CS 130 WFV

Long Exam 1

August 2, 2013

General Instructions

- Answer the items completely. Show your solutions.
- Write as legibly as possible. Illegible or unreadable answers and solutions may not merit any points.
- Refrain from making unnecessary motions and sounds during the exam. Any suspicious behavior will be dealt with accordingly.
- Direct all questions to the proctor.
- If you need to go to the CR, hand your questionnaire, answer sheet, and scratch paper to the proctor before heading out. Only one person at any given time is allowed to go out.
- Once you're done with the exam (one way or the other), place your scratch papers and the questionnaire inside your blue book.

Questions

Given the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & -2 & 3 & 5 \\ -1 & -2 & -1 & -7 \end{pmatrix}$$

1. What is the rank of A?
   ANSWER: 2
2. Provide a basis for the *column space* of A.
   ANSWER: $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \right\}$
3. Provide a basis for the *row space* of A. *(HINT: Try transposing the matrix.)*
   ANSWER: $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \\ 5 \end{pmatrix} \right\}$
4. What are the eigenvalues for A? Provide the corresponding eigenspace for each eigenvalue of A.
   ANSWER: There are no eigenvalues (and hence, no eigenvectors) for A since A is not a square matrix.

Let V be a set of functions of the form $\alpha e^x \cos x + \beta e^x \sin x,\ \alpha, \beta \in \mathbb{R}$.

5. Prove that V is a vector space under addition and (constant) multiplication.
   ANSWER:
   Let $v_1 = \alpha_1 e^x \cos x + \beta_1 e^x \sin x,\ v_2 = \alpha_2 e^x \cos x + \beta_2 e^x \sin x$, and $v_3 = \alpha_3 e^x \cos x + \beta_3 e^x \sin x$ be elements in V, and let $\gamma_1, \gamma_2 \in \mathbb{R}$

   - $v_1 + v_2 = \alpha_1 e^x \cos x + \beta_1 e^x \sin x + \alpha_2 e^x \cos x + \beta_2 e^x \sin x = (\alpha_1 + \alpha_2) e^x \cos x + (\beta_1 + \beta_2) e^x \sin x \in V$
     (a) $v_1 + v_2 = (\alpha_1 + \alpha_2) e^x \cos x + (\beta_1 + \beta_2) e^x \sin x = (\alpha_2 + \alpha_1) e^x \cos x + (\beta_2 + \beta_1) e^x \sin x = v_2 + v_1$
     (b) $(v_1 + v_2) + v_3 = ((\alpha_1 + \alpha_2) + \alpha_3) e^x \cos x + ((\beta_1 + \beta_2) + \beta_3) e^x \sin x = (\alpha_1 + (\alpha_2 + \alpha_3)) e^x \cos x + (\beta_1 + (\beta_2 + \beta_3)) e^x \sin x = v_1 + (v_2 + v_3)$
     (c) $0 = 0 e^x \cos x + 0 e^x \sin x \in V$. Also, $v_1 + 0 = \alpha_1 e^x \cos x + \beta_1 e^x \sin x = v_1$
     (d) $-v_1 = -\alpha_1 e^x \cos x - \beta_1 e^x \sin x \in V$. Also, $v_1 + (-v_1) = (\alpha_1 - \alpha_1) e^x \cos x + (\beta_1 - \beta_1) e^x \sin x = 0$
   - $\gamma v_1 = \gamma(\alpha_1 e^x \cos x + \beta_1 e^x \sin x) = \gamma \alpha_1 e^x \cos x + \gamma \beta_1 e^x \sin x \in V$
     (a) $\gamma_1(\gamma_2 v_1) = \gamma_1(\gamma_2 \alpha_1) e^x \cos x + \gamma_1(\gamma_2 \beta_1) e^x \sin x = (\gamma_1 \gamma_2) \alpha_1 e^x \cos x + (\gamma_1 \gamma_2) \beta_1 e^x \sin x = (\gamma_1 \gamma_2) v_1$
     (b) $(\gamma_1 + \gamma_2) v_1 = (\gamma_1 + \gamma_2) \alpha_1 e^x \cos x + (\gamma_1 + \gamma_2) \beta_2 e^x \sin x = (\gamma_1 \alpha_1 + \gamma_2 \alpha_1) e^x \cos x + (\gamma_1 \beta_1 + \gamma_2 \beta_2) e^x \sin x = \gamma_1 v_1 + \gamma_2 v_1$
     (c) $\gamma_1(v_1 + v_2) = \gamma_1(\alpha_1 + \alpha_2) e^x \cos x + \gamma_1(\beta_1 + \beta_2) e^x \sin x = (\gamma_1 \alpha_1 + \gamma_1 \alpha_2) e^x \cos x + (\gamma_1 \beta_1 + \gamma_1 \beta_2) e^x \sin x = \gamma_1 v_1 + \gamma_1 v_2$
     (d) $1 v_1 = 1 \alpha_1 e^x \cos x + 1 \beta_1 e^x \sin x = \alpha_1 e^x \cos x + \beta_1 e^x \sin x = v_1$

6. Suppose $L : V \to V,\ L(v) = v' + v,\ \forall v \in V$. Prove that L is a linear transformation.
   (HINT: $\cos' x = -\sin x,\ \sin' x = \cos x,\ e'^x = e^x$*)*
   ANSWER:
   To expand the result of applying L to a vector in V, we let $v = \alpha e^x \cos x + \beta e^x \sin x \in V$:
   $L(v) = \alpha e^x \cos x - \alpha e^x \sin x + \beta e^x \cos x + \beta e^x \sin x + \alpha e^x \cos x + \beta e^x \sin x = (2\alpha + \beta) e^x \cos x + (2\beta - \alpha) e^x \sin x$

   Let $v_1 = \alpha_1 e^x \cos x + \beta_1 e^x \sin x$ and $v_2 = \alpha_2 e^x \cos x + \beta_2 e^x \sin x$ be elements in V, and let $\gamma \in \mathbb{R}$

   Now, we need to prove that $L(\gamma v_1 + v_2) = \gamma L(v_1) + L(v_2)$ for L to be a linear transformation. The left hand side of the equation is evaluated as follows:
   $L(\gamma v_1 + v_2) = L((\gamma \alpha_1 + \alpha_2) e^x \cos x + (\gamma \beta_1 + \beta_2) e^x \sin x) = (2(\gamma \alpha_1 + \alpha_2) + (\gamma \beta_1 + \beta_2)) e^x \cos x + (2(\gamma \beta_1 + \beta_2) - (\gamma \alpha_1 + \alpha_2)) e^x \sin x = ((2\gamma \alpha_1 + 2\alpha_2) + (\gamma \beta_1 + \beta_2)) e^x \cos x + ((2\gamma \beta_1 + 2\beta_2) - (\gamma \alpha_1 + \alpha_2)) e^x \sin x$

   The right hand side of the equation is evaluated as follows:
   $L(v_1) = (2\alpha_1 + \beta_1) e^x \cos x + (2\beta_1 - \alpha_1) e^x \sin x$
   $L(v_2) = (2\alpha_2 + \beta_2) e^x \cos x + (2\beta_2 - \alpha_2) e^x \sin x$
   $\gamma L(v_1) + L(v_2) = (2\gamma \alpha_1 + \gamma \beta_1) e^x \cos x + (2\gamma \beta_1 - \gamma \alpha_1) e^x \sin x + (2\alpha_2 + \beta_2) e^x \cos x + (2\beta_2 - \alpha_2) e^x \sin x = ((2\gamma \alpha_1 + 2\alpha_2) + (\gamma \beta_1 + \beta_2)) e^x \cos x + ((2\gamma \beta_1 + 2\beta_2) - (\gamma \alpha_1 + \alpha_2)) e^x \sin x$