CS 161 **Problem Set 3**
Fall 2025 **Due:** Friday October 24 at 11:59pm on Gradescope.

Style guide and expectations: Please see the top of the "Homework" page on the course webpage for guidance on what we are looking for in homework solutions. We will grade according to these standards.

Make sure to look at the "**We are expecting**" blocks below each problem to see what we will be grading for in each problem!

Collaboration policy: You may do the HW in groups of size up to three. Please submit one HW for your whole group on Gradescope. (Note that there is an option to submit as a group). See the "Policies" section of the course website for more on the collaboration policy.

LLM policy: Check out the course webpage for best practices on how to productively use LLMs on homework, if you use them at all.

Exercises

We recommend you do the exercises on your own before collaborating with your group. The point is to check your understanding.

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1. **(6 pt.)** In this exercise, we'll explore different types of randomized algorithms. We say that a randomized algorithm is a **Las Vegas Algorithm** if it is always correct, but the running time is a random variable. We say that a randomized algorithm is a **Monte Carlo Algorithm** if there is some probability that it is incorrect. For example, QuickSort (with a random pivot) is a Las Vegas algorithm, since it always produces a sorted array (but if we get very unlucky QuickSort may be slow).

   Suppose that there is a population of n ducks. Half of the ducks are green, and half are brown, but it's dark outside and you can't tell the difference until you catch one and look at it with a flashlight. Assume that catching a random duck and looking at it takes time $O(1)$.

   The algorithms given in Figure 1 all attempt to find a single green duck. Fill in the chart below. You may use asymptotic notation for the running times; give the best big-Oh bound that you can. For the probability of returning a correct duck, do not use asymptotic notation; give the best bound that you can.

Algorithm	Monte Carlo or Las Vegas?	Expected running time	Worst-case running time	Probability of returning a green duck
Algorithm 1				
Algorithm 2				
Algorithm 3				

   Note that the LaTeX code for the table is available in the source file.

[**Hint:** *Remember (see Lecture 5 and the pre-lecture exercise) that the expectation of a geometric random variable with probability p is equal to $\frac{1}{p}$*]

[**We are expecting:** *A filled-in table. No justification is required.*]

Algorithm 1: findGreenDuck1

Input: A flock of n ducks
while *true* **do**
    Choose a random duck from the flock;
    if *That duck is green* **then**
        return *that duck*
    else
        Release the duck back into the flock

Algorithm 2: findGreenDuck2

Input: A flock of n ducks
for *100 iterations* **do**
    Choose a random duck from the flock;
    if *That duck is green* **then**
        return *that duck*
    else
        Release the duck back into the flock
Choose a random duck from the flock;
return *That duck*

Algorithm 3: findGreenDuck3

Input: A flock of n ducks
Put the ducks in a line, in a random order ;
/* Assume it takes time $O(n)$ to put the n ducks in a random order; and assume that they stay put once ordered. */
for $i = 0, \ldots, n-1$ **do**
    if *Duck i is green* **then**
        return *Duck i*

Figure 1: Three algorithms for finding a green duck

SOLUTION:

Algorithm	Monte Carlo or Las Vegas?	Expected running time	Worst-case running time	Probability of returning a green duck
Algorithm 1	Las Vegas	$O(1)$	∞	1
Algorithm 2	Monte Carlo	$O(1)$	$O(1)$	$1 - 1/2^{101}$
Algorithm 3	Las Vegas	$O(n)$	$O(n)$	1

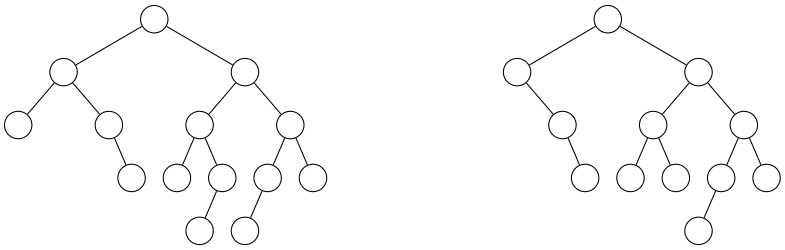
2. **(4 pt.)** [**Coloring RB Trees.**]

Can you color in the nodes of the trees below to make legitimate red-black trees? For each tree, either color the nodes to make a valid red-black tree, or say that no such coloring exists.

Note: the LaTeX code to make these trees is included in the template. If you'd like to use this code, you can color a node by editing it like:

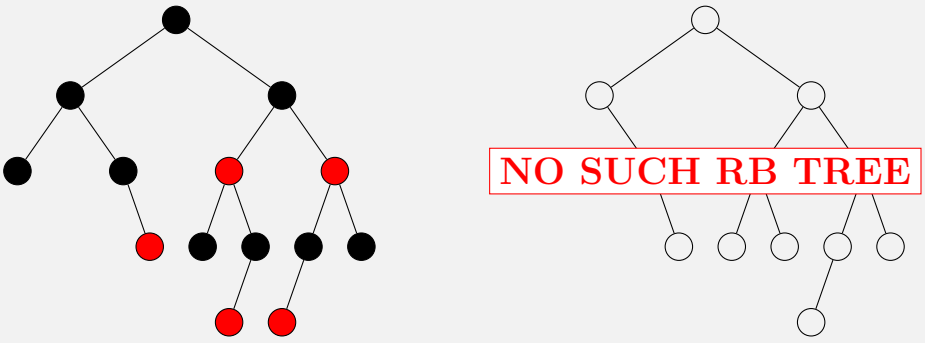
`\node[draw,circle,fill=red]` or `\node[draw,circle,fill=black]`.

Or you can just color the nodes in your favorite drawing program and include the image with `\includegraphics{my_image.png}`.



[**We are expecting:** *For each tree, either an image of a colored-in red-black tree or a statement "No such red-black tree." No justification is required.*]

SOLUTION:



3. **(5 pt.)** This exercise references the IPython notebook `HW3.ipynb`, available on the course website along with this problem set.

   In our implementation of radixSort in class, we used bucketSort to sort each digit. Why did we use bucketSort and not some other sorting algorithm? There are several reasons, and we'll explore one of them in this exercise.

   (a) **(2 pt.)** One reason we chose bucketSort was that it makes radixSort work correctly! In `HW3.ipynb`, we've implemented four different sorting algorithms—bucketSort, quickSort, and two versions of mergeSort—as well as radixSort.

   Note: The IPython notebook is long, but just because it implements many different sorting algorithms. Don't get scared!

   There is a `TODO` statement in the IPython notebook where you can change the code to use different sorting algorithms; you just have to make sure that the sorting algorithm you want to use is the one that is not commented out. No programming necessary!

   Modify the code for radixSort to use each one of these four algorithms within radixSort, and test it out on the examples suggested.

   Which sorting algorithms seem to be correct as "inner sorting algorithms" for radixSort?

   - Does using bucketSort always work correctly?
   - Does using quickSort always work correctly?
   - Does using mergeSort (with merge1) always work correctly?
   - Does using mergeSort (with merge2) always work correctly?

   [**We are expecting:** *Yes or no for each part.*]

   (b) **(3 pt.)** Explain what you saw above. What was special about the algorithms which worked? Why does this special thing matter? (You may wish to play around with `HW3.ipynb` to "debug" the incorrect cases.)

   [**We are expecting:** *A clear definition of the special property that the correct algorithms have, and a few sentences explaining why it matters. A minimal example of what might go wrong is a great way to explain why this property matters.*

   You do not need to justify why each of the algorithms do or do not have the property.]

   Note: yes, this property is alluded to in the reading. That's why this is an exercise and not a problem! To get the most out of this exercise, play around with the examples in the code and make sure you really understand what's going on.

SOLUTION:

(a) bucketSort works, quickSort doesn't, mergeSort1 doesn't work and mergeSort 2 does work.

(b) The special thing is that these algorithms are **stable.** That is, if we have two items x and y with the same keys, and x comes before y in the original list, then x comes before y after the sort has been run. As we saw in class, this was crucial for the correctness proof of radixSort: once we had sorted on the first digit for example, we needed that work not to be un-done when we sorted on the second digit. More precisely, suppose that we are

radixSorting [12, 13]. First we sort on the LSD, and the array doesn't change. Then we sort on the MSD, which is 1 for both 12 and 13. If the sort isn't stable, then we might switch the order of the elements in the case of a tie, and end up with [13, 12] rather than [12, 13].

Problems

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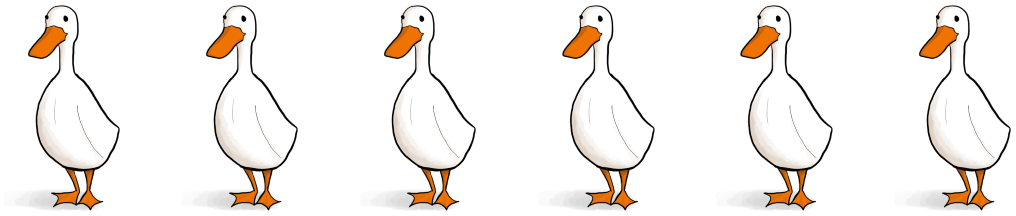
4. **(6 pt.)** **[Ducks.]** Suppose that n ducks are standing in a line.



Each duck has a political leaning: left, right, or center. You'd like to sort the ducks so that all the left-leaning ones are on the left, the right-leaning ones are on the right, and the centrist ducks are in the middle. You can only do two sorts of operations on the ducks:

Operation	Result
`poll(j)`	Ask the duck in position j about its political leanings
`swap(i,j)`	Swap the duck in position j with the duck in position i

However, in order to do either operation, you need to pay the ducks to cooperate: each operation costs one piece of duckweed. Also, you didn't bring a piece of paper or a pencil (or your smartphone or tablet or tablet or whatever you use to take notes) so you can't write anything down and have to rely on your memory! Like many humans, your memory is limited, and you can only remember up to seven[1] integers between 0 and $n-1$ at a time (i.e. you can use at most seven integer-valued variables at a time in your algorithm).

Design an algorithm to sort the ducks, which costs $O(n)$ pieces of duckweed, and uses no extra memory other than storing at most seven[2] integers between 0 and $n-1$.

[**Hint:** *Does this task look like anything we've seen in class?*]

[**We are expecting:** *Pseudocode* ***AND*** *a short English description of your algorithm;* ***AND*** *a short explanation of why it uses only $O(n)$ pieces of duckweed and never uses more than seven numbers of memory.*]

SOLUTION:

We can use a variant of the algorithm which we used for QuickSort to partition elements in-place:

```
sortDucks( ducks ):
   i = 0
```

[1] `https://en.wikipedia.org/wiki/The_Magical_Number_Seven,_Plus_or_Minus_Two`

[2] You don't need to use all seven storage spots, but you can if you want to. Can you do it with only two?

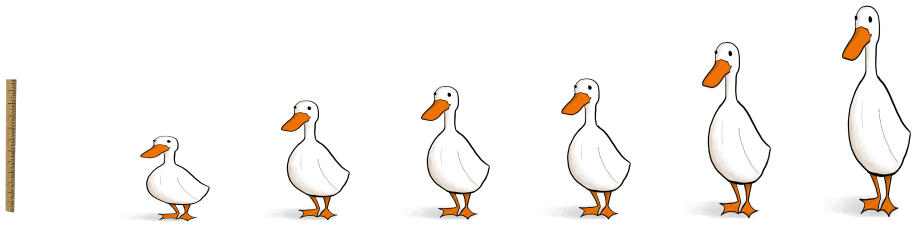
```
# first, go through all the ducks and put the
# left-leaning ones on the left.
for j = 0, ..., n-1:
   if poll(j) == "left":
       swap( i, j )
       i += 1
# next, go through and put the center-leaning ones
# to the left of the right-leaning ones.
for j = 0, ..., n-1:
    if poll(j) == "center":
       swap( i, j )
       i += 1
# now we should be all done!
```

This takes two passes through the ducks, and in each pass, for each j, it does two operations: a `poll` operation and a `swap` operation. Thus, the total cost is at most $4n = O(n)$ pieces of duckweed. Moreover, we only need to remember i and j, so we just need 2 integers worth of memory.

To see that it works, consider what happens in the first for loop. We'll maintain the loop invariant that ducks $0, \ldots, i-1$ are left-leaning, and none of ducks $i, \ldots, j-1$ are left-leaning . For the base case, when $i = j = 0$, this is trivially true. Now assume it's true for $j-1$. We advance j until the j'th duck is left-leaning, and we swap it with the duck in the i'th position. By assumption, duck i was not left-leaning, so it is true that none of the ducks $i+1, \ldots, j$ are left-leaning. Also since duck j (before we swapped) was left-leaning, we now have ducks $0, \ldots, i$ are left-leaning. After incrementing i and j, this establishes the loop invariant for the next round.

Thus, at the end of the first loop, ducks $0, \ldots, i-1$ are left-leaning, and there are no left-leaning ducks left among $i, \ldots, n-1$. The logic for the second loop is exactly the same, and it puts all of the centrist ducks to the left of the right-leaning ducks. Thus, at the end of the day the ducks are sorted.

5. **[Ducks.] (6 pt.)** Suppose that n ducks of distinct heights are standing in a line, ordered from shortest to tallest.



You have a measuring stick of a certain height, and you would like to identify a duck which is the same height as the stick, or else report that there is no such duck. The only operation you are allowed to perform is `compareToStick(f)`, where f is a duck (that is, you cannot directly access the heights of each duck). `compareToStick(f)` returns `taller` if f is taller than the stick, `shorter` if f is shorter than the stick, and `the same` if f is the same height as the stick. You've still forgotten to bring a paper and a pencil and so you can only store up to seven integers in $\{0, \ldots, n-1\}$ at a time. And you have to pay a duck one piece of duckweed every time you perform `compareToStick` on it.

(a) **(2 pt.)** Give an algorithm in this model of computation which either finds a duck the same height as the stick, or else returns "No such duck," and uses $O(\log(n))$ pieces of duckweed.

[**We are expecting:** *Pseudocode* ***AND*** *an English description. You do not need to justify the correctness or duckweed usage.*]

(b) **(4 pt.)** Prove that any algorithm in this model of computation must use $\Omega(\log(n))$ pieces of duckweed.

[**We are expecting:** *A short but convincing argument.*]

SOLUTION:

(a) This is an application of binary search:

```
findDuck( ducks ):
    if len( ducks ) == 0:
        return "no such duck."
    j = floor( len(ducks)/2 )
    if compareToStick( ducks[j] ) == "the same":
        return ducks[j]
    else if compareToStick( ducks[j] ) == "shorter":
        return findDuck( ducks[j+1:] )
    else if compareToStick( ducks[j] ) == "taller":
        return findDuck( ducks[:j] )
```

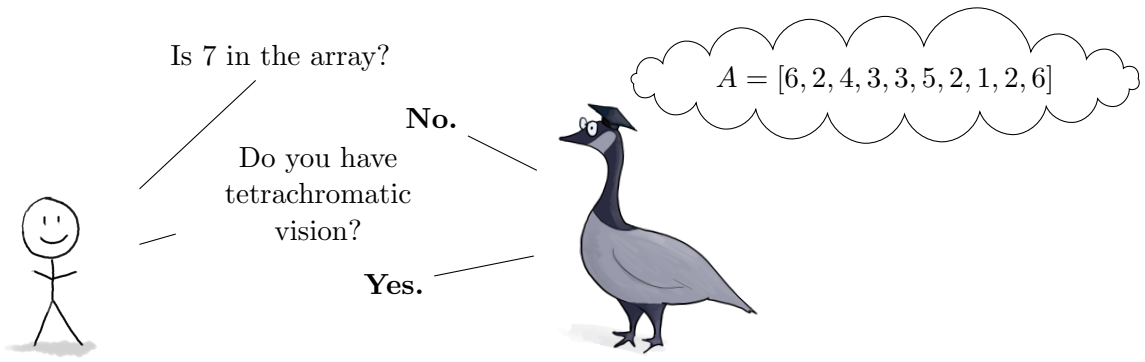
That is, at each step we divide the line in half. If the current duck is too short, we recurse on the taller (right) half; if the current duck is too tall, we recurse on the shorter

(left) half. Since we divide the input in half each time, the total number of calls to `findDuck` is $\log(n)$, and each call uses only one call to `compareToStick`, so the total number of brine shrimp used is $O(\log(n))$.

(b) We use the decision tree method. Any algorithm in this model works by comparing ducks to the stick, and thus can be written as a tri-nary decision tree: at each step, there are one of three possible answers, bigger, smaller, or the same. There are $n + 1$ possible outcomes (each duck could be the right height, or none of them could be) so the decision tree has at least $n+1$ leaves. Thus it has height at least $\log_3(n+1) = \Omega(\log(n))$.

6. **[Goose!] (5 pt.)** A wise goose has knowledge of an array A of length n, such that $A[i] \in \{1, \ldots, k\}$ for all i. (Note that the elements of A are not necessarily distinct). You don't have direct access to A, but you can ask the wise goose *any* yes/no questions about it. For example, you could ask "If I remove $A[5]$ and swap $A[7]$ with $A[8]$, would the array be sorted?" or "can some geese fly as high as 29,000ft?"

   Unlike in the previous problems, this time you did bring a paper and pencil, and your job is to write down all of the elements of A in sorted order.[3] The wise goose charges one piece of duckweed per question.[4]



   (a) **(5 pt.)** Give a procedure that outputs a sorted version of A which uses $O(k \log(n))$ pieces of duck weed. You may assume that you know n and k, although this is not necessary.

   [**We are expecting:** *Pseudocode AND a short English description of your algorithm;* ***AND*** *a brief explanation of why it uses $O(k \log(n))$ pieces of duckweed.*]

   (b) **(1 BONUS pt.)** Prove that any procedure to solve this problem must use $\Omega(k \log(n/k))$ pieces of duckweed.

   [**We are expecting:** *Nothing; this part is not required.*]

SOLUTION:

(a) First note that there can be between 0 and n elements in A with value c, for each number $c \in \{1, \ldots, k\}$. So our approach will work by finding the exact amount of elements in A for each value c and writing down that many c's.

To do so, the algorithm will do a binary search from 0 to n on each of the numbers $\{1, \ldots, k\}$, to find their frequencies. To do binary search, we can ask questions of the nature "are there $n/2$ or less elements with value i", or "are there between $3n/8$ and $n/2$ elements with value i", depending on the binary search's current lower and upper bounds. Then we write down that many copies of the value i on our notepad.

[3] Note that you don't have any ability to change the array A itself, you can only ask the wise goose about it.
[4] Despite the name, it turns out that geese also eat duckweed.

```
for i = 1, ..., k:
    # binary search
    num_copies = goose_binary_search( 0, n, i )
    write down num_copies copies of i on your notepad

def goose_binary_search( low, high, k):
    mid = (low + high)//2
    ask the wise goose if there are exactly mid copies of k in the list.
    if so,
        return mid
    ask the wise goose if there are < mid copies of k in the list.
    if so,
        return goose_binary_search( low, mid, k )
    return goose_binary_search( mid+1, high, k )
```

This algorithm uses $O(k \log(n))$ brine shrimp because each binary search asks $O(\log(n))$ questions and we perform a binary search for each of the k numbers.

(b) We use the decision tree method from class: imagine that each question we ask the goose is a node in the decision tree, and the leaves are the answers (the possible sorted lists). Consider the number of distinct sorted lists that the wise goose may be thinking of. Each list corresponds to a list of numbers $x_1, \ldots, x_k$, so that $\sum_i x_i = n$. Each such list means "There are x_1 copies of 0, x_2 copies of 1," etc. How many of these lists are there? It turns out there are

$$X = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

of them. (This is the number of ways to put n balls into k bins). Thus, the decision tree has at least X leaves, and so the depth of the tree (number of questions asked) is at least $\log_2(X)$, which is

$$\log_2 \binom{n+k-1}{k-1} \geq \log_2 \left(\left(\frac{n}{k}\right)^{k-1} \right) = \Omega(k \log(n/k)).$$