nla hw02

cs

October 7, 2025

1 Question 1

2 Question 2

2.1 (i)

2.2 (ii)

We show that if $\mathbf{A}$ is a $n \times n$ invertible matrix, then it's conditional number must be greater or equal to one:

$$\kappa_p(\mathbf{A}) \geq 1$$

where this is true for any p-norm.

By definition, we want to show that:

$$||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq 1$$

where by the definition of the matrix p-norm, $\forall \mathbf{x}, \mathbf{y} \neq \mathbf{0}$:

$$||\mathbf{A}||_p = \max_{\mathbf{x} \neq \mathbf{0}} \left\{ \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \right\}$$

$$||\mathbf{A}^{-1}||_p = \max_{\mathbf{y} \neq \mathbf{0}} \left\{ \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} \right\}$$

and so we can bound the left hand side of our goal by:

$$||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p}$$

for any non-zero $\mathbf{x}$ and $\mathbf{y}$. But because matrix $\mathbf{A}$ is invertible, it actually defines a bijective linear map between $\mathbf{x}$ and $\mathbf{y}$ if we define relation $\mathbf{A}\mathbf{x} = \mathbf{y}$, which allow us to conveniently write:

$$\begin{aligned} ||\mathbf{A}||_p ||\mathbf{A}^{-1}||_p \geq \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} &= \frac{||\mathbf{y}||_p}{||\mathbf{A}^{-1}\mathbf{y}||_p} \frac{||\mathbf{A}^{-1}\mathbf{y}||_p}{||\mathbf{y}||_p} \\ &= 1 \end{aligned}$$