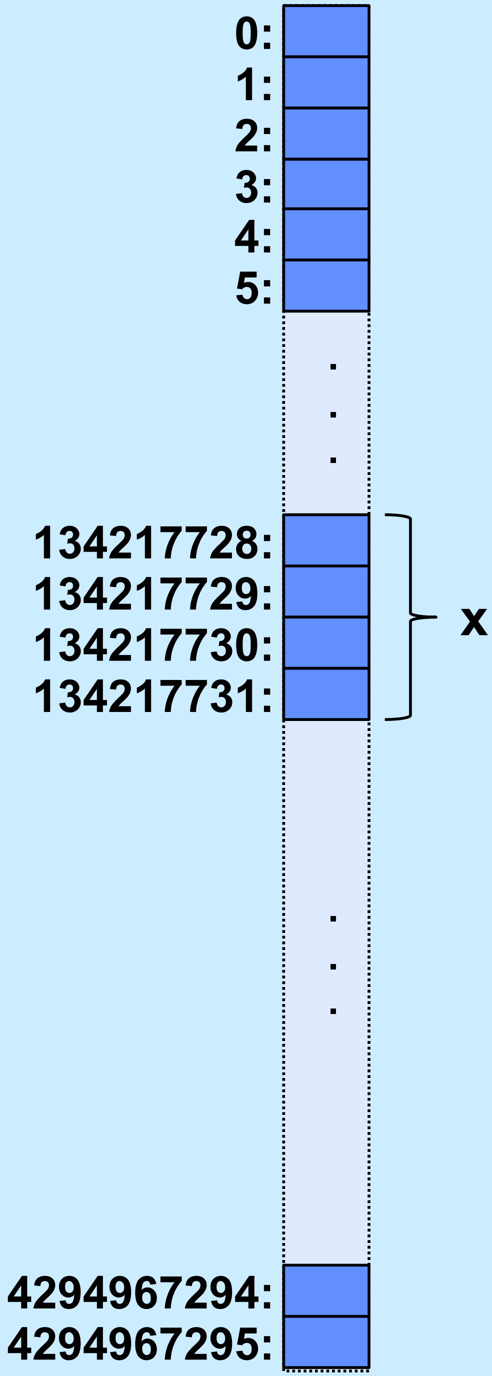
CS 33

Data Representation, Part 1

Representing Data in Memory

- **x is a 4-byte integer**
  - how do the 32 bits represent its value?



Unsigned Integers

b_{w-1}	b_{w-2}	b_{w-3}	…	b_2	b_1	b_0

$$\text{value} = \sum_{i=0}^{w-1} b_i \cdot 2^i$$

Signed Integers

- **Sign-magnitude**

$$\underbrace{\boxed{b_{w-1}}}_{\text{sign}}\underbrace{\boxed{b_{w-2}}\boxed{b_{w-3}}\boxed{\quad\ldots\quad}\boxed{b_2}\boxed{b_1}\boxed{b_0}}_{\text{magnitude}}$$

$$\text{value} = (-1)^{b_{w-1}} \cdot \sum_{i=0}^{w-2} b_i \cdot 2^i$$

- two representations of zero!
  - computer must have two sets of instructions
    - one for signed arithmetic, one for unsigned

Signed Integers

- Ones' complement
  - negate a number by forming its bit-wise complement
    - » e.g., (-1)·01101011 = 10010100

$b_{w-1} = 0 \Rightarrow$ non-negative number

value = $$\sum_{i=0}^{w-2} b_i \cdot 2^i$$

$b_{w-1} = 1 \Rightarrow$ negative number

value = $$\sum_{i=0}^{w-2} (b_i\text{-}1) \cdot 2^i$$

two zeros!

Signed Integers

- **Two's complement**

  $b_{w-1} = 0 \Rightarrow$ non-negative number

  $$\text{value} = \sum_{i=0}^{w-2} b_i \cdot 2^i$$

  $b_{w-1} = 1 \Rightarrow$ negative number

  $$\text{value} = (-1) \cdot 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

one zero!

Example

- **w = 4**

```
0000: 0        1000: -8
0001: 1        1001: -7
0010: 2        1010: -6
0011: 3        1011: -5
0100: 4        1100: -4
0101: 5        1101: -3
0110: 6        1110: -2
0111: 7        1111: -1
```

Signed Integers

- **Negating two's complement**

$$value = -b_{w-1}2^{w-1} + \sum_{i=0}^{w-2} b_i 2^i$$

  – **how to compute –*value*?**
    (~value)+1

Signed Integers

- Negating two's complement (continued)

$$\textit{value} + (\sim\textit{value} + 1)$$

$$= (\textit{value} + \sim\textit{value}) + 1$$

$$= (2^w - 1) + 1$$

$$= 2^w$$

$$= \boxed{1}\overbrace{\boxed{0}\boxed{0}\boxed{0}\boxed{\quad\ldots\quad}\boxed{0}\boxed{0}\boxed{0}}^{w}$$

Quiz 1

- **We have a computer with 4-bit words that uses two's complement to represent negative numbers. What is the result of subtracting 0010 (2) from 0001 (1)?**
  a) 1111
  b) 1110
  c) 1001
  d) 0111

Signed vs. Unsigned in C

- **char, short, int, and long**
  - signed integer types
  - right shift (>>) is arithmetic
- **unsigned char, unsigned short, unsigned int, unsigned long**
  - unsigned integer types
  - right shift (>>) is logical

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* **=** $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* **=** $2^{w-1} - 1$
    011…1
- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
`UMax`	**65535**	`FF FF`	`11111111 11111111`
`TMax`	**32767**	`7F FF`	`01111111 11111111`
`TMin`	**-32768**	`80 00`	`10000000 00000000`
`-1`	**-1**	`FF FF`	`11111111 11111111`
`0`	**0**	`00 00`	`00000000 00000000`

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**

  $|TMin| = TMax + 1$

  » Asymmetric range

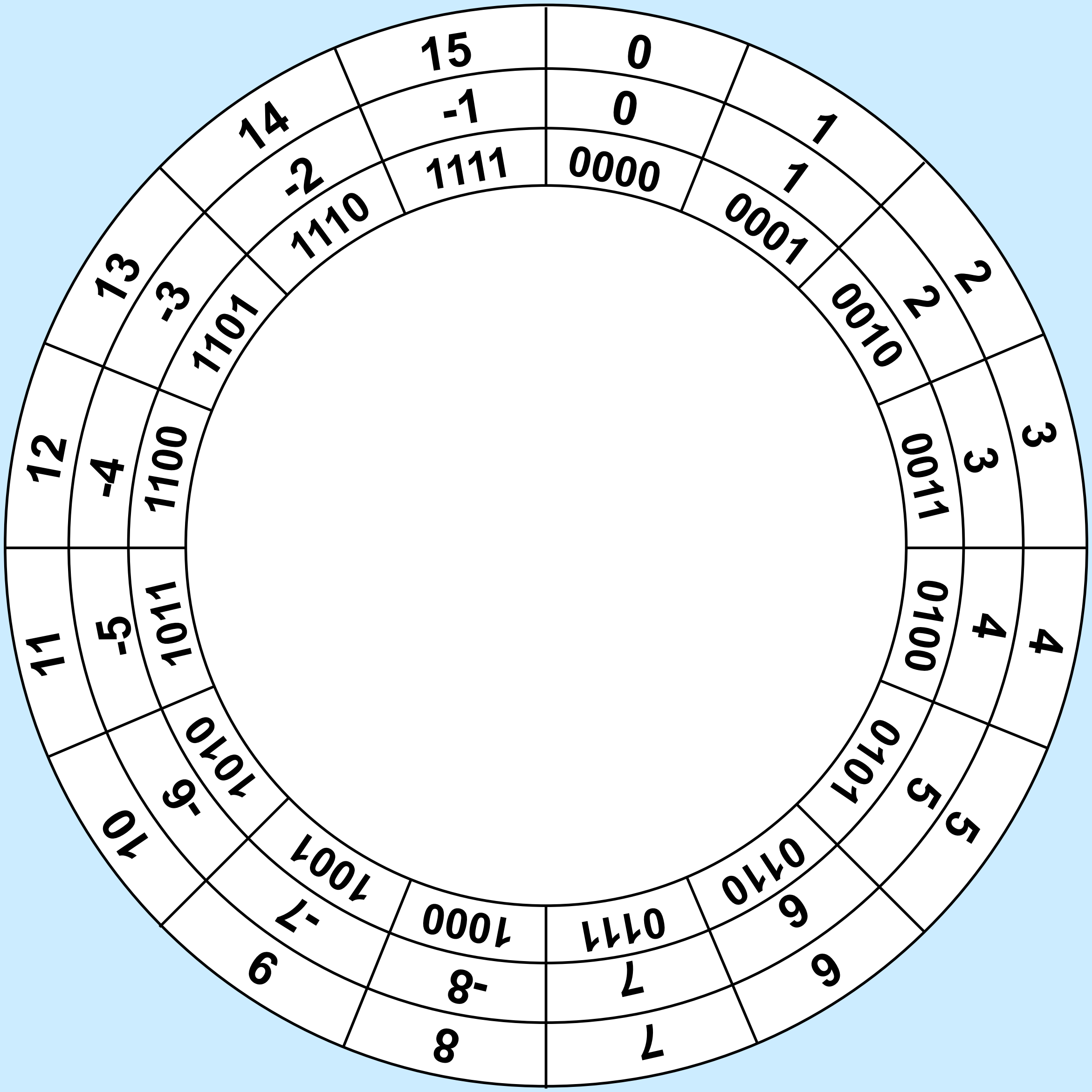
  $UMax = 2 * TMax + 1$

- **C Programming**
  - **#include** <limits.h>
  - declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - values platform-specific

Quiz 2

- **What is –TMin (assuming two's complement signed integers)?**
  a) 0
  b) 1
  c) TMin
  d) TMax

4-Bit Computer Arithmetic



Signed vs. Unsigned in C

- **Constants**
  - by default are considered to be signed integers
  - unsigned if have "U" as suffix

    `0U, 4294967259U`

- **Casting**
  - explicit casting between signed & unsigned

```
int tx, ty;
unsigned ux, uy; // "unsigned" means "unsigned int"
tx = (int) ux;
uy = (unsigned int) ty;
```

  - implicit casting also occurs via assignments and function calls

```
tx = ux;
uy = ty;
```

Casting Surprises

- Expression evaluation
  - if there is a mix of unsigned and signed in single expression, *signed values implicitly cast to unsigned*
  - including comparison operations <, >, ==, <=, >=
  - examples for *W* = 32: TMIN = -2,147,483,648 , TMAX = 2,147,483,647

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int)2147483648U	>	signed

Quiz 3

What is the value of

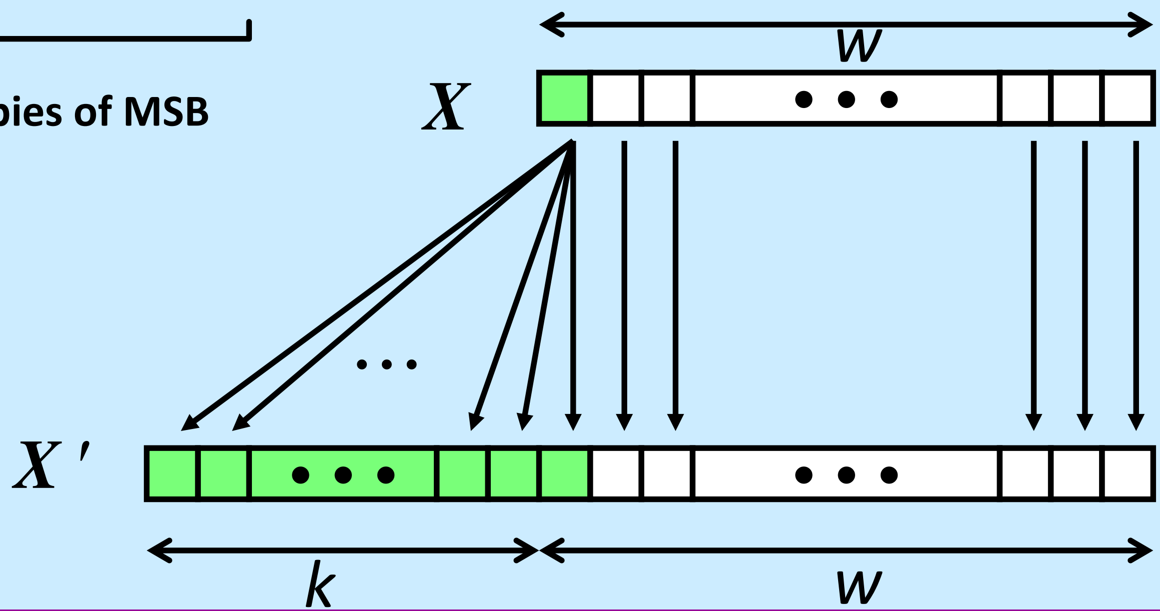
```
(unsigned long)-1 - (long)ULONG_MAX
```

???

a) 0
b) -1
c) 1
d) ULONG_MAX

Sign Extension

- **Task:**
  - given *w*-bit signed integer *x*
  - convert it to *w*+*k*-bit integer with same value
- **Rule:**
  - make *k* copies of sign bit:
  - $X' = \underbrace{x_{w-1},\ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2},\ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
  - C automatically performs sign extension

Does it Work?

$$val_w = -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$\begin{aligned} val_{w+1} &= -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i \\ &= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i \end{aligned}$$

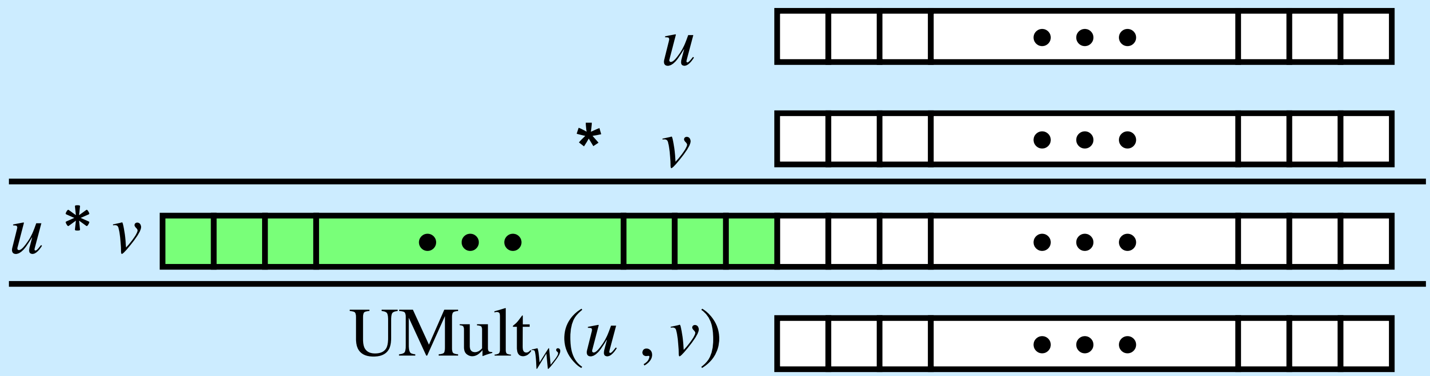
$$\begin{aligned} val_{w+2} &= -2^{w+1} + 2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i \\ &= -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i \\ &= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i \end{aligned}$$

Unsigned Multiplication

Operands: w bits

True Product: $2*w$ bits

Discard w bits: w bits



- **Standard multiplication function**
  - **ignores high order *w* bits**
- **Implements modular arithmetic**

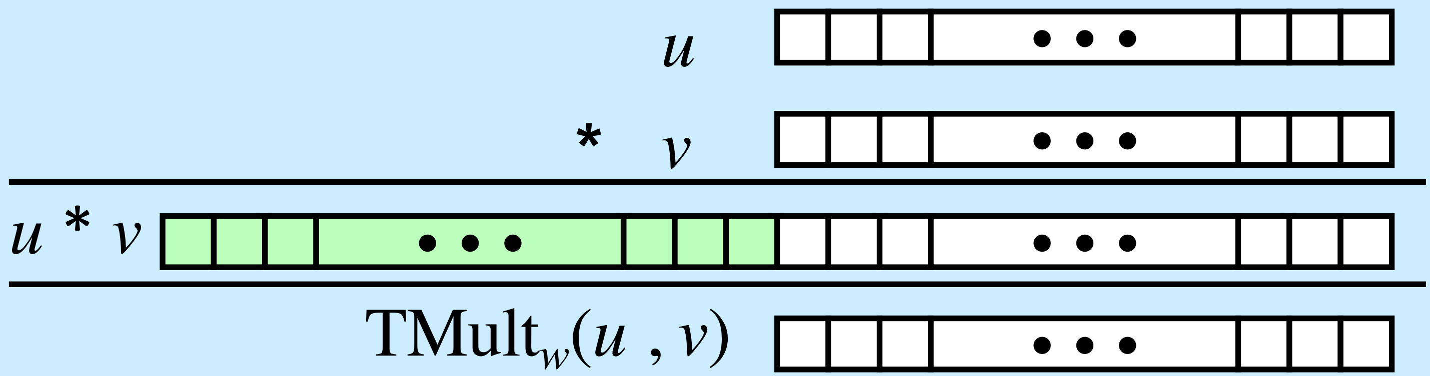
$\mathrm{UMult}_w(u, v) \quad = \quad u \cdot v \bmod 2^w$

Signed Multiplication

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard multiplication function**
  - **ignores high order *w* bits**
  - **some of which are different from those of unsigned multiplication**
  - **lower bits are the same**

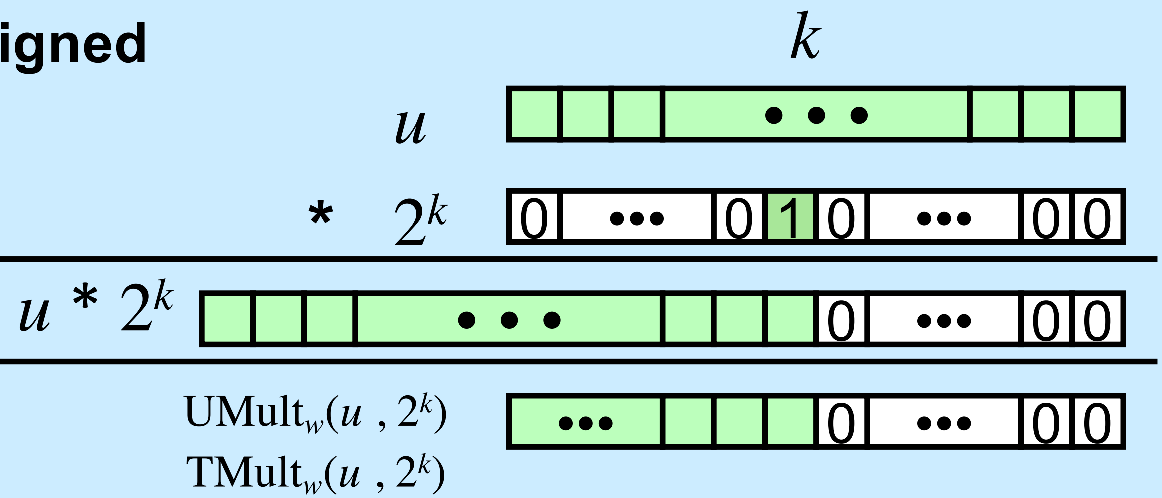
Power-of-2 Multiply with Shift

- **Operation**
  - **`u << k` gives `u * 2`^{k}**
  - **both signed and unsigned**

operands: *w* bits

true product: *w*+*k* bits

discard *k* bits: *w* bits



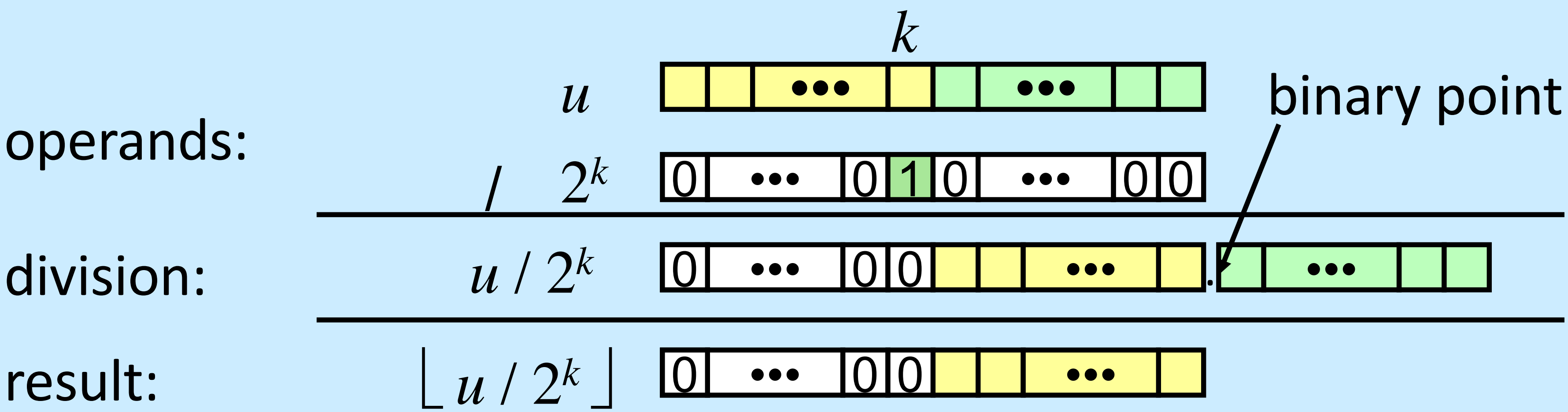
- **Examples**

```
u << 3 ==     u * 8
u << 5 - u << 3 == u * 24
```

  - **most machines shift and add faster than multiply**
    - » **compiler generates this code automatically**

Unsigned Power-of-2 Divide with Shift

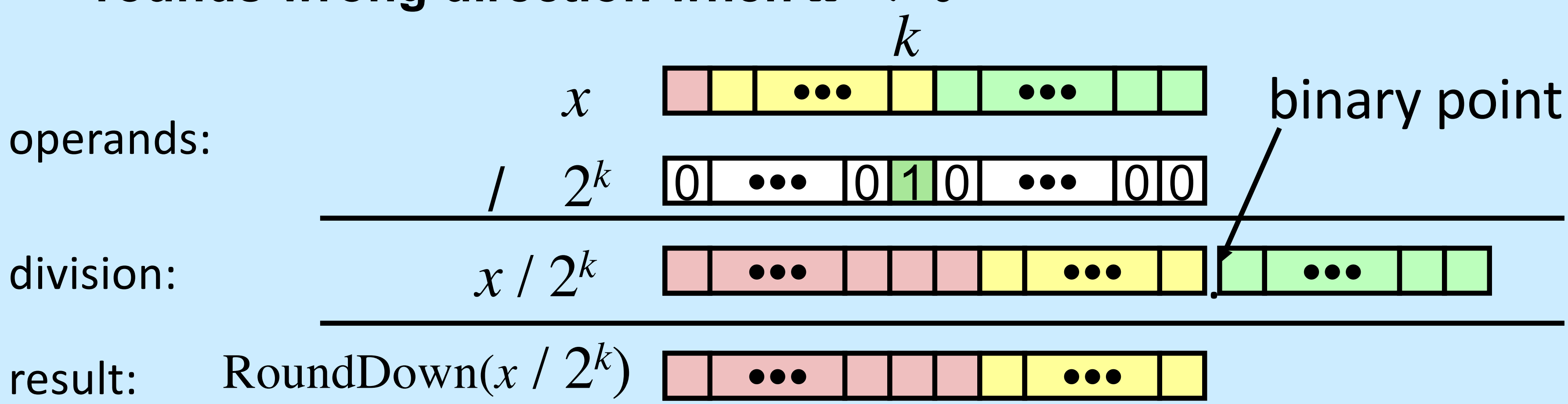
- **Quotient of unsigned by power of 2**
  - `u >> k` gives $\lfloor$ `u / ` $2^k \rfloor$
  - uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Signed Power-of-2 Divide with Shift

- **Quotient of signed by power of 2**
  - `x >> k` gives $\lfloor$ `x / ` $2^k \rfloor$
  - uses arithmetic shift
  - rounds wrong direction when `x < 0`

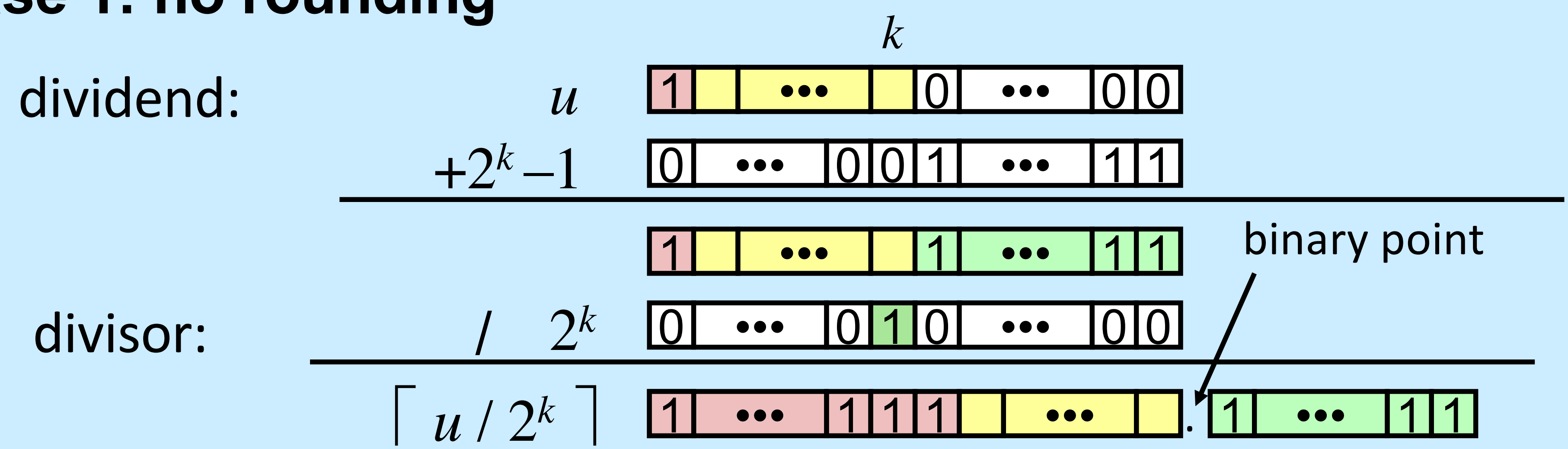


	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

Correct Power-of-2 Divide

- **Quotient of negative number by power of 2**
  - want $\lceil \mathtt{x} \;/\; 2^k \rceil$ (round toward 0)
  - compute as $\lfloor (\mathtt{x}+2^k-\mathtt{1}) / \; 2^k \rfloor$
    - in C: `(x + (1<<k)-1) >> k`
    - biases dividend toward 0

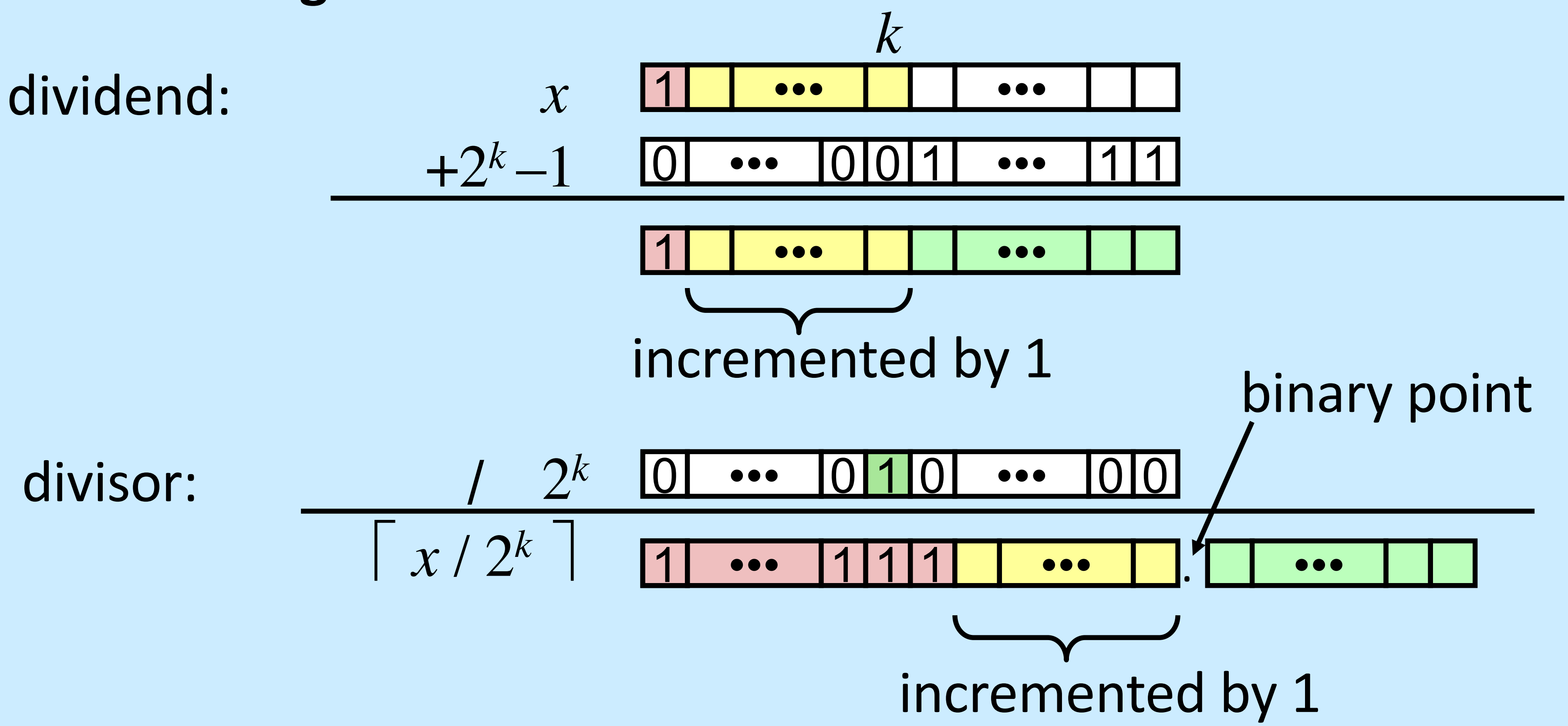
Case 1: no rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: rounding



Biasing adds 1 to final result

Why Should I Use Unsigned?

- *Don't* use just because number nonnegative
  - easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- *Do* use when using bits to represent sets
  - logical right shift, no sign extension