Deterministic Perturbations For Simultaneous Perturbation Methods Using Circulant Matrices.

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***Abstract*— We consider the problem of finding optimal parameters under simulation optimization setup. For a p-dimensional parameter optimization, the classical Kiefer-Wolfowitz Finite Difference Stochastic Approximation (FDSA) scheme uses $p+1$ or $2p$ simulations of the system feedback for one-sided and two-sided gradient estimates respectively. The dependence on the dimension p makes FDSA impractical for high dimensional problems. An alternative approach for gradient estimation in high dimensional problems is the simultaneous perturbation technique that appears in [1],[2]. The main idea in this approach is to estimate the gradient by using only two settings of the p-dimensional parameter being optimized. The two settings of the parameter are obtained by simultaneously perturbing all the components of the parameter by adding a random direction. A drawback of using random directions for the gradient estimate is the very large or possibly infinite range of these random directions (for e.g. ± 1 symmetric Bernoulli perturbations typically used in 1SPSA algorithm has a range of cardinality 2^p). In this article we consider deterministic perturbations with a range of cardinality $p+1$ to improve the convergence of these algorithms. A novel construction of deterministic perturbations based on specially chosen circulant matrix is proposed. Convergence analysis of the proposed algorithms is presented along with numerical experiments.**

I. INTRODUCTION

Simulation optimization problems frequently arise in engineering disciplines like transportation systems, machine learning, service systems, manufacturing etc. Practical limitations, lack of model information and the large dimensionality of these problems prohibit analytic solution of these problems and simulation is often employed to evaluate a system in these disciplines. Hence simulation budget becomes critical and one aims to converge to optimal parameters using as few simulations as possible.

To state the problem formally, consider a system where noise-corrupted feedback of the system is available i.e., given the system parameter θ the feedback that is available is $h(\theta,\xi)$ where ξ is the noise term inherent in the system, pictorially shown in Figure 1. The objective in these problems is to find a parameter vector that gives the optimal expected performance of the system. Suppose $J(\theta) = \mathbb{E}_\xi[h(\theta,\xi)]$ then $h(\theta,\xi) = J(\theta) + M(\theta,\xi)$ where $M(\theta,\xi) = h(\theta,\xi) - \mathbb{E}_\xi[h(\theta,\xi)]$ is a mean zero term. Hence, one needs to solve the following optimization problem:

$$\text{Find } \theta^* = \arg\min_{\theta\in\mathbb{R}^p} J(\theta). \tag{1}$$

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[⋆]Supported by Robert Bosch Center for Cyber-Physical Systems, IISc.

Analogous to deterministic optimization problems, where explicit analytic gradient of the objective function is available, a solution approach could be to devise an algorithm that mimics the familiar gradient descent algorithm. However in our setting only noise corrupted samples of the objective are available. So one essentially needs to estimate the gradient of the objective function using simulation samples.

This adaptive system optimization framework has its roots in the work by Robbins and Monro [3] and Kiefer and Wolfowitz [4]. The work by Kiefer and Wolfowitz uses stochastic approximation framework by Robbins and Monro to optimize an objective using noisy samples. In their work Kiefer and Wolfowitz [4] use one-sided or two-sided approximation of the gradient. This method has the drawback of using $p+1$ simulations for one-sided approximation and $2p$ simulations for two-sided approximation of the gradient for a p-dimensional problem. Later works [2],[1] have replaced the gradient approximation using finite differences with random perturbations.

Simultaneous perturbation is a useful technique to estimate the gradient from the function samples, especially in high dimensional problems - see [5], [6] for a comprehensive treatment of this subject matter. The first-order Simultaneous Perturbation Stochastic Approximation algorithm for simulation optimization, henceforth referred to as 1SPSA-2R, was proposed in [1]. 1SPSA-2R uses two simulations per iteration and random perturbations to perturb the parameter vector. A one measurement variant of [1] is proposed in [7] which we refer here as 1SPSA-1R. A closely related algorithm is the first-order Random Directions Stochastic Approximation henceforth referred to as 1RDSA-2R that appears in [2, pp. 58-60]. The algorithm 1RDSA-2R differs from 1SPSA-2R, both in the construction as well as convergence analysis and performs poorly compared to 1SPSA-2R, see [8] for a detailed comparative study.

To enhance the performance of 1SPSA-2R and 1SPSA-1R, [9] proposed deterministic perturbations based on Hadamard matrices. We refer here the variants of 1SPSA-2R and 1SPSA-1R that use Hadamard perturbations as 1SPSA-2H and 1SPSA-1H. Analogous to their work we propose deterministic perturbations for 1RDSA-2R and its one simulation variant (referred as 1RDSA-1R here) based on a specially chosen circulant matrix.

II. MOTIVATION

Here we motivate the necessary conditions that deterministic perturbations should satisfy. Analogous properties for Hadamard matrix based perturbations are well motivated in

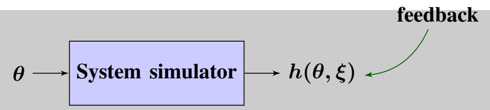

Fig. 1: Simulation Optimization Model

[5]. Both 1SPSA-2R and 1RDSA-2R iteratively update the parameter vector and a typical update step is of the following form:

$$\theta_{n+1} = \theta_n - a_n \widehat{\nabla} J(\theta_n), \tag{2}$$

where a_n is the step-size that satisfies standard stochastic approximation conditions (see (A2) in section IV) and $\widehat{\nabla} J(\theta)$ is an estimate of the gradient of the objective function J. Thus, (2) can be considered as the stochastic version of the well-known gradient descent method for optimization.

Now consider the Taylor series expansion of $J(\theta_n + \delta_n d_n)$ around θ_n, we have

$$J(\theta_n + \delta_n d_n) = J(\theta_n) + \delta_n d_n^T \nabla J(\theta_n) + o(\delta_n). \tag{3}$$

Similarly, an expansion of $J(\theta_n - \delta_n d_n)$ around θ_n gives

$$J(\theta_n - \delta_n d_n) = J(\theta_n) - \delta_n d_n^T \nabla J(\theta_n) + o(\delta_n). \tag{4}$$

From (3) and (4), we have

$$\frac{J(\theta_n + \delta_n d_n) - J(\theta_n - \delta_n d_n)}{2\delta_n} d_n - \nabla J(\theta_n) \tag{5}$$
$$= (d_n d_n^T - I)\nabla J(\theta_n) + o(\delta_n). \tag{6}$$

Note that $(d_n d_n^T - I)\nabla J(\theta_n)$ constitutes the bias in the gradient estimate and from the assumption (A3) in section IV

$$\mathbb{E}\left[(d_n d_n^T - I)\nabla J(\theta_n) \middle| \theta_n \right] = 0. \tag{7}$$

In the case of a one-simulation algorithm with parameter $\theta_n + \delta_n d_n$, a similar Taylor series expansion gives

$$\frac{J(\theta_n + \delta_n d_n)}{\delta_n} d_n - \nabla J(\theta_n) \tag{8}$$
$$= \frac{J(\theta_n)}{\delta_n} d_n + (d_n d_n^T - I)\nabla J(\theta_n) + O(\delta_n). \tag{9}$$

One expects the following to hold in addition to (7) in the case of random perturbations for one simulation algorithms:

$$\mathbb{E}\left[\frac{J(\theta_n)}{\delta_n} d_n \middle| \theta_n \right] = 0. \tag{10}$$

Notice that (7) and (10) are achieved asymptotically for a random perturbation direction d_n, for example with ± 1 symmetric Bernoulli components, d_n has to cycle through 2^p possible vectors in the range to achieve bias cancellation. Clearly one is motivated to look for perturbations that satisfy similar properties and have a smaller cycle length for faster bias cancellation and thereby improve the performance of the algorithm. Analogous to (7) and (10) we expect deterministic sequence of perturbations to satisfy the following properties:

Algorithm 1 Basic structure of 1RDSA-2C and 1RDSA-1C algorithms.

Input:
- $\theta_0 \in \mathbb{R}^p$, initial parameter vector;
- $\delta_n, n \geq 0$, a sequence to approximate gradient;
- Matrix of perturbations

$$Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u],$$

with $u = [1, 1, \cdots, 1]^T$;
- noisy measurement of cost objective J;
- $a_n, n \geq 0$, step-size sequence satisfying assumption **(A2)** of section IV;

for $n = 1, 2, \ldots n_{\text{end}}$ **do**
Let d_n be $\text{mod}(n, p+1)^{\text{th}}$ column of Q.
Update the parameter as follows:

$$\theta_{n+1} = \theta_n - a_n \widehat{\nabla} J(\theta_n), \tag{11}$$

where $\widehat{\nabla} J(\theta_n)$ is chosen according to (13) or (14).
end for
Return $\theta_{n_{\text{end}}}$.

(P1) Let $D_n := d_n d_n^T - I$. For any $s \in \mathbb{N}$ there exists a $P \in \mathbb{N}$ such that $\sum_{n=s+1}^{s+P} D_n = 0$ and,

(P2) $\sum_{n=s+1}^{s+P} d_n = 0$.

In the following sections we construct deterministic perturbations which satisfy (P1) and (P2) and provide convergence results for the resulting 1RDSA-2C and 1RDSA-1C algorithms.

The rest of the paper is organized as follows: In section III, we describe the various ingredients in algorithm 1 and the construction of perturbations d_n. In section IV, we present the convergence results for 1RDSA-2C and 1RDSA-1C algorithms. In section V, we present the results from numerical experiments and finally, in section VI, we provide the concluding remarks.

III. Structure of the Algorithm

Algorithm 1 presents the basic structure of 1RDSA-2C and 1RDSA-1C. We describe the individual components of algorithm 1 below.

A. Deterministic Pertubations

Let $\delta_n, n \geq 0$ denote a sequence of diminishing positive real numbers satisfying assumption A2 in section IV and $d_n = (d_n^1, \ldots, d_n^p)^\mathsf{T}$ denote $(p+1)$ mod n column of Q. where Q is constructed as follows. Let H be the $p \times p$ dimensional matrix defined as:

$$H = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix}. \tag{12}$$

Observe that $H = I + uu^T$, where $u = [1, 1, \cdots, 1]^T$, is a positive definite matrix. Hence $H^{-1/2}$ is well defined. Define $p \times (p+1)$ matrix Q as $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. The columns of Q are used as perturbations d_n.

B. Gradient estimation

Let y_n^+, y_n^- be defined as below. $y_n^+ = J(\theta_n + \delta_n d_n) + M_n^+$ and $y_n^- = J(\theta_n - \delta_n d_n) + M_n^-$, where the noise terms M_n^+, M_n^- satisfy $\mathbb{E}[M_n^+ - M_n^- | \mathcal{F}_n] = 0$ with $\mathcal{F}_n = \sigma(\theta_m, m \leq n)$ denoting the underlying sigma-field. The estimate of the gradient $\nabla J(\theta_n)$ is given by

$$\widehat{\nabla} J(\theta_n) = \left[\frac{(y_n^+ - y_n^-) d_n}{2\delta_n} \right] \text{ for 1RDSA-2C} \tag{13}$$

$$\widehat{\nabla} J(\theta_n) = \left[\frac{(y_n^+) d_n}{\delta_n} \right] \text{ for 1RDSA-1C} \tag{14}$$

Observe that while 1RDSA-2C algorithm uses two function samples y_n^+ and y_n^- at $\theta_n + \delta_n d_n$ and $\theta_n - \delta_n d_n$, 1RDSA-1C algorithm uses only one function sample y_n^+ at $\theta_n + \delta_n d_n$.

IV. CONVERGENCE ANALYSIS

We make the same assumptions as those used in the analysis of [1], with a few minor alterations. The assumptions are listed below. Also $\|.\|$ denotes 2-norm.

(A1) The map $J : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous and is differentiable with bounded second order derivatives. Further, the map $L : \mathbb{R}^p \to \mathbb{R}$ defined as $L(\theta) = -\nabla J(\theta)$ is Lipschitz continuous.

(A2) The step-size sequences $a_n, \delta_n > 0, \forall n$ satisfy

$$a_n, \delta_n \to 0, \sum_n a_n = \infty, \sum_n \left(\frac{a_n}{\delta_n} \right)^2 < \infty.$$

Further, $\frac{a_j}{a_n} \to 1$ as $n \to \infty$, for all $j \in \{n, n+1, n+2 \cdots, n+M\}$ for any given $M > 0$ and $b_n = \frac{a_n}{\delta_n}$ is such that $\frac{b_j}{b_n} \to 1$ as $n \to \infty$, for all $j \in \{n, n+1, n+2, \cdots, n+M\}$.

(A3) $\max_n \|d_n\| = C, \max_n \|D_n\| = \bar{C}$.

(A4) The iterates θ_n remain uniformly bounded almost surely, i.e.,

$$\sup_n \|\theta_n\| < \infty, a.s.$$

(A5) The ODE $\dot{\theta}(t) = -\nabla J(\theta(t))$ has a compact set $G \subset \mathbb{R}^p$ as its set of asymptotically stable equilibria (i.e., the set of local minima of J is compact).

(A6) The sequences $(M_n^+, \mathcal{F}_n), (M_n^-, \mathcal{F}_n), n \geq 0$ form martingale difference sequences. Further, $(M_n^+, M_n^-, n \geq 0)$ are square integrable random variables satisfying

$$\mathbb{E}[\|M_{n+1}^{\pm}\|^2 | \mathcal{F}_n] \leq K(1 + \|\theta_n\|^2) \text{ a.s., } \forall n \geq 0,$$

for a given constant $K \geq 0$.

The following lemma is useful in obtaining the negative square root of H i.e., $H^{-1/2}$. Also note that it takes only $O(p)$ operations to compute $H^{-1/2}$ using the lemma and circulant structure of $H^{-1/2}$.

Lemma 1. *Let I be a $p \times p$ identity matrix and $u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ be a $p \times 1$ vector of 1s, then*

$$(I + uu^T)^{-1/2} = I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}}.$$

Proof. It is enough to show that

$$(I + uu^T) \left[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}} \right]^2 = I.$$

Using $\|u\|^2 = u^T u = p$ we have

$$\begin{aligned}
&(I + uu^T) \left[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}} \right]^2 \\
&= (I + uu^T) \Bigg[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}} \\
&- \frac{uu^T}{p} + \frac{uu^T uu^T}{p^2} - \frac{uu^T uu^T}{p^2\sqrt{(1+p)}} \\
&+ \frac{uu^T}{p\sqrt{(1+p)}} - \frac{uu^T uu^T}{p^2\sqrt{(1+p)}} + \frac{uu^T uu^T}{p^2(1+p)} \Bigg] \\
&= (I + uu^T) \Bigg[I - \frac{uu^T}{p} + \frac{uu^T}{p\sqrt{(1+p)}} \\
&- \frac{uu^T}{p} + \frac{uu^T}{p} - \frac{uu^T}{p\sqrt{(1+p)}} \\
&+ \frac{uu^T}{p\sqrt{(1+p)}} - \frac{uu^T}{p\sqrt{(1+p)}} + \frac{uu^T}{p(1+p)} \Bigg] \\
&= (I + uu^T) \left[I - uu^T \left(\frac{1}{p} - \frac{1}{p(1+p)} \right) \right] \\
&= (I + uu^T) \left[I - \frac{uu^T}{1+p} \right] \\
&= I + uu^T - \frac{uu^T}{1+p} - \frac{puu^T}{1+p} \\
&= I
\end{aligned}$$

proving the lemma. ∎

Let H defined as in (12) and $Q = \sqrt{p+1}[H^{-1/2}, -H^{-1/2}u]$. Let the pertubations d_n be the columns of Q.

Lemma 2. *The perturbations d_n chosen as columns of Q satisfy properties (P1) and (P2).*

Proof. Let $P = p+1$. Observe that as n goes through one cycle from 1 to $p+1$ we have $\sum_{n=1}^{p+1} d_n d_n^T = QQ^T$ and $\sum_{n=1}^{p+1} d_n = Q \begin{bmatrix} u \\ 1 \end{bmatrix}$. Now it is enough to show that $\sum_{n=s+1}^{s+P} D_n = 0$

and $\sum_{n=s+1}^{s+P} d_n = 0$ i.e., $Q \begin{bmatrix} u \\ 1 \end{bmatrix} = 0$ for the choice $s = 0$. Consider

$$\begin{aligned}
\sum_{n=0}^{P} D_n &= \sum_{n=1}^{P} (d_n d_n^T - I) \\
&= \sum_{n=1}^{P} (d_n d_n^T) - (p+1)I \\
&= QQ^T - (p+1)I \\
&= (p+1)([H^{-1/2}, -H^{-1/2}u][H^{-1/2}, -H^{-1/2}u]^T - I) \\
&= (p+1)([H^{-1} + H^{-1/2}uu^T H^{-1/2}] - I) \\
&= (p+1)(H^{-1/2}(I + uu^T)H^{-1/2} - I) \\
&= (p+1)(H^{-1/2}(H)H^{-1/2} - I) \\
&= 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
Q \begin{bmatrix} u \\ 1 \end{bmatrix} &= \sqrt{p+1}([H^{-1/2}, -H^{-1/2}u]) \begin{bmatrix} u \\ 1 \end{bmatrix} \\
&= \sqrt{p+1}(H^{-1/2}u - H^{-1/2}u) \\
&= 0
\end{aligned}$$

proving the lemma. □

Lemma 3. *Given any fixed integer $P > 0$, $\|\theta_{m+k} - \theta_m\| \to 0$ w.p.1, as $m \to \infty$, for all $k \in \{1, \cdots, P\}$*

Proof. Fix a $k \in \{1, \cdots, P\}$. Now

$$\begin{aligned}
\theta_{n+k} = \theta_n &- \sum_{j=n}^{n+k-1} a_j \left(\frac{J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)}{2\delta_j} \right) d_j \\
&- \sum_{j=n}^{n+k-1} a_j M_{j+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\theta_{n+k} - \theta_n\| \leq &\sum_{j=n}^{n+k-1} a_j \left| \frac{J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j)}{2\delta_j} \right| \|d_j\| \\
&+ \sum_{j=n}^{n+k-1} a_j \|M_{j+1}\|
\end{aligned}$$

Now clearly,

$$N_n = \sum_{j=0}^{n-1} a_j M_{j+1}, n \geq 1,$$

forms a martingale sequence with respect to the filtration $\{\mathcal{F}_n\}$. Further, from the assumption (A6) we have,

$$\begin{aligned}
\sum_{m=0}^{n} \mathbb{E}[\|N_{m+1} - N_m\|^2 | \mathcal{F}_m] &= \sum_{m=0}^{n} \mathbb{E}[a_m^2 \|M_{m+1}\|^2 | \mathcal{F}_m] \\
&\leq \sum_{m=0}^{n} a_m^2 K(1 + \|\theta_m\|^2)
\end{aligned}$$

From the assumption (A4), the quadratic variation process of $N_n, n \geq 0$ converges almost surely. Hence by the martingale convergence theorem, it follows that $N_n, n \geq 0$ converges almost surely. Hence $\| \sum_{j=n}^{n+k-1} a_j M_{j+1} \| \to 0$ almost surely as $n \to \infty$. Moreover

$$\begin{aligned}
&\left\| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j) \Big) d_j \right\| \\
&\leq \left| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j) \Big) \right| \|d_j\| \\
&\leq C \Big(|J(\theta_j + \delta_j d_j)| + |J(\theta_j - \delta_j d_j)| \Big),
\end{aligned}$$

since $\|d_j\| \leq C, \forall j \geq 0$. Note that

$$\begin{aligned}
|J(\theta_j + \delta_j d_j)| - |J(0)| &\leq |J(\theta_j + \delta_j d_j) - J(0)| \\
&\leq \hat{B} \|\theta_j + \delta_j d_j\|,
\end{aligned}$$

where $\hat{B}$ is the Lipschitz constant of the function $J(.)$. Hence,

$$|J(\theta_j + \delta_j d_j)| \leq \tilde{B}(1 + \|\theta_j + \delta_j d_j\|),$$

for $\tilde{B}$ =max$(|J(0)|, \hat{B})$. Similarly,

$$|J(\theta_j - \delta_j d_j)| \leq \tilde{B}(1 + \|\theta_j - \delta_j d_j\|).$$

From assumption (A1), it follows that

$$\sup_j \left\| \Big(J(\theta_j + \delta_j d_j) - J(\theta_j - \delta_j d_j) \Big) d_j \right\| \leq \tilde{K} < \infty,$$

for some $\tilde{K} > 0$. Thus,

$$\|\theta_{n+k} - \theta_n\| \leq \tilde{K} \sum_{j=n}^{n+k-1} \frac{a_j}{\delta_j} + \| \sum_{j=n}^{n+k-1} a_j M_{j+1} \|$$

$$\to 0 \text{ a.s. with } n \to \infty$$

proving the lemma. □

Lemma 4.

For any $m \geq 0$,

$$\left\| \sum_{n=m}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) \right\| \to 0 \textit{ and}$$

$$\left\| \sum_{n=m}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) \right\| \to 0, \textit{almost surely, as } m \to \infty.$$

Proof. From lemma 3, it can be seen that $\|\theta_{m+s} - \theta_m\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Also from assumption (A1), we have $\|\nabla J(\theta_{m+s}) - \nabla J(\theta_m)\| \to 0$ as $m \to \infty$, for all $s = 1, \cdots, P$. Now from lemma 2, $\sum_{n=m}^{m+P-1} D_n = 0$

$\forall m \geq 0$. Hence $D_m = -\sum\limits_{n=m+1}^{m+P-1} D_n$. Thus we have,

$$
\begin{aligned}
&\Big\| \sum_{n=m}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) + D_m \nabla J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{a_n}{a_m} D_n \nabla J(\theta_n) - \sum_{n=m+1}^{m+P-1} D_n \nabla J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} D_n \Big(\frac{a_n}{a_m} \nabla J(\theta_n) - \nabla J(\theta_m) \Big) \Big\| \\
&\leq \bar{C} \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{a_n}{a_m} \nabla J(\theta_n) - \nabla J(\theta_m) \Big) \Big\| \\
&= \bar{C} \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{a_n}{a_m} - 1 \Big) \nabla J(\theta_n) + \Big(\nabla J(\theta_n) - \nabla J(\theta_m) \Big) \Big\| \\
&\leq \bar{C} \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{a_n}{a_m} - 1 \Big) \nabla J(\theta_n) \Big\| + \Big\| \nabla J(\theta_n) - \nabla J(\theta_m) \Big\|
\end{aligned}
$$

The claim now follows from assumptions (A1) and (A2). Now observe that $\|J(\theta_{m+k}) - J(\theta_m)\| \to 0$ as $m \to \infty$, for all $k \in \{1, \cdots, P\}$ as a consequence of (A1) and lemma3. Moreover from $d_m = -\sum_{n=m+1}^{m+P-1} d_n$ we have

$$
\begin{aligned}
&\Big\| \sum_{n=m}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) + d_m J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} \frac{b_n}{b_m} d_n J(\theta_n) - \sum_{n=m+1}^{m+P-1} d_n J(\theta_m) \Big\| \\
&= \Big\| \sum_{n=m+1}^{m+P-1} d_n \Big(\frac{b_n}{b_m} J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&\leq \sum_{n=m+1}^{m+P-1} \|d_n\| \Big\| \Big(\frac{b_n}{b_m} J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&\leq C \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{b_n}{b_m} J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&= C \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{b_n}{b_m} - 1 \Big) J(\theta_n) + \Big(J(\theta_n) - J(\theta_m) \Big) \Big\| \\
&\leq C \sum_{n=m+1}^{m+P-1} \Big\| \Big(\frac{b_n}{b_m} - 1 \Big) J(\theta_n) \Big\| + \Big\| \Big(J(\theta_n) - J(\theta_m) \Big) \Big\|
\end{aligned}
$$

The claim now follows as a consequence of assumptions (A1) and (A2). □

Theorem 5. *$\theta_n, n \geq 0$ obtained from 1RDSA-2C satisfy $\theta_n \to G$ almost surely.*

Proof. Note that

$$
\theta_{n+P} = \theta_n - \sum_{l=n}^{n+P-1} a_l \Big(\frac{J(\theta_l + \delta_l d_l) - J(\theta_l - \delta_l d_l)}{2\delta_l} d_l + M_{l+1} \Big).
$$

It follows that

$$
\begin{aligned}
\theta_{n+P} = \theta_n &- \sum_{l=n}^{n+P-1} a_l \nabla J(\theta_l) \\
&- \sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) \\
&- \sum_{l=n}^{n+P-1} a_l \xi^1(\delta_l) - \sum_{l=n}^{n+P-1} a_l M_{l+1}
\end{aligned}
$$

Now the third term on the RHS can be written as

$$
a_n \sum_{l=n}^{n+P-1} \frac{a_l}{a_n} D_l \nabla J(\theta_l) = a_n \xi_n,
$$

where $\xi_n = o(1)$ from lemma 4. Thus, the algorithm is asymptotically analogous to

$$
\theta_{n+1} = \theta_n - a_n (\nabla J(\theta_n) + o(\delta) + M_{n+1}).
$$

Hence from chapter 2 of [10] we have that $\theta_n, n \geq 0$ converge to local minima of the function J. □

Theorem 6. *$\theta_n, n \geq 0$ obtained from 1RDSA-1C satisfy $\theta_n \to G$ almost surely.*

Proof. Note that

$$
\theta_{n+P} = \theta_n - \sum_{l=n}^{n+P-1} a_l \Big(\frac{J(\theta_l + \delta_l d_l)}{2\delta_l} \Big) d_l - \sum_{l=n}^{n+P-1} a_l M_{l+1}.
$$

It follows that

$$
\begin{aligned}
\theta_{n+P} = \theta_n &- \sum_{l=n}^{n+P-1} a_l \nabla J(\theta_l) \\
&- \sum_{l=n}^{n+P-1} a_l \frac{J(\theta_l)}{\delta_l} d_l - \sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) \\
&- \sum_{l=n}^{n+P-1} a_l O(\delta_l) - \sum_{l=n}^{n+P-1} a_l M_{l+1}
\end{aligned}
$$

Now we observe that

$$
\begin{aligned}
\sum_{l=n}^{n+P-1} a_l \frac{J(\theta_l)}{\delta_l} d_l &= \sum_{l=n}^{n+P-1} b_l J(\theta_l) d_l \\
&= b_n \sum_{l=n}^{n+P-1} \frac{b_l}{b_n} \frac{J(\theta_l)}{\delta_l} d_l = b_n \xi_n^1,
\end{aligned}
$$

where $\xi_n^1 = o(1)$ by lemma 4. Similarly

$$
\sum_{l=n}^{n+P-1} a_l (d_l d_l^T - I) \nabla J(\theta_l) = a_n \xi_n^2,
$$

with $\xi_n^2 = o(1)$ by lemma 4. The rest follows as explained in Theorem 5. □

V. SIMULATION EXPERIMENTS

A. Implementation

We compare the performance of 1SPSA-2R, 1SPSA-2H and 1RDSA-2C in the case of two simulation algorithms. In the case of one simulation algorithms we compare the performance of 1SPSA-1R, 1SPSA-1H and 1RDSA-1C. We chose i.i.d Bernoulli ± 1-valued perturbations for 1SPSA-2R and 1SPSA-1R. [1]

For the empirical evaluations, we use the following two loss functions in $p = 10$ dimensions:

a) Quadratic loss:

$$J(\theta) = \theta^{\mathsf{T}} A \theta + b^{\mathsf{T}} \theta. \tag{15}$$

For this particular choice of p the optimum θ^* for the above J is a 10×1 column vector with each entry equal to -0.9091, and $J(\theta^*) = -4.55$.

b) Fourth-order loss:

$$J(\theta) = \theta^{\mathsf{T}} A^{\mathsf{T}} A \theta + 0.1 \sum_{j=1}^{N} (A\theta)_j^3 + 0.01 \sum_{j=1}^{N} (A\theta)_j^4. \tag{16}$$

The optimum θ^* for the above J is $\theta^* = 0$, with $J(\theta^*) = 0$.

In both functions, A is such that pA is an upper triangular matrix with each nonzero entry equal to one, b is the N-dimensional vector of ones and the noise structure is similar to that used in [11]. For any θ, the noise is $[\theta^{\mathsf{T}}, 1]z$, where $z \approx \mathcal{N}(0, \sigma^2 I_{11 \times 11})$. We perform experiments for noisy as well as noise-less settings, with $\sigma = 0.01$ for the noisy case. For all algorithms, we chose step sizes to have the form $\delta_n = c/(n+1)^{\gamma}$ and $a_n = 1/(n+B+1)^{\alpha}$. We set $\alpha = 0.602$ and $\gamma = 0.101$. These values for α and γ have been used before (see [11]) and have demonstrated good finite-sample performance empirically, while satisfying the theoretical requirements needed for asymptotic convergence. For all the algorithms, the initial point θ_0 is the p-dimensional vector of ones.

B. Results

We use Normalized Mean Square Error (NMSE) as the performance metric for evaluating the algorithms. NMSE is the ratio $\left\|\theta_{n_{\text{end}}} - \theta^*\right\|^2 / \left\|\theta_0 - \theta^*\right\|^2$. Here n_{end} denotes the iteration number at the end of simulation budget.

Tables I–II present the normalized mean square values observed for the three algorithms - 1SPSA-2R, 1SPSA-2H and 1RDSA-2C for the quadratic and fourth-order loss functions respectively. Tables III–IV present the NMSE values observed for the three algorithms - 1SPSA-1R, 1SPSA-1H and 1SPSA-1C with quadratic and fourth order loss functions respectively. The results in Table I are obtained after 2000 simulations and the results in Table II are obtained after 10000 simulations. The results in Tables –III–IV are obtained after running the one simulation algorithms with a budget of 20000 simulations.

Figures 2 and 3 show plots of $\log_{10}$(NMSE) as a function of number of iterations with quadratic and fourth-order loss objectives respectively, with $\sigma = 0.01$, when 2-simulation methods are used.

[1]The implementation is available at `https://github.com/cs1070166/1RDSA-2Cand1RDSA-1C/`

TABLE I: NMSE values of two simulation methods for quadratic objective (15) with and without noise for 2000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA-2R	$5.762 \times 10^{-3} \pm 2.473 \times 10^{-3}$
1SPSA-2H	$4.012 \times 10^{-5} \pm 1.654 \times 10^{-5}$
1RDSA-2C	$2.188 \times 10^{-5} \pm 9.908 \times 10^{-6}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA-2R	$5.755 \times 10^{-3} \pm 2.460 \times 10^{-3}$
1SPSA-2H	$1.601 \times 10^{-5} \pm 2.724 \times 10^{-20}$
1RDSA-2C	$2.474 \times 10^{-8} \pm 1.995 \times 10^{-23}$

TABLE II: NMSE values of two simulation methods for fourth order objective (16) with and without noise for 10000 simulations: standard error from 100 replications shown after $\pm$ symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA-1R	$2.762 \times 10^{-2} \pm 1.415 \times 10^{-2}$
1SPSA-1H	$3.958 \times 10^{-3} \pm 4.227 \times 10^{-4}$
1RDSA-1C	$3.598 \times 10^{-3} \pm 4.158 \times 10^{-4}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA-1R	$2.747 \times 10^{-2} \pm 1.413 \times 10^{-2}$
1SPSA-1H	$3.901 \times 10^{-3} \pm 4.359 \times 10^{-18}$
1RDSA-1C	$3.535 \times 10^{-3} \pm 1.743 \times 10^{-18}$

Figures 4 and 5 plot the $\log_{10}$(NMSE) as a function of the iterations with quadratic and fourth-order loss objectives respectively, with $\sigma = 0.01$, when 1-simulation methods are used.

From the results in Tables I,II, III,IV and plots 2,3,4, 5, we make the following observations:

Observation1 In the case of two simulation algorithms, 1RDSA-2C is slightly better than 1SPSA-2H, while both of them outperform 1SPSA-2R.

Observation2 In the case of one simulation algorithms, 1RDSA-1C is better than both 1SPSA-1H and 1SPSA-1R.

TABLE III: NMSE values of one simulation methods for quadratic objective (15) with and without noise for 20000 simulations: standard error from 100 replications shown after ± symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA-1R	$8.582 \times 10^{-2} \pm 3.691 \times 10^{-2}$
1SPSA-1H	$2.774 \times 10^{-2} \pm 2.578 \times 10^{-4}$
1RDSA-1C	$8.225 \times 10^{-3} \pm 5.959 \times 10^{-5}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA-1R	$8.584 \times 10^{-2} \pm 3.681 \times 10^{-2}$
1SPSA-1H	$2.770 \times 10^{-2} \pm 3.836 \times 10^{-17}$
1RDSA-1C	$8.225 \times 10^{-3} \pm 1.569 \times 10^{-17}$

TABLE IV: NMSE values of one simulation methods for fourth order objective (16) with and without noise for 20000 simulations: standard error from 100 replications shown after ± symbol

Noise parameter $\sigma = 0.01$	
Method	**NMSE**
1SPSA-1R	$3.240 \times 10^{-1} \pm 1.836 \times 10^{-1}$
1SPSA-1H	$8.916 \times 10^{-2} \pm 1.896 \times 10^{-2}$
1RDSA-1C	$4.972 \times 10^{-2} \pm 9.812 \times 10^{-3}$
Noise parameter $\sigma = 0$	
Method	**NMSE**
1SPSA-1R	$3.192 \times 10^{-1} \pm 1.991 \times 10^{-1}$
1SPSA-1H	$8.173 \times 10^{-2} \pm 1.255 \times 10^{-16}$
1RDSA-1C	$4.403 \times 10^{-2} \pm 9.066 \times 10^{-17}$

VI. CONCLUSIONS

We presented a novel construction of deterministic perturbations for 1RDSA-2R and 1RDSA-1R algorithms and showed that the resulting algorithms 1RDSA-2C and 1RDSA-1C are provably convergent. The advantage with our deterministic perturbation construction is that the same set of perturbations can be used for both two simulation and one simulation variants. These perturbations also have a smaller cycle length compared to Hadmard matrix based perturbations. Numerical experiments demonstrated that 1RDSA-2C (1RDSA-1C) outperforms 1SPSA-2R (1SPSA-1R) and 1SPSA-2H (1SPSA-1H). As future work, it would be interesting to derive similar methods in the context of 2RDSA. A challenging future direction would be to derive weak convergence results when deterministic perturbations are used.

Fig. 2: $\log_{10}$(NMSE)vs No of iterations for quadratic objective (15) with noise ($\sigma = 0.01$) using 2 simulation methods.

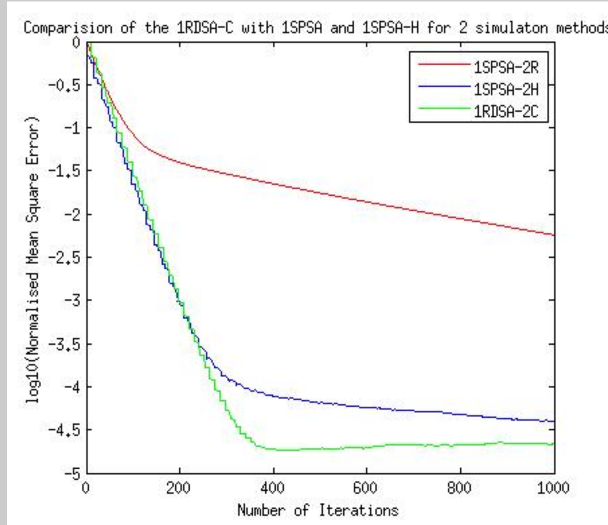

Fig. 3: $\log_{10}$(NMSE)vs No of iterations for fourth order objective objective (16) with noise ($\sigma = 0.01$) using 2 simulation methods.

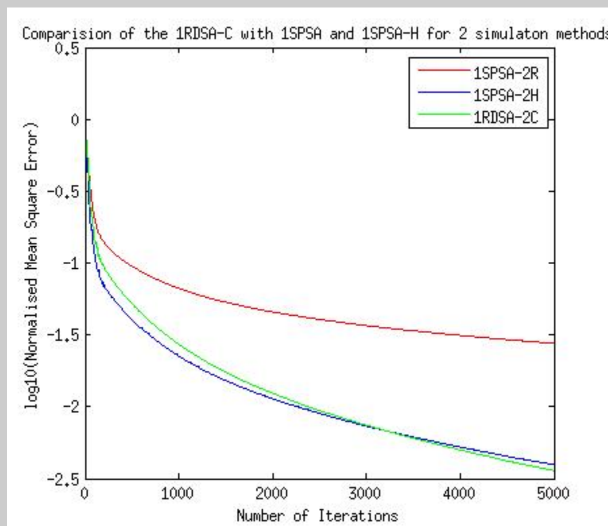

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Fig. 4: $\log_{10}$(NMSE)vs No of iterations for quadratic order objective objective (15) with noise ($\sigma = 0.01$) using 1 simulation methods.

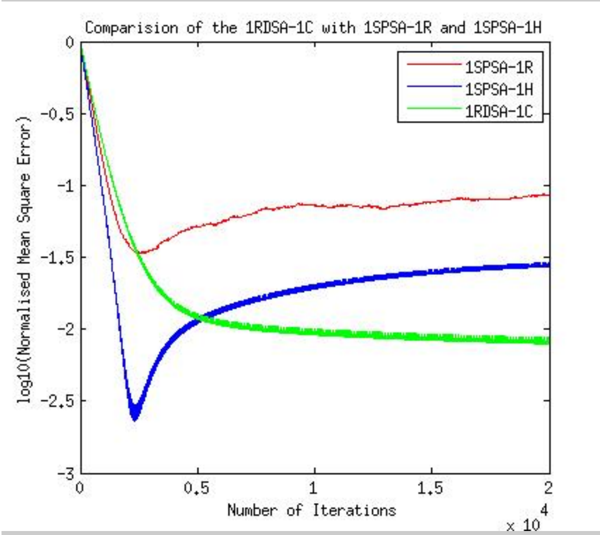
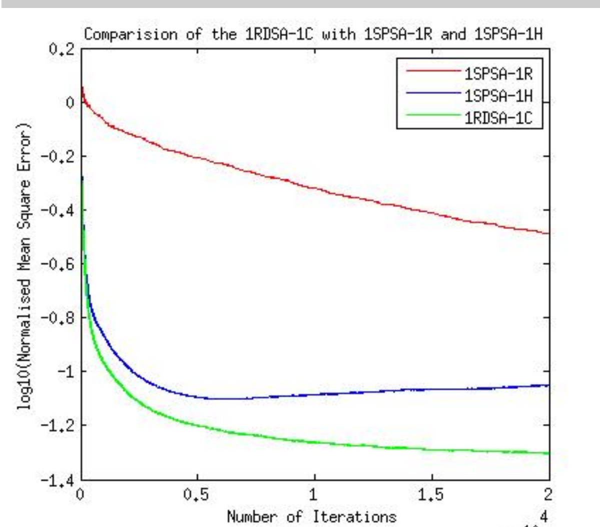


Fig. 5: $\log_{10}$(NMSE)vs No of iterations for fourth order objective objective (16) with noise ($\sigma = 0.01$) using 1 simulation methods.



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