0x102	0x103	



Read “Gulliver’s Travels” by Jonathan Swift for an explanation of the egg.



Here we have a four-byte integer one. In the big-endian representation, the address of the integer is the address of the byte containing its most-significant bits (the big end), while in the little-endian representation, the address of the integer is the address of the byte containing its least-significant bits (the little end). Suppose we pass a pointer to this integer to some procedure. However, in a type-mismatch, the procedure assumes that what is passed it is a two-byte integer. On a big-endian system, it would think it was passed a zero, but on a little-endian system, it would think it was passed a one.

This is not an argument in favor of either approach, but simply an observation that behaviors could be different.

Quiz 1

```
int main() {
  long x=1;
  func((int *)&x);
  return 0;
}

void func(int *arg) {
  printf("%d\n", *arg);
}
```

What value is printed on a big-endian 64-bit computer?

a) 1
b) 0
c) 2^{32}
d) 2^{32}-1

Which Byte Ordering Do We Use?

```
int main() {
    unsigned int x = 0x03020100;
    unsigned char *xarray = (unsigned char *)&x;
    for (int i=0; i<4; i++) {
            printf("%02x", xarray[i]);
    }
    printf("\n");
    return 0;
}
```

Possible results:

```
00010203
03020100
```

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This code prints out the value of x, one byte at a time, starting with the byte at the lowest address (little end). On x86-based computers, it will print:

00010203

which means that the address of an int is the address of the byte containing its least significant digits (little endian).

Fractional binary numbers

- What is 1011.101_2?

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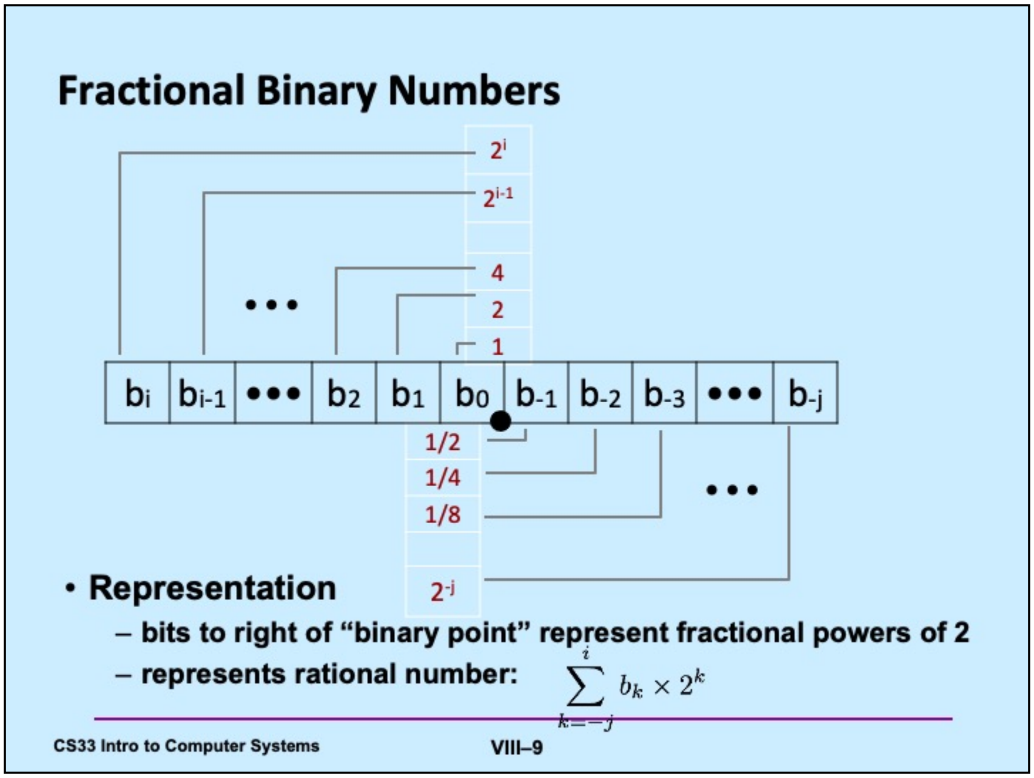

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Representable Numbers

- **Limitation #1**
  - can exactly represent only numbers of the form $n/2^k$
    - » other rational numbers have repeating bit representations
  - value — representation
    - » 1/3 `0.0101010101[01]…2`
    - » 1/5 `0.001100110011[0011]…2`
    - » 1/10 `0.0001100110011[0011]…2`
- **Limitation #2**
  - just one setting of decimal point within the *w* bits
    - » limited range of numbers (very small values? very large?)

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IEEE Floating Point

- **IEEE Standard 754**
  - established in 1985 as uniform standard for floating point arithmetic
    - » before that, many idiosyncratic formats
  - supported on all major CPUs
- **Driven by numerical concerns**
  - nice standards for rounding, overflow, underflow
  - hard to make fast in hardware
    - » numerical analysts predominated over hardware designers in defining standard

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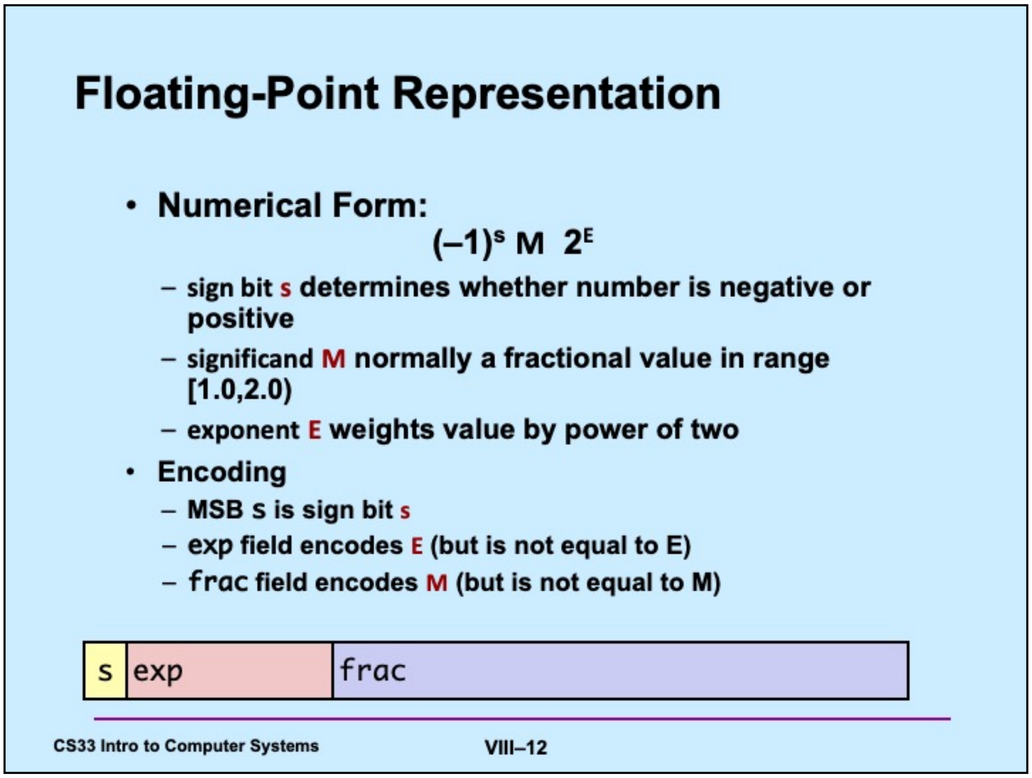
IEEE is the Institute for Electrical and Electronics Engineers (pronounced "eye triple e").



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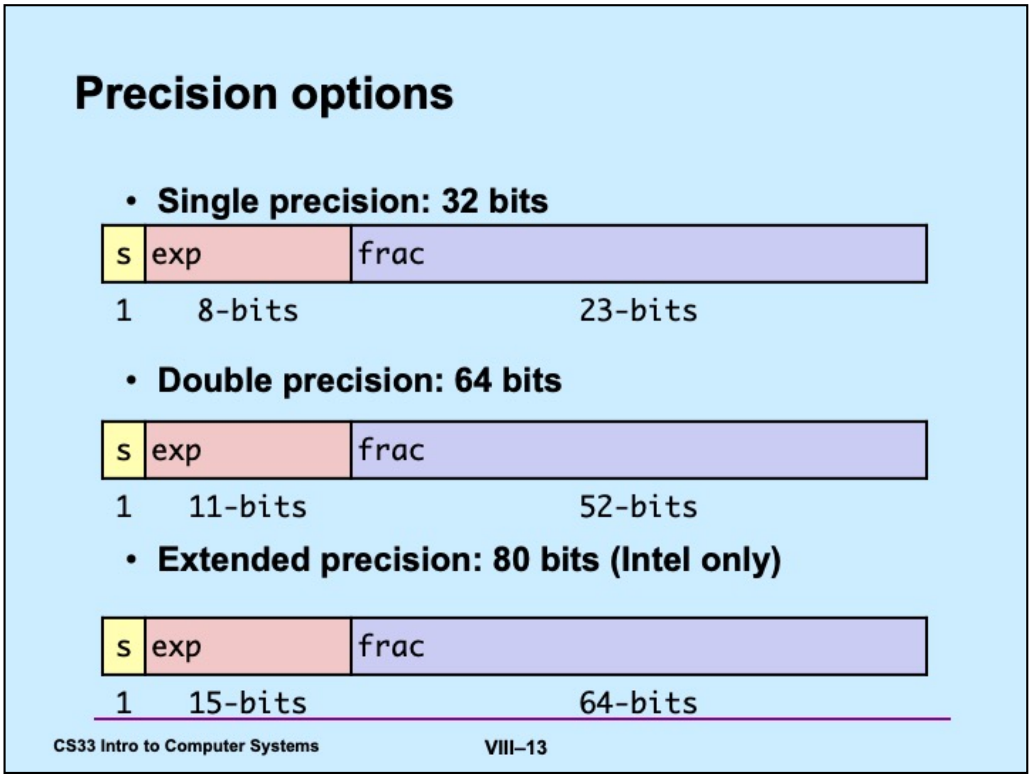

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On x86 hardware, all floating-point arithmetic is done with 80 bits, then reduced to either 32 or 64 as required.

“Normalized” Values

- When: exp ≠ 000…0 and exp ≠ 111…1
- Exponent coded as biased value: E = Exp – Bias
  - exp: unsigned value exp
  - bias = 2^{k-1} - 1, where k is number of exponent bits
    - » single precision: 127 (Exp: 1…254, E: -126…127)
    - » double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac
  - minimum when frac=000…0 (M = 1.0)
  - maximum when frac=111…1 (M = 2.0 – ε)
  - get extra leading bit for “free”

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Normalized Encoding Example

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**
  - M = $1.\underline{1101101101101}_2$
  - `frac` = $\underline{1101101101101}0000000000_2$

- **Exponent**
  - E = **13**
  - *bias* = **127**
  - *exp* = **140** = 10001100_2

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

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Denormalized Values

- Condition: exp = `000…0`
- Exponent value: E = –Bias + 1 (instead of E = 0 – Bias)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = `000…0`, `frac` = `000…0`
    - » represents zero value
    - » note distinct values: +0 and –0 (why?)
  - `exp` = `000…0`, `frac` ≠ `000…0`
    - » numbers closest to 0.0
    - » equispaced

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Special Values

- **Condition:** `exp` = 111…1
- **Case:** `exp` = 111…1, `frac` = 000…0
  - represents value ∞ (infinity)
  - operation that overflows
  - both positive and negative
  - e.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- **Case:** `exp` = 111…1, `frac` ≠ 000…0
  - not-a-number (NaN)
  - represents case when no numeric value can be determined
  - e.g., sqrt(–1), ∞ − ∞, ∞ × 0

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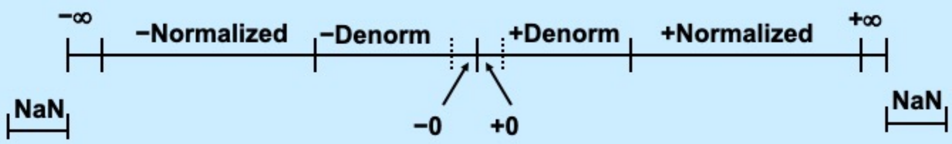
Visualization: Floating-Point Encodings



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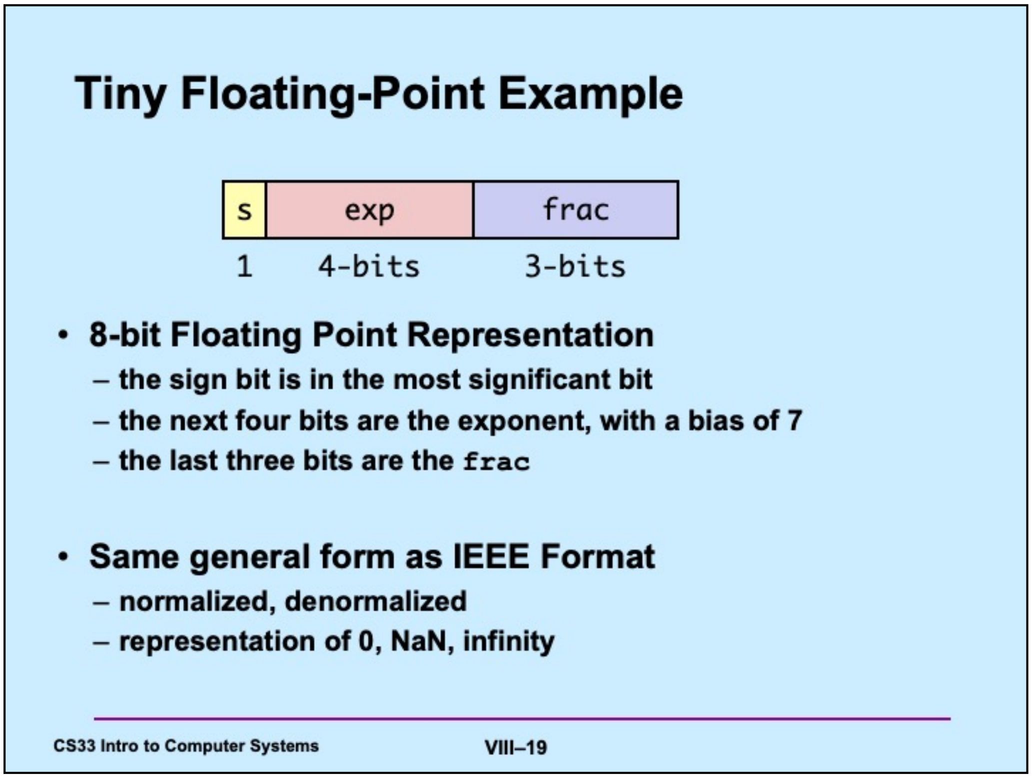

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Dynamic Range (Positive Only)

	s exp frac	E	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

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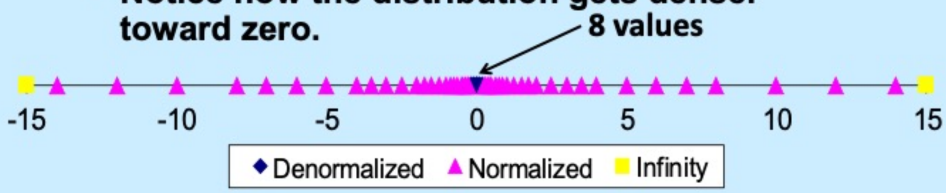
Distribution of Values

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is $2^{3-1}-1 = 3$**

s	exp	frac
1	3-bits	2-bits

- **Notice how the distribution gets denser toward zero.**



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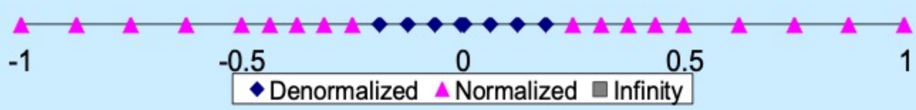
Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - **e = 3 exponent bits**
  - **f = 2 fraction bits**
  - **bias is 3**

s	exp	frac
1	3-bits	2-bits



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Quiz 1

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - bias is 3

s	exp	frac
1	3-bits	2-bits

What number is represented by 0 010 10?

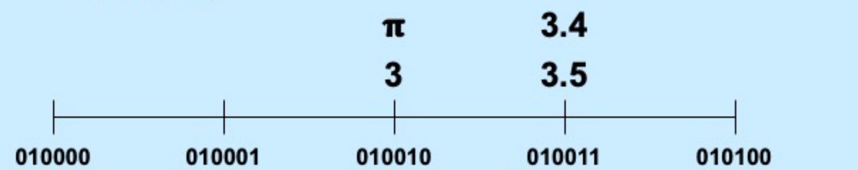
a) 3
b) 1.5
c) .75
d) none of the above

Mapping Real Numbers to Float

- The real number 3 is represented as
  0 100 10
- The real number 3.5 is represented as
  0 100 11
- How is the real number 3.4 represented?

  0 100 11
- How is the real number π represented?

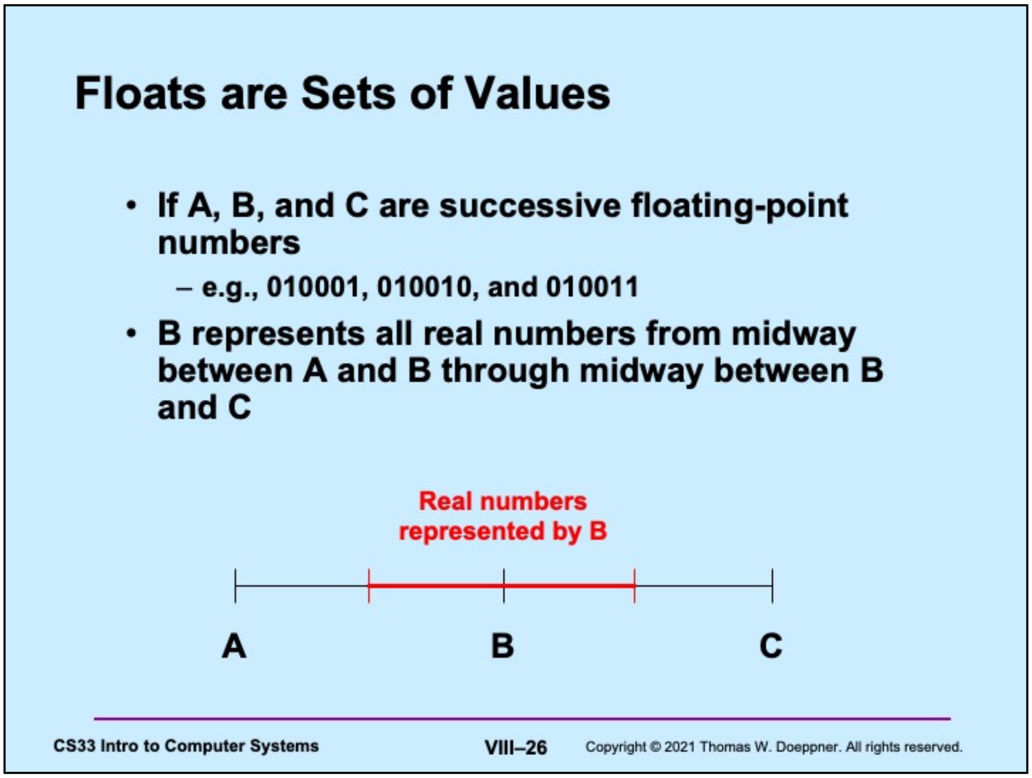
  0 100 10



We're assuming here the six-bit floating-point format.

Mapping Real Numbers to Float

- If R is a real number, it's mapped to the floating-point number whose value is closest to R
- What if it's midway between two values?
  - rounding rules coming up soon!



Note that we still have to discuss rounding so as to accommodate values that are equidistant from A and B or from B and C.

A special case is 0. Positive 0 represents a range of values that are greater than or equal to 0. Negative 0 represents a range of values that are less than or equal to zero.

Significance

- **Normalized numbers**
  - for a particular exponent value E and an S-bit significand, the range from 2^E up to 2^{E+1} is divided into 2^S equi-spaced floating-point values
    - » thus each floating-point value represents $1/2^S$ of the range of values with that exponent
    - » all bits of the signifcand are important
    - » we say that there are S significant bits – for reasonably large S, each floating-point value covers a rather small part of the range
      - high accuracy
      - for S=23 (32-bit float), accurate to one in 2^{23} (.0000119% accuracy)

Significance

- **Unnormalized numbers**
  - high-order zero bits of the significand aren't important
  - in 8-bit floating point, 0 0000 001 represents 2^{-9}
    - » it is the only value with that exponent: 1 significant bit (either 2^{-9} or 0)
  - 0 0000 010 represents 2^{-8}
    0 0000 011 represents $1.5*2^{-8}$
    - » only two values with exponent -8: 2 significant bits (distinguishing those two values, as well as 2^{-9} and 0)
  - fewer significant bits mean less accuracy
  - 0 0000 001 represents a range of values from $.5*2^{-9}$ to $1.5*2^{-9}$
  - 50% accuracy

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Recall that the bias for the exponent of 8-bit IEEE FP is 7, thus for unnormalized numbers the actual exponent is -6 (-bias+1). The significand has an implied leading 0, thus 0 0000 001 represents $2^{-6} * 2^{-3}$.

With 8-bit IEEE FP. the value 0 0000 001 is interpreted as 2^{-9}, But the number represented could be 50% less or 50% more. The value 0 0000 010 is interpreted as 2^{-8}. But the number could be 25% less or 25% more.

Floating-Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x ×f y = Round(x × y)`
- Basic idea
  - first compute exact result
  - make it fit into desired precision
    - » possibly overflow if exponent too large
    - » possibly round to fit into `frac`

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Rounding

- Rounding modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
towards zero	$1	$1	$1	$2	–$1
round down (−∞)	$1	$1	$1	$2	–$2
round up (+∞)	$2	$2	$2	$3	–$1
nearest integer	$1	$2	?	?	?
nearest even (default)	$1	$2	$2	$2	–$2

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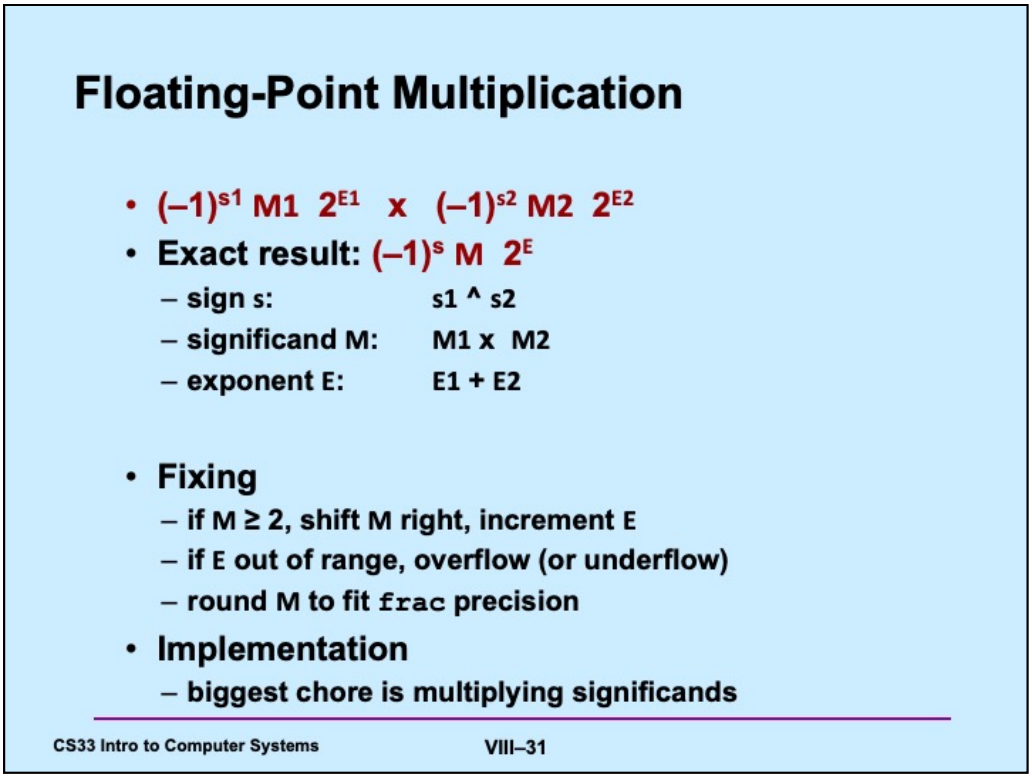
IEEE floating point uses the **nearest even** approach to rounding: if a value to be encoded is exactly half-way between two floating-point values, it is rounded to whichever value's rightmost bit is zero).



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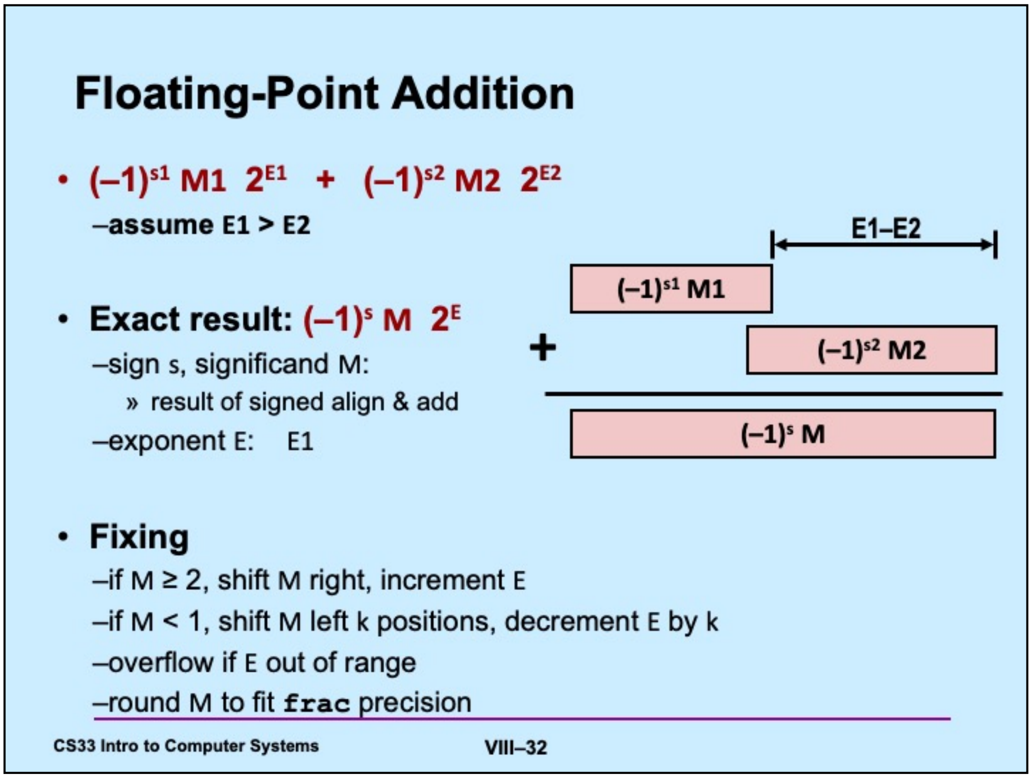
Note that to compute E, one must first convert exp_1 and exp_2 to E_1 and E_2, then add them them together and check for underflow or overflow (corresponding to -∞ and +∞), and then convert to exp.



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Note that, by default, overflow results in either +∞ or -∞.