Solving Linear Systems

CS 130 - Mathematical Methods in Computer Science

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Topics

System of Linear Equations

- System of linear equations: $Ax = b$

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- Assume first that A is invertible

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$$\Downarrow$$

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\begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ -3 \end{pmatrix}
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Topics

Cramer's Rule

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$$A_2 = \begin{pmatrix} 2 & 8 & -1 \\ -3 & -11 & 2 \\ 2 & -3 & 2 \end{pmatrix}$$

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$$A_2 = \begin{pmatrix} 2 & 8 & -1 \\ -3 & -11 & 2 \\ 2 & -3 & 2 \end{pmatrix}$$

$$det(A_2) = -3,\ y = \frac{det(A_2)}{det(A)} = \frac{-3}{-1} = 3$$

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$$A_3 = \begin{pmatrix} 2 & 1 & 8 \\ -3 & -1 & -11 \\ 2 & 1 & -3 \end{pmatrix}$$

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$$det(A) = -1$$

$$A_3 = \begin{pmatrix} 2 & 1 & 8 \\ -3 & -1 & -11 \\ 2 & 1 & -3 \end{pmatrix}$$

$$det(A_3) = 1, \; y = \frac{det(A_3)}{det(A)} = \frac{1}{-1} = -1$$

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Gaussian Elimination Method

- Form the augmented matrix $A|b$

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- Transform matrix A in the augmented matrix to an upper triangular matrix

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- Form the augmented matrix $A|b$
- Transform matrix A in the augmented matrix to an upper triangular matrix
- Backward substitution
  - The bottommost row contains the solution for x_n.
  - Go up one row, substitute the solution for x_n, and solve for x_{n-1}.
  - Repeat until x_1 is solved.

Gaussian Elimination Method

- Form the augmented matrix $A|b$

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- Transform matrix A in the augmented matrix to an upper triangular matrix

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$$\begin{pmatrix} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 2 & -3 \end{pmatrix}$$

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$$2x + y - z = 8 \Rightarrow 2x + 3 - (-1) = 8 \Rightarrow x = 2$$

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$$\begin{pmatrix} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Gauss-Jordan Reduction Method

- Backward Substitution

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow x = 2, y, 3, z = -1$$

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  - If matrix A is invertible, then... ???

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What if...

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- A is not square?

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$$\begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

$$z = t \in \mathbb{R}, y = -1 - 3t, x = 2 - 5t$$

Do solutions to nonhomogenous systems always exist?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 1 \\ 1 & 0 & 1 & -2 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

END OF LESSON 2