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अनुक्रमांक Entry No.		अनुवर्ती पुस्तिका संख्या CONTINUATION BOOK No.	
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यह अनुवर्ती पुस्तिका तभी जांची जायेगी जब सभी उपरोक्त प्रविष्टियाँ पूर्ण होंगी। सभी पृष्ठों पर लिखें।
This continuation book will be evaluated only if all the above entries are complete. Write on all pages.

1. Let the projector project from $\mathbb{R}^m$ to subspace $\mathbb{R}^n$ (W).
Now let us assume that we have two projections of a vector v onto the subspace W which are w_1 and w_2.

$w_1 \in W$ and $w_2 \in W \Rightarrow w_1 - w_2 \in W$

$w_1 = P_1 v \qquad w_2 = P_2 v \qquad P_1, P_2$ are two projectors

$(w_1 - v) \in W^\perp$

$(w_2 - v) \in W^\perp$

$\Rightarrow (w_1 - w_2) \in W^\perp$ as $W^\perp$ is a subspace (by closure)

But $(w_1 - w_2) \in W$ contradiction

So $w_1 = w_2$ for ~~all~~ any v

Thus $P_1 = P_2$ as ~~they~~ all the images are same for any ~~data~~ element in $\mathbb{R}^m$.

Thus orthogonal projector is unique which projects to subspace W.

2. $Ax = b$ is the problem we wanted to solve.
But since $b \notin \text{Range}(A)$ we introduced an error term r s.t.
$b + r \in \text{Range}(A)$ and then we had to minimize r.

$\min \|r\|_2 \qquad Ax = b + r$

$\Rightarrow \min \|Ax - b\|$

Here we assumed error in b only ~~hence~~

P

$[A|b]$ is $m \times (n+1)$ matrix. We wanted this to be n rank matrix with $b \in \text{span} \langle a_1, \ldots, a_n \rangle$ a_i's are columns of A.

i.e. $[A|(b+r)]$ to be rank n matrix.

Now we assume that error can be in A as well as b so we want rank $[(A+E)|(b+r)] = n$ s.t. $b+r \in \text{span} \langle x_1, \ldots, x_n \rangle$ x_i's are columns of $A+E$.

E is error term in A. $E_{m \times n}$, $A_{m \times n}$, $b_{m \times 1}$, $r_{m \times 1}$, $X_{n \times 1}$

Total error is then $\|[E|r]\|_2$. Thus the problem is

$$\min_{b+r \in \text{Range}(A+E)} \|[E|r]\|_2^2$$

$$[A|b] = [U_A|U_b] \begin{bmatrix} \Sigma_A & 0 \\ 0 & \Sigma_b \end{bmatrix} \begin{bmatrix} V_{AA} & V_{Ab} \\ V_{bA} & V_{bb} \end{bmatrix}^T$$ be the SVD of $[A|b]$

To make $b+r \in$ colspace $(A+E)$

we set the singular values corresponding to $b = 0$

$$[(A+E)|(b+r)] = [U_A|U_b] \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{AA} & V_{Ab} \\ V_{bA} & V_{bb} \end{bmatrix}^T$$

$$\Rightarrow [E|r] = -[U_A|U_b] \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_b \end{bmatrix} \begin{bmatrix} V_{AA} & V_{Ab} \\ V_{bA} & V_{bb} \end{bmatrix}^T$$

$$= -U_b \Sigma_b \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix}^T = -[A|b] \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix} \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix}^T$$

$\Rightarrow [E|r] \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix} = -[A|b] \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix}$ Right Multiplying both sides by $\begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix}$

$$\Rightarrow [(A+E)|(b+r)] \begin{bmatrix} V_{Ab} \\ V_{bb} \end{bmatrix} = 0$$

$$\Rightarrow (A+E) V_{Ab} = (b+r) V_{bb}$$

$\Rightarrow (A+E)(-V_{Ab} V_{bb}^{-1}) = b+r$ Right multiply both sides by V_{bb}^{-1} (which is invertible).

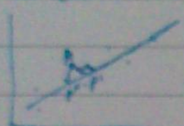
$$\Rightarrow X = -V_{Ab} V_{bb}^{-1}$$

Thus by starting with best rank n approximation of $[A|b]$ we get $X = -V_{Ab} V_{bb}^{-1}$ by assuming error in both A and b.

This solution will always exist as all steps are always possible in proof. One thing is that since b was $m \times 1$ so V_{bb} was 1×1 so always invertible. However if V_{bb} is not invertible in general case of b, as $m \times k$ then this solution would not exist.

In line fitting $A = [e \ x] = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}_{n\times 2}$ $X = \begin{bmatrix} c \\ m \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$

Problem is to solve $AX = Y$. When we allow error in A as well as Y to be allowed we consider error $= \|AX - Y\|_2$ with perpendicular distance



So error is $P-P'$ and not $P-P''$.

3-(a) $F = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}$ $J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ $c = \cos\theta$ $s = \sin\theta$.

J is a rotation matrix which rotates x, y by θ angle in clockwise direction.
As for $\theta = 90°$ $J\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$F = \begin{bmatrix} -c & s \\ s & c \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{B}$ A is reflection along y axis
B is rotation anti-clockwise through θ.

So F is rotation in anticlockwise by θ then reflection along y axis.

(b) In Householder method of QR factorisation, we take a vector x and apply Q on it such that $Qx = \begin{bmatrix} \|x\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Here Q is a <u>reflection</u> transformation about an axis such that all but first dimension disappear.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ now we construct $Q_1 = \begin{bmatrix} \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{(n-2)\times(n-2)} & \underline{0} \\ \underline{0} & \begin{matrix} c & -s \\ s & c \end{matrix} \end{bmatrix}_{n\times n}$ here $c = \cos\theta$ $s = \sin\theta$
$\theta = \arctan\left(-\frac{x_n}{x_{n-1}}\right)$

Now we notice that $Q_1 X = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ 0 \end{bmatrix}_{n\times 1} = X'$. Continuing this way

$Q_2 = \begin{bmatrix} I_{n-3} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & \underline{0} & & 0 \end{bmatrix}_{n\times n}$ $Q_2 X' = \begin{bmatrix} x_1'' \\ \vdots \\ x_{n-2}'' \\ 0 \\ 0 \end{bmatrix}_{n\times 1}$ i.e. $Q_2 Q_1 X = \begin{bmatrix} x_1'' \\ \vdots \\ x_{n-2}'' \\ 0 \\ 0 \end{bmatrix}$

We go on like this to construct $Q = Q_{n-1} Q_{n-2} \dots Q_2 Q_1$ s.t.
$QX = \begin{bmatrix} \|X\| \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$ only 1st dimension remains.

Once this is done then we use Householder method to find R ~~[illegible]~~ of $A = QR$.

$$A = \left[a_1 \,|\, \cdots \,|\, a_n \right]$$

Find Q_1 s.t. $Q_1 a_1 = \begin{bmatrix} \|a_1\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and apply to A.

$$Q_1 A = \left[\begin{matrix} \|a_1\|_2 \\ 0 \\ \vdots \\ 0 \end{matrix} \,\middle|\, a_2' \,\middle|\, a_3' \,\middle|\, \cdots \,\middle|\, a_n' \right]$$

Next we take a_2' and do similar steps except that instead of diminishing last $n-1$ dimensions we diminish last $n-2$. Let the transformation matrix we found be Q_2 then

$$Q_2 Q_1 A = \left[\begin{matrix} \|a_1\|_2 \\ 0 \\ \vdots \\ 0 \end{matrix} \,\middle|\, \begin{matrix} \alpha \\ \beta \\ 0 \\ 0 \end{matrix} \,\middle|\, a_3'' \,\middle|\, \cdots \,\middle|\, a_n'' \right]$$

where $\sqrt{\alpha^2 + \beta^2} = \|a_2'\|_2 = \|a_2\|_2$

We continue this way to get $Q_1, Q_2, \ldots, Q_{n-1}$

Then $Q_{n-1} Q_{n-2} \cdots Q_2 Q_1 A = R$ where R is upper triangular matrix.

Also $Q' = Q_{n-1} Q_{n-2} \cdots Q_2 Q_1$ is orthogonal matrix which applied nothing but rotations.

$A = Q'^{-1} R$ Let $Q = Q'^{-1}$

$\Rightarrow A = QR$ Q is also orthogonal matrix.

So we used Householder algorithm and just modified the way we zero out the entries in A to get R.

This gives us QR factorisation of matrix A.