CSL 356, Tutorial Sheet 5

1. Show how to implement Lagrange's interpolation formula in $O(n^2)$ operations.

2. Describe an efficient algorithm to evaluate a degree n univariate polynomial $P(x)$ at n *arbitrary* points $x_1, x_2 \ldots x_n$ (not necessarily roots of unity). You may assume that polynomial division takes the same asymptotic time as polynomial multiplication.
   Hint : Use the following observation which is similar to the remainder theorem. Let $D_{i,j} = \Pi(x - x_i)(x - x_{i+1}) \cdots (x - x_j)$ and let $P(x) = Q_{1,1}(x)D_{1,n} + R_{1,1}(x)$. Then $P(x_i) = R_{1,1}(x_i)$ where the degree of $R_{1,1}$ is less than $D_{1,n}$. To compute $R_{1,1}(x_i)$, we can apply a similar decomposition, once with $D_{1,n/2}$ and $D_{n/2+1,n}$ recursively. This defines a tree where at each node we do a polynomial division (of degree $n/2^i$ at distance i from the root). At the leaf nodes, we have the answers.

3. Let y be a string of length m. A substring x of y is called a period of y if $y = (x^k)x'$, where (x^k) is the string x repeated k times and x' is a prefix of x. The period is the shortest period of y. Design an efficient algorithm to determine the period of a string of length n.
   Hint: Prove that a string X is a period of a string Y iff Y is a prefix of XY

4. If p and q are periods (not the shortest) and $|p| + |q| < m$ then there is a period of length $|p| - |q|$ (assuming p is larger).
   (This is the equivalent of Euclid's algorithm for strings).

5. Consider an intuitive extension to KMP algorithm for handling wild-card (single character) in pattern. To compute the failure function, we can assign $f(i) = f(i-1) + 1$ where $X_i = *$ since $*$ can match any character. Similarly if $f(j-1) = i-1$, then $f(j) = i$ as X_j can also match $X_i = *$.
   Now argue if this is consistent.
   Hint: Consider the pattern $a \cdot b \cdot a \cdot * \cdot a \cdot a \cdot b \cdot a$.

6. **Graph theory**
   (i) Show that in any graph there are at least two vertices of the same degree.
   (ii) Given a degree sequence $d_1, D_2 \ldots d_n$ such that $\sum_i d_i = 2n - 2$, construct a tree whose vertices have the above degrees.
   (iii) Show that in a complete graph of six vertices where edges are colored red or blue, there is either a red or a blue triangle.

7. Given a directed acyclic graph, that has maximal path length k, design an efficient algorithm that partitions the vertices into $k+1$ sets such that there is no path between any pair of vertices in a set.

8. Given an undirected graph, describe an algorithm to determine if it contains an even-length cycle. Can you do the same for odd-length cycle ?

9. Given an undirected connected graph G, define the Biconnected Component Graph H as follows. For each BCC of G and articulation point of G, there is a vertex in H. There is an edge between vertices x and y in H if x is articulation point in the BCC y.
   (a) Prove that H is a tree.
   (b) Using H (or otherwise), design an efficient algorithm that adds the minimal number of edge to G to make it biconnected.

10. A directed graph is Eulerian if the in degree equals out degree for every vertex. Show that an Eulerian graph admits a tour where every edge is visited exactly once. Design an efficient (linear time) algorithm to find such a tour.

11. An n-vertex undirected graph is a scorpion if it has a vertex of degree 1 (the string) connected to a vertex of degree 2 (the tail) connected to a vertex of degree $n-2$ (the body) which is connected to the remaining $n-3$ vertices (the feet). Some of the feet may be connected among themselves. Give an $O(n)$ algorithm to check if a given $n \times n$ adjacency matrix represents a scorpion.

12. * Instead of a DFS tree, starting from an arbitrary spanning tree, redesign the bi-connectivity algorithm. Your algorithm should run in linear time.

13. Given an undirected graph, orient the edges so that the resulting graph is strongly connected. When is it possible ? Design a linear time algorithm for this problem.

14. Find a maximum subgraph of $G = (V, E)$ that has degrees of each vertex is at least k.

15. Describe an efficient algorithm to find the *girth* of a given undirected graph. The *girth is defined as the length of the smallest cycle.*