Deterministic Perturbations For Simultaneous Perturbation Methods Using Circulant Matrices

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Abstract— **We consider the problem of finding optimal parameters under simulation optimization setup. For a p-dimensional parameter optimization, the classical Kiefer-Wolfowitz Finite Difference Stochastic Approximation (FDSA) scheme uses $p+1$ or $2p$ simulations of the system feedback for one-sided and two-sided gradient estimates respectively. The dependence on the dimension p makes FDSA impractical for high dimensional problems. An elegant approach for gradient estimation in high dimensional problems is the simultaneous perturbation technique. The main idea in this approach is to estimate the gradient by using only two settings of the p-dimensional parameter being optimized. The two settings of the parameter are obtained by simultaneously perturbing all the components of the parameter by adding a random direction. In this article we propose a novel construction of deterministic perturbation directions based on a specially chosen circulant matrix as an alternative to random directions. Also convergence analysis of the proposed optimization algorithms with constructed perturbation directions is presented along with numerical experiments.**

I. INTRODUCTION

Simulation optimization problems frequently arise in engineering disciplines like transportation systems, machine learning, service systems, manufacturing etc. Practical limitations, lack of model information and the large dimensionality of these problems prohibit analytic solution of these problems and simulation is often employed to evaluate a system in these disciplines. Hence simulation budget becomes critical and one aims to converge to optimal parameters using as few simulations as possible.

To state the problem formally, consider a system where noise-corrupted feedback of the system is available i.e., given the system parameter θ the feedback that is available is $h(\theta, \xi)$ where ξ is the noise term inherent in the system, pictorially shown in Figure 1. The objective in these problems is to find a parameter vector that gives the optimal expected performance of the system. Define $J(\theta) := \mathbb{E}_\xi[h(\theta, \xi)]$ and $N(\theta, \xi) := h(\theta, \xi) - \mathbb{E}_\xi[h(\theta, \xi)]$ then clearly $J(\theta)$ is the expected performance of the system and $N(\theta, \xi)$ is a mean zero additive random noise term and we have $h(\theta, \xi) = J(\theta) + N(\theta, \xi)$. Hence, one needs to solve the following optimization problem:

$$\text{Find } \theta^* = \arg\min_{\theta \in \mathbb{R}^p} J(\theta). \tag{1}$$

Analogous to deterministic optimization problems, where explicit analytic gradient of the objective function is available, a solution approach could be to devise an algorithm that mimics the familiar gradient descent algorithm. However in our setting only noise corrupted samples of the objective, $h(\theta, \xi) = J(\theta) + N(\theta, \xi)$, are available. So one essentially needs to estimate the gradient of the objective function using simulation samples.

This adaptive system optimization framework has its roots in the work by Robbins and Monro [1] and Kiefer and Wolfowitz [2]. The work by Kiefer and Wolfowitz uses stochastic approximation framework by Robbins and Monro to optimize an objective using noisy samples. In their work Kiefer and Wolfowitz [2] propose both one-sided and two-sided approximation of the gradient. This method has the drawback of using $p+1$ simulations for one-sided approximation and $2p$ simulations for two-sided approximation of the gradient for a p-dimensional problem. Later works [3],[4] have replaced the gradient approximation using finite differences with random perturbations.

Simultaneous perturbation is a useful technique to estimate the gradient from the function samples, especially in high dimensional problems - see [5], [6] for a comprehensive treatment of this subject matter. The first-order Simultaneous Perturbation Stochastic Approximation algorithm for simulation optimization, henceforth referred to as 1SPSA-2R, was proposed in [4]. 1SPSA-2R uses two simulations per iteration and random perturbations to perturb the parameter vector. A one measurement variant of [4] is proposed in [7] which we refer here as 1SPSA-1R. A closely related algorithm is the first-order Random Directions Stochastic Approximation henceforth referred to as 1RDSA-2R that appears in [3, pp. 58-60]. The algorithm 1RDSA-2R differs from 1SPSA-2R, both in the construction as well as convergence analysis and performs poorly compared to 1SPSA-2R, see [8] for a detailed comparative study. Recents works on second order Random Directions Stochastic Approximation (2RDSA) [9],[10] have arisen interest in these algorithms. Our work primarily concerns first order Random Directions Stochastic Approximation.

As an alternative to typically used symmetric Bernoulli random perturbations used in 1SPSA-2R and 1SPSA-1R, [11] proposes deterministic perturbations based on Hadamard matrices. We refer here the variants of 1SPSA-2R and 1SPSA-1R that use Hadamard perturbations as 1SPSA-2H and 1SPSA-1H. In a similar manner we propose deterministic perturbations for 1RDSA-2R and its one simulation variant (referred as 1RDSA-1R here) based on a specially chosen circulant matrix.

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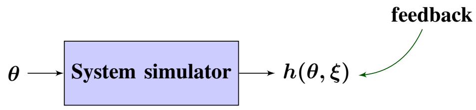

Fig. 1: Simulation Optimization Model

II. MOTIVATION

Here we motivate the necessary conditions that deterministic perturbations for 1RDSA-2R and 1RDSA-1R should satisfy. Analogous properties for Hadamard matrix based perturbations for 1SPSA-2R and 1SPSA-1R are well motivated in [5]. Now 1RDSA-2R iteratively updates the parameter vector and a typical update step is of the following form:

$$\theta_{n+1} = \theta_n - a_n \widehat{\nabla} J(\theta_n), \tag{2}$$

where a_n is the step-size that satisfies standard stochastic approximation conditions (see (A2) in section V) and $\widehat{\nabla} J(\theta)$ is an estimate of the gradient of the objective function J. Thus, (2) can be considered as the stochastic version of the well-known gradient descent method for optimization.

Suppose $\mathcal{F}_n = \sigma(\theta_m, 0 \leq m \leq n)$ and d_n is a random vector with distribution $\mathcal{N}(0, I)$ and independent of $\mathcal{F}_n$. Suppose δ_n is as in (A2) of V and $y_n^+ = J(\theta_n + \delta_n d_n) + N_n^+$ and $y_n^- = J(\theta_n - \delta_n d_n) + N_n^-$ then the gradient estimate of 1RDSA-2R is

$$\widehat{\nabla} J(\theta_n) = \left[\frac{y_n^+ - y_n^-}{2\delta_n} \right] d_n.$$

Assume that $\mathbb{E}[N_n^{\pm} | \mathcal{F}_n, d_n] = 0$ a.s. With additional boundedness assumptions on the higher order derivaties of J and iterates, θ_n, of the algorithm, convergence properties of the algorithm is shown. See [3] for details.

Now to motivate properties of deterministic perturbations, consider the Taylor series expansion of $J(\theta_n \pm \delta_n d_n)$ around θ_n, we have

$$J(\theta_n \pm \delta_n d_n) = J(\theta_n) \pm \delta_n d_n^T \nabla J(\theta_n) + o(\delta_n). \tag{3}$$

From (3), we have

$$\begin{aligned} &\frac{J(\theta_n + \delta_n d_n) - J(\theta_n - \delta_n d_n)}{2\delta_n} d_n - \nabla J(\theta_n) \\ &= (d_n d_n^T - I) \nabla J(\theta_n) + o(\delta_n). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\widehat{\nabla} J(\theta_n) - \nabla J(\theta_n) \\ &= (d_n d_n^T - I) \nabla J(\theta_n) + o(\delta_n) + \left[\frac{N_n^+ - N_n^-}{2\delta_n} \right] d_n \end{aligned}$$

From the assumption $\mathbb{E}[N_n^{\pm} | \mathcal{F}_n, d_n] = 0$ a.s. we have $\mathbb{E}\left[\left[\frac{N_n^+ - N_n^-}{2\delta_n} \right] d_n | \mathcal{F}_n \right] = 0$. Since d_n is distributed $\mathcal{N}(0, I)$ and independent of $\mathcal{F}_n$, we have $\mathbb{E}[(d_n d_n^T - I) \nabla J(\theta_n) | \mathcal{F}_n] = \mathbb{E}[(d_n d_n^T - I)] \nabla J(\theta_n) = 0$. So we have

$$\mathbb{E}[\widehat{\nabla} J(\theta_n) - \nabla J(\theta_n) | \mathcal{F}_n] = o(\delta_n). \tag{4}$$

Now the gradient estimate in the case of one simulation version, that is 1RDSA-1R, is

$$\nabla J(\theta_n) = \frac{J(\theta_n + \delta_n d_n)}{\delta_n}.$$

Consider the Taylor series expansion for J with parameter $\theta_n + \delta_n d_n$. We have

$$\begin{aligned} &\frac{J(\theta_n + \delta_n d_n)}{\delta_n} d_n - \nabla J(\theta_n) \\ &= \frac{J(\theta_n)}{\delta_n} d_n + (d_n d_n^T - I) \nabla J(\theta_n) + O(\delta_n). \end{aligned}$$

Since d_n is distributed $\mathcal{N}(0, I)$ and independent of $\mathcal{F}_n$, in addition to $\mathbb{E}[(d_n d_n^T - I) \nabla J(\theta_n) | \mathcal{F}_n] = 0$ we also have $\mathbb{E}[\frac{J(\theta_n)}{\delta_n} d_n | \mathcal{F}_n] = \frac{J(\theta_n)}{\delta_n} \mathbb{E}[d_n] = 0$. With the assumption $\mathbb{E}[N_n^+ | \mathcal{F}_n, d_n] = 0$, we have

$$\mathbb{E}[\widehat{\nabla} J(\theta_n) - \nabla J(\theta_n) | \mathcal{F}_n] = O(\delta_n). \tag{5}$$

Properties (4) and (5) are used in the literature crucially to prove the convergence of 1RDSA-2R and 1RDSA-1R respectively.

Notice that (4) and (5) depend on distribution properties of random perturbation direction d_n, inparticular on $\mathbb{E}[d_n] = 0$ and $\mathbb{E}[d_n d_n^T] = I$. A natural question to consider is "Are there deterministic d_n that give convergence?". A related work [11] that considers deterministic pertubations the case of 1SPSA-2R and 1SPSA-1R shows that such considerations can improve the performance of the algorithm. Clearly one is motivated to look for deterministic perturbations for 1RDSA-2R and 1RDSA-1R also. Motivated by the properties of distribution of d_n, inparticular $\mathbb{E}[d_n] = 0$ and $\mathbb{E}[d_n d_n^T] = I$, we prose the following properties for deterministic perturbation sequence d_n:

(P1) Let $D_n := d_n d_n^T - I$. For any $s \in \mathbb{N}$ there exists a $P \in \mathbb{N}$ such that $\sum_{n=s+1}^{s+P} D_n = 0$ and,

(P2) $\sum_{n=s+1}^{s+P} d_n = 0$.

III. CONTRIBUTION AND ORGANISATION

In the following sections we construct deterministic perturbations which satisfy (P1) and (P2) and provide convergence results for the resulting 1RDSA-2C and 1RDSA-1C algorithms. Our results shows that the proposed properties (P1) and (P2) for deterministic perturbations are sufficient to derive the convergence properties that are enjoyed by random perturbations. To the best of our knowledge the statement and proof of lemma 1 is original and may be of independent interest. We believe that defining properties (P1) and (P2) and proving convergence theorems in a manner analogous to the results in [11] is novel.

The rest of the paper is organized as follows: In section IV, we describe the various ingredients in algorithm 1 and the construction of perturbations d_n. In section V, we present the convergence results for the proposed 1RDSA-2C and 1RDSA-1C algorithms. In section VI, we present the results from numerical experiments and finally, in section VII, we provide the concluding remarks.

Algorithm 1 Basic structure of 1RDSA-2C and 1RDSA-1C algorithms.

Input:
- $\theta_0 \in \mathbb{R}^p$, initial parameter vector;
- $\delta_n, n \geq 0$, a sequence to approximate gradient;
- Matrix of perturbations

$$Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u],$$

with $u = [1, 1, \cdots, 1]^T$;
- noisy measurement of cost objective J;
- $a_n, n \geq 0$, step-size sequence satisfying assumption **(A2)** of section V;

for $n = 1, 2, \ldots n_{\text{end}}$ **do**

Let d_n be mod$(n, p+1)^{\text{th}}$ column of Q.

Update the parameter as follows:

$$\theta_{n+1} = \theta_n - a_n \widehat{\nabla} J(\theta_n)$$

where $\widehat{\nabla} J(\theta_n)$ is chosen according to (7) or (8).

end for

Return $\theta_{n_{\text{end}}}$.

IV. STRUCTURE OF THE ALGORITHM

Algorithm 1 presents the basic structure of 1RDSA-2C and 1RDSA-1C. We describe the individual components of algorithm 1 below.

A. Deterministic Pertubations

Let $\delta_n, n \geq 0$ denote a sequence of diminishing positive real numbers satisfying assumption (A2) in section V and $d_n = (d_n^1, \ldots, d_n^p)^{\intercal}$ denote $(p+1)$ mod n column of Q. where Q is constructed as follows. Let H be the $p \times p$ dimensional matrix defined as:

$$H = \begin{bmatrix} 2 \ 1 \ 1 \cdots 1 \\ 1 \ 2 \ 1 \cdots 1 \\ \vdots \vdots \vdots \vdots \\ 1 \ 1 \ 1 \cdots 2 \end{bmatrix}. \tag{6}$$

Observe that $H = I + uu^T$, where $u = [1, 1, \cdots, 1]^T$, is a positive definite matrix. Hence $H^{-1/2}$ is well defined. Define $p \times (p+1)$ matrix Q as $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. The columns of Q are used as perturbations d_n.

B. Gradient estimation

Let y_n^+, y_n^- be defined as below. $y_n^+ = J(\theta_n + \delta_n d_n) + N_n^+$ and $y_n^- = J(\theta_n - \delta_n d_n) + N_n^-$, where the noise terms N_n^+, N_n^- are assumed to satisfy $\mathbb{E}[N_n^+ - N_n^- | \mathcal{F}_n] = 0$ with $\mathcal{F}_n = \sigma(\theta_m, m \leq n)$ denoting the underlying sigma-field. The estimate of the gradient $\nabla J(\theta_n)$ is given by

$$\widehat{\nabla} J(\theta_n) = \left[\frac{y_n^+ - y_n^-}{2\delta_n}\right] d_n \text{ for 1RDSA-2C} \tag{7}$$

$$\widehat{\nabla} J(\theta_n) = \left[\frac{y_n^+}{\delta_n}\right] d_n \text{ for 1RDSA-1C} \tag{8}$$

Observe that while 1RDSA-2C algorithm uses two function samples y_n^+ and y_n^- at $\theta_n + \delta_n d_n$ and $\theta_n - \delta_n d_n$, 1RDSA-1C algorithm uses only one function sample y_n^+ at $\theta_n + \delta_n d_n$.

V. CONVERGENCE ANALYSIS

We make the same assumptions as those used in the analysis of [4], with a few minor alterations. The assumptions are listed below. Also $\|.\|$ denotes 2-norm. More detailed proofs of the results are presented in the archived version [12].

(A1) The map $J : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous and is differentiable with bounded second and third order derivatives. Further, the map $L : \mathbb{R}^p \to \mathbb{R}^p$ defined as $L(\theta) = -\nabla J(\theta)$ is Lipschitz continuous.

(A2) The step-size sequences $a_n, \delta_n > 0, \forall n$ satisfy

$$a_n, \delta_n \to 0, \sum_n a_n = \infty, \sum_n \left(\frac{a_n}{\delta_n}\right)^2 < \infty.$$

Further, $\frac{a_j}{a_n} \to 1$ as $n \to \infty$, for all $j \in \{n, n+1, n+2 \cdots, n+M\}$ for any given $M > 0$ and $b_n = \frac{a_n}{\delta_n}$ is such that $\frac{b_j}{b_n} \to 1$ as $n \to \infty$, for all $j \in \{n, n+1, n+2, \cdots, n+M\}$.

(A3) $\max_n \|d_n\| = C, \max_n \|D_n\| = \bar{C}$.

(A4) The iterates θ_n remain uniformly bounded almost surely, i.e., $\sup_n \|\theta_n\| < \infty$, a.s.

(A5) The ODE $\dot{\theta}(t) = -\nabla J(\theta(t))$ has a compact set $G \subset \mathbb{R}^p$ as its set of asymptotically stable equilibria (i.e., the set of local minima of J is compact).

(A6) The noise sequences $(N_n^{\pm}, \mathcal{F}_n), n \geq 0$ form martingale difference sequences. Further, $(N_n^{\pm}, n \geq 0)$ are square integrable random variables satisfying

$$\mathbb{E}[\|N_n^{\pm}\|^2 | \mathcal{F}_n] \leq K(1 + \|\theta_n\|^2) \text{ a.s.}, \forall n \geq 0,$$

for a given constant $K \geq 0$.

The following lemma is useful in obtaining the negative square root of H i.e., $H^{-1/2}$. Also note that it takes only $O(p^2)$ operations to compute $H^{-1/2}$ using the lemma and circulant structure of $H^{-1/2}$. To the best of our knowledge this lemma is original and may be of independent interest.

Lemma 1. *Let I be a $p \times p$ identity matrix and $u = [1, 1, \cdots 1]^T$ be a $p \times 1$ column vector of 1s, then*

$$(I + uu^T)^{-1/2} = I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}.$$

Proof. It is enough to show that

$$(I + uu^T)\left[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}\right]^2 = I.$$

Using $\|u\|^2 = u^T u = p$ in the expansion of $\Big[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}\Big]^2$ gives the result. $\square$

Let H defined as in (6) and $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. Let the pertubations d_n be the columns of Q.

Lemma 2. *The perturbations d_n chosen as columns of Q satisfy properties (P1) and (P2).*

Proof. Let $P = p+1$. Observe that as n goes through one cycle from 1 to $p+1$ we have $\sum_{n=1}^{p+1} d_n d_n^T = QQ^T$ and $\sum_{n=1}^{p+1} d_n = Q\begin{bmatrix} u \\ 1 \end{bmatrix}$. Now it is enough to show that $\sum_{n=s+1}^{s+P} D_n = 0$ and $\sum_{n=s+1}^{s+P} d_n = 0$ i.e., $Q\begin{bmatrix} u \\ 1 \end{bmatrix} = 0$ for the choice $s = 0$. Consider

$$\begin{aligned}
&\sum_{n=0}^{P} D_n = \sum_{n=1}^{P}(d_n d_n^T - I) = \sum_{n=1}^{P}(d_n d_n^T) - (p+1)I \\
&= QQ^T - (p+1)I \\
&= (p+1)([H^{-1/2}, -H^{-1/2}u][H^{-1/2}, -H^{-1/2}u]^T - I) \\
&= 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
Q\begin{bmatrix} u \\ 1 \end{bmatrix} &= \sqrt{p+1}([H^{-1/2}, -H^{-1/2}u])\begin{bmatrix} u \\ 1 \end{bmatrix} \\
&= \sqrt{p+1}(H^{-1/2}u - H^{-1/2}u) = 0,
\end{aligned}$$

proving the lemma. □

Lemma 3. *Given any fixed integer $P > 0$, $\|\theta_{m+k} - \theta_m\| \to 0$ $w.p.1$, as $m \to \infty$, for all $k \in \{1, \cdots, P\}$*

Proof. Fix a $k \in \{1, \cdots, P\}$. Now

$$\begin{aligned}
\theta_{n+k} = \theta_n - &\sum_{j=n}^{n+k-1} a_j \left(\frac{J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)}{2\delta_j}\right) d_j \\
- &\sum_{j=n}^{n+k-1} a_j N_j.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\theta_{n+k} - \theta_n\| \leq &\sum_{j=n}^{n+k-1} a_j \left|\frac{J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)}{2\delta_j}\right| \|d_j\| \\
+ &\sum_{j=n}^{n+k-1} a_j \|N_j\|
\end{aligned}$$

Now clearly, $M_n = \sum_{j=0}^{n-1} a_j N_j, n \geq 1$, forms a martingale sequence with respect to the filtration $\{\mathcal{F}_n\}$. Further, from the assumption (A6) we have,

$$\begin{aligned}
\sum_{m=0}^{n} \mathbb{E}[\|M_{m+1} - M_m\|^2 | \mathcal{F}_m] &= \sum_{m=0}^{n} \mathbb{E}[a_m^2 \|N_m\|^2 | \mathcal{F}_m] \\
&\leq \sum_{m=0}^{n} a_m^2 K(1 + \|\theta_m\|^2)
\end{aligned}$$

From the assumption (A4), the quadratic variation process of $M_n, n \geq 0$ converges almost surely. Hence by the martingale convergence theorem, it follows that $M_n, n \geq 0$ converges almost surely. Hence $\| \sum_{j=n}^{n+k-1} a_j N_j \| \to 0$ almost surely as $n \to \infty$. Moreover

$$\begin{aligned}
&\left\| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)\Big) d_j \right\| \\
&\leq \left| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)\Big) \right| \|d_j\| \\
&\leq C\Big(|J(\theta_j + \delta_j d_j)| + |J(\theta_j - \delta_j d_j)|\Big),
\end{aligned}$$

since $\|d_j\| \leq C, \forall j \geq 0$. Note that

$$\begin{aligned}
|J(\theta_j + \delta_j d_j)| - |J(0)| &\leq |J(\theta_j + \delta_j d_j) - J(0)| \\
&\leq \hat{B}\|\theta_j + \delta_j d_j\|,
\end{aligned}$$

where $\hat{B}$ is the Lipschitz constant of the function $J(.)$. Hence,

$$|J(\theta_j + \delta_j d_j)| \leq \tilde{B}(1 + \|\theta_j + \delta_j d_j\|),$$

for $\tilde{B}$ =max$(|J(0)|, \hat{B})$. Similarly,

$$|J(\theta_j - \delta_j d_j)| \leq \tilde{B}(1 + \|\theta_j - \delta_j d_j\|).$$

From assumption (A1), it follows that

$$\sup_j \left\| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)\Big) d_j \right\| \leq \tilde{K} < \infty,$$

for some $\tilde{K} > 0$. Thus, $\|\theta_{n+k} - \theta_n\| \leq \tilde{K} \sum_{j=n}^{n+k-1} \frac{a_j}{\delta_j} + \| \sum_{j=n}^{n+k-1} a_j N_j \|$ $\to 0$ a.s. with $n \to \infty$, proving the lemma. □

Lemma 4. *For any $m \geq 0$, $\left\| \sum_{n=m}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) \right\|$ and $\left\| \sum_{n=m}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) \right\| \to 0$, almost surely, as $m \to \infty$.*

Proof. From lemma 3, it can be seen that $\|\theta_{m+s} - \theta_m\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Also from assumption (A1), we have $\|\nabla J(\theta_{m+s}) - \nabla J(\theta_m)\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Now from lemma 2, $\sum_{n=m}^{m+P-1} D_n = 0$ $\forall m \geq 0$. Hence $D_m = -\sum_{n=m+1}^{m+P-1} D_n$. Thus we have,

$$\begin{aligned}
&\left\| \sum_{n=m}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) \right\| \\
&= \left\| \sum_{n=m+1}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) + D_m \nabla J(\theta_m) \right\| \\
&= \left\| \sum_{n=m+1}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) - \sum_{n=m+1}^{m+P-1} D_n \nabla J(\theta_m) \right\| \\
&= \left\| \sum_{n=m+1}^{m+P-1} D_n \Big(\frac{a_n}{a_m} \nabla J(\theta_n) - \nabla J(\theta_m)\Big) \right\| \\
&\leq \sum_{n=m+1}^{m+P-1} \|D_n\| \left\| \Big(\frac{a_n}{a_m} \nabla J(\theta_n) - \nabla J(\theta_m)\Big) \right\| \\
&\leq \bar{C} \sum_{n=m+1}^{m+P-1} \left\| \Big(\frac{a_n}{a_m} - 1\Big) \nabla J(\theta_n) \right\| + \left\| \nabla J(\theta_n) - \nabla J(\theta_m) \right\|
\end{aligned}$$

The claim now follows from assumptions (A1) and (A2). Now observe that $\|J(\theta_{m+k}) - J(\theta_m)\| \to 0$ as $m \to \infty$, for all $k \in \{1, \cdots, P\}$ as a consequence of (A1) and lemma3. Moreover from $d_m = -\sum_{n=m+1}^{m+P-1} d_n$ we have

$$
\begin{aligned}
&\Big\| \sum_{n=m}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) + d_m J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) - \sum_{n=m+1}^{m+P-1} d_n J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} d_n \Big(\frac{b_n}{b_m} J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&\leq \sum_{n=m+1}^{m+P-1} \|d_n\| \Big\| \Big(\frac{b_n}{b_m} J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&\leq C \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{b_n}{b_m} - 1 \Big) J(\theta_n) \Big\| + \Big\| \Big(J(\theta_n) - J(\theta_m) \Big) \Big\|
\end{aligned}
$$

The claim now follows as a consequence of assumptions (A1) and (A2). □

Theorem 5. *Let G be as in assumption (A5). $\theta_n, n \geq 0$ obtained from 1RDSA-2C satisfy $\theta_n \to G$ almost surely.*

Proof. Note that

$$
\theta_{n+P} = \theta_n - \sum_{l=n}^{n+P-1} a_l \Big[\frac{J(\theta_l + \delta_l d_l) - J(\theta_l - \delta_l d_l)}{2\delta_l} d_l + N_l \Big].
$$

It follows that

$$
\begin{aligned}
\theta_{n+P} = \theta_n &- \sum_{l=n}^{n+P-1} a_l \nabla J(\theta_l) - \sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) \\
&- \sum_{l=n}^{n+P-1} a_l \xi^1(\delta_l) - \sum_{l=n}^{n+P-1} a_l N_l
\end{aligned}
$$

Now the third term on the RHS can be written as

$$
a_n \sum_{l=n}^{n+P-1} \frac{a_l}{a_n} D_l \nabla J(\theta_l) = a_n \xi_n,
$$

where $\xi_n = o(1)$ from lemma 4. Thus, the algorithm is asymptotically analogous to

$$
\theta_{n+1} = \theta_n - a_n (\nabla J(\theta_n) + o(\delta) + N_n).
$$

Hence from chapter 2 of [13] we have that $\theta_n, n \geq 0$ converge to local minima of the function J. □

Theorem 6. *Let G be as in assumption (A5). $\theta_n, n \geq 0$ obtained from 1RDSA-1C satisfy $\theta_n \to G$ almost surely.*

Proof. Note that

$$
\theta_{n+P} = \theta_n - \sum_{l=n}^{n+P-1} a_l \Big(\frac{J(\theta_l + \delta_l d_l)}{2\delta_l} \Big) d_l - \sum_{l=n}^{n+P-1} a_l N_l.
$$

It follows that

$$
\begin{aligned}
\theta_{n+P} = \theta_n &- \sum_{l=n}^{n+P-1} a_l \nabla J(\theta_l) - \sum_{l=n}^{n+P-1} a_l \frac{J(\theta_l)}{\delta_l} d_l \\
&- \sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) - \sum_{l=n}^{n+P-1} a_l O(\delta_l) \\
&- \sum_{l=n}^{n+P-1} a_l N_l
\end{aligned}
$$

Now we observe that the third term on the RHS is

$$
\begin{aligned}
\sum_{l=n}^{n+P-1} a_l \frac{J(\theta_l)}{\delta_l} d_l &= \sum_{l=n}^{n+P-1} b_l J(\theta_l) d_l \\
&= b_n \sum_{l=n}^{n+P-1} \frac{b_l}{b_n} \frac{J(\theta_l)}{\delta_l} d_l = b_n \xi_n^1,
\end{aligned}
$$

where $\xi_n^1 = o(1)$ by lemma 4. Similarly

$$
\sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) = a_n \xi_n^2,
$$

with $\xi_n^2 = o(1)$ by lemma 4. The rest follows as explained in Theorem 5. □

VI. SIMULATION EXPERIMENTS

A. Implementation

We compare the performance of 1SPSA-2R, 1SPSA-2H and 1RDSA-2C in the case of two simulation algorithms. We compare 1RDSA-2C with 1SPSA-2R instead of 1RDSA-2R as 1SPSA-2R performs better compared to 1RDSA-2R [8]. In the case of one simulation algorithms we compare the performance of 1SPSA-1R, 1SPSA-1H and 1RDSA-1C. We chose i.i.d Bernoulli ± 1-valued perturbations for 1SPSA-2R and 1SPSA-1R. [1]

The following loss functions have been widely used in the literature [14], [10] for the empirical evaluations. We use these loss functions in $p = 10$ dimensions:

a) Quadratic loss:

$$
J(\theta) = \theta^{\mathsf{T}} A \theta + b^{\mathsf{T}} \theta. \tag{9}
$$

For this particular choice of p the optimum θ^* for the above J is a 10×1 column vector with each entry equal to -0.9091, and $J(\theta^*) = -4.55$.

b) Fourth-order loss:

$$
J(\theta) = \theta^{\mathsf{T}} A^{\mathsf{T}} A \theta + 0.1 \sum_{j=1}^{N} (A\theta)_j^3 + 0.01 \sum_{j=1}^{N} (A\theta)_j^4. \tag{10}
$$

The optimum θ^* for the above J is $\theta^* = 0$, with $J(\theta^*) = 0$. In both functions, A is such that pA is an upper triangular matrix with each nonzero entry equal to one, b is the p-dimensional vector of ones and the noise structure is similar to that used in [14]. For any θ, the noise is $[\theta^{\mathsf{T}}, 1]z$, where $z \approx \mathcal{N}(0, \sigma^2 I_{11\times 11})$. We perform experiments for noisy as

[1] The implementation is available at `https://github.com/cs1070166/1RDSA-2Cand1RDSA-1C/`

TABLE I: NMSE values of two simulation methods for quadratic objective (9) with and without noise for 2000 simulations: standard error from 100 replications shown after $\pm$ symbol

Method	NMSE
Noise parameter $\sigma = 0.01$	
1SPSA-2R	$5.762 \times 10^{-3} \pm 2.473 \times 10^{-3}$
1SPSA-2H	$4.012 \times 10^{-5} \pm 1.654 \times 10^{-5}$
1RDSA-2C	$\mathbf{2.188 \times 10^{-5} \pm 9.908 \times 10^{-6}}$
Noise parameter $\sigma = 0$	
1SPSA-2R	$5.755 \times 10^{-3} \pm 2.460 \times 10^{-3}$
1SPSA-2H	$1.601 \times 10^{-5} \pm 2.724 \times 10^{-20}$
1RDSA-2C	$\mathbf{2.474 \times 10^{-8} \pm 1.995 \times 10^{-23}}$

TABLE II: NMSE values of two simulation methods for fourth order objective (10) with and without noise for 10000 simulations: standard error from 100 replications shown after $\pm$ symbol

Method	NMSE
Noise parameter $\sigma = 0.01$	
1SPSA-2R	$2.762 \times 10^{-2} \pm 1.415 \times 10^{-2}$
1SPSA-2H	$3.958 \times 10^{-3} \pm 4.227 \times 10^{-4}$
1RDSA-2C	$\mathbf{3.598 \times 10^{-3} \pm 4.158 \times 10^{-4}}$
Noise parameter $\sigma = 0$	
1SPSA-2R	$2.747 \times 10^{-2} \pm 1.413 \times 10^{-2}$
1SPSA-2H	$3.901 \times 10^{-3} \pm 4.359 \times 10^{-18}$
1RDSA-2C	$\mathbf{3.535 \times 10^{-3} \pm 1.743 \times 10^{-18}}$

TABLE III: NMSE values of one simulation methods for quadratic objective (9) with and without noise for 20000 simulations: standard error from 100 replications shown after $\pm$ symbol

Method	NMSE
Noise parameter $\sigma = 0.01$	
1SPSA-1R	$8.582 \times 10^{-2} \pm 3.691 \times 10^{-2}$
1SPSA-1H	$2.774 \times 10^{-2} \pm 2.578 \times 10^{-4}$
1RDSA-1C	$\mathbf{8.225 \times 10^{-3} \pm 5.959 \times 10^{-5}}$
Noise parameter $\sigma = 0$	
1SPSA-1R	$8.584 \times 10^{-2} \pm 3.681 \times 10^{-2}$
1SPSA-1H	$2.770 \times 10^{-2} \pm 3.836 \times 10^{-17}$
1RDSA-1C	$\mathbf{8.225 \times 10^{-3} \pm 1.569 \times 10^{-17}}$

TABLE IV: NMSE values of one simulation methods for fourth order objective (10) with and without noise for 20000 simulations: standard error from 100 replications shown after $\pm$ symbol

Method	NMSE
Noise parameter $\sigma = 0.01$	
1SPSA-1R	$3.240 \times 10^{-1} \pm 1.836 \times 10^{-1}$
1SPSA-1H	$8.916 \times 10^{-2} \pm 1.896 \times 10^{-2}$
1RDSA-1C	$\mathbf{4.972 \times 10^{-2} \pm 9.812 \times 10^{-3}}$
Noise parameter $\sigma = 0$	
1SPSA-1R	$3.192 \times 10^{-1} \pm 1.991 \times 10^{-1}$
1SPSA-1H	$8.173 \times 10^{-2} \pm 1.255 \times 10^{-16}$
1RDSA-1C	$\mathbf{4.403 \times 10^{-2} \pm 9.066 \times 10^{-17}}$

well as noise-less settings, with $\sigma = 0.01$ for the noisy case. For all algorithms, we chose step sizes to have the form $\delta_n = c/(n+1)^\gamma$ and $a_n = 1/(n+B+1)^\alpha$. We set $\alpha = 0.602$ and $\gamma = 0.101$. These values for α and γ have been used before (see [14]) and have demonstrated good finite-sample performance empirically, while satisfying the theoretical requirements needed for asymptotic convergence. For all the algorithms, the initial point θ_0 is the p-dimensional vector of ones.

B. Results

We use Normalized Mean Square Error (NMSE) as the performance metric for evaluating the algorithms. NMSE is the ratio $\|\theta_{n_{\text{end}}} - \theta^*\|^2 / \|\theta_0 - \theta^*\|^2$. Here n_{end} denotes the iteration number at the end of simulation budget.

From the results in tables I,II, III and IV (see also, plots in archived version [12]) we make the following observations:

Observation1 In the case of two simulation algorithms, 1RDSA-2C is slightly better than 1SPSA-2H, while both of them outperform 1SPSA-2R.

Observation2 In the case of one simulation algorithms, 1RDSA-1C is better than both 1SPSA-1H and 1SPSA-1R.

VII. CONCLUSIONS

We presented a novel construction of deterministic perturbations for 1RDSA-2R and 1RDSA-1R algorithms and showed that the resulting algorithms 1RDSA-2C and 1RDSA-1C are provably convergent. The advantage with our deterministic perturbation construction is that, unlike in 1SPSA-2H and 1SPSA-1H, the same set of perturbations can be used for both two simulation and one simulation variants. Numerical experiments demonstrated that 1RDSA-2C (1RDSA-1C) performs better than 1SPSA-2R (1SPSA-1R) and 1SPSA-2H (1SPSA-1H). On more advantage albeit minor is the savings in computation that is incurred in generating random d_n. Once constructed deterministic d_n can be stored and used multiple times. Though methods with deterministic d_n perform better in experiments [11], Unlike in random d_n case, it appears that there are no rate of convergence results. A challenging future direction would be to derive rate of convergence results when deterministic perturbations are used. Also as another future work, it would be interesting to derive similar methods in the context of 2RDSA.

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