# Introduction to Matrices and Vectors

## CS 130 - Mathematical Methods in Computer Science

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1. Basic Concepts

2. Matrix Operations
   - Elementary Row Operations
   - Operations on Matrices, Vectors, and/or Scalars

3. Inverse of a Matrix
   - Matrix Inverse
   - Matrix Pseudoinverse

4. Determinant of a Matrix
   - Determinant
   - Some Notes
   - Cofactor Expansion

# Topics

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# A System of Linear Equations

$$\begin{aligned} 2x + y - z &= 2 \\ x + 3y + 3z &= 1 \\ -5x - 6y + 2z &= -3 \end{aligned}$$

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$$\Downarrow$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ -5 & -6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

# Definitions

- Matrix:

$$A_{mxn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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- Square matrix (of order n):

$$A_{nxn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

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- Diagonal Matrix:

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- Scalar Matrix:

$$A_{nxn} = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$$

## Definitions

- Identity Matrix:

$$A_{nxn} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

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- Scalar: $c \in \mathbb{R}$

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- Upper Triangular Matrix:

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- Symmetric Matrix:

$$A_{nxn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

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$$y = 2, x = 3$$

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- Row swap/interchange:

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- Row (scalar multiple) addition:

$$\begin{pmatrix} 3 & -2 \\ -3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -2 \\ 0 & -5 \end{pmatrix}$$

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# Matrix operations

- Matrix addition: $A_{mxn} + B_{mxn} = C_{mxn}, c_{ij} = a_{ij} + b_{ij}$

$$\begin{pmatrix} 3 & 4 \\ 2 & -9 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 3 & -8 \end{pmatrix}$$

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  - $(A + B) + C = A + (B + C)$
  - $A_{mxn} + 0_{mxn} = 0_{mxn} + A_{mxn} = A_{mxn}$
  - $A + D = 0 \rightarrow D = -A$

# Matrix operations

- Scalar multiplication: $rA_{mxn} = B_{mxn}, b_{ij} = ra_{ij}$

$$5\begin{pmatrix} -7 & 5 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -35 & 25 \\ 10 & 0 \end{pmatrix}$$

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- Dot product (of vectors): $a_{1xn(nx1)} \cdot b_{nx1(1xn)}$ $\sum_{k=1}^{n} a_i b_i$

$$\begin{pmatrix} 1 \\ -3 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 32$$

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$$\begin{pmatrix} 3 & 4 \\ 2 & -9 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & -4 & 5 \\ 1 & 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 17 & -4 & 11 \\ -5 & -12 & -26 & 19 \\ 6 & 11 & 4 & 1 \end{pmatrix}$$

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# Matrix operations

- Matrix transposition: $A^T_{mxn} = B_{nxm}, b_{ij} = a_{ji}$

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  - $(A^T)^{-1} = (A^{-1})^T$

# How to Find the Inverse (Basic Method)

- Matrix augmentation: $A_{mxn}|B_{mxp} = C_{mx(n+p)}$

$$\begin{pmatrix} 3 & 2 & -8 \\ 2 & -9 & 0 \end{pmatrix} | \begin{pmatrix} 2 & 11 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -8 & 2 & 11 \\ 2 & -9 & 0 & 1 & -1 \end{pmatrix}$$

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  - For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.
  - If a column contains a leading 1, then all other entries in that column are 0.

# How to Find the Inverse (Basic Method)

- Reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -8 & 0 & 11 \\ 0 & 1 & 9 & 0 & -1 \\ 0 & 0 & 0 & 1 & 22 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 13 & 0 \\ 0 & 1 & 99 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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- Using elementary row operations, transform the augmented matrix to RREF.
- Once in RREF, if left submatrix is $I_{nxn}$, then the right one is $A^{-1}$ . If not, then A is singular.

# How to Find the Inverse (Basic Method)

- Augment square matrix $A_{nxn}$ with identity matrix $I_{nxn}$

$$A = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$$

# How to Find the Inverse (Basic Method)

- Augment square matrix $A_{nxn}$ with identity matrix $I_{nxn}$

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# How to Find the Inverse (Basic Method)

- Using elementary row operations, transform the augmented matrix to RREF.

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$$A^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{pmatrix}$$

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$$A^+ = (A^TA)^{-1}A^T$$

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# How to Compute the Determinant

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  - 312 is an even permutation (2 inversions: 31 and 32).
  - 321 is an odd permutation (3 inversions: 32, 31, and 21).

# How to Compute the Determinant (Basic Method)

- $det(A) = \sum_{perm(S_n)} \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$
  - If the permutation $j_1 j_2 \cdots j_n$ is even, then sign is plus.
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$$\begin{aligned} det(A) &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ &= -15 - (-15) - (-10) + (-5) + (-12) - (-6) \\ &= -1 \end{aligned}$$

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## Cofactor expansion

- The cofactor of entry $a_{ij}$ of square matrix A:
  $A_{ij} = (-1)^{i+j} det(M_{ij})$
  - $M_{ij}$ is the (n-1) x (n-1) submatrix derived by deleting the $i^{th}$ row and $j^{th}$ column of A

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- The adjoint of A:

$$adj(A) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

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- $Aadj(A) = adj(A)A = det(A)I_n$
- $A^{-1} = \frac{adj(A)}{det(A)}$

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$$adj(A) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -5 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

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$$A^{-1} = \frac{adj(A)}{det(A)} = \begin{pmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{pmatrix}$$

**END OF LESSON 1**