CS 130 - Linear Algebra: Examples

1 Matrices and Linear Systems

1. Find the inverse for the following matrix[1]:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

SOLUTION:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & -2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -1 \\ -2 & 1 & -2 \end{pmatrix}$$

2. What is the determinant of the matrix A in number 1?

SOLUTION: Convert A to an upper (or lower) triangular matrix, then get the product of the diagonal entries. Keep in mind the row operations performed towards the transformation.

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Since the operations used were all addition of row scalar multiples, the determinant of the resulting upper triangular matrix is just the same as A, hence

$$det(A) = a_{11} * a_{22} * a_{33} = -1$$

3. Solve the following linear system of equations using Gaussian Elimination Method[1]:

$$\begin{pmatrix} 1 & 4 & -2 \\ -1 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -17 \end{pmatrix}$$

[1] http://www.cliffsnotes.com/study_guide/Using-Elementary-Row-Operations-to-Determine-A1.topicArticleId-20807,articleId-20784.html

SOLUTION:

$$\begin{pmatrix} 1 & 4 & -2 & 1 \\ -1 & 1 & -1 & 7 \\ 3 & 0 & 1 & -17 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 5 & -3 & 8 \\ 3 & 0 & 1 & -17 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 5 & -3 & 8 \\ 0 & -12 & 7 & -20 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 5 & -3 & 8 \\ 0 & 0 & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

Backward substitution:

$$-\frac{1}{5}z = -\frac{4}{5} \Rightarrow z = 4$$
$$5y - 3z = 8 \Rightarrow 5y - 12 = 8 \Rightarrow 5y = 20 \Rightarrow y = 4$$
$$x + 4y - 2z = 1 \Rightarrow x + 16 - 8 = 1 \Rightarrow x + 8 = 1 \Rightarrow x = -7$$

4. Solve the same system in 4 but using Gauss-Jordan Reduction Method.

SOLUTION:

$$\begin{pmatrix} 1 & 4 & -2 & 1 \\ -1 & 1 & -1 & 7 \\ 3 & 0 & 1 & -17 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 5 & -3 & 8 \\ 3 & 0 & 1 & -17 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 5 & -3 & 8 \\ 0 & -12 & 7 & -20 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1 & 4 & -2 & 1 \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & -12 & 7 & -20 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{27}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & -12 & 7 & -20 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{27}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{27}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & 0 & 1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & 0 & 1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$
$$\Rightarrow x = -7, y = 4, z = 4$$

2 Vector Spaces

1. Let V be the set of all polynomials with degree 2 or less, with operations standard polynomial addition and scalar multiplication. Is V a vector space?

SOLUTION: The standard form for such polynomials is $a_2x^2 + a_1x + a_0$, where $a_0, a_1, a_2 \in \mathbb{R}$. If $a_2 \neq 0$, then the polynomial is of degree 2. If $a_2 = 0$ but $a_1 \neq 0$, then the polynomial is of degree 1. If $a_2 = 0$ and $a_1 = 0$, then the polynomial is of degree 0 regardless of the value of a_0. Now let's see if it is indeed a vector space. Let $b_2x^2 + b_1x + b_0, d_2x^2 + d_1x + d_0$ and $f_2x^2 + f_1x + f_0$ be elements from V, and c and e be scalars:

- Closure under addition: $\left(b_2x^2 + b_1x + b_0\right) + \left(d_2x^2 + d_1x + d_0\right) = \left(b_2 + d_2\right)x^2 + \left(b_1 + d_1\right)x + \left(b_0 + d_0\right) \in V$
- Commutativity under addition: $\left(b_2x^2 + b_1x + b_0\right) + \left(d_2x^2 + d_1x + d_0\right) = \left(b_2 + d_2\right)x^2 + \left(b_1 + d_1\right)x + \left(b_0 + d_0\right) = \left(d_2 + b_2\right)x^2 + \left(d_1 + b_1\right)x + \left(d_0 + b_0\right) = \left(d_2x^2 + d_1x + d_0\right) + \left(b_2x^2 + b_1x + b_0\right)$
- Associativity under addition: $\left(\left(b_2x^2 + b_1x + b_0\right) + \left(d_2x^2 + d_1x + d_0\right)\right) + \left(f_2x^2 + f_1x + f_0\right) = \left(b_2 + d_2\right)x^2 + \left(b_1 + d_1\right)x + \left(b_0 + d_0\right) + \left(f_2x^2 + f_1x + f_0\right) = \left(b_2 + d_2 + f_2\right)x^2 + \left(b_1 + d_1 + f_1\right)x + \left(b_0 + d_0 + f_0\right) = \left(b_2x^2 + b_1x + b_0\right) + \left(d_2 + f_2\right)x^2 + \left(d_1 + f_1\right)x + \left(d_0 + f_0\right) = \left(b_2x^2 + b_1x + b_0\right) + \left(\left(d_2x^2 + d_1x + d_0\right) + \left(f_2x^2 + f_1x + f_0\right)\right)$

- Additive Identity: $0 \in V$ (polynomial of degree 0), $b_2x^2 + b_1x + b_0 + 0 = b_2x^2 + b_1x + b_0$
- Additive Inverse: $-b_2x^2 - b_1x - b_0 \in V$, $b_2x^2 + b_1x + b_0 + -b_2x^2 - b_1x - b_0 = 0$
- Closure under scalar multiplication: $c(b_2x^2 + b_1x + b_0) = cb_2x^2 + cb_1x + cb_0 \in V$
- Associativity under scalar multiplication: $c\left(e\left(b_2x^2 + b_1x + b_0\right)\right) = c\left(eb_2x^2 + eb_1x + eb_0\right) = ceb_2x^2 + ceb_1x + ceb_0 = (ce)\,b_2x^2 + (ce)\,b_1x + (ce)\,b_0 = (ce)\left(b_2x^2 + b_1x + b_0\right)$
- Distributivity of sum of scalars: $(c+e)\left(b_2x^2 + b_1x + b_0\right) = (c+e)\,b_2x^2 + (c+e)\,b_1x + (c+e)\,b_0 = cb_2x^2 + cb_1x + cb_0 + eb_2x^2 + eb_1x + eb_0 = c\left(b_2x^2 + b_1x + b_0\right) + e\left(b_2x^2 + b_1x + b_0\right)$
- Distributivity scalar multiplication to sum of polynomials: $c\left(\left(b_2x^2 + b_1x + b_0\right) + \left(d_2x^2 + d_1x + d_0\right)\right) = c\left((b_2+d_2)\,x^2 + (b_1+d_1)\,x + (b_0+d_0)\right) = (cb_2+cd_2)\,x^2 + (cb_1+cd_1)\,x + (cb_0+cd_0) = \left(cb_2x^2 + cb_1x + cb_0\right) + \left(cd_2x^2 + cd_1x + cd_0\right) = c\left(b_2x^2 + b_1x + b_0\right) + c\left(d_2x^2 + d_1x + d_0\right)$
- Scalar Multiplicative Identity: $1 \in V$ (polynomial of degree 0), $1\left(b_2x^2 + b_1x + b_0\right) = b_2x^2 + b_1x + b_0$

Since V is closed under polynomial addition and scalar multiplication, and the relative properties for both operations are preserved, therefore V is a vector space.

2. Consider W, the subset of V in number 1 that contains only polynomials of exactly degree 2. Is W a subspace of V?

   SOLUTION: W contains polynomials $a_2x^2 + a_1x + a_0$, where $a_0, a_1, a_2 \in \mathbb{R}$ and furthermore $a_2 \neq 0$. Let $b_2x^2 + b_1x + b_0$ and $d_2x^2 + d_1x + d_0$ be elements from W, $b_2, d_2 \neq 0$, and c be a scalar. We then form
   $$c\left(b_2x^2 + b_1x + b_0\right) + d_2x^2 + d_1x + d_0 = (cb_2 + d_2)\,x^2 + (cb_1 + d_1)\,x + (cb_0 + d_0)$$
   Note that even though $b_2, d_2 \neq 0$, if $cb_2 = -d_2$ then $cb_2 + d_2 = 0$, zeroing out the term with x^2, then making the polynomial have degree less than 2, which would mean that the resulting polynomial would not be contained in W. Hence, W is not a subspace of V.

3. Prove that the set $X = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

   SOLUTION: To prove that X is a basis, we need to show that: 1) it spans $\mathbb{R}^2$ and 2) the vectors in X are linearly independent. To prove 1), we need to show that we can express any 2-vector as a linear combination of the vectors in X, or in matrix form, we show that the linear system Ax = b has a unique solution, where the columns of A are the vectors in X (i.e. $\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$), and $b = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$. Using Gauss-Jordan reduction:
   $$\left(\begin{array}{ccc} 1 & 1 & a \\ 3 & -2 & b \end{array}\right) \Rightarrow \left(\begin{array}{ccc} 1 & 0 & \frac{2a+b}{5} \\ 0 & 1 & \frac{3a-b}{5} \end{array}\right) \Rightarrow x = \frac{2a+b}{5}, y = \frac{3a-b}{5}$$
   Notice that for any real number a and b, a unique solution exists as we did not need to introduce another variable other than a and b that would represent another arbitrary value. Hence, X spans $\mathbb{R}^2$. To prove 2), we then solve the homogenous system Ax = 0. If the solution is unique and trivial, then the columns of A (and ergo the vectors of X) are linearly independent. Again using Gauss-Jordan reduction:
   $$\left(\begin{array}{ccc} 1 & 1 & 0 \\ 3 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \Rightarrow x = 0, y = 0$$
   Since there is only the trivial solution for the homogenous system, the vectors of X are linearly independent. And by proving 1) and 2), we then say that X is a basis for $\mathbb{R}^2$

4. Assume that X in the previous item is now an ordered basis. What is the coordinate vector of $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with respect to X?

   SOLUTION: Looking for $[v]_X$ would have us solve the linear system $X\,[v]_X = v$, where X is the matrix formed by augmenting the vectors of the ordered basis X. Using Gauss-Jordan to solve the linear system, we have:

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & \frac{1}{5} \end{pmatrix} \Rightarrow [v]_X = \begin{pmatrix} \frac{4}{5} \\ \frac{1}{5} \end{pmatrix}$$

5. Let $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be an ordered basis for $\mathbb{R}^2$. What is the transition matrix from B to X ($P_{B \to X}$)?

   SOLUTION: We first form the augmented matrix $X|B$, then transform to RREF:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{5} \end{pmatrix} \Rightarrow P_{B \to X} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

6. Obtain an orthonormal basis from X.

   SOLUTION: First we get the orthogonal basis U via Gram-Schmidt:

$$u_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$u_2 = x_2 - proj_{u_1}(x_2) = x_2 - \frac{u_1 \cdot x_2}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{-5}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$$

   Then normalizing the two orthogonal vectors, we have the orthonormal basis:

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{pmatrix}, \begin{pmatrix} \frac{3\sqrt{2}}{2\sqrt{5}} \\ \frac{\sqrt{2}}{2\sqrt{5}} \end{pmatrix} \right\}$$

3 Eigenvectors and Eigenvalues

Find the eigenvalues and associated eigenvectors of the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

SOLUTION: We set up the matrix $\lambda I - A$ as follows:

$$\lambda I - A = \begin{pmatrix} \lambda - 1 & 1 & -2 \\ -2 & \lambda & -3 \\ 0 & -1 & \lambda + 1 \end{pmatrix}$$

We now need to get the determinant of the matrix above to set up the characteristic equation. One way to do so is to convert the matrix into an upper triangular one:

$$\begin{pmatrix} \lambda - 1 & 1 & -2 \\ -2 & \lambda & -3 \\ 0 & -1 & \lambda + 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda - 1 & 1 & -2 \\ 0 & \lambda + \frac{2}{\lambda - 1} & -3 - \frac{4}{\lambda - 1} \\ 0 & -1 & \lambda + 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda - 1 & 1 & -2 \\ 0 & \frac{\lambda^2 - \lambda + 2}{\lambda - 1} & -\frac{3\lambda + 1}{\lambda - 1} \\ 0 & 0 & \lambda + 1 - \frac{3\lambda + 1}{\lambda^2 - \lambda + 2} \end{pmatrix}$$

Note that the operations used to transform the matrix were all addition of row scalar multiples. The determinant is then just the product of the diagonal entries. Hence the characteristic equation is:

$$(\lambda-1)\left(\frac{\lambda^2-\lambda+2}{\lambda-1}\right)\left(\frac{(\lambda+1)\left(\lambda^2-\lambda+2\right)-3\lambda-1}{\lambda^2-\lambda+2}\right)=0$$

$$\Rightarrow (\lambda+1)\left(\lambda^2-\lambda+2\right)-3\lambda-1=0 \Rightarrow \lambda^3-\lambda^2+2\lambda+\lambda^2-\lambda+2-3\lambda-1=0$$
$$\Rightarrow \lambda^3-2\lambda+1=0$$

By inspection, we can say that $\lambda=1$ is a solution to the characteristic equation above. Given this, we then divide (both sides of) the characteristic equation by $(\lambda-1)$ to try and obtain the other 2 roots:

$$\frac{\lambda^3-2\lambda+1}{\lambda-1}=\frac{0}{\lambda-1}\Rightarrow \lambda^2+\lambda-1=0$$

Using the quadratic formula, we get the other two roots/eigenvalues $\lambda=\frac{-1\pm\sqrt{5}}{2}$. To get the corresponding eigenvectors, we solve the homogenous system $(\lambda I-A)\,x=0$ for each eigenvalue. For $\lambda=1$, we have

$$\begin{pmatrix} 0 & 1 & -2 \\ -2 & 1 & -3 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\Rightarrow\begin{pmatrix} 0 & 1 & -2 & 0 \\ -2 & 1 & -3 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}\Rightarrow\begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -\frac{r}{2} \\ 2r \\ r \end{pmatrix}, r\in\mathbb{R}$$

For $\lambda=\frac{-1-\sqrt{5}}{2}$, we have

$$\begin{pmatrix} \frac{-3-\sqrt{5}}{2} & 1 & -2 \\ -2 & \frac{-1-\sqrt{5}}{2} & -3 \\ 0 & -1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\Rightarrow\begin{pmatrix} \frac{-3+\sqrt{5}}{2} & 1 & -2 & 0 \\ -2 & \frac{-1+\sqrt{5}}{2} & -3 & 0 \\ 0 & -1 & \frac{1+\sqrt{5}}{2} & 0 \end{pmatrix}\Rightarrow\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{-1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -s \\ \frac{1+\sqrt{5}}{2}s \\ s \end{pmatrix}, s\in\mathbb{R}$$

For $\lambda=\frac{-1+\sqrt{5}}{2}$, we have

$$\begin{pmatrix} \frac{-3+\sqrt{5}}{2} & 1 & -2 \\ -2 & \frac{-1+\sqrt{5}}{2} & -3 \\ 0 & -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\Rightarrow\begin{pmatrix} \frac{-3-\sqrt{5}}{2} & 1 & -2 & 0 \\ -2 & \frac{-1-\sqrt{5}}{2} & -3 & 0 \\ 0 & -1 & \frac{1-\sqrt{5}}{2} & 0 \end{pmatrix}\Rightarrow\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{-1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -t \\ \frac{1-\sqrt{5}}{2}t \\ t \end{pmatrix}, t\in\mathbb{R}$$

4 Linear Transformation

1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}$$

   Is L a linear transformation?

   SOLUTION: First, get two 2-vectors, $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then a scalar c. We then check if $L(ca + b) = cL(a) + L(b)$.

$$L(ca+b) = L\left(c\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = L\left(\begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \end{pmatrix}\right) = \begin{pmatrix} ca_1 + b_1 - ca_2 - b_2 \\ ca_1 + b_1 + ca_2 + b_2 \end{pmatrix}$$

$$cL(a)+L(b) = c\begin{pmatrix} a_1 - a_2 \\ a_1 + a_2 \end{pmatrix} + \begin{pmatrix} b_1 - b_2 \\ b_1 + b_2 \end{pmatrix} = \begin{pmatrix} ca_1 - ca_2 \\ ca_1 + ca_2 \end{pmatrix} + \begin{pmatrix} b_1 - b_2 \\ b_1 + b_2 \end{pmatrix} = \begin{pmatrix} ca_1 - ca_2 + b_1 - b_2 \\ ca_1 + ca_2 + b_1 + b_2 \end{pmatrix}$$

   We see that $L(ca + b)$ is indeed equal to $cL(a) + L(b)$ and therefore conclude that L is a linear transformation.

2. What is the kernel and range of L in number 1?

   SOLUTION: To find ker(L), we look for values x_1 and x_2 such that $x_1 - x_2 = 0$ and $x_1 + x_2 = 0$. This is similar to having the homogenous system:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Solving using Gauss-Jordan, we have

$$\left(\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

   This means that the system only has the trivial solution, and that the only values for x_1 and x_2 such that the image of the vector under L result to the zero vector is $x_1 = 0$ and $x_2 = 0$. Therefore $ker(L) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$. To find range(L), we compute for the possible values that can be formed with $x_1 - x_2$ and $x_1 + x_2$. To do this, let $\begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$ be the vector that is equal to the image of a 2-vector under L. We then have:

$$\begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

   Solving using Gauss-Jordan to get the range, we have

$$\left(\begin{array}{ccc} 1 & -1 & a \\ 1 & 1 & b \end{array}\right) \Rightarrow \left(\begin{array}{ccc} 1 & 0 & \frac{\frac{3a+b}{4}}{} \\ 0 & 1 & \frac{b-a}{2} \end{array}\right)$$

   a and b can then be (uniquely) represented by the linear system represented by $\begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. It means that L can cover all possible vectors in the co-domain of L, i.e. $\mathbb{R}^2$. Therefore, $range(L) = \mathbb{R}^2$. As a further note, the fact that ker(L) only contains the zero vector means that L is one-to-one, and the fact that range(L) is the whole co-domain means that L is onto.

3. What is the corresponding matrix for L?

   SOLUTION: The matrix of L has columns that are the image of the natural basis vectors of the domain, which in this case is $\mathbb{R}^2$. The order by which the image of the natural basis vectors should be placed in the matrix as columns should be in the same order as how the natural basis vectors appear as columns of the identity matrix, i.e.

$$\left(L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \right) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$