Ordinary Differential Equations: First Order

CS 130 - Mathematical Methods in Computer Science

Jan Michael C. Yap

Department of Computer Science
University of the Philippines Diliman

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1. Preliminaries
   - Ordinary Differential Equations
   - First Order Ordinary Differential Equations
2. Solving First Order ODEs
   - Exact Differential Equations
   - Separation of Variables
   - Homogeneous Equations and the Standard Method
   - Linear Differential Equations
   - Bernoulli's Equation

Topics

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Ordinary Differential Equations

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$$\frac{dy}{dx} = x^2 + 2x + 1$$

$$x\frac{d^2y}{dx^2} + x^3 + y = 5$$

Some preliminaries

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  - Particular solutions are obtained by defining boundary conditions

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  - Particular solutions are obtained by defining boundary conditions
  - If the boundary conditions specifies the independent variable to be set to zero, then an initial condition is defined on the ODE to be solved

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First Order ODEs

$$\frac{dy}{dx} = x^2 + 2x + 1$$

$$\sin x + \sin y + x \cos y \frac{dy}{dx} = 0$$

First Order ODEs

$$\frac{dy}{dx} = x^2 + 2x + 1$$

$$\sin x + \sin y + x \cos y \frac{dy}{dx} = 0$$

- First order ODEs can be solved using one integration (but with possibly multiple "preprocessing" steps)

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Exact Differential Equations

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$$x^2 \frac{dy}{dx} + 2xy = 3x^2$$

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$$x^2\frac{dy}{dx} + 2xy = 3x^2 \Rightarrow \frac{d\left(x^2y\right)}{dx} = 3x^2$$

$$\Rightarrow d\left(x^2y\right) = 3x^2dx \Rightarrow \int d\left(x^2y\right) = \int 3x^2dx$$

$$\Rightarrow x^2y = x^3 + C$$

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$$\sin x + \sin y + x\cos y\frac{dy}{dx} = 0 \Rightarrow \sin y + x\cos y\frac{dy}{dx} = -\sin x$$

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$$x\frac{dy}{dx} = y \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \Rightarrow \ln y = \ln x + C$$

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$$x(4-x)\frac{dy}{dx} - y = 0 \Rightarrow x(4-x)\frac{dy}{dx} = y \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x(4-x)}$$

Separation of Variables

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$$x(4-x)\frac{dy}{dx} - y = 0 \Rightarrow x(4-x)\frac{dy}{dx} = y \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x(4-x)}$$

$$\Rightarrow \ln y = \frac{\ln x}{4} - \frac{\ln(4-x)}{4} + C$$

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Homogeneous Equations

- Homogeneous polynomial

$$x^2 - 3xy + 4y^2$$

$$2y^3 - 4x^2y + x^3 - 3xy^2$$

$$\sqrt{x+y}$$

$$2y^3 \exp\left(\frac{y}{x}\right) - \frac{x^4}{x+3y}$$

Homogeneous Equations

- We can "re-express" equations as $f(x, y)$

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$$f(\lambda x, \lambda y) = \lambda^2 x^2 - 3\lambda^2 xy + 4\lambda^2 y^2 = \lambda^2 \left(x^2 - 3xy + 4y^2\right)$$

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$$f(\lambda x, \lambda y) = \lambda^2 x^2 - 3\lambda^2 xy + 4\lambda^2 y^2 = \lambda^2 \left(x^2 - 3xy + 4y^2\right)$$

$$f(\lambda x, \lambda y) = \lambda^k f(x, y)$$

Standard Method

- Suppose we have

$$F(x, y)\frac{dy}{dx} = G(x, y)$$

where F and G are homogeneous and have the same degree

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$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

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$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$x\frac{dy}{dx} = x + 2y$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$x\frac{dy}{dx} = x + 2y \Rightarrow x\left(v + x\frac{dv}{dx}\right) = x + 2vx$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$x\frac{dy}{dx} = x + 2y \Rightarrow x\left(v + x\frac{dv}{dx}\right) = x + 2vx \Rightarrow v + x\frac{dv}{dx} = 1 + 2v$$

Standard Method

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$$x\frac{dy}{dx} = x + 2y \Rightarrow x\left(v + x\frac{dv}{dx}\right) = x + 2vx \Rightarrow v + x\frac{dv}{dx} = 1 + 2v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v$$

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$$x\frac{dy}{dx} = x + 2y \Rightarrow x\left(v + x\frac{dv}{dx}\right) = x + 2vx \Rightarrow v + x\frac{dv}{dx} = 1 + 2v$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v \Rightarrow \frac{1}{1+v}\frac{dv}{dx} = \frac{1}{x}$$

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$$\Rightarrow x\frac{dv}{dx} = 1 + v \Rightarrow \frac{1}{1+v}\frac{dv}{dx} = \frac{1}{x} \Rightarrow \ln(1+v) = \ln x + C$$

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$$\Rightarrow \ln\left(1 + \frac{y}{x}\right) = \ln x + C$$

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$$2xy\frac{dy}{dx} + x^2 + y^2 = 0$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$2xy\frac{dy}{dx} + x^2 + y^2 = 0 \Rightarrow 2vx^2\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 = 0$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$2xy\frac{dy}{dx} + x^2 + y^2 = 0 \Rightarrow 2vx^2\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 = 0$$

$$\Rightarrow 2v\left(v + x\frac{dv}{dx}\right) + 1 + v^2 = 0$$

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$$\Rightarrow 2v\left(v + x\frac{dv}{dx}\right) + 1 + v^2 = 0 \Rightarrow v + x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$

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$$2xy\frac{dy}{dx} + x^2 + y^2 = 0 \Rightarrow 2vx^2\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 = 0$$

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$$\Rightarrow x\frac{dv}{dx} = \frac{-1 - v^2}{2v} - v$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$2xy\frac{dy}{dx} + x^2 + y^2 = 0 \Rightarrow 2vx^2\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 = 0$$

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$$\Rightarrow x\frac{dv}{dx} = \frac{-1-v^2}{2v} - v \Rightarrow \frac{2v}{-1-3v^2}\frac{dv}{dx} = \frac{1}{x}$$

Standard Method

$$\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$2xy\frac{dy}{dx} + x^2 + y^2 = 0 \Rightarrow 2vx^2\left(v + x\frac{dv}{dx}\right) + x^2 + v^2x^2 = 0$$

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$$\Rightarrow -\frac{1}{3}\ln\left(-1-3\left(\frac{y}{x}\right)^2\right) = \ln x + C$$

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Linear Differential Equations

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3$$

Linear Differential Equations

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3 \Rightarrow x\frac{dy}{dx} + y = 2x^4$$

Linear Differential Equations

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3 \Rightarrow x\frac{dy}{dx} + y = 2x^4 \Rightarrow \frac{d\left(xy\right)}{dx} = 2x^4$$

Linear Differential Equations

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3 \Rightarrow x\frac{dy}{dx} + y = 2x^4 \Rightarrow \frac{d\left(xy\right)}{dx} = 2x^4$$

$$xy = \frac{2x^5}{5} + C$$

Linear Differential Equations

- Linear differential equation

$$\frac{dy}{dx} + F(x)\,y = G(x)$$

Linear Differential Equations

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$$\frac{dy}{dx} + F(x)\,y = G(x)$$

- Integrating factor

$$e^{\int F(x)dx}$$

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$$\Rightarrow e^{\int F(x)dx}\frac{dy}{dx} + e^{\int F(x)dx}F(x)\,y = e^{\int F(x)dx}G(x)$$

$$\Rightarrow \frac{d\left(ye^{\int F(x)dx}\right)}{dx} = e^{\int F(x)dx}G(x)$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3 \Rightarrow e^{\int \frac{1}{x} dx}$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = 2x^3 \Rightarrow e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + 2xy = 2e^{-x^2}$$

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$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{\int 2xdx}$$

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$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = 2$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{\int 2xdx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = 2 \Rightarrow \frac{d\left(ye^{x^2}\right)}{dx} = 2$$

Linear Differential Equations

$$\boxed{e^{\int F(x)dx}}$$

$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{\int 2xdx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \Rightarrow e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy = 2 \Rightarrow \frac{d\left(ye^{x^2}\right)}{dx} = 2$$

$$\Rightarrow ye^{x^2} = 2x + C$$

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- Linear differential equation

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Bernoulli's Equation

- Linear differential equation

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- Bernoulli's equation

$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k$$

Bernoulli's Equation

- Linear differential equation

$$\frac{dy}{dx} + F(x)\,y = G(x)$$

- Bernoulli's equation

$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k \Rightarrow \frac{1}{y^k}\frac{dy}{dx} + y^{1-k}F(x) = G(x)$$

Bernoulli's Equation

- Linear differential equation

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$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k \Rightarrow \frac{1}{y^k}\frac{dy}{dx} + y^{1-k}F(x) = G(x)$$

$$z = y^{1-k}$$

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$$\frac{dy}{dx} + F(x)\,y = G(x)$$

- Bernoulli's equation

$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k \Rightarrow \frac{1}{y^k}\frac{dy}{dx} + y^{1-k}F(x) = G(x)$$

$$z = y^{1-k} \Rightarrow \frac{dz}{dx} = (1-k)\,\frac{1}{y^k}\frac{dy}{dx}$$

Bernoulli's Equation

- Linear differential equation

$$\frac{dy}{dx} + F(x)\,y = G(x)$$

- Bernoulli's equation

$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k \Rightarrow \frac{1}{y^k}\frac{dy}{dx} + y^{1-k}F(x) = G(x)$$

$$z = y^{1-k} \Rightarrow \frac{dz}{dx} = (1-k)\frac{1}{y^k}\frac{dy}{dx} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}$$

Bernoulli's Equation

- Linear differential equation

$$\frac{dy}{dx} + F(x)\,y = G(x)$$

- Bernoulli's equation

$$\frac{dy}{dx} + F(x)\,y = G(x)\,y^k \Rightarrow \frac{1}{y^k}\frac{dy}{dx} + y^{1-k}F(x) = G(x)$$

$$z = y^{1-k} \Rightarrow \frac{dz}{dx} = (1-k)\frac{1}{y^k}\frac{dy}{dx} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}$$

$$\frac{1}{y^k}\frac{dy}{dx} + \frac{1}{y^{1-k}}F(x) \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} + zF(x)$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)} \frac{dz}{dx} = \frac{1}{y^k} \frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2} \frac{dz}{dx} + xz = e^{-x^2}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)} \frac{dz}{dx} = \frac{1}{y^k} \frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2} \frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2}\frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2} \Rightarrow e^{\int 2x dx} = e^{x^2}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2}\frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2} \Rightarrow e^{\int 2x dx} = e^{x^2}$$

$$e^{x^2}\frac{dz}{dx} + 2e^{x^2}xz = 2$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2}\frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2} \Rightarrow e^{\int 2xdx} = e^{x^2}$$

$$e^{x^2}\frac{dz}{dx} + 2e^{x^2}xz = 2 \Rightarrow \frac{d\left(e^{x^2}z\right)}{dx} = 2$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2}\frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2} \Rightarrow e^{\int 2x dx} = e^{x^2}$$

$$e^{x^2}\frac{dz}{dx} + 2e^{x^2}xz = 2 \Rightarrow \frac{d\left(e^{x^2}z\right)}{dx} = 2 \Rightarrow e^{x^2}z = 2x + C$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$xy - \frac{dy}{dx} = y^3 e^{-x^2} \Rightarrow -\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = e^{-x^2} \Rightarrow \frac{1}{2}\frac{dz}{dx} + xz = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2e^{-x^2} \Rightarrow e^{\int 2x dx} = e^{x^2}$$

$$e^{x^2}\frac{dz}{dx} + 2e^{x^2} xz = 2 \Rightarrow \frac{d\left(e^{x^2} z\right)}{dx} = 2 \Rightarrow e^{x^2} z = 2x + C$$

$$\Rightarrow \frac{e^{x^2}}{y^2} = 2x + C$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \Rightarrow e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \Rightarrow e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -1$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \Rightarrow e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -1 \Rightarrow \frac{d\left(\frac{z}{x}\right)}{dx} = -1$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \Rightarrow e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -1 \Rightarrow \frac{d\left(\frac{z}{x}\right)}{dx} = -1 \Rightarrow \frac{z}{x} = -x + C$$

Bernoulli's Equation

$$\boxed{z = y^{1-k} \Rightarrow \frac{1}{(1-k)}\frac{dz}{dx} = \frac{1}{y^k}\frac{dy}{dx}}$$

$$\frac{dy}{dx} + \frac{y}{x} = xy^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \Rightarrow -\frac{dz}{dx} + \frac{z}{x} = x$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \Rightarrow e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -1 \Rightarrow \frac{d\left(\frac{z}{x}\right)}{dx} = -1 \Rightarrow \frac{z}{x} = -x + C$$

$$\Rightarrow \frac{1}{xy} = -x + C$$

END OF LESSON 7