Ordinary Differential Equations: Simultaneous Equations

CS 130 - Mathematical Methods in Computer Science

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Topics

Simulataneous Ordinary Differential Equations

- System of first order linear differential equations with two dependent variables

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- System of first order linear differential equations with two dependent variables
- Solving simultaneous equations would involve solving second order linear ODEs

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  - Substitution method

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- System of first order linear differential equations with two dependent variables
- Solving simultaneous equations would involve solving second order linear ODEs
  - More basic and linear algebra and a bit of differentiation.
- Two methods
  - Substitution method
  - Matrix method

Topics

1. Solving Simultaneous ODEs
2. Substitution Method
3. Matrix Method
    - Homogenous System of ODEs
    - Non-Homogenous System of ODEs

System with first order linear ODEs

$$\alpha_1 \frac{dy}{dx} + \beta_1 \frac{dz}{dx} + \gamma_1 y + \delta_1 z = f_1(x)$$

$$\alpha_2 \frac{dy}{dx} + \beta_2 \frac{dz}{dx} + \gamma_2 y + \delta_2 z = f_2(x)$$

$$\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$$

Substitution Method

- If both equations have two derivative terms, eliminate one derivative term
  - If one equation has only one derivative term, use that equation for the next step

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  - If one equation has only one derivative term, use that equation for the next step
- Express one dependent variable as an equation of the remaining terms
- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable
- Perform back substitution

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$
$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

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$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$
$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

$$\Downarrow$$

$$-9\frac{dy}{dx} - 12y + 6z = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$
$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Express one dependent variable as an equation of the remaining terms

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Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$
$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Express one dependent variable as an equation of the remaining terms

$$-9\frac{dy}{dx} - 12y + 6z = 0 \Rightarrow z = \frac{3}{2}\frac{dy}{dx} + 2y$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$z = \frac{3}{2}\frac{dy}{dx} + 2y$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$
$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$z = \frac{3}{2}\frac{dy}{dx} + 2y \Rightarrow \frac{dy}{dx} + 4\frac{d\left(\frac{3}{2}\frac{dy}{dx} + 2y\right)}{dx} + 6\left(\frac{3}{2}\frac{dy}{dx} + 2y\right) = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

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$$\Rightarrow \frac{dy}{dx} + 6\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 9\frac{dy}{dx} + 12y = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$z = \frac{3}{2}\frac{dy}{dx} + 2y \Rightarrow \frac{dy}{dx} + 4\frac{d\left(\frac{3}{2}\frac{dy}{dx} + 2y\right)}{dx} + 6\left(\frac{3}{2}\frac{dy}{dx} + 2y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} + 6\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 9\frac{dy}{dx} + 12y = 0 \Rightarrow 6\frac{d^2y}{dx^2} + 18\frac{dy}{dx} + 12y = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$6\frac{d^2y}{dx^2} + 18\frac{dy}{dx} + 12y = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$6\frac{d^2y}{dx^2}+18\frac{dy}{dx}+12y = 0 \Rightarrow 6m^2+18m+12 = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$6\frac{d^2y}{dx^2}+18\frac{dy}{dx}+12y = 0 \Rightarrow 6m^2+18m+12 = 0 \Rightarrow (m+1)(m+2) = 0$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$6\frac{d^2y}{dx^2}+18\frac{dy}{dx}+12y = 0 \Rightarrow 6m^2+18m+12 = 0 \Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow y = Ae^{-x} + Be^{-2x}$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

- Perform back substitution

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- Perform back substitution

$$y = Ae^{-x} + Be^{-2x} \Rightarrow \frac{dy}{dx} = -Ae^{-x} - 2Be^{-2x}$$

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$$y = Ae^{-x} + Be^{-2x} \Rightarrow \frac{dy}{dx} = -Ae^{-x} - 2Be^{-2x}$$

$$z = \frac{3}{2}\frac{dy}{dx} + 2y \Rightarrow z = -\frac{3}{2}Ae^{-x} - 3Be^{-2x} + 2Ae^{-x} + 4Be^{-2x}$$

Substitution Method

$$\frac{dy}{dx} + 4\frac{dz}{dx} + 6z = 0$$

$$5\frac{dy}{dx} + 2\frac{dz}{dx} + 6y = 0$$

$$\Downarrow$$

$$y = Ae^{-x} + Be^{-2x}$$

$$z = -\frac{3}{2}Ae^{-x} - 3Be^{-2x} + 2Ae^{-x} + 4Be^{-2x}$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$

$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$
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- If one equation has only one derivative term, use that equation for the next step

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$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

- Express one dependent variable as an equation of the remaining terms

$$\frac{dy}{dx} + 8y - 3z = 5e^{-x} \Rightarrow z = \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$

$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$z = \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)$$

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- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$z = \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)$$

$$\Rightarrow 5\frac{dy}{dx} - 2\frac{d\left(\frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)\right)}{dx} + 4y - \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right) = e^{-x}$$

Substitution Method

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$$\Rightarrow \frac{d^2y}{dx} + \frac{dy}{dx} - 2y = -4e^{-x}$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$
$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

- Perform substitution of variables on one equation, then solve the resulting differential equation for the other variable

$$5\frac{dy}{dx} - 2\frac{d\left(\frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)\right)}{dx} + 4y - \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right) = e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx} + \frac{dy}{dx} - 2y = -4e^{-x} \Rightarrow y = 2e^{-x} + Ae^{x} + Be^{-2x}$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$
$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

- Perform back substitution
$$z = \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right)$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$
$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

- Perform back substitution

$$z = \frac{1}{3}\left(\frac{dy}{dx} + 8y - 5e^{-x}\right) \Rightarrow z = 3e^{-x} + 3Ae^{x} + 2Be^{-2x}$$

Substitution Method

$$5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x}$$

$$\frac{dy}{dx} + 8y - 3z = 5e^{-x}$$

$$\Downarrow$$

$$y = 2e^{-x} + Ae^{x} + Be^{-2x}$$

$$z = 3e^{-x} + 3Ae^{x} + 2Be^{-2x}$$

Topics

System of two simultaneous ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

System of two simultaneous ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

Topics

Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

$$\Downarrow$$

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Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{dY}{dx} = MY$$

Solving A Homogenous System of ODEs

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$$\Rightarrow Y = Ke^{\lambda x}, K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, k_1, k_2, \lambda \in \mathbb{R}$$

Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{dY}{dx} = MY$$

$$\Rightarrow Y = Ke^{\lambda x}, K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, k_1, k_2, \lambda \in \mathbb{R} \Rightarrow \frac{dY}{dx} = \lambda K e^{\lambda x}$$

Solving A Homogenous System of ODEs

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$$\lambda K e^{\lambda x} = MY$$

Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{dY}{dx} = MY$$

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$$\lambda K e^{\lambda x} = MY \Rightarrow \lambda K e^{\lambda x} = MKe^{\lambda x}$$

Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{dY}{dx} = MY$$

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$$\lambda Ke^{\lambda x} = MY \Rightarrow \lambda Ke^{\lambda x} = MKe^{\lambda x} \Rightarrow \lambda K = MK$$

Solving A Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \frac{dY}{dx} = MY$$

$$\Rightarrow Y = Ke^{\lambda x}, K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, k_1, k_2, \lambda \in \mathbb{R} \Rightarrow \frac{dY}{dx} = \lambda Ke^{\lambda x}$$

$$\lambda Ke^{\lambda x} = MY \Rightarrow \lambda Ke^{\lambda x} = MKe^{\lambda x} \Rightarrow \lambda K = MK$$

- ZOMG! Solving eigenvalues and eigenvectors (again)!

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

$$\Downarrow$$

$$|M - \lambda I| = 0$$

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

$$\Downarrow$$

$$|M - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} \alpha_1 - \lambda & \beta_1 \\ \alpha_2 & \beta_2 - \lambda \end{pmatrix} \right| = 0$$

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

$$\Downarrow$$

$$|M - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} \alpha_1 - \lambda & \beta_1 \\ \alpha_2 & \beta_2 - \lambda \end{pmatrix} \right| = 0$$

- Quadratic equation in λ

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

$$\Downarrow$$

$$|M - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} \alpha_1 - \lambda & \beta_1 \\ \alpha_2 & \beta_2 - \lambda \end{pmatrix} \right| = 0$$

- Quadratic equation in λ
- Three possible forms of the roots
  - $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$
  - $\lambda = \lambda \in \mathbb{R}$
  - $\lambda = p \pm qi, p, q \in \mathbb{R}, i = \sqrt{-1}$

Solving A Homogenous System of ODEs

$$\lambda K = MK$$

$$\Downarrow$$

$$|M - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} \alpha_1 - \lambda & \beta_1 \\ \alpha_2 & \beta_2 - \lambda \end{pmatrix} \right| = 0$$

- Quadratic equation in λ
- Three possible forms of the roots
  - $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$
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Solving A Homogenous System of ODEs

- Next, solve for K

Solving A Homogenous System of ODEs

- Next, solve for K

$$MK = \lambda K \Rightarrow MK - \lambda K = 0 \Rightarrow (M - \lambda I)K = 0$$

Solving A Homogenous System of ODEs

- Next, solve for K

$$MK = \lambda K \Rightarrow MK - \lambda K = 0 \Rightarrow (M - \lambda I)K = 0$$

  - Simplification: let k_1 in K $= 1$ *always*

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

Solving A Homogenous System of ODEs

- Next, solve for K

$$MK = \lambda K \Rightarrow MK - \lambda K = 0 \Rightarrow (M - \lambda I)K = 0$$

  - Simplification: let k_1 in K $= 1$ *always*

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$$

  - Just down to solving k!

Solving A Homogenous System of ODEs: Templates

- Let A and B be arbitrary constants:
  - $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$

$$Y = A \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} e^{\lambda_1 x} + B \begin{pmatrix} 1 \\ k_{\lambda_2} \end{pmatrix} e^{\lambda_2 x}$$

Solving A Homogenous System of ODEs: Templates

- Let A and B be arbitrary constants:
  - $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$

$$Y = A \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} e^{\lambda_1 x} + B \begin{pmatrix} 1 \\ k_{\lambda_2} \end{pmatrix} e^{\lambda_2 x}$$

  - $\lambda = \lambda \in \mathbb{R}$

$$Y = \left\{ (Ax + B) \begin{pmatrix} 1 \\ k \end{pmatrix} + \frac{A}{\beta_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} e^{\lambda x}$$

Solving A Homogenous System of ODEs: Templates

- Let A and B be arbitrary constants:
  - $\lambda = \lambda_1, \lambda_2 \in \mathbb{R}$

$$Y = A\begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} e^{\lambda_1 x} + B\begin{pmatrix} 1 \\ k_{\lambda_2} \end{pmatrix} e^{\lambda_2 x}$$

  - $\lambda = \lambda \in \mathbb{R}$

$$Y = \left\{(Ax+B)\begin{pmatrix} 1 \\ k \end{pmatrix} + \frac{A}{\beta_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} e^{\lambda x}$$

  - $\lambda = p \pm qi, p, q \in \mathbb{R}, i = \sqrt{-1}$

$$Y = e^{px}\left\{\cos(qx)\begin{pmatrix} A \\ mA+nB \end{pmatrix} + \sin(qx)\begin{pmatrix} B \\ mB-nA \end{pmatrix}\right\},$$

$$k = m \pm ni$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$$\Downarrow$$

$$M = \begin{pmatrix} -4 & 5 \\ -1 & 2 \end{pmatrix}$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$$\Downarrow$$

$$M = \begin{pmatrix} -4 & 5 \\ -1 & 2 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (-4-\lambda) & 5 \\ -1 & (2-\lambda) \end{pmatrix} \right| = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$$\Downarrow$$

$$M = \begin{pmatrix} -4 & 5 \\ -1 & 2 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (-4-\lambda) & 5 \\ -1 & (2-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$$\Downarrow$$

$$M = \begin{pmatrix} -4 & 5 \\ -1 & 2 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (-4-\lambda) & 5 \\ -1 & (2-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

$$\Rightarrow \lambda_1 = -3, \lambda_2 = 1$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_1 = -3 :$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_1 = -3$:

$$(M - \lambda_1 I)K = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_1 = -3$:

$$(M - \lambda_1 I)K = 0 \Rightarrow \begin{pmatrix} (-4-\lambda_1) & 5 \\ -1 & (2-\lambda_1) \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_1 = -3:$

$$(M - \lambda_1 I)K = 0 \Rightarrow \begin{pmatrix} (-4-\lambda_1) & 5 \\ -1 & (2-\lambda_1) \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 5 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_1 = -3:$

$$(M - \lambda_1 I)K = 0 \Rightarrow \begin{pmatrix} (-4-\lambda_1) & 5 \\ -1 & (2-\lambda_1) \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 5 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_1} \end{pmatrix} = 0 \Rightarrow k_{\lambda_1} = \frac{1}{5}$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_2 = 1:$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_2 = 1:$

$$(M - \lambda_2 I)K = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_2 = 1$:

$$(M - \lambda_2 I)K = 0 \Rightarrow \begin{pmatrix} -5 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_2} \end{pmatrix} = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$\lambda_2 = 1$:

$$(M - \lambda_2 I)K = 0 \Rightarrow \begin{pmatrix} -5 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k_{\lambda_2} \end{pmatrix} = 0 \Rightarrow k_{\lambda_2} = 1$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

Solving A Homogenous System of ODEs

$$\frac{dy}{dx} = -4y + 5z$$
$$\frac{dz}{dx} = -y + 2z$$

$$\Downarrow$$

$$\begin{pmatrix} y \\ z \end{pmatrix} = A \begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix} e^{-3x} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{x}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -1 \\ 1 & (3-\lambda) \end{pmatrix} \right| = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -1 \\ 1 & (3-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -1 \\ 1 & (3-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$

$$\frac{dy_2}{dx} = y_1 + 3y_2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$(M - \lambda I)K = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$

$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$(M - \lambda I)K = 0 \Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$

$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$(M - \lambda I)K = 0 \Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 0 \Rightarrow k = -1$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - y_2$$
$$\frac{dy_2}{dx} = y_1 + 3y_2$$

$$\Downarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left\{ (Ax + B) \begin{pmatrix} 1 \\ -1 \end{pmatrix} - A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} e^{2x}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$

$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -5 \\ 2 & (3-\lambda) \end{pmatrix} \right| = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -5 \\ 2 & (3-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 - 4\lambda + 13 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$$\Downarrow$$

$$M = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix} \Rightarrow \left| \begin{pmatrix} (1-\lambda) & -5 \\ 2 & (3-\lambda) \end{pmatrix} \right| = 0 \Rightarrow \lambda^2 - 4\lambda + 13 = 0$$

$$\Rightarrow \lambda = 2 \pm 3i$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 + 3i$:

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 + 3i$:

$(M - \lambda_2 I)K = 0$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 + 3i$:

$$(M - \lambda_2 I)K = 0 \Rightarrow \begin{pmatrix} -1-3i & -5 \\ 2 & 1-3i \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 + 3i$:

$$(M - \lambda_2 I)K = 0 \Rightarrow \begin{pmatrix} -1-3i & -5 \\ 2 & 1-3i \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 0 \Rightarrow k = \frac{-1-3i}{5}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 - 3i$:

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$

$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$\lambda = 2 - 3i$:

$$(M - \lambda_2 I)K = 0 \Rightarrow \begin{pmatrix} -1+3i & -5 \\ 2 & 1+3i \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = 0 \Rightarrow k = \frac{-1+3i}{5}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_1 - 5y_2$$
$$\frac{dy_2}{dx} = 2y_1 + 3y_2$$

$$\Downarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{2x}\left\{\cos(3x)\begin{pmatrix} A \\ \frac{-A+3B}{5} \end{pmatrix} + \sin(3x)\begin{pmatrix} B \\ \frac{-B-3A}{5} \end{pmatrix}\right\}$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$

$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$
$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

- Eliminating $\frac{dy_1}{dx}$:

$$3\frac{dy_2}{dx} - 9y_1 - 3y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$
$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

- Eliminating $\frac{dy_1}{dx}$:

$$3\frac{dy_2}{dx} - 9y_1 - 3y_2 = 0 \Rightarrow \frac{dy_2}{dx} = 3y_1 + y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$
$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

- Eliminating $\frac{dy_1}{dx}$:

$$3\frac{dy_2}{dx} - 9y_1 - 3y_2 = 0 \Rightarrow \frac{dy_2}{dx} = 3y_1 + y_2 = 0$$

- Eliminating $\frac{dy_2}{dx}$:

$$3\frac{dy_1}{dx} - 3y_1 - 9y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$
$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

- Eliminating $\frac{dy_1}{dx}$:

$$3\frac{dy_2}{dx} - 9y_1 - 3y_2 = 0 \Rightarrow \frac{dy_2}{dx} = 3y_1 + y_2 = 0$$

- Eliminating $\frac{dy_2}{dx}$:

$$3\frac{dy_1}{dx} - 3y_1 - 9y_2 = 0 \Rightarrow \frac{dy_1}{dx} = y_1 + 3y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$

$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

Solving A Homogenous System of ODEs

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} + 2y_1 - 2y_2 = 0$$

$$\frac{dy_1}{dx} + 2\frac{dy_2}{dx} - 7y_1 - 5y_2 = 0$$

$$\Downarrow$$

$$\frac{dy_1}{dx} = y_1 + 3y_2 = 0$$

$$\frac{dy_2}{dx} = 3y_1 + y_2 = 0$$

Topics

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

- At least one of either f_1 or f_2 is not zero.

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

- At least one of either f_1 or f_2 is not zero.
- As with second order ODEs, the general solution is the sum of a complementary function and a particular integral.

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

- At least one of either f_1 or f_2 is not zero.
- As with second order ODEs, the general solution is the sum of a complementary function and a particular integral.
    - The complementary function (as always) is solved by making the system homogenous, then solving it using the technique discussed in previous section.

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = \alpha_1 y_1 + \beta_1 y_2 + f_1(x)$$
$$\frac{dy_2}{dx} = \alpha_2 y_1 + \beta_2 y_2 + f_2(x)$$

- At least one of either f_1 or f_2 is not zero.
- As with second order ODEs, the general solution is the sum of a complementary function and a particular integral.
  - The complementary function (as always) is solved by making the system homogenous, then solving it using the technique discussed in previous section.
  - The trial form of particular integrals is based on f_1 and/or f_2.

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -4y_1 - 5y_2 + f(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -4y_1 - 5y_2 + f(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -4y_1 - 5y_2 + f(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

Complementary Function: $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + B\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4x}$

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -4y_1 - 5y_2 + f(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -4y_1 - 5y_2 + f(x)$$

$$\Downarrow$$

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$\frac{dY}{dx} = MY + Nx$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$\frac{dY}{dx} = MY + Nx$$

Trial solution: $Y = P + Qx$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$\frac{dY}{dx} = MY + Nx$$

Trial solution: $Y = P + Qx \Rightarrow \dfrac{dY}{dx} = Q$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$\frac{dY}{dx} = MY + Nx$$

Trial solution: $Y = P + Qx \Rightarrow \frac{dY}{dx} = Q$

$$\frac{dY}{dx} = MY + Nx \Rightarrow Q = M(P + Qx) + Nx$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$\frac{dY}{dx} = MY + Nx$$

Trial solution: $Y = P + Qx \Rightarrow \frac{dY}{dx} = Q$

$$\frac{dY}{dx} = MY + Nx \Rightarrow Q = M(P + Qx) + Nx \Rightarrow Q = MP + MQx + Nx$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

  $$Q = MP + MQx + Nx$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

  $$Q = MP + MQx + Nx \Rightarrow Q = MP + (MQ + N)x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$Q = MP + MQx + Nx \Rightarrow Q = MP + (MQ + N)x$$

$$\Rightarrow Q = MP, MQ + N = \mathbf{0}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$MQ + N = \mathbf{0}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$MQ + N = \mathbf{0} \Rightarrow Q = -M^{-1}N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$MQ + N = \mathbf{0} \Rightarrow Q = -M^{-1}N \Rightarrow Q = -\begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$MQ + N = \mathbf{0} \Rightarrow Q = -M^{-1}N \Rightarrow Q = -\begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

  $$Q = MP$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$Q = MP \Rightarrow P = M^{-1}Q$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

  $$Q = MP \Rightarrow P = M^{-1}Q \Rightarrow P = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

$$Q = MP \Rightarrow P = M^{-1}Q \Rightarrow P = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} -\frac{5}{16} \\ \frac{1}{4} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

  Particular Integral: $\begin{pmatrix} -\frac{5}{16} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} x$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = x$

Particular Integral: $\begin{pmatrix} -\frac{5}{16} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} x$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + B\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4x} + \begin{pmatrix} -\frac{5}{16} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\frac{dY}{dx} = MY + Ne^{2x}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\frac{dY}{dx} = MY + Ne^{2x}$$

Trial solution: $Y = Pe^{2x}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\frac{dY}{dx} = MY + Ne^{2x}$$

Trial solution: $Y = Pe^{2x} \Rightarrow \frac{dY}{dx} = 2Pe^{2x}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\frac{dY}{dx} = MY + Ne^{2x}$$

$$\text{Trial solution: } Y = Pe^{2x} \Rightarrow \frac{dY}{dx} = 2Pe^{2x}$$

$$\frac{dY}{dx} = MY + Ne^{2x} \Rightarrow 2Pe^{2x} = MPe^{2x} + Ne^{2x}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\frac{dY}{dx} = MY + Ne^{2x}$$

$$\text{Trial solution: } Y = Pe^{2x} \Rightarrow \frac{dY}{dx} = 2Pe^{2x}$$

$$\frac{dY}{dx} = MY + Ne^{2x} \Rightarrow 2Pe^{2x} = MPe^{2x} + Ne^{2x} \Rightarrow 2P = MP + N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

  $2P = MP + N$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

  $$2P = MP + N \Rightarrow 2P - MP = N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

  $$2P = MP + N \Rightarrow 2P - MP = N \Rightarrow (2I - M)P = N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

  $$2P = MP + N \Rightarrow 2P - MP = N \Rightarrow (2I - M)P = N$$

  $$\Rightarrow P = (2I - M)^{-1}N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$2P = MP + N \Rightarrow 2P - MP = N \Rightarrow (2I - M)P = N$$

$$\Rightarrow P = (2I - M)^{-1}N \Rightarrow P = \begin{pmatrix} \frac{7}{18} & \frac{1}{18} \\ -\frac{4}{2} & \frac{2}{18} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$2P = MP + N \Rightarrow 2P - MP = N \Rightarrow (2I - M)P = N$$

$$\Rightarrow P = (2I - M)^{-1}N \Rightarrow P = \begin{pmatrix} \frac{7}{18} & \frac{1}{18} \\ -\frac{4}{2} & \frac{2}{18} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} \frac{1}{18} \\ \frac{2}{18} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{2x}$

$$\text{Particular Integral:} \quad \begin{pmatrix} \frac{1}{18} \\ \frac{2}{18} \end{pmatrix} e^{2x}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + B\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4x} + \begin{pmatrix} \frac{1}{18} \\ \frac{2}{18} \end{pmatrix} e^{2x}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$\frac{dY}{dx} = MY + N\sin x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$\frac{dY}{dx} = MY + N\sin x$$

Trial solution: $Y = P\sin x + Q\cos x$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$\frac{dY}{dx} = MY + N\sin x$$

Trial solution: $Y = P\sin x + Q\cos x \Rightarrow \frac{dY}{dx} = P\cos x - Q\sin x$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$\frac{dY}{dx} = MY + N\sin x$$

Trial solution: $Y = P\sin x + Q\cos x \Rightarrow \dfrac{dY}{dx} = P\cos x - Q\sin x$

$$\frac{dY}{dx} = MY + N\sin x \Rightarrow P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

$$\Rightarrow P\cos x - Q\sin x = MQ\cos x + (MP + N)\sin x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

$$\Rightarrow P\cos x - Q\sin x = MQ\cos x + (MP + N)\sin x$$

$$\Rightarrow P = MQ, -Q = MP+N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

$$\Rightarrow P\cos x - Q\sin x = MQ\cos x + (MP + N)\sin x$$

$$\Rightarrow P = MQ, -Q = MP + N \Rightarrow -Q = M(MQ) + N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P\cos x - Q\sin x = M(P\sin x + Q\cos x) + N\sin x$$

$$\Rightarrow P\cos x - Q\sin x = MQ\cos x + (MP + N)\sin x$$

$$\Rightarrow P = MQ, -Q = MP+N \Rightarrow -Q = M(MQ)+N \Rightarrow -Q = M^2Q+N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

  $-Q = M^2Q + N$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

  $-Q = M^2Q + N \Rightarrow M^2Q + Q = -N$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

  $-Q = M^2 Q + N \Rightarrow M^2 Q + Q = -N \Rightarrow (M^2 + I)Q = -N$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$-Q = M^2Q + N \Rightarrow M^2Q + Q = -N \Rightarrow (M^2 + I)Q = -N$$

$$\Rightarrow Q = -(M^2 + I)^{-1}N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$-Q = M^2Q + N \Rightarrow M^2Q + Q = -N \Rightarrow (M^2 + I)Q = -N$$

$$\Rightarrow Q = -(M^2 + I)^{-1}N \Rightarrow Q = -\begin{pmatrix} -\frac{22}{34} & -\frac{5}{34} \\ \frac{20}{34} & \frac{3}{34} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

  $$-Q = M^2Q + N \Rightarrow M^2Q + Q = -N \Rightarrow (M^2 + I)Q = -N$$

  $$\Rightarrow Q = -(M^2 + I)^{-1}N \Rightarrow Q = -\begin{pmatrix} -\frac{22}{34} & -\frac{5}{34} \\ \frac{20}{34} & \frac{3}{34} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  $$\Rightarrow Q = \begin{pmatrix} -\frac{5}{34} \\ \frac{3}{34} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P = MQ$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P = MQ \Rightarrow P = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} -\frac{5}{34} \\ \frac{3}{34} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

$$P = MQ \Rightarrow P = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} -\frac{5}{34} \\ \frac{3}{34} \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} \frac{3}{34} \\ \frac{5}{34} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = \sin x$

Particular Integral: $\begin{pmatrix} \frac{3}{34} \\ \frac{5}{34} \end{pmatrix} \sin x + \begin{pmatrix} -\frac{5}{34} \\ \frac{3}{34} \end{pmatrix} \cos x$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + B\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4x} +$$

$$\begin{pmatrix} \frac{3}{34} \\ \frac{5}{34} \end{pmatrix} \sin x + \begin{pmatrix} -\frac{5}{34} \\ \frac{3}{34} \end{pmatrix} \cos x$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

$$\frac{dY}{dx} = MY + Ne^{-x}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

$$\frac{dY}{dx} = MY + Ne^{-x}$$

Trial solution: $Y = (P + Qx)e^{-x}$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

$$\frac{dY}{dx} = MY + Ne^{-x}$$

Trial solution: $Y = (P + Qx)e^{-x} \Rightarrow \dfrac{dY}{dx} = Qe^{-x} - (P + Qx)e^{-x}$

Solving Non-Homogenous System of ODEs

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Trial solution: $Y = (P + Qx)e^{-x} \Rightarrow \frac{dY}{dx} = Qe^{-x} - (P + Qx)e^{-x}$

$$\frac{dY}{dx} = MY + Ne^{-x} \Rightarrow Qe^{-x} - (P+Qx)e^{-x} = M((P+Qx)e^{-x}) + Ne^{-x}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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$$\frac{dY}{dx} = MY + Ne^{-x} \Rightarrow Qe^{-x} - (P + Qx)e^{-x} = M((P + Qx)e^{-x}) + Ne^{-x}$$

$$\Rightarrow -Q = MQ, Q - P = MP + N$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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- $f(x) = e^{-x}$

$$-Q = MQ \Rightarrow Q = \begin{pmatrix} k \\ -k \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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Solving Non-Homogenous System of ODEs

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Solving Non-Homogenous System of ODEs

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Solving Non-Homogenous System of ODEs

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Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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- $f(x) = e^{-x}$

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Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

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$$\Rightarrow Q = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

$$k = \frac{1}{3}$$

$$\Rightarrow Q = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} r \\ \frac{1}{3} - r \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

$$\Rightarrow \frac{dY}{dx} = MY + Nf(x)$$

- $f(x) = e^{-x}$

$$k = \frac{1}{3}$$

$$\Rightarrow Q = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} r \\ \frac{1}{3} - r \end{pmatrix} \Rightarrow r = 0$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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$$\Rightarrow P = \begin{pmatrix} r \\ \frac{1}{3} - r \end{pmatrix} \Rightarrow r = 0 \Rightarrow P = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$$

Solving Non-Homogenous System of ODEs

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x)$$

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Particular Integral: $\left(\begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} x\right) e^{-x}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x} + B\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4x} + \left(\begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} x\right) e^{-x}$$

Solving Non-Homogenous System of ODEs: Final Note

- If f_1 and f_2 are both non-zero functions:

Solving Non-Homogenous System of ODEs: Final Note

- If f_1 and f_2 are both non-zero functions:

$$\frac{dY}{dx} = MY + N_1 f_1(x) + N_2 f_2(x)$$

$$N_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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$$\text{Trial Solution: } X = \text{Trial Solution}_{f_1} + \text{Trial Solution}_{f_2}$$

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$$f_1(x) = x, f_2(x) = e^{2x}$$

Solving Non-Homogenous System of ODEs: Final Note

- If f_1 and f_2 are both non-zero functions:

$$\frac{dY}{dx} = MY + N_1 f_1(x) + N_2 f_2(x)$$

$$N_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Trial Solution: $X = \text{Trial Solution}_{f_1} + \text{Trial Solution}_{f_2}$

$$f_1(x) = x, f_2(x) = e^{2x} \Rightarrow X = P + Qx + Re^{2x}$$

END OF LESSON 9