# Applied Cryptography and Network Security
# CS 1653

Summer 2023

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(Slides are adapted from Prof. Adam Lee's CS1653 slides.)

# Announcements

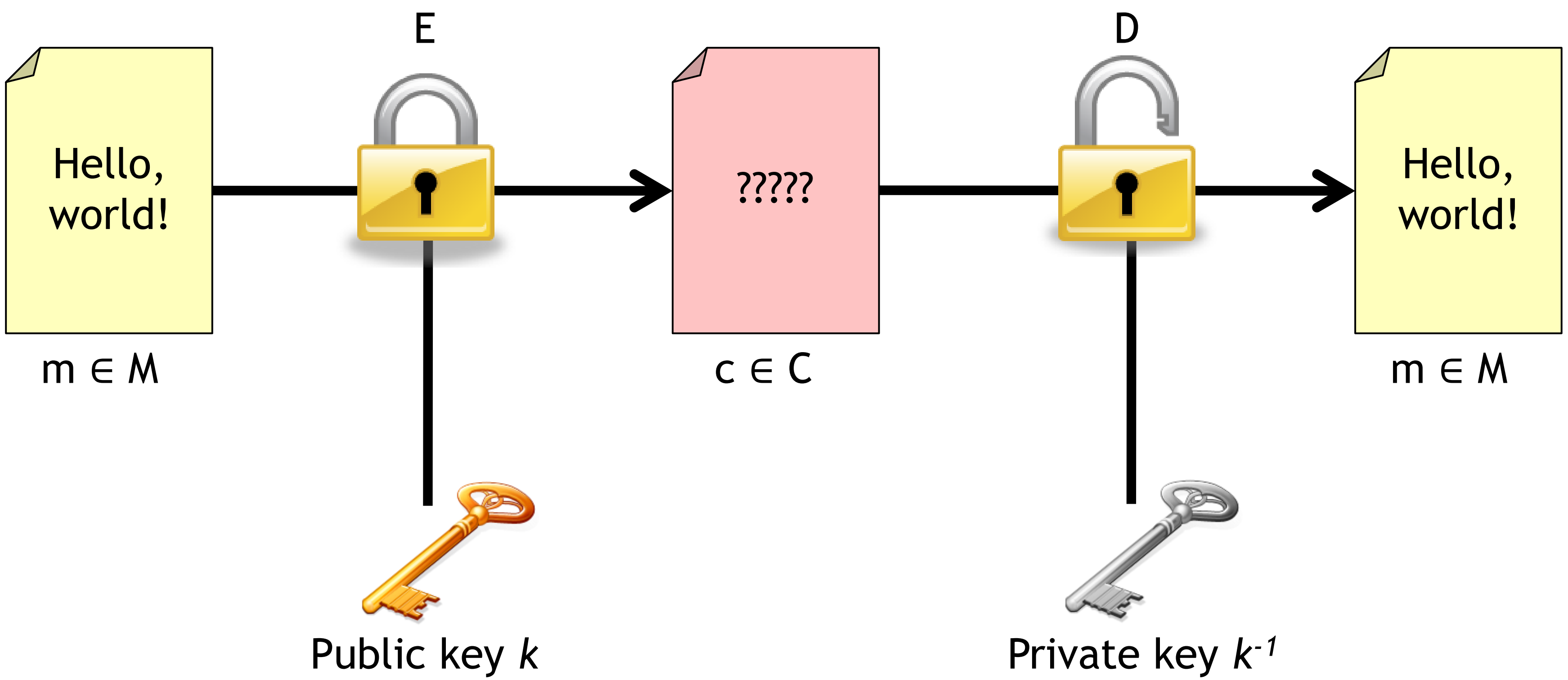
- Homework 3 due this Friday @ 11:59 pm
- Phase 2 of Project
  - due on Friday 6/30 @ 11:59 pm
- Makeup lecture on Friday 6/16 @ 11:00 am

# Public key cryptosystems

Formally, a cryptosystem can be represented as the 5-tuple (E, D, M, C, K)

- M is a message space
- K is a key space
- E : M × K → C is an encryption function
- C is a ciphertext space
- D : C × K → M is a decryption function

Note: Each "key" in K is actually a pair of keys, $(k, k^{-1})$



# What can we do with public key cryptography?

First, we need some way of finding a user's public key

Print it in
the newspaper

Post it on your
webpage

A trusted
keyserver (PKI)

Important: It is critical to verify the authenticity of any public key!
(How?)

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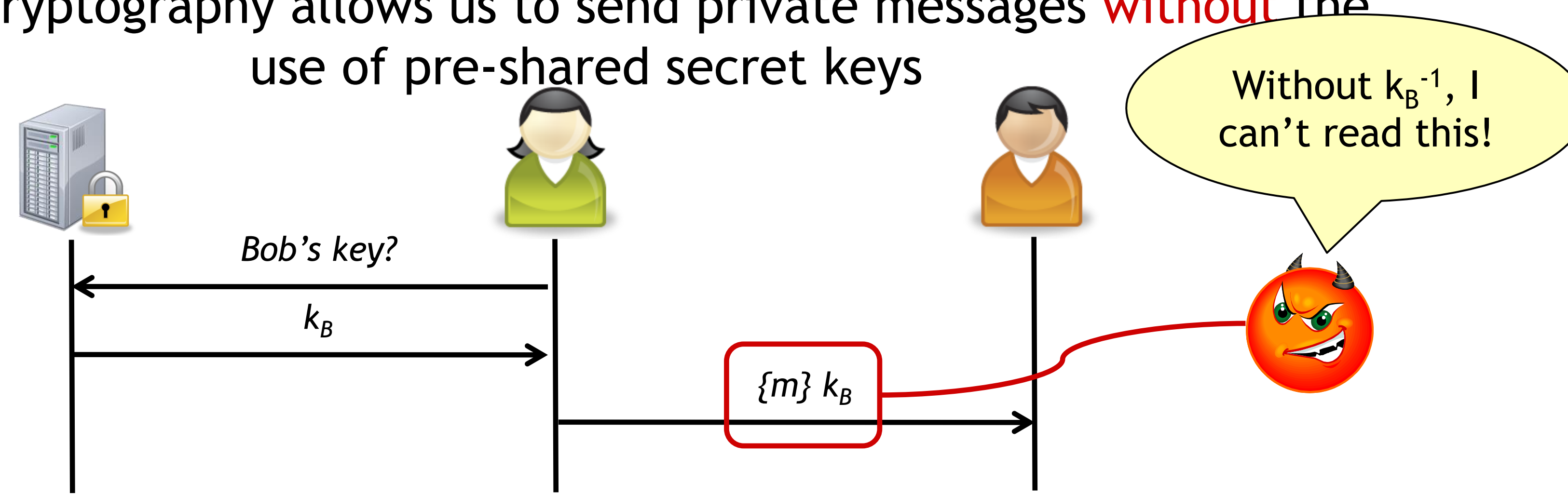
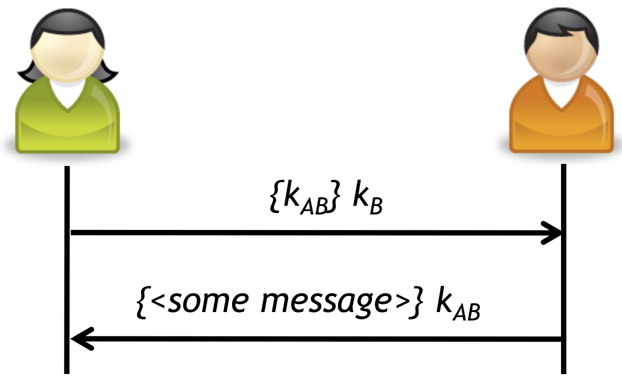
Public key cryptography allows us to send private messages without the use of pre-shared secret keys



# Public key cryptography help exchange symmetric keys

1. Generate a key $k_{AB}$
2. Encrypt $k_{AB}$ using Bob's public key $k_B$
3. Transmit




1. Decrypt $k_{AB}$ using private key $k_B^{-1}$
2. Encrypt message using $k_{AB}$
3. Transmit

**Note:** Only Bob can decode $k_{AB}$, since only he knows $k_B^{-1}$

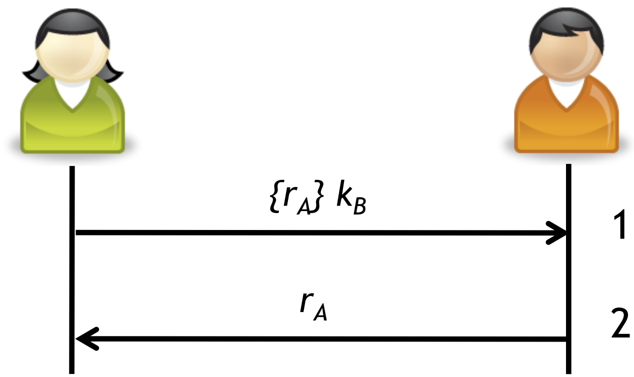
- Unfortunately, Bob doesn't know who this key is from
- Key exchange is not quite this easy in practice, but it isn't *much* harder

**Question:** Why do we want to exchange symmetric keys?!

- Public key cryptography is usually pretty slow…
  - Based on "fancy" math, not bit shifting
  - Symmetric key algorithms are orders of magnitude faster
- It's always a good idea to change keys periodically

# Public key cryptography to authenticate users

1. Pick $r_A$ at random
2. Encrypt $r_A$ using Bob's public key $k_B$
3. Transmit




1. Decrypt $r_A$ using private key $k_B^{-1}$
2. Transmit

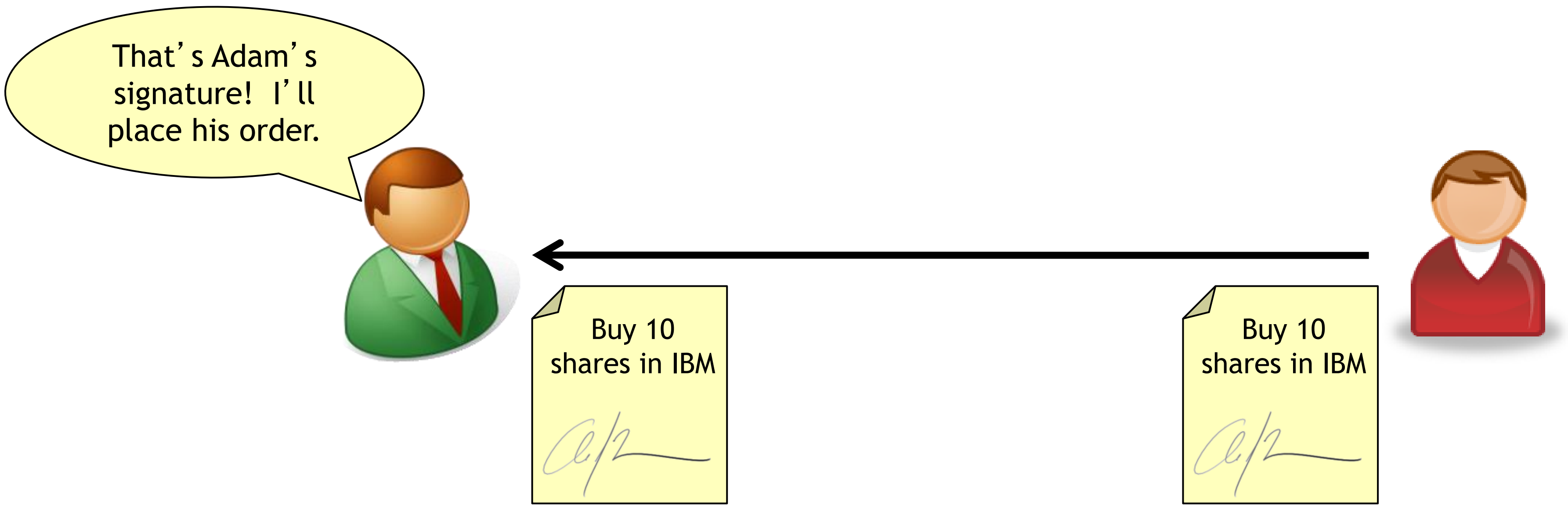
Note: As in the previous key exchange, only Bob can decrypt $r_A$, since only Bob knows $k_B^{-1}$.

It is of absolute importance that the random numbers used during this type of protocol are not predictable and are never reused (Why?)

- Unpredictable:
  - The security of this protocol is a proof of possession of $k_B^{-1}$
  - If predictable, an adversary can guess the "challenge" without decrypting!
  - (This is bad news)
- Reusing challenges may* lead to replay attacks (When?)

# Public key systems also let us  create digital signatures

Goal:  If Bob is given a message *m* and a signature S(*m*) supposedly computed by Adam, he can determine whether Adam wrote m




For this to occur, we require that

- The signature S(*m*) must be unforgeable
- The signature S(*m*) must be verifiable

Question: *How can we do this?*

# In many public key cryptosystems, encryption and decryption are commutative

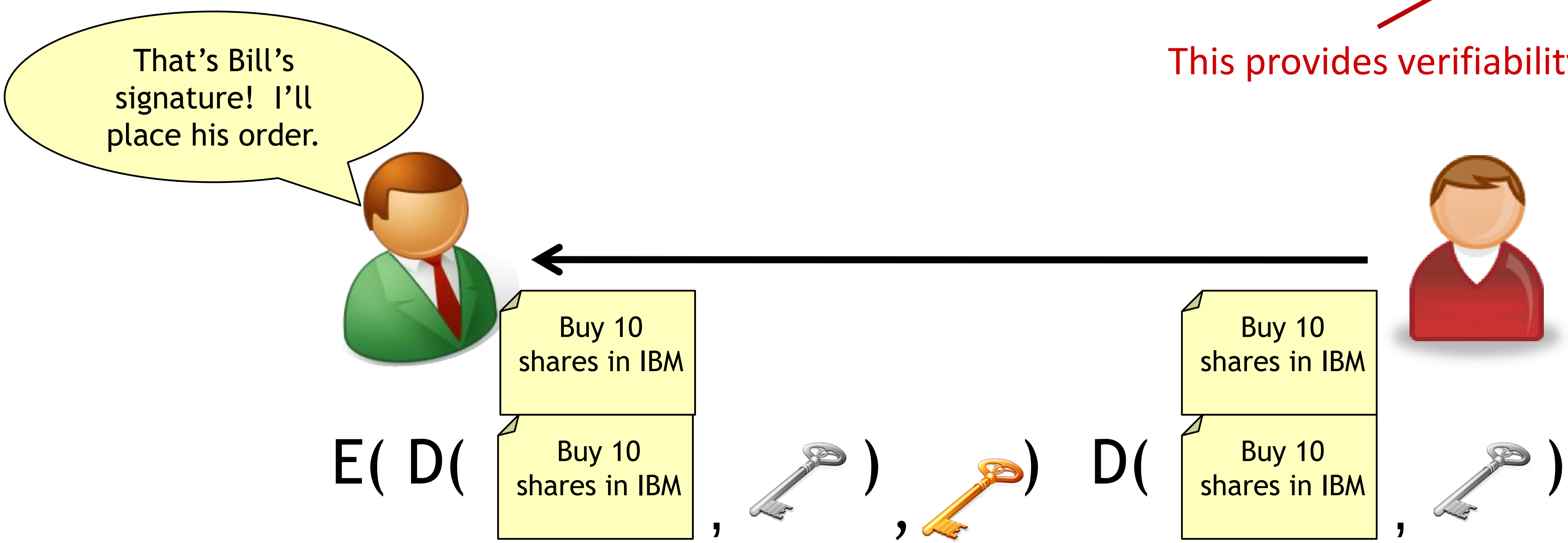
That is, $D(E(m, k), k^{-1}) = E(D(m, k^{-1}), k) = m$

In such a system, we can use digital signatures as follows:

- To sign a message, compute $D(m, k^{-1})$
- Transmit $m$ and $D(m, k^{-1})$ to the recipient
- The recipient uses the sender's public key to verify that $E(D(m, k^{-1}), k) = m$

This is unforgeable

This provides verifiability




Question: Does encryption with a shared key have the same properties?

# Features and Requirements

These features all require that for a given key pair ($k$, $k^{-1}$), $k$ can be made public and $k^{-1}$ must remain secret

So, in a public key cryptosystem it must be

1. Computationally easy to encipher or decipher a message
2. Computationally infeasible to derive the private key from the public key
3. Computationally infeasible to determine the private key using a **chosen plaintext attack**

Informally, easy means "polynomial complexity", while infeasible means "no easier than a brute force search"

*How do public key cryptosystems work?*

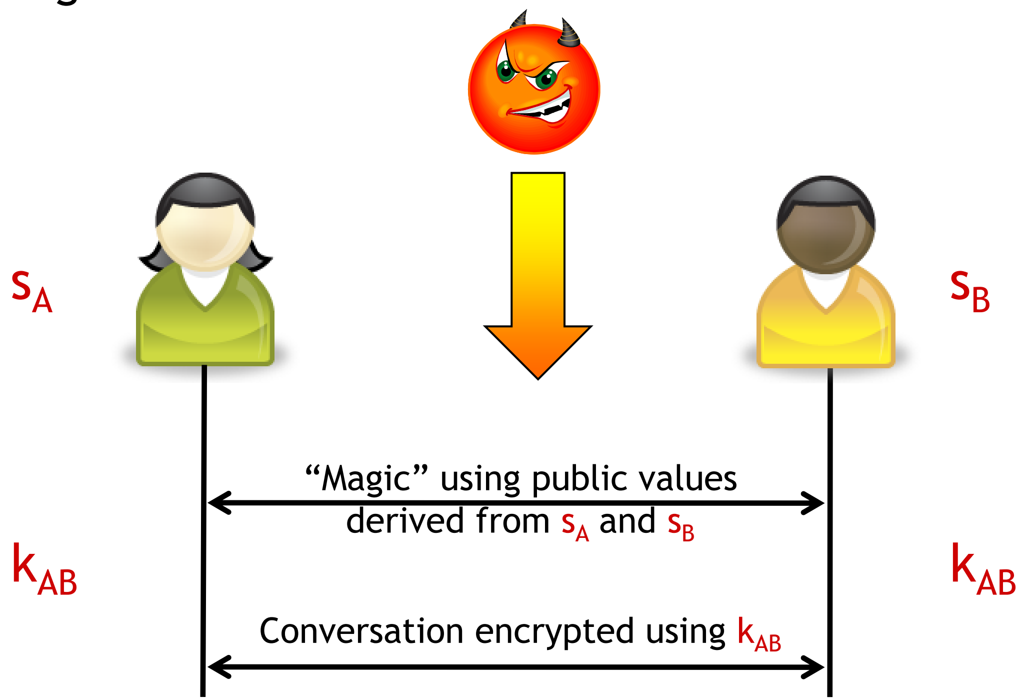
# Diffie and Hellman proposed* the notion of public key cryptography

Diffie and Hellman did not succeed in developing a full-fledged public key cryptosystem

- i.e., their system cannot be used to encrypt/decrypt documents directly
- Rather, it allows two parties to agree on a shared secret using an entirely public channel

Question: Why is this an interesting problem to solve?

- Key exchange!



# Diffie and Hellman proposed their system in 1976

Seminal paper: Whitfield Diffie and Martin E. Hellman, "New Directions in Cryptography," IEEE Transactions on Information Theory (22)6 : 644 – 654, Nov. 1976

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***Problem:*** The widening use of telecommunications coupled with the key distribution problems inherent with secret key cryptography point to the fact that current solutions are not scalable!

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This paper accomplishes many things:

- Clearly articulates why the key distribution problem must be solved
- Motivates the need for digital signatures
- Presents the first public key cryptographic algorithm
- Opened the "challenge" of designing a general-purpose public key cryptosystem

Variants of the Diffie-Hellman key exchange algorithm are widely used today!

# How does the Diffie-Hellman protocol work?

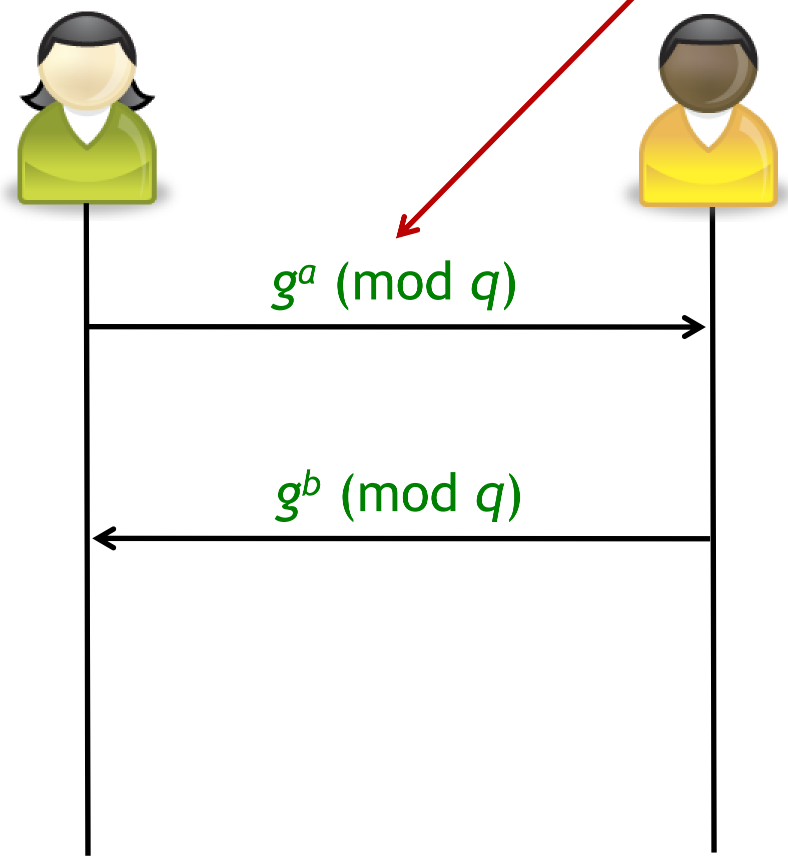
***Step 0:*** Alice and Bob agree on a finite cyclic group G of (large) prime order $q$, and a generator $g$ for this group. This information is all public.

$a$ is Alice's private key

$g^a \pmod{q}$ is Alice's public key

***Step 1:***

- Randomly choose $a \in \{1, 2, \ldots, q\text{-}1\}$
- Compute $g^a \pmod{q}$
- Send $g^a \pmod{q}$




***Step 2:***

- Randomly choose $b \in \{1, 2, \ldots, q\text{-}1\}$
- Compute $g^b \pmod{q}$
- Send $g^b \pmod{q}$

***Step 3:***

- Compute $(g^b \pmod{q})^a \pmod{q}) = g^{ba} \pmod{q} = K_{ab}$

***Step 3':***

- Compute $(g^a \pmod{q})^b \pmod{q}) = g^{ab} \pmod{q} = K_{ab}$

# Why is the Diffie-Hellman key exchange protocol safe?

Recall: We need to show that it is hard for a "bad guy" to learn any of the secret information generated by this protocol, assuming that they know all public information

***Public information:*** G, $g$, $q$, $g^a$ (mod $q$), $g^b$ (mod $q$)

***Private information:*** $a$, $b$, $K_{ab} = g^{ab}$ (mod $q$)

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Tactic 1: Can we get $g^{ab}$ (mod $q$) from $g^a$ (mod $q$) and $g^b$ (mod $q$)?

- We can get $g^{am+bn}$ (mod $q$) for arbitrary $m$ and $n$, but this is no help…

Tactic 2: Can we get $a$ from $g^a$ (mod $q$)?

- This called taking the discrete logarithm of $g^a$ (mod $q$)
- The discrete logarithm problem is **widely believed** to be very hard to solve in **certain types** of cyclic groups

***Conclusion:*** If solving the discrete logarithm problem is hard, then the Diffie-Hellman key exchange is secure!

# Hm, interesting…

Recall from previous lectures that:

- Block ciphers secure data through confusion and diffusion
- Designing block cipher mechanisms is equal parts art and science
- The security of a block cipher is typically accepted over time (Assurance!)
  - Recall the initial skepticism over DES
  - The NIST competitions promote this as well

In public key cryptography, the relationship between $k$ and $k^{-1}$ is intrinsically mathematical

Result: The security of these systems is also rooted in mathematical relationships, and proofs of security involve reductions to mathematically "hard" problems

- e.g., Diffie-Hellman safe **if** the discrete logarithm is hard

# The RSA cryptosystem picks up where Diffie and Hellman left off

RSA was proposed* by Ron Rivest, Adi Shamir, and Leonard Adelman in 1978. It can be used to encrypt/decrypt and digitally sign arbitrary data!

Key generation:

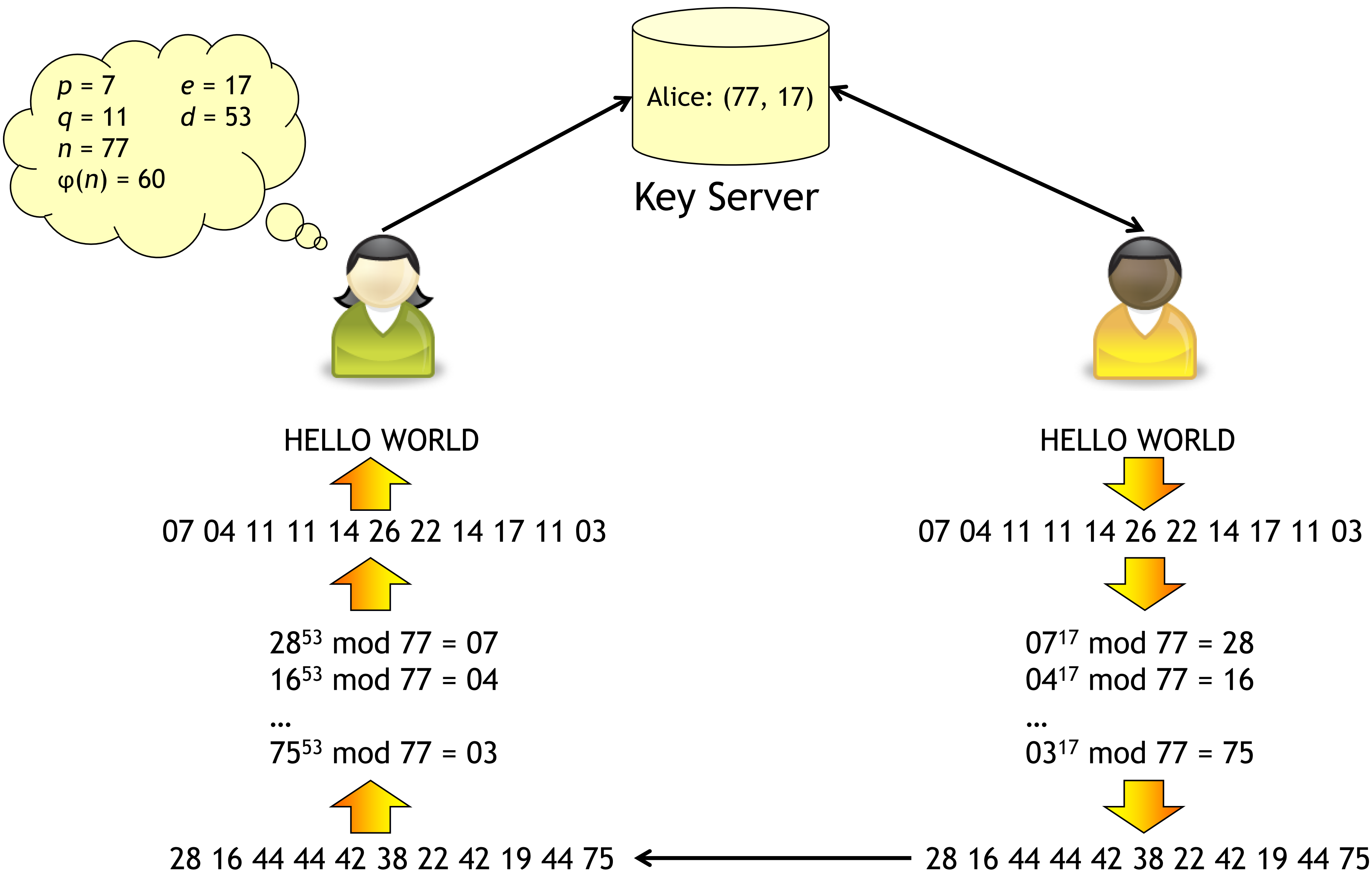
- Choose two large prime numbers $p$ and $q$, compute $n = pq$
- Compute $\varphi(n) = (p-1)(q-1)$
- Choose an integer $e$ such that $\gcd(e, \varphi(n)) = 1$
- Calculate $d$ such that $ed \equiv 1 \pmod{\varphi(n)}$
- Public key: $n$, $e$
- Private key: $p$, $q$, $d$

We'll discuss how to do these steps and why they work later today

Usage:

- Encryption: $M^e \pmod{n}$
- Decryption: $C^d \pmod{n} = M^{ed} \pmod{n} = M^{k\phi(n) + 1} \pmod{n} = M^1 \pmod{n} = M$

# An RSA Example



# What is involved in breaking RSA?

To break RSA, an attacker would need to derive the decryption exponent *d* from the public key (*n*, *e*)

Mathematicians think that this is a hard problem

This is conjectured to be as hard as factoring *n* into *p* and *q*. Why?

- Given *p*, *q*, and *e*, we can compute $\phi(n)$
- This allows us to compute *d* easily!

But what if there is some entirely unrelated way to derive *d* from the public key (*n*, *e*)?

Question: Should this make you uneasy? Why or why not?

My Answer: Probably not, since this bizarre new attack would also have applications to factoring large numbers.

# As always, nothing is really that easy…

The bad news: Naive implementations of RSA are vulnerable to chosen ciphertext attacks

The good news: These attacks can be prevented by using a padding scheme like OAEP prior to encryption

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*Don't implement cryptography yourself! Use a standardized implementation, and verify that it is standards compliant.*

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Lastly, don't forget that implementations can be subjected to attacks

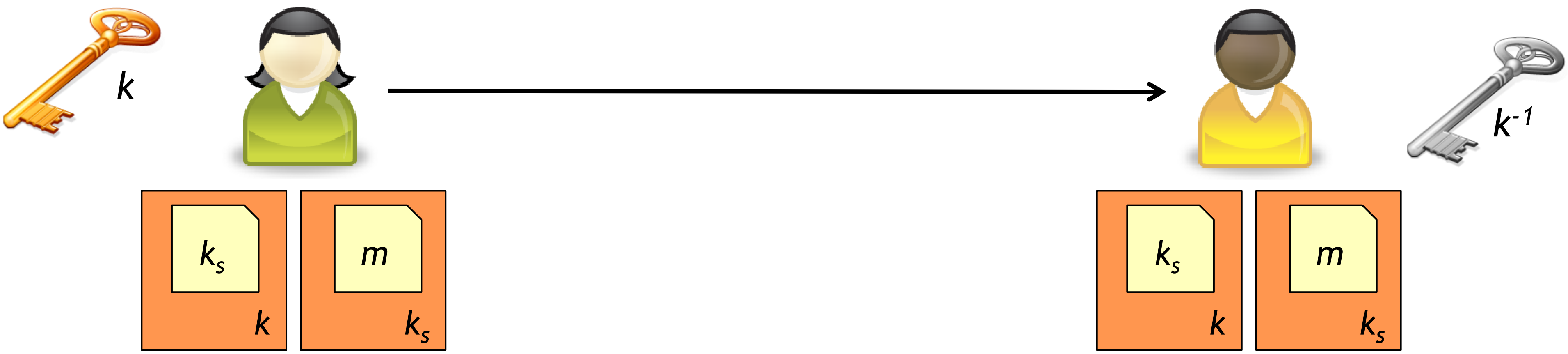
- Timing attacks
- Power consumption attacks
- Etc…

More on this later

# Unfortunately, RSA is slow when compared to symmetric key algorithms like AES or HMAC-X

Using RSA as part of a hybrid cryptosystem can speed up encryption

- Generate a symmetric key $k_s$
- Encrypt $m$ with $k_s$
- Use RSA to encrypt $k_s$ using public key $k$
- Transmit $E_{ks}(m)$, $E_k(k_s)$




Using hash functions can help speed up signing operations

- Intuition: $H(m) << m$, so signing $H(m)$ takes far less time than signing $m$
- Why is this safe? H's preimage resistance property!

# Some public-key systems have an interesting property known as malleability

Informally, a malleable cryptosystem allows meaningful modifications to be made to ciphertexts without revealing the underlying plaintext

- E.g., $E(x) \otimes E(y) = E(x + y)$

See: Pascal Paillier, Public-Key Cryptosystems Based on Composite Degree Residuosity Classes, EUROCRYPT 1999, pages 223-238.

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MacKenzie et al. define a tag-based cryptosystem

- Messages encrypted relative to a key and a tag
- Only messages with the same tag can be combined

For example:

- $E(m, t) \otimes E(m', t) = E(mm', t)$
- $E(m, t) \otimes E(m', t') =$ <garbage>

See: Philip MacKenzie, Michael K. Reiter, and Ke Yang, "Alternatives to Non-malleability: Definitions, Constructions, and Applications (Extended Abstract)", TCC 2004, pages 171-190.

# Discussion

**Question 1:** Why might malleability be an interesting property for a cryptosystem to have?

- Tallying electronic votes
- Aggregating private values
- A primitive for privacy-preserving computation
- …

**Question 2:** Why might this be bad?

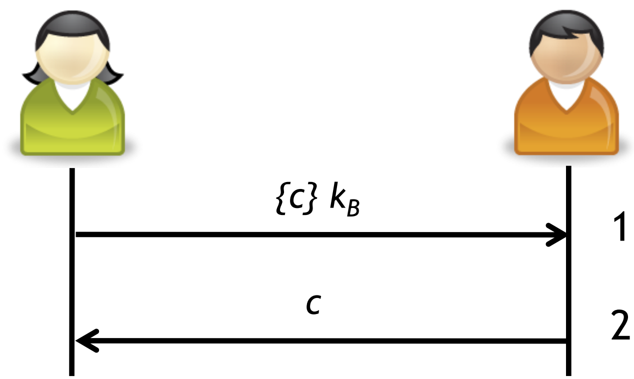
- Modifications by an active attacker!
- ***Example:*** Modifying an encrypted payment

*In short, these types of cryptosystems have interesting properties, but require care to use properly.*

# Note that public key cryptography allows us to prove knowledge of a secret without revealing that secret

***Example:*** Decrypting a challenge

1. Pick challenge c at random
2. Encrypt c using Bob's public key $k_B$
3. Transmit



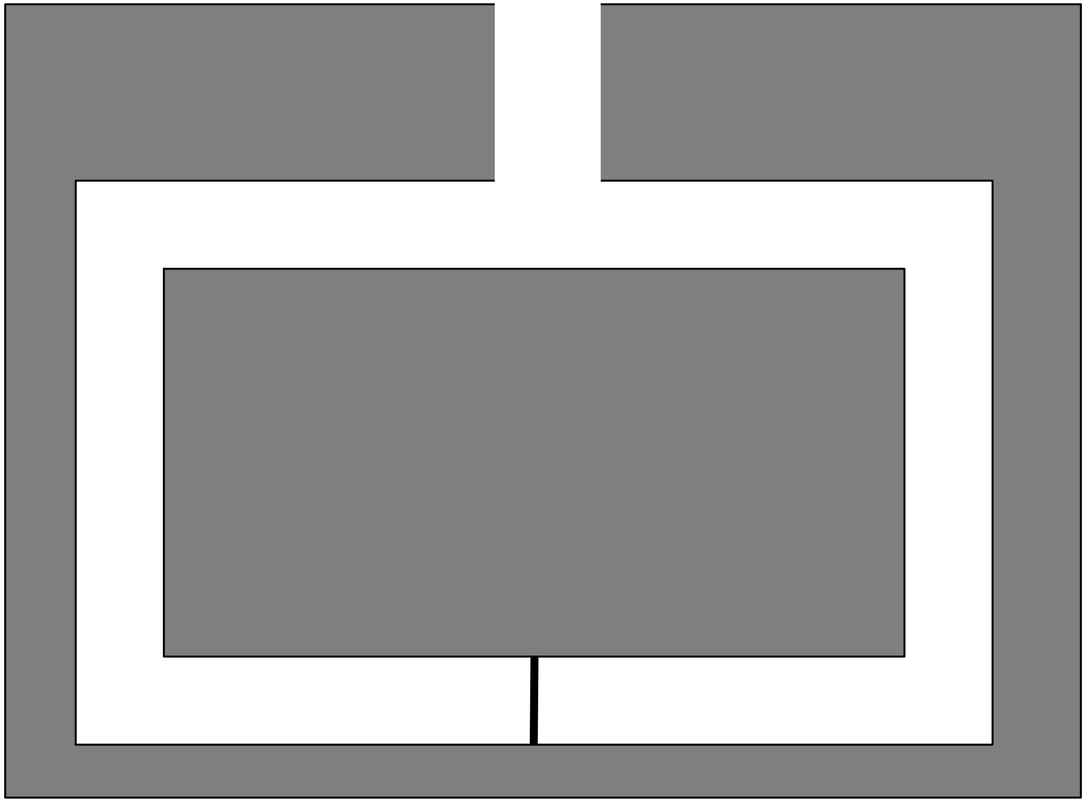

1. Decrypt c using private key $k_B^{-1}$
2. Transmit

Note: Revealing the challenge, c, does not leak information about the private key $k_B^{-1}$, yet Alice is (correctly) convinced that Bob knows $k_B^{-1}$

This type of protocol is called a zero knowledge protocol

# Zero-knowledge proofs are easy to understand: A children's puzzle

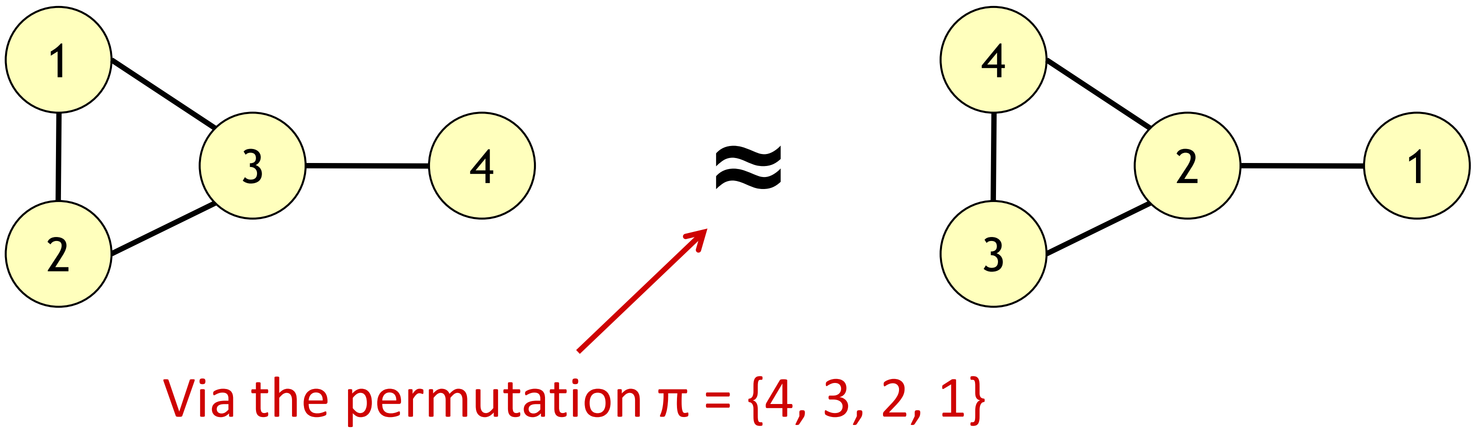
***Example:*** The secret cave



Note: To ensure correctness, this "protocol" needs to be run multiple times (Why?)

# We can construct a more realistic ZK system based on the (hard) problem of determining graph isomorphism

Informally, two graphs are isomorphic if the only difference between them is the names of their nodes




Determining whether two graphs are isomorphic is in NP, but the best known algorithm is $2^{O(\sqrt{(n \log n)})}$. This means that if the graphs are large, solving this problem will take a long time, but checking a solution is very easy.

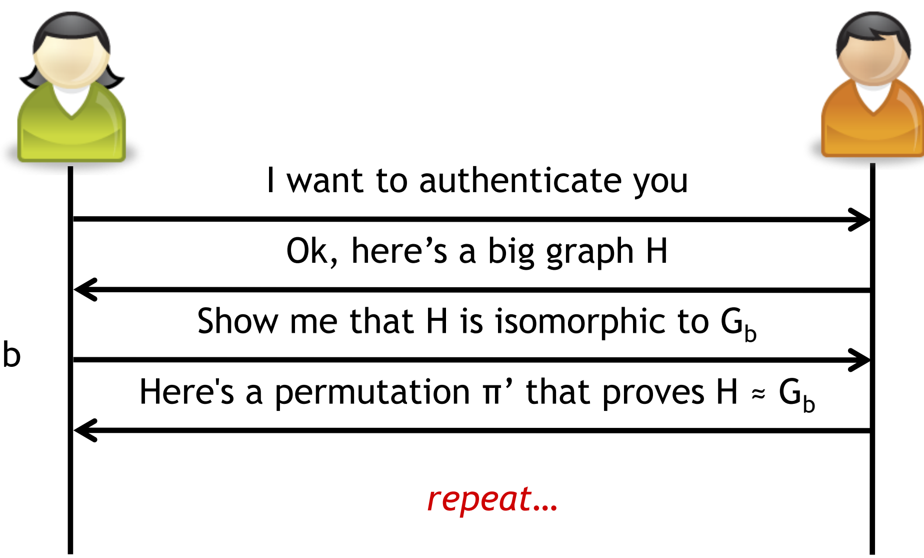
# Authenticating via Graph Isomorphism

How do we find these efficiently?

Our protocol has fairly simple parameters

- Public key: Two (big) isomorphic graphs $G_0$ and $G_1$
- Private key: The permutation mapping $G_0 \rightarrow G_1$

The protocol:




Generate a random bit b

Assume $G_0$ related to $G_1$ via permutation π

Generate a random bit b'
Generate permutation $\pi_H$ that maps $G_{b'}$ to a new graph H

If b = b', $\pi' = \pi_H$
If b <> b', $\pi' = \pi(\pi_H)$

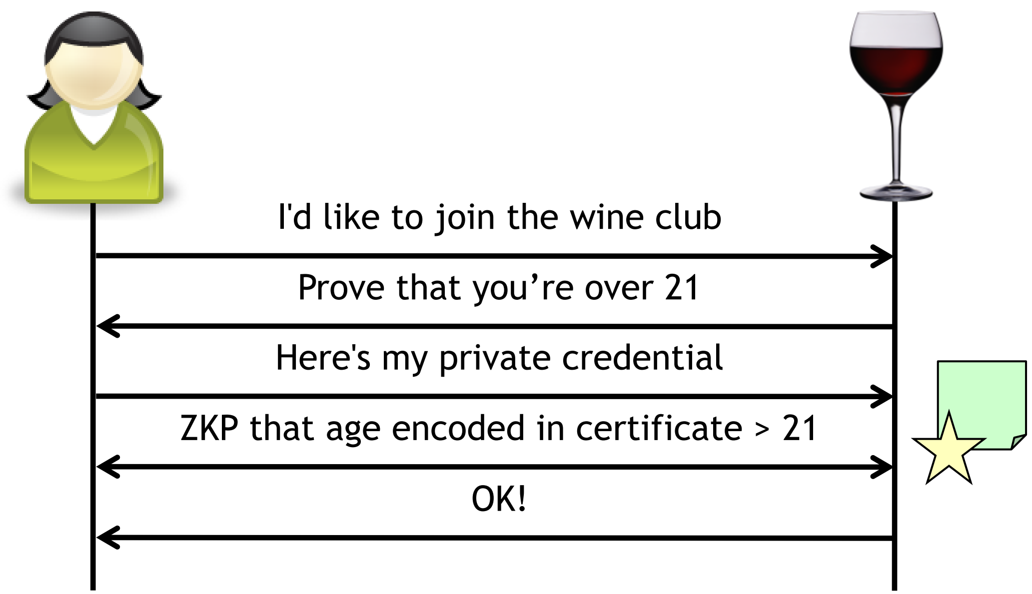
Why does this work?

- Answering this once means that Bob knows (at least) the permutation mapping from $G_b$ to H.
- Doing this m times means that Bob knows the mapping between $G_0$ and $G_1$ with probability $1 - 0.5^m$
- Note that this leaks no information regarding the permutation π (Why?)

# Zero knowledge proofs of knowledge can be used to solve a variety of interesting authorization problems

Private/Anonymous credential systems allow users to prove that they have certain attributes without actually revealing these attributes

***Example:*** Purchasing wine over the Internet




The private credential scheme proposed by Stefan Brands enables many types of attribute properties to be checked in a zero-knowledge fashion

Stefan Brands, Rethinking Public Key Infrastructures and Digital Certificates, MIT Press (2000)

# Summary so far …

Secret key cryptography has a key distribution problem

Public key cryptography overcomes this problem!

- Public encryption key
- Private decryption key

Digital signatures provide both integrity protection and non-repudiation

Malleable cryptosystems are useful, but their usage entails certain risks

Zero knowledge proof systems have many interesting applications

Next: *Really* understanding RSA

# Didn’t we learn about RSA?

We saw what RSA does and learned a little bit about how we can use those features

Our goal will be to explore

- Why RSA actually works
- Why RSA is efficient* to use
- Why it is reasonably safe to use RSA

In short, it’s time for more details …

Note: Efficiency is a relative term ☺

# RSA Overview / Roadmap

How do we choose large, pseudo-random primes?!

Key generation:

- Choose two large prime numbers $p$ and $q$, compute $n = pq$
- Compute $\phi(n) = (p-1)(q-1)$
- Choose an integer $e$ such that $\gcd\big(e, \phi(n)\big) = 1$
- Calculate *d* such that $ed \equiv 1 \pmod{\phi(n)}$
- Public key: $n, e$
- Private key: $p, q, d$

Why is $\phi(n) = (p-1)(q-1)$

How can we do this?

This seems tricky, too

Isn't this expensive?

Usage:

- Encryption: $M^e \pmod{n}$
- Decryption: $C^d \pmod{n} = M^{ed} \pmod{n} = M^{k\phi(n)+1} \pmod{n} = M^1 \pmod{n} = M$

Why does this work?

# Before we can do anything, we need a few large, pseudo-random primes

If our numbers are small, primality testing is pretty easy

- Try to divide n by all numbers less than $\sqrt{n}$
- The Sieve of Eratosthenes is a general extension of this principle

RSA requires big primes, so brute force testing is not an option (Why?)

To choose the types of numbers that RSA needs, we instead use a probabilistic primality testing method test : $Z \times Z \rightarrow \{T, F\}$

- test(n, a) = F means that n is composite based on the witness a
- test(n, a) = T means that n is probably prime based on the witness a

Often with probability ≈ 1/2

To test a number n for primality:

1. Randomly choose a witness a
2. if test(n, a) = F, n is composite
3. if test(n, a) = T, loop until we're reasonably certain that n is prime

k repetitions means P[n composite] = $2^{-k}$

# Fermat's little theorem can help us!

Fermat's little theorem: Given a prime number $p$ and a natural number $a$ such that $1 \leq a < p$, then $\mathrm{a}^{\mathrm{p}-1} \equiv 1 \bmod p$

How does this help with primality testing?

- If $\mathrm{a}^{\mathrm{p}-1} \neq 1 \bmod p$, then $p$ is definitely composite
- If $\mathrm{a}^{\mathrm{p}-1} \equiv 1 \bmod p$, then $p$ is probably prime

Note: Some composite numbers will always pass this test (Yikes!)

- These are called Carmichael numbers
- Carmichael numbers are rare, but may still be found
- Other primality tests (e.g., Miller-Rabin) avoid detecting these numbers

This helps us test whether some number is prime. But how exactly does this help us generate RSA keys?

# Putting it all together…

```
foundPrime = false
while (!foundPrime)
    let r = some large, odd, random number
    foundPrime = true
    for (iters = 0; iters < k; iters++)
        let a = random number less than r
        if (a^(r-1) ≠ 1 mod r)
            foundPrime = false
            break
return r
```

The prime number theorem tells us that, on average, the number of primes less than n is approximately n/ln(n)

- That is, P[n prime] ≈ 1/ln(n)
- Searching for a prime is hard, but not ridiculously so

# RSA Overview / Roadmap

How do we choose large, pseudo-random primes?!

Key generation:

- ✔ Choose two large prime numbers $p$ and $q$, compute $n = pq$
- Compute $\phi(n) = (p-1)(q-1)$
- Choose an integer $e$ such that $\gcd\big(e, \phi(n)\big) = 1$
- Calculate *d* such that $ed \equiv 1 \pmod{\phi(n)}$
- Public key: $n, e$
- Private key: $p, q, d$

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Why does this work?

# $\varphi(n)$ is called Euler's totient function

Definition: The totient function, $\varphi(n)$, counts the number of elements less than n that are relatively prime to n

For an RSA modulus $n = pq$, calculating $\varphi(n)$ is actually pretty simple

Consider each of the $pq$ numbers $\leq n$

- All multiples of $p$ share a common factor with $n$
  - There are $q$ such numbers $\{p, 2p, 3p, \dots, qp\}$
- Similarly, all multiples of $q$ share a common factor with $n$
  - There are $p$ such numbers $\{q, 2q, 3q, \dots, pq\}$
- So, we have that $\varphi(n) = pq - p - q + 1$
- As a result, $\varphi(n) = pq - p - q + 1 = (p-1)(q-1)$

The +1 controls for subtracting pq twice

Note: Calculating $\varphi(n)$ is easy because we know how to factor $n$!

# RSA Overview / Roadmap

Key generation:

✔ • Choose two large prime numbers $p$ and $q$, compute $n = pq$

✔ • Compute $\phi(n) = (p-1)(q-1)$

- Choose an integer $e$ such that $\gcd\big(e, \phi(n)\big) = 1$
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Why does this work?

# Review of greatest common divisors

Definition: Let $a$ and $b$ be integers, not both zero. The largest integer $d$ such that $\mathrm{d} \mid a$ and $d \mid b$ is called the greatest common divisor of $a$ and $b$, and is denoted by $\gcd(a, b)$

Note: We can (naively) find GCDs by comparing the common divisors of two numbers.

***Example:*** What is the GCD of 24 and 36?

- Factors of 24: 1, 2, 3, 4, 6, 12
- Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18
- ∴ gcd(24, 36) = 12

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*Wait. Aren't we dealing with numbers that are hard to factor?*

# Luckily, computing GCDs is not all that hard…

Intuition: Rather than computing the GCD of two big numbers, we can instead compute the GCD of smaller numbers that have the same GCD!

Interesting observation: $\gcd(a, b)$ is the same as $\gcd(a - b, b)$

Wait, what?

Divides

First, we must show that $d|a \land d|b \to d|(a - b)$

- If $d|a$ and $d|b$, then $a = kd$ and $b = jd$
- Then $a - b = kd - jd = (k - j)d$
- So, $d|a \land d|b \to d|(a - b)$

Ok, so $d$ is a divisor of $(a - b)$, but is it the greatest divisor?

- The **relevant** divisors of $(a - b)$ are a subset of the divisors of $a$ and the divisors of $b$
- Since $d = \gcd(a, b)$, it is the greatest of the remaining divisors

# Euclid's algorithm optimizes this process!

Euclid's algorithm finds $\gcd(a, b)$ as follows:

- Set $r_{-1} = a,\ r_{-2} = b,\ n = 0$
- While $r_{n-1} \neq 0$
  - divide $r_{n-2}$ by $r_{n-1}$ to find $q_n$ and $r_n$ such that $r_{n-2} = q_n r_{n-1} + r_n$
  - $n = n + 1$
- $\gcd(a, b) = r_{n-2}$

***Example:*** Computing gcd(414, 662)

| $n$ | $q_n$ | $r_n$ |
|---|---|---|
| -2 | - | 662 |
| -1 | - | 414 |
| 0 | 1 | 248 |
| 1 | 1 | 166 |
| 2 | 1 | 82 |
| 3 | 2 | 2 |
| 4 | 41 | 0 |

# That's all fine and good, but how does this help us compute $e \equiv 1 \bmod \varphi(n)$?

Method 1: Use Euclid's algorithm

- Choose a random $e$
- Use Euclid's algorithm to determine whether $\gcd\left(e, \phi(n)\right) = 1$
- Repeat as needed

Method 2: We can just choose a large prime number, $r > \max(p, q)$

Why does method 2 work?

- $r$ is a prime number, so it has no divisors other than itself and 1
- $r$ is larger than $p$ and $q$, so $r \neq p$ and $r \neq q$

# RSA Overview / Roadmap

Key generation:

✔ • Choose two large prime numbers $p$ and $q$, compute $n = pq$

✔ • Compute $\phi(n) = (p-1)(q-1)$

✔ • Choose an integer $e$ such that $\gcd\big(e, \phi(n)\big) = 1$

- Calculate *d* such that $ed \equiv 1 \pmod{\phi(n)}$
- Public key: $n, e$
- Private key: $p, q, d$

How can we do this?

This seems tricky, too

Isn't this expensive?

Usage:

- Encryption: $M^e \pmod{n}$
- Decryption: $C^d \pmod{n} = M^{ed} \pmod{n} = M^{k\phi(n)+1} \pmod{n} = M^1 \pmod{n} = M$

Why does this work?

# It turns out that Euclid's algorithm can help us compute $d \equiv e^{-1}$, too

If we maintain a little extra state, we can figure out numbers $u_n$ and $v_n$ such that $r_n = u_n a + v_n b$ when calculating $\gcd(a, b)$

If $a$ and $b$ are relatively prime, this will allow us to calculate $a^{-1}$

- $1 = u_n a + v_n b$ // If $a$ and $b$ are relatively prime, $r_n = 1$
- $u_n a = 1 - v_n b$ // Subtract $v_n b$ from both sides
- $u_n a \equiv 1 \bmod b$ // Definition of congruence
- So $u_n = a^{-1}$ // Definition of inverse

This makes $r_n = u_n a + v_n b$ for $n = -1$ and $n = -2$

The extended Euclid's algorithm works as follows:

- Set $r_{-1} = b,\ r_{-2} = a,\ n = 0,\ u_{-2} = 1,\ v_{-2} = 0,\ u_{-1} = 0,\ v_{-1} = 1$
- While $r_{n-1} \neq 0$
  - divide $r_{n-2}$ by $r_{n-1}$ to find $q_n$ and $r_n$ such that $r_{n-2} = q_n r_{n-1} + r_n$
  - $u_n = u_{n-2} - q_n u_{n-1}$
  - $v_n = v_{n-2} - q_n v_{n-1}$
  - $n = n + 1$
- $\gcd(a, b) = r_{n-2} = u_{n-2} a + v_{n-2} b$

# How about an example?

***Example:*** Find the inverse of 797 mod 1047

| $n$ | $q_n$ | $r_n$ | $u_n$ | $v_n$ |
|---|---|---|---|---|
| -2 | | 797 | 1 | 0 |
| -1 | | 1047 | 0 | 1 |
| 0 | 0 | 797 | 1 | 0 |
| 1 | 1 | 250 | -1 | 1 |
| 2 | 3 | 47 | 4 | -3 |
| 3 | 5 | 15 | -21 | 16 |
| 4 | 3 | 2 | 67 | -51 |
| 5 | 7 | 1 | -490 | 373 |

So, 1 = -490*797 + 373*1047

- -490*797 = 1 + (-373)1047
- -490*797 ≡ 1 mod 1047
- In other words, -490 is the inverse of 797 mod 1047

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Why does this work?

# Isn't exponentiation really expensive?

Exponentiation can be sped up using a trick called successive squaring

```
int pow(int m, int e)
    if(e is even)
        return pow(m*m, e/2)
    else
        return m * pow(m, e - 1)
```

For example, consider computing $2^{15}$

- Naive method: $2 * 2 * 2 * \ldots * 2 = 32{,}768$
- Fast method: $2^{15} = 2 * 4^7$

$= 2 * 4 * 4^6$

$= 2 * 4 * 16^3$

$= 2 * 4 * 16 * 16^2$

$= 2 * 4 * 16 * 256$

$= 32{,}768$

O(e) multiplications

O(log(e)) multiplications

This only gets us partway there. Various other algorithmic tricks enable modulo exponentiation to be efficient!

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Why does this work?

# Why *does* decryption work?

Note: Decryption will work if and only if $C^d \bmod n = M$

$$\begin{aligned}
C^d \bmod n &= M^{ed} \bmod n && // \; C = M^e \bmod n \\
&= M^{k\phi(n)+1} \bmod n && // \; ed \equiv 1 \bmod \phi(n), \text{ so } ed = k\phi(n) + 1 \\
&= M^1 \bmod n && // \; ?!? \\
&= M \bmod n && // \; M^1 = M
\end{aligned}$$

The only hitch in showing the correctness of the decryption process is proving that $M^{k\phi(n)+1} \bmod n = M^1 \bmod n$

Fortunately, two smart guys can help us out with this…

Pierre de Fermat
160? - 1665

Leonhard Euler
1707 - 1783

# First, we need to learn about the set $Z_n^*$

Definition: $Z_n^*$ is the set of all integers relatively prime to n

***Example:*** $Z_{10}^*$

| $\times$ | 1 | 3 | 7 | 9 |
|---|---|---|---|---|
| 1 | 1 | 3 | 7 | 9 |
| 3 | 3 | 9 | 1 | 7 |
| 7 | 7 | 1 | 9 | 3 |
| 9 | 9 | 7 | 3 | 1 |

Interesting note: $\forall a, b \in Z_n^*: ab \in Z_n^*$

- $a$ relatively prime to $n$ means that $\exists u_1, v_1: u_1 a + v_1 n = 1$
- $b$ relatively prime to n means that $\exists u_2, v_2: u_2 b + v_2 n = 1$
- Multiplying gives us $(u_1 u_2)ab + (u_1 v_2 a + v_1 u_2 b + v_1 v_2 n)n = 1$

The above states that $Z_n^*$ is closed under multiplication

# This leads us to something called Euler's theorem

Theorem: $\forall a \in Z_n^*: a^{\phi(n)} \equiv 1 \bmod n$

Proof:

- Multiply all $\phi(n)$ elements of $Z_n^*$ together, calling the product $x$
- Note that $x \in Z_n^*$, and has an inverse $x^{-1}$
- Now, multiply each element of $Z_n^*$ by $a$ and multiply each of the resulting elements together. This will give us $a^{\phi(n)}x$
- Multiplying each element of $Z_n^*$ actually just rearranges these elements.
- As a result, we have that $a^{\phi(n)}x = x$
- If we divide both sides of the equation by $x$, we get that $a^{\phi(n)} = 1$ ❑

*Ok, so what does Euler's theorem have to do with RSA?*

# We can restate Euler's theorem so that it more clearly connects to RSA math

Theorem: $\forall a \in Z_n^*, k \in Z^+ : a^{k\phi(n)+1} \equiv a \bmod n$

Proof: $a^{k\phi(n)+1} = a^{k\phi(n)}a = a^{\phi(n)k}a = 1^k a = a$ ❑

**From Euler's theorem!**

---

Now, in RSA

- All of our math is done $\bmod n$
- Our message space is chosen from elements of $Z_n^*$
- $ed \equiv 1 \bmod \phi(n)$, so $ed = k\phi(n) + 1$ for some $k$
- $\therefore M^{ed} \bmod n = M^{k\phi(n)+1} \bmod n = M^1 \bmod n = M$

**Decryption works!**

# RSA Overview / Roadmap

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But why is RSA *safe* to use?

# Now, why exactly is RSA safe to use?

In the original RSA paper*, the authors identify four avenues for attacking the mathematics behind RSA

1. Factoring $n$ to find $p$ and $q$
2. Determining $\phi(n)$ without factoring $n$
3. Determining $d$ without factoring $n$ or learning $\phi(n)$
4. Learning to take $e^{\text{th}}$ roots modulo $n$

As it turns out, all of these attacks are thought to be hard to do

- But you shouldn't take my word for it…
- Let's see why!

*R.L. Rivest, A. Shamir, and L. Adleman, A Method for Obtaining Digital Signatures and Public-Key Cryptosystems, Communications of the ACM 21(2): 120-126, Feb. 1978.

# It turns out that factoring is a hard* problem

First of all, why is factoring an issue?

- $n$ is the public modulus of the RSA algorithm
- If we can factor $n$ to find $p$ and $q$, we can compute $\phi(n)$
- Given $\phi(n)$ and $e$, we can easily compute the decryption exponent $d$

Fortunately, mathematicians believe that factoring numbers is a very difficult problem. History backs up this belief.

The fastest general-purpose algorithm for integer factorization is called the general number field sieve. This algorithm has running time:

$$O(e^{(c+o(1))(\log n)^{\frac{1}{3}}(\log \log n)^{\frac{2}{3}}})$$

Note: This running time is sub-exponential

- i.e., Factoring can be done faster than brute force
- This explains why RSA keys are larger than AES keys
  - RSA: Typically 2048-4096 bits
  - AES: Typically 128 bits

# What about computing $\phi(n)$ without factoring?

Question: Why would the ability to compute $\phi(n)$ be a bad thing?

- It would allow us to easily compute $d$, since $ed \equiv 1 \bmod \phi(n)$

Good news: If we can compute $\phi(n)$, it will allow us to factor n

- Note 1: $\phi(n) = n - p - q + 1$
  $= n - (p + q) + 1$
- Rewriting gives us $(p + q) = n - \phi(n) + 1$
- Note 2: $(p - q) = \sqrt{(p + q)^2 - 4n}$
- Note 3: $(p + q) - (p - q) = 2q$
- Finally, given $q$ and $n$, we can easily compute $p$

$$
\begin{aligned}
(p + q)^2 - 4n &= p^2 + 2pq + q^2 - 4n \\
&= p^2 + 2pq + q^2 - 4pq \\
&= p^2 - 2pq + q^2 \\
&= (p - q)^2
\end{aligned}
$$

What does this mean?

- If factoring is actually hard, then so is computing $\phi(n)$ without factoring
- (Recall the concept of reduction)

# What about computing $d$ without factoring $n$ or knowing $\phi(n)$?

As it turns out, if we can figure out $d$ without knowing $\phi(n)$ and without factoring $n$, $d$ can be used to help us factor $n$

---

Given $d$, we can compute $ed - 1$, since we know $e$

Note: $ed - 1$ is a multiple of $\phi(n)$

- $ed \equiv 1 \bmod \phi(n)$
- $ed = 1 + k\phi(n)$
- $ed - 1 = k\phi(n)$ ✔

It has been shown that $n$ can be efficiently factored using any multiple of $\phi(n)$. As such, if we know $e$ and $d$, we can efficiently factor $n$.

# Are there any other attacks that we need to worry about?

Recall: $C = M^e \bmod n$

- $e$ is part of the public key, so the adversary knows this
- If we could compute $e^{\text{th}}$ roots mod $n$, we could decrypt without $d$

It is not known whether breaking RSA yields an efficient factoring algorithm, but the inventors conjecture that this is the case

- This conjecture was made in 1978
- To date, it has either been proved or disproved

---

*Conclusion: Odds are that breaking RSA efficiently implies that factoring can be done efficiently. Since factoring is hard, RSA is probably safe to use.*

# RSA Wrap Up

Hopefully you now have a better understanding of RSA

- How each step of the process works
- How these steps can be made reasonably efficient
- Why RSA is safe to use

Unfortunately, this is not the end of the story…

- Although theoretically secure, implementations can be broken
- We'll revisit this in a later lecture

Next time: Secret sharing and threshold cryptography