Homework 1

Solution Key

This homework must be typed in LaTeX and handed in via Gradescope.

Please ensure that your solutions are complete, concise, and communicated clearly. Use full sentences and plan your presentation before you write. Except in the rare cases where it is indicated otherwise, consider every problem as asking you to prove your result.

Problem 1

Provide a simple proof (no more than 3 lines for each) of whether each of these statements is true or false:

1. $n^{2/3} \in o(n^2)$
2. $10^{1000}n \in O(n \log n)$
3. $5000n \in \omega(n)$

First, let's recall the formal definitions for big-O, little-O, and little-ω notation.

For two functions $f(x)$, $g(x)$ we say that $f(x) \in O(g(x))$ if there exist positive constants C, k such that $|f(x)| \leq C|g(x)|$ for all $x \geq k$.

We instead say that $f(x) \in o(g(x))$ if for every constant $C > 0$, there exists a constant $k > 0$ such that $|f(x) < C|g(x)|$ for all $x \geq k$.

Finally, we say that $f(x) \in \omega(g(x))$ if for every constant $C > 0$, there exists a constant $k > 0$ such that $|f(x)| > C|g(x)|$ for all $x \geq k$.

1. Fix a constant $C > 0$. Choose $k = C^{-3/4}$. For $n > k$ we have that:

$$\begin{aligned} C^{-3/4} &< n && \text{Given } n > k \\ 1 &< Cn^{4/3} && \text{Exponentiate both sides by } \frac{4}{3} \text{ and multiply by } C > 0 \\ n^{2/3} &< Cn^2 && \text{Multiply both sides by } n^{2/3} > 0 \end{aligned}$$

   Therefore, for every positive constant C, there exists a choice k such that $n^{2/3} < Cn^2$ for $n > k \implies n^{2/3} \in o(n^2)$.

   Note: If you are wondering how we chose k, try solve the problem backwards. Begin with $n^{2/3} < Cn^2$ and solve for what values n must be larger than to satisfy the inequality. Then simply reverse those steps to get the above proof!

2. Choose $C = 10^{10000}$ and $k = 10$. Then for $n > k$ we have that:

$$10^{1000}n < 10^{10000}n = 10^{10000}n \cdot 1 < 10^{10000}n \log n$$

   Thus, there exist constants $C, k > 0$ such that $10^{1000}n < Cn \log n$ for $n > k \implies 10^{1000}n \in O(n \log n)$.

3. We proceed via contradiction. Suppose that $5000n \in \omega(n)$. Thus, there exists a choice of $k > 0$ such that for all $n > k$, $5000n > 5001n$ (here we choose $C = 5001$). Therefore, we have that

$$5000(k+1) > 5001(k+1) \iff 5000k + 5000 > 5001k + 5001 \iff -1 > k$$

But, $k > 0 > -1$. Therefore, we have arrived a contradiction $\implies 5000n \notin \omega(n)$.

Problem 2

Argue whether the next statement is true or false: "Every positive integer is equal to the sum of two integer squares".

Solution: 3 cannot be written as a sum of two integer squares.

Problem 3

Provide a simple proof for the following formula using induction:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution Using induction, we want to prove the statement $P(n)$: $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for every integer $n \geq 1$.

Base Case: $P(1)$ is true since $1^2 = 1 = \frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2+1)$.

Inductive Hypothesis: For a particular k, $P(k)$ is true, that is $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{1}{6}k(k+1)(2k+1)$.

Inductive Case: We now want to prove $P(k+1)$. Consider and algebraically simplify the left hand side of the equation, $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 = \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) = \frac{1}{6}(k+1)(2k^2 + 7k + 6) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$. Hence, the statement $P(k+1)$ is true.

Therefore, the base case is true and for all k, $P(k)$ implies $P(k+1)$ so we can conclude that this statement is in fact true.

Problem 4

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions. Prove the following statement: "If $g \circ f$ is bijective, then f is injective and g is surjective".

Proving f is injective: Assume for the sake of contradiction, that $g \circ f$ is bijective but f is *not* injective. This means that for some a_1 and a_2 such that $a_1 \neq a_2$, $f(a_1) = f(a_2)$. This implies that $g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2)$. However, since $g \circ f$ is bijective, it must hold that $a_1 = a_2$, but this is a contradiction.
Proving g is surjective: Assume for the sake of contradiction, that $g \circ f$ is bijective but g is *not* surjective. This means there exists some $c_1 \in C$ such that for all $b \in B, g(b) \neq c_1$. However, since $g \circ f$ is bijective, there must be some a such that $g \circ f(a) = g(f(a)) = c_1$. Thus, since $f(a) \in B$ and $g(f(a)) = c_1$, we have a contradiction.

Problem 5

Prove by contraposition that if $5n + 5$ is an odd integer, then n is an even integer.

Contrapositive statement: if n is not an even (and therefore odd) integer, then $5n+5$ is not an odd (and thus even) integer. Since n is odd, $5n$ must also be odd because the product of two odd integers is odd. Then, $5n + 5$ must be even because the sum of two odd integers is even. Hence, we proved the contrapositive statement, thus if $5n + 5$ is odd, then n is even.