Deterministic Perturbations For Simultaneous Perturbation Methods Using Circulant Matrices.

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***Abstract*— We consider the problem of finding optimal parameters under simulation optimization setup. For a p-dimensional parameter vector optimization, the classical Kiefer-Wolfowitz Finite Difference Stochastic Approximation (FDSA) scheme uses $p+1$ or $2p$ simulations of the system feedback for one-sided and symmetric difference gradient estimates respectively. The dependence on the dimension p makes FDSA impractical for high dimensional problems. An alternative approach for gradient estimation in high dimensional problems is the simultaneous pertubation technique that appears in [1],[2]. The main idea in this approach is to estimate the gradient by using only two settings of the p-dimensional parameter vector being optimized. The two settings of the parameter vector are obtained by simulataneously perturbing all the components of the parameter vector by adding a random direction. In this article we consider deterministic pertubation vectors to improve the convergence of these algorithms. A novel construction of deterministic pertubations based on specially chosen circulant matrix is proposed. Convergence analysis of the proposed algorithms is presented along with numerical experiments.**

I. INTRODUCTION

Simulation optimization problems frequently arise in engineering disciplines like transportation systems, machine learning, service systems, manufacturing etc. Practical limitations, lack of model information and the large dimensionality of these problems prohibit analytic solution of these problems and simulation is often employed to evaluate a system in these disciplines. Hence simulation budget becomes critical and one aims to converge to optimal parameters using as few simulations as possible.

To state the problem formally, consider a system where noise-corrupted feedback of the system is available i.e., given the system parameter vector θ the feedback that is available is $h(\theta, \xi)$ where ξ is noise term inherent in the system, pictorially shown in Figure 1 . The objective in these problems is to find a parameter vector that gives the optimal expected performance of the system. Suppose $J(\theta) = \mathbb{E}[h(\theta, \xi)]$ then $h(\theta, \xi) = J(\theta) + M(\theta, \xi)$ where $M(\theta, \xi) = h(\theta, \xi) - \mathbb{E}[h(\theta, \xi)]$ is a mean zero term. The following optimization problem needs to be solved.

$$\text{Find } \theta^* = \arg\min_{\theta \in \mathbb{R}^p} J(\theta). \tag{1}$$

Analogous to optimization problems with explicit analytic gradient of the objective function, a solution approach could be to devise an algorithm that mimics the familiar gradient descent algorithm. However in our setting only noise corrupted samples of the objective are available. So one essentially needs to estimate the gradient of the objective function using simulation samples.

This adaptive system optimization framework has its roots in the work by Robbins and Monro [3] and Kiefer and Wolfowitz [4]. The work by Kiefer and Wolfowitz uses stochastic approximation framework by Robbins and Monro to optimize an objective using noisy samples. In their work Kiefer and Wolfowitz [4] use one-sided or symmetric difference approximation of the gradient. This method has the drawback of using $p+1$ simulations for one-sided approximation and $2p$ simulations for symmetric difference approximation of the gradient for a problem of dimension p. Later works [2],[1] have replaced the gradient approximation using finite differences by the popular simulataneous perturbation technique.

Simultaneous perturbation is a useful technique to estimate the gradient from the function samples, especially in high dimensional problems - see [5], [6] for a comprehensive treatment of this subject matter. The first-order Simultaneous Perturbation Stochastic Approximation algorithm for simulation optimization, henceforth referred to as 1SPSA-2R, was proposed in [1]. 1SPSA-2R uses 2 simulations per iteration and random perturbations to perturb the parameter vector. A one measurement variant of [1] is proposed in [7] which we refer here as 1SPSA-1R. A closely related algorithm is the first-order Random Directions Stochastic Approximation henceforth referred to as 1RDSA-2R that appears in [2, pp. 58-60]. The algorithm 1RDSA-2R differs from 1SPSA-2R, both in the construction as well as convergence analysis and performs poorly compared to 1SPSA-2R -see [8] for detailed comparative study.

To enhance the performance of 1SPSA-2R and 1SPSA-1R , Shalabh et al., in [9] propose deterministic pertubations based on Hadamard matrices. We refer the variants of 1SPSA-2R and 1SPSA-1R that use Hadamard pertubations as 1SPSA-2H and 1SPSA-1H. Analogous to their work we propose deterministic perturbations for 1RDSA-2R and its one simulation variant (refered as 1RDSA-1R here) based on specially chosen circulant matrix.

II. MOTIVATION

Here we motivate the necessary conditions that deterministic perturbations should satisfy. Analogous properties for Hadamard matrix based pertubations are well motivated in [5]. Now both 1SPSA-2R and 1RDSA-2R iteratively update

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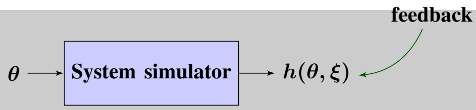

Fig. 1: Simulation Optimization Model

the parameter vector and a typical update step is of the following form.

$$\theta(n+1) = \theta(n) - a(n)\widehat{\nabla} f(\theta(n)), \tag{2}$$

where $a(n)$ is the step-size that satisfies standard stochastic approximation conditions (see (A2) in Section IV), $\widehat{\nabla} f(\theta)$ is an estimate of the gradient of the objective function f. Thus, (2) can be considered as the stochastic version of the well-known gradient descent method for optimization.

Now consider the Taylor series expansion of $J(\theta(n) + \delta d(n))$ around $\theta(n)$, we have

$$J(\theta(n)+\delta d(n)) = J(\theta(n)) + \delta d(n)^T \nabla J(\theta(n)) + o(\delta) \tag{3}$$

Similarly, an expansion of $J(\theta(n)-\delta d(n))$ around $\theta(n)$ gives

$$J(\theta(n)-\delta d(n)) = J(\theta(n)) - \delta d(n)^T \nabla J(\theta(n)) + o(\delta) \tag{4}$$

From (3) and (4), we have

$$\frac{J(\theta(n)+\delta d(n)) - J(\theta(n)-\delta d(n))}{2\delta} d(n) - \nabla J(\theta(n)) \tag{5}$$

$$= (d(n)d(n)^T - I)\nabla J(\theta(n)) + o(\delta) \tag{6}$$

Note that $(d(n)d(n)^T - I)\nabla J(\theta(n))$ constitutes the bias in the gradient estimate and from the assumption A3

$$\mathbb{E}\left[(d(n)d(n)^T - I)\nabla J(\theta(n)) \middle| \theta(n) \right] = 0. \tag{7}$$

In the case of a one-simulation algorithm with parameter $\theta(n) + \delta d(n)$, a similar Taylor series expansion gives

$$\frac{J(\theta(n)+\delta d(n))}{\delta} d(n) - \nabla J(\theta(n)) \tag{8}$$

$$= \frac{J(\theta(n))}{\delta} d(n) + (d(n)d(n)^T - I)\nabla J(\theta(n)) + O(\delta) \tag{9}$$

One expects the following to hold in addition to (7) in the case of random perturbations for one simulation algorithms

$$\mathbb{E}\left[\frac{J(\theta(n))}{\delta} d(n) \middle| \theta(n) \right] = 0 \tag{10}$$

Notice that (7) and (10) are achieved asymptotically. Clearly one is motivated to look for pertubations that satisfy similar properties in finite time. So one expects a deterministic sequence of perturbations to satisfy the following properties.

P1. Let $D(n) := d(n)d(n)^T - I$. There exists a $P \in \mathbb{N}$ such that $\sum_{n=s}^{s+P} D(n) = 0$.

Algorithm 1 Basic structure of 1RDSA-2C and 1RDSA-1C algorithms.

Input:
- $\theta_0 \in \mathbb{R}^p$, initial parameter vector;
- $\delta(n)$, sequence to approximate gradient;
- Matrix of perturbations

$$Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u];$$

- noisy measurement of cost objective J;
- $a(n)$, step-size sequence satisfying assumption (**A2**.)

for $n = 1, 2, \ldots N$ **do**

Let $d(n)$ be mod$(n, p+1)^{\text{th}}$ column of Q.

Update the parameter as follows:

$$\theta(n+1) = \theta(n) - a(n)(\widehat{\nabla} f(\theta(n)) \tag{11}$$

where $\widehat{\nabla} f(\theta)$ is chosen according to (12) or (13).

end for

Return $\theta(n)$.

P2. $\sum_{n=s}^{s+P} d(n) = 0$.

The rest of the paper is organized as follows: In Section III, we describe the various ingredients in algorithm 1. In Section IV, we present the convergence results for 1RDSA-2C and 1RDSA-1C algorithms. In Section V, we present the results from numerical experiments and finally, in Section VI, provide the concluding remarks.

III. Structure of the Algorithm

Algorithm 1 presents the basic structure of 1RDSA-2C and 1RDSA-1C. We describe the individual components of 1 below.

A. Deterministic Pertubations

Let $\delta(n), n \geq 0$ denote a sequence of diminishing positive real numbers and $d(n) = (d(n)^1, \ldots, d(n)^p)^{\mathsf{T}}$ denote mod $(n, p+1)^{\text{th}}$ column of Q. where Q is constructed as follows. Let

$$H = \begin{bmatrix} 2 & 1 & 1 \cdots 1 \\ 1 & 2 & 1 \cdots 1 \\ \vdots & \vdots & \vdots \; \vdots \; \vdots \\ 1 & 1 & 1 \cdots 2 \end{bmatrix}.$$

Observe that $H = I + uu^T$ is a positive definite matrix. Hence $H^{-1/2}$ is well defined. Define $p \times (p+1)$ matrix Q as $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. The columns of Q are used as perturbations $d(n)$.

B. Gradient estimation

Let $y(n)^+$, $y(n)^-$ be defined as below. $y(n)^+ = J(\theta(n) + \delta(n)d(n)) + \xi(n)^+$ and $y(n)^- = J(\theta(n) - \delta(n)d(n)) + \xi(n)^-$, where the noise terms $\xi(n)^+, \xi(n)^-$ satisfy $\mathbb{E}\left[\xi(n)^+ - \xi(n)^- \,|\, \mathcal{F}(n)\right] = 0$ with $\mathcal{F}(n) =$

$\sigma(\theta(m), m \leq n)$ denoting the underlying sigma-field. The estimate of the gradient $\nabla J(\theta(n))$ is given by

$$\widehat{\nabla} f(\theta(n)) = \left[\frac{(y(n)^+ - y(n)^-)d(n)}{2\delta(n)}\right] \quad \text{for 1RDSA-2C} \tag{12}$$

$$\widehat{\nabla} f(\theta(n)) = \left[\frac{(y(n)^+)d(n)}{\delta(n)}\right] \quad \text{for 1RDSA-1C} \tag{13}$$

Observe that while 1RDSA-2C algorithm uses two function samples $y(n)^+$ and $y(n)^-$ at $\theta(n) + \delta(n)d(n)$ and $\theta(n) - \delta(n)d(n)$, 1RDSA-1C algorithm uses one function sample $y(n)^+$ at $\theta(n) + \delta(n)d(n)$.

IV. CONVERGENCE ANALYSIS

We make the same assumptions as those used in the analysis of [1], with a few minor alterations. The assumptions are listed below.

A1. The map $J : \mathbb{R}^p \rightarrow \mathbb{R}$ is Lipschitz continuous and is differentiable with bounded second order derivatives. Further, the map $L : \mathbb{R}^p \rightarrow \mathbb{R}$ defined as $L(\theta) = -\nabla J(\theta)$ is Lipschitz continuous.

A2. The step-size sequences $a(n), \delta(n) > 0, \forall n$ satisfy

$$a(n), \delta(n) \text{ as } n \rightarrow \infty, \sum_n a(n) = \infty, \sum_n \left(\frac{a(n)}{\delta(n)}\right)^2 < \infty.$$

Further, $\frac{a(j)}{a(n)} \rightarrow 1$ as $n \rightarrow \infty$, for all $j \in \{n, n+1, \cdots, n+M\}$ for any given $M > 0$ and $b(n) = \frac{a(n)}{\delta(n)}$ is such that $\frac{b(j)}{b(n)} \rightarrow 1$ as $n \rightarrow \infty$, for all $j \in \{n, n+1, \cdots, n+M\}$

A3. The random vectors $d(n)$, $n \geq 0$ are mutually independent and identitcally distributed. Moreover $\mathbb{E}[d(n)d(n)^T] = I$

A4. The iterates $\theta(n)$ remain uniformly bounded almost surely, i.e,

$$\sup_n \|\theta(n)\| < \infty, a.s.$$

A5. The set H containing the globally asymptotically stable equilibria of the ODE $\dot{\theta}(t) = -\nabla J(\theta(t))$ (i.e., the local minima of J) is a compact subset of $\mathbb{R}^N$.

The following lemma is useful in obtaining the negative square root of H i.e., $H^{-1/2}$. Also note that it takes only $O(p)$ operations to compute $H^{-1/2}$ using the lemma.

Lemma 1. *Let I be a $p \times p$ identity matrix and $u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ be a $p \times 1$ all ones vector then*

$$(I + uu^T)^{-1/2} = I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}.$$

Proof. It is enough to show that

$$(I + uu^T)\left[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}\right]^2 = I.$$

Using $\|u\|^2 = u^T u = p$ we have

$$\begin{aligned}
&(I + uu^T)\left[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}\right]^2 \\
&= (I + uu^T)\Bigg[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}} \\
&- \frac{uu^T}{p} + \frac{uu^T}{p} - \frac{uu^T}{p\sqrt{(1+p)}} \\
&+ \frac{uu^T}{p\sqrt{(1+p)}} - \frac{uu^T}{p\sqrt{(1+p)}} + \frac{uu^T}{p(1+p)}\Bigg] \\
&= (I + uu^T)\left[I - uu^T(\frac{1}{p} - \frac{1}{p(1+p)})\right] \\
&= (I + uu^T)\left[I - \frac{uu^T}{1+p}\right] \\
&= I + uu^T - \frac{uu^T}{1+p} - \frac{puu^T}{1+p} \\
&= I
\end{aligned}$$

proving the lemma. □

As defined earlier let

$$H = \begin{bmatrix} 2 & 1 & 1 \cdots 1 \\ 1 & 2 & 1 \cdots 1 \\ \vdots & \vdots & \vdots \vdots \\ 1 & 1 & 1 \cdots 2 \end{bmatrix}$$

and $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. Let the pertubations $d(n)$ be the columns of Q.

Lemma 2. *The perturbations $d(n)$ choosen as columns of Q satisfy properties P1 and P2.*

Proof. Let $P = p + 1$. Observe that as n goes through one cycle from 1 to $p+1$ we have $\sum_{n=1}^{p+1} d(n)d(n)^T = QQ^T$ and $\sum_{n=1}^{p+1} d(n) = Q\begin{bmatrix} u \\ 1 \end{bmatrix}$. Now it is enough to show that $\sum_{n=s}^{s+P} D_n = 0$ and $\sum_{n=s}^{s+P} d(n) = 0$ i.e $Q\begin{bmatrix} u \\ 1 \end{bmatrix}$=0.

$$\begin{aligned}
&\sum_{n=s}^{s+P} D_n \\
&= \sum_{n=s}^{s+P} (d(n)d(n)^T - I) \\
&= \sum_{n=s}^{s+P} (d(n)d(n)^T - (p+1)I \\
&= QQ^T - (p+1)I \\
&= (p+1)([H^{-1/2}, -H^{-1/2}u][H^{-1/2}, -H^{-1/2}u]^T - I) \\
&= (p+1)([H^{-1} + H^{-1/2}uu^T H^{-1/2}] - I) \\
&= (p+1)(H^{-1/2}(I + uu^T)H^{-1/2} - I) \\
&= 0.
\end{aligned}$$

In addition

$$
\begin{aligned}
&Q\begin{bmatrix} u \\ 1 \end{bmatrix} \\
&= \sqrt{p+1}([H^{-1/2}, -H^{-1/2}u])\begin{bmatrix} u \\ 1 \end{bmatrix} \\
&= \sqrt{p+1}(H^{-1/2}u - H^{-1/2}u) = 0
\end{aligned}
$$

proving the lemma. □

Lemma 3. *Given any fixed integer $P > 0$, $\|\theta(m+k) - \theta(m)\| \to 0$ $w.p.1$, as $m \to \infty$, for all $k \in 1, \cdots, P$*

Proof. Fix a $k \in \{1, \cdots, P\}$. Now

$$
\begin{aligned}
&\theta(n+k) = \theta(n) - \\
&\sum_{j=n}^{j=n+k-1} a(j)\Big(\frac{J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))}{2\delta(j)}\Big)d(j) \\
&- \sum_{j=n}^{j=n+k-1} a(j)M(j+1)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\theta(n+k) - \theta(n)\| \leq \\
&\sum_{j=n}^{j=n+k-1} a(j)\Big|\Big(\frac{J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))}{2\delta(j)}\Big)\Big|d(j) \\
&- \sum_{j=n}^{n+k-1} a(j)M(j+1)
\end{aligned}
$$

Now clearly,

$$
N(n) = \sum_{j=0}^{n-1} a(j)M(j+1), n \geq 1,
$$

forms a martingale sequence. Further, from the *****assumptions***** we have,

$$
\begin{aligned}
&\sum_{m=0}^{n} \mathbb{E}[\|N(m+1) - N(m)\|^2 | \mathcal{F}(m)] \\
&= \sum_{m=0}^{n} \mathbb{E}[a(m)^2 \|M(m+1)\|^2 | \mathcal{F}(m)] \\
&\leq \sum_{m=0}^{n} a(m)^2 K(1 + \|\theta(n)\|^2)
\end{aligned}
$$

From the assumptions the quadratic variation process of $N(n), n \geq 0$ converges almost surely. Hence by the martingale convergence theorem, it follows that $N(n), n \geq 0$ converges almost surely. Hence $\|\sum_{j=n}^{n+k-1} a(j)M(j+1)\| \to 0$ almost surely as $n \to \infty$. Moreover

$$
\begin{aligned}
&\Big\|\Big(J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))\Big)d(j)\Big\| \\
&\leq \Big|\Big(J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))\Big)\Big|\|d(j)\| \\
&\leq K\Big(|J(\theta(j)+\delta d(j))| + |J(\theta(j)-\delta d(j))|\Big),
\end{aligned}
$$

since $\|d(j)\| \leq K, \forall j \geq 0$. Note that

$$
\begin{aligned}
|J(\theta(j)+\delta d(j))| - |J(0)| &\geq |J(\theta(j)+\delta d(j)) - J(0)| \\
&\geq \hat{B}\|\theta(j)+\delta d(j)\|,
\end{aligned}
$$

where $\hat{B}$ is the Lipschitz constant of the function $J(.)$. Hence,

$$
|J(\theta(j)+\delta d(j))| \leq \tilde{B}(1 + \|\theta(j)+\delta d(j)\|),
$$

for $\tilde{B}$ =max$(|J(0)|, \hat{B})$. Similarly,

$$
|J(\theta(j)-\delta d(j))| \leq \tilde{B}(1 + \|\theta(j)-\delta d(j)\|).
$$

From assumptions it follows that

$$
\sup_j \Big\|\Big(J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))\Big)d(j)\Big\| \leq \tilde{K} < \infty,
$$

for some $\tilde{K} > 0$. Thus,

$$
\begin{aligned}
&\|\theta(n+k) - \theta(n)\| \\
&\leq \tilde{K}\sum_{j=n}^{n+k-1}\frac{a(j)}{\delta(j)} + \|\sum_{j=n}^{n+k-1} a(j)M(j+1)\| \\
&\to 0 \text{ a.s. with } n \to \infty.
\end{aligned}
$$

The claim follows. □

Lemma 4. *For any $m \geq 0$,*

$$
\Big\|\sum_{n=m}^{m+P-1}\frac{a(n)}{a(m)}D(n)\nabla J(\theta(n))\Big\| \to 0 \textit{ and}
$$

$$
\Big\|\sum_{n=m}^{m+P-1}\frac{b(n)}{b(m)}d(n)J(\theta(n))\Big\| \to 0,
$$

almost surely, as $m \to \infty$.

Proof. From lemma 3, it can be seen that $\|\theta(m+s) - \theta(m)\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Also from assumption A1, we have $\|\nabla J(\theta(m+s)) - \nabla J(\theta(m))\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Now from lemma 2

$\sum_{n=m}^{m+P-1} D(n) = 0 \ \forall m \geq 0$. Thus,

$$\Big\|\sum_{n=m}^{m+P-1} \frac{a(n)}{a(m)} D(n)\nabla J(\theta(n))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} \frac{a(n)}{a(m)} D(n)\nabla J(\theta(n)) + D(m)\nabla J(\theta(m))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} \frac{a(n)}{a(m)} D(n)\nabla J(\theta(n)) - \sum_{n=m+1}^{m+P-1} D(n)\nabla J(\theta(m))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} D(n)\Big(\frac{a(n)}{a(m)}\nabla J(\theta(n)) - \nabla J(\theta(m))\Big)\Big\|$$
$$\leq \bar{K}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\frac{a(n)}{a(m)}\nabla J(\theta(n)) - \nabla J(\theta(m))\Big)\Big\|$$
$$= \bar{K}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\Big(\frac{a(n)}{a(m)} - 1\Big)\nabla J(\theta(n))$$
$$+ \Big(\nabla J(\theta(n)) - \nabla J(\theta(m))\Big)\Big)\Big\|$$
$$\leq \bar{K}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\Big(\frac{a(n)}{a(m)} - 1\Big)\nabla J(\theta(n))\Big)\Big\|$$
$$+ \sum_{n=m+1}^{m+P-1} \Big\|\Big(\nabla J(\theta(n)) - \nabla J(\theta(m))\Big)\Big\|$$

The claim now follows as
a consequence of the assumptions.***polish
Now observe that $\|J(\theta(m+k)) - J(\theta(m))\| \to 0$ as $m \to \infty$, for all $k \in \{1, \cdots, P\}$.

$$\Big\|\sum_{n=m}^{m+P-1} \frac{b(n)}{b(m)} d(n)J(\theta(n))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} \frac{b(n)}{b(m)} d(n)J(\theta(n)) + d(m)J(\theta(m))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} \frac{b(n)}{b(m)} d(n)J(\theta(n)) - \sum_{n=m+1}^{m+P-1} d(n)J(\theta(m))\Big\|$$
$$= \Big\|\sum_{n=m+1}^{m+P-1} d(n)\Big(\frac{b(n)}{b(m)}J(\theta(n)) - J(\theta(m))\Big)\Big\|$$
$$\leq \sum_{n=m+1}^{m+P-1} \Big\|d(n)\Big(\frac{b(n)}{b(m)}J(\theta(n)) - J(\theta(m))\Big)\Big\|$$
$$\leq \bar{\bar{K}}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\frac{b(n)}{b(m)}J(\theta(n)) - J(\theta(m))\Big)\Big\|$$
$$= \bar{\bar{K}}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\Big(\frac{b(n)}{b(m)} - 1\Big)J(\theta(n)) + \Big(J(\theta(n)) - J(\theta(m))\Big)\Big)\Big\|$$
$$\leq \bar{\bar{K}}\sum_{n=m+1}^{m+P-1} \Big\|\Big(\Big(\frac{b(n)}{b(m)} - 1\Big)J(\theta(n))\Big)\Big\|$$
$$+ \sum_{n=m+1}^{m+P-1} \Big\|\Big(J(\theta(n)) - J(\theta(m))\Big)\Big\|$$

The claim now follows as a consequence of the assumptions
*************polish ************* □

Theorem 5. *$\theta(n), n \geq 0$ obtained from 1RDSA-2C satisfy $\theta(n) \to H$ almost surely.*

Proof. Note that

$$\theta(n+P) = \theta(n) - \sum_{l=n}^{l=n+P-1} a(l)\Big(\frac{J(\theta(j)+\delta d(j)) - J(\theta(j)-\delta d(j))}{2\delta(l)}d(j) + M(l+1)\Big)$$

It follows that

$$\theta(n+P) = \theta(n) - \sum_{l=n}^{n+P-1} a(l)\nabla J(\theta(l)) - \sum_{l=n}^{n+P-1} a(l)(d(l)d(l)^T - I)\nabla J(\theta(l)) - \sum_{l=n}^{n+P-1} a(l)o(\delta(l)) - \sum_{l=n}^{n+P-1} a(l)M(l+1)$$

Now the third term on the RHS can be written as

$$a(n)\sum_{l=n}^{l=n+P-1} \frac{a(l)}{a(n)} D(l)\nabla J(\theta(l)) = a(n)\xi(n),$$

where $\xi(n) = o(1)$ from the lemmas.
******* to be polished ********
Thus, the algorithm is asymptotically analogous to

$$\theta(n+1) = \theta(n) - a(n)(\nabla J(\theta(n)) + o(\delta) + M(n+1)).$$

Also from the convergence of the martingale sequence $N(n)$, it follows that $\sum_{l=n}^{\infty} \to 0$ as $n \to \infty$, almost surely.
******* to be polished ******** □

Theorem 6. *$\theta(n), n \geq 0$ obtained from 1RDSA-1C satisfy $\theta(n) \to H$ almost surely.*

Proof. Note that

$$\theta(n+P) = \theta(n) - \sum_{l=n}^{n+P-1} a(l)\Big(\frac{J(\theta(j)+\delta d(j))}{2\delta(l)}\Big)d(j) - \sum_{l=n}^{n+P-1} a(l)M(l+1)$$

It follows that

$$\theta(n+P) = \theta(n) - \sum_{l=n}^{n+P-1} a(l)\nabla J(\theta(l))$$
$$- \sum_{l=n}^{n+P-1} a(l)\frac{J(\theta(l))}{\delta(l)} d(l)$$
$$- \sum_{l=n}^{n+P-1} a(l)(d(l)d(l)^T - I)\nabla J(\theta(l))$$
$$- \sum_{l=n}^{n+P-1} a(l)O(\delta(l)) - \sum_{l=n}^{l=n+P-1} a(l)M(l+1)$$

Now we observe that

$$\sum_{l=n}^{n+P-1} a(l)\frac{J(\theta(l))}{\delta(l)} d(l) = \sum_{l=n}^{n+P-1} b(l)J(\theta(l))d(l)$$
$$= b(n)\sum_{l=n}^{n+P-1} \frac{b(l)}{b(n)}\frac{J(\theta(l))}{\delta(l)} d(l) = b(n)\xi^1(n),$$

where $\xi^1(n) = o(1)$ by lemmas. Similarly

$$\sum_{l=n}^{n+P-1} a(l)(d(l)d(l)^T - I)\nabla J(\theta(l)) = a(n)\xi^2(n),$$

with $\xi^2(n) = o(1)$. The rest follows as in Theorem 5. $\square$

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V. SIMULATION EXPERIMENTS

A. Implementation

We compare the performance of 1SPSA-2R, 1SPSA-2H and 1RDSA-2C in the case of 2 simulation algorithms. In the case of 1 simulation algorithms we compare the performance 1SPSA-1R, 1SPSA-1H and 1RDSA-1C . We chose i.i.d Bernoulli ± 1-valued perturbations for 1SPSA-2R and 1SPSA-1R. [1].

For the empirical evaluations, we use the following two loss functions in $N = 10$ dimensions:

a) Quadratic loss:

$$f(x) = x^{\mathsf{T}} A x + b^{\mathsf{T}} x. \tag{14}$$

For this particular choice of p the optimum x^* for the above f is a 10×1 column vector with each entry equal to -0.9091, and $f(x^*) = -4.55$.

b) Fourth-order loss:

$$f(x) = x^{\mathsf{T}} A^{\mathsf{T}} A x + 0.1\sum_{j=1}^{N}(Ax)_j^3 + 0.01\sum_{j=1}^{N}(Ax)_j^4. \tag{15}$$

The optimum x^* for above f is $x^* = 0$, with $f(x^*) = 0$.

In both functions, A is such that pA is an upper triangular matrix with each nonzero entry equal to one, b is the N-dimensional vector of ones and the noise structure is similar to that used in [10]. For any x, the noise is $[x^{\mathsf{T}}, 1]z$, where $z \approx \mathcal{N}(0, \sigma^2 I_{11\times 11})$. We perform experiments for noisy as well as noise-less settings, with $\sigma = 0.01$ for the noisy case. For all algorithms, we set $\delta_n = 3.8/n^{0.101}$ and $a_n = 1/(n+A+1)^{0.602}$. These choices have been used before (see [10]) and have demonstrated good finite-sample performance empirically, while satisfying the theoretical requirements needed for asymptotic convergence. For all the algorithms, the initial point x_0 is the p-dimensional vector of ones.

[1]The implementation is available at `https://github.com/****/1CPSA-M/archive/master.zip`.

TABLE I: NMSE values of 2 simulation methods for quadratic objective (14) with and without noise for 2000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA	$5.762 \times 10^{-3} \pm 2.473 \times 10^{-3}$
1SPSA-Hadamard Pertubations	$4.012 \times 10^{-5} \pm 1.654 \times 10^{-5}$
1RDSA-D	$2.188 \times 10^{-5} \pm 9.908 \times 10^{-6}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA	$5.755 \times 10^{-3} \pm 2.460 \times 10^{-3}$
1SPSA-Hadamard Perturbations	$1.601 \times 10^{-5} \pm 2.724 \times 10^{-20}$
1RDSA-D	$2.474 \times 10^{-8} \pm 1.995 \times 10^{-23}$

TABLE II: NMSE values of 2 simulation methods for fourth order objective (15) with and without noise for 10000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA	$2.762 \times 10^{-2} \pm 1.415 \times 10^{-2}$
1SPSA-Hadamard Pertubations	$3.958 \times 10^{-3} \pm 4.227 \times 10^{-4}$
1RDSA-D	$3.598 \times 10^{-3} \pm 4.158 \times 10^{-4}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA	$2.747 \times 10^{-2} \pm 1.413 \times 10^{-2}$
1SPSA-Hadamard Perturbations	$3.901 \times 10^{-3} \pm 4.359 \times 10^{-18}$
1RDSA-D	$3.535 \times 10^{-3} \pm 1.743 \times 10^{-18}$

B. Results

We use Normalized Mean Square Error (NMSE) as performance metric for evaluating the algorithms. NMSE is the ratio $\|\theta(n_{\text{end}}) - \theta^*\|^2 / \|\theta(0) - \theta^*\|^2$. Here n_{end} denotes the iteration number at the end of simulation budget.

Tables I–II present the normalized mean square values observed for the three algorithms - 1SPSA, 1SDSA-H and

TABLE III: NMSE values of 1 simulation methods for quadratic objective (14) with and without noise for 20000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA	$8.582 \times 10^{-2} \pm 3.691 \times 10^{-2}$
1SPSA-Hadamard Pertubations	$2.774 \times 10^{-2} \pm 2.578 \times 10^{-4}$
1RDSA-D	$8.225 \times 10^{-3} \pm 5.959 \times 10^{-5}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA	$8.584 \times 10^{-2} \pm 3.681 \times 10^{-2}$
1SPSA-Hadamard Perturbations	$2.770 \times 10^{-2} \pm 3.836 \times 10^{-17}$
1RDSA-D	$8.225 \times 10^{-3} \pm 1.569 \times 10^{-17}$

TABLE IV: NMSE values of 1 simulation methods for fourth order objective (15) with and without noise for 20000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA	$3.240 \times 10^{-1} \pm 1.836 \times 10^{-1}$
1SPSA-Hadamard Pertubations	$8.916 \times 10^{-2} \pm 1.896 \times 10^{-2}$
1RDSA-D	$4.972 \times 10^{-2} \pm 9.812 \times 10^{-3}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA	$3.192 \times 10^{-1} \pm 1.991 \times 10^{-1}$
1SPSA-Hadamard Perturbations	$8.173 \times 10^{-2} \pm 1.255 \times 10^{-16}$
1RDSA-D	$4.403 \times 10^{-2} \pm 9.066 \times 10^{-17}$

1RDSA-D for the fourth-order and quadratic loss functions, respectively for 2 simulation algorithms. Table IV–III presents the NMSE values obtained for the 1 simulation algorithms with the fourth order and quadratic loss. The results in Table I is after 2000 simulations and the results in Table II is after 10000 simulations.

The results in Table IV–III are obtained after running the 1 simulation algorithms with a budget of 20000 function evaluations.

Figure 2 and 3 plots the $\log_{10}$ (NMSE) as a function of the iterations with quadratic and fourth-order loss objectives respectively, with $\sigma = 0.01$

Figure 4 and 5 plots the $\log_{10}$ (NMSE) as a function of the iterations with quadratic and fourth-order loss objectives respectively, with $\sigma = 0.01$

From the results in Tables I,II III,IV and plots 2 3 4 5, we

Fig. 2: $\log_{10}$(NMSE)vs No of iterations for quadratic objective (14) with noise ($\sigma = 0.01$) using 2 simulation methods.

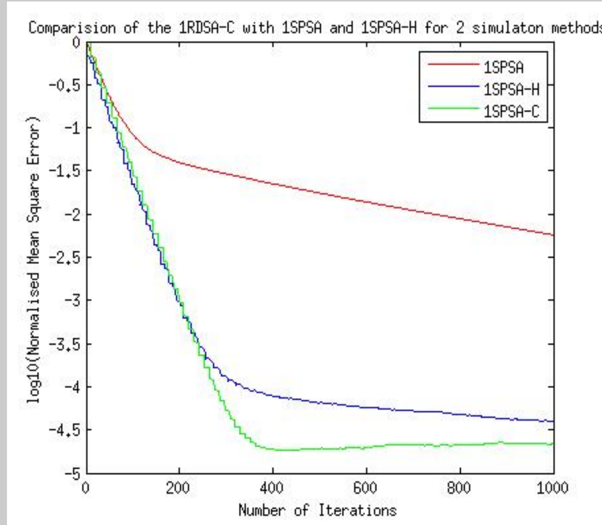
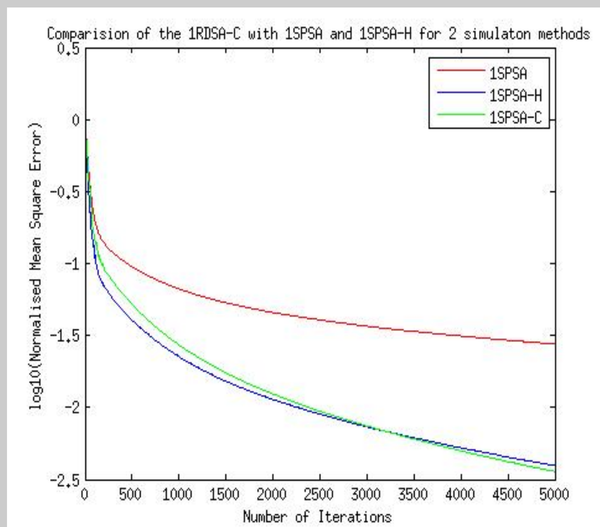


Fig. 3: $\log_{10}$(NMSE)vs No of iterations for fourth order objective objective (15) with noise ($\sigma = 0.01$) using 2 simulation methods.



make the following observations:

Observation In two simulation algorithms 1RDSA-D outperforms both 1SPSA-H and 1SPSA.

VI. CONCLUSIONS

We presented a novel construction of deterministic pertubations for 1RDSA-2R and 1RDSA-1R algorithms and showed that the resulting algorithms 1RDSA-2C and 1RDSA-1C are provably convergent. The advantage with deterministic pertubations is that it uses same pertubations for both two simulation and one simulation variants. It also has smaller cycle length. Numerical experiments demonstrated that 1RDSA-2C (1RDSA-1C) outperforms 1SPSA-2R (1SPSA-1R) and 1SPSA-2H (1SPSA-1H). As future work, it would be interesting to derive similar methods in

Fig. 4: $\log_{10}$(NMSE)vs No of iterations for quadratic order objective objective (14) with noise ($\sigma = 0.01$) using 1 simulation methods.

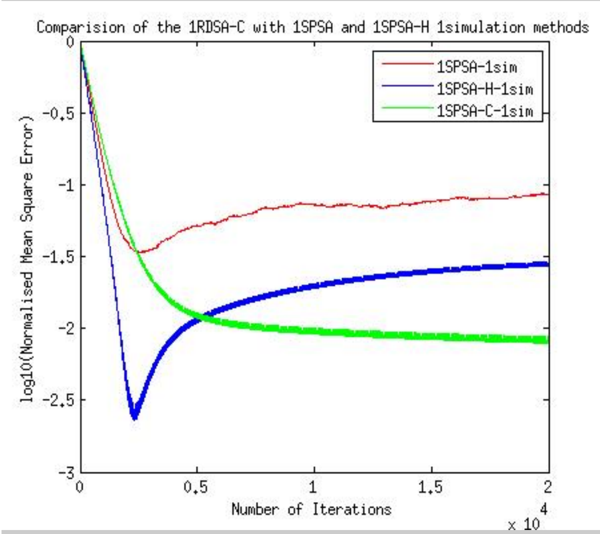
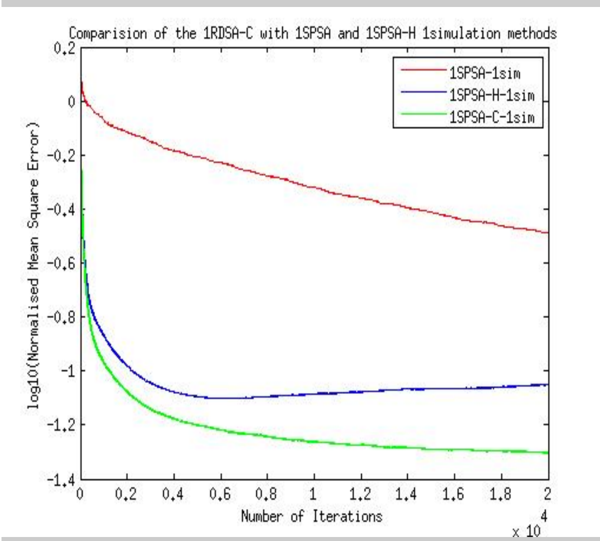


Fig. 5: $\log_{10}$(NMSE)vs No of iterations for fourth order objective objective (15) with noise ($\sigma = 0.01$) using 1 simulation methods.



the context of 2RDSA. A challenging future work could be to derive weak convergence results when deterministic pertubations are used.

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