

Lecture 10

Finding strongly connected components

Announcements

- HW4 Due Friday
- Next Week: Midterm 2!
 - Thursday 11/6, 9am-10:20am
 - 60 minutes for the exam, same as last time
 - Keep an eye out for a logistics post on Ed
 - Covers through Lecture 11 (Thursday)
 - Emphasis on stuff since the last midterm
 - We are aware that you won't have a chance to do HW on Lectures 10 or 11 yet
 - We'll post a practice exam soon

We hear your feedback about the first midterm!

- We will try to make the second midterm shorter (in terms of number/length of problems)
- It will have short questions, rather than a big long question worth a lot of points
- Unfortunately, we need to stick to 60 minutes for the midterm
 - Our class is 80 minutes long
 - We need time for ID-checking at the end (AIWG says we must check IDs)
 - Multiple out-of-class midterms are logistically challenging for a class of this size

Last time

- Breadth-first and depth-first search
- Plus, applications!
 - Topological sorting
 - In-order traversal of BSTs
 - Finding Bacon numbers
 - ~~Testing for bipartiteness~~ (we skipped this due to time, but either go back to the slides or figure out on your own how to use BFS to do this!)
- The key was paying attention to the structure of the tree that these search algorithms implicitly build.

Today

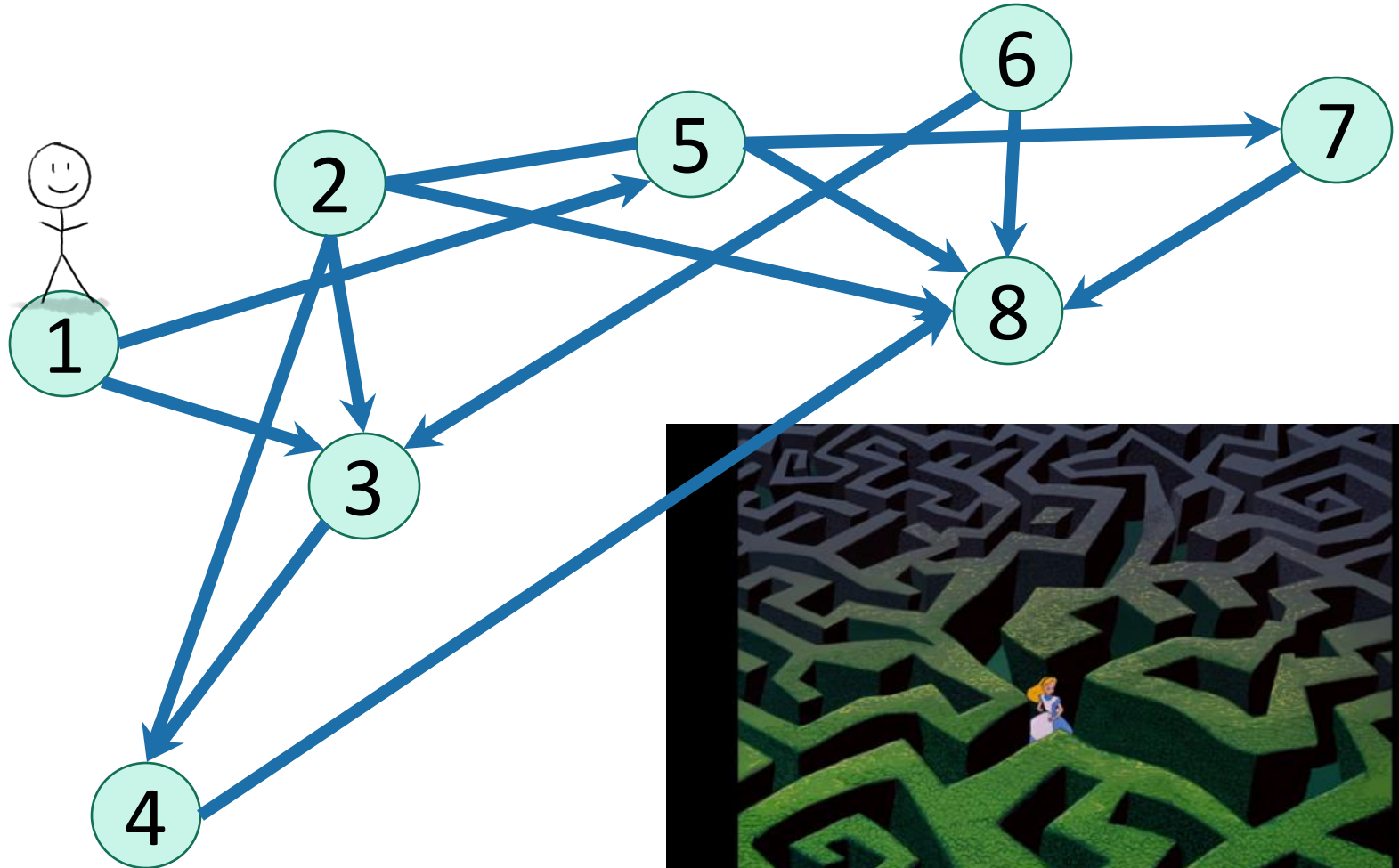
- One more application of DFS:

Finding **Strongly Connected Components**

Today, all graphs are **directed**!
Check that the things we did
last week still all work!

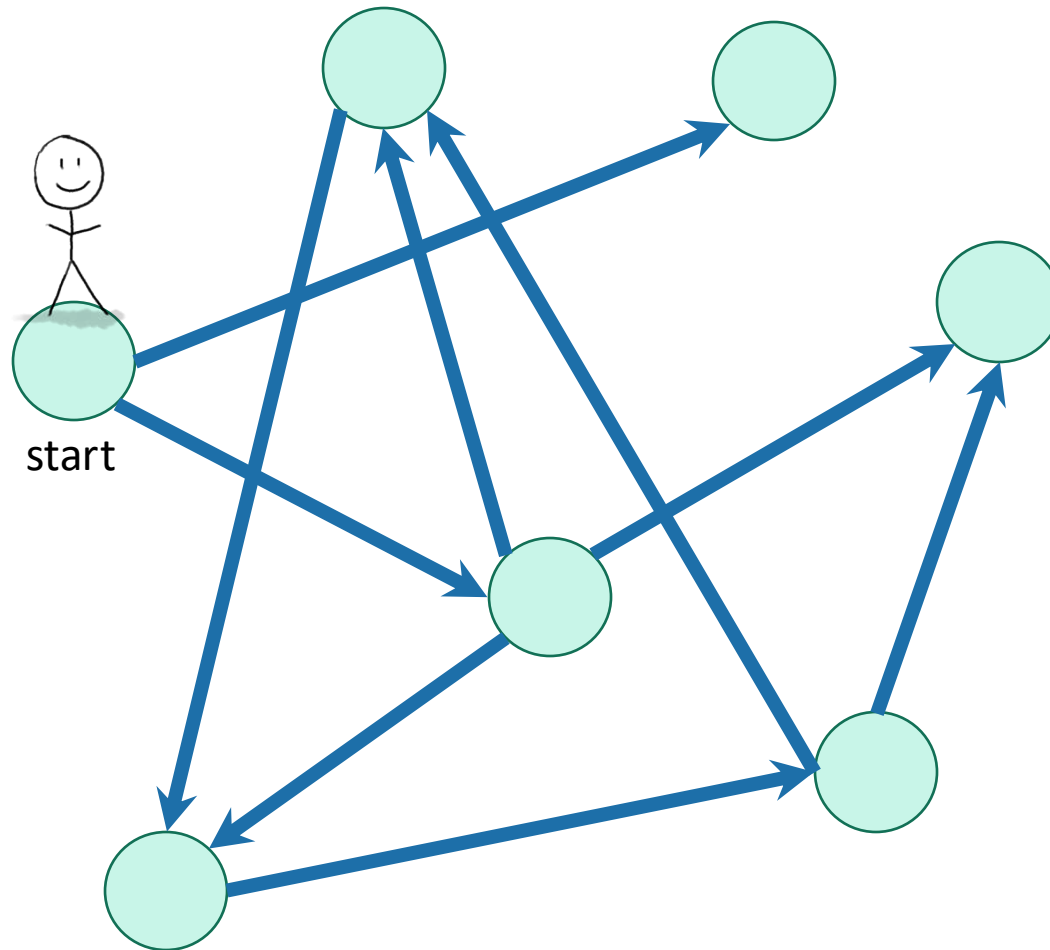
Recall: DFS




It's how you'd explore a labyrinth with chalk and a piece of string.



Depth First Search

Exploring a labyrinth with chalk and a piece of string

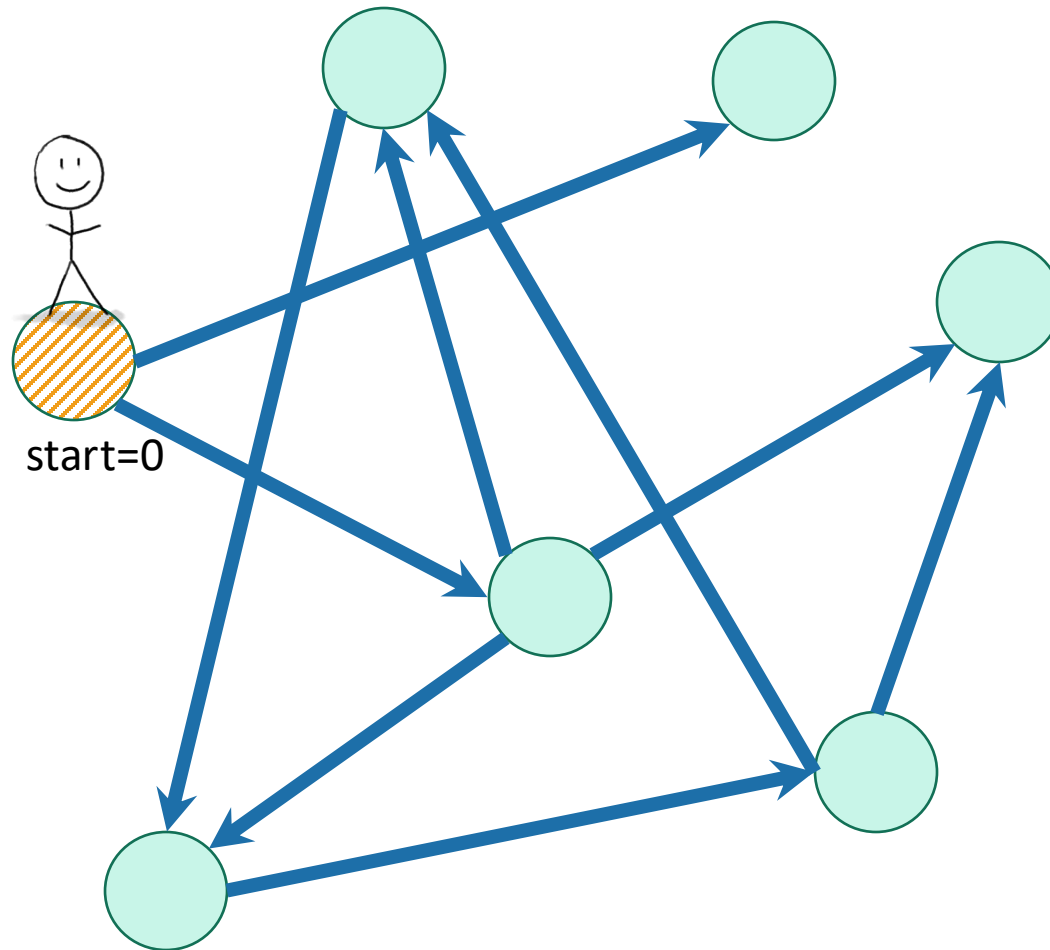





-  Not been there yet
-  Been there, haven't explored all the paths out.
-  Been there, have explored all the paths out.

This is the same picture we had last time, except I've directed all of the edges. Notice that there **ARE** directed cycles. 8

Depth First Search

Exploring a labyrinth with chalk and a piece of string



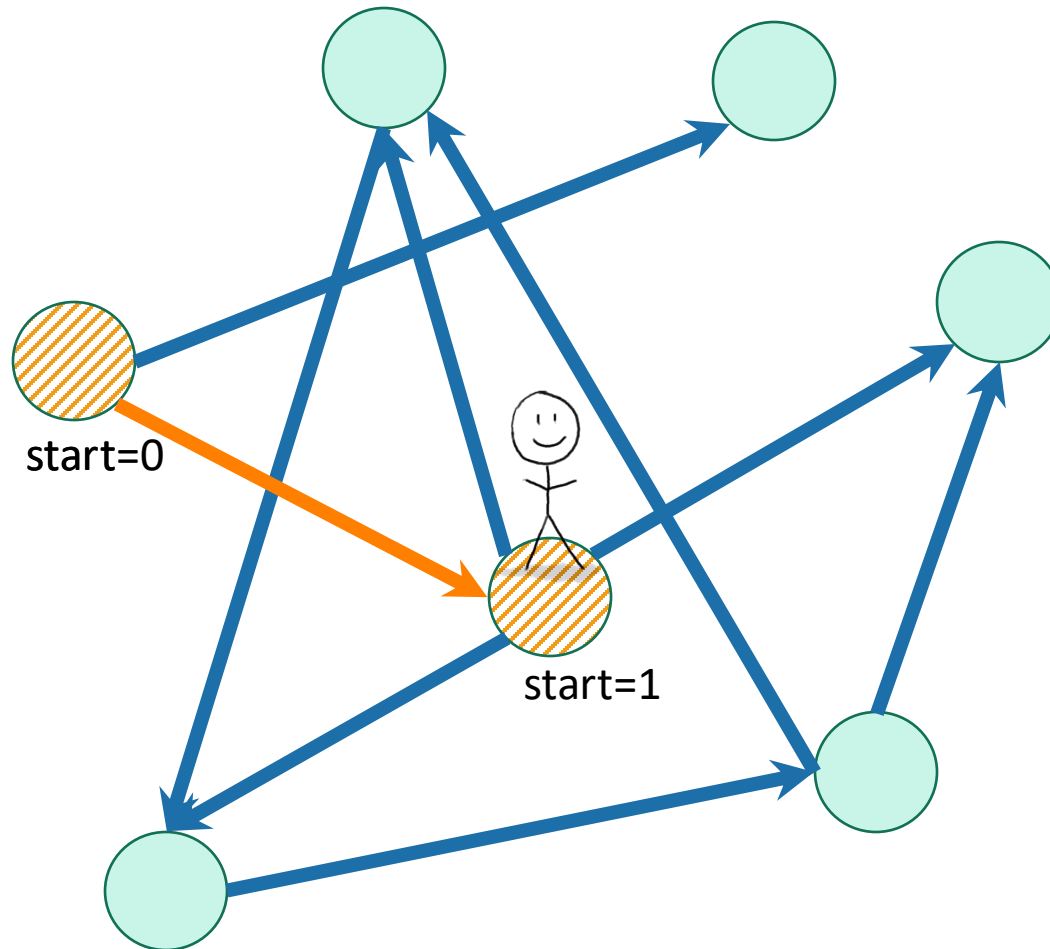
-  Not been there yet
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Recall we also keep track of **start** and **finish** times for every node.

Depth First Search

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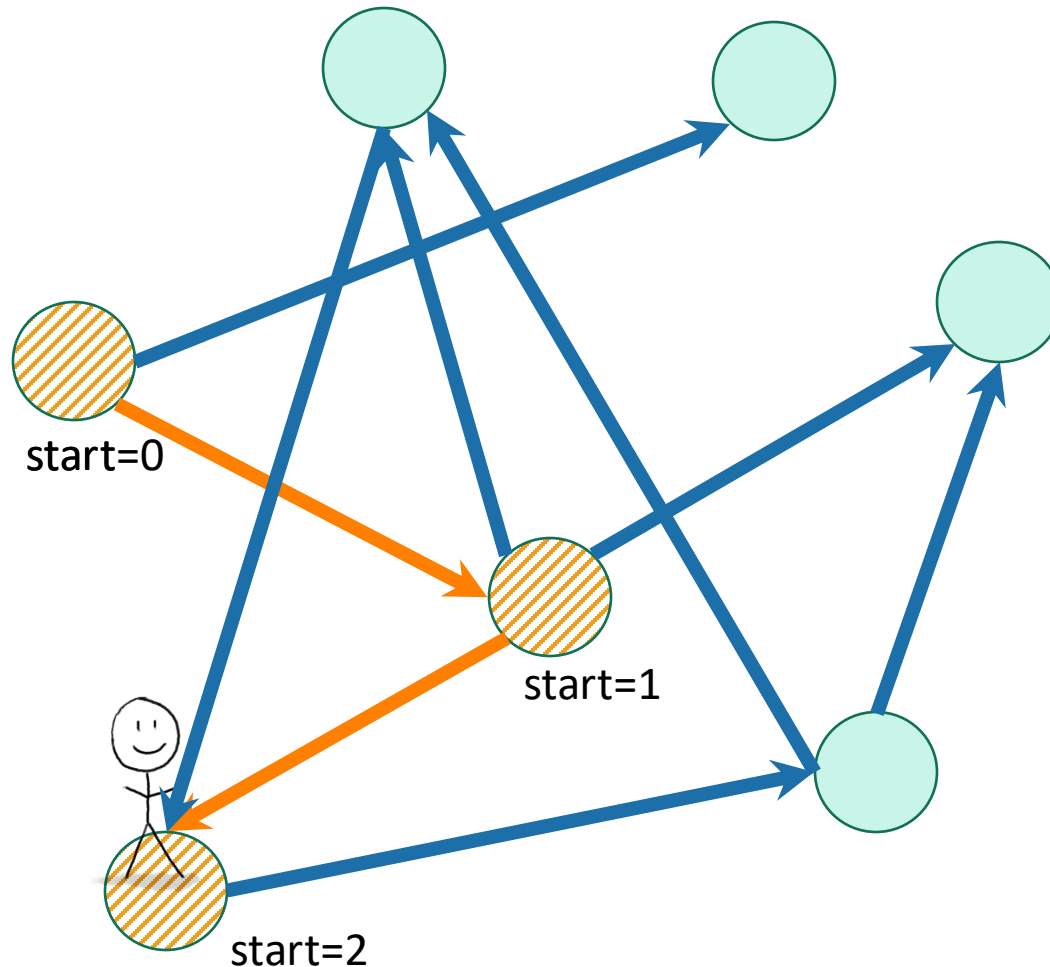
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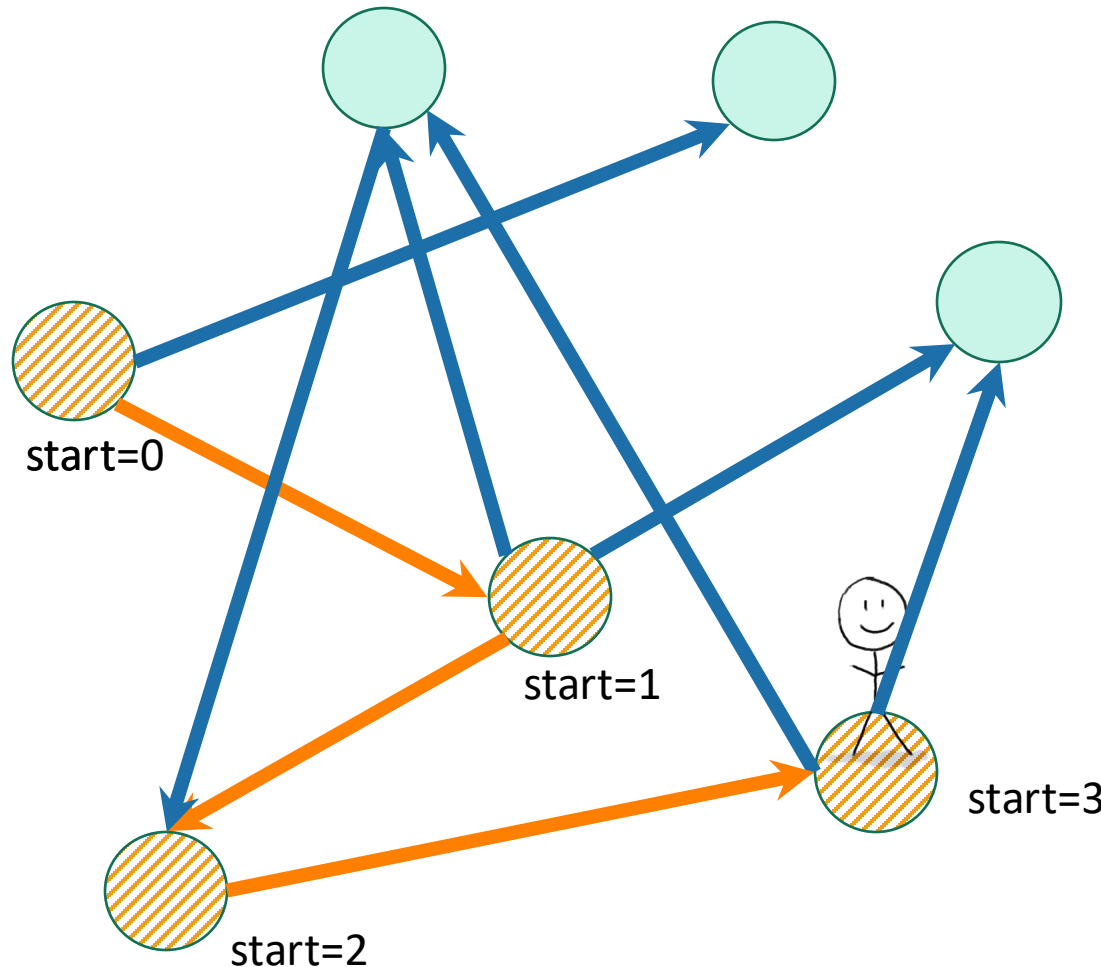
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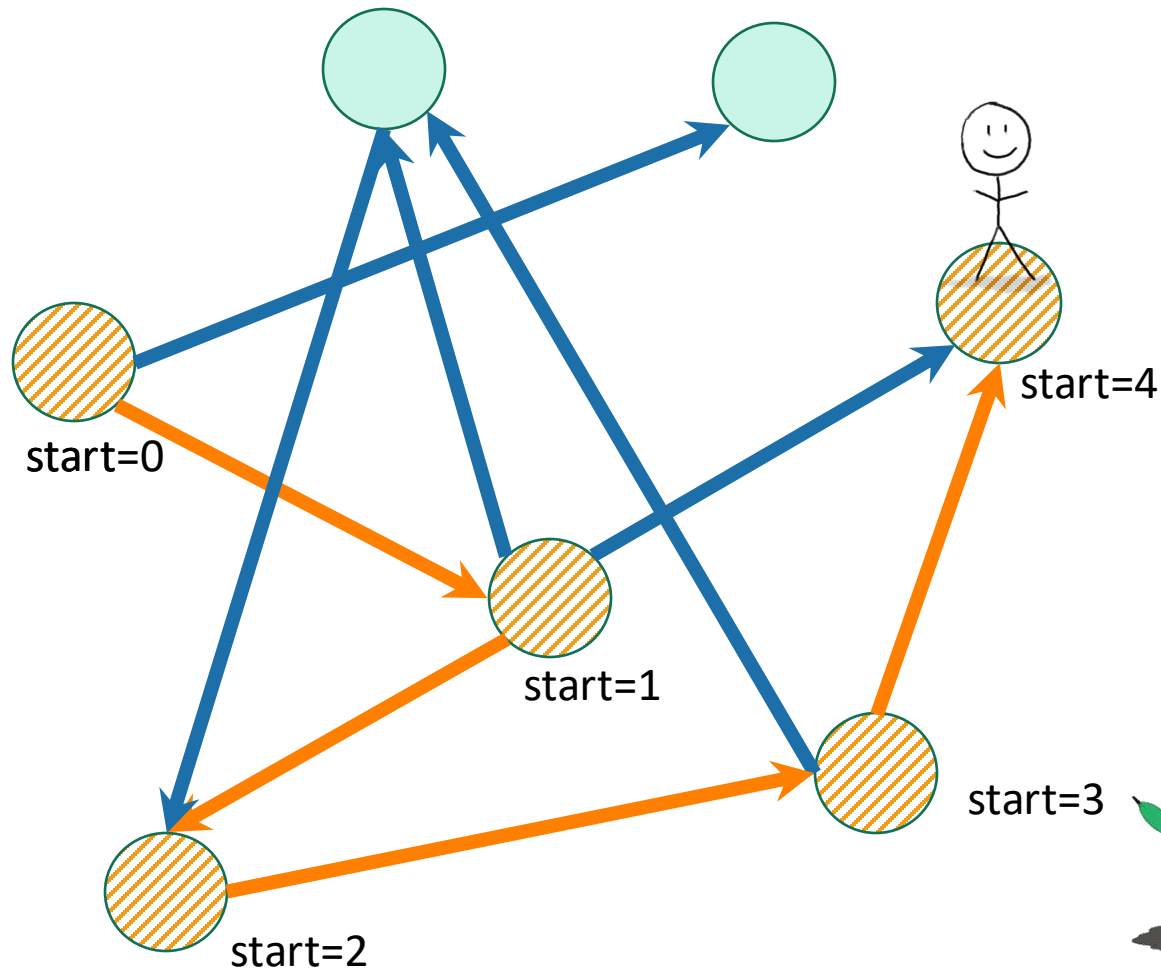
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Depth First Search

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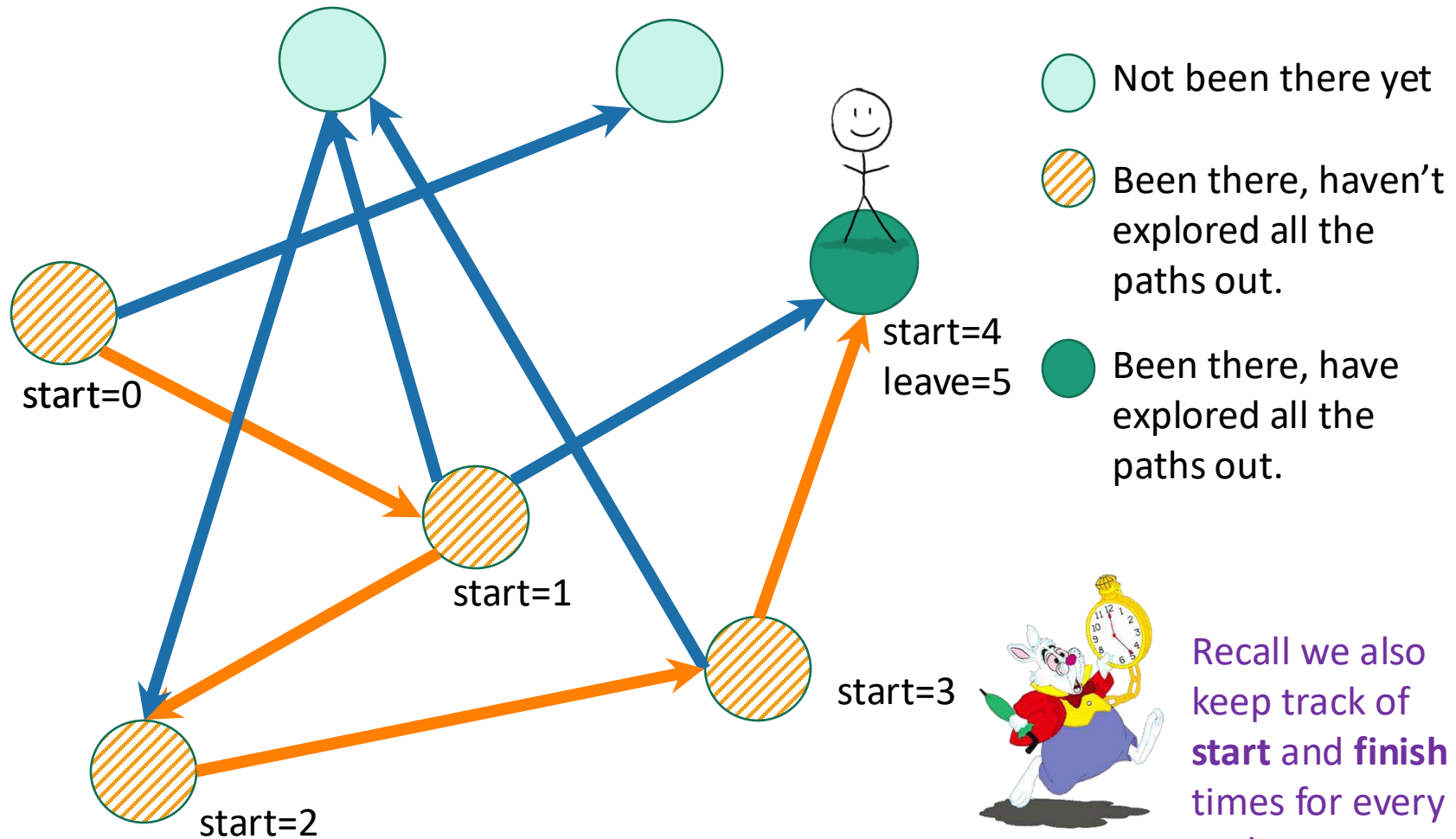
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Recall we also keep track of **start** and **finish** times for every node. 13

Depth First Search

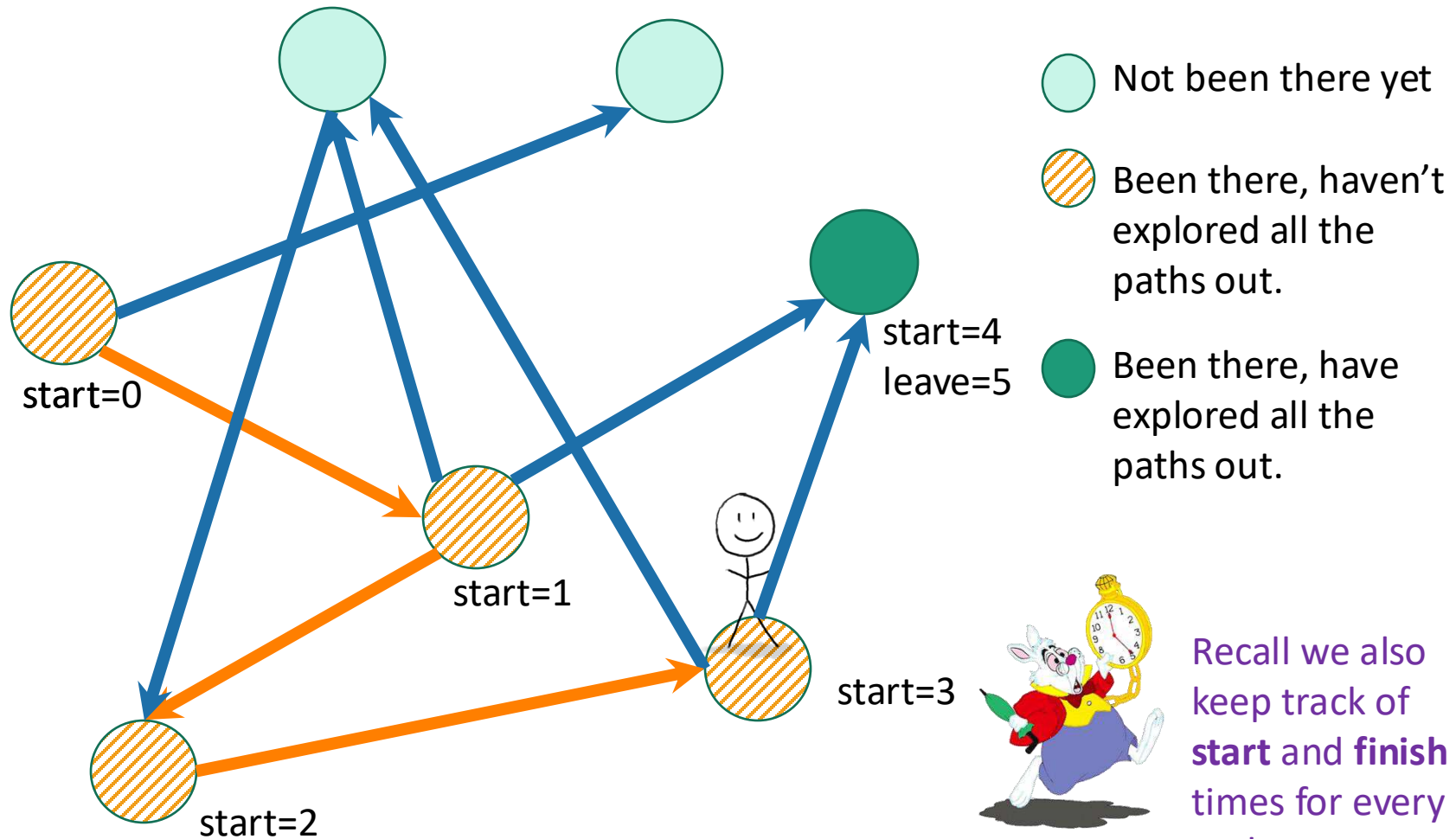
Exploring a labyrinth with chalk and a piece of string



Recall we also keep track of **start** and **finish** times for every node.

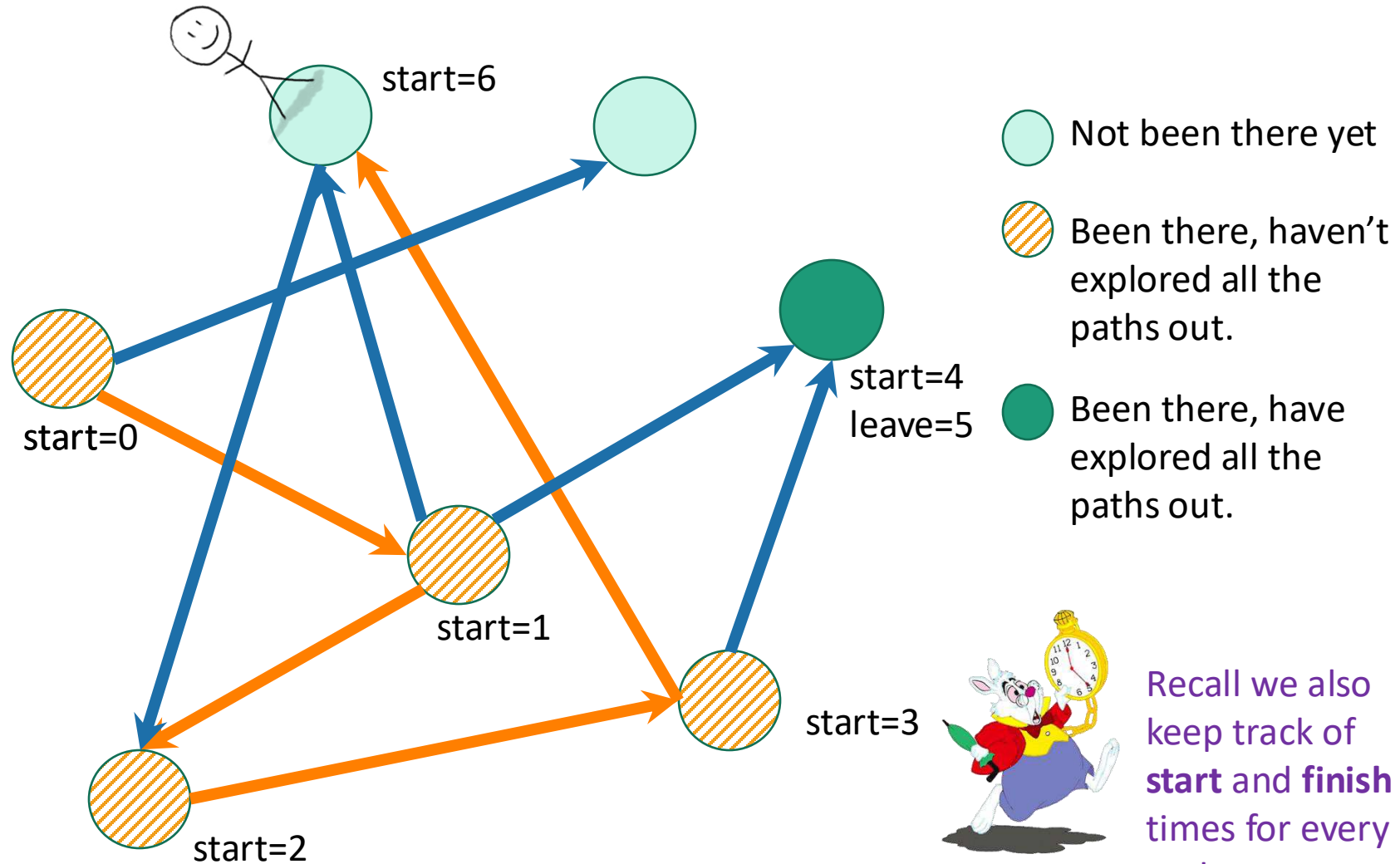
Depth First Search

Exploring a labyrinth with chalk and a piece of string



Depth First Search

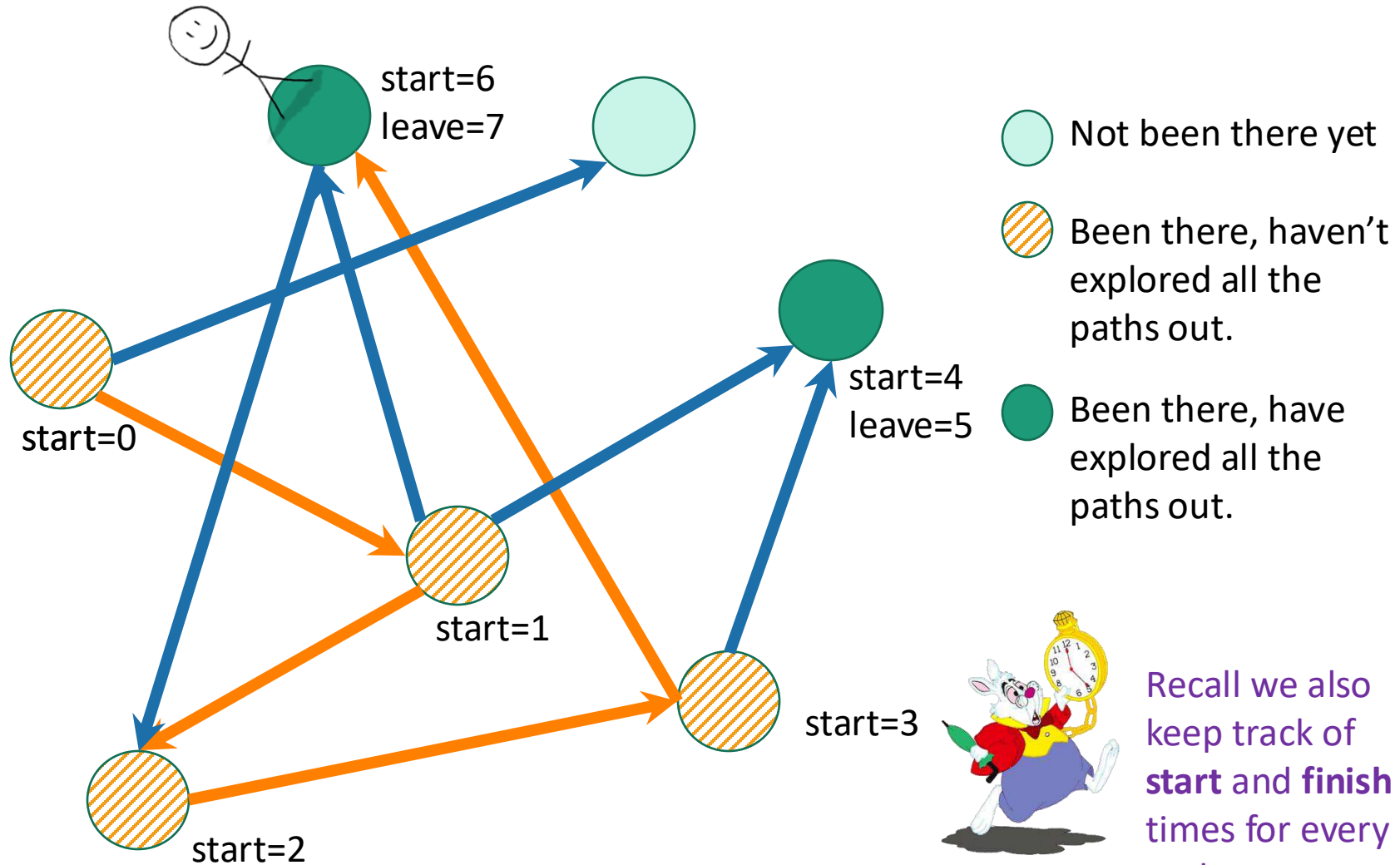
Exploring a labyrinth with chalk and a piece of string



Recall we also keep track of **start** and **finish** times for every node.

Depth First Search

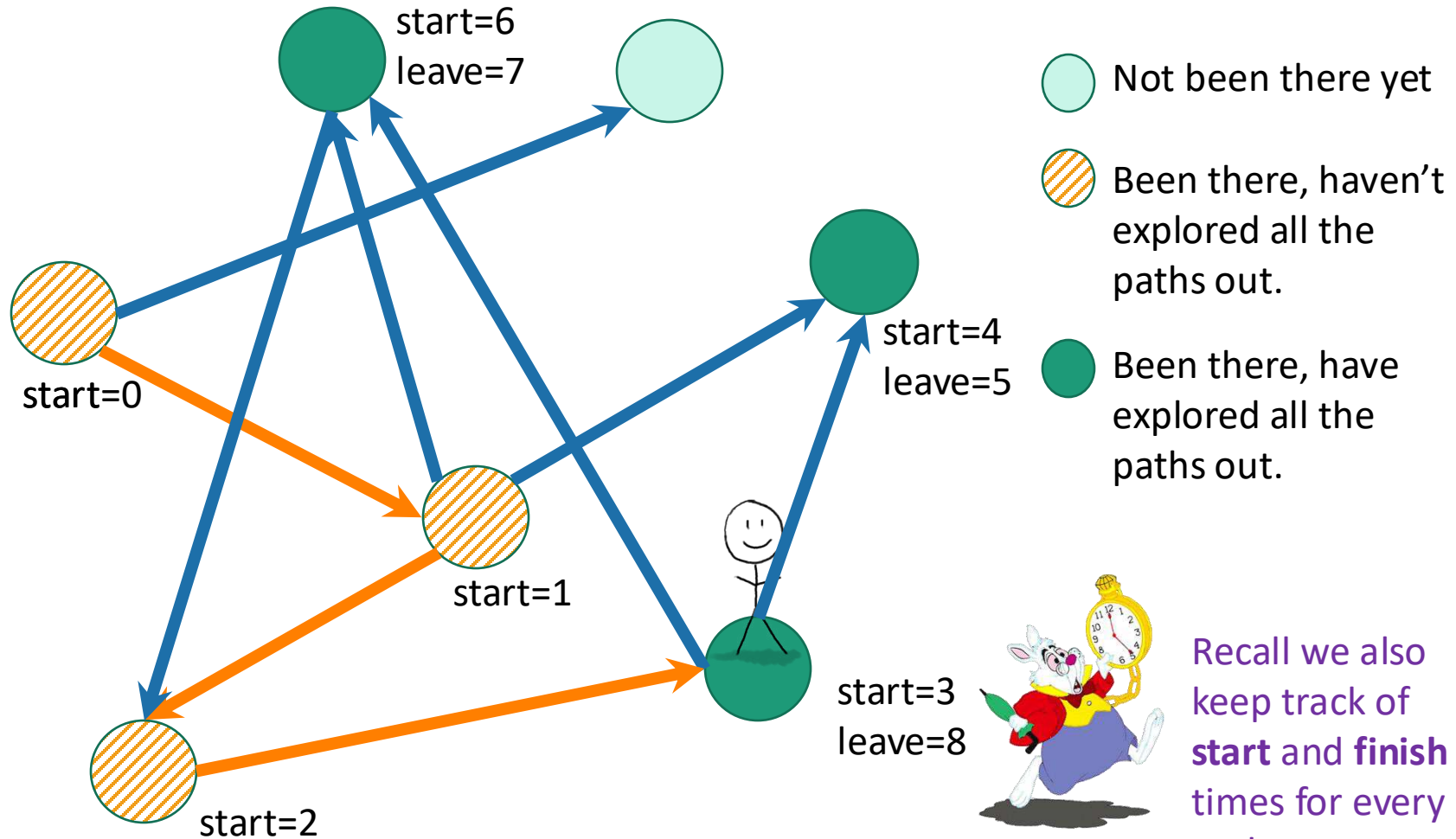
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Recall we also keep track of **start** and **finish** times for every node.

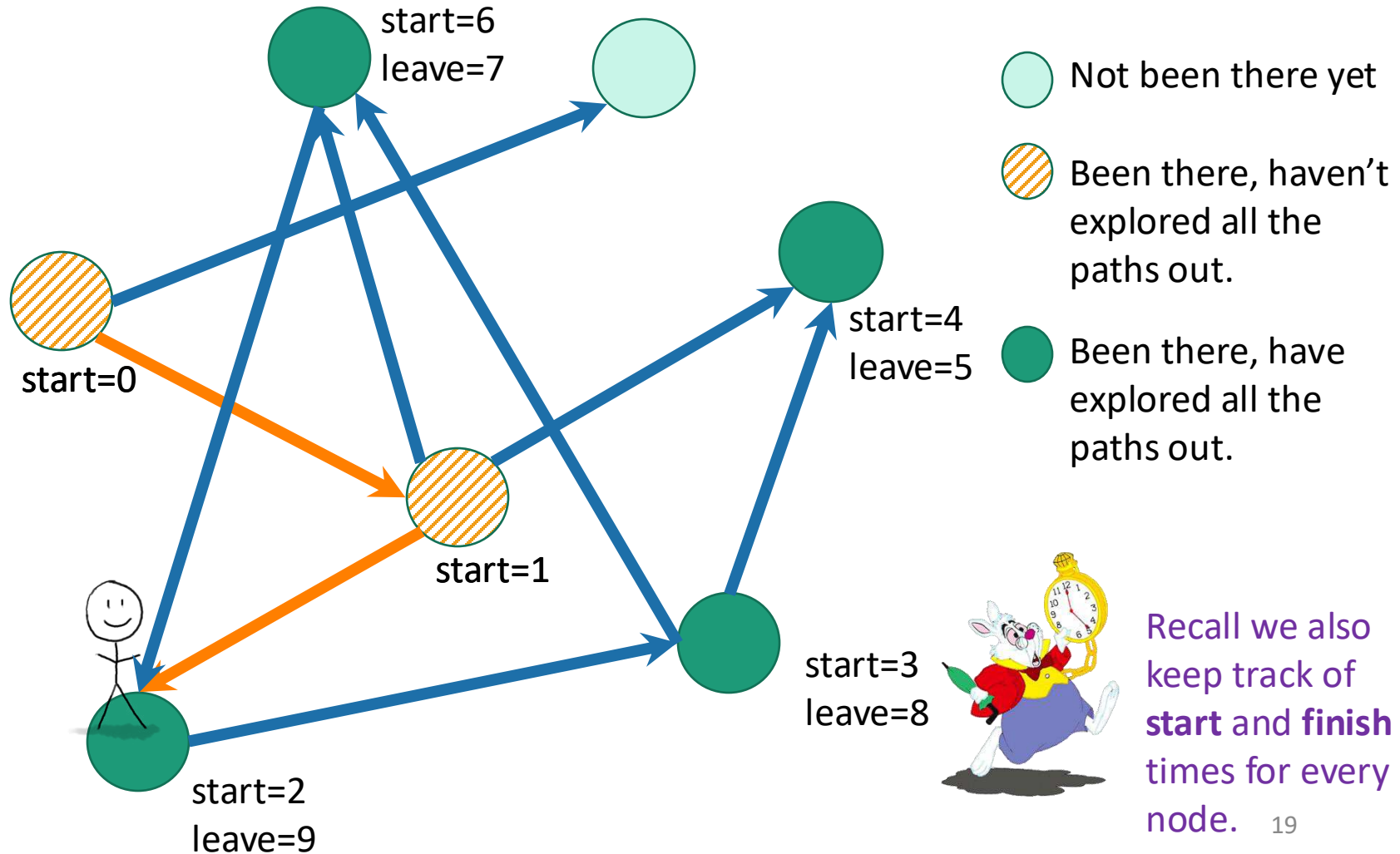
Depth First Search

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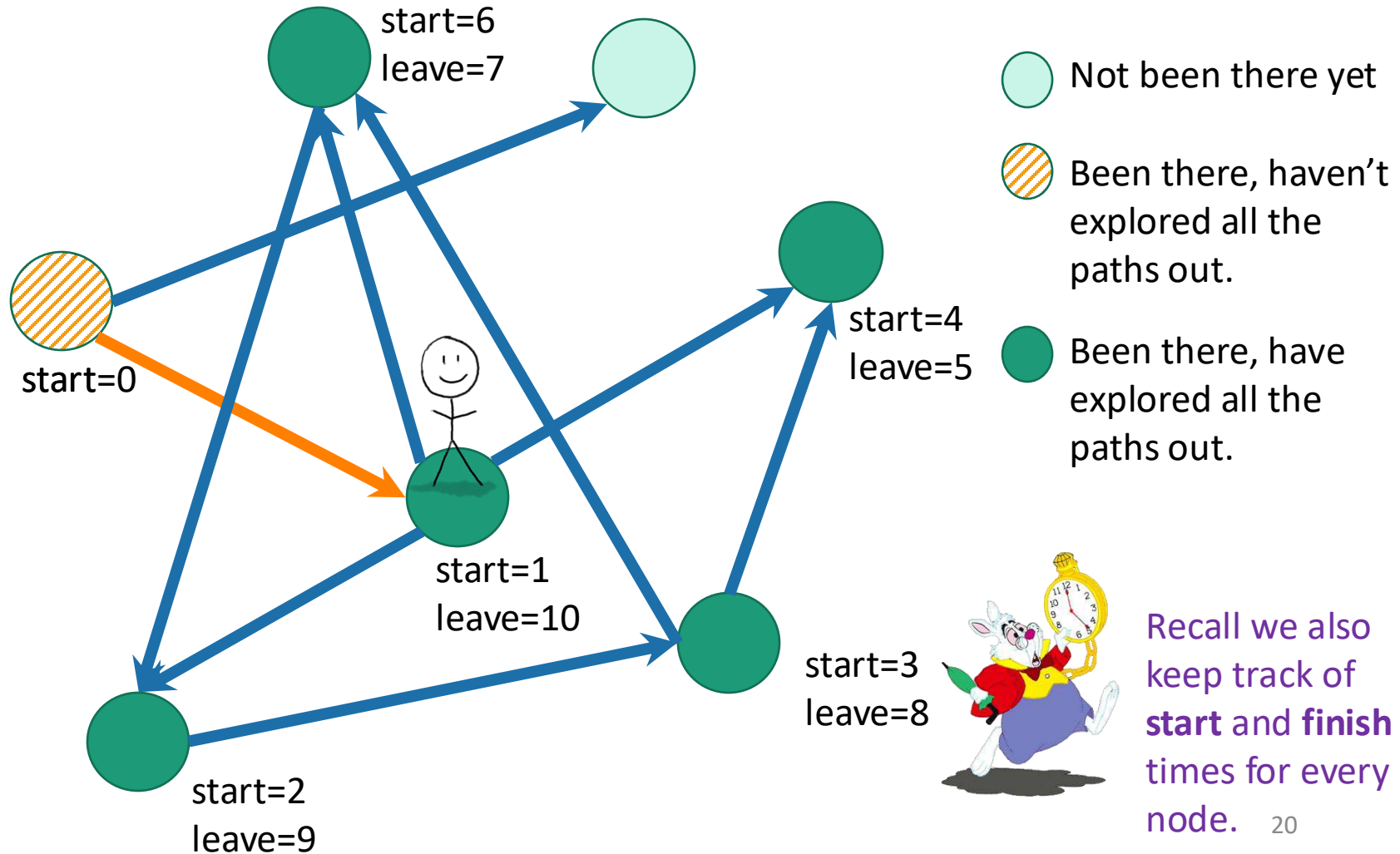
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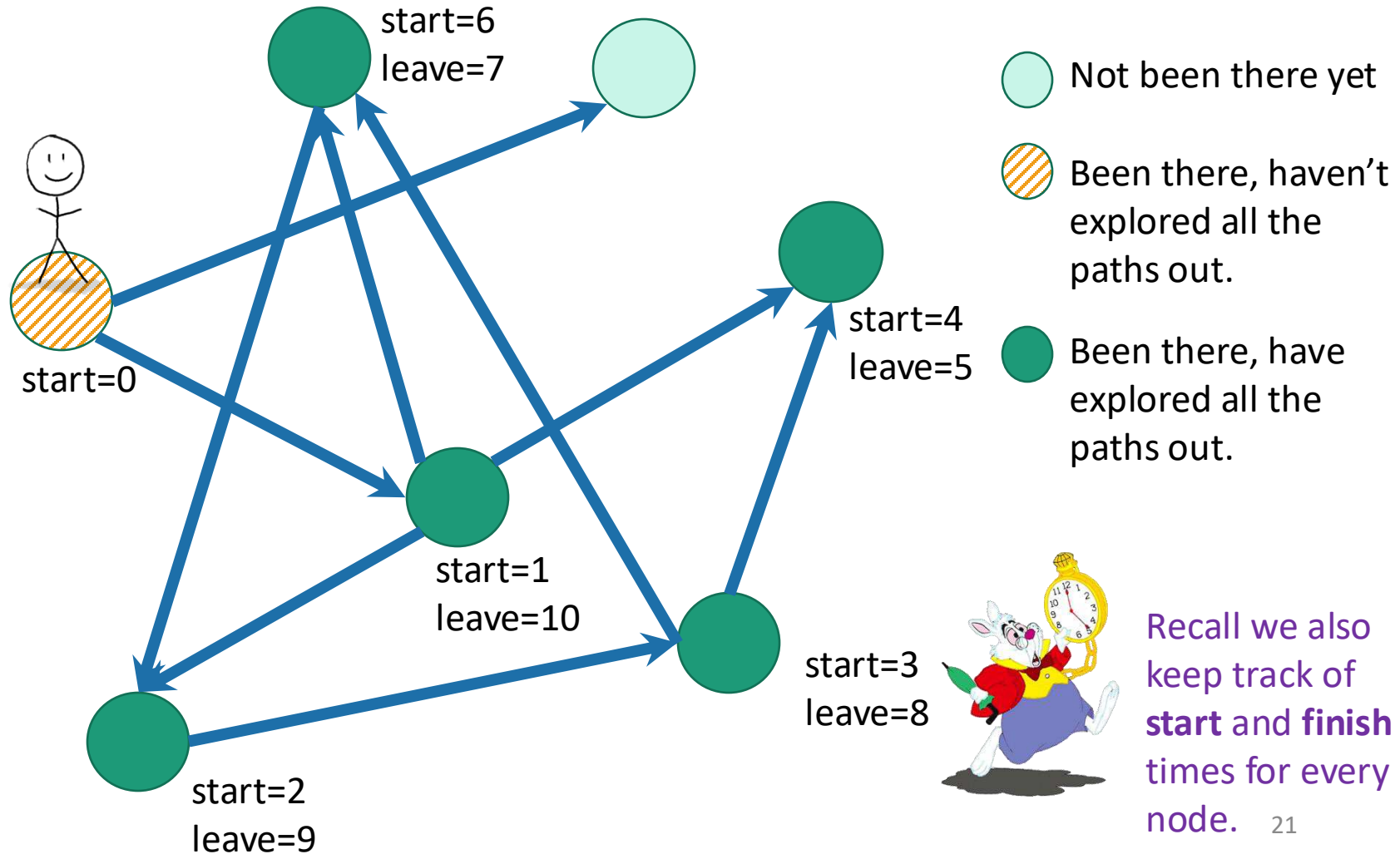
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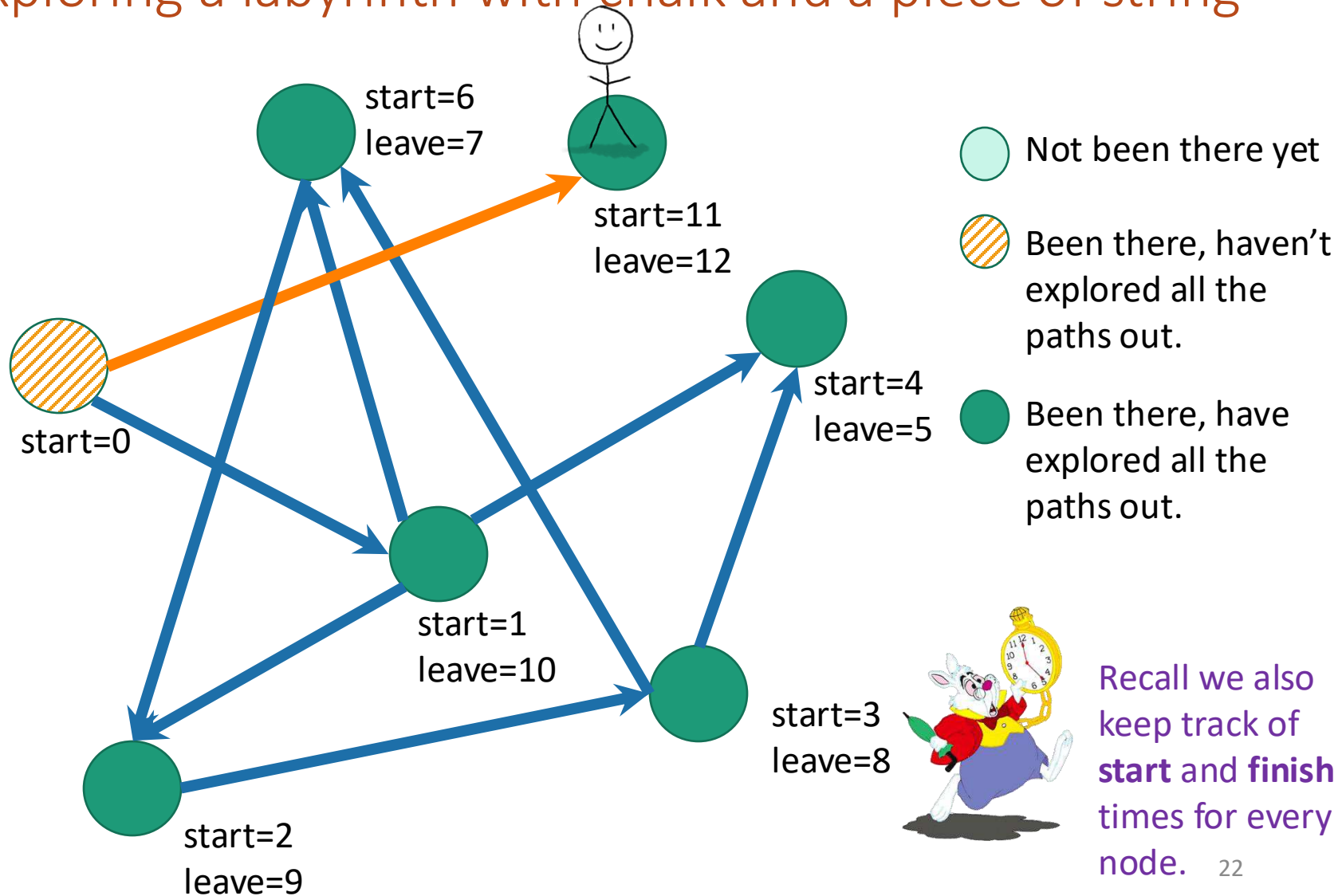
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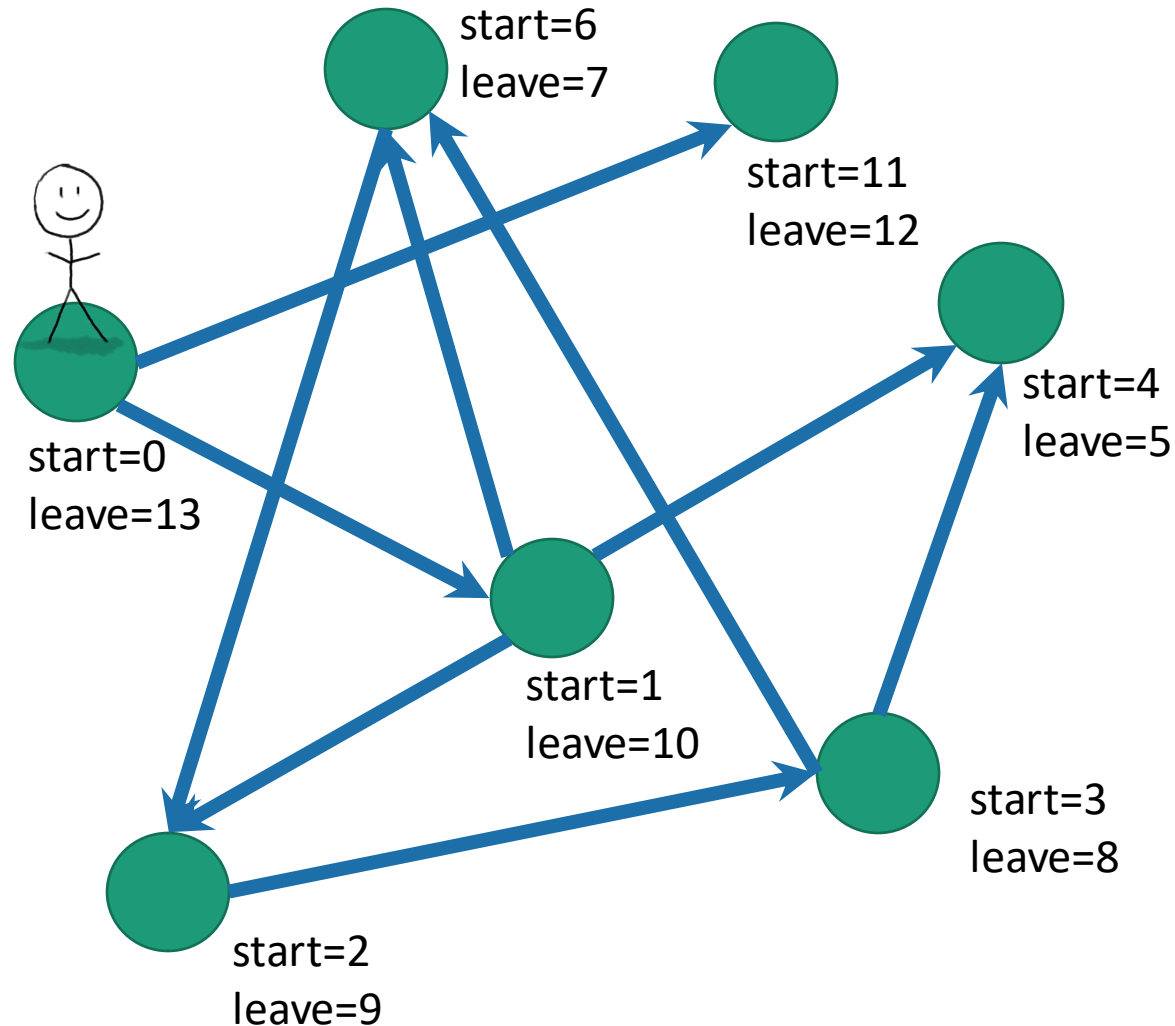
Depth First Search




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Depth First Search

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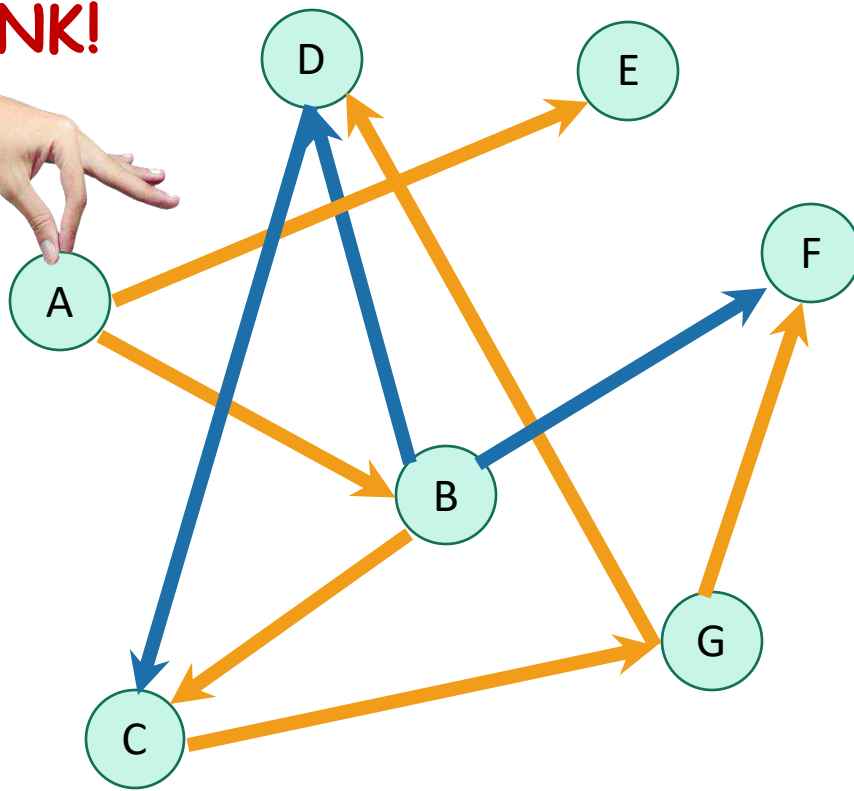
-  Not been there yet
-  Been there, haven't explored all the paths out.
-  Been there, have explored all the paths out.

*Labyrinth:
explored!*

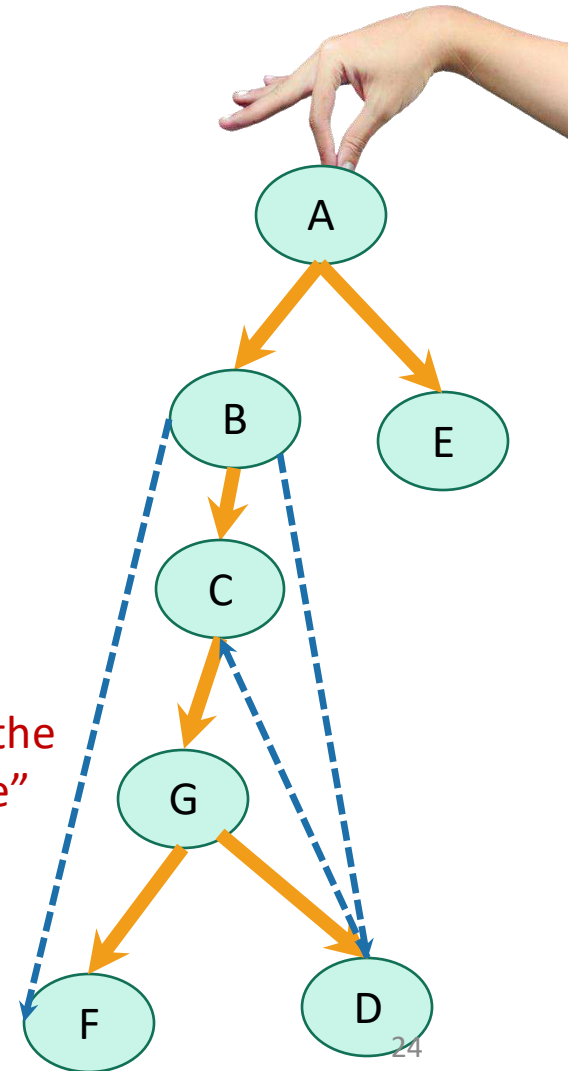
Depth first search

implicitly creates a tree on everything you can reach

YOINK!

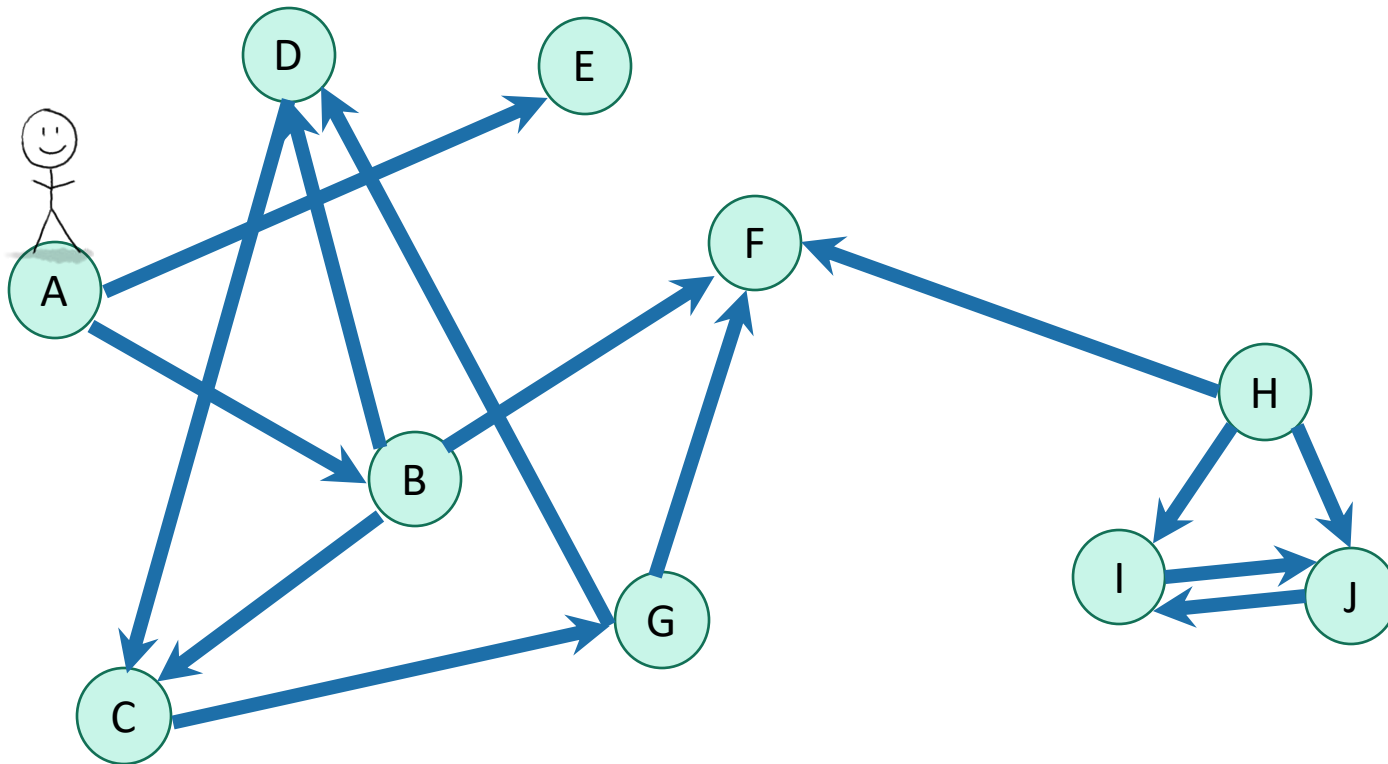


Call this the
"DFS tree"



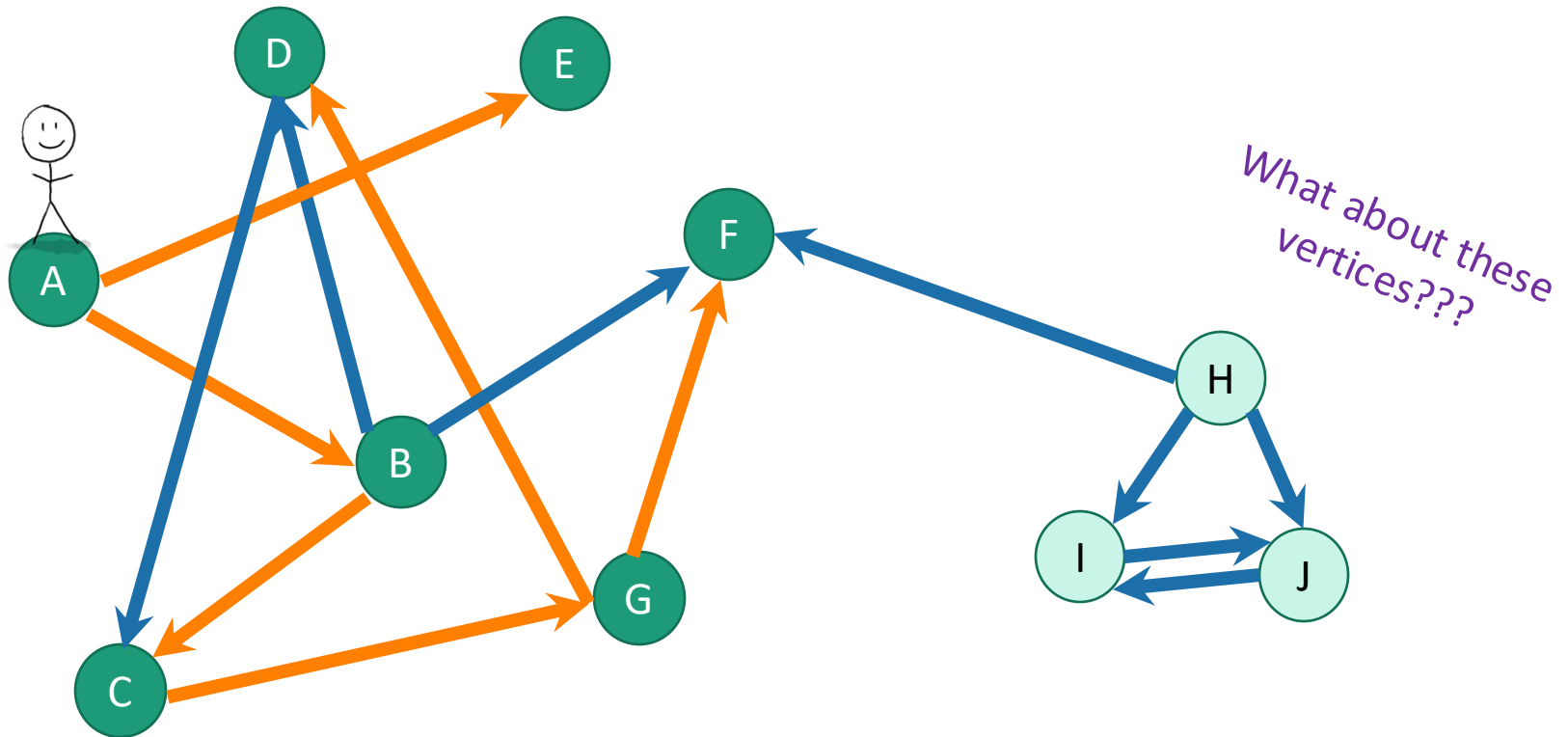
When you can't reach everything

- Run DFS repeatedly to get a **depth-first forest**



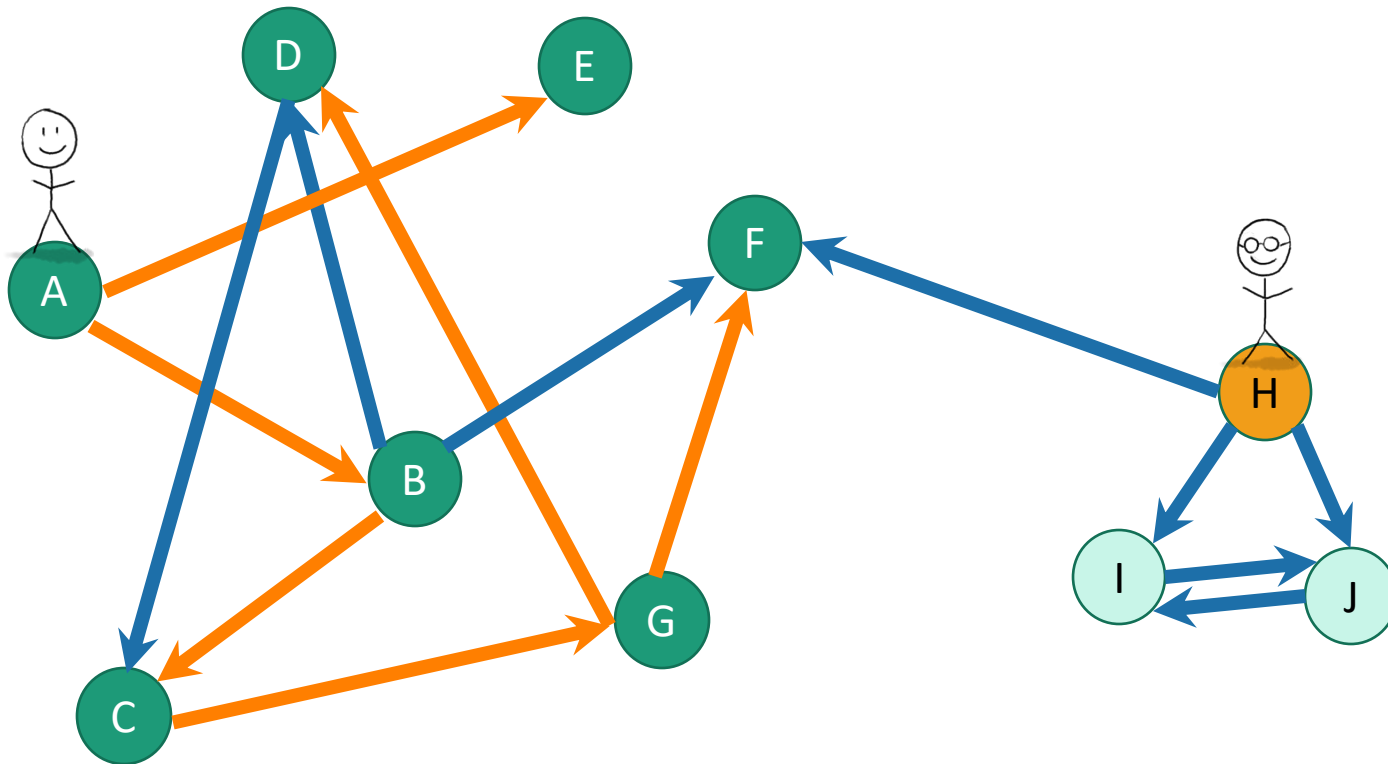
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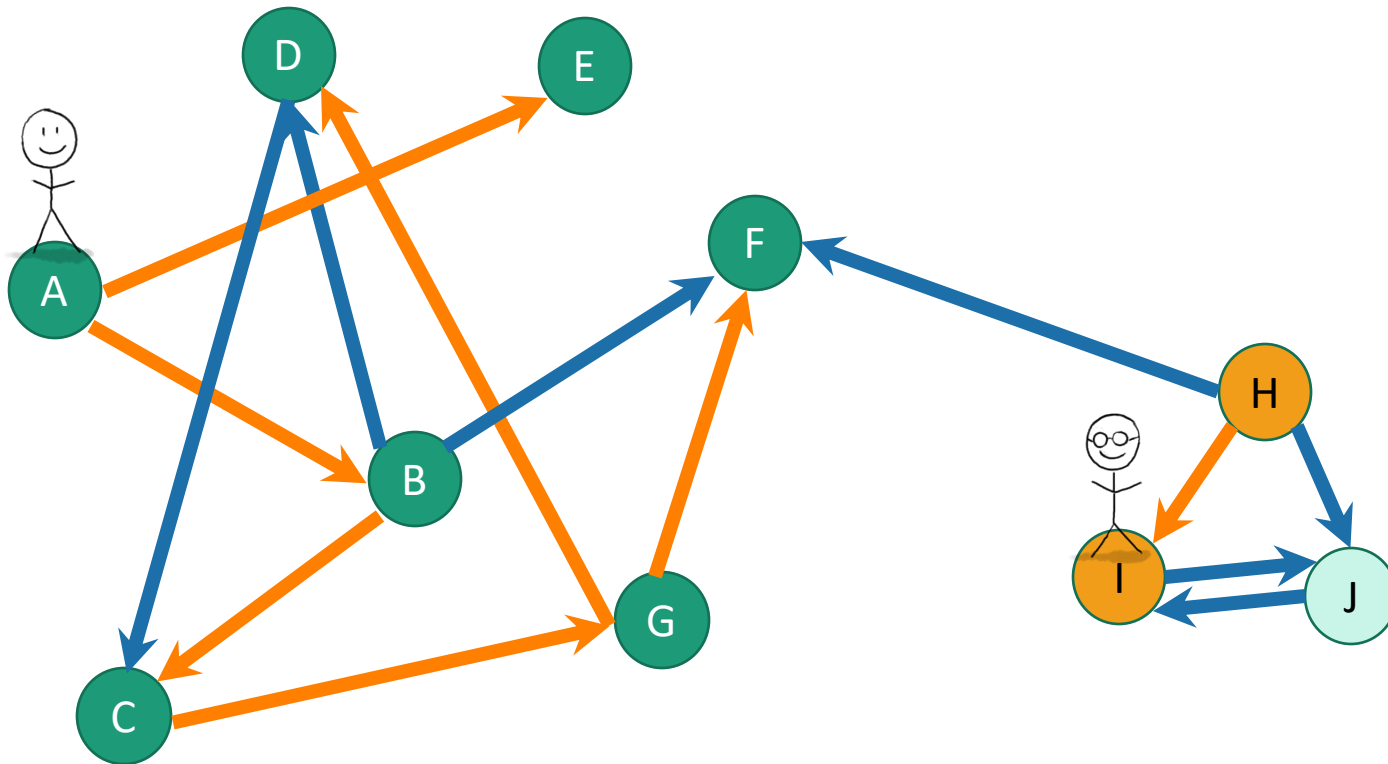
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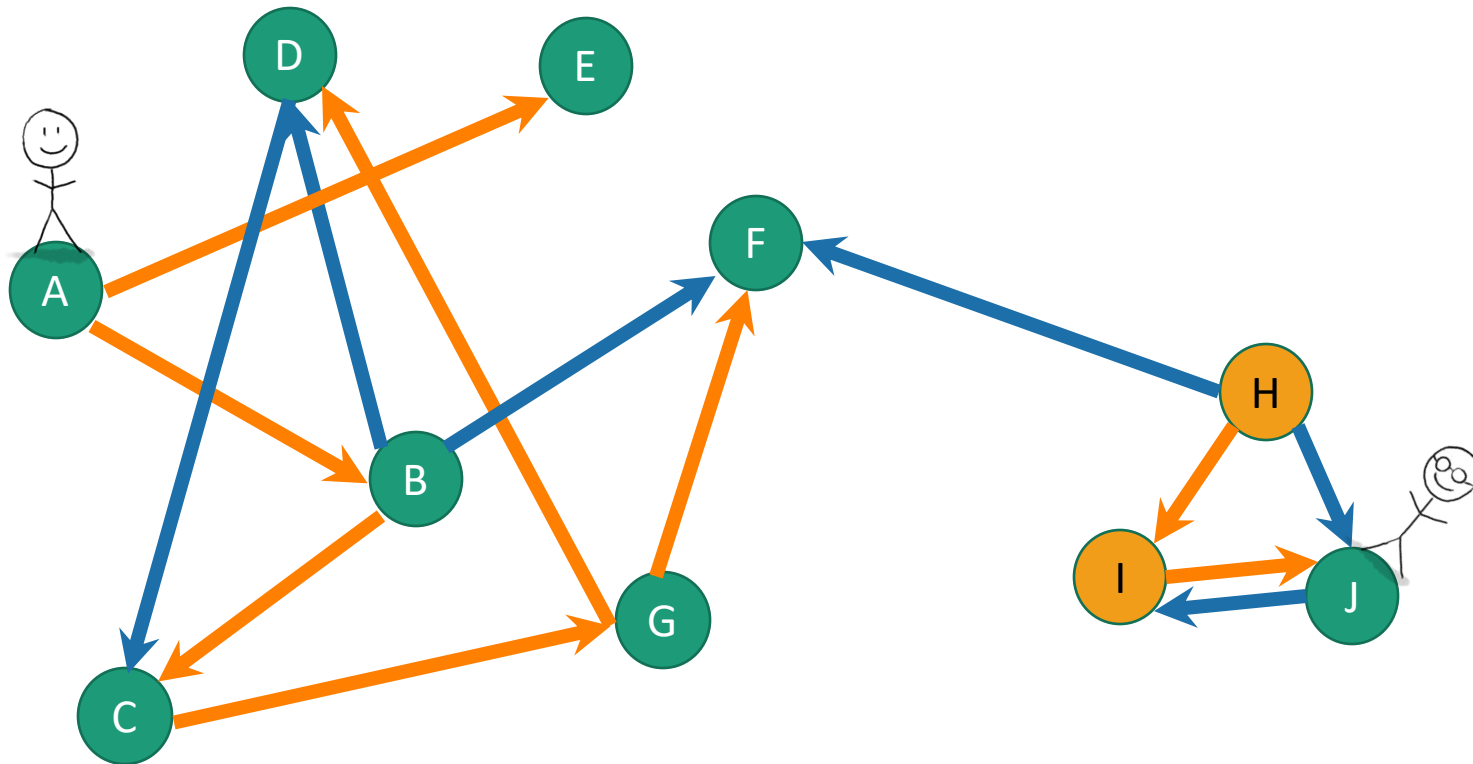
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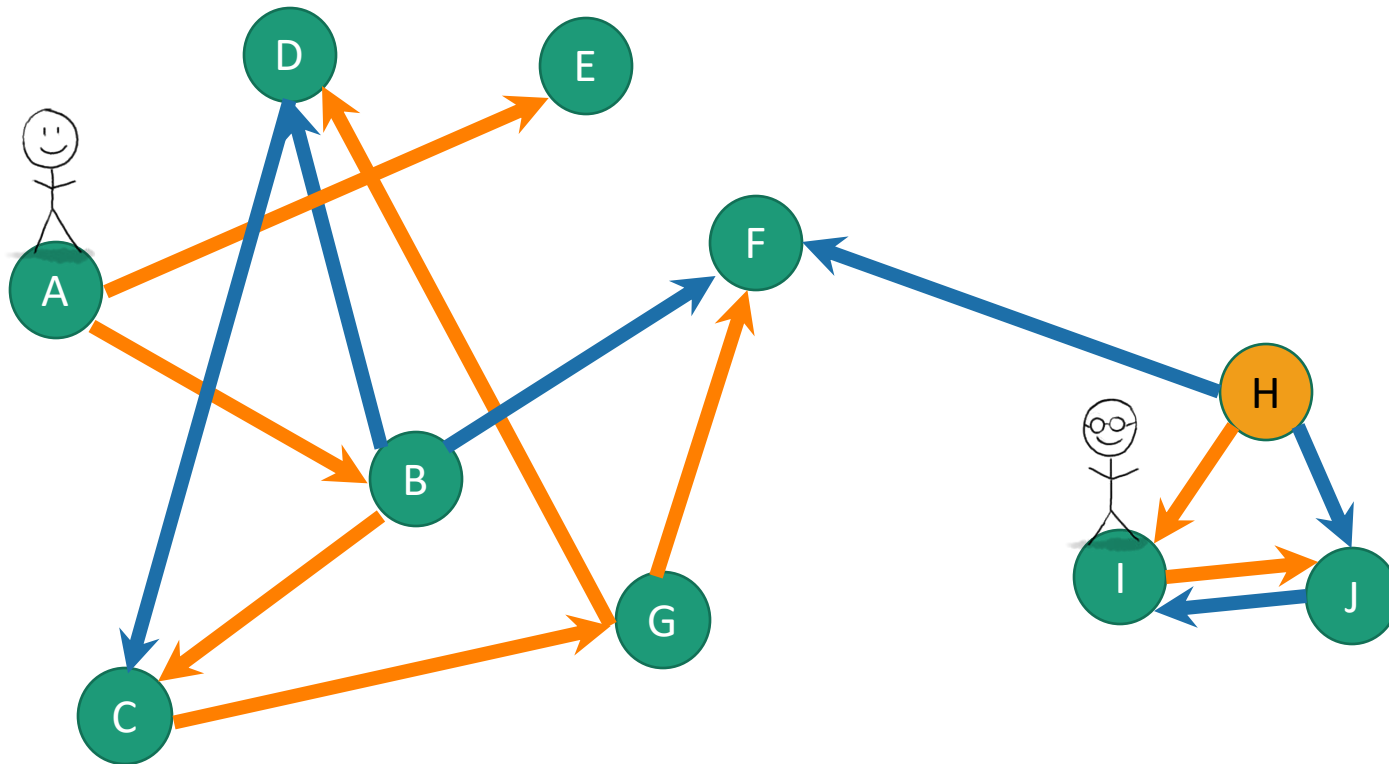
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- Run DFS repeatedly to get a **depth-first forest**



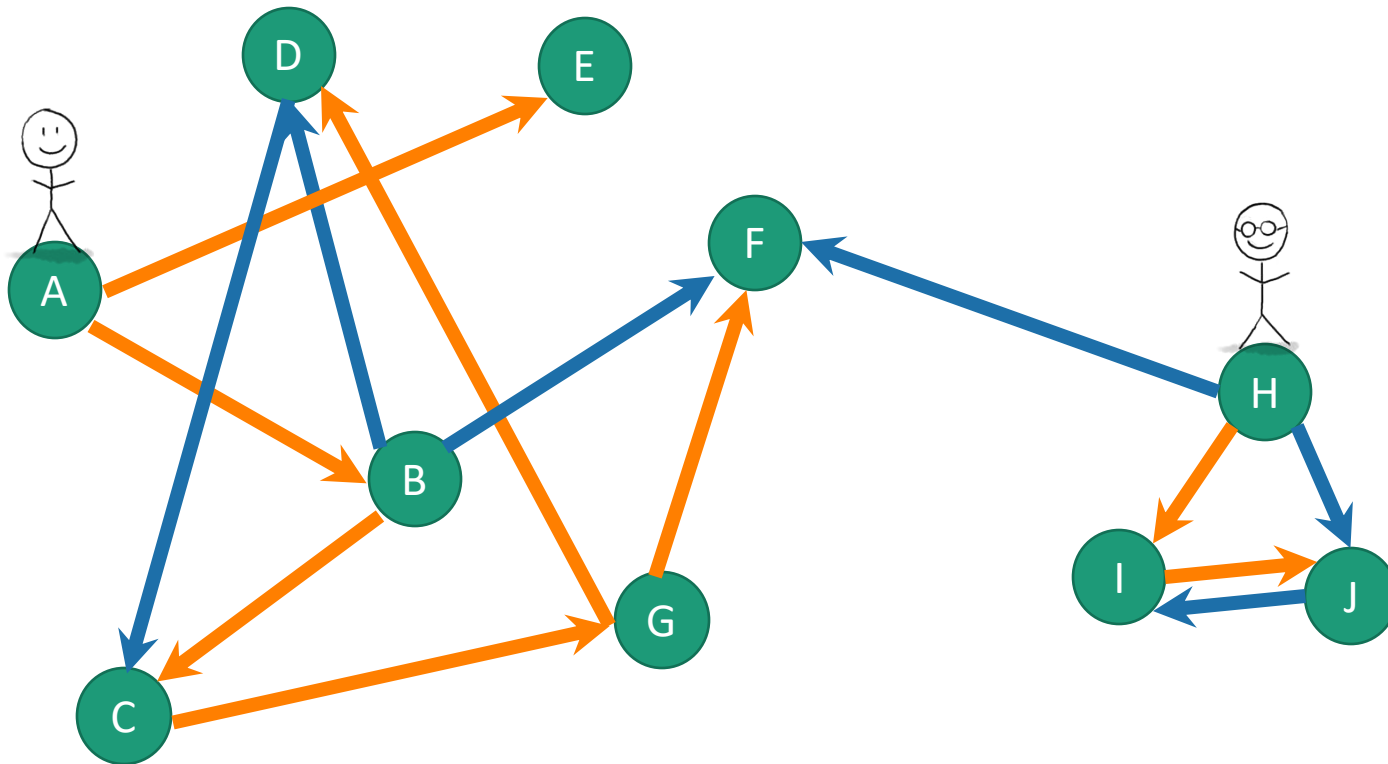
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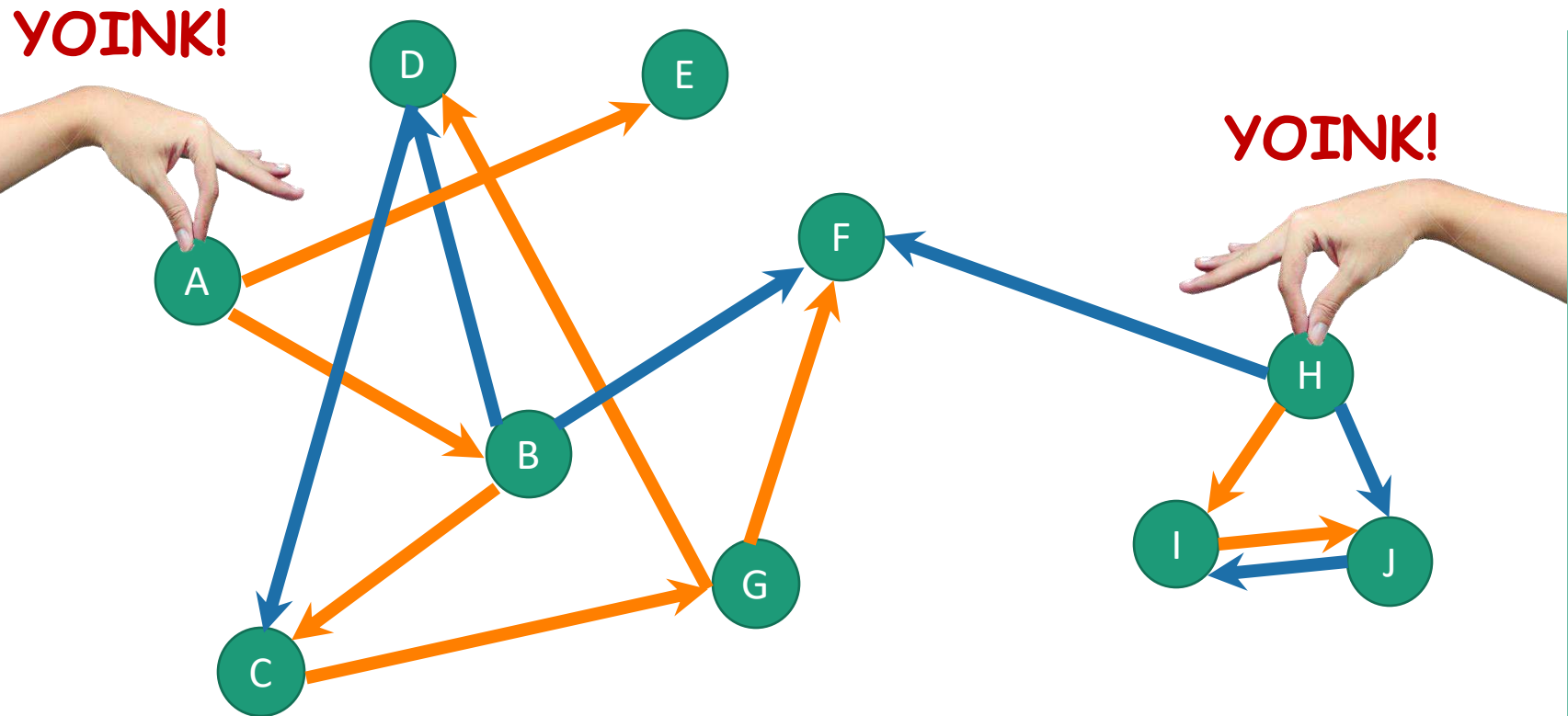
When you can't reach everything

- Run DFS repeatedly to get a **depth-first forest**



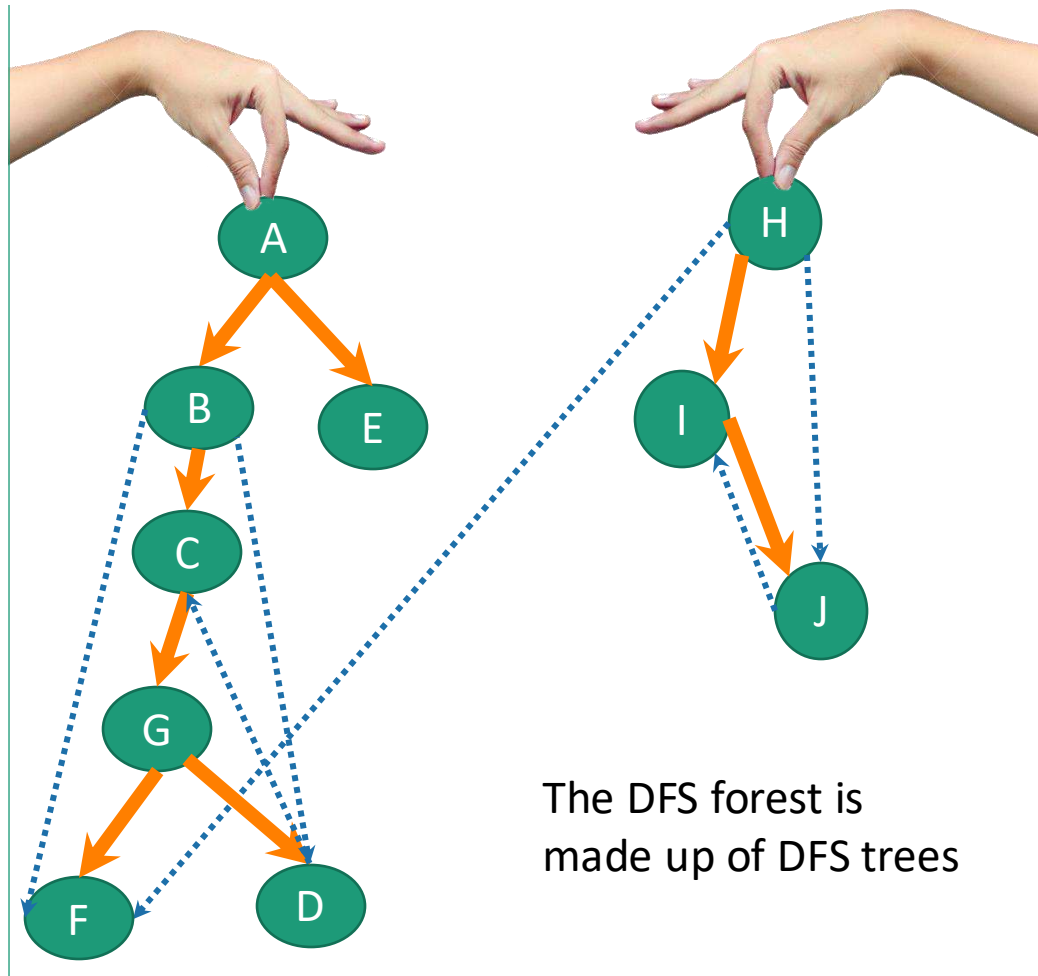
When you can't reach everything

- Run DFS repeatedly to get a **depth-first forest**



When you can't reach everything

- Run DFS repeatedly to get a **depth-first search forest**



Recall:

(Works the same with DFS forests)

- If v is a descendent of w in this tree:



- If w is a descendent of v in this tree:



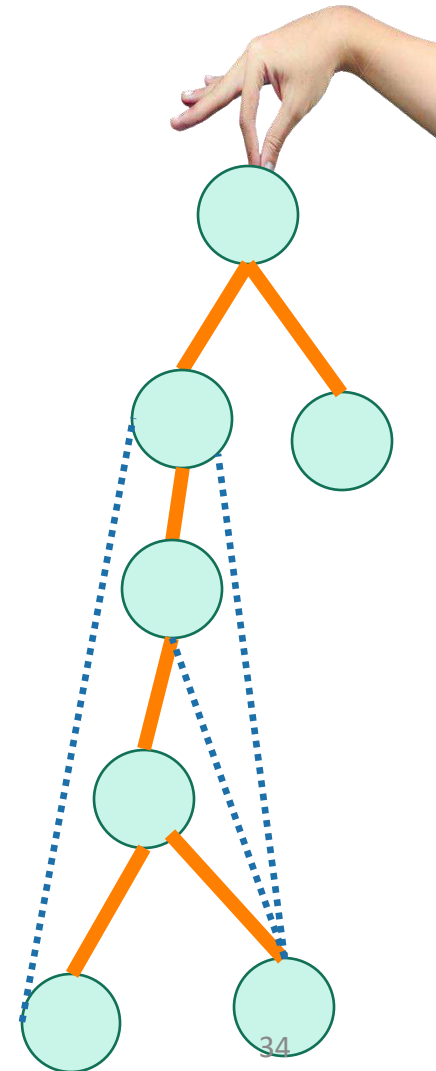
- If neither are descendants of each other:



(or the other way around)

If v and w are in different trees, it's always this last one (or the other way around).

DFS tree

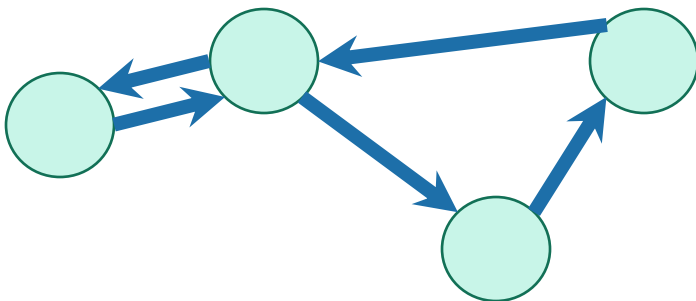


Enough of review

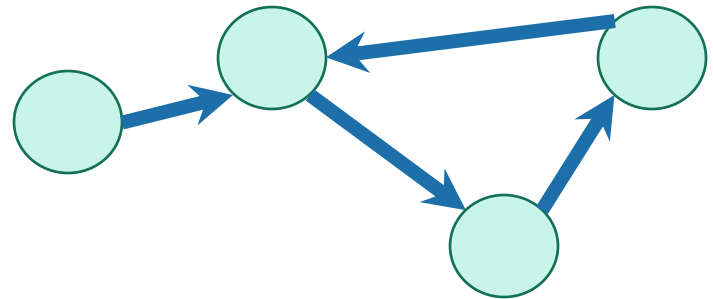
Strongly connected components

Strongly connected components

- A directed graph $G = (V, E)$ is **strongly connected** if:
- for all v, w in V :
 - there is a path from v to w and
 - there is a path from w to v .



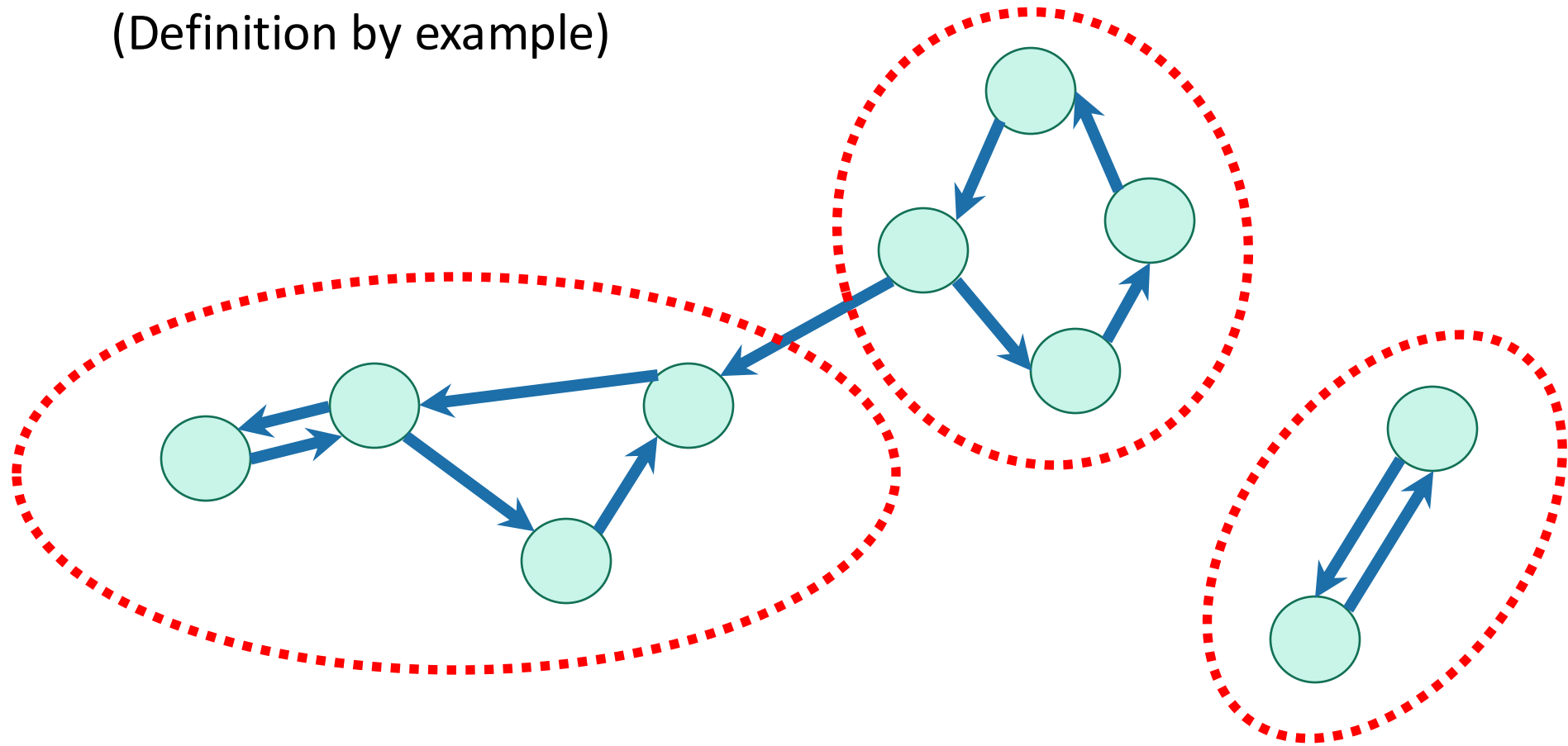
strongly connected



not strongly connected

We can decompose a graph into **strongly connected components** (SCCs)

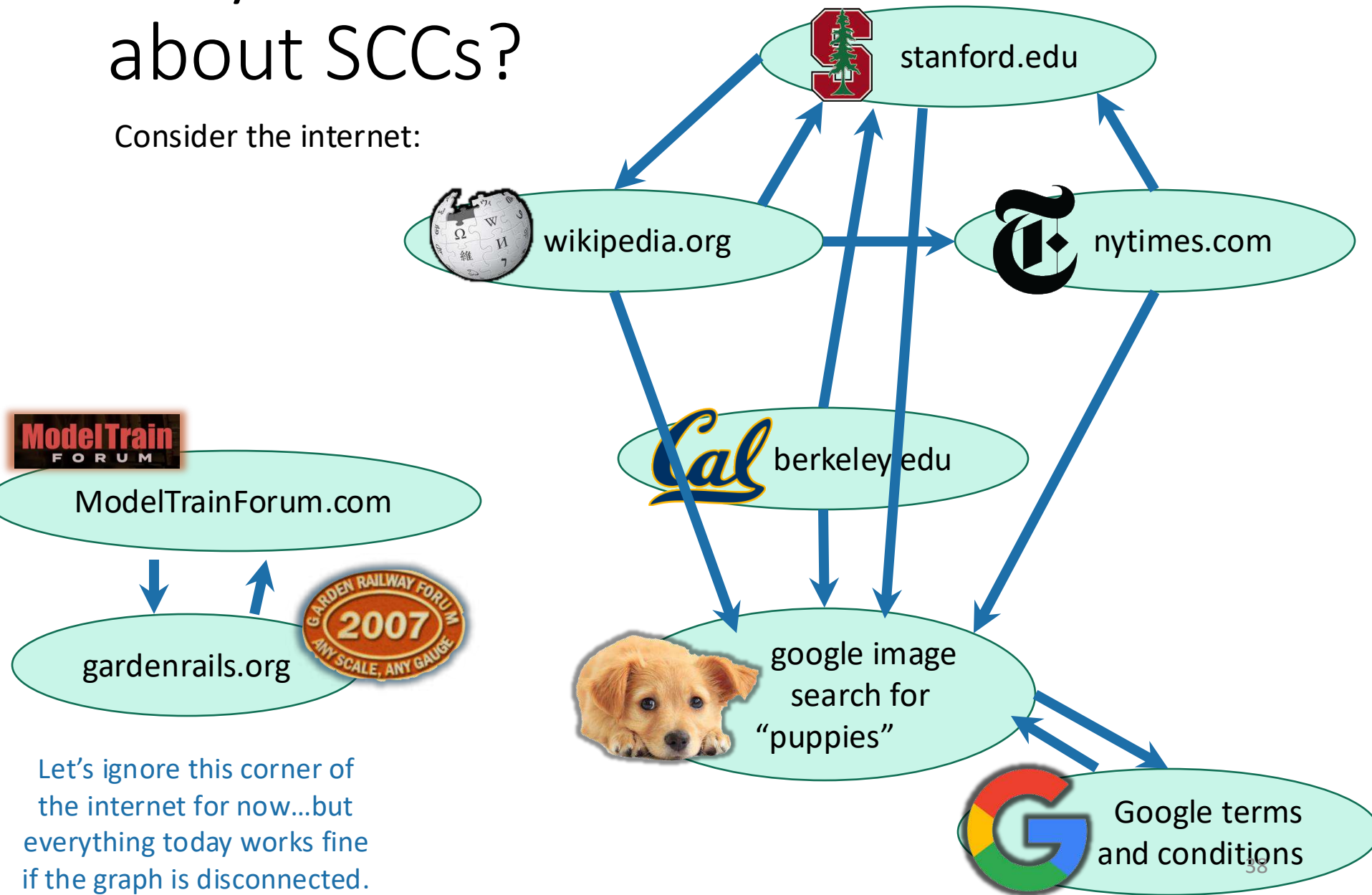
(Definition by example)



Definition by definition: The SCCs are the equivalence classes under the “are mutually reachable” equivalence relation.

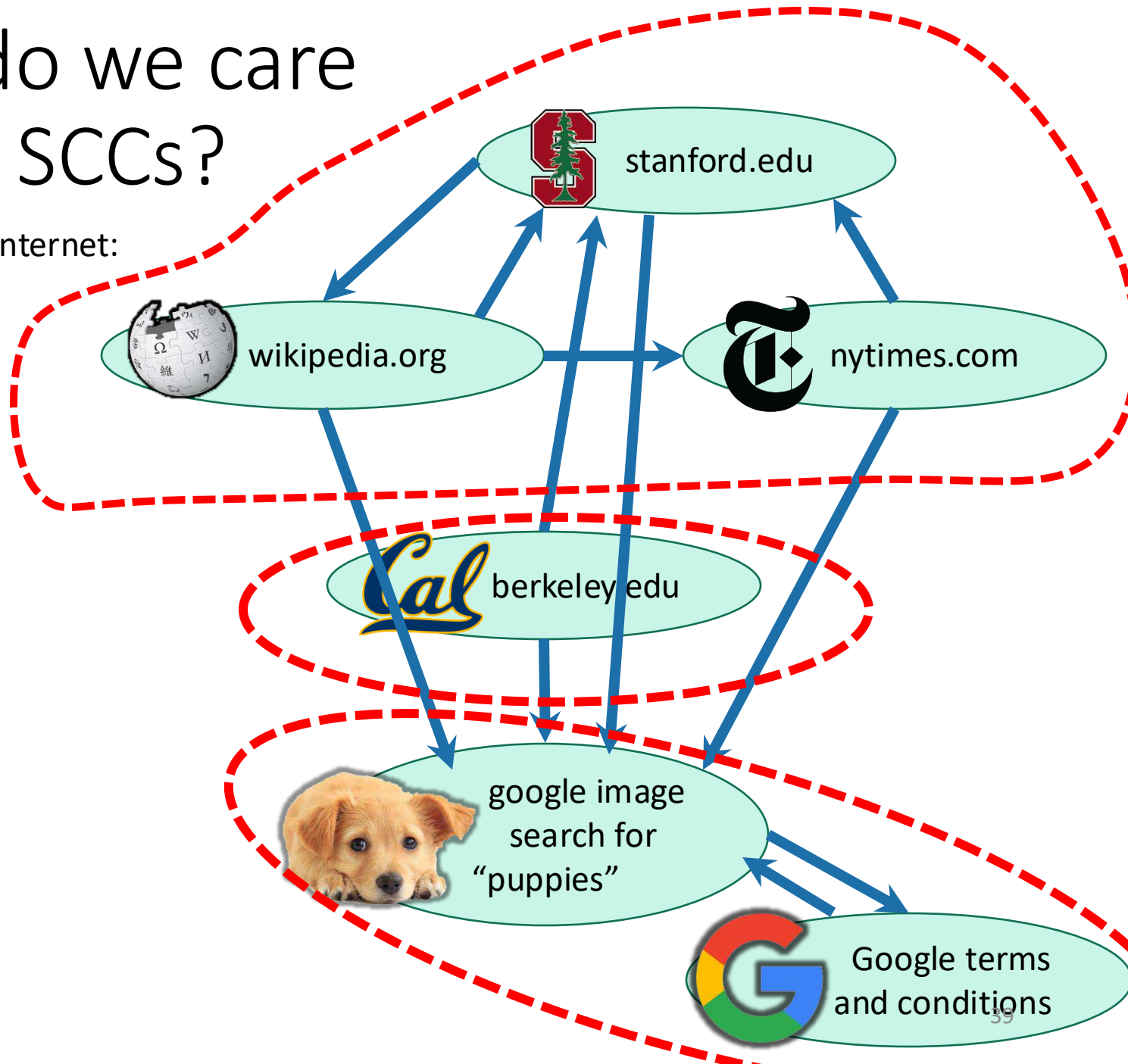
Why do we care about SCCs?

Consider the internet:



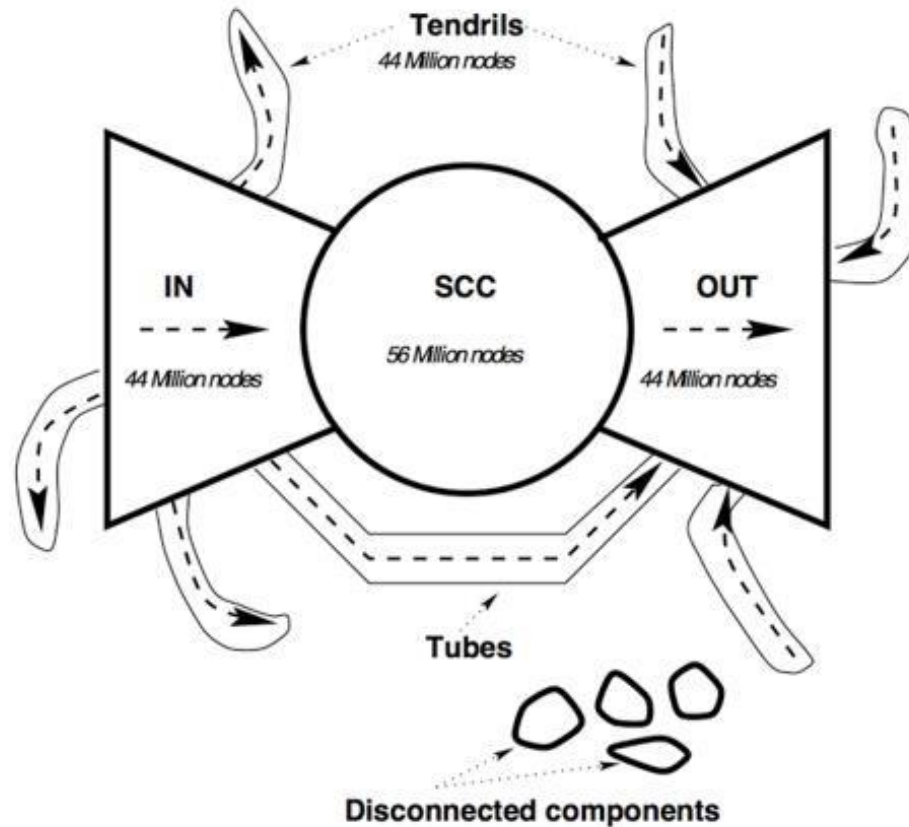
Why do we care about SCCs?

Consider the internet:



Aside: what the internet graph actually looks like

- Circa 2000^[1]:

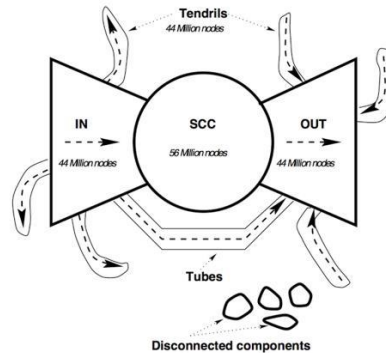


[1] "Graph Structure in the Web." Broder et al. *Computer Networks*. 2000.

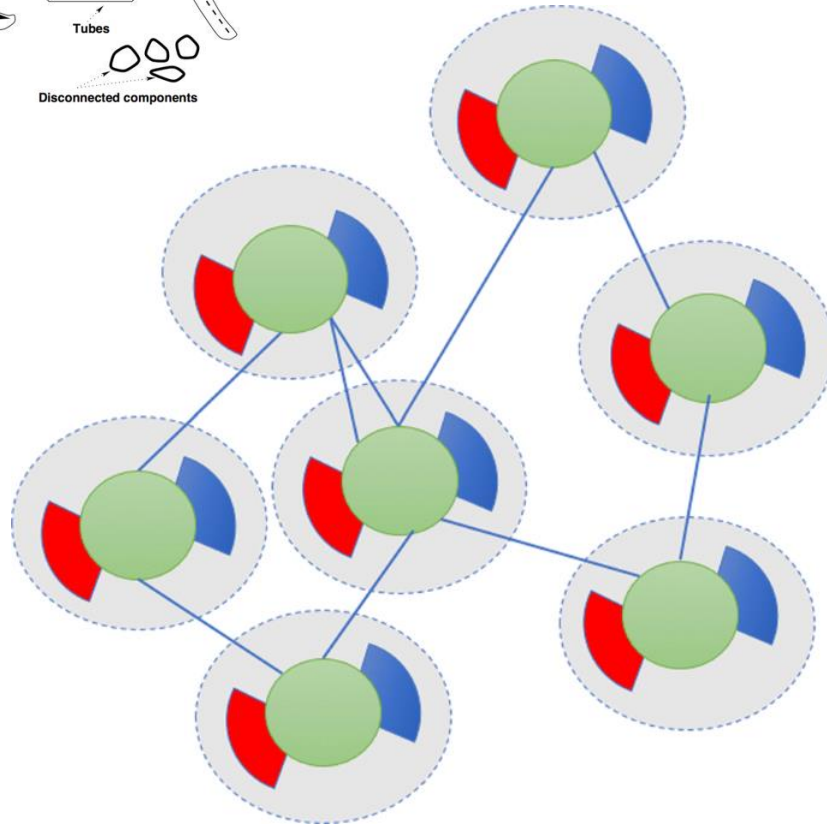
[2] "Local Bow-tie Structure of the Web." Fujita et al. *Applied Network Science*. 2019.

Aside: what the internet graph actually looks like

- Circa 2000^[1]:



- Circa 2019^[2]:

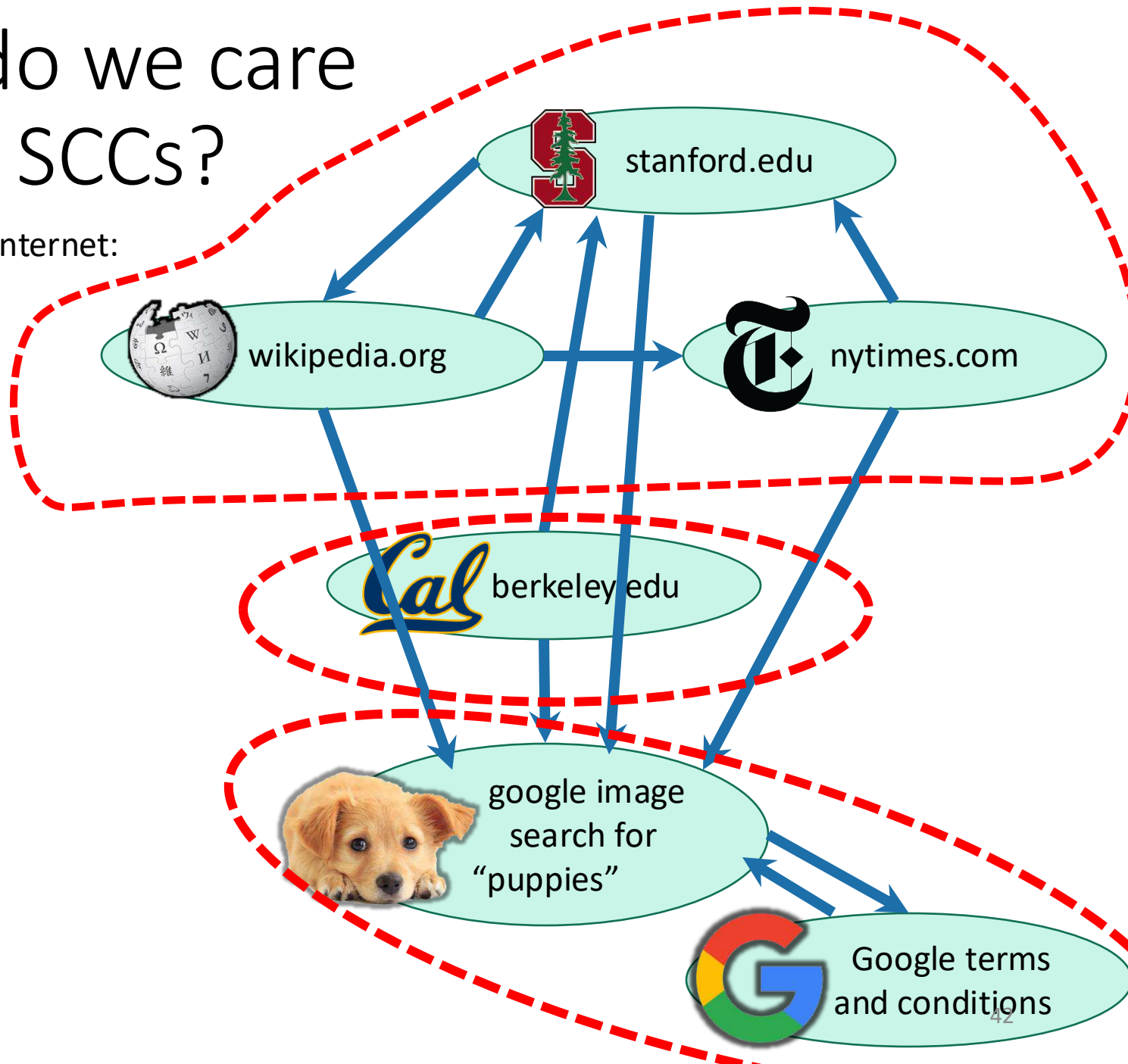


[1] "Graph Structure in the Web." Broder et al. *Computer Networks*. 2000.

[2] "Local Bow-tie Structure of the Web." Fujita et al. *Applied Network Science*. 2019.

Why do we care about SCCs?

Consider the internet:



Why do we care about SCCs?

- Strongly connected components tell you about **communities**.
- Lots of graph algorithms only make sense on SCCs, so we first want to find SCCs and run those algorithms on each part.
 - E.g., “Eigenvector Centrality” (related to PageRank)
- We will see later that the SCCs themselves form a DAG (the “SCC DAG” or “condensation graph”), which can also be useful to analyze!

How to find SCCs?

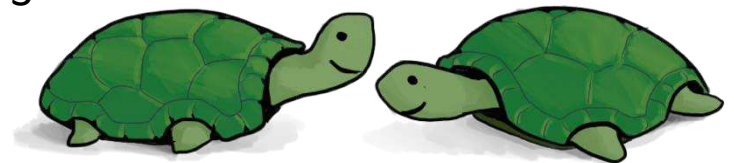
Try 1:

- Consider all possible decompositions and check.

Try 2:

- Something like...
 - Run DFS a bunch to find out which u 's and v 's belong in the same SCC
 - Aggregate that information to figure out the SCCs

Come up with a straightforward way to use DFS to find SCCs. What's the running time?
More than n^2 or less than n^2 ?



One straightforward solution

- SCCs = []
- For each u:
 - Run DFS from u
 - For each vertex v that u can reach:
 - If v is in an SCC we've already found:
 - Run DFS from v to see if you can reach u
 - If so, add u to v's SCC
 - Break
 - If we didn't break, create a new SCC which just contains u.

This will not be our final solution so don't worry too much about it...



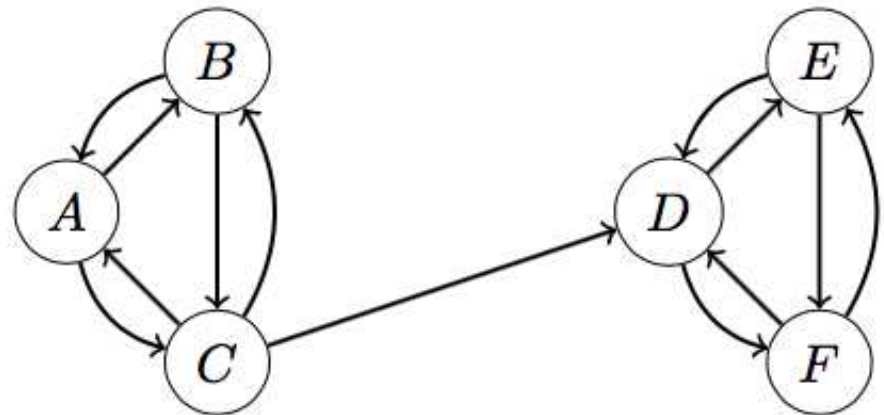
Running time AT LEAST $\Omega(n^2)$, no matter how smart you are about implementing the rest of it...

Today

- We will see how to find strongly connected components in time $O(n+m)$
- !!!!!
- This is called Kosaraju's algorithm.
 - Heads up: the textbook has a slightly different presentation.

Pre-Lecture exercise

- What do you get when you run DFS from A?
- What about from D?



- Suggested algorithm: run DFS from the “right” place to identify SCCs
- Issue: what is the “right” place?

Algorithm

Running time: $O(n + m)$

- Do DFS to create a DFS forest.
 - Choose starting vertices in any order.
 - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
 - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest**.



Look, it works!

- (See IPython notebook)

```
In [4]: print(G)
```

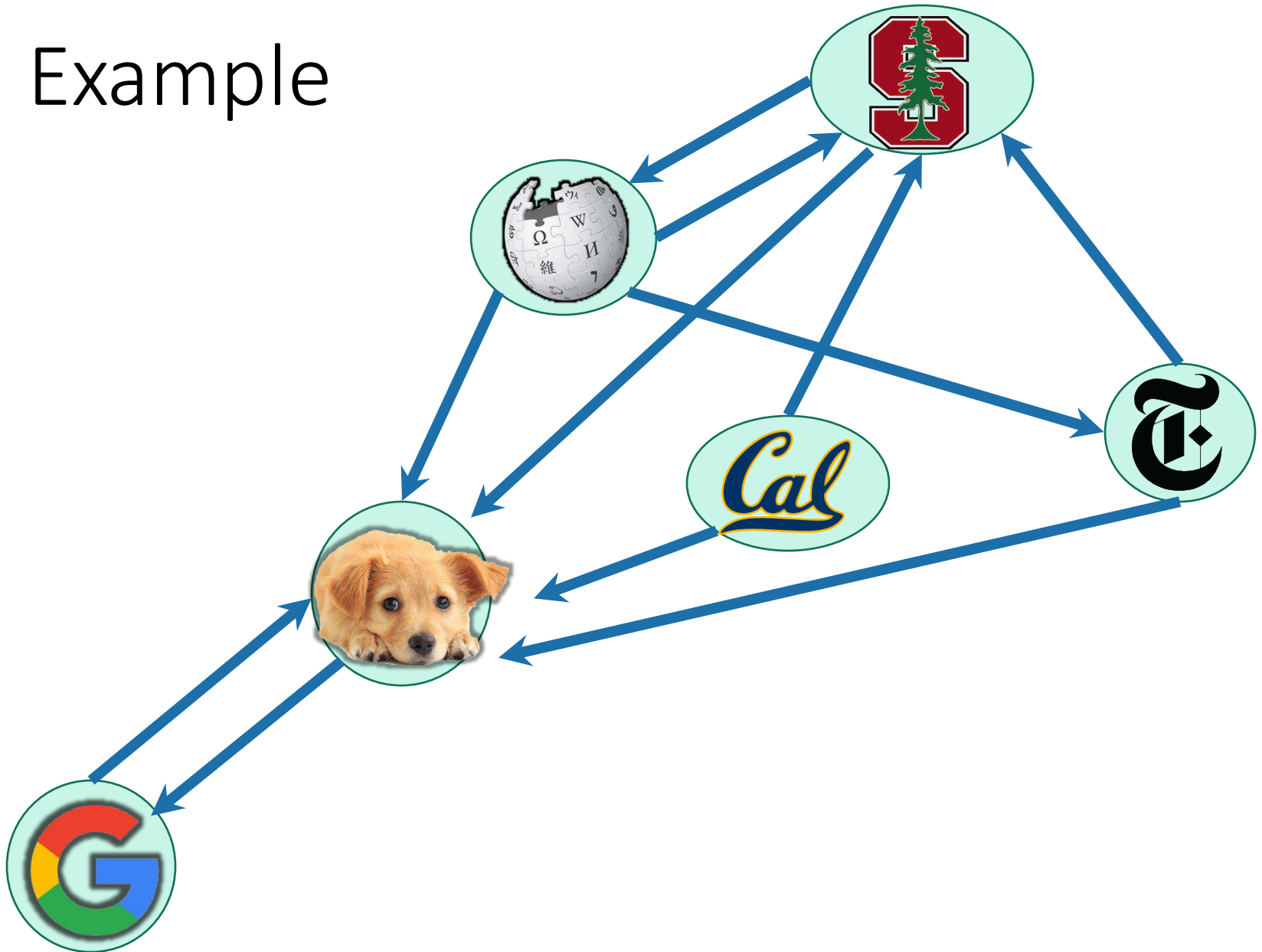
```
CS161Graph with:
  Vertices:
    Stanford,Wikipedia,NYTimes,Berkeley,Puppies,Google,
  Edges:
    (Stanford,Wikipedia) (Stanford,Puppies) (Wikipedia,Stanford) (Wikipedia,NYTimes) (Wikipedia,Puppies) (NYTimes,Stanford) (NYTimes,Puppies) (Berkeley,Stanford) (Berkeley,Puppies) (Puppies,Google) (Google,Puppies)
```

```
In [5]: SCCs = SCC(G, False)
for X in SCCs:
    print ([str(x) for x in X])
```

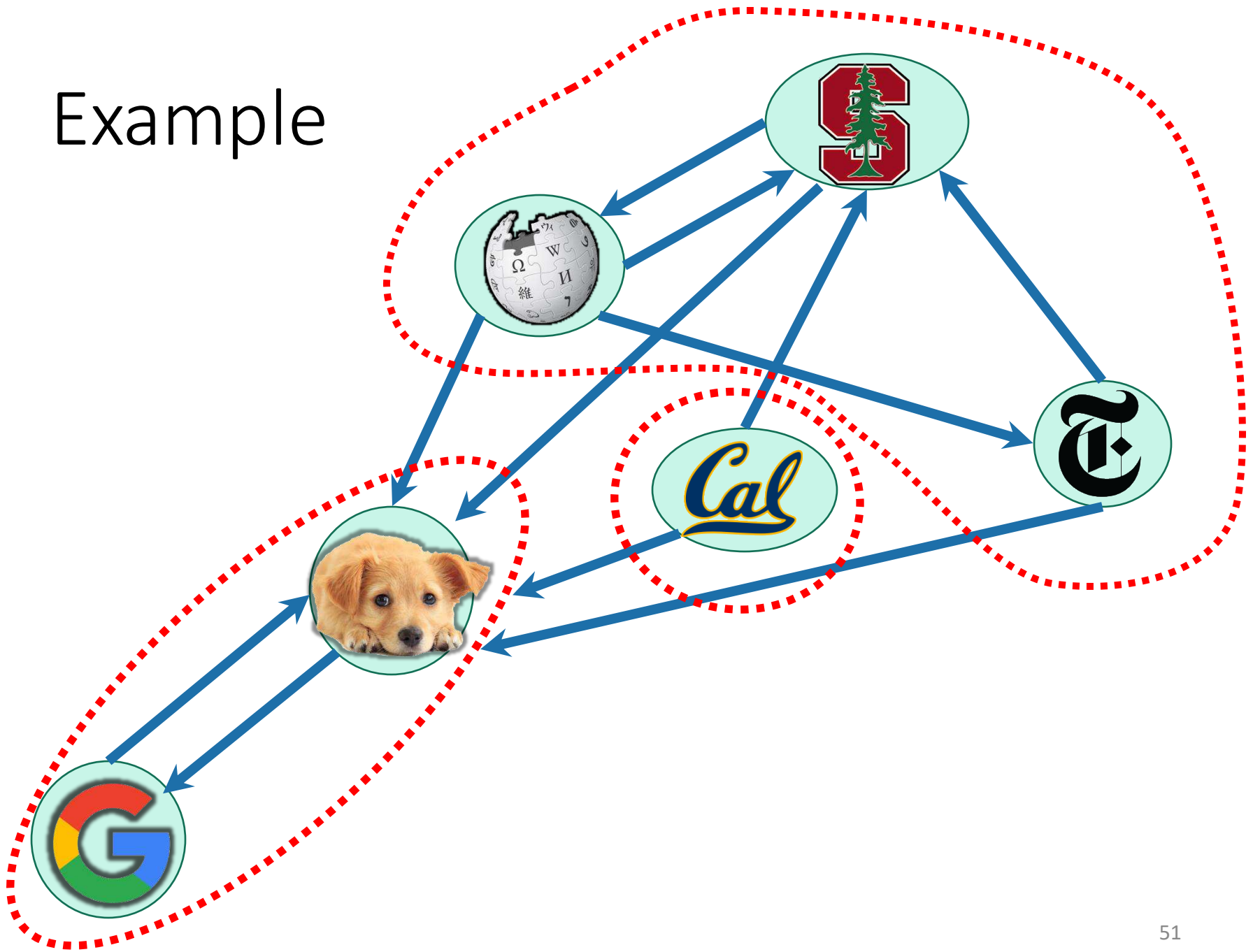
```
['Berkeley']
['Stanford', 'NYTimes', 'Wikipedia']
['Puppies', 'Google']
```

But let's dig into it...

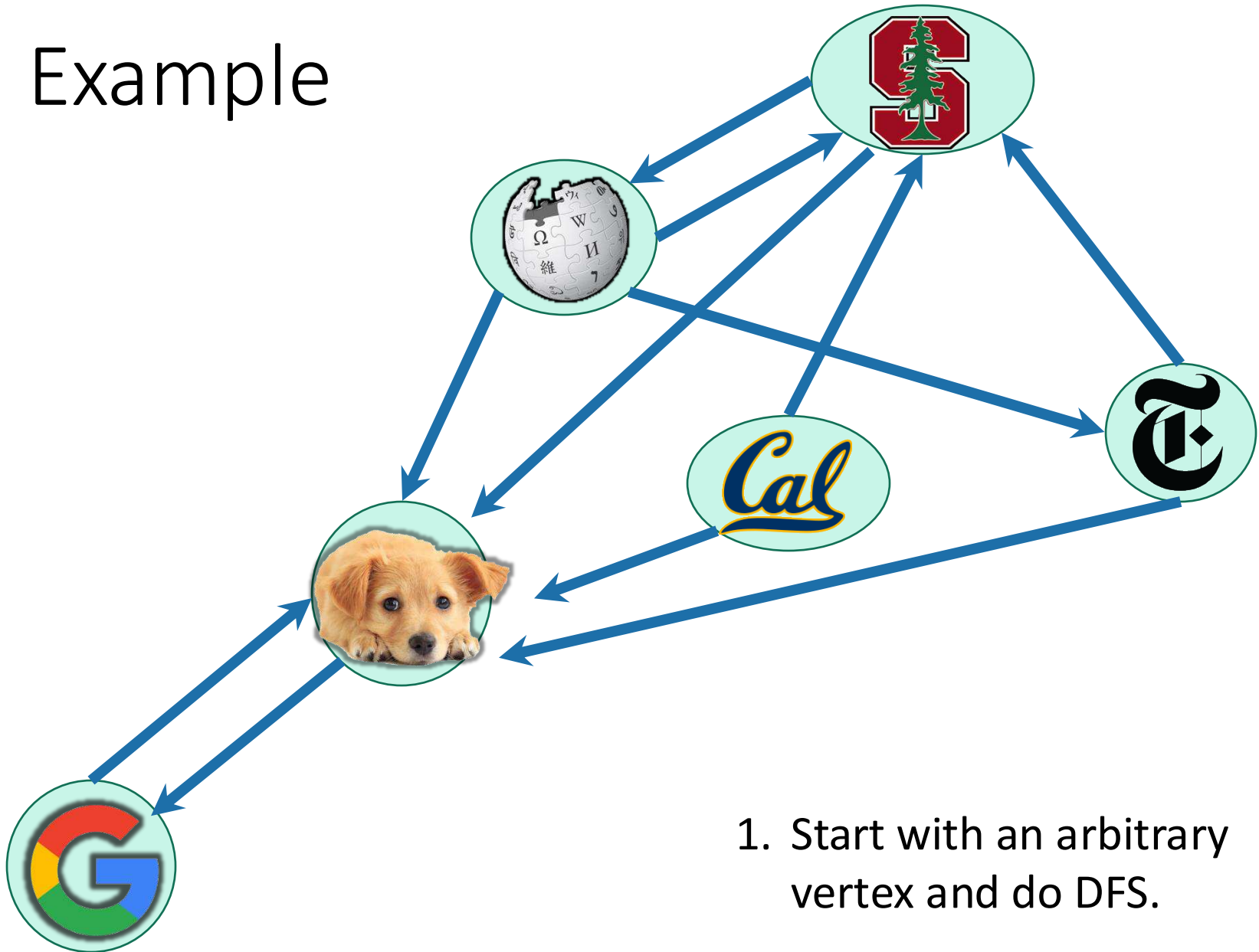
Example



Example

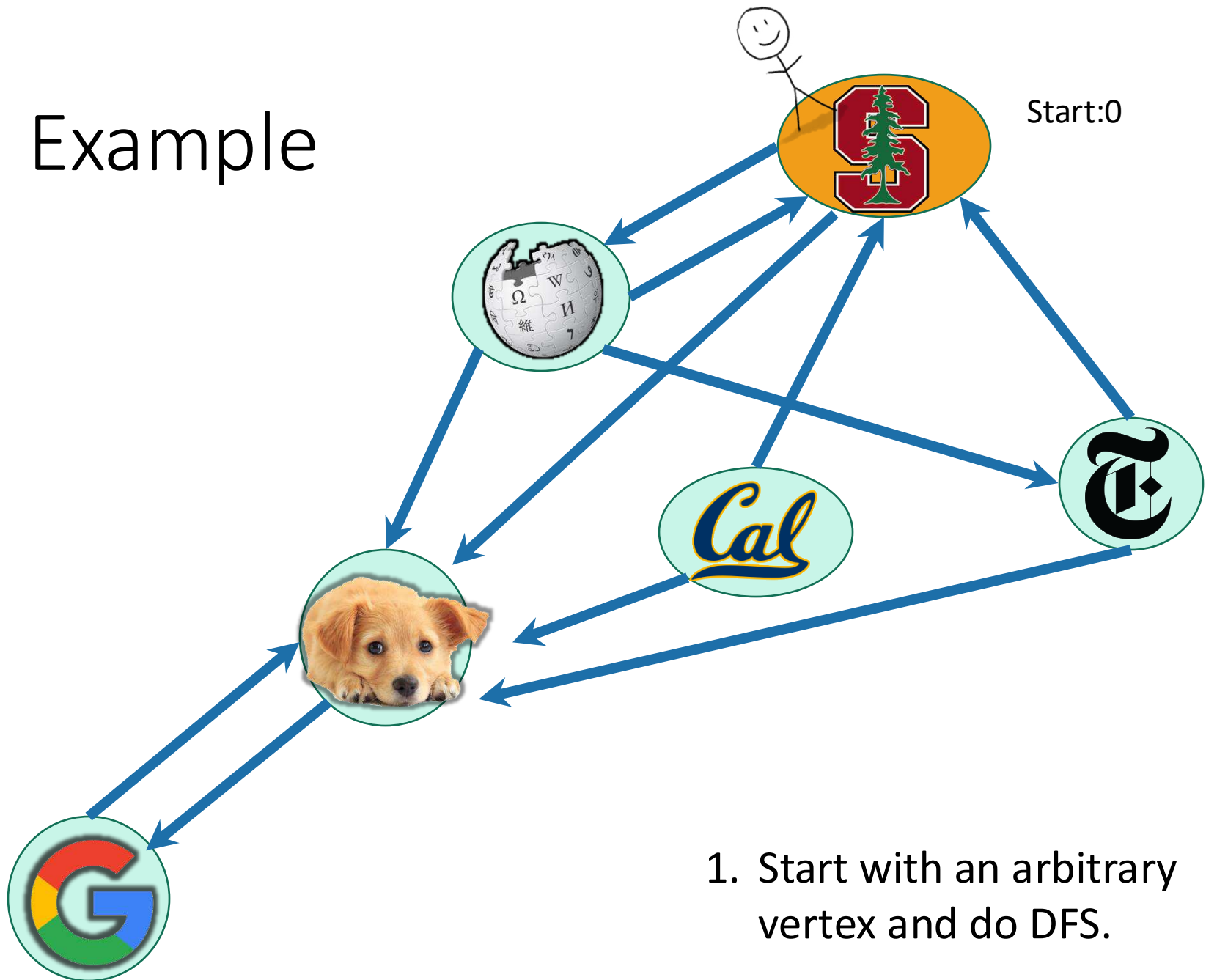


Example



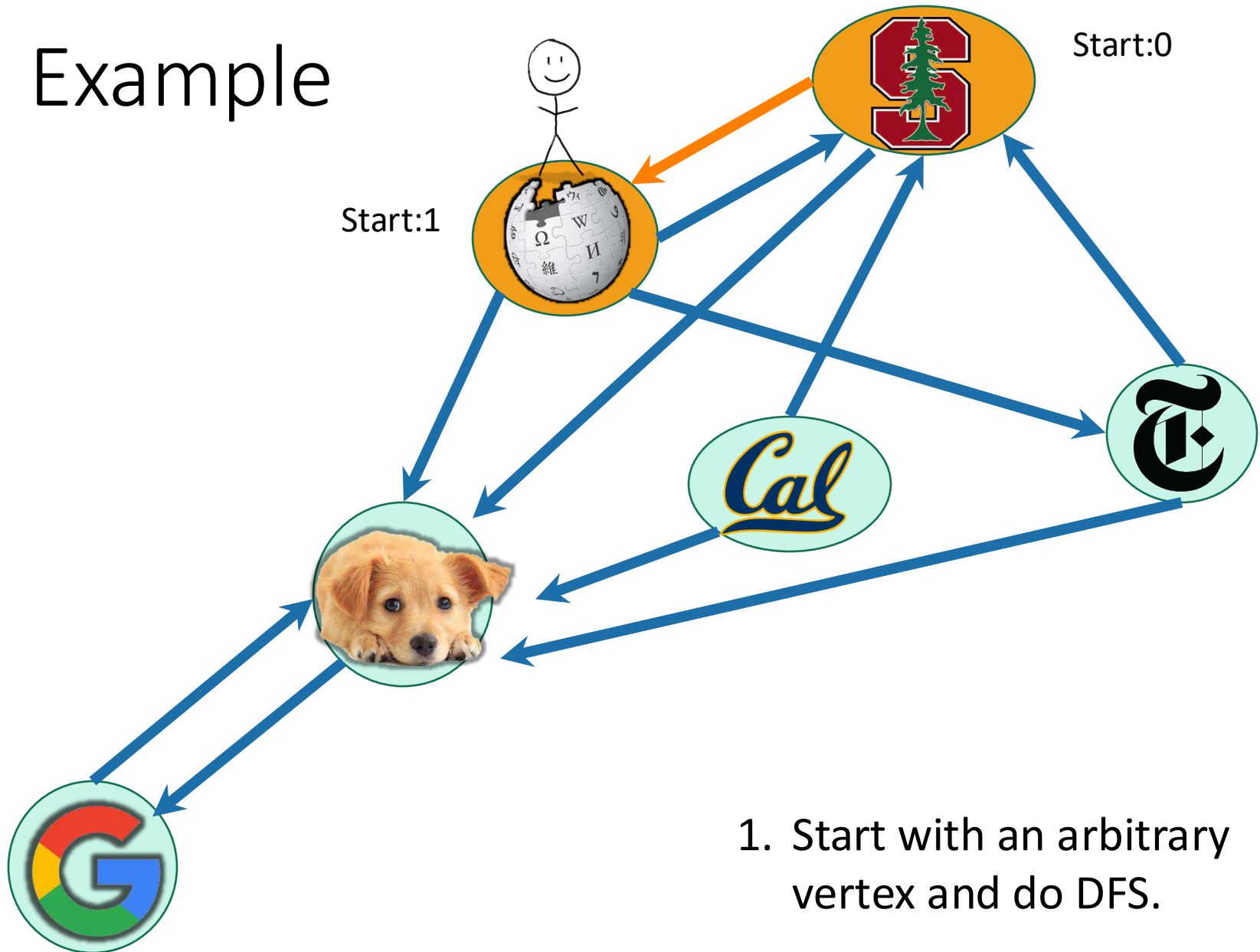
1. Start with an arbitrary vertex and do DFS.

Example



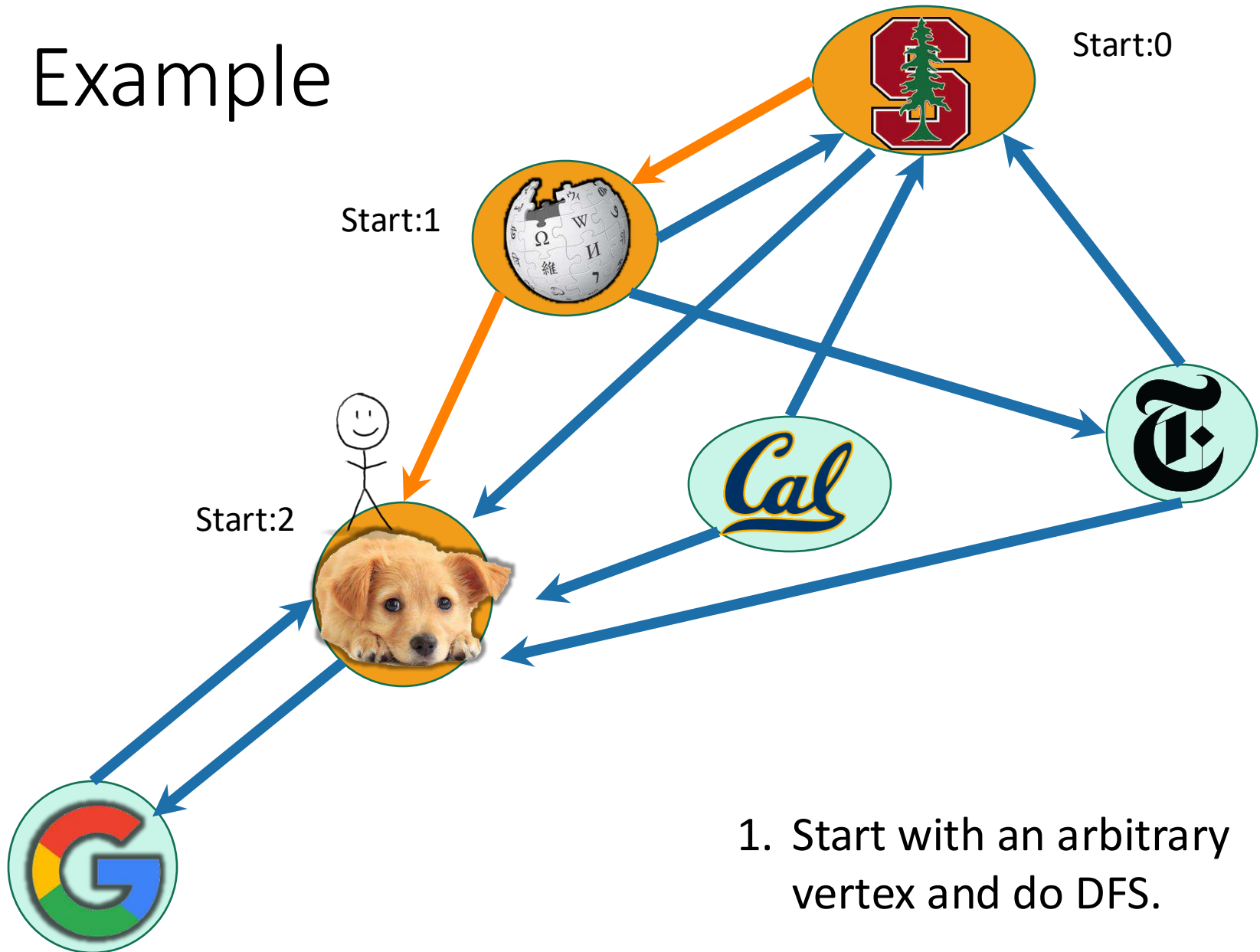
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Example



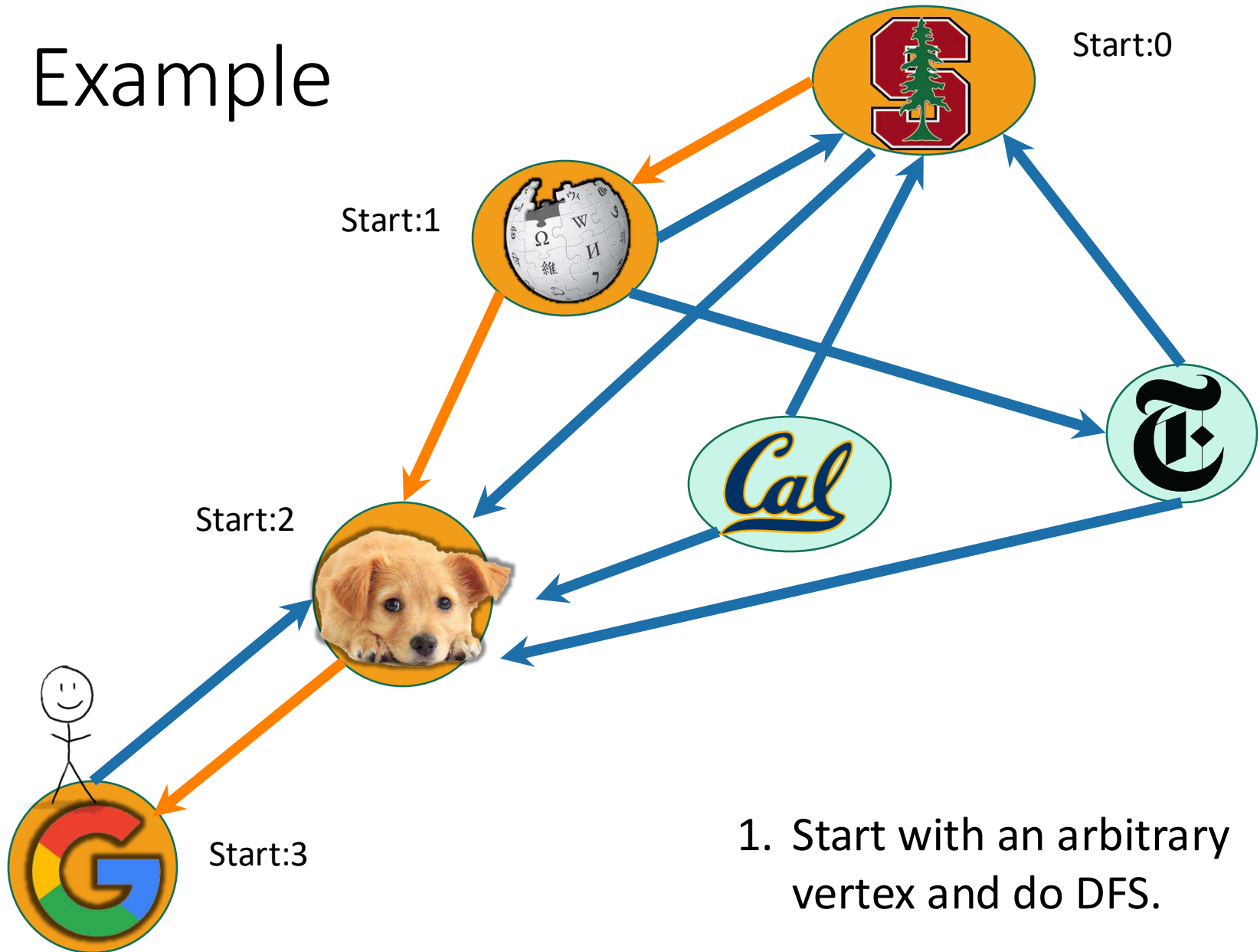
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Example



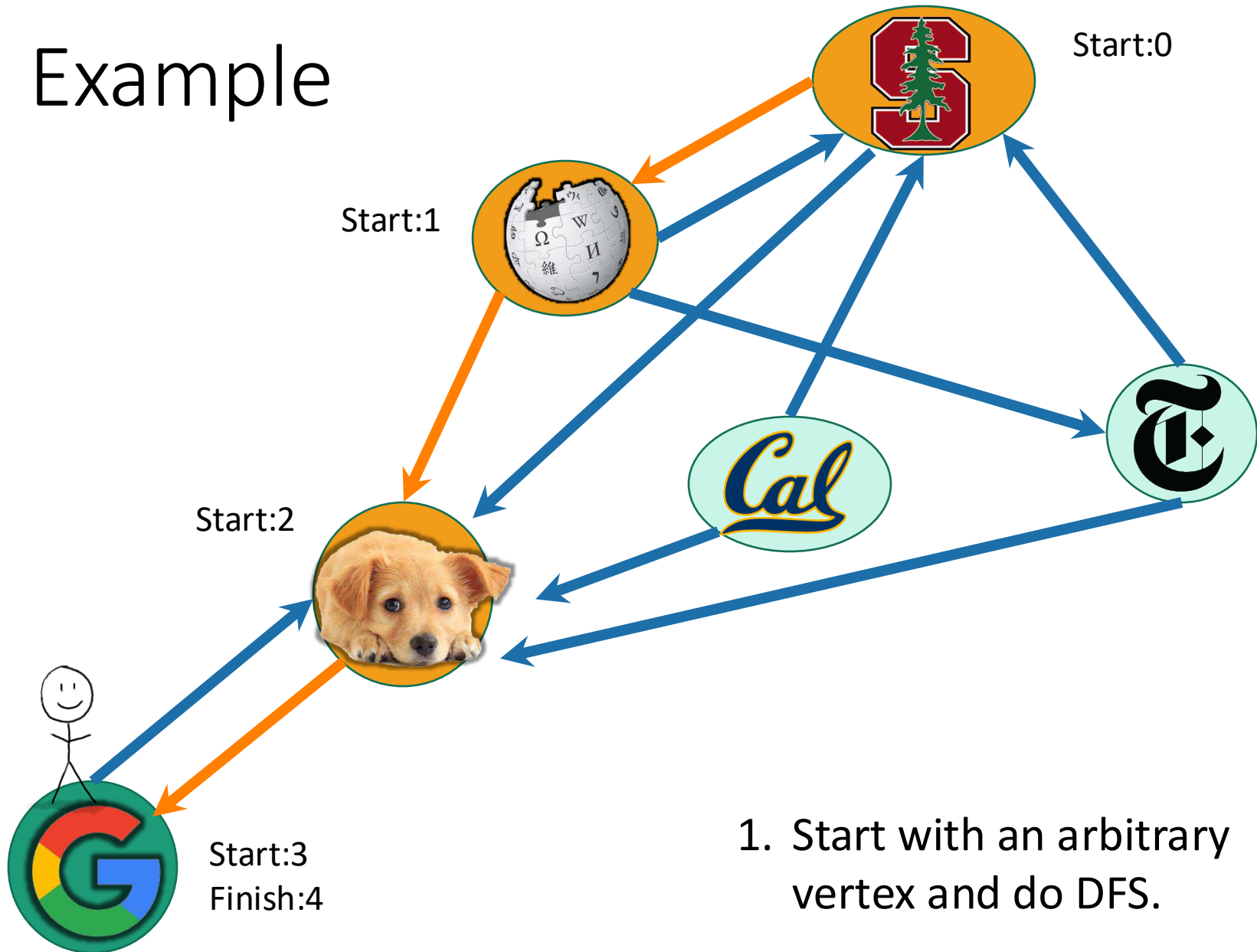
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Example



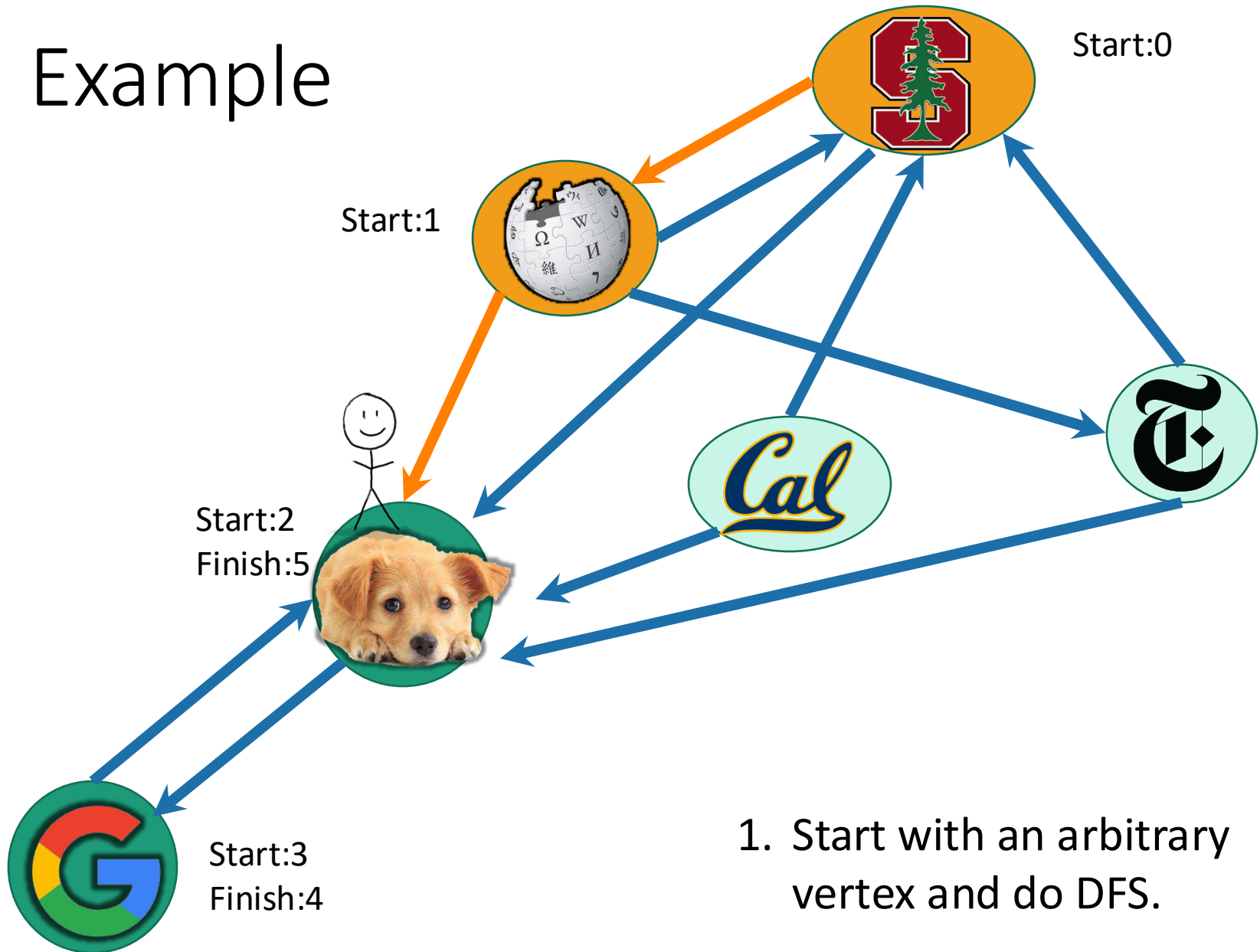
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Example



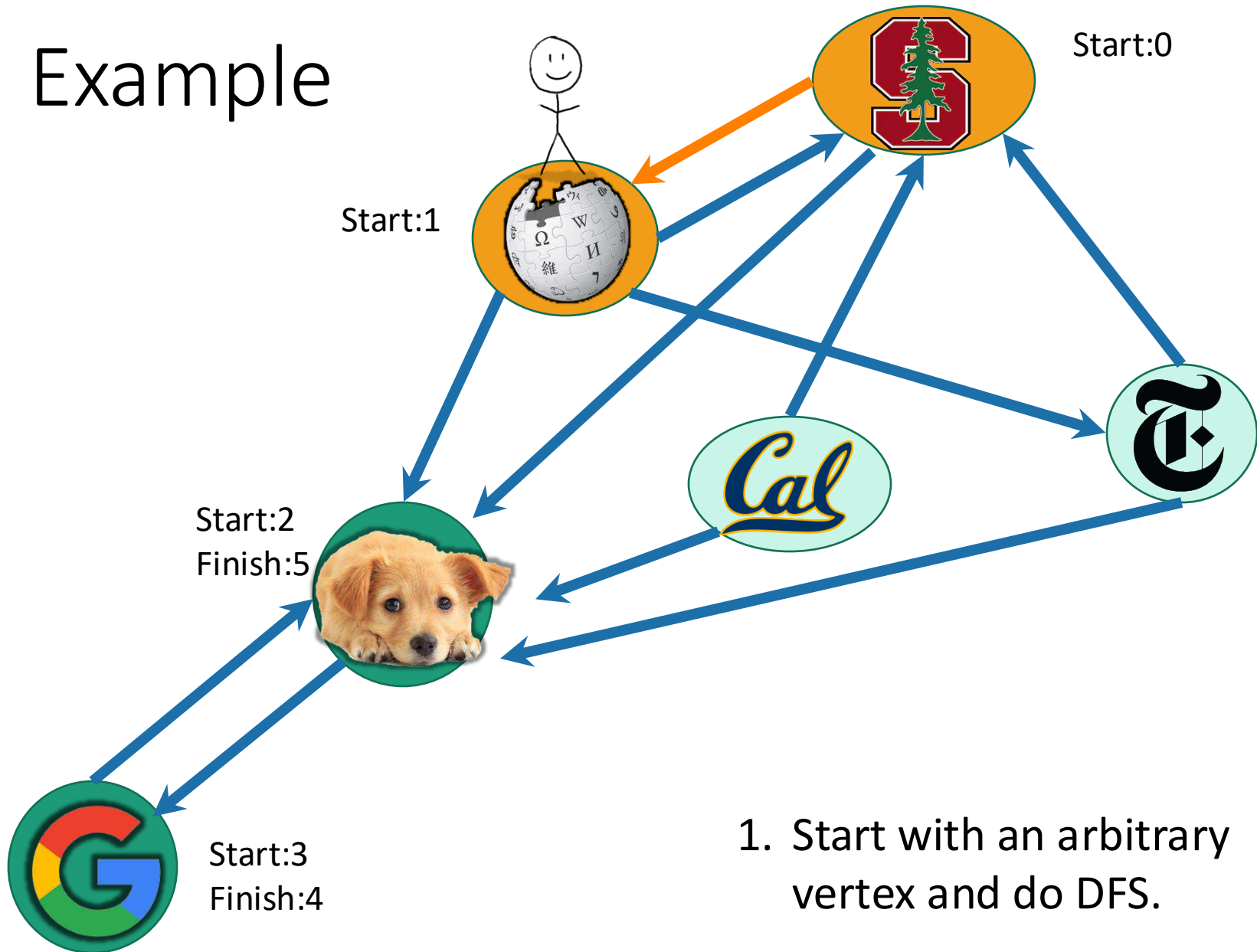
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Example

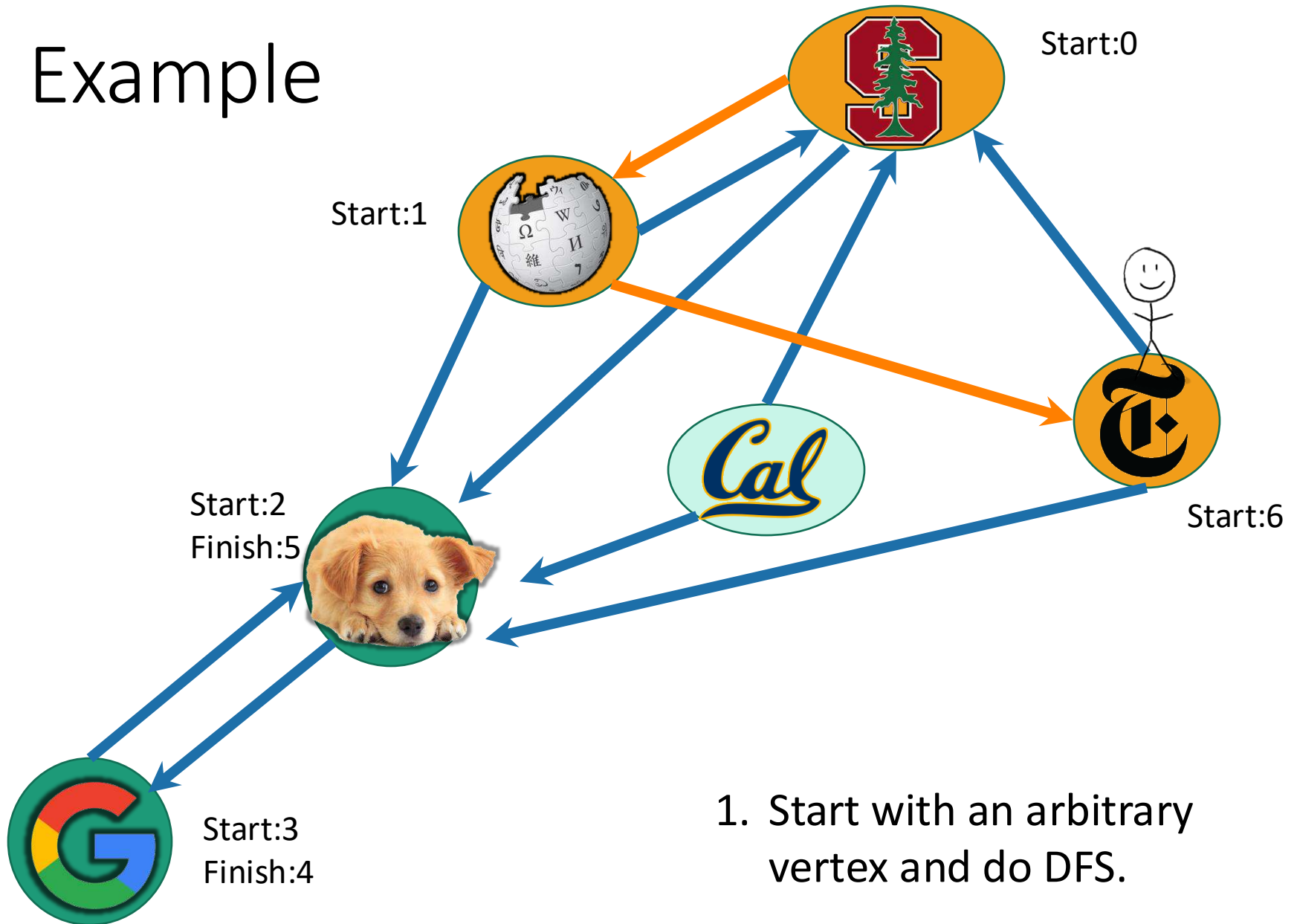


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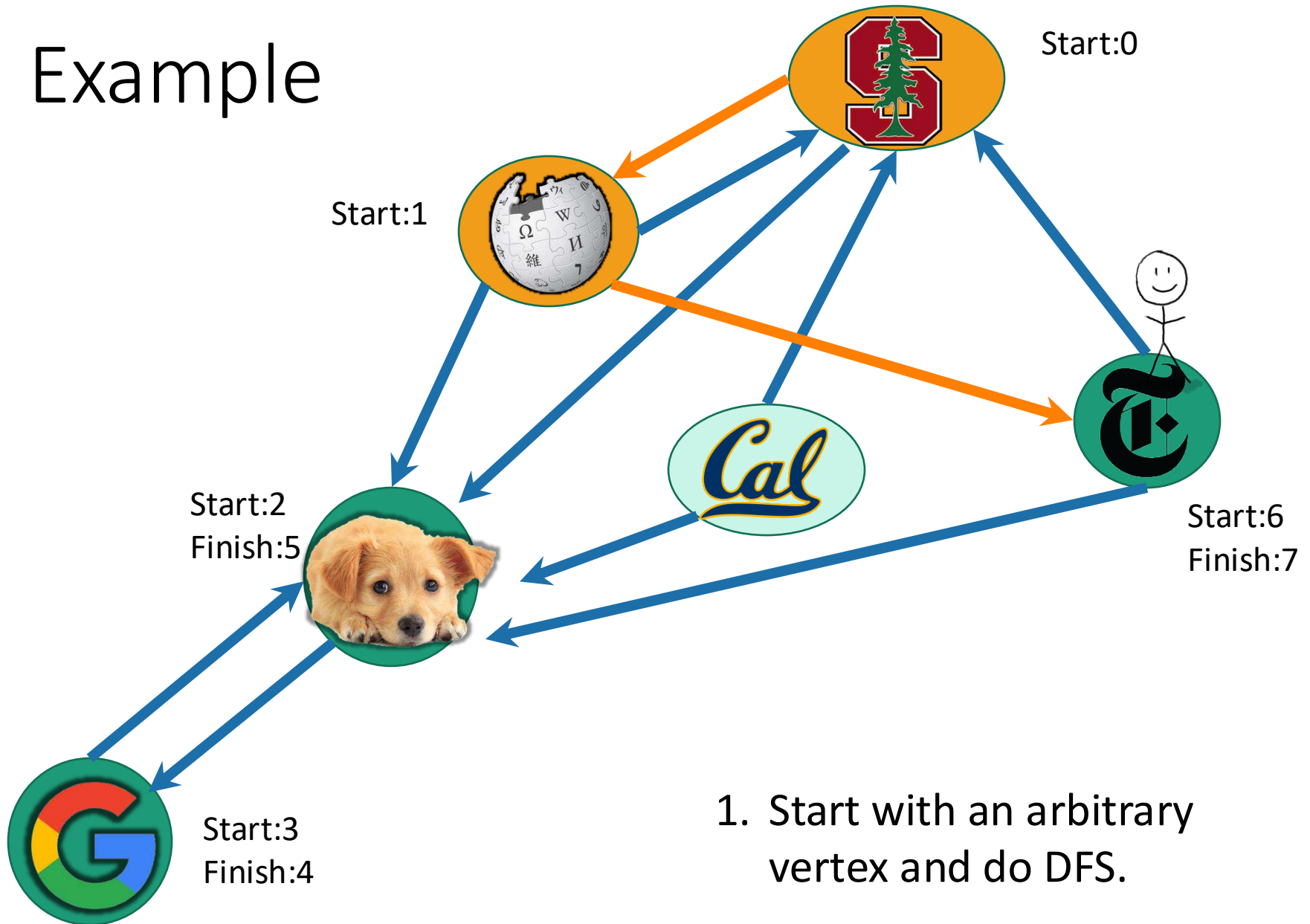
Example



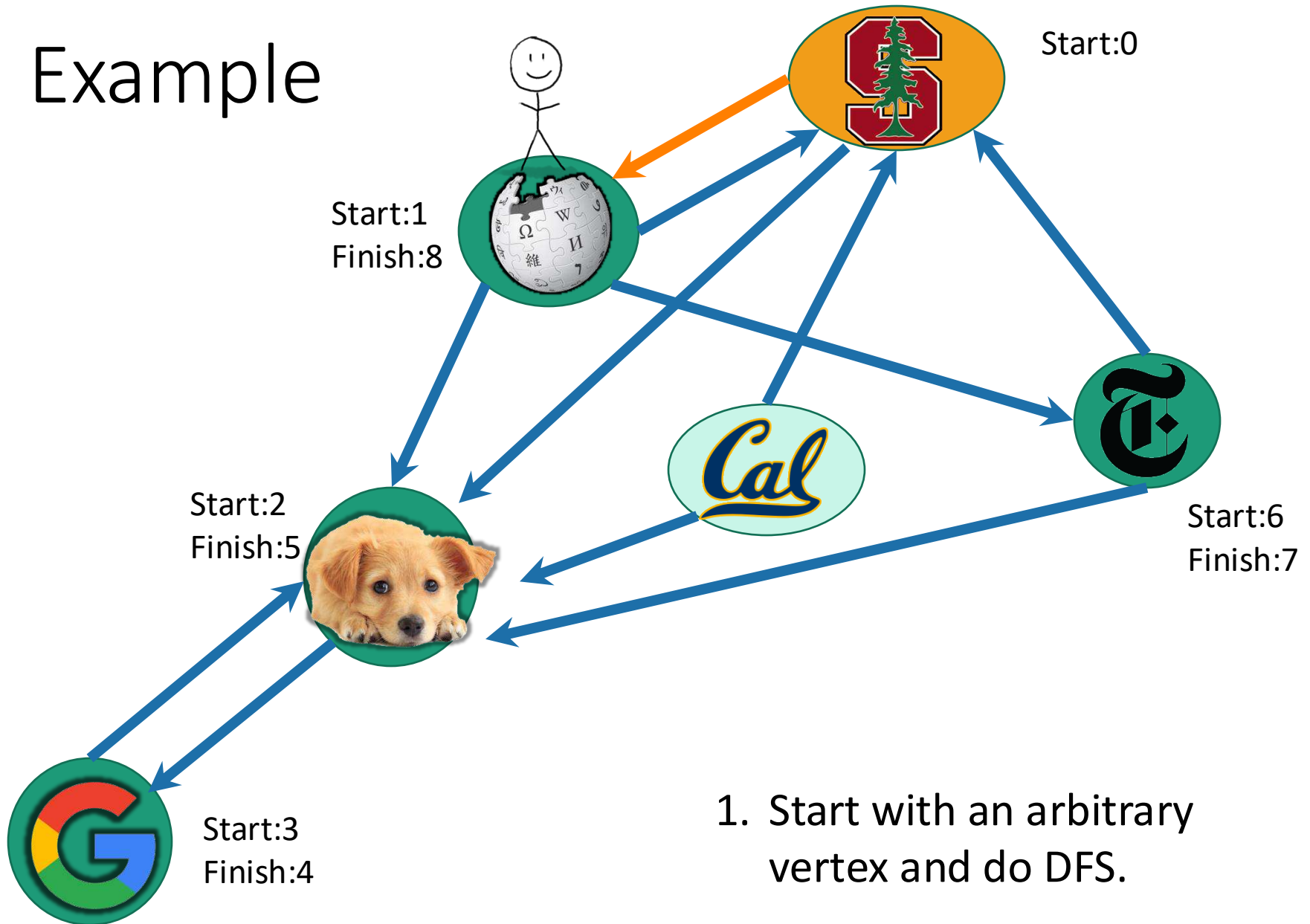
Example



Example

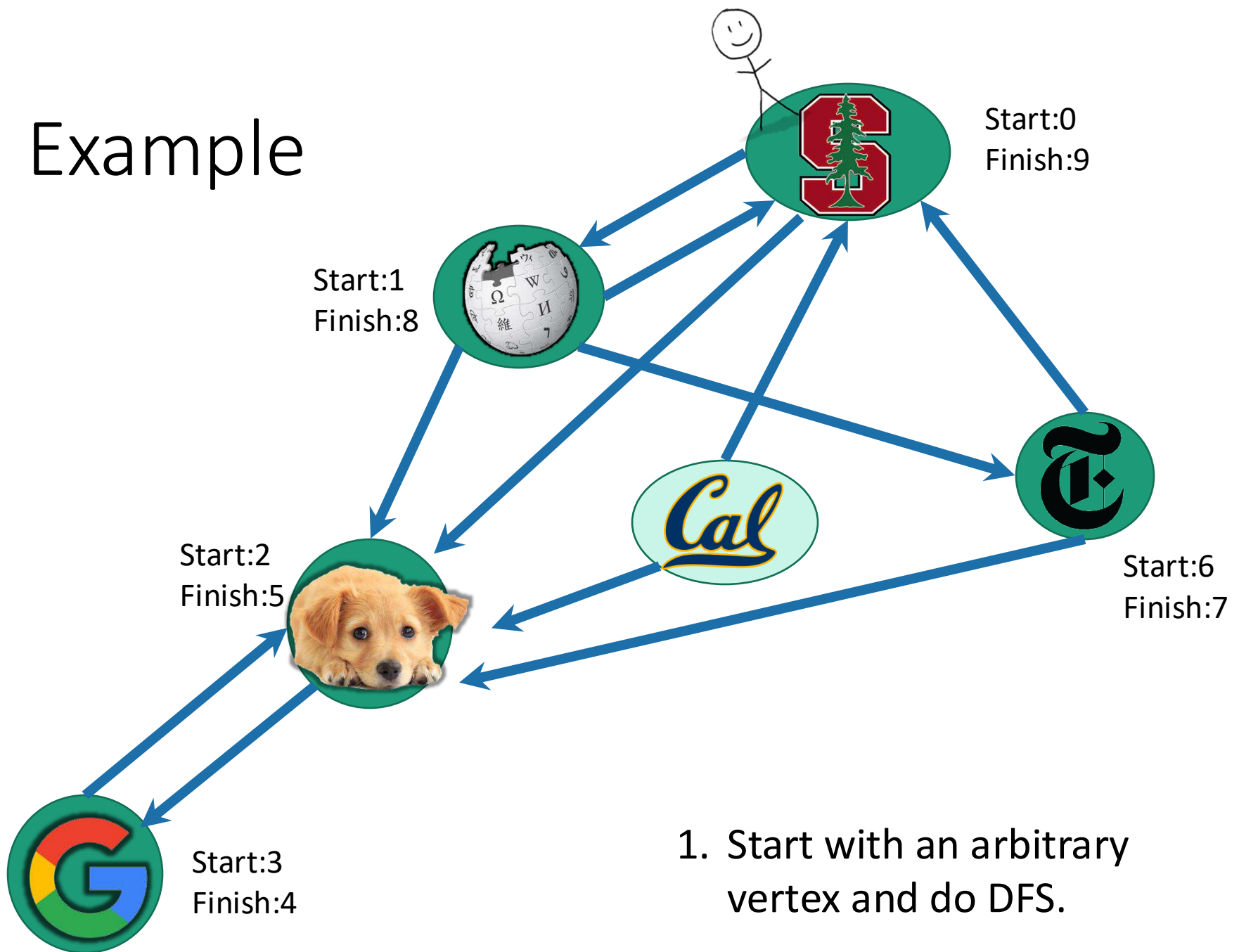


Example

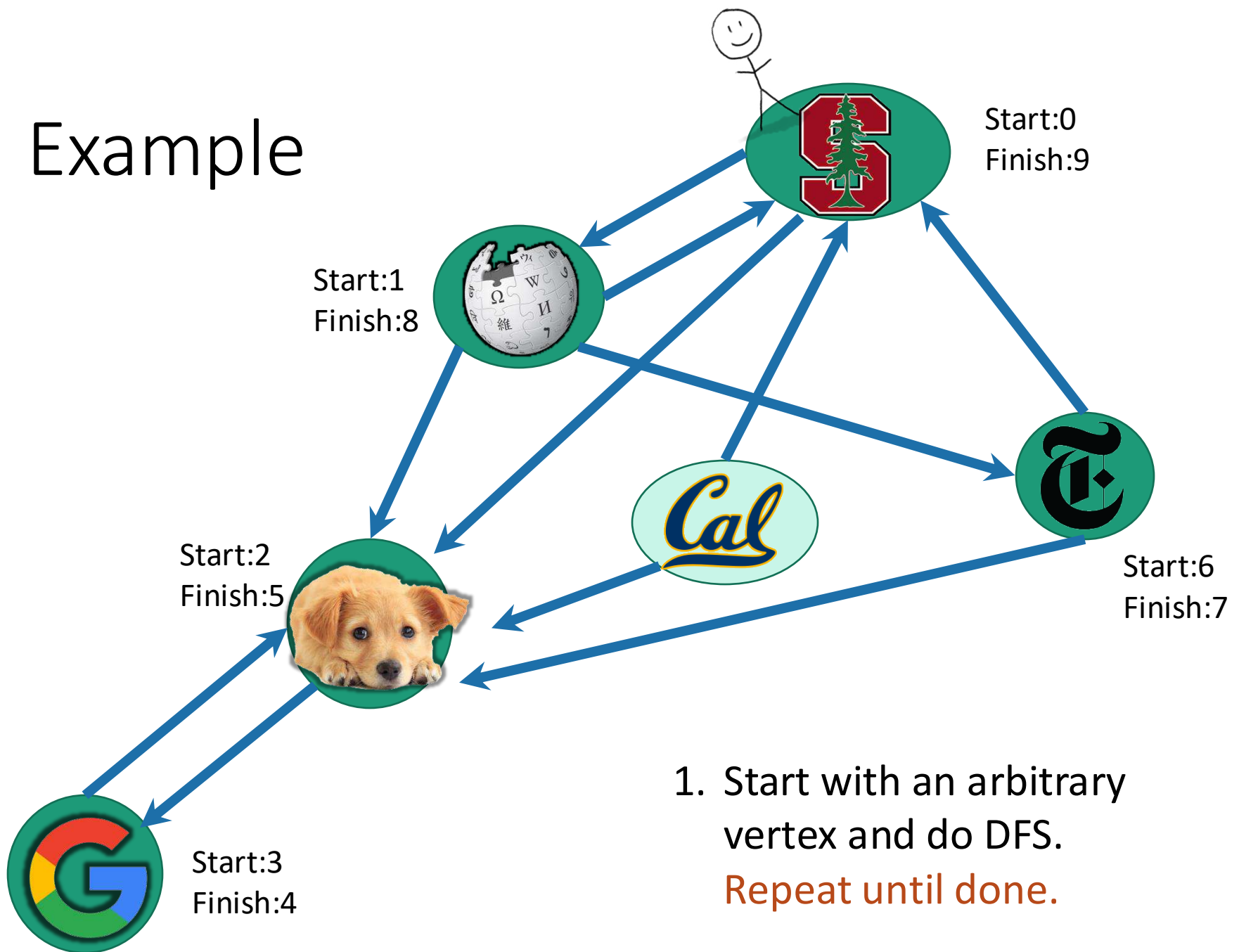


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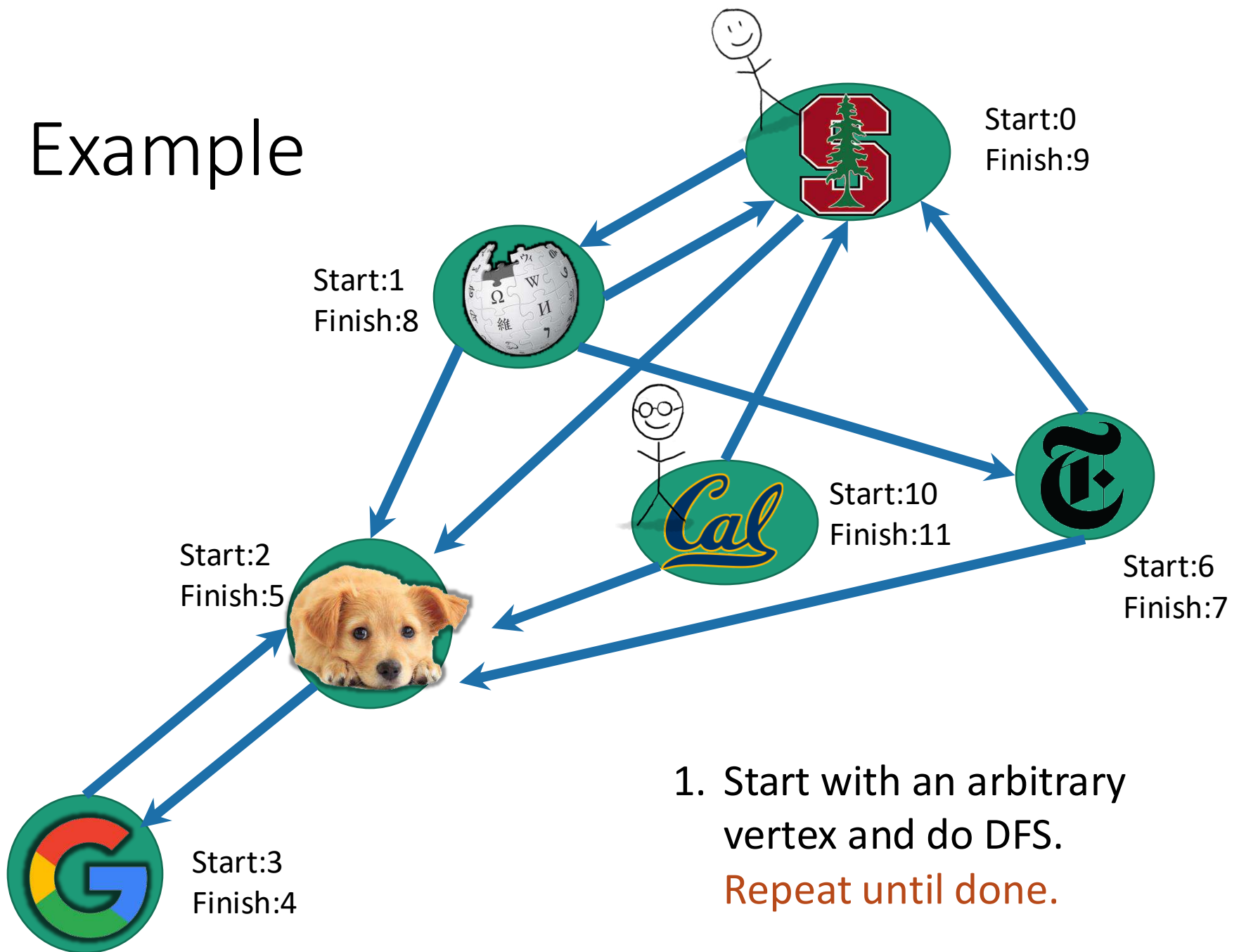
Example



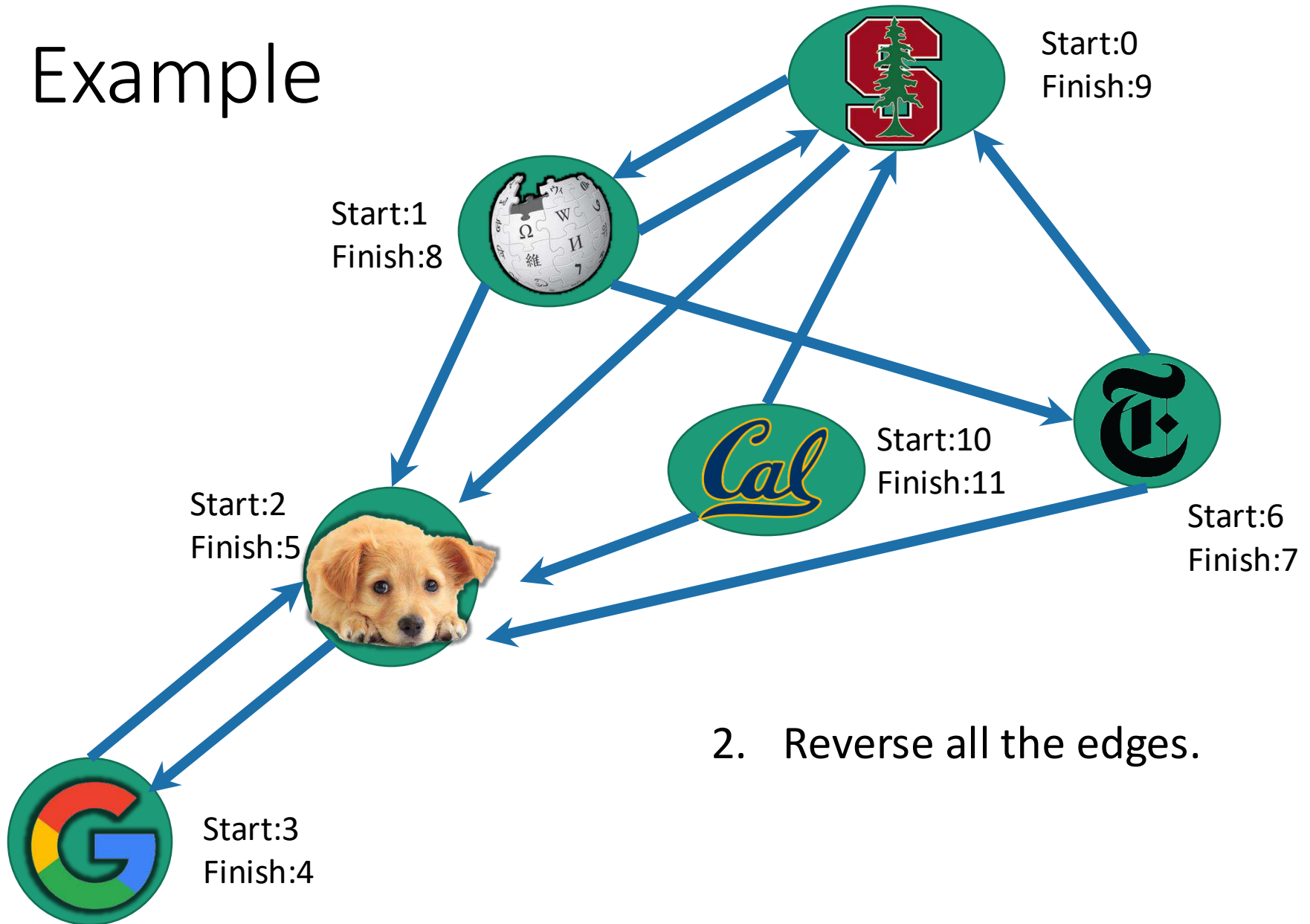
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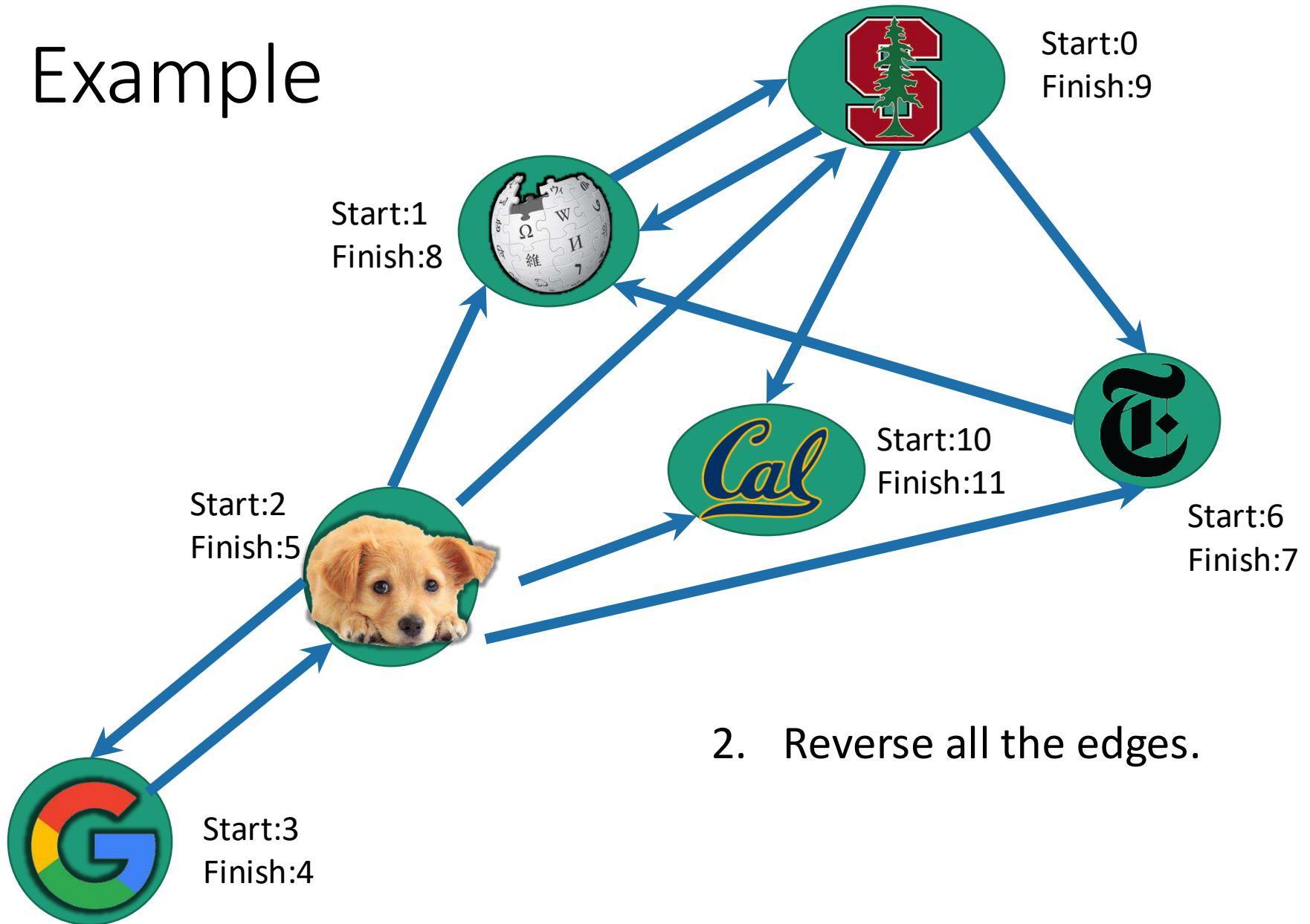
Example



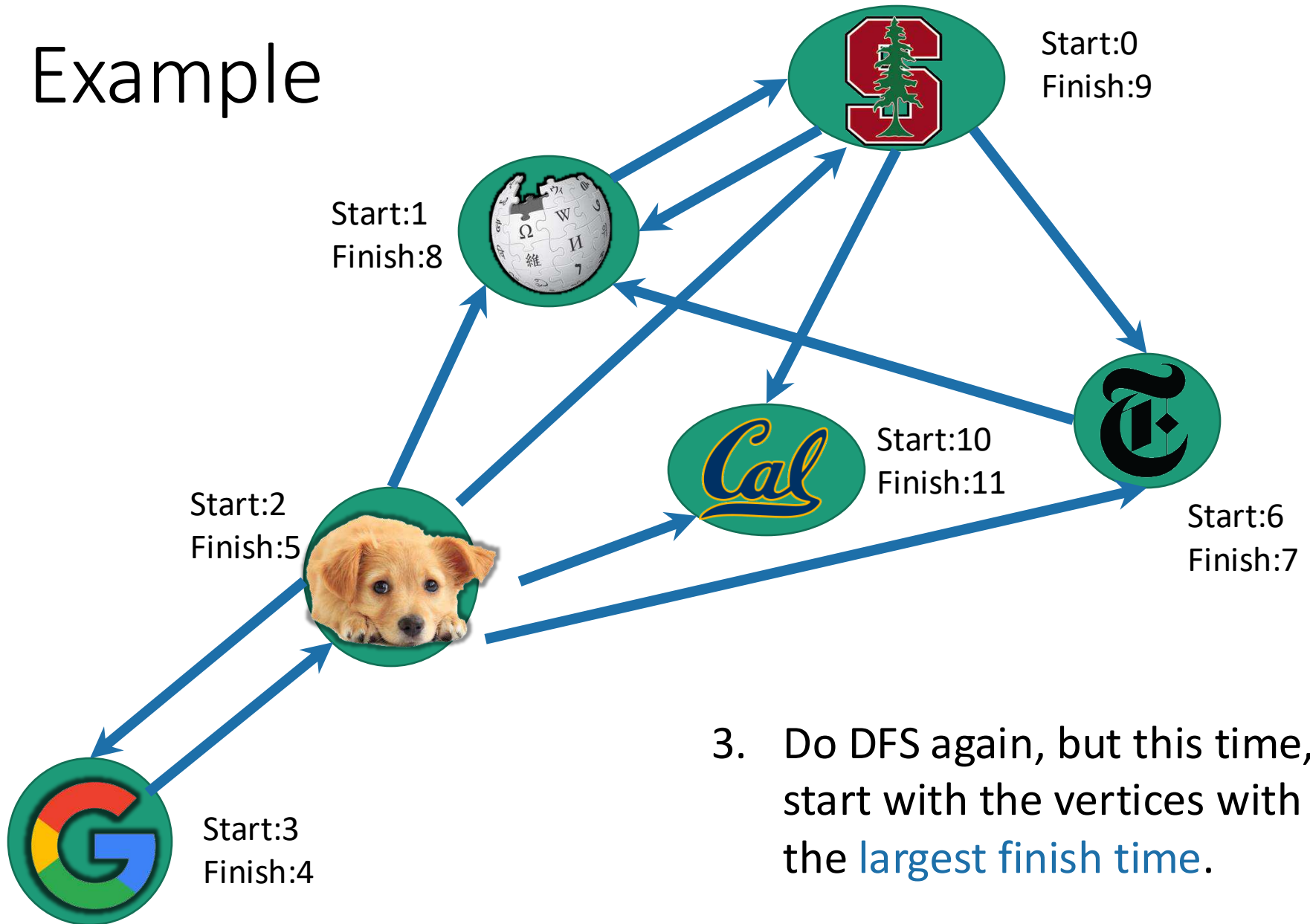
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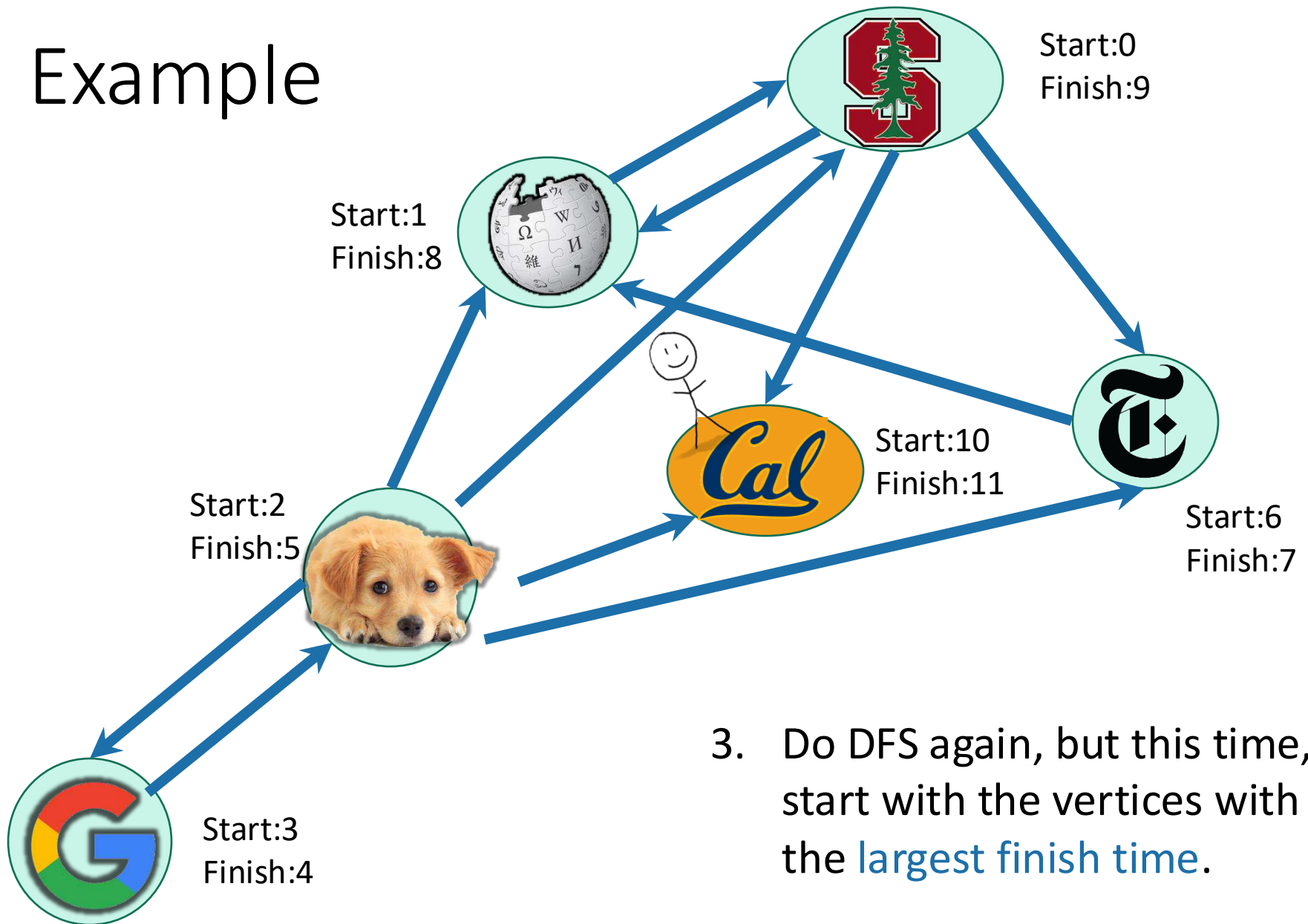


Example

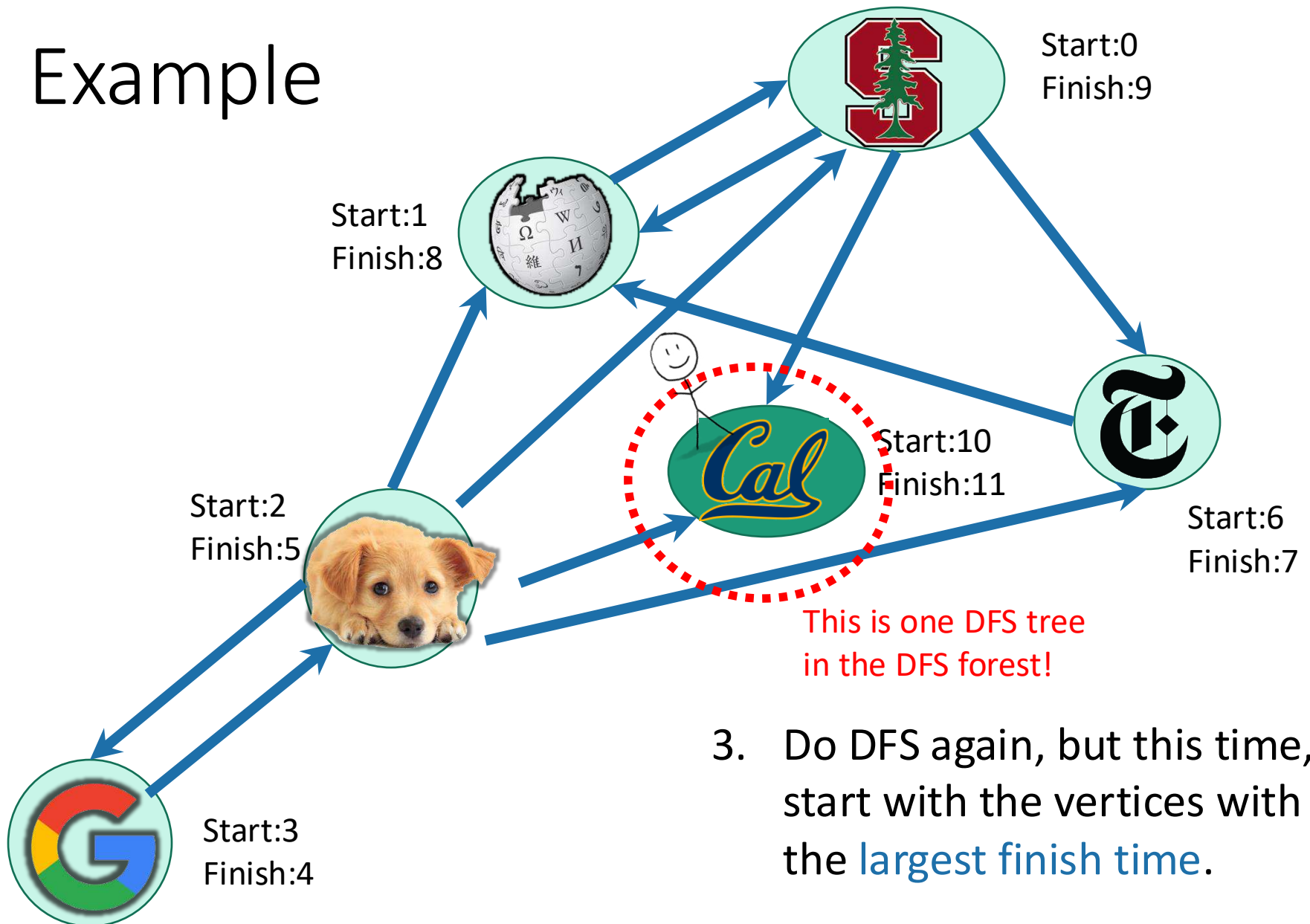


3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

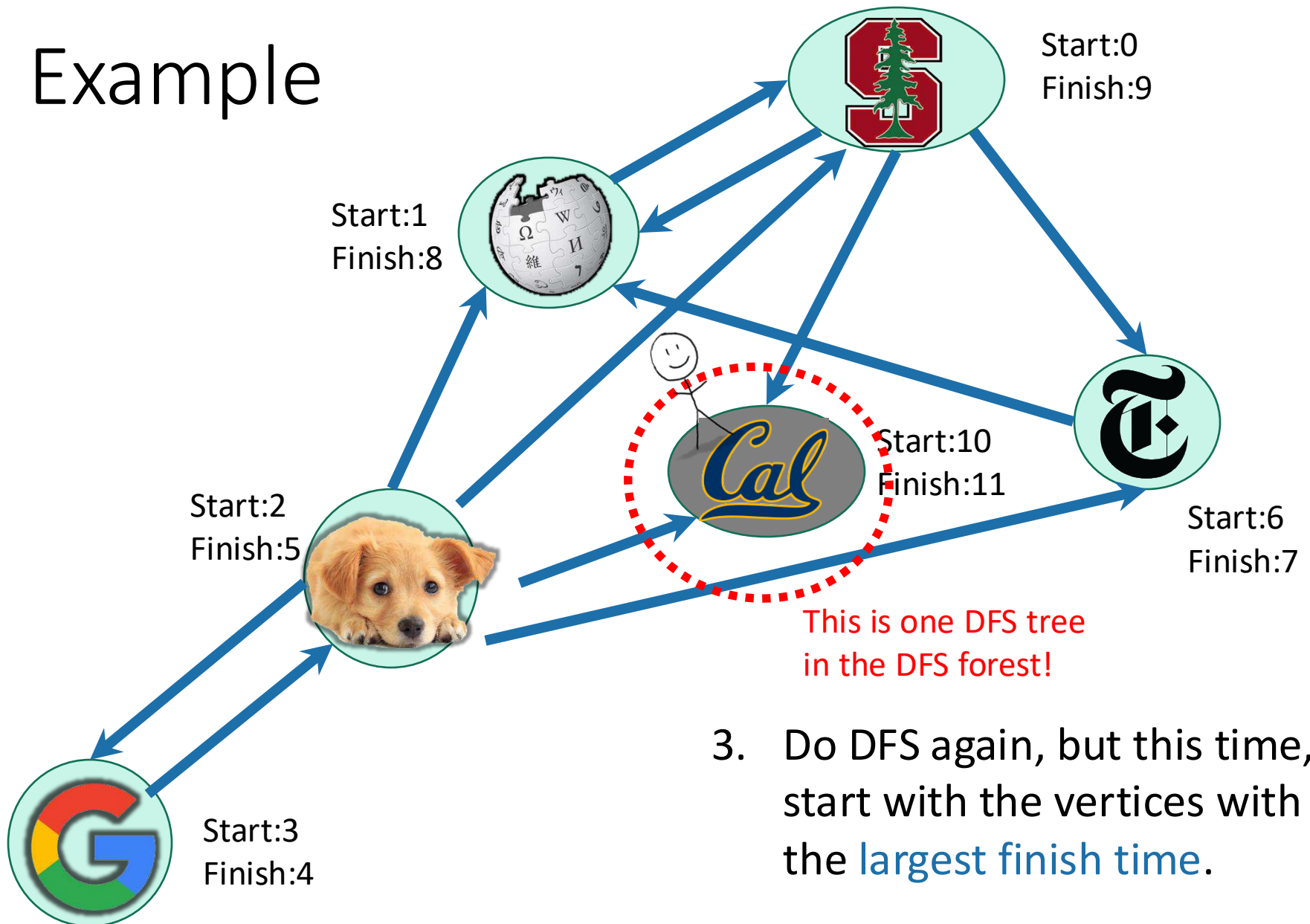
Example



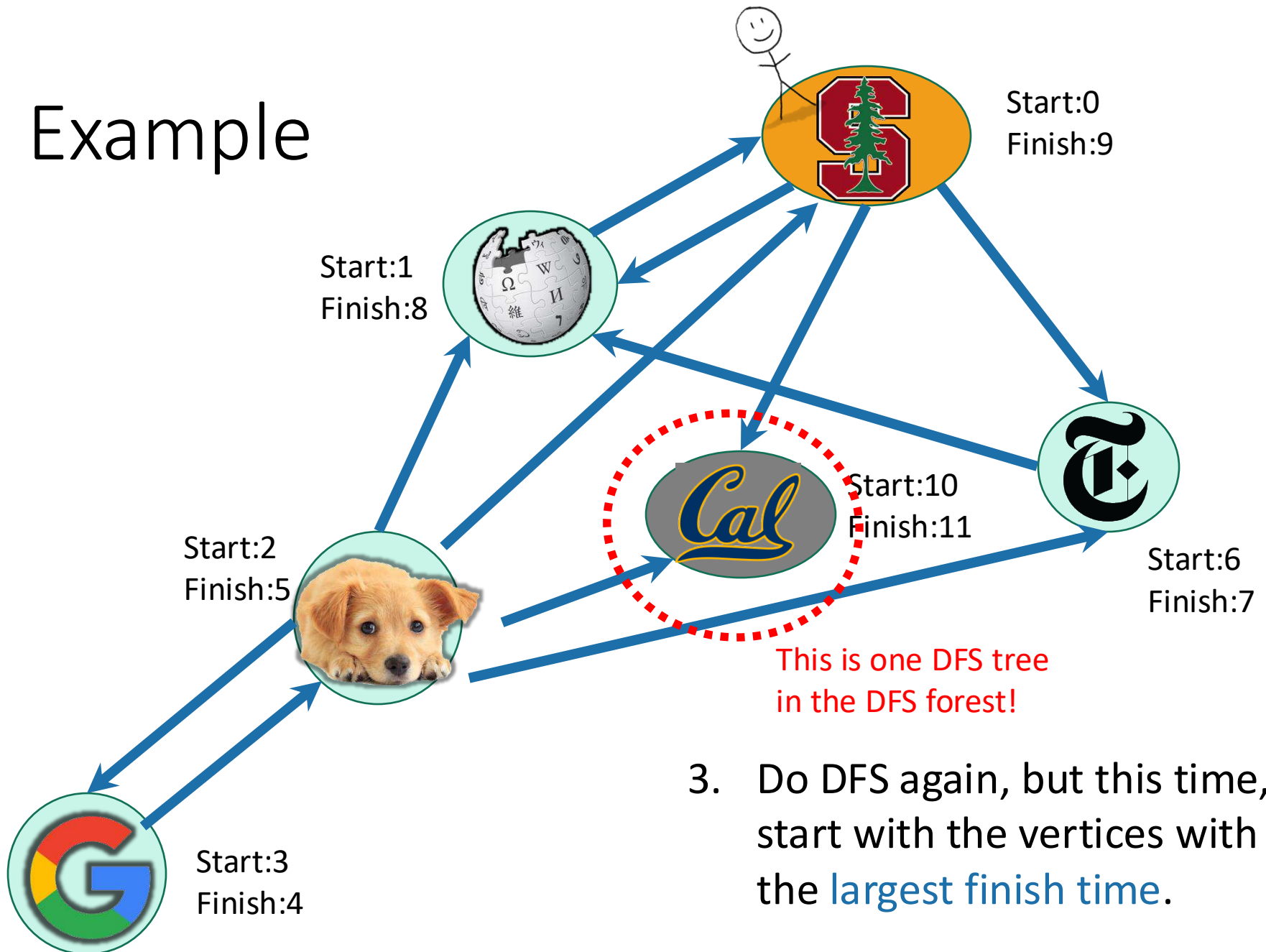
Example



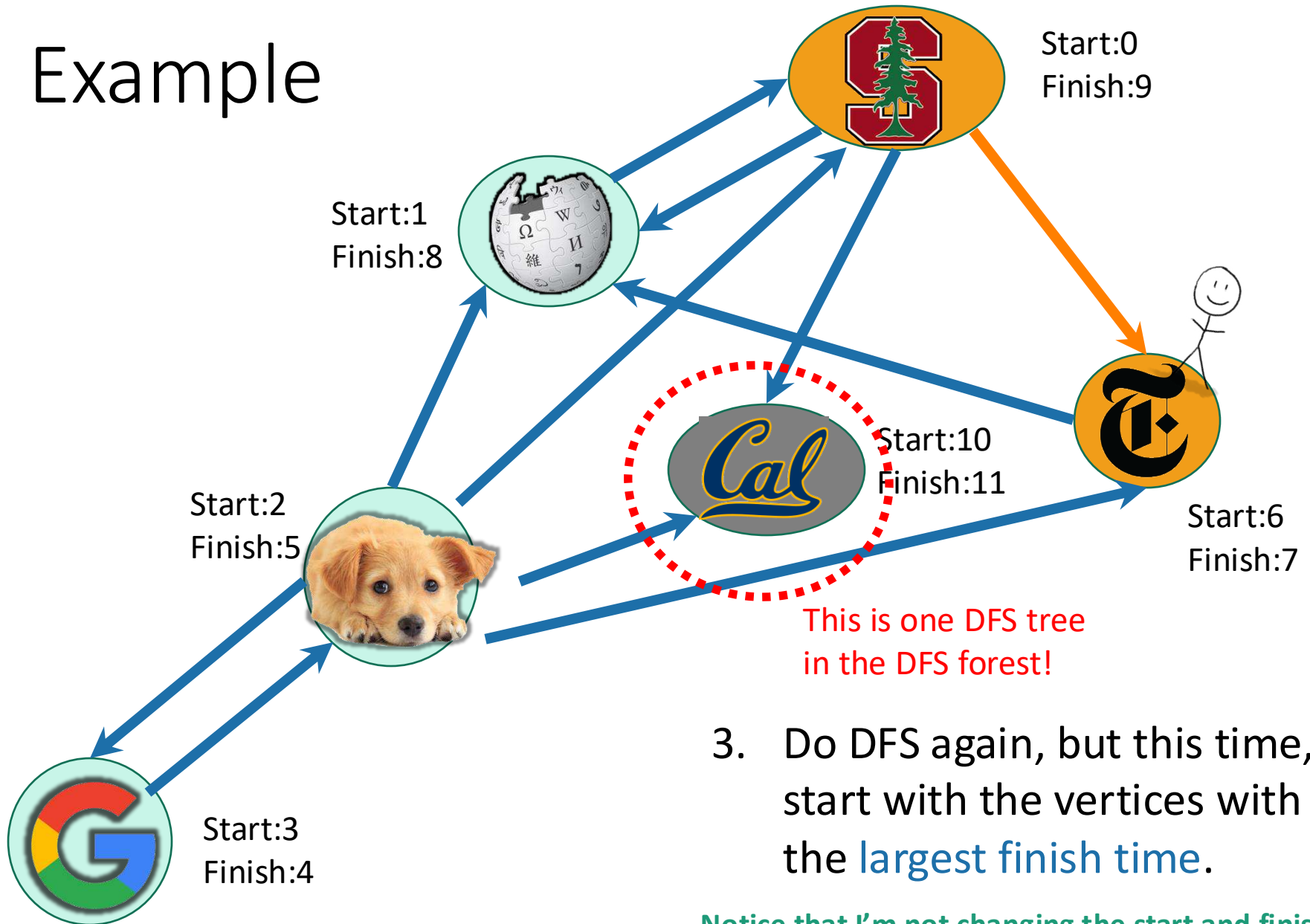
Example



Example



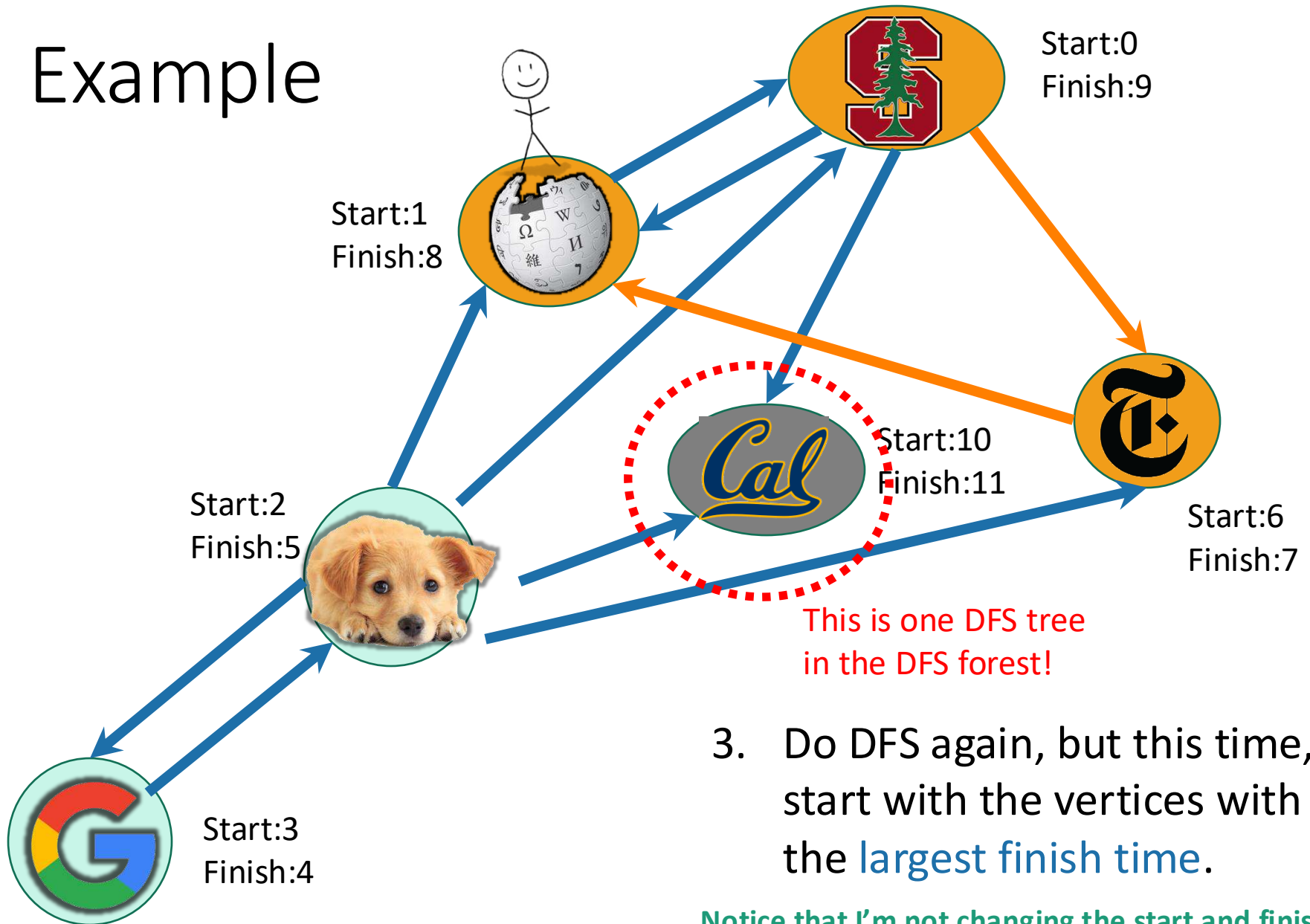
Example



3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

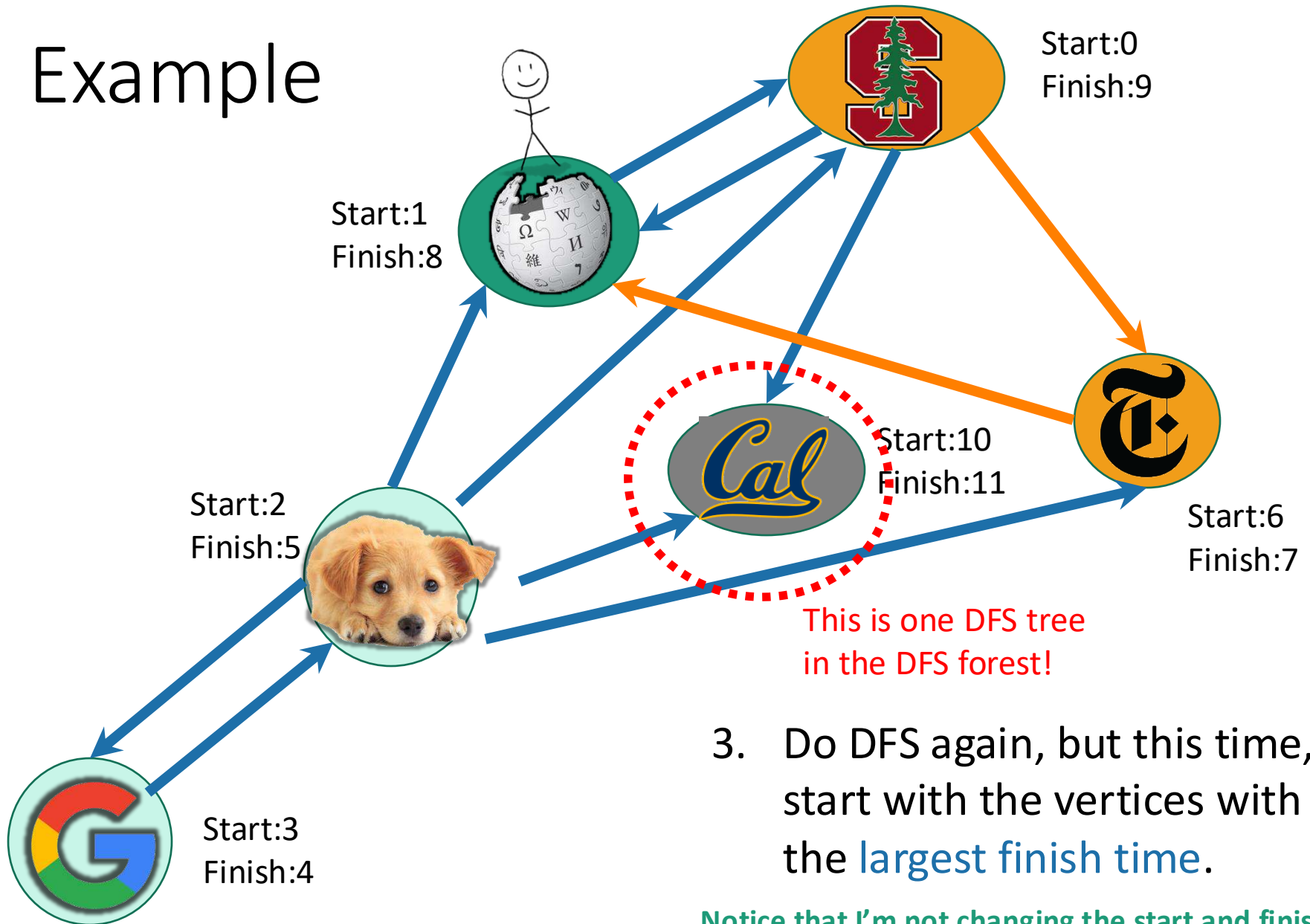
Example



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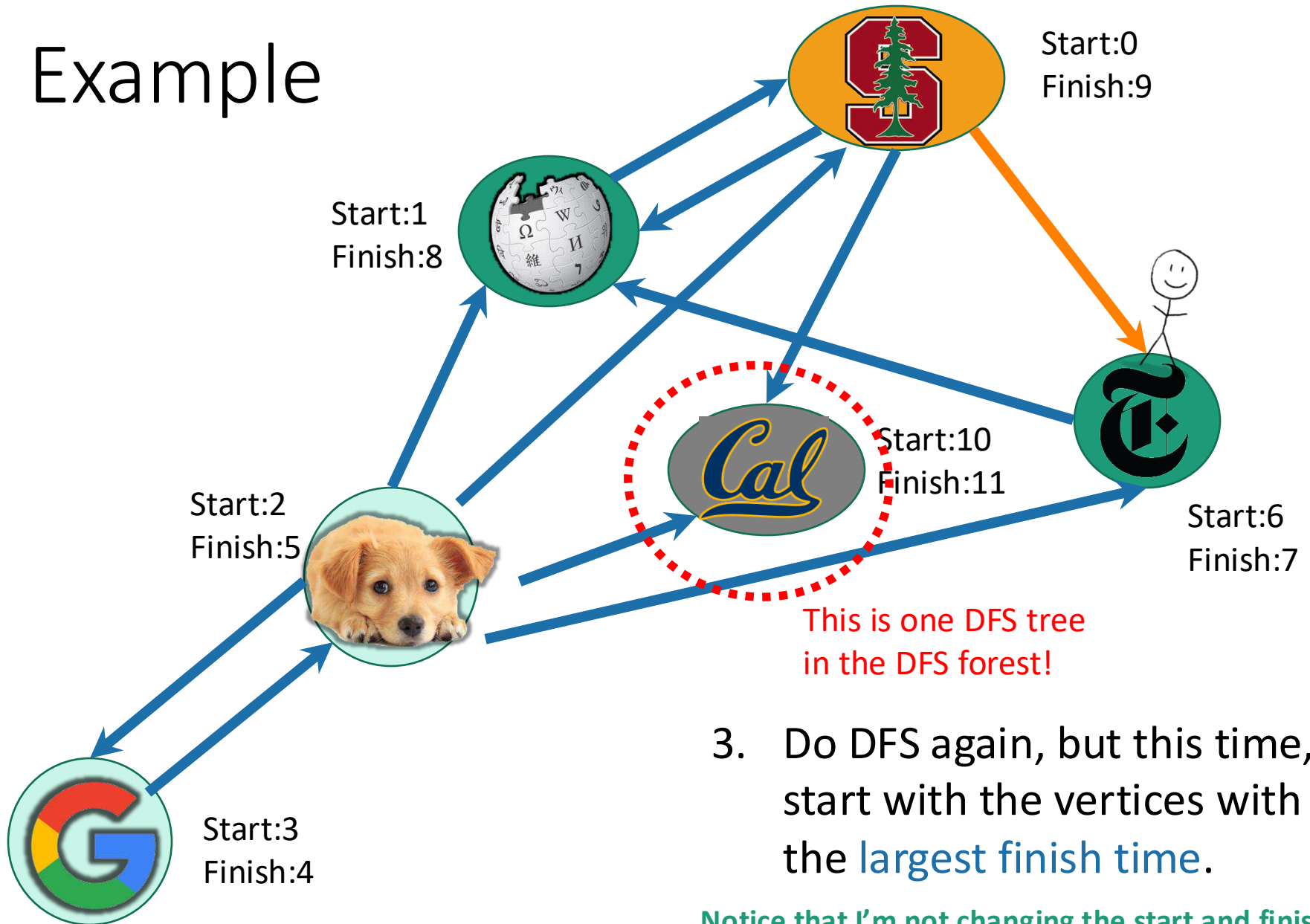
Example



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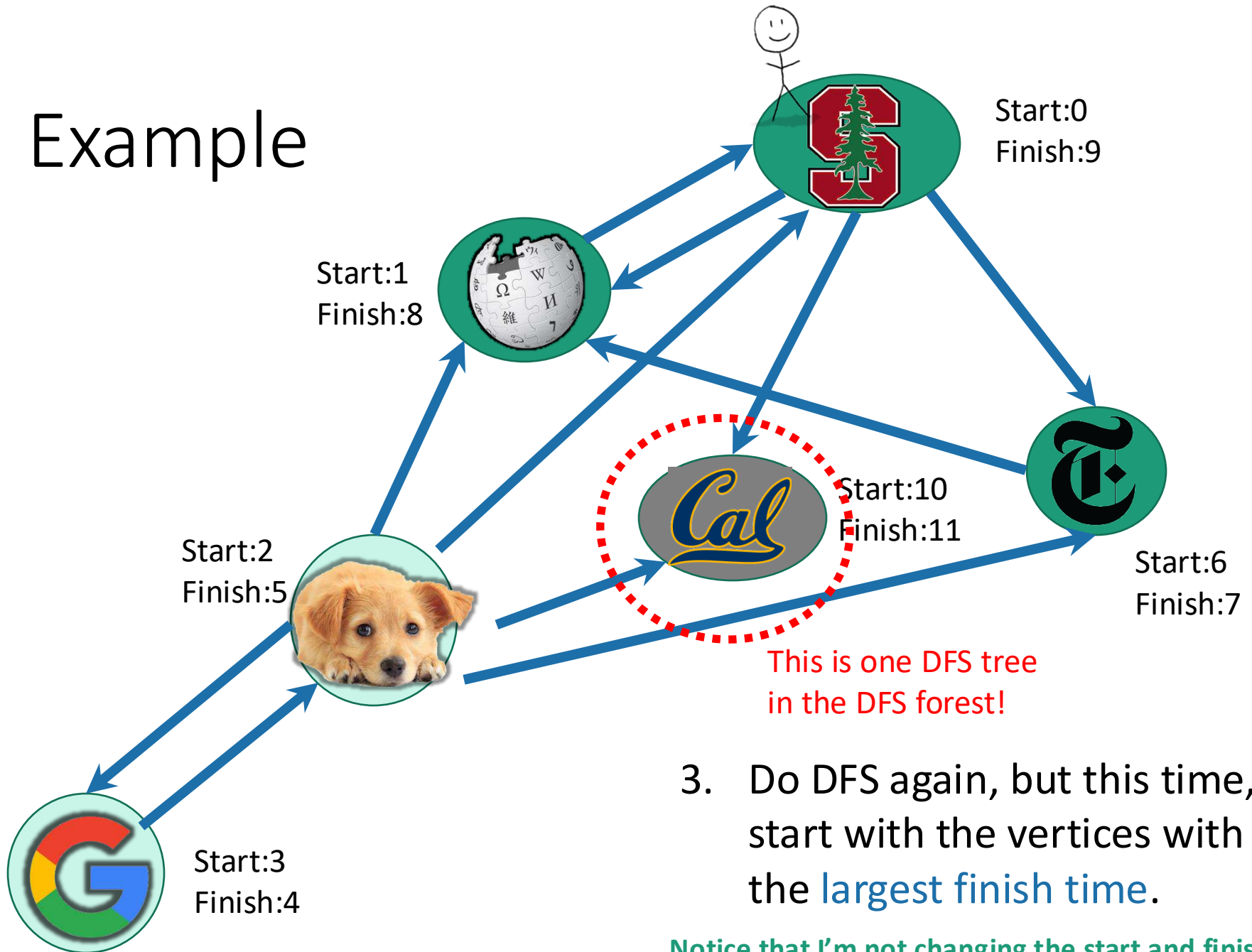
Example



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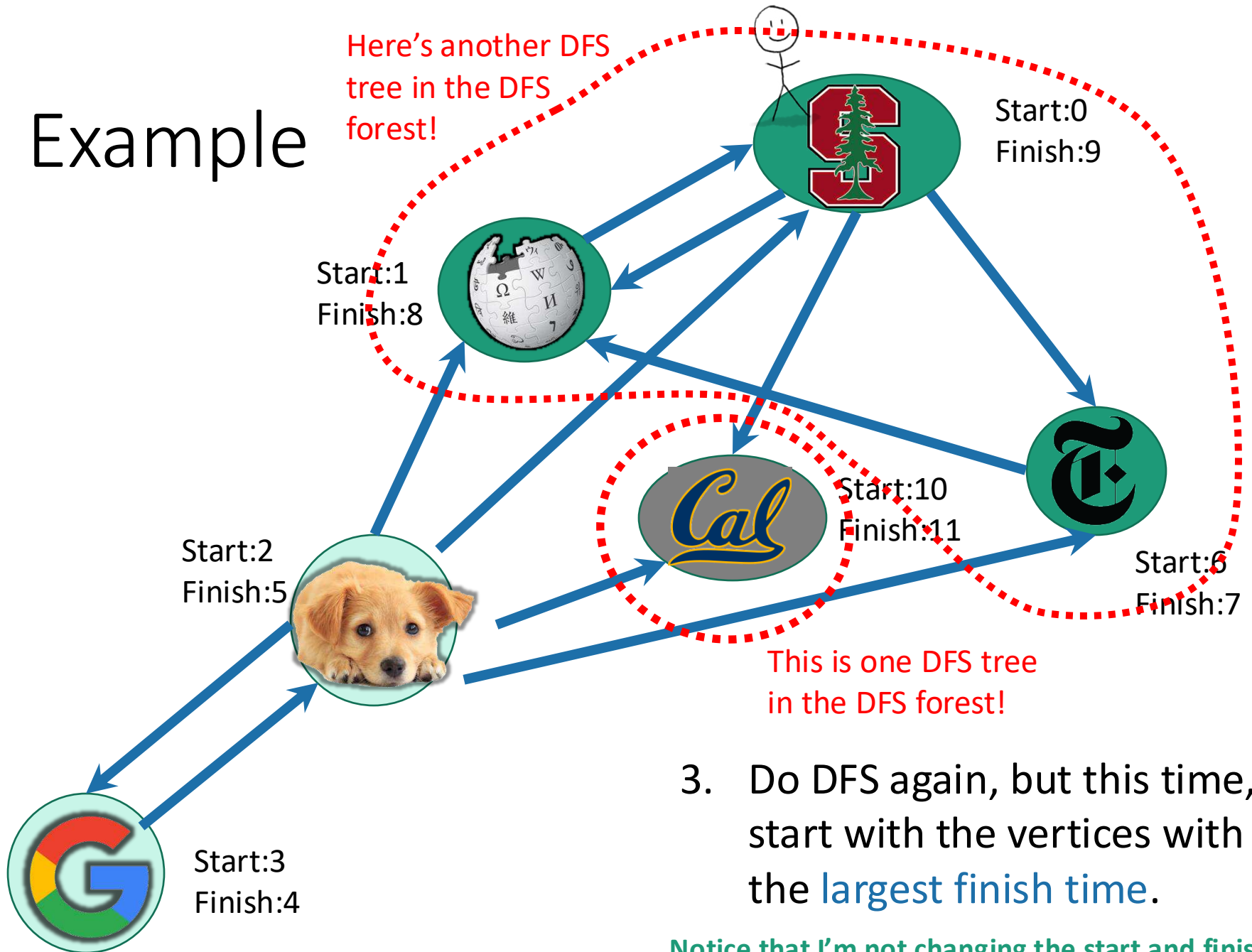
Example



3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

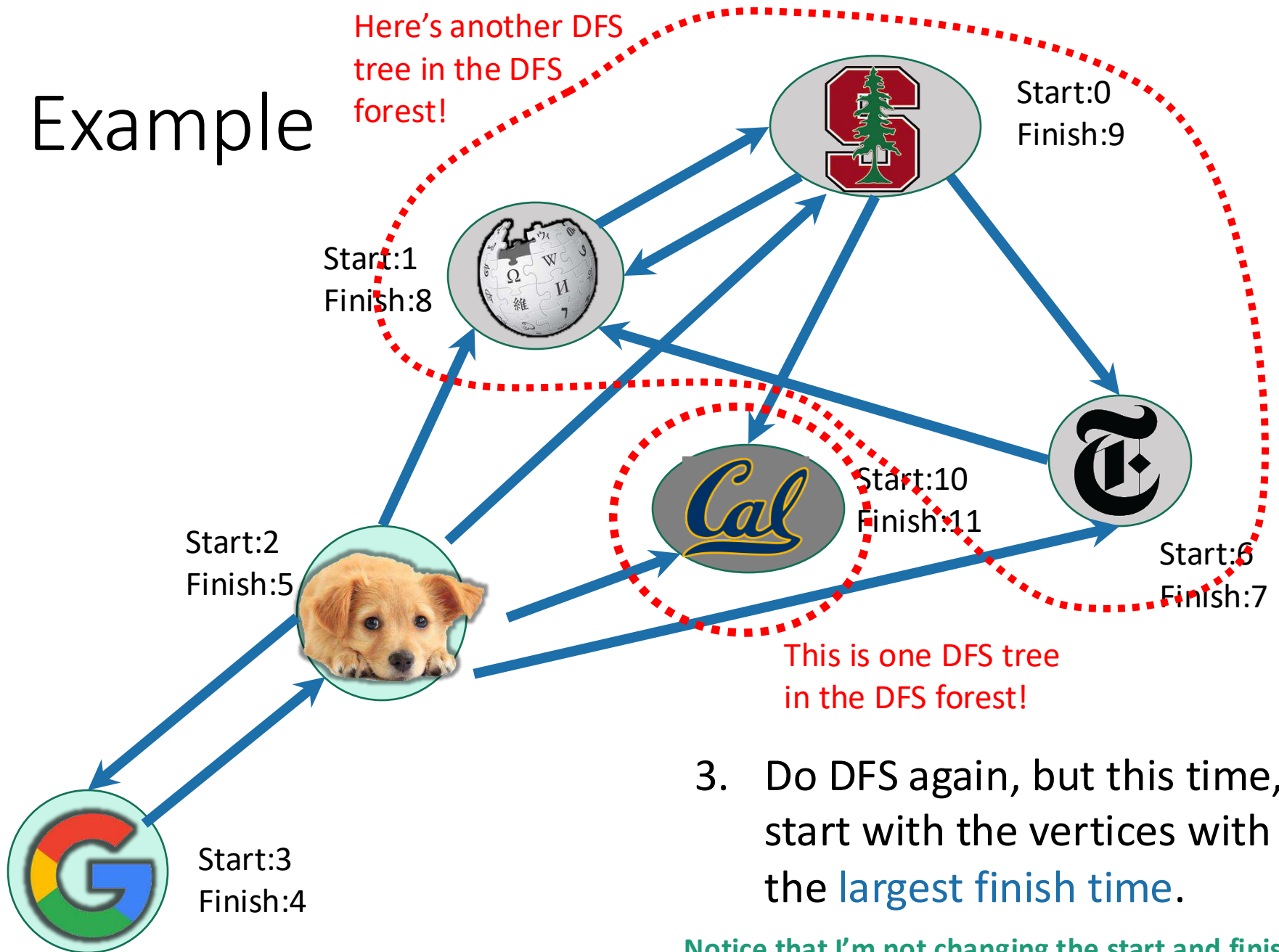
Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example



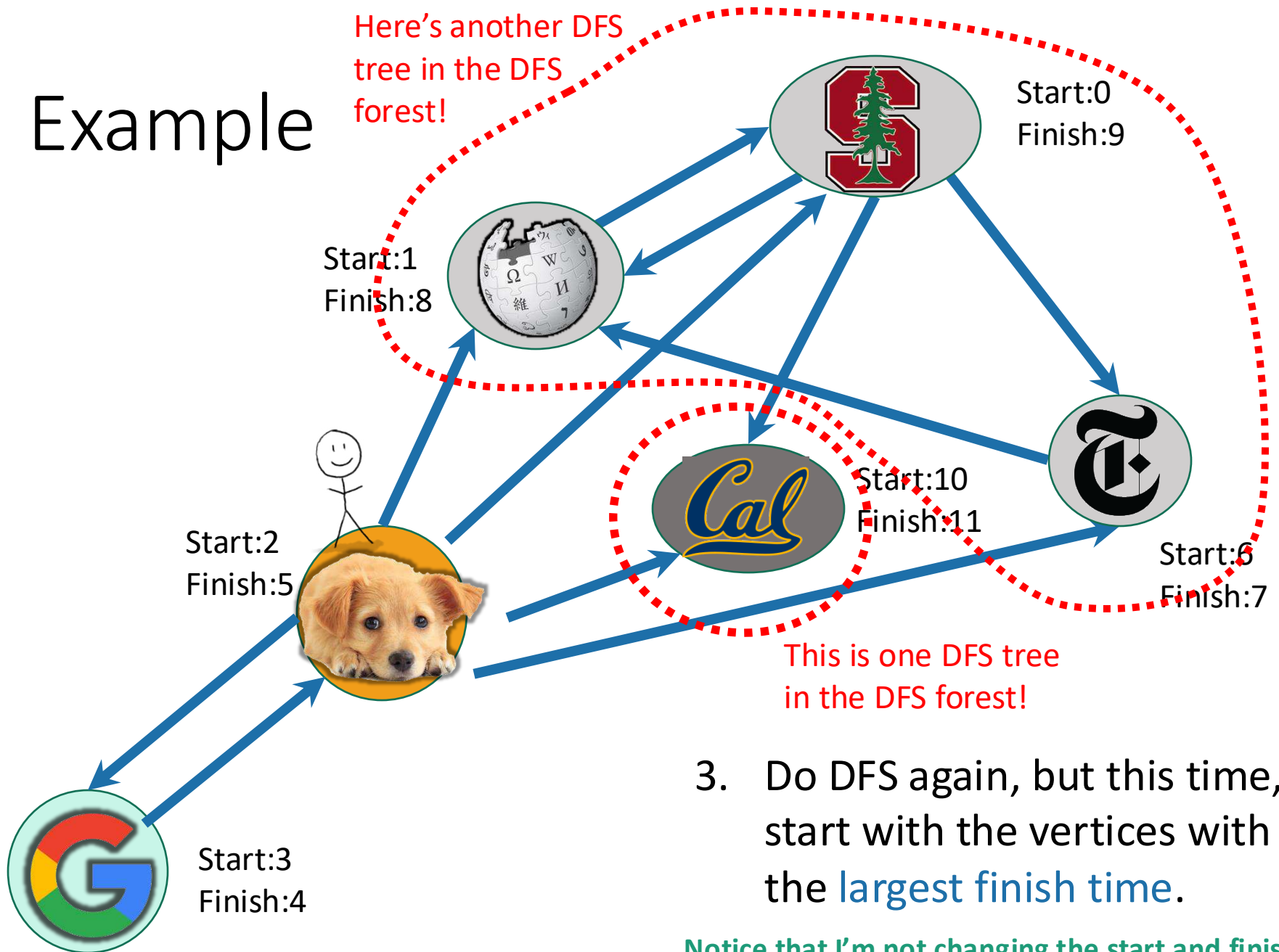
Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example



Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example

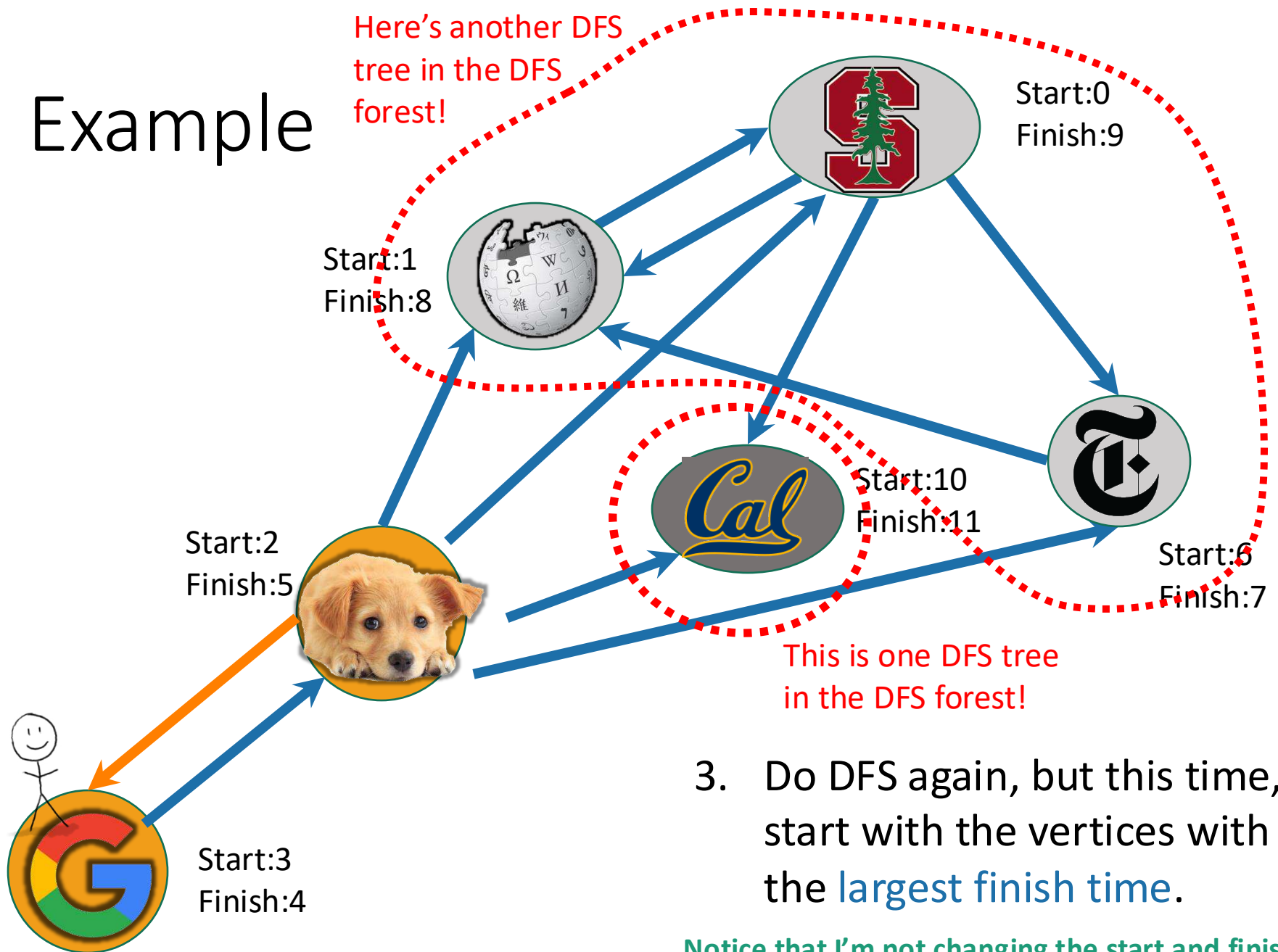


3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example

Here's another DFS tree in the DFS forest!

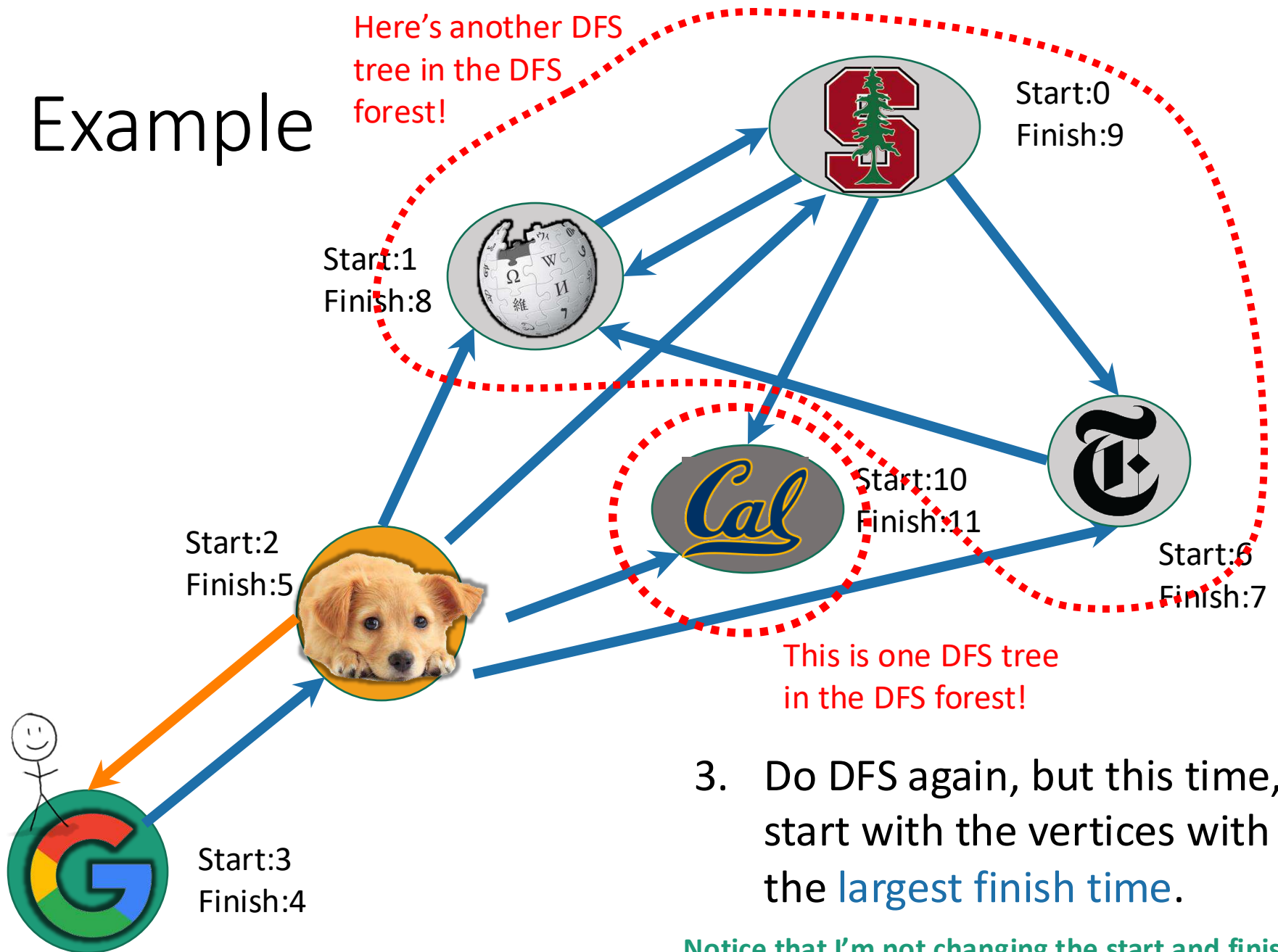


3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example

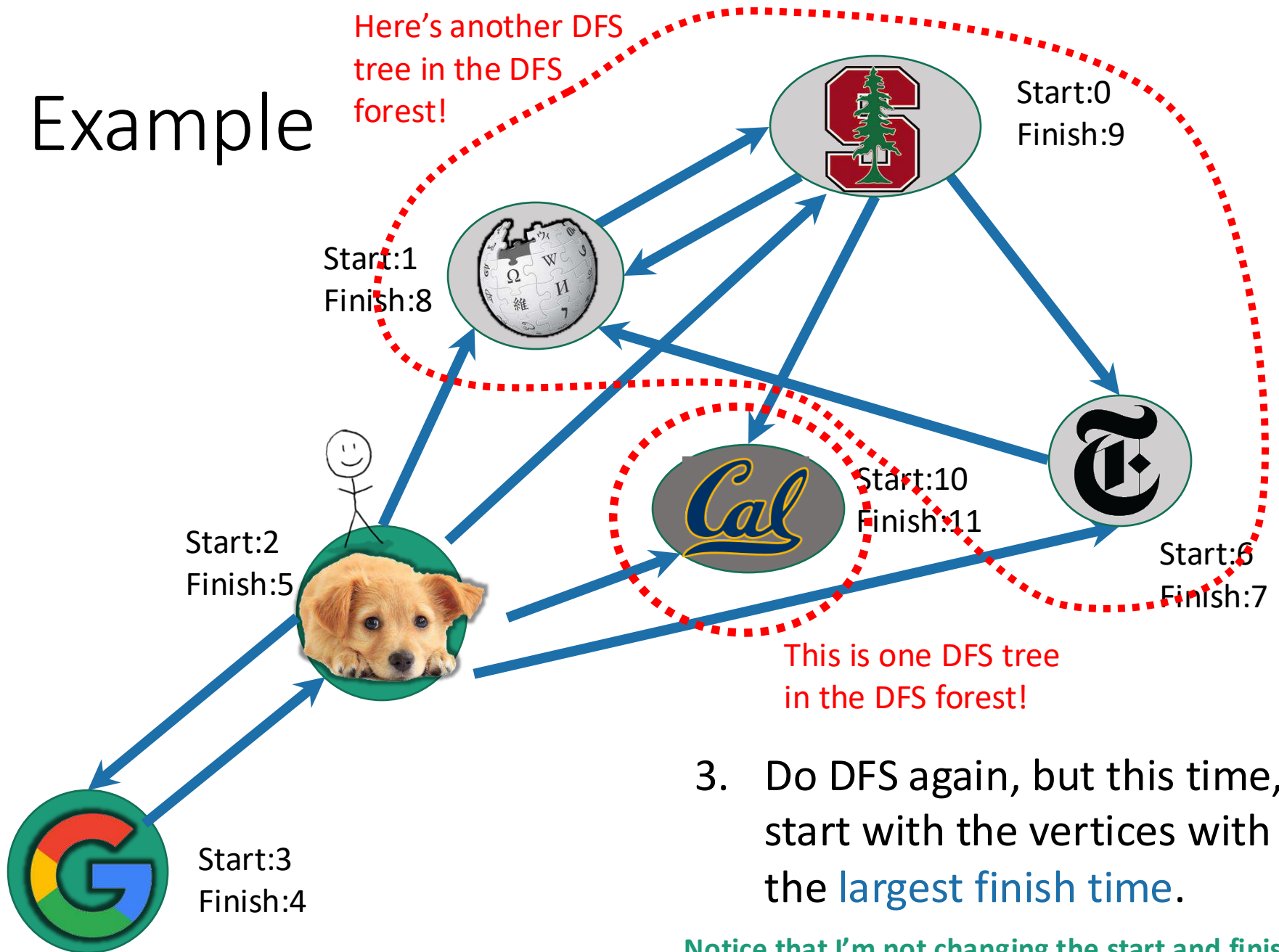
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Example

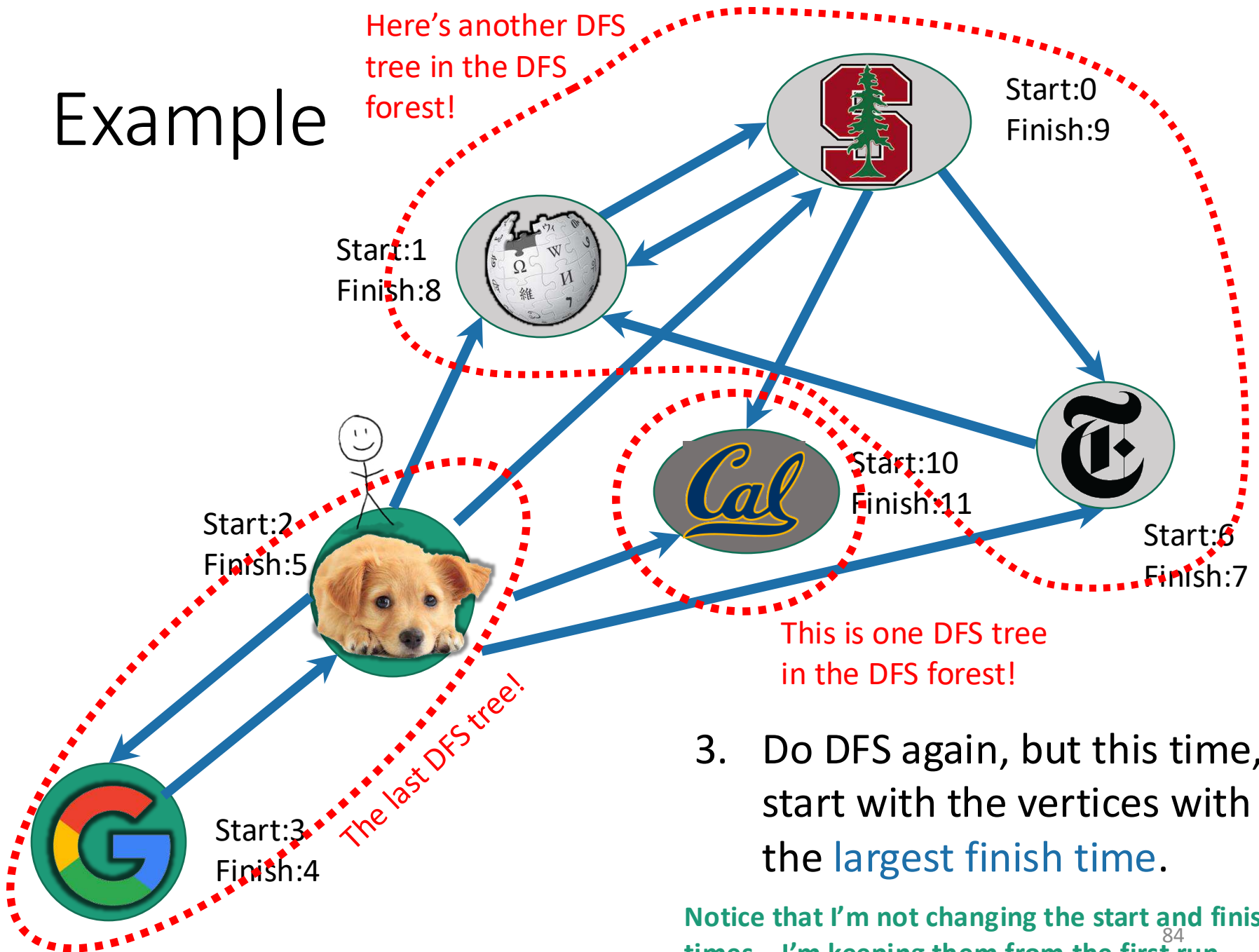


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Example

Here's another DFS tree in the DFS forest!

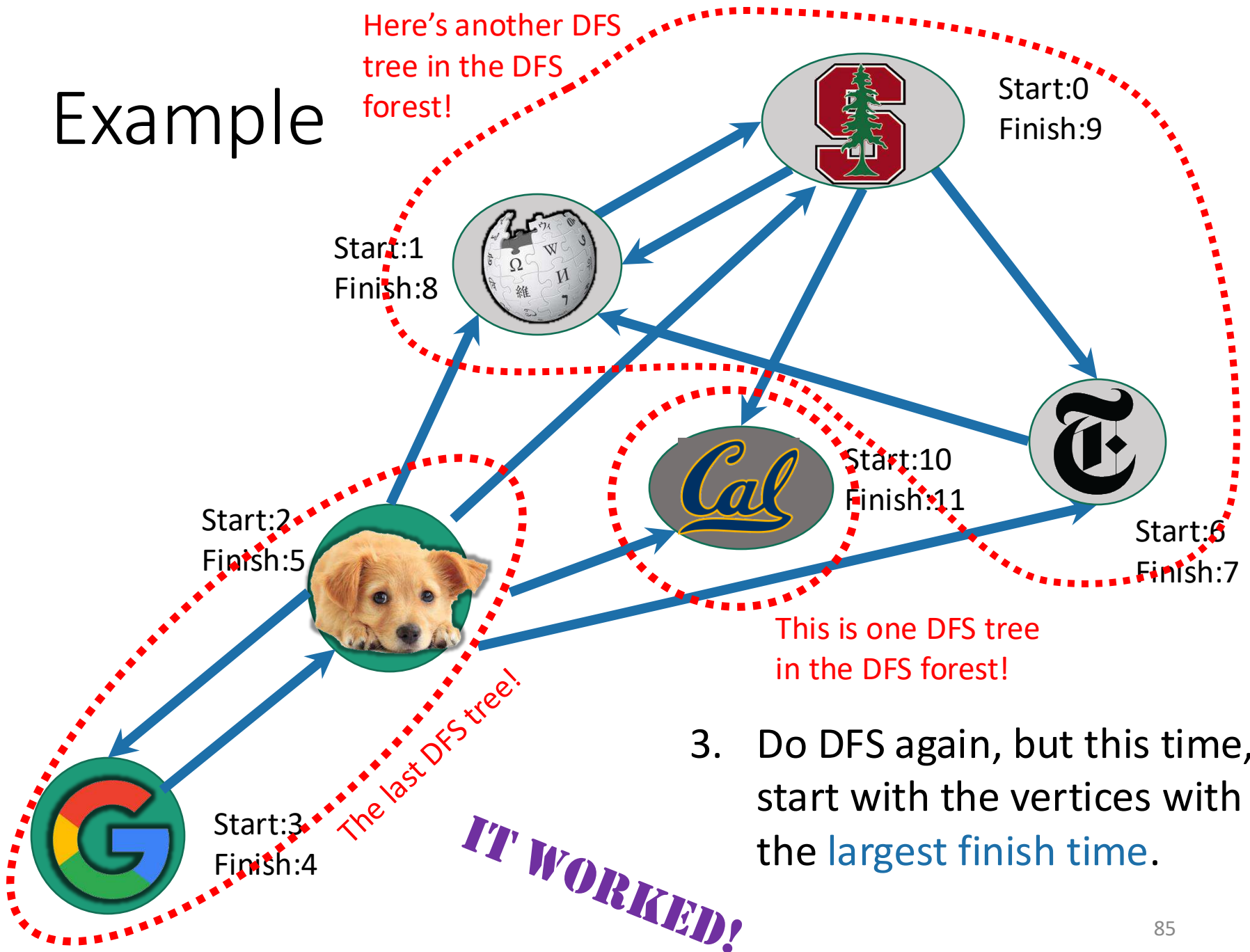


3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

Example

Here's another DFS tree in the DFS forest!

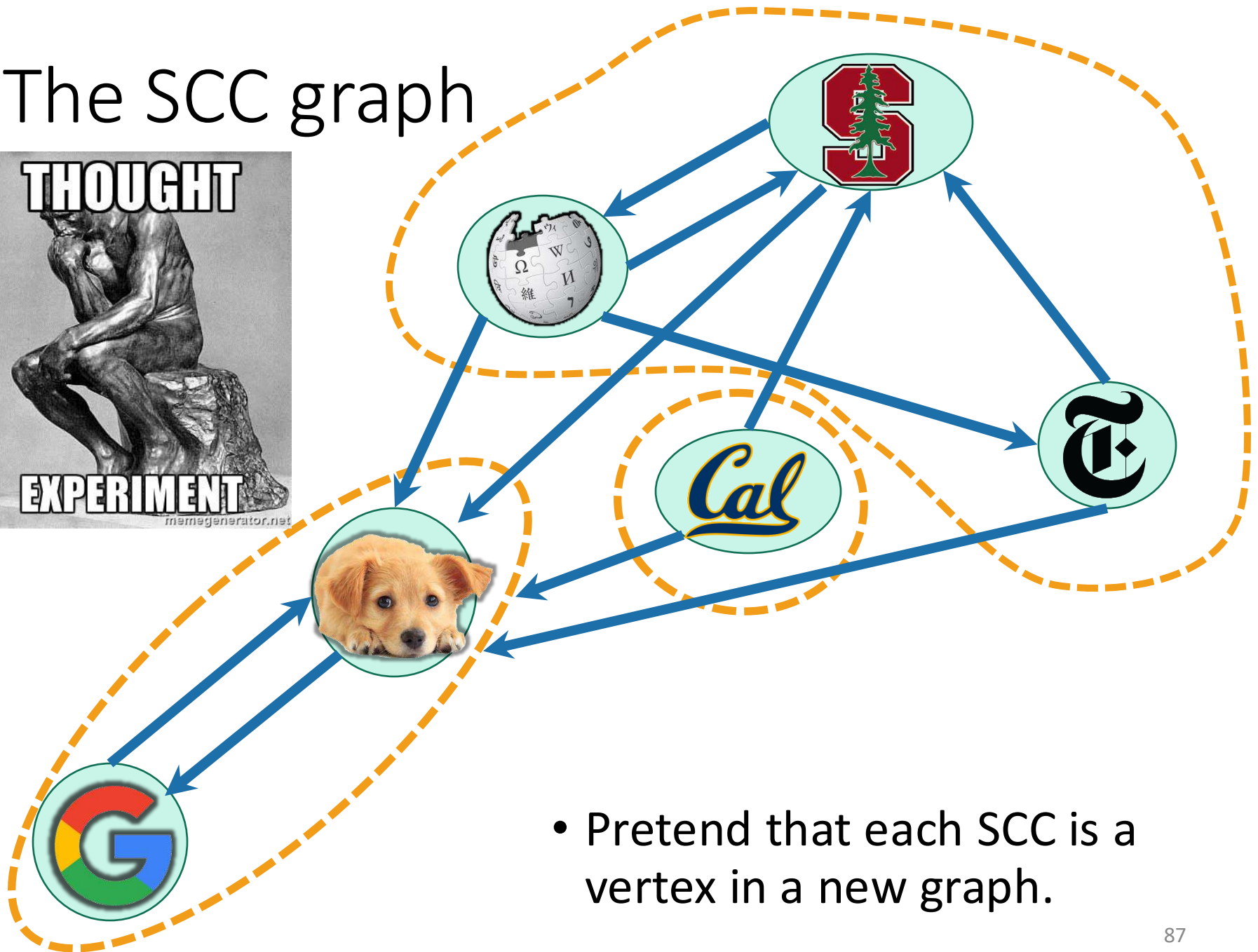
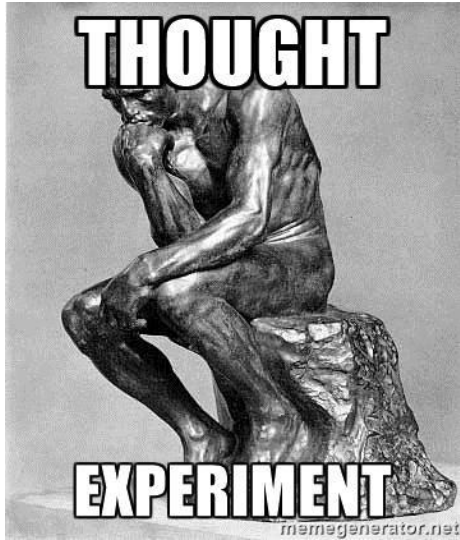


3. Do DFS again, but this time, start with the vertices with the **largest finish time**.

One question



The SCC graph



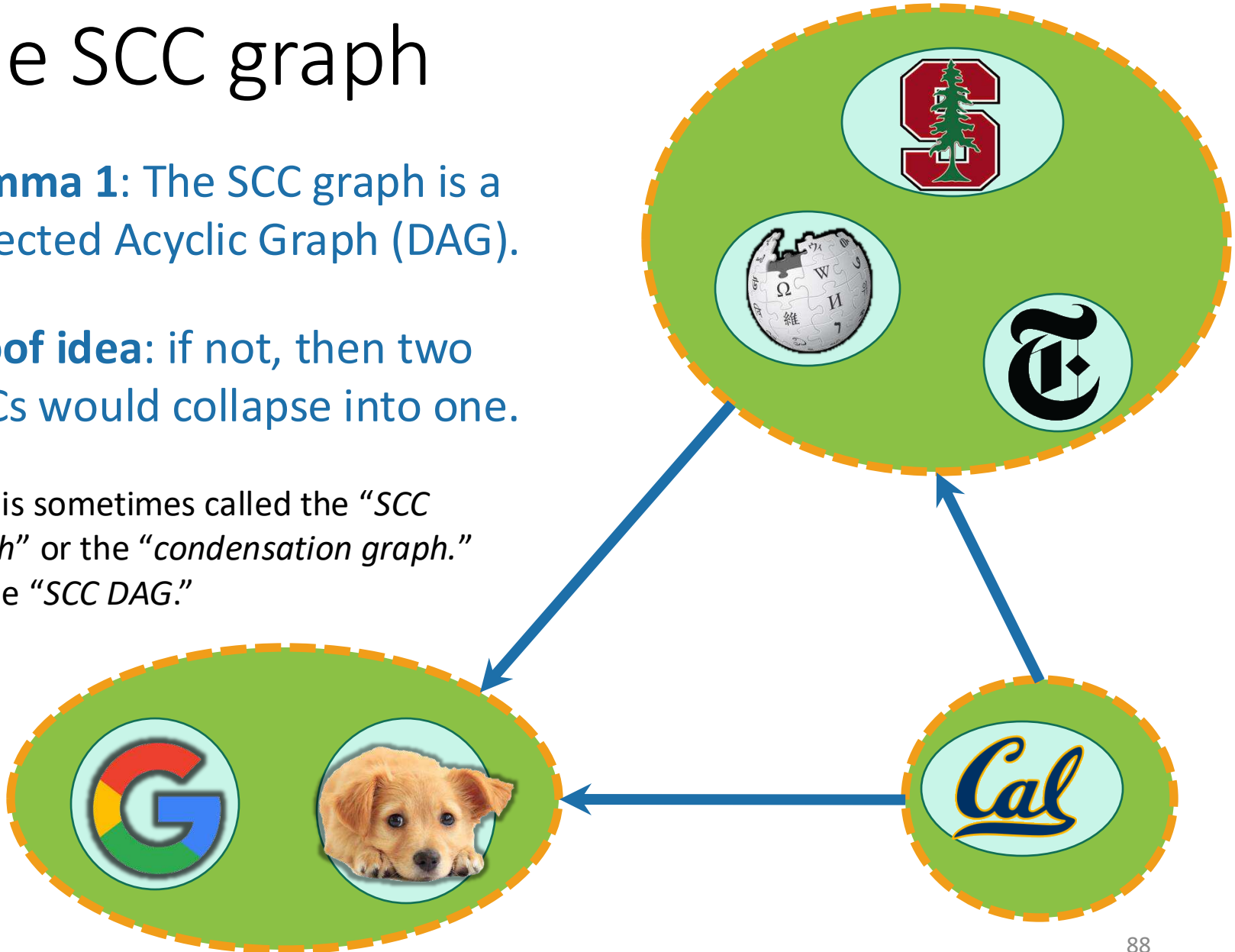
- Pretend that each SCC is a vertex in a new graph.

The SCC graph

Lemma 1: The SCC graph is a Directed Acyclic Graph (DAG).

Proof idea: if not, then two SCCs would collapse into one.

This graph is sometimes called the “SCC meta-graph” or the “condensation graph.”
I’ll call it the “SCC DAG.”

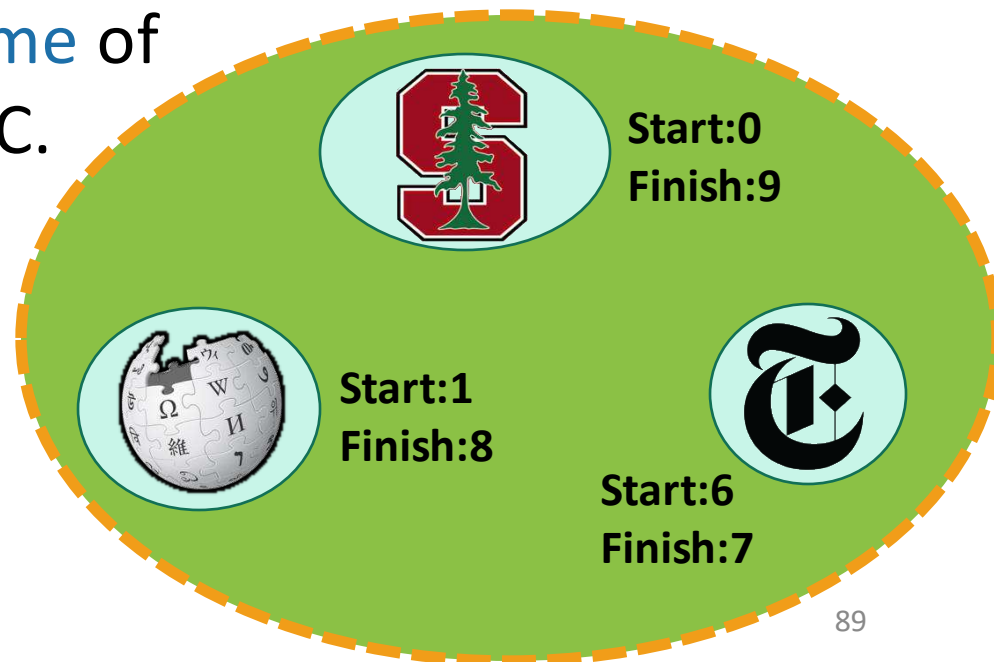


Starting and finishing times in a SCC

Definitions:

- The **finishing time** of a SCC is the **largest finishing time** of any element of that SCC.
- The **starting time** of a SCC is the **smallest starting time** of any element of that SCC.

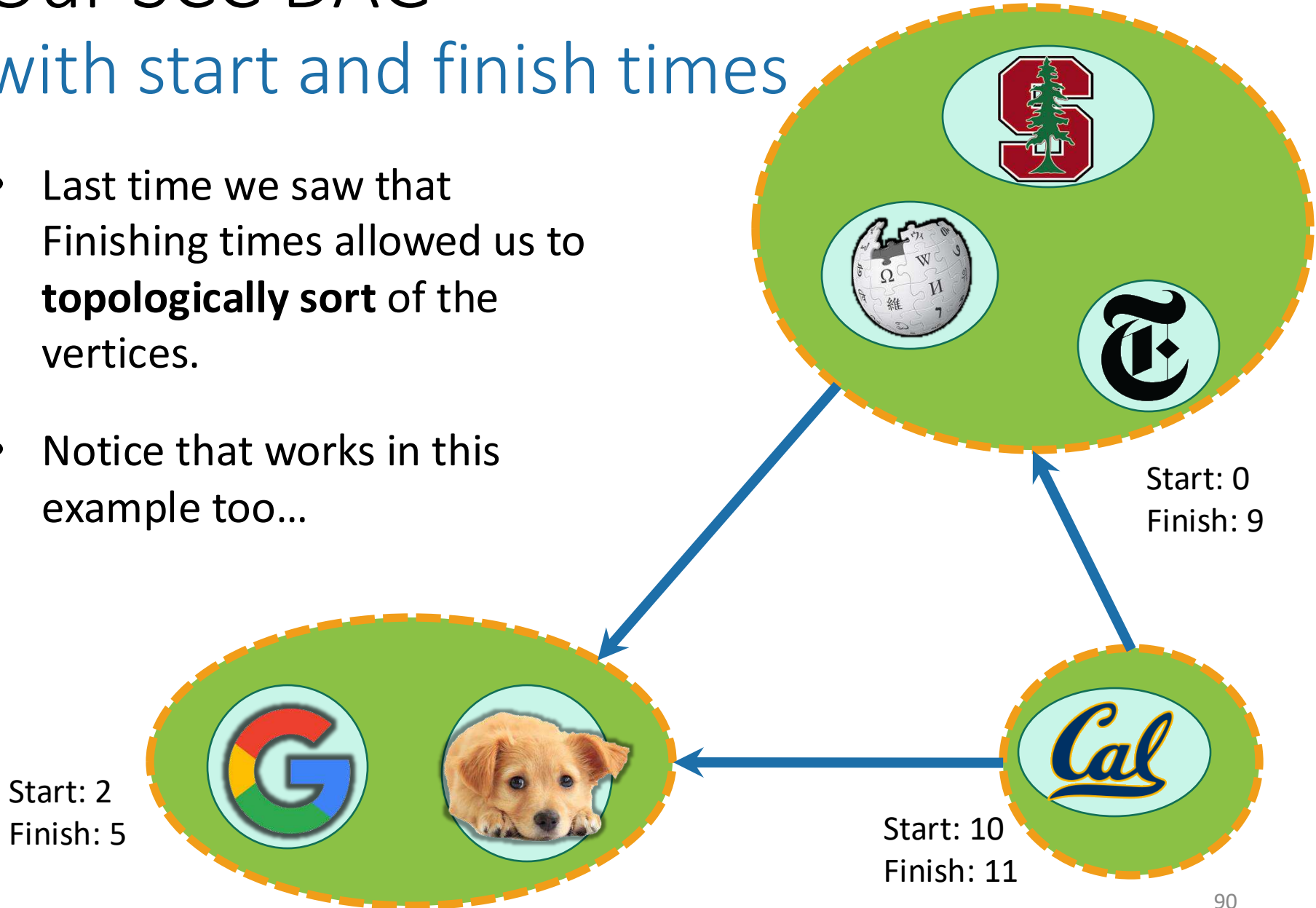
Start: 0
Finish: 9



Our SCC DAG

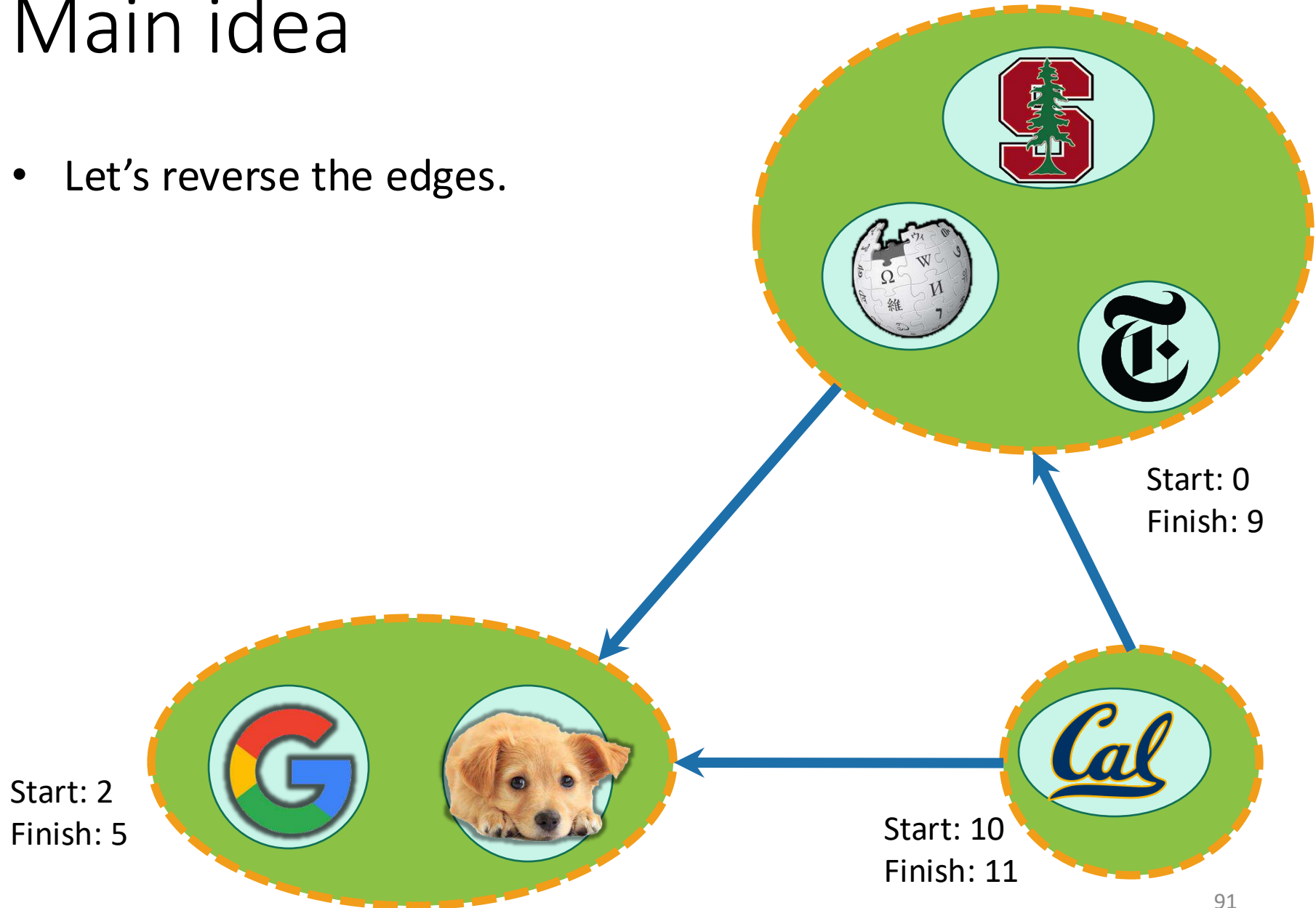
with start and finish times

- Last time we saw that Finishing times allowed us to **topologically sort** of the vertices.
- Notice that works in this example too...



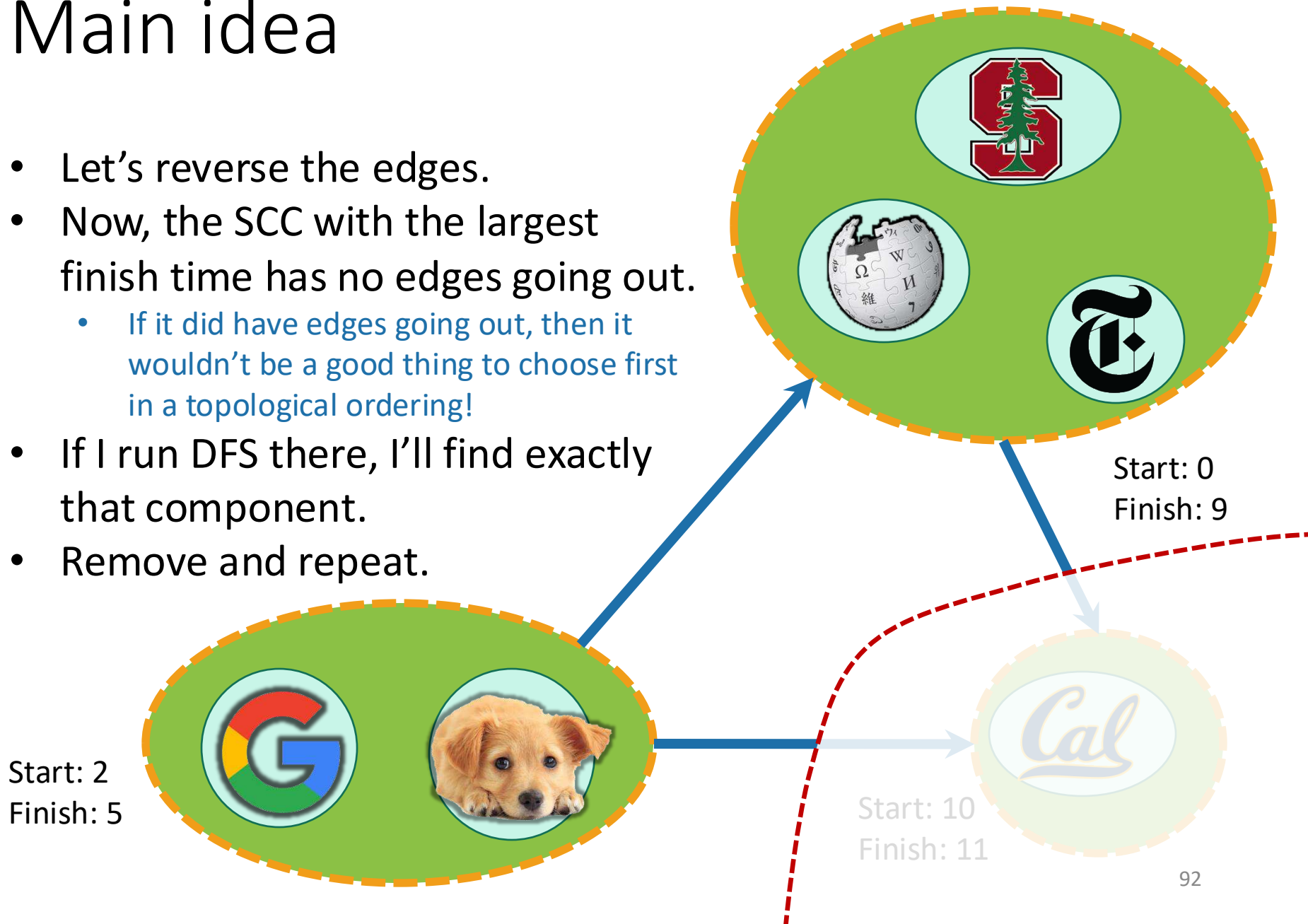
Main idea

- Let's reverse the edges.



Main idea

- Let's reverse the edges.
- Now, the SCC with the largest finish time has no edges going out.
 - If it did have edges going out, then it wouldn't be a good thing to choose first in a topological ordering!
- If I run DFS there, I'll find exactly that component.
- Remove and repeat.



Let's make this idea formal.

Recall

- If v is a descendent of w in this tree:



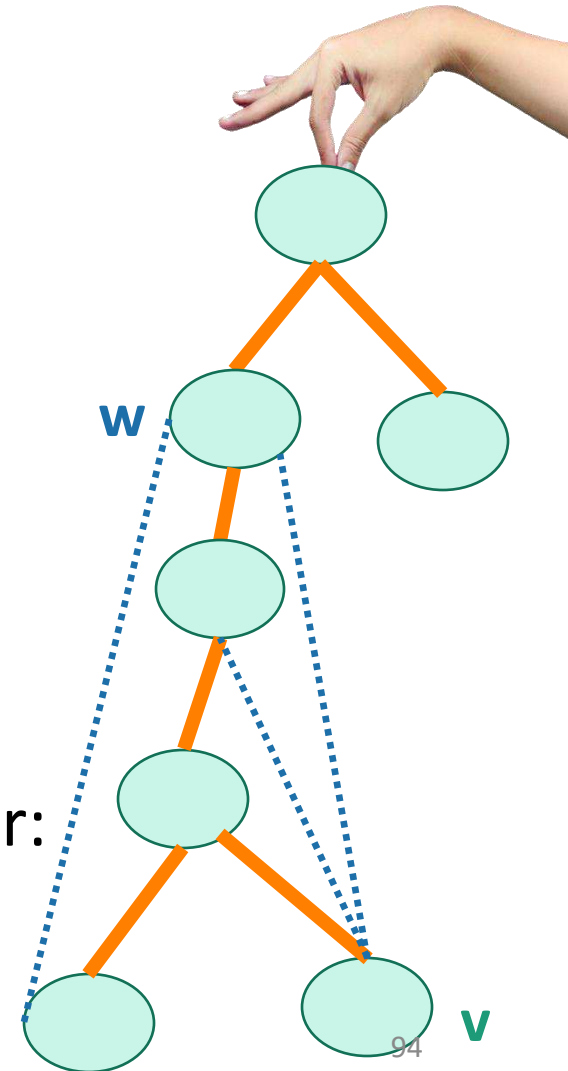
- If w is a descendent of v in this tree:



- If neither are descendants of each other:



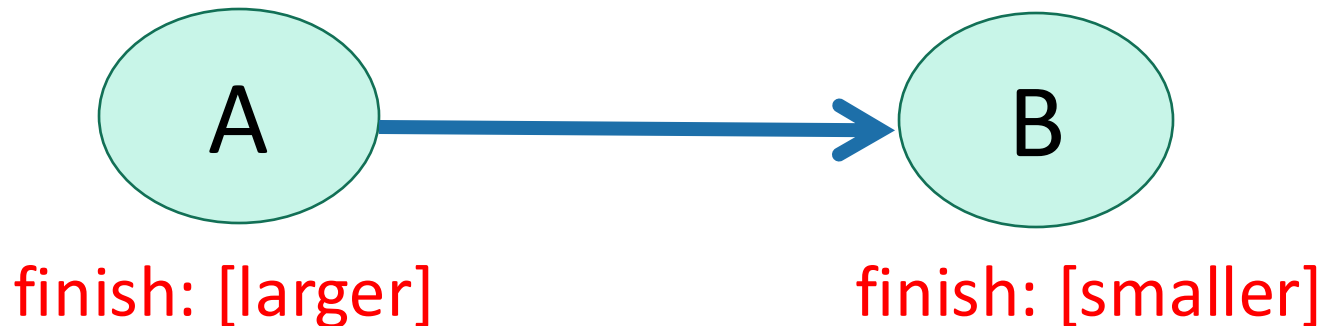
(or the other way around)



* The version of DFS where we keep restarting until we explore the whole graph

As we saw last time...

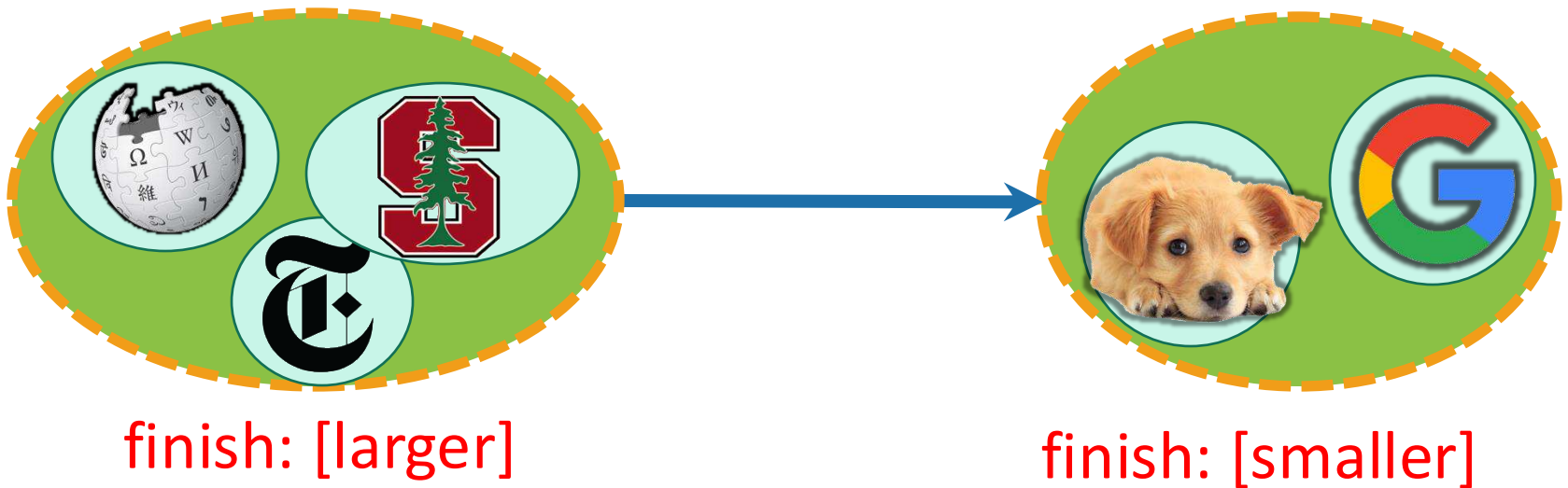
Claim: If we run DFS on a DAG:



That is, if we run DFS* on a DAG and if there is an edge from A to B, $A.\text{finish} > B.\text{finish}$.

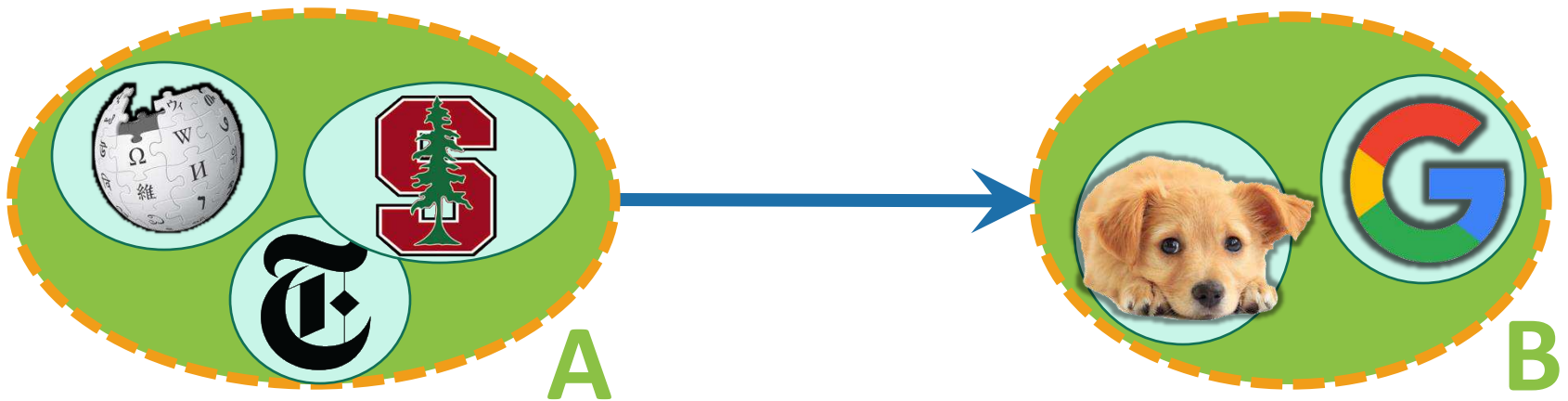
Same thing, in the SCC DAG.

- **Claim:** we'll always have



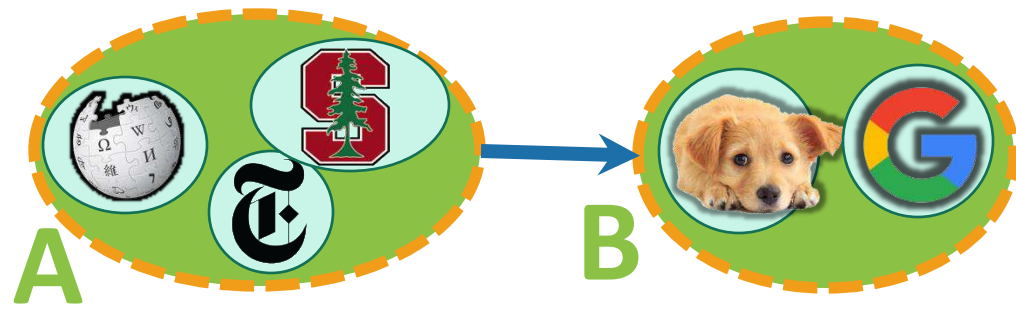
Let's call it Lemma 2

- If there is an edge like this:



- Then $A.\text{finish} > B.\text{finish}$.

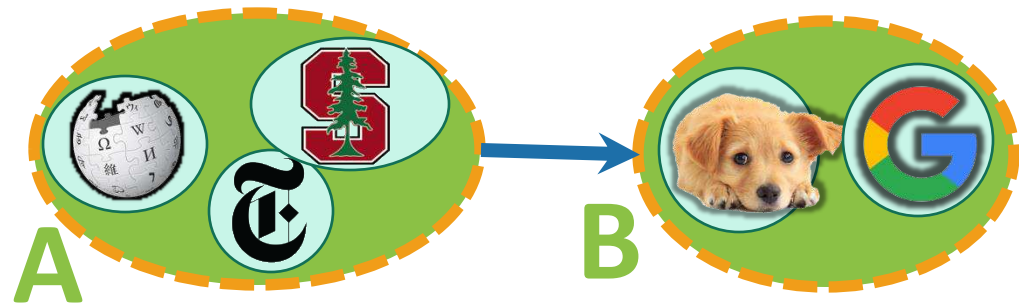
Proof idea



Want to show $A.\text{finish} > B.\text{finish}$.

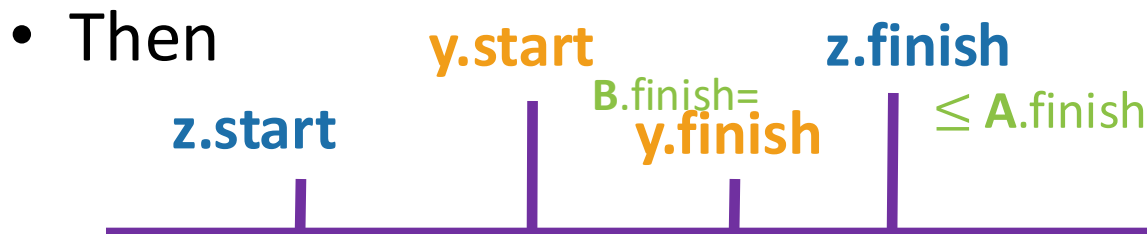
- **Two cases:**
 - We reached **A** before **B** in our first DFS.
 - We reached **B** before **A** in our first DFS.

Proof idea



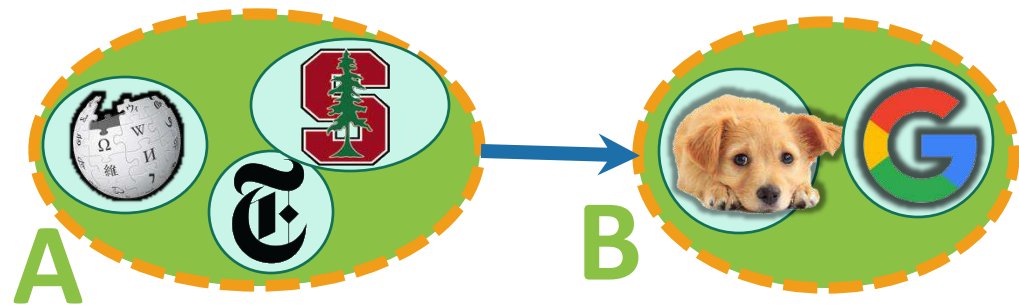
Want to show $A.\text{finish} > B.\text{finish}$.

- **Case 1:** We reached **A** before **B** in our first DFS.
- Say that:
 - **y** has the largest finish in **B**; $B.\text{finish} = y.\text{finish}$
 - **z** was discovered first in **A**; $A.\text{finish} \geq z.\text{finish}$
- Then:
 - Reach **A** before **B**
 - \Rightarrow we will discover **y** via **z**
 - \Rightarrow **y** is a descendant of **z** in the DFS forest.



aka,
 $A.\text{finish} > B.\text{finish}$

Proof idea

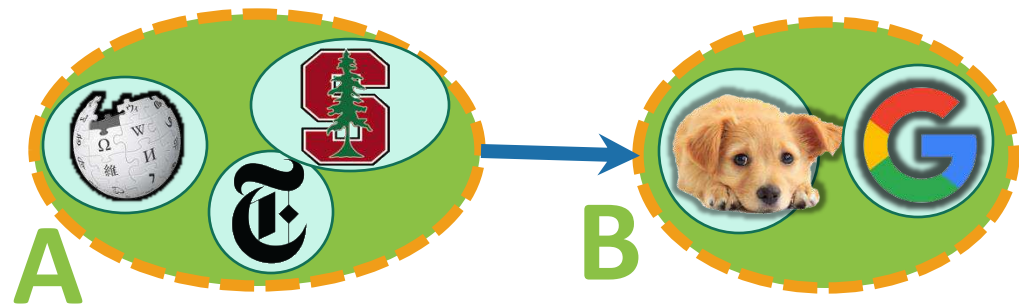


Want to show $A.\text{finish} > B.\text{finish}$.

- **Case 2:** We reached **B** before **A** in our first DFS.
- There are no paths from B to A
 - because the SCC graph has no cycles
- So we completely finish exploring B and never reach A.
- A is explored later after we restart DFS.

aka,
 $A.\text{finish} > B.\text{finish}$

Proof idea



Want to show $A.\text{finish} > B.\text{finish}$.

- **Two cases:**
 - We reached **A** before **B** in our first DFS.
 - We reached **B** before **A** in our first DFS.
- In either case:

$A.\text{finish} > B.\text{finish}$

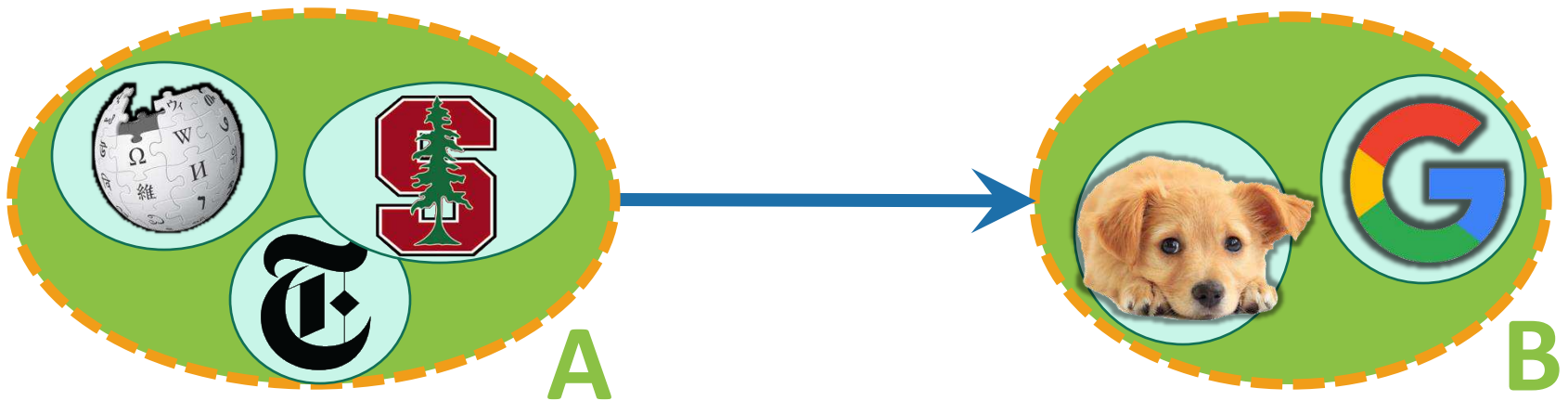
which is what we wanted to show.



Notice: this is exactly the same two-case argument that we did last time for topological sorting, just with the SCC DAG!

This establishes:
Lemma 2

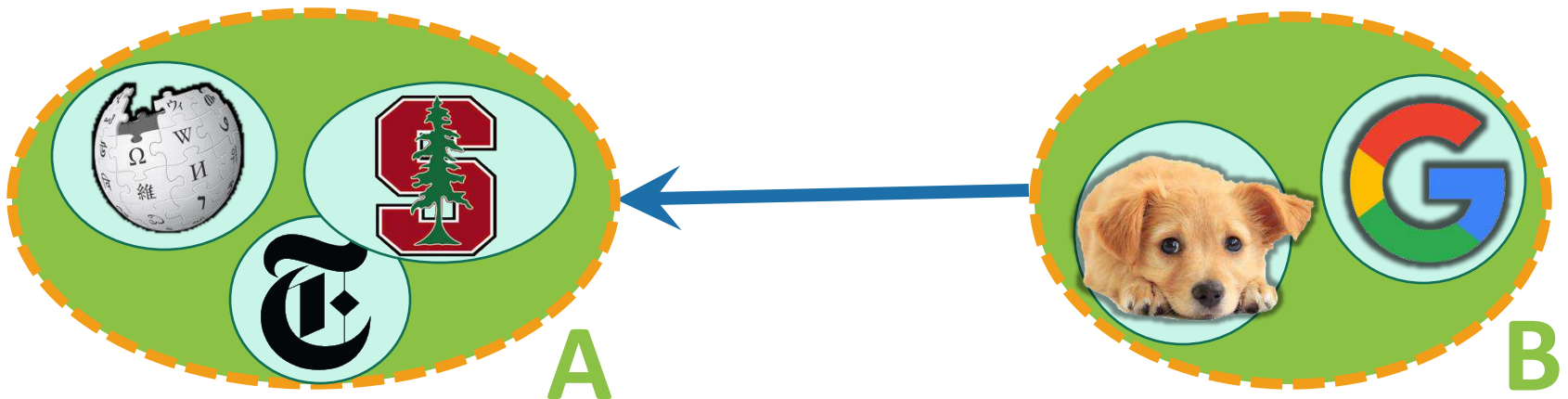
- If there is an edge like this:



- Then $A.\text{finish} > B.\text{finish}$.

This establishes:
Corollary 1

- If there is an edge like this in the **reversed SCC DAG**:



- Then $A.\text{finish} > B.\text{finish}$.

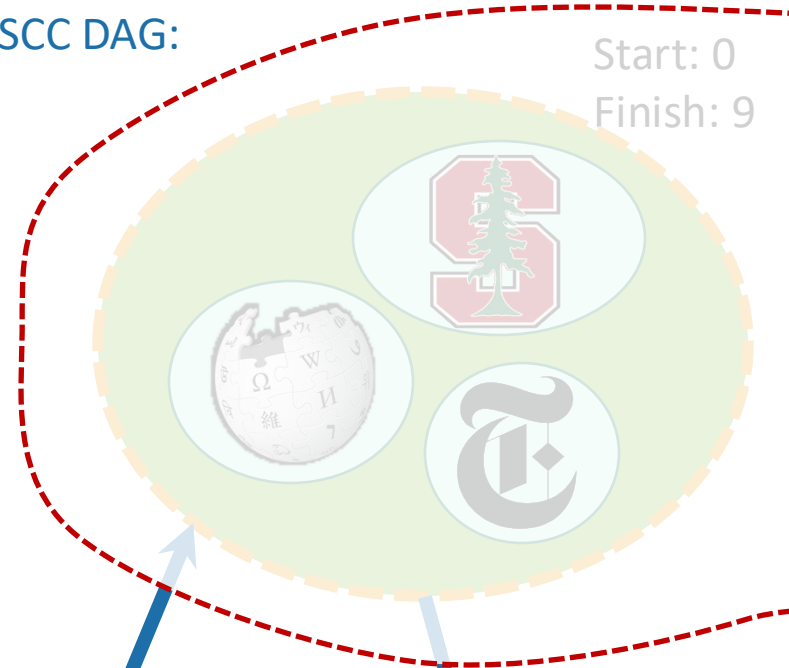
Now we see why this finds SCCs.

Remember that after the first round of DFS, and after we reversed all the edges, we ended up with this SCC DAG:

- The Corollary says that **all blue arrows point towards larger finish times**.
 - So if we start with the largest finish time, **all blue arrows lead in**.
 - Thus, that connected component, and only that connected component, are reachable by the second round of DFS
-
- Now, we've deleted that first component.
 - The next one has the **next biggest finishing time**.
 - So all remaining blue arrows lead in.
 - Repeat.



Start: 2
Finish: 5



Start: 0
Finish: 9



Start: 10
Finish: 11

Formally, we prove it by...

Induction!

Formally, we prove it by induction

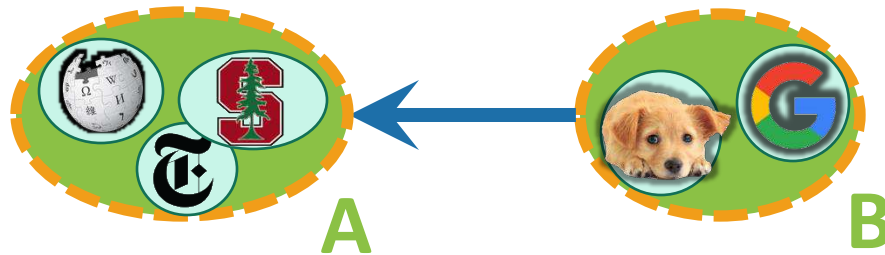
- **Theorem:** The algorithm we saw before will correctly identify strongly connected components.
- **Inductive hypothesis:**
 - The first t DFS trees found in the second (reversed) DFS forest are the t SCCs with the largest finish times.
- **Base case: ($t=0$)**
 - The first 0 DFS trees found in the second (reversed) DFS forest are the 0 SCCs with the largest finish times. **(TRUE)**
- **Inductive step:**
 - Fun exercise! [See skipped slide; we already did the intuition]
- **Conclusion:**
 - The IH holds when $t = \text{\#SCCs}$, aka the second (reversed) DFS forest contains all the SCCs as its trees!

Inductive step

- **Assume by induction that the first t trees are the last-finishing SCCs.**
- Consider the $(t+1)^{\text{st}}$ tree produced, suppose the root is **x**.
- Suppose that **x** lives in the SCC **A**.
- Then **A.finish** > **B.finish** for all remaining SCCs **B**.
 - This is because we chose **x** to have the largest finish time.
- Then there are no edges leaving **A** in the remaining SCC DAG.
 - This follows from the Corollary.
- Then DFS started at **x** recovers exactly **A**.
 - It doesn't recover any more since nothing else is reachable.
 - It doesn't recover any less since A is strongly connected.
 - (Notice that we are using that A is still strongly connected when we reverse all the edges).
- **So the $(t+1)^{\text{st}}$ tree is the SCC with the $(t+1)^{\text{st}}$ biggest finish time.**

Recap of proof idea

- The finish times of the first DFS basically do topological sort on the SCC DAG.
- Thus, when we reverse all the edges we have:
 - If there is an edge like this in the reversed graph:



- Then $A.\text{finish} > B.\text{finish}$.
- Then when we do DFS again in reverse order of finish time, we pick off the SCCs.

Punchline:
we can find SCCs in time $O(n + m)$

Algorithm:

- Do DFS to create a **DFS forest**.
 - Choose starting vertices in any order.
 - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
 - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest**.



(Clearly it wasn't obvious since it took all class to do! But hopefully it is less mysterious now.)

Recap

- Depth First Search reveals a very useful structure!
 - We saw last week that this structure can be used to do **Topological Sorting** in time $O(n+m)$
 - Today we saw that it can also find **Strongly Connected Components** in time $O(n + m)$
 - This was a non-obvious algorithm!
 - (at least, it was non-obvious 80 minutes ago)

Next time

- Dijkstra's algorithm!

BEFORE Next time

- Pre-lecture exercise: weighted graphs!