Vector Spaces

CS 130 - Mathematical Methods in Computer Science

Jan Michael C. Yap

Department of Computer Science
University of the Philippines Diliman

November 22, 2012

Topics

(n-)Vector

- (n-)Vector

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Vector Operations

- Vector addition: a + b

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Vector Operations

- Vector addition: a + b

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

- Let a, b and c be vectors:
  - $a + b = b + a$
  - $(a + b) + c = a + (b + c)$
  - $a + 0 = a$
  - $a + -a = 0$

Vector Operations

- Scalar multiplication: Let c be a scalar and a be a vector,

$$c\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}$$

Vector Operations

- Scalar multiplication: Let c be a scalar and a be a vector,

$$c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}$$

- Let c and d be scalars and u and v be vectors:
  - $c(du) = (cd)u$
  - $(c + d)u = cu + du$
  - $c(u + v) = cu + cv$
  - $1u = u$

Topics

Vector Space

- A (real) vector space V is a set of elements with (vector) addition and scalar multiplication (i.e. vector operations) such that any combination of vector operations (with their corresponding properties preserved) applied to any vectors in V result to another vector still in V

Vector Space

- A (real) vector space V is a set of elements with (vector) addition and scalar multiplication (i.e. vector operations) such that any combination of vector operations (with their corresponding properties preserved) applied to any vectors in V result to another vector still in V
  - $V_1 = \mathbb{R}^n$, with standard vector addition and standard scalar multiplication

Vector Space

- A (real) vector space V is a set of elements with (vector) addition and scalar multiplication (i.e. vector operations) such that any combination of vector operations (with their corresponding properties preserved) applied to any vectors in V result to another vector still in V
  - $V_1 = \mathbb{R}^n$, with standard vector addition and standard scalar multiplication
  - $V_2 =$ the set consisting of all m x n matrices with standard matrix addition and standard scalar multiplication

Vector Space

- A (real) vector space V is a set of elements with (vector) addition and scalar multiplication (i.e. vector operations) such that any combination of vector operations (with their corresponding properties preserved) applied to any vectors in V result to another vector still in V
  - $V_1 = \mathbb{R}^n$, with standard vector addition and standard scalar multiplication
  - $V_2 =$ the set consisting of all m x n matrices with standard matrix addition and standard scalar multiplication
  - $V_3 =$ set of all $\mathbb{R}^2$ vectors with standard vector addition and scalar multiplication defined as $c\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ 0 \end{pmatrix}$

Vector Space

- A (real) vector space V is a set of elements with (vector) addition and scalar multiplication (i.e. vector operations) such that any combination of vector operations (with their corresponding properties preserved) applied to any vectors in V result to another vector still in V
  - $V_1 = \mathbb{R}^n$, with standard vector addition and standard scalar multiplication
  - $V_2 =$ the set consisting of all m x n matrices with standard matrix addition and standard scalar multiplication
  - $V_3 =$ set of all $\mathbb{R}^2$ vectors with standard vector addition and scalar multiplication defined as $c\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ 0 \end{pmatrix}$
  - $V_4 =$ set of purely second-degree polynomials with standard polynomial addition and standard scalar multiplication

Topics

Subspace

- A subspace W of a (real) vector space V is a subset of V such that W is also a vector space with respect to the vector operations defined in V

Subspace

- A subspace W of a (real) vector space V is a subset of V such that W is also a vector space with respect to the vector operations defined in V
- Enough to show that given $W \subseteq V$, and for vectors $u, v \in W$ and a scalar c, $cu + v \in W$

Subspace

- A subspace W of a (real) vector space V is a subset of V such that W is also a vector space with respect to the vector operations defined in V
- Enough to show that given $W \subseteq V$, and for vectors $u, v \in W$ and a scalar c, $cu + v \in W$
  - $W_1 =$ subset of $\mathbb{R}^3$, where all vectors are of the form $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $a \in \mathbb{R}$ with standard vector operations

Subspace

- A subspace W of a (real) vector space V is a subset of V such that W is also a vector space with respect to the vector operations defined in V
- Enough to show that given $W \subseteq V$, and for vectors $u, v \in W$ and a scalar c, $cu + v \in W$
  - $W_1 =$ subset of $\mathbb{R}^3$, where all vectors are of the form $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $a \in \mathbb{R}$ with standard vector operations
  - $W_2 =$ the subset of m x n matrices consisting of all n x n diagonal matrices with standard matrix addition and scalar multiplication

Subspace

- A subspace W of a (real) vector space V is a subset of V such that W is also a vector space with respect to the vector operations defined in V
- Enough to show that given $W \subseteq V$, and for vectors $u, v \in W$ and a scalar c, $cu + v \in W$
    - $W_1 =$ subset of $\mathbb{R}^3$, where all vectors are of the form $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $a \in \mathbb{R}$ with standard vector operations
    - $W_2 =$ the subset of m x n matrices consisting of all n x n diagonal matrices with standard matrix addition and scalar multiplication
    - $W_3 =$ subset of $\mathbb{R}^3$, where all vectors are of the form $\begin{pmatrix} a \\ a^2 \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$ with standard vector operations

Topics

Topics

1. Preliminaries
2. Vector Space
3. Subspace
4. Basis
   - Basis
   - Dimension and Matrix Rank
   - Coordinates and Change of Basis
   - Orthonormal Basis

Linear Independence

- Linear Independence: Let c_i be scalars and x_i be n-vectors,

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0 \rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Linear Independence

- Linear Independence: Let c_i be scalars and x_i be n-vectors,

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0 \rightarrow c_1 = c_2 = \cdots = c_n = 0$$

  - If not all $c_i = 0$, then the vectors have linear dependence, i.e. one or more vectors can be expressed as a linear combination of the other vectors

Linear Independence

- Linear Independence: Let c_i be scalars and x_i be n-vectors,

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0 \rightarrow c_1 = c_2 = \cdots = c_n = 0$$

  - If not all $c_i = 0$, then the vectors have linear dependence, i.e. one or more vectors can be expressed as a linear combination of the other vectors

$$\left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

Linear Independence

- Linear Independence: Let c_i be scalars and x_i be n-vectors,

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0 \rightarrow c_1 = c_2 = \cdots = c_n = 0$$

  - If not all $c_i = 0$, then the vectors have linear dependence, i.e. one or more vectors can be expressed as a linear combination of the other vectors

$$\left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ -10 \end{pmatrix} \right\}$$

Linear Combination

- Linear Combination

$$\begin{pmatrix} 1 \\ 12 \\ -5 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + -10 \begin{pmatrix} \frac{-3}{10} \\ 0 \\ \frac{13}{10} \end{pmatrix}$$

Linear Combination

- Linear Combination

$$\begin{pmatrix} 1 \\ 12 \\ -5 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + -10 \begin{pmatrix} \frac{-3}{10} \\ 0 \\ \frac{13}{10} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ -5 & -6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow x \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + y \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} + z \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Spanning Set

- A spanning set of vectors $S = \{v_1, v_2, ..., v_k\}$ of a vector (sub)space W is a set of vectors such that all other vectors in W can be expressed as a linear combination of the elements of S
    - span(S) = W

Spanning Set

- A spanning set of vectors $S = \{v_1, v_2, ..., v_k\}$ of a vector (sub)space W is a set of vectors such that all other vectors in W can be expressed as a linear combination of the elements of S
  - span(S) = W

$$span\left(\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\\frac{1}{2}\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right) = \mathbb{R}^3$$

Basis

- A basis B for vector (sub)space W is a spanning set of vectors whose elements have linear independence

Basis

- A basis B for vector (sub)space W is a spanning set of vectors whose elements have linear independence
$$\left\{ \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

Basis

- A basis B for vector (sub)space W is a spanning set of vectors whose elements have linear independence
  $\left\{ \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$
  - Standard or natural basis for $\mathbb{R}^n$: column vectors of an identity matrix of order n

Basis

- A basis B for vector (sub)space W is a spanning set of vectors whose elements have linear independence

$$\left\{ \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} \text{ is a basis for } \mathbb{R}^3$$

  - Standard or natural basis for $\mathbb{R}^n$: column vectors of an identity matrix of order n

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ is a natural basis for } \mathbb{R}^2$$

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix
- Transform the augmented matrix to RREF

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix
- Transform the augmented matrix to RREF
- The vectors corresponding to the columns with leading 1s form a basis

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix
- Transform the augmented matrix to RREF
- The vectors corresponding to the columns with leading 1s form a basis

$$\left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix

$$\left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

How to find a basis given a spanning set

- Augment the vectors in the spanning set to form a matrix

$$\left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix}$$

How to find a basis given a spanning set

- Transform the augmented matrix to RREF

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix}$$

How to find a basis given a spanning set

- Transform the augmented matrix to RREF

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

How to find a basis given a spanning set

- The vectors corresponding to the columns with leading 1s form a basis

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

How to find a basis given a spanning set

- The vectors corresponding to the columns with leading 1s form a basis

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Topics

Dimension and Matrix Rank

- The dimension of a vector space V is the number of vectors in a basis for V.

Dimension and Matrix Rank

- The dimension of a vector space V is the number of vectors in a basis for V.
  - The dimension of $\mathbb{R}^n$ is n

Dimension and Matrix Rank

- The dimension of a vector space V is the number of vectors in a basis for V.
  - The dimension of $\mathbb{R}^n$ is n
- The rank of a matrix is the dimension of the vector space corresponding to the rows (or columns) of the matrix.

Dimension and Matrix Rank

- The dimension of a vector space V is the number of vectors in a basis for V.
  - The dimension of $\mathbb{R}^n$ is n
- The rank of a matrix is the dimension of the vector space corresponding to the rows (or columns) of the matrix.

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{pmatrix}$$

Dimension and Matrix Rank

- The dimension of a vector space V is the number of vectors in a basis for V.
  - The dimension of $\mathbb{R}^n$ is n
- The rank of a matrix is the dimension of the vector space corresponding to the rows (or columns) of the matrix.

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{pmatrix} \Rightarrow rank(A) = 3$$

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix}$$

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$rank(A) = 3,$

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$rank(A) = 3, nullity(A) = 1$

Some Notes on Matrix Rank

- The dimensions of the vector spaces corresponding to the rows and columns of a matrix are equal.
- Given A_{mxn}, if $rank(A) < n$ then $n - rank(A)$ is called the nullity of A, denoted nullity(A)
  - The nullity of A is the dimension of the null space of A

$$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 5 & 3 & 11 & -1 \\ 3 & 2 & 7 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$rank(A) = 3, nullity(A) = 1 \Rightarrow rank(A) + nullity(A) = 4$$

Topics

Ordered Basis

- Same concept as basis, but order of the vector matters

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Ordered Basis

- Same concept as basis, but order of the vector matters

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Ordered Basis

- Same concept as basis, but order of the vector matters

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$B_3 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Coordinates

- Coordinate vector of v w.r.t. ordered basis B:

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, v = \sum_{i=1}^{n} c_i bi, c_i \in \mathbb{R}, b_i \in B$$

Coordinates

- Coordinate vector of v w.r.t. ordered basis B:

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, v = \sum_{i=1}^{n} c_i bi, c_i \in \mathbb{R}, b_i \in B$$

  - The entries of $[v]_B$ are called coordinates

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Coordinates

- Coordinate vector of v w.r.t. ordered basis B:

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, v = \sum_{i=1}^{n} c_i bi, c_i \in \mathbb{R}, b_i \in B$$

  - The entries of $[v]_B$ are called coordinates

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow [v]_{B_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Coordinates

- Coordinate vector of v w.r.t. ordered basis B: $[v]_B$:

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, v = \sum_{i=1}^{n} c_i bi, c_i \in \mathbb{R}, b_i \in B$$

  - The entries of $[v]_B$ are called coordinates

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Coordinates

- Coordinate vector of v w.r.t. ordered basis B: $[v]_B$:

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, v = \sum_{i=1}^{n} c_i bi, c_i \in \mathbb{R}, b_i \in B$$

  - The entries of $[v]_B$ are called coordinates

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow [v]_{B_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Change of Basis

$$v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Change of Basis

$$v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow [v]_{B_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

Change of Basis

$$v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow [v]_{B_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \Rightarrow [v]_{B_4} = \begin{pmatrix} \frac{1}{9} \\ \frac{2}{9} \\ \frac{20}{9} \end{pmatrix}$$

Change of Basis

- Transition matrix from basis S to basis T: $P_{S \to T}$

Change of Basis

- Transition matrix from basis S to basis T: $P_{S\to T}$

$$P_{B_2\to B_4} = \begin{pmatrix} \frac{5}{9} & -\frac{2}{9} & \frac{1}{3} \\ -\frac{8}{9} & \frac{5}{9} & -\frac{1}{3} \\ \frac{1}{9} & \frac{5}{9} & -\frac{1}{3} \end{pmatrix}$$

Change of Basis

- Transition matrix from basis S to basis T: $P_{S \to T}$

$$P_{B_2 \to B_4} = \begin{pmatrix} \frac{5}{9} & -\frac{2}{9} & \frac{1}{3} \\ -\frac{8}{9} & \frac{5}{9} & -\frac{1}{3} \\ \frac{1}{9} & \frac{5}{9} & -\frac{1}{3} \end{pmatrix}$$

$$P_{B_2 \to B_4} [v]_{B_2} = \begin{pmatrix} \frac{1}{9} \\ \frac{2}{9} \\ \frac{20}{9} \end{pmatrix} = [v]_{B_4}$$

Change of Basis

- Form the augmented matrix $T|S$

Change of Basis

- Form the augmented matrix $T|S$
- Transform augmented matrix to RREF

Change of Basis

- Form the augmented matrix $T|S$
- Transform augmented matrix to RREF
- The square matrix on the right hand side of the RREF matrix is $P_{S \to T}$

Change of Basis

- Form the augmented matrix $T|S$

$$B_4|B_2 = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & 0 \\ 5 & 3 & -1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Change of Basis

- Transform augmented matrix to RREF

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 0 & 0 \\ 5 & 3 & -1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Change of Basis

- Transform augmented matrix to RREF

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 0 & 0 \\ 5 & 3 & -1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{5}{9} & -\frac{2}{9} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{8}{9} & \frac{5}{9} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{9} & \frac{5}{9} & -\frac{1}{3} \end{pmatrix}$$

Change of Basis

- The square matrix on the right hand side of the RREF matrix is $P_{S\to T}$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{5}{9} & -\frac{2}{9} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{8}{9} & \frac{5}{9} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{9} & \frac{5}{9} & -\frac{1}{3} \end{pmatrix}$$

Topics

Orthonormal Vectors

- Orthogonal vectors

Orthonormal Vectors

- Orthogonal vectors

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ \frac{10}{3} \end{pmatrix} = 0$$

Orthonormal Vectors

- Orthogonal vectors

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ \frac{10}{3} \end{pmatrix} = 0$$

- Euclidean norm of vector v: $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$

Orthonormal Vectors

- Orthogonal vectors

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ \frac{10}{3} \end{pmatrix} = 0$$

- Euclidean norm of vector v: $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$

$$\| \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \| = \sqrt{13}$$

Orthonormal Vectors

- Normalizing a vector v: $\frac{1}{\|v\|}v$

Orthonormal Vectors

- Normalizing a vector v: $\frac{1}{\|v\|}v$

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

Orthonormal Vectors

- Normalizing a vector v: $\frac{1}{\|v\|}v$

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

- Orthonormal (set of) vectors
$\{v_i, 1 \leq i \leq k : v_a \cdot v_b = 0, a \neq b, 1 \leq a, b \leq k \ AND \ \|v_i\| = 1, \forall i\}$

Orthonormal Vectors

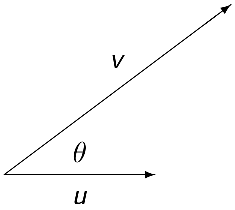
- Normalizing a vector v: $\frac{1}{\|v\|}v$

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$$

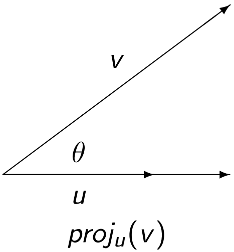
- Orthonormal (set of) vectors $\{v_i, 1 \leq i \leq k : v_a \cdot v_b = 0, a \neq b, 1 \leq a, b \leq k \ AND \ \|v_i\| = 1, \forall i\}$

$$\left\{ \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \right\}$$

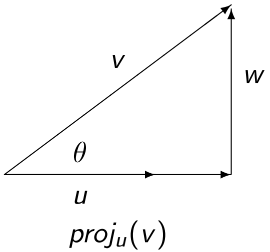
Vector Projection and Orthogonal Vectors



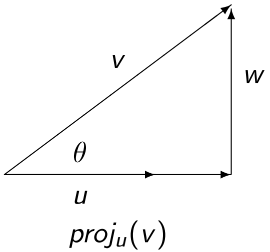
Vector Projection and Orthogonal Vectors



Vector Projection and Orthogonal Vectors

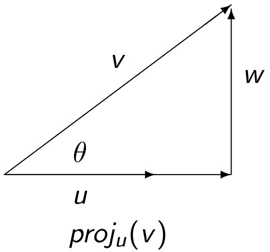


Vector Projection and Orthogonal Vectors



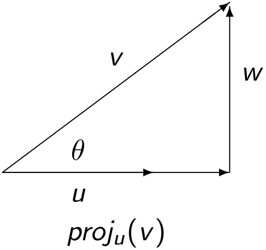
- Projection of v on u: $proj_u(v)$

Vector Projection and Orthogonal Vectors



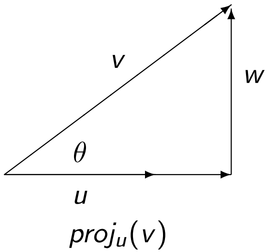
- Projection of v on u: $proj_u(v)$
  - $proj_u(v) = \frac{\|v\| \cos(\theta)}{\|u\|} u$

Vector Projection and Orthogonal Vectors



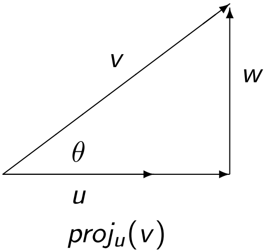
- Projection of v on u: $proj_u(v)$
  - $proj_u(v) = \frac{\|v\| \cos(\theta)}{\|u\|} u = \frac{\|v\| \|u\| \cos(\theta)}{\|u\| \|u\|} u$

Vector Projection and Orthogonal Vectors



- Projection of v on u: $proj_u(v)$
  - $proj_u(v) = \frac{\|v\| \cos(\theta)}{\|u\|} u = \frac{\|v\|\|u\| \cos(\theta)}{\|u\|\|u\|} u = \frac{v \cdot u}{u \cdot u} u$

Vector Projection and Orthogonal Vectors



- Projection of v on u: $proj_u(v)$
  - $proj_u(v) = \frac{\|v\|\cos(\theta)}{\|u\|}u = \frac{\|v\|\|u\|\cos(\theta)}{\|u\|\|u\|}u = \frac{v \cdot u}{u \cdot u}u$
- $w = v - proj_u(v)$

Gram-Schmidt Process

- Given a basis V for W, $V = \{v_1, v_2, \cdots, v_n\}$, an orthogonal basis $U = \{u_1, u_2, \cdots, u_n\}$ is derived as follows:

Gram-Schmidt Process

- Given a basis V for W, $V = \{v_1, v_2, \cdots, v_n\}$, an orthogonal basis $U = \{u_1, u_2, \cdots, u_n\}$ is derived as follows:
  - $u_1 = v_1$

Gram-Schmidt Process

- Given a basis V for W, $V = \{v_1, v_2, \cdots, v_n\}$, an orthogonal basis $U = \{u_1, u_2, \cdots, u_n\}$ is derived as follows:
  - $u_1 = v_1$
  - $u_k = v_k - (\sum_{i=1}^{k-1} proj_{u_i}(v_k))$

Gram-Schmidt Process

- Given a basis V for W, $V = \{v_1, v_2, \cdots, v_n\}$, an orthogonal basis $U = \{u_1, u_2, \cdots, u_n\}$ is derived as follows:
  - $u_1 = v_1$
  - $u_k = v_k - (\sum_{i=1}^{k-1} proj_{u_i}(v_k))$
- The orthonormal basis is then formed by normalizing the vectors in U, i.e. $\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \cdots, \frac{u_n}{\|u_n\|} \right\}$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_1 = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_2 = v_2 - proj_{u_1}(v_2)$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \frac{21}{34} \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_2 = v_2 - proj_{u_1}(v_2) = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \frac{21}{34} \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3)$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_3 = v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \left(-\frac{2}{34}\right) \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} - \left(-\frac{884}{1190}\right) \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$B_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$u_3 = \begin{pmatrix} \frac{9}{35} \\ -\frac{27}{35} \\ \frac{45}{35} \end{pmatrix}$$

Gram-Schmidt Process

- Find an orthogonal basis

$$U_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}, \begin{pmatrix} \frac{9}{35} \\ -\frac{27}{35} \\ \frac{45}{35} \end{pmatrix} \right\}$$

Gram-Schmidt Process

- Normalize the computed orthogonal basis to form the orthonormal basis

$$U_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}, \begin{pmatrix} \frac{9}{35} \\ -\frac{27}{35} \\ \frac{45}{35} \end{pmatrix} \right\}$$

Gram-Schmidt Process

- Normalize the computed orthogonal basis to form the orthonormal basis

$$U_4 = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}, \begin{pmatrix} \frac{9}{35} \\ -\frac{27}{35} \\ \frac{45}{35} \end{pmatrix} \right\}$$

$$\Downarrow$$

$$O_4 = \left\{ \sqrt{\frac{1}{34}} \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \sqrt{\frac{1156}{1190}} \begin{pmatrix} -1 \\ -\frac{3}{34} \\ \frac{5}{34} \end{pmatrix}, \sqrt{\frac{1225}{2835}} \begin{pmatrix} \frac{9}{35} \\ -\frac{27}{35} \\ \frac{45}{35} \end{pmatrix} \right\}$$

END OF LESSON 3