nla hw02

cs

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1 Question 1

1.1 (i)

Let $0 < \epsilon \leq 1$, and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

We compute the condition numbers in 1, 2, and ∞-norms. So:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\epsilon \end{bmatrix}.$$

The 1-norm is max column sum:

$$||\mathbf{A}||_1 = \max\{1, \epsilon\} = 1$$

$$||\mathbf{A}^{-1}||_1 = \max\{1, 1/\epsilon\} = 1/\epsilon$$

while the 2-norm is the square root of the eigenvalue of $\mathbf{A}^T\mathbf{A}$ with the largest magnitude:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix}$$

$$||\mathbf{A}||_2 = 1$$

$$(\mathbf{A}^{-1})^T\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\epsilon^2 \end{bmatrix}$$

$$||\mathbf{A}^{-1}||_2 = 1/\epsilon.$$

Finally the ∞-norm is the max row sum:

$$||\mathbf{A}||_\infty = 1$$

$$||\mathbf{A}^{-1}||_\infty = 1/\epsilon.$$

We are then able to compute the conditon numbers:

$$\kappa_1(\mathbf{A}) = ||\mathbf{A}||_1||\mathbf{A}^{-1}||_1 = 1/\epsilon$$

$$\kappa_2(\mathbf{A}) = ||\mathbf{A}||_2||\mathbf{A}^{-1}||_2 = 1/\epsilon$$

$$\kappa_\infty(\mathbf{A}) = ||\mathbf{A}||_\infty||\mathbf{A}^{-1}||_\infty = 1/\epsilon.$$

1.2 (ii)

The condition numbers go to infinity as $\epsilon \to 0$. This is in compliance with the definition of condition numbers, since when $\epsilon \to 0$:

$$\mathbf{A} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

where matrix $\mathbf{A}$ becomes not invertible, and all $\kappa_p(\mathbf{A}) = \infty$.

2 Question 2

2.1 (i)

Let $\mathbf{I}$ be the $n \times n$ identity matrix. Show that:

$$||\mathbf{I}||_p = 1$$

for all p-norms.

By definition, we have that:

$$\begin{aligned} ||\mathbf{I}||_p &= \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{||\mathbf{I}\mathbf{x}||_p}{||\mathbf{x}||_p} \right\} \\ &= \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{||\mathbf{x}||_p}{||\mathbf{x}||_p} \right\} \\ &= 1 \end{aligned}$$

and this applies for any p.

2.2 (ii)

We show that if $\mathbf{A}$ is a $n \times n$ invertible matrix, then it's conditional number must be greater or equal to one:

$$\kappa_p(\mathbf{A}) \geq 1$$

where this is true for any p-norm.

By definition, we want to show that:

$$||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq 1$$

where by the definition of the matrix p-norm $\mathbf{y} \neq \mathbf{0}$:

$$||\mathbf{A}||_p = \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \right\}$$

$$||\mathbf{A}^{-1}||_p = \max_{\mathbf{y} \neq \mathbf{0}} \left\{ \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} \right\}$$

and so we can bound the left hand side of our goal by:

$$||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p}$$

for any non-zero $\mathbf{x}$ and $\mathbf{y}$. But because matrix $\mathbf{A}$ is invertible, it actually defines a bijective linear map between $\mathbf{x}$ and $\mathbf{y}$ if we define relation $\mathbf{A}\mathbf{x} = \mathbf{y}$, which allows us to conveniently write:

$$\begin{aligned} ||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} &= \frac{||\mathbf{y}||_p}{||\mathbf{A}^{-1}\mathbf{y}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} \\ &= 1. \end{aligned}$$