Eigenvectors and Eigenvalues

CS 130 - Mathematical Methods in Computer Science

Jan Michael C. Yap

Department of Computer Science
University of the Philippines Diliman

Topics

What does a matrix do to a vector?

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- A "transforms" x by rotating and scaling the vector.

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    - General notation: λ is an is an eigenvalue of A and x is the associated eigenvector for the eigenvalue λ.

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- Characteristic equation: $|(\lambda I - A)| = 0$

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$$\lambda I - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 11 & 2 \\ -9 & 0 \end{pmatrix} = \begin{pmatrix} \lambda - 11 & -2 \\ 9 & \lambda \end{pmatrix}$$

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$$|\lambda I - A| = 0 \Rightarrow \lambda^2 - 11\lambda + 18 = 0$$

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$$\lambda = 9, \lambda = 2$$

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- For each eigenvalue, solve for the associated eigenvector by solving the homogenous system $(\lambda I - A)x = 0$
  $\lambda = 9$:

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$$x = \begin{pmatrix} c \\ -c \end{pmatrix}, c \in \mathbb{R}$$

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  $$\lambda = 2 : x = \begin{pmatrix} -\frac{2c}{9} \\ c \end{pmatrix}, c \in \mathbb{R}$$

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  - $\prod_{i=1}^{n} \lambda_i = det(A)$

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Diagonalization

$$A = \begin{pmatrix} 11 & 2 \\ -9 & 0 \end{pmatrix}, P = \begin{pmatrix} -1 & 2 \\ 1 & -9 \end{pmatrix}, P^{-1} = \begin{pmatrix} -\frac{9}{7} & -\frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} \end{pmatrix}$$

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$$D = P^{-1}AP = \begin{pmatrix} -\frac{9}{7} & -\frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 11 & 2 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -9 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 2 \end{pmatrix}$$

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  - The columns of P (the matrix used for diagonalization) are the n linearly independent eigenvectors

END OF LESSON 4