CS 130 Exercises

Fourier Series

March 17, 2013

Compute the coefficients of the Fourier series of the following functions:

1. $f(x) = \sin x, x \in [0, L]$

   - Coefficients for Fourier cosine series:

$$A_0 = \frac{1}{L}\int_0^L \sin x dx = \frac{1}{L}\left[-\cos x\right]_0^L = \frac{1}{L}\left(-\cos L + 1\right)$$

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L \sin x \cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \left(\sin\left(x + \frac{n\pi x}{L}\right) + \sin\left(x - \frac{n\pi x}{L}\right)\right)dx \\
&= \frac{1}{L}\int_0^L \left(\sin\left(\left(1 + \frac{n\pi}{L}\right)x\right) + \sin\left(\left(1 - \frac{n\pi}{L}\right)x\right)\right)dx \\
&= \frac{1}{L}\left[-\frac{L}{L + n\pi}\cos\left(\left(1 + \frac{n\pi}{L}\right)x\right) - \frac{L}{L - n\pi}\cos\left(\left(1 - \frac{n\pi}{L}\right)x\right)\right]_0^L \\
&= \frac{1}{L}\left(-\frac{L}{L + n\pi}\cos\left(L + n\pi\right) + \frac{L}{L + n\pi} - \frac{L}{L - n\pi}\cos\left(L - n\pi\right) + \frac{L}{L - n\pi}\right) \\
&= -\frac{1}{L + n\pi}\cos\left(L + n\pi\right) + \frac{1}{L + n\pi} - \frac{1}{L - n\pi}\cos\left(L - n\pi\right) + \frac{1}{L - n\pi}
\end{aligned}$$

   - Coefficient for Fourier sine series:

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L \sin x \sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \left(\cos\left(x - \frac{n\pi x}{L}\right) - \cos\left(x + \frac{n\pi x}{L}\right)\right)dx \\
&= \frac{1}{L}\left[\frac{L}{L - n\pi}\sin\left(\left(1 - \frac{n\pi}{L}\right)x\right) - \frac{L}{L + n\pi}\sin\left(\left(1 + \frac{n\pi}{L}\right)x\right)\right]_0^L \\
&= \frac{1}{L}\left(\frac{L}{L - n\pi}\sin\left(L - n\pi\right) - \frac{L}{L + n\pi}\sin\left(L + n\pi\right)\right) \\
&= \frac{1}{L - n\pi}\sin\left(L - n\pi\right) - \frac{1}{L + n\pi}\sin\left(L + n\pi\right)
\end{aligned}$$

2. $f(x) = L - x, x \in [-L, L]$

- Coefficients for Fourier cosine series:

$$A_0 = \frac{1}{2L}\int_{-L}^{L}(L-x)dx = \frac{1}{2L}\left[Lx - \frac{x^2}{2}\right]_{-L}^{L} = L$$

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx - \frac{1}{L}\int_{-L}^{L}x\cos\left(\frac{n\pi x}{L}\right)dx \\ &= 2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)dx - 0 = 2\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L = 0 \end{aligned}$$

- Coefficient for Fourier sine series:

$$B_n = \frac{1}{L}\int_{-L}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx - \frac{1}{L}\int_{-L}^{L}x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 0 - \frac{2}{L}\int_{0}^{L}x\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2}{L}\left(\left[-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\int_0^L\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= -\frac{2}{L}\left(-\frac{L^2}{n\pi}\cos\left(n\pi\right) - 0 + \frac{L}{n\pi}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L\right) = -\frac{2}{L}\left(-\frac{L^2}{n\pi}\cos\left(n\pi\right)\right) = \frac{2L}{n\pi}(-1)^n$$

3. $f(x) = \begin{cases} 0 & x \in [-L, 0] \\ 1 & x \in [0, L] \end{cases}$

- Coefficients for Fourier cosine series:

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0}0dx + \int_0^L 1dx\right) = 0 + \frac{1}{2L}\left[x\right]_0^L = \frac{1}{2}$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{0}0\cos\left(\frac{n\pi x}{L}\right)dx + \int_0^L 1\cos\left(\frac{n\pi x}{L}\right)dx\right) = \frac{1}{L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L = 0$$

- Coefficient for Fourier sine series:

$$\begin{aligned} B_n &= \frac{1}{L}\left(\int_{-L}^{0}0\sin\left(\frac{n\pi x}{L}\right)dx + \int_0^L 1\sin\left(\frac{n\pi x}{L}\right)dx\right) = \frac{1}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L \\ &= \frac{1}{L}\left(-\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\right) = \frac{1}{n\pi}\left(-(-1)^n + 1\right) = \frac{1}{n\pi}\left((-1)^{(n+1)} + 1\right) \end{aligned}$$

4. $f(x) = e^x, x \in [-L, L]$

- Coefficients for Fourier cosine series:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} e^x dx = \frac{1}{2L}\left[e^x\right]_{-L}^{L} = \frac{1}{2L}\left(e^L - e^{-L}\right)$$

$$A_n = \frac{1}{L}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\left(\left[e^x \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} e^x \sin\left(\frac{n\pi x}{L}\right) dx\right)$$

$$= \frac{1}{L}\left(0 - \frac{L}{n\pi}\left(\left[-e^x \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)\right)$$

$$= \frac{1}{L}\left(-\frac{L}{n\pi}\left(-e^L \frac{L}{n\pi}\cos(n\pi) + e^{-L}\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)\right)$$

$$= \frac{1}{L}\left(e^L \frac{L^2}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L^2}{n^2\pi^2}\cos(n\pi) - \frac{L^2}{n^2\pi^2}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)$$

$$= e^L \frac{L}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L}{n^2\pi^2}\cos(n\pi) - \frac{L^2}{n^2\pi^2}\left(\frac{1}{L}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)$$

$$\Rightarrow A_n = e^L \frac{L}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L}{n^2\pi^2}\cos(n\pi) - \frac{L^2}{n^2\pi^2}A_n$$

$$\Rightarrow A_n + \frac{L^2}{n^2\pi^2}A_n = e^L \frac{L}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L}{n^2\pi^2}\cos(n\pi)$$

$$\Rightarrow \left(1 + \frac{L^2}{n^2\pi^2}\right)A_n = e^L \frac{L}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L}{n^2\pi^2}\cos(n\pi)$$

$$\Rightarrow A_n = \frac{n^2\pi^2}{n^2\pi^2 + L^2}\left(e^L \frac{L}{n^2\pi^2}\cos(n\pi) - e^{-L}\frac{L}{n^2\pi^2}\cos(n\pi)\right) = \frac{L}{n^2\pi^2 + L^2}(-1)^n(e^L - e^{-L})$$

- Coefficient for Fourier sine series:

$$B_n = \frac{1}{L}\int_{-L}^{L} e^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\left(\left[-e^x \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)$$

$$= \frac{1}{L}\left(-e^L \frac{L}{n\pi}\cos(n\pi) + e^{-L}\frac{L}{n\pi}\cos(n\pi)\right) + \frac{L}{n\pi}\left(\frac{1}{L}\int_{-L}^{L} e^x \cos\left(\frac{n\pi x}{L}\right) dx\right)$$

$$= \frac{1}{n\pi}(-1)^n(-e^L + e^{-L}) + \frac{L}{n\pi}A_n = \frac{1}{n\pi}(-1)^n(-e^L + e^{-L}) + \frac{L}{n\pi}\left(\frac{L}{n^2\pi^2 + L^2}(-1)^n(e^L - e^{-L})\right)$$

$$= \frac{1}{n\pi}(-1)^n\left((-e^L + e^{-L}) + \frac{L^2}{n^2\pi^2 + L^2}(e^L - e^{-L})\right) = \frac{1}{n\pi}(-1)^n(e^L - e^{-L})\left(-1 + \frac{L^2}{n^2\pi^2 + L^2}\right)$$