Stat 110 Homework 1, Fall 2023

Due: Friday 9/15 at 5:00 pm, submitted as a PDF via Gradescope. Some information on how to use Gradescope can be found at this link. *Make sure to assign to each question all the pages with your work on that question.* (You can assign multiple pages to the same question, or the same page to multiple questions.) After submitting your homework, *check your submission* to make sure everything you want graded is present, easily legible, and correctly assigned. No submissions on paper or by email will be accepted, and no extensions will be granted aside from the Monday extensions described in the syllabus.

Your work can be typeset, written using a tablet, or scanned from handwritten work on paper, but must be clear, easily legible, and correctly rotated. If you would like to typeset your work, I recommend using LaTeX. There are some LaTeX resources on the course webpage under Supplementary Links, and a nice online interface for LaTeX is Overleaf. If you would like to handwrite and scan your work using a phone, I recommend the free app Adobe Scan.

Please show your work and give clear, careful, convincing justifications (using *words and sentences* to explain your logic, not just formulas). When a numerical answer is a messy expression, please give both an exact answer and a numerical approximation, unless otherwise specified. For example, if the answer is

$$\frac{\binom{13}{2}\cdot\binom{4}{2}^2\cdot 44}{\binom{52}{5}} \approx 0.0475,$$

then please give *both* the left-hand side (an exact answer in terms of binomial coefficients) and the right-hand side (a numerical approximation). For calculations, you may find the free tools WolframAlpha and RStudio useful.

1. Alvin and Bella are each dealt 7 cards from a standard, well-shuffled deck of cards.

(a) Find the probability that Alvin has 3 pairs in his hand (where a pair means 2 cards of the same rank) but not 3 or more cards of the same rank. For example, a favorable outcome would be if Alvin has a pair of 5's, a pair of 8's, a pair of Queens, and a 10.

(b) Find the probability that Alvin has 3 pairs in his hand but not 3 or more cards of the same rank, Bella also has 3 pairs in her hand but not 3 or more cards of the same rank, and that the ranks of Alvin's pairs are exactly the same as the ranks of Bella's pairs.

2. There are n kids and n toys. Instead of deterministically giving one toy to each kid, the allocation is completely randomized: each toy is given to a random kid (with all kids equally likely), independently.

(a) Find the probability that every kid receives a toy.

(b) Find the probability that exactly one kid does not receive any toys.

3. Give a story proof that, for all nonnegative integers k and n with $k < n$, we have

$$\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k} = 2^{n-k}\binom{n}{k}.$$

4. The Cambridge House of Waffles (CHOW) sells delicious waffles.[1] They have three types of waffle: Brussels, Liege, and Pandan. There are ten toppings available: apple, caramel, elderberry syrup, fudge, maple syrup, oatmeal cookie crumbs, strawberry, toffee, ube jam, and yogurt. A customer can choose any set of toppings for their waffle.

As an abbreviation, the staff use the first letters of the type and toppings to denote a waffle. For example, a Brussels waffle with apple, yogurt, elderberry syrup, and strawberry is abbreviated as BAYES. (The first letter is always B, L, or P to indicate the waffle type, and then the toppings can be listed in any order, e.g., BAYES is the same as BESAY.)

(a) Without doing any calculations, show in the context of CHOW that

$$\sum_{j=0}^{10} \binom{10}{j} = 2^{10}.$$

(b) Three friends go to CHOW and sit at a table together. They will each get one waffle. There is a touchscreen at the table, through which they jointly submit the order for the table. The touchscreen doesn't have a way to indicate which waffle is for which friend. The chef then receives an alphabetized list of the waffles for the table. For example, if the three friends want a Pandan waffle with maple syrup and fudge, a Liege waffle with oatmeal cookie crumbs, toffee, ube jam, and strawberry, and a Brussels waffle with apple, yogurt, elderberry syrup, and strawberry, then the chef receives the list (BAYES, LOTUS, PMF).

How many possibilities are there for the list that the chef receives?

(c) If the three friends each choose completely randomly what to get, are all the possibilities from (b) equally likely? If so, explain why. If not, give an explicit example of a possibility that is as unlikely as possible and an explicit example of a possibility that is as likely as possible.

5. The password requirements for accounts on a certain website are as follows:

- the password must be exactly 6 characters;
- each character can be a lowercase letter (a through z), uppercase letter (A through Z), or numeric (0 through 9);
- there must be at least one lowercase letter, at least one uppercase letter, and at least one numeric character.

A 6-character string is generated completely randomly (with each character being a lowercase letter, uppercase letter, or numeric, and with all strings of length 6 equally likely). Find the probability that the string is allowable as a password.

6. Matilda loves reading books. She goes to the library and finds that there are 3 new books in each of her 6 favorite genres. She chooses 9 of these books to borrow, completely randomly (so all sets of 9 out of the 18 books are equally likely). Find the probability that there are books from all 6 genres among the books she borrows.

7. Show that for any events A and B,

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A \cup B) \leq P(A) + P(B).$$

[1] CHOW is fictional. Sadly, Zinneken's declined Joe's proposal to give him a lifetime supply of free waffles in exchange for making Zinneken's the subject of this problem.