fo hw01

cs

October 2025

1 Question 5

1.1 5.1

We first show that the epigraph of the following function is a polyhedron:

$$f(x) = \max_{i=1,\ldots,m} \{(a^i)^T x + b_i\}.$$

Using the definition of the epigraph set:

$$\text{epi}(f) = \{(x, z) \in \text{dom}(f) \times \mathbb{R} \mid z \geq f(x)\}$$

we see that the condition $z \geq f(x)$ unpacks into:

$$z \geq \max_i \{(a^i)^T x + b_i\}$$

which in turn defines a sequence of inequalities:

$$\begin{gathered} z \geq (a^1)^T x + b_1 \\ z \geq (a^2)^T x + b_2 \\ \cdots \\ z \geq (a^m)^T x + b_m. \end{gathered}$$

Specifying more than one condition requires the intersection of their respective partitions; each inequality of linear form defines a halfspace, and so we have an intersection of a finite number of halfspaces, exactly the definition of a polyhedron.

1.2 5.2

We can use the fact that the function f is convex if and only if the epigraph of f, epi(f), is convex (this is proposition 3.1 in the notes). Yet we have shown previously that epi(f) is a polyhedron, and all polyhedrons are by default convex (remark 6.2). We proceed to verify this for our epi(f). We need that $\forall \lambda \in [0, 1]$ and $\forall (x, t), (y, s) \in \text{epi}(f)$, that $\lambda(x, t) + (1 - \lambda)(y, s) \in \text{epi}(f)$, or that:

$$\lambda t + (1 - \lambda)s \geq f\Big(\lambda x + (1 - \lambda)y\Big)$$

which after substitution, our goal unpacks into:

$$\lambda t + (1 - \lambda)s \geq \max_i \{\lambda(a^i)^T x + (1 - \lambda)(a^i)^T y + b_i\}$$

$$\lambda t + (1 - \lambda)s \geq \max_i \{\lambda\Big((a^i)^T x + b_i\Big) + (1 - \lambda)\Big((a^i)^T y + b_i\Big)\}.$$

This form is useful, since we already know that:

$$t \geq \max_i \{(a^i)^T x + b_i\}$$

$$s \geq \max_i \{(a^i)^T y + b_i\}$$

and using the fact that $\max a + \max b \geq \max{(a + b)}$:

$$\begin{aligned} \lambda t + (1 - \lambda)s &\geq \lambda \max_i \{(a^i)^T x + b_i\} + (1 - \lambda) \max_i \{(a^i)^T y + b_i\} \\ &\geq \max_i \{\lambda\Big((a^i)^T x + b_i\Big) + (1 - \lambda)\Big((a^i)^T y + b_i\Big)\} \end{aligned}$$

which is exactly our initial goal. Therefore epi(f) is convex, and so f is a convex function.