Ordinary Differential Equations: Second Order

CS 130 - Mathematical Methods in Computer Science

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Topics

1. Preliminaries
2. Solving Second Order ODEs
   - Homogenous Equations and the Auxiliary Function
   - Non-Homogenous Equations and the Particular Integral

Linear Second Order ODEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$a, b, c \in \mathbb{R}$$

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- Solution depends on coefficients and right-hand side function

Linear Second Order ODEs with Constant Coefficients

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$$a, b, c \in \mathbb{R}$$

- Solution depends on coefficients and right-hand side function
- Solution will generate two **arbitrary** constants

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$$a, b, c \in \mathbb{R}$$

- Solution depends on coefficients and right-hand side function
- Solution will generate two **arbitrary** constants

$$\frac{d^2y}{dx^2} = 0$$

Linear Second Order ODEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$a, b, c \in \mathbb{R}$$

- Solution depends on coefficients and right-hand side function
- Solution will generate two **arbitrary** constants

$$\frac{d^2y}{dx^2} = 0 \Rightarrow y = Ax + B$$

Topics

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1. Preliminaries
2. Solving Second Order ODEs
   - Homogenous Equations and the Auxiliary Function
   - Non-Homogenous Equations and the Particular Integral

Homogenous Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Homogenous Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$7\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

$$\frac{d^2y}{dx^2} = -5\frac{dy}{dx} + 3y$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx}$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx} \Rightarrow e^{\int \frac{c}{b}dx} = e^{\frac{cx}{b}}$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx} \Rightarrow e^{\int \frac{c}{b}dx} = e^{\frac{cx}{b}}$$

$$\Downarrow$$

$$e^{\frac{c}{b}x}\frac{dy}{dx} + e^{\frac{c}{b}x}\frac{c}{b}y = 0$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx} \Rightarrow e^{\int \frac{c}{b}dx} = e^{\frac{cx}{b}}$$

$$\Downarrow$$

$$e^{\frac{c}{b}x}\frac{dy}{dx} + e^{\frac{c}{b}x}\frac{c}{b}y = 0 \Rightarrow \frac{d\left(ye^{\frac{c}{b}x}\right)}{dx} = 0$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx} \Rightarrow e^{\int \frac{c}{b}dx} = e^{\frac{cx}{b}}$$

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$$e^{\frac{c}{b}x}\frac{dy}{dx} + e^{\frac{c}{b}x}\frac{c}{b}y = 0 \Rightarrow \frac{d(ye^{\frac{c}{b}x})}{dx} = 0 \Rightarrow ye^{\frac{c}{b}x} = A$$

Auxiliary Function

$$b\frac{dy}{dx} + cy = 0 \Rightarrow \frac{dy}{dx} + \frac{c}{b}y = 0$$

$$\Downarrow$$

$$e^{\int F(x)dx} \Rightarrow e^{\int \frac{c}{b}dx} = e^{\frac{cx}{b}}$$

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$$e^{\frac{c}{b}x}\frac{dy}{dx} + e^{\frac{c}{b}x}\frac{c}{b}y = 0 \Rightarrow \frac{d(ye^{\frac{c}{b}x})}{dx} = 0 \Rightarrow ye^{\frac{c}{b}x} = A$$

$$\Downarrow$$

$$y = Ae^{-\frac{c}{b}x}$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x}$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx}$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx}$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx} \Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx} \Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx} \Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \Rightarrow aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$$

Auxiliary Function

$$y = Ae^{-\frac{c}{b}x} \Rightarrow y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx} \Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \Rightarrow aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$$

$$\Downarrow$$

$$am^2 + bm + c = 0$$

Auxiliary Function

- The roots of the auxiliary equation, m_1, m_2

$$y = Ae^{m_1 x}, y = Be^{m_2 x}$$

Auxiliary Function

- The roots of the auxiliary equation, m_1, m_2

$$y = Ae^{m_1 x}, y = Be^{m_2 x}$$

- BUT, ODE was necessarily linear, hence:

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

Auxiliary Function

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Auxiliary Function

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \Rightarrow m^2 + 5m + 6 = 0$$

Auxiliary Function

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \Rightarrow m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0$$

Auxiliary Function

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \Rightarrow m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0$$

$$\Downarrow$$

$$y = Ae^{-2x} + Be^{-3x}$$

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m - r)^2 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m - r)^2 = 0 \Rightarrow y = Ae^{rx} + Be^{rx}$$

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m - r)^2 = 0 \Rightarrow y = Ae^{rx} + Be^{rx}$$

$$\Rightarrow y = Ce^{rx}, C = A + B$$

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m - r)^2 = 0 \Rightarrow y = Ae^{rx} + Be^{rx}$$

$$\Rightarrow y = Ce^{rx}, C = A + B$$

- The resulting "general" solution is **NOT** general, since constants A and B are not arbitrary!

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m - r)^2 = 0 \Rightarrow y = Ae^{rx} + Be^{rx}$$

$$\Rightarrow y = Ce^{rx}, C = A + B$$

- The resulting "general" solution is **NOT** general, since constants A and B are not arbitrary!
  - Problem lies in the constants, not in roots

Auxiliary Function: Dealing with Similar Roots

$$am^2 + bm + c = 0 \Rightarrow a(m-r)^2 = 0 \Rightarrow y = Ae^{rx} + Be^{rx}$$

$$\Rightarrow y = Ce^{rx}, C = A + B$$

- The resulting "general" solution is **NOT** general, since constants A and B are not arbitrary!
  - Problem lies in the constants, not in roots
- Change "trial" solution to something else, instead of $y = Ae^{rx}$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx}, z = g(x)$$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx}, z = g(x)$$

$$\Downarrow$$

$$y = ze^{rx}$$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx}, z = g(x)$$

$$\Downarrow$$

$$y = ze^{rx} \Rightarrow \frac{dy}{dx} = zre^{rx} + e^{rx}\frac{dz}{dx}$$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx}, z = g(x)$$

$$\Downarrow$$

$$y = ze^{rx} \Rightarrow \frac{dy}{dx} = zre^{rx} + e^{rx}\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}$$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx} \Rightarrow \frac{dy}{dx} = zre^{mx} + e^{rx}\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}$$

Auxiliary Function: Dealing with Similar Roots

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$$\Rightarrow \frac{d^2y}{dx^2} = zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Auxiliary Function: Dealing with Similar Roots

$$y = ze^{rx} \Rightarrow \frac{dy}{dx} = zre^{mx} + e^{rx}\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}$$

$$\Downarrow$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$\Rightarrow a(zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}) + b(zre^{rx} + e^{rx}\frac{dz}{dx}) + c(ze^{rx}) = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow a(zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}) + b(zre^{rx} + e^{rx}\frac{dz}{dx}) + c(ze^{rx}) = 0$$

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$$\Rightarrow e^{rx}\left\{a(zr^2 + 2r\frac{dz}{dx} + \frac{d^2z}{dx^2}) + b(zr + \frac{dz}{dx}) + cz\right\} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow a(zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}) + b(zre^{rx} + e^{rx}\frac{dz}{dx}) + c(ze^{rx}) = 0$$

$$\Rightarrow e^{rx}\left\{a(zr^2 + 2r\frac{dz}{dx} + \frac{d^2z}{dx^2}) + b(zr + \frac{dz}{dx}) + cz\right\} = 0$$

$$\Rightarrow azr^2 + 2ar\frac{dz}{dx} + a\frac{d^2z}{dx^2} + bzr + b\frac{dz}{dx} + cz = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow a(zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}) + b(zre^{rx} + e^{rx}\frac{dz}{dx}) + c(ze^{rx}) = 0$$

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$$\Rightarrow azr^2 + 2ar\frac{dz}{dx} + a\frac{d^2z}{dx^2} + bzr + b\frac{dz}{dx} + cz = 0$$

$$\Rightarrow azr^2 + bzr + cz + 2ar\frac{dz}{dx} + b\frac{dz}{dx} + a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow a(zr^2e^{rx} + 2re^{rx}\frac{dz}{dx} + e^{rx}\frac{d^2z}{dx^2}) + b(zre^{rx} + e^{rx}\frac{dz}{dx}) + c(ze^{rx}) = 0$$

$$\Rightarrow e^{rx}\left\{a(zr^2 + 2r\frac{dz}{dx} + \frac{d^2z}{dx^2}) + b(zr + \frac{dz}{dx}) + cz\right\} = 0$$

$$\Rightarrow azr^2 + 2ar\frac{dz}{dx} + a\frac{d^2z}{dx^2} + bzr + b\frac{dz}{dx} + cz = 0$$

$$\Rightarrow azr^2 + bzr + cz + 2ar\frac{dz}{dx} + b\frac{dz}{dx} + a\frac{d^2z}{dx^2} = 0$$

$$\Rightarrow z(ar^2 + br + c) + \frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow z(ar^2 + br + c) + \frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow z(ar^2 + br + c) + \frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$ar^2 + br + c = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\Rightarrow z(ar^2 + br + c) + \frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$ar^2 + br + c = 0$$

$$\Downarrow$$

$$z(ar^2+br+c)+\frac{dz}{dx}(2ar+b)+a\frac{d^2z}{dx^2} = 0 \Rightarrow \frac{dz}{dx}(2ar+b)+a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$a(m - r)^2 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$a(m - r)^2 = 0 \Rightarrow a(m^2 - 2mr + r^2) = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$a(m - r)^2 = 0 \Rightarrow a(m^2 - 2mr + r^2) = 0 \Rightarrow am^2 - 2amr + ar^2 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar+b)+a\frac{d^2z}{dx^2}=0$$

$$\Downarrow$$

$$a(m-r)^2=0 \Rightarrow a(m^2-2mr+r^2)=0 \Rightarrow am^2-2amr+ar^2=0$$

$$am^2-2amr+ar^2=am^2+bm+c$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$a(m - r)^2 = 0 \Rightarrow a(m^2 - 2mr + r^2) = 0 \Rightarrow am^2 - 2amr + ar^2 = 0$$

$$am^2 - 2amr + ar^2 = am^2 + bm + c \Rightarrow b = -2ar$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar + b) + a\frac{d^2z}{dx^2} = 0$$

$$\Downarrow$$

$$a(m - r)^2 = 0 \Rightarrow a(m^2 - 2mr + r^2) = 0 \Rightarrow am^2 - 2amr + ar^2 = 0$$

$$am^2 - 2amr + ar^2 = am^2 + bm + c \Rightarrow b = -2ar$$

$$\Rightarrow 2ar + b = 2ar + (-2ar) = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{dz}{dx}(2ar+b)+a\frac{d^2z}{dx^2}=0$$

$$\Downarrow$$

$$a(m-r)^2=0 \Rightarrow a(m^2-2mr+r^2)=0 \Rightarrow am^2-2amr+ar^2=0$$

$$am^2-2amr+ar^2=am^2+bm+c \Rightarrow b=-2ar$$

$$\Rightarrow 2ar+b=2ar+(-2ar)=0$$

$$\Downarrow$$

$$\frac{dz}{dx}(2ar+b)+a\frac{d^2z}{dx^2}=0 \Rightarrow a\frac{d^2z}{dx^2}=0$$

Auxiliary Function: Dealing with Similar Roots

$$a\frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$a\frac{d^2z}{dx^2} = 0 \Rightarrow \frac{d^2z}{dx^2} = 0$$

Auxiliary Function: Dealing with Similar Roots

$$a\frac{d^2z}{dx^2} = 0 \Rightarrow \frac{d^2z}{dx^2} = 0 \Rightarrow z = Ax + B$$

Auxiliary Function: Dealing with Similar Roots

$$a\frac{d^2z}{dx^2} = 0 \Rightarrow \frac{d^2z}{dx^2} = 0 \Rightarrow z = Ax + B$$

$$\Downarrow$$

$$y = ze^{rx} \Rightarrow y = (Ax + B)e^{rx}$$

Auxiliary Function: Dealing with Similar Roots

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

Auxiliary Function: Dealing with Similar Roots

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0 \Rightarrow 4m^2 + 4m + 1 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0 \Rightarrow 4m^2 + 4m + 1 = 0 \Rightarrow (2m+1)^2 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0 \Rightarrow 4m^2 + 4m + 1 = 0 \Rightarrow (2m+1)^2 = 0$$

$$\Downarrow$$

$$y = (Ax + B)e^{-\frac{1}{2}x}$$

Auxiliary Function: Dealing with Complex Roots

$$am^2 + bm + c = 0 \Rightarrow m = \alpha \pm \beta i$$

Auxiliary Function: Dealing with Complex Roots

$$am^2 + bm + c = 0 \Rightarrow m = \alpha \pm \beta i$$

$$\Downarrow$$

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x}$$

Auxiliary Function: Dealing with Complex Roots

$$am^2 + bm + c = 0 \Rightarrow m = \alpha \pm \beta i$$

$$\Downarrow$$

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x}$$

$$\Downarrow$$

$$e^{ix} = \cos x + i \sin x$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x}$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \Rightarrow y = Ae^{\alpha x}e^{\beta xi} + Be^{\alpha x}e^{-\beta xi}$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \Rightarrow y = Ae^{\alpha x}e^{\beta xi} + Be^{\alpha x}e^{-\beta xi}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\left[\cos(\beta x) + i\sin(\beta x)\right] + B\left[\cos(\beta x) - i\sin(\beta x)\right]\right\}$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \Rightarrow y = Ae^{\alpha x}e^{\beta x i} + Be^{\alpha x}e^{-\beta x i}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\left[\cos(\beta x) + i\sin(\beta x)\right] + B\left[\cos(\beta x) - i\sin(\beta x)\right]\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\cos(\beta x) + B\cos(\beta x) + Ai\sin(\beta x) - Bi\sin(\beta x)\right\}$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \Rightarrow y = Ae^{\alpha x}e^{\beta xi} + Be^{\alpha x}e^{-\beta xi}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\left[\cos(\beta x) + i\sin(\beta x)\right] + B\left[\cos(\beta x) - i\sin(\beta x)\right]\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\cos(\beta x) + B\cos(\beta x) + Ai\sin(\beta x) - Bi\sin(\beta x)\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{(A+B)\cos(\beta x) + (A-Bi)\sin(\beta x)\right\}$$

Auxiliary Function: Dealing with Complex Roots

$$y = Ae^{(\alpha+\beta i)x} + Be^{(\alpha-\beta i)x} \Rightarrow y = Ae^{\alpha x}e^{\beta x i} + Be^{\alpha x}e^{-\beta x i}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\left[\cos(\beta x) + i\sin(\beta x)\right] + B\left[\cos(\beta x) - i\sin(\beta x)\right]\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{A\cos(\beta x) + B\cos(\beta x) + Ai\sin(\beta x) - Bi\sin(\beta x)\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{(A+B)\cos(\beta x) + (A-Bi)\sin(\beta x)\right\}$$

$$\Rightarrow y = e^{\alpha x}\left\{P\cos(\beta x) + Q\sin(\beta x)\right\}$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0 \Rightarrow m^2 - 6m + 13 = 0$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0 \Rightarrow m^2 - 6m + 13 = 0 \Rightarrow m = 3 \pm 2i$$

Auxiliary Function: Dealing with Similar Roots

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0 \Rightarrow m^2 - 6m + 13 = 0 \Rightarrow m = 3 \pm 2i$$

$$\Downarrow$$

$$y = e^{3x}(P\cos 2x + Q\sin 2x)$$

Topics

1. Preliminaries
2. Solving Second Order ODEs
   - Homogenous Equations and the Auxiliary Function
   - Non-Homogenous Equations and the Particular Integral

Non-Homogenous Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Non-Homogenous Equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 4$$

$$4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 2x + 3x^2$$

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- Let $y = u(x)$ be a particular solution to $f(x)$:

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- Let $y = u(x)$ be a particular solution to $f(x)$:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x)$$

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- Let $y = u(x)$ be a particular solution to $f(x)$:

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x)$$

- u(x) is termed as the particular integral

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- Now, let $y = u(x) + v(x)$. Substituting to above ODE:

Complementary Function and Particular Integral

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

- Now, let $y = u(x) + v(x)$. Substituting to above ODE:

$$\Rightarrow a\frac{d^2(u+v)}{dx^2} + b\frac{d(u+v)}{dx} + c(u+v) = f(x)$$

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- The corresponding solution obtained from the auxiliary equation, v(x), is called the complementary function

Solving Non-Homogenous Second Order ODEs

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- Problem now lies with obtaining the particular integral.
  - But there are clues on how to solve for it, particularly on the form of $f(x)$.

Determining Particular Integrals

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$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 20$$

Determining Particular Integrals

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- $u(x) = \frac{20}{10} = 2$

  $$y = 2 + Ae^{-2x} + Be^{-5x}$$

Determining Particular Integrals

$$f(x) = k \Rightarrow u(x) = \frac{k}{c} \Rightarrow \frac{du}{dx} = 0 \Rightarrow \frac{d^2u}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 6$$

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- $u(x) = \frac{6}{-2} = -3$

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$$y = -3 + Ae^{2x} + Be^{-x}$$

Determining Particular Integrals

$$f(x) = px + q$$

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$$\Rightarrow m^2 - 11m + 28 = 0 \Rightarrow (m-4)(m-7) = 0 \Rightarrow Ae^{4x} + Be^{7x}$$

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- $v(x) = Ae^{4x} + Be^{7x}$
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$$\Rightarrow 28\alpha x = 84$$

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$$u(x) = 3x + 1$$

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$$y = 3x + 1 + Ae^{4x} + Be^{7x}$$

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$$f(x) = px + q \Rightarrow u(x) = \alpha x + \beta \Rightarrow \frac{du}{dx} = \alpha \Rightarrow \frac{d^2u}{dx^2} = 0$$

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = x + 1$$

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$$\Rightarrow \alpha = -1 \Rightarrow -2\alpha - \beta = 1 \Rightarrow \beta = 1$$

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$$u(x) = -x + 1$$

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$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = x + 1$$

- $v(x) = Ae^{x} + Be^{-\frac{1}{3}x}$
- $u(x) = -x + 1$

$$y = -x + 1 + Ae^{x} + Be^{-\frac{1}{3}x}$$

Determining Particular Integrals

$$f(x) = px^2+qx+r$$

Determining Particular Integrals

$$f(x) = px^2 + qx + r \Rightarrow u(x) = \alpha x^2 + \beta x + \gamma \Rightarrow \frac{du}{dx} = 2\alpha x + \beta \Rightarrow \frac{d^2u}{dx^2} = 2\alpha$$

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$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 4y = 4x^2 + 10x - 23$$

Determining Particular Integrals

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- $v(x) = Ae^{4x} + Be^{-\frac{1}{2}x}$

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- $v(x) = Ae^{4x} + Be^{-\frac{1}{2}x}$
- $u(x) = -x^2 + x + 3$

Determining Particular Integrals

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- $v(x) = Ae^{4x} + Be^{-\frac{1}{2}x}$
- $u(x) = -x^2 + x + 3$

$$y = -x^2 + x + 3 + Ae^{4x} + Be^{-\frac{1}{2}x}$$

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$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 4x^2 + 2x - 4$$

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- $v(x) = Ae^{-x} + Be^{-4x}$
- $u(x) = x^2 - 2x + 1$

Determining Particular Integrals

$$f(x) = px^2 + qx + r \Rightarrow u(x) = \alpha x^2 + \beta x + \gamma \Rightarrow \frac{du}{dx} = 2\alpha x + \beta \Rightarrow \frac{d^2u}{dx^2} = 2\alpha$$

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- $u(x) = x^2 - 2x + 1$

$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x}$$

Determining Particular Integrals

$$f(x) = p \sin kx + q \cos kx$$

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$$f(x) = p \sin kx + q \cos kx$$

$$\Rightarrow u(x) = \alpha \sin kx + \beta \cos kx \Rightarrow \frac{du}{dx} = \alpha k \cos kx - \beta k \sin kx$$

$$\Rightarrow \frac{d^2u}{dx^2} = -\alpha k^2 \sin kx - \beta k^2 \cos kx$$

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$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 8\cos 3x - 19 \sin 3x$$

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- $v(x) = e^x(A \cos x + B \sin x)$

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- $v(x) = e^x(A \cos x + B \sin x)$
- $u(x) = \sin 3x - 2 \cos 3x$

Determining Particular Integrals

$$f(x) = p \sin kx + q \cos kx$$

$$\Rightarrow u(x) = \alpha \sin kx + \beta \cos kx \Rightarrow \frac{du}{dx} = \alpha k \cos kx - \beta k \sin kx$$

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$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 8 \cos 3x - 19 \sin 3x$$

- $v(x) = e^x(A \cos x + B \sin x)$
- $u(x) = \sin 3x - 2 \cos 3x$

$$y = \sin 3x - 2 \cos 3x + e^x(A \cos x + B \sin x)$$

Determining Particular Integrals

$$f(x) = p\sin kx + q\cos kx$$

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- $u(x) = 27e^{-x}$

$$y = 27e^{-x} + (Ax + B)e^{-\frac{3}{2}x}$$

Solving Non-Homogenous Equations: Notes

- If f(x) is a sum of 2 or more types of functions, then particular integrals for each type may be added together towards an overall particular integral.

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$$y = 3x - 8 + 2\cos x + \sin x + Ae^{-\frac{1}{2}x} + Be^{-\frac{3}{2}x}$$

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$$u(x) = f(x) \Rightarrow 4\alpha e^{2x} - 3(2\alpha e^{2x}) + 2(\alpha e^{2x}) = e^{2x}$$

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$$\Rightarrow 4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} = e^{2x} \Rightarrow 0 = e^{2x}(!!!!)$$

Solving Non-Homogenous Equations: Notes

- Note that what were discussed are forms of **"trial"** solutions - which means they might not work at times!
  - In this case, multiply an extra x to u(x)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$$

$$u(x) = \alpha x e^{kx} \Rightarrow \frac{du}{dx} = \alpha(kxe^{kx} + e^{kx}) \Rightarrow \frac{d^2u}{dx^2} = \alpha(k^2xe^{kx} + 2ke^{kx})$$

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$$u(x) = \alpha x e^{2x} \Rightarrow \frac{du}{dx} = \alpha(2xe^{2x} + e^{2x}) \Rightarrow \frac{d^2u}{dx^2} = \alpha(4xe^{2x} + 4e^{2x})$$

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$$u(x) = \alpha x e^{2x} \Rightarrow \frac{du}{dx} = \alpha(2xe^{2x} + e^{2x}) \Rightarrow \frac{d^2u}{dx^2} = \alpha(4xe^{2x} + 4e^{2x})$$

$$u(x) = f(x) \Rightarrow \alpha(4xe^{2x} + 4e^{2x}) - 3(\alpha(2xe^{2x} + e^{2x})) + 2(\alpha x e^{2x}) = e^{2x}$$

END OF LESSON 8