Fourier Series

CS 130 - Mathematical Methods in Computer Science

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Topics

1. Preliminaries
   - Periodic Functions
   - Even and Odd Functions
   - Orthogonal Functions
2. Fourier Series
   - Fourier Sine Series
   - Fourier Cosine Series
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  - Quick question: What is the period P for $\sin(kx)$ and $\cos(kx)$? $P = \frac{2\pi}{k}$

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- The integration facts are only guaranteed to hold on a "symmetric" interval, $[-a, a]$

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$$2 \int_0^L dx = 2L$$

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$$2 \int_0^L \cos^2\left(\frac{k\pi x}{L}\right) dx = \int_0^L \left(1 + \cos\left(\frac{2k\pi x}{L}\right)\right) dx$$

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    - Why did we not begin at $n = 0$?

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    - NOTE: each element in the set union is of a different index, hence j can never be equal to k in the case where we get the integral of the product of a sine and a cosine function.

Summary

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$$\int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right) dx = \begin{cases} 2L & \text{if } j = k = 0 \\ L & \text{if } j = k \neq 0 \\ 0 & \text{if } j \neq k \end{cases}$$

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$$\int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx = 0$$

Topics

Main idea of Fourier series

- Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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  - Concept of odd and even functions is relevant in this case.
- Instead of derivatives, we express f(x) as a sum of (weighted) sine and cosine functions on that interval.
  - Choice of sines and cosines, apart from their obvious periodicity, is because the functions form a mutually orthogonal set of functions, which limit the number of variables we need to solve.

Topics

Fourier sine series

- Fourier sine series of f(x) on the interval $[-L, L]$

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Examples

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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$$1 + x^2 = 0$$

Fourier sine series on a “half” interval

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- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

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$$B_n = \frac{L}{n\pi}\left(1 + (-1)^{(n+1)} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)$$

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Topics

Fourier cosine series

- Fourier cosine series of f(x) on the interval $[-L, L]$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

  - On the interval $[0, L]$:

$$A_n = \begin{cases} \frac{1}{L}\int_{0}^{L} f(x)dx & n = 0 \\ \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

Examples

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$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} xdx = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$$x = 0$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [0, L]$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [0, L]$

$$A_0 = \frac{1}{L}\int_0^L x dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [0, L]$

$$A_0 = \frac{1}{L}\int_0^L xdx = \frac{L}{2}$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [0, L]$

$$A_0 = \frac{1}{L}\int_0^L xdx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

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$$A_0 = \frac{1}{L}\int_0^L xdx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{n^2\pi^2}(1 - (-1)^n)$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x, x \in [0, L]$

$$A_0 = \frac{1}{L}\int_0^L x dx = \frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2L}{n^2\pi^2}(1-(-1)^n)$$

$$x = \frac{L}{2} + \frac{2L}{n^2\pi^2}\sum_{n=1}^{\infty}(1-(-1)^n)\cos\left(\frac{n\pi x}{L}\right)$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\int_{0}^{L} x^2 dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\int_0^L x^2 dx = \frac{L^2}{3}$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\int_0^L x^2 dx = \frac{L^2}{3}$$

$$A_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\int_{0}^{L} x^2 dx = \frac{L^2}{3}$$

$$A_n = \frac{2}{L}\int_{0}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{4L^2(-1)^n}{n^2\pi^2}$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x^2 dx = \frac{1}{L}\int_0^L x^2 dx = \frac{L^2}{3}$$

$$A_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{4L^2(-1)^n}{n^2\pi^2}$$

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

$$A_0 = \frac{1}{L}\int_0^{\frac{L}{2}} \frac{L}{2}dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} \left(x - \frac{L}{2}\right) dx$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

$$A_0 = \frac{1}{L}\int_0^{\frac{L}{2}} \frac{L}{2}dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} \left(x - \frac{L}{2}\right) dx$$

$$A_0 = \frac{3L}{8}$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

$$A_n = \frac{2}{L}\int_0^{\frac{L}{2}} \frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{\frac{L}{2}}^{L}\left(x - \frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{2L}{n^2\pi^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right)$$

Examples

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n > 0 \end{cases}$$

- $f(x) = \begin{cases} \frac{L}{2} & x \in [0, \frac{L}{2}] \\ x - \frac{L}{2} & x \in [\frac{L}{2}, L] \end{cases}$

$$A_n = \frac{2}{L}\int_0^{\frac{L}{2}} \frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{\frac{L}{2}}^{L}\left(x - \frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{2L}{n^2\pi^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$f(x) = \frac{3L}{8} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

Topics

Fourier series

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

  - $f(x) = x, x \in [-L, L]$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = x, x \in [-L, L]$

$$A_0 = 0$$

$$A_n = 0$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = x, x \in [-L, L]$

$$A_0 = 0$$

$$A_n = 0$$

$$B_n = \frac{(-1)^{(n+1)}2L}{n\pi}$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = x, x \in [-L, L]$

$$A_0 = 0$$

$$A_n = 0$$

$$B_n = \frac{(-1)^{(n+1)} 2L}{n\pi}$$

$$x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

  - $f(x) = x^2, x \in [-L, L]$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

  - $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{L^2}{3}$$

$$A_n = \frac{4L^2(-1)^n}{n^2\pi^2}$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

  - $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{L^2}{3}$$

$$A_n = \frac{4L^2(-1)^n}{n^2\pi^2}$$

$$B_n = 0$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = x^2, x \in [-L, L]$

$$A_0 = \frac{L^2}{3}$$

$$A_n = \frac{4L^2(-1)^n}{n^2\pi^2}$$

$$B_n = 0$$

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = \begin{cases} L & x \in [-L, 0] \\ 2x & x \in [0, L] \end{cases}$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = \begin{cases} L & x \in [-L, 0] \\ 2x & x \in [0, L] \end{cases}$

$$A_0 = L$$

$$A_n = \frac{2L^2}{n^2\pi^2}\left(-1 + (-1)^n\right)$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = \begin{cases} L & x \in [-L, 0] \\ 2x & x \in [0, L] \end{cases}$

$$A_0 = L$$

$$A_n = \frac{2L^2}{n^2\pi^2}\left(-1 + (-1)^n\right)$$

$$B_n = -\frac{L}{n\pi}\left(1 + (-1)^n\right)$$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = \begin{cases} L & x \in [-L, 0] \\ 2x & x \in [0, L] \end{cases}$

Examples

- Fourier series of f(x) on the interval $[-L, L]$ (or $[0, L]$)

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

- $f(x) = \begin{cases} L & x \in [-L, 0] \\ 2x & x \in [0, L] \end{cases}$

$$x^2 = L + \frac{2L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1 + (-1)^n) \cos\left(\frac{n\pi x}{L}\right)$$

$$-\frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (1 + (-1)^n) \sin\left(\frac{n\pi x}{L}\right)$$

END OF LESSON 11