Homework 1

Due: Oct. 1, 2020

This homework must be typed in LaTeX and handed in via Gradescope.

Please ensure that your solutions are complete, concise, and communicated clearly. Use full sentences and plan your presentation before you write. Except in the rare cases where it is indicated otherwise, consider every problem as asking you to prove your result.

Problem 1

Provide a simple proof (no more than 3 lines for each) of whether each of these statements is true or false:

1. $n^{2/3} \in o(n^2)$
2. $10^{1000}n \in O(n \log n)$
3. $5000n \in \omega(n)$

First, let's recall the formal definition of Big-O notation: for two functions $f(n)$ and $g(n)$, we have that $f(n)$ is $O(g(n))$ if there exist positive constants c and k such that $0 \leq f(n) \leq cg(n)$ for all $n \geq k$.
Now, let's recall how little-o differs from Big-O: for two functions $f(n)$ and $g(n)$, we have that $f(n)$ is $o(g(n))$ if for every choice of a constant $c > 0$, we can find a constant k such that the inequality $0 <= f(n) < cg(n)$ holds for all $n > k$.
Finally, let's recall the definition of little omega: For two functions $f(n)$ and $g(n)$, we have that $f(n)$ is $\omega(g(n))$ if for every choice of a constant $c > 0$, we can find a constant k such that the inequality $f(n) > cg(n)$ holds for all $n > k$.

1. $$n^{2/3} < c * n^2$$
   $$\frac{1}{c^2} < n$$
   No matter how small c becomes, the inequality will hold as for all $n > k$ long as we choose k large enough. i.e. $k = \frac{1}{c^2} + 1$
2. $$10^{1000}n \leq c * n \log n$$
   $$\frac{10^{1000}}{c} \leq \log n$$
   We can easily satisfy the inequality for all $n > k$ by setting $c = 10^{100000}$ and $k = 1$

3.

$$5000n > c * n$$

$$5000 > c$$

Thus we see that the statement is false, as the above inequality does not hold for all values of c.

Problem 2

Argue whether the next statement is true or false: “Every positive integer is equal to the sum of two integer squares”.

Solution Counterexample: 3 cannot be written as a sum of two integer squares.

Problem 3

The Fibonacci numbers have a number of cute properties. Prove the following:

(a) F_{4n} is divisible by 3 for every $n \geq 1$

Solution We want to prove that $F_{4n} \equiv 0 \pmod 3$ is true for every integer $n \geq 1$. Using induction, our base where $n = 1$ is true since $F_{4(1)} \equiv 0 \pmod 3$ is true. Our inductive hypothesis is that for a particular k, $P(k)$ is true, that is $F_{4k} \equiv 0 \pmod 3$. We now want to prove $P(k+1)$, that is $F_{4k+4} \equiv 0 \pmod 3$. At this step, the student can use modular arithmetic to expand $F_{4k} \equiv 0 \pmod 3$ to $F_{4k+4} \equiv 0 \pmod 3$. Therefore the base case is true and for all k, $P(k)$ implies $P(k+1)$ so we can conclude that this statement is in fact true.

(b) $1 < F_{n+1}/F_n < 2$ for all $n > 2$

Solution Let $P(k)$ be the predicate that $1 < F_{k+1}/F_k < 2$ is true for an integer $k > 2$. The base case is true since when $k = 3$, $1 < F_4/F_3 < 2$. We now assume that for a particular k, $P(k)$ is true. In our inductive step, we must prove that $P(k+1)$ is also true, that is $1 < F_{k+2}/F_{k+1} < 2$. We see that this is equivalent to $1 < (F_{k+1} + F_k)/F_{k+1} < 2$, which is equal to $1 < 1 + F_k/F_{k+1} < 2$. We know that $F_k/F_{k+1} < 1$ so we have proved the inductive step. Therefore the base case is true and for all k, $P(k)$ implies $P(k+1)$ so we can conclude that this statement is in fact true.

Problem 4

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions. If $g \circ f$ is bijective, then f is injective and g is surjective.

Proving f is injective: Assume for the sake of contradiction, that $g \circ f$ is bijective but f is *not* injective. This means that for some a_1 and a_2 such that $a_1 \neq a_2$, $f(a_1) = f(a_2)$. This implies that $g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2)$. However, since $g \circ f$ is bijective, it must hold that $a_1 = a_2$, but this is a contradiction.
Proving g is surjective: Assume for the sake of contradiction, that $g \circ f$ is bijective but g is *not* surjective. This means there exists some $c_1 \in C$ such that for all $b \in B, g(b) \neq c_1$. However, since $g \circ f$ is bijective, there must be some a such that $g \circ f(a) = g(f(a)) = c_1$. Thus, since $f(a) \in B$ and $g(f(a)) = c_1$, we have a contradiction.

Problem 5

Prove by contraposition that If mn is odd, then m and n are odd.

Solution Assume that it is not true that both m and n are odd. Then either m and n or both are even. Since the product with any even number is even, mn is even. We have shown that if it is not true that both m and n are odd, then it is not true that mn is odd. Therefore, we have shown that if mn is odd, then m is odd and n is odd.