Linear Transformations

CS 130 - Mathematical Methods in Computer Science

Jan Michael C. Yap

Department of Computer Science
University of the Philippines Diliman

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Topics

Topics

Scaling and projecting a vector

$$f : \mathbb{R}^3 \to \mathbb{R}^2, f\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

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$$f\left(\begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

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Topics

Some Linear Transformations

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- Dilation

$$L : \mathbb{R}^n \to \mathbb{R}^n, L\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = r\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, r > 1$$

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- Rotation in $\mathbb{R}^2$

$$L : \mathbb{R}^n \to \mathbb{R}^n, L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Some Linear Transformations

- Reflection about the main axis in $\mathbb{R}^2$

$$L_1 : \mathbb{R}^n \to \mathbb{R}^n, L_1\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$L_2 : \mathbb{R}^n \to \mathbb{R}^n, L_2\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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- $L(u - v) = L(u) - L(v)$
- If $B = \{b_1, b_2, \cdots, b_n\}$ is a basis for the n-dimensional vector space V and $u \in V$, then $L(u) = \sum_{i=1}^{n} c_i L(b_i)$

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- If ker(L) contains only 0_V, then L is one-to-one

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  - $dim(range(L)) + dim(ker(L)) = n = dim(V)$

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$$A = \left(f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \right)$$

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  - The square matrix on the right hand side of the RREF matrix is $P_{S \to T}$
- Extend this to work with linearly transformed vectors across different vector spaces

Generalizing the change of basis process

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ with $S = \{v_1, v_2, \cdots, v_n\}$ and $T = \{w_1, w_2, \cdots, w_m\}$ be bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then the matrix of the linear transformation L w.r.t. S and T, denoted as $P^L_{S \to T}$, is obtained as follows:

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$$S = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

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$$\Downarrow$$

$$T|L(S) = \begin{pmatrix} 2 & -1 & 0 & -2 & 2 \\ 1 & 1 & 10 & 6 & -2 \end{pmatrix}$$

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$$\left(\begin{array}{ccccc} 2 & -1 & 0 & -2 & 2 \\ 1 & 1 & 10 & 6 & -2 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc} 1 & 0 & \frac{10}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{20}{3} & \frac{14}{3} & -2 \end{array}\right)$$

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Generalizing the change of basis process

- !!!

$$\begin{pmatrix} 1 & 0 & \frac{10}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix} \Rightarrow P^L_{S\to T} = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix}$$

General change of basis

$$S = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

$$x = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}, f(x) = \begin{pmatrix} 16 \\ 18 \end{pmatrix}$$

General change of basis

$$S = \left\{ \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

$$x = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}, f(x) = \begin{pmatrix} 16 \\ 18 \end{pmatrix}$$

$$\Downarrow$$

$$[x]_S = \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}, [f(x)]_T = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix}$$

General change of basis

$$[x]_S = \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}, [f(x)]_T = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix}, P^L_{S \to T} = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix}$$

General change of basis

$$[x]_S = \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}, [f(x)]_T = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix}, P^L_{S \to T} = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix}$$

$$\Downarrow$$

$$P^L_{S \to T}[x]_S$$

General change of basis

$$[x]_S = \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}, [f(x)]_T = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix}, P^L_{S \to T} = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix}$$

$$\Downarrow$$

$$P^L_{S \to T}[x]_S = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix} \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}$$

General change of basis

$$[x]_S = \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix}, [f(x)]_T = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix}, P^L_{S\to T} = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix}$$

$$\Downarrow$$

$$P^L_{S\to T}[x]_S = \begin{pmatrix} \frac{10}{3} & \frac{4}{3} & 0 \\ \frac{20}{3} & \frac{14}{3} & -2 \end{pmatrix} \begin{pmatrix} \frac{47}{9} \\ -\frac{41}{9} \\ \frac{31}{9} \end{pmatrix} = \begin{pmatrix} \frac{34}{3} \\ \frac{20}{3} \end{pmatrix} = [f(x)]_T$$

END OF LESSON 5