Assignment 6

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***Abstract*— This Document demonstrate a method to find the foot perpendicular from a given point to a given plane using Singular Value Decomposition.**

Download all python codes from https://github.com/cs19resch11004/hari
Download all Latex-tikz codes from https://github.com/cs19resch11004/hari

I. Problem

Write the equation of a plane through the point A $(-3, 4, -1)$ and perpendicular to the line

$$\frac{x+2}{-3} = \frac{y-2}{1} = \frac{z-4}{2} \tag{1}$$

II. Solution

Let the equation of plane is

$$ax + by + cz + d = 0 \tag{2}$$

Direction ratio of the line (1) is given as

$$D = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \tag{3}$$

Now let consider

$$A = \begin{pmatrix} -3 & 4 & -1 \end{pmatrix} \tag{4}$$

Since plane is passing through the point A (-3, 4, -1) and perpendicular to the line (1), hence

$$AD + d = 0 \tag{5}$$
$$\implies d = -11 \tag{6}$$

Hence equation of the plane is

$$-3x + y + 2z - 11 = 0 \tag{7}$$
$$\implies -3x + y + 2z = 11 \tag{8}$$

equation (8) can written as :

$$\begin{pmatrix} -3 & 1 & 2 \end{pmatrix} x = 11 \tag{9}$$

For finding the foot of perpendicular from point $D(-2, 2, 4)$ to the plane $\begin{pmatrix} -3 & 1 & 2 \end{pmatrix} x = 11$ using SVD, we need to represent the equation of plane in parametric form,

$$Q = p + \alpha_1 m_1 + \alpha_2 m_2 \tag{10}$$

Here $p(-3, 4, -1)$ is a point on plane and m_1, m_2 are two vectors parallel to plane and hence $\perp$ to n. First we find orthogonal vectors m_1 and m_2 to the vector n. Let, $m = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$m^T n = 0$$
$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0$$
$$\implies -3a + b + 2c = 0 \tag{11}$$

Putting a=1 and b=0 we get,

$$m_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} \tag{12}$$

Putting a=0 and b=1 we get,

$$m_2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \tag{13}$$

Let Q be the point on plane with shortest distance to point D. Q can be expressed in (10) form as

$$Q = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \frac{-1}{2} \end{pmatrix} \tag{14}$$

Computation of Pseudo Inverse using SVD in order to determine the value of α_1 and α_2 :

$$\begin{pmatrix}-3\\4\\-1\end{pmatrix} + \alpha_1\begin{pmatrix}1\\0\\\frac{3}{2}\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\\frac{-1}{2}\end{pmatrix} = \begin{pmatrix}-2\\2\\4\end{pmatrix} \tag{15}$$

$$\alpha_1\begin{pmatrix}1\\0\\\frac{3}{2}\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\\frac{-1}{2}\end{pmatrix} = \begin{pmatrix}1\\-2\\5\end{pmatrix} \tag{16}$$

$$\begin{pmatrix}1 & 0\\0 & 1\\\frac{3}{2} & \frac{-1}{2}\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}1\\-2\\5\end{pmatrix} \tag{17}$$

$$Mx = b \tag{18}$$

Now, to solve (17), we perform Singular Value Decomposition on M as follows,

$$M = USV^T \tag{19}$$

Where the columns of V are the eigen vectors of M^TM ,the columns of U are the eigen vectors of MM^T and S is diagonal matrix of singular value of eigenvalues of M^TM.

$$M^TM = \begin{pmatrix}\frac{13}{4} & -\frac{3}{4}\\-\frac{3}{4} & \frac{5}{4}\end{pmatrix} \tag{20}$$

$$MM^T = \begin{pmatrix}1 & 0 & \frac{3}{2}\\0 & 1 & -\frac{1}{2}\\\frac{3}{2} & -\frac{1}{2} & \frac{5}{2}\end{pmatrix} \tag{21}$$

Putting(19) in (18) we get,

$$USV^Tx = b \tag{22}$$

$$\implies x = VS_+U^Tb \tag{23}$$

Where S_+ is Moore-Penrose Pseudo-Inverse of S.Now, calculating eigen value of MM^T,

$$\mid MM^T - \lambda I \mid = 0 \tag{24}$$

$$\begin{vmatrix}1-\lambda & 0 & \frac{3}{2}\\0 & 1-\lambda & \frac{-1}{2}\\\frac{3}{2} & \frac{-1}{2} & \frac{5}{2}-\lambda\end{vmatrix} = 0 \tag{25}$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{7}{2}) = 0 \tag{26}$$

Hence eigen values of MM^T are,

$$\lambda_1 = \frac{7}{2} \tag{27}$$

$$\lambda_2 = 1 \tag{28}$$

$$\lambda_3 = 0 \tag{29}$$

Hence the eigen vectors of MM^T are,

$$u_1 = \begin{pmatrix}\frac{3}{5}\\-\frac{1}{5}\\1\end{pmatrix}, u_2 = \begin{pmatrix}\frac{1}{3}\\1\\0\end{pmatrix}, u_3 = \begin{pmatrix}-\frac{3}{2}\\\frac{1}{2}\\1\end{pmatrix}, \tag{30}$$

Normalizing the eigen vectors we get,

$$u_1 = \begin{pmatrix}\frac{3}{\sqrt{35}}\\-\frac{1}{\sqrt{35}}\\\frac{5}{\sqrt{35}}\end{pmatrix}, u_2 = \begin{pmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\\0\end{pmatrix}, u_3 = \begin{pmatrix}-\frac{3}{\sqrt{14}}\\\frac{1}{\sqrt{14}}\\\frac{2}{\sqrt{14}}\end{pmatrix} \tag{31}$$

Hence we obtain U of (19) as follows,

$$U = \begin{pmatrix}\frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}}\\-\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}}\\\frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}}\end{pmatrix} \tag{32}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get S of (19) as follows,

$$S = \begin{pmatrix}\sqrt{\frac{7}{2}} & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{33}$$

Now, calculating eigen value of M^TM,

$$\mid M^TM - \lambda I \mid = 0 \tag{34}$$

$$\begin{vmatrix}\frac{13}{4}-\lambda & \frac{-3}{4}\\\frac{-3}{4} & \frac{5}{4}-\lambda\end{vmatrix} = 0 \tag{35}$$

$$\implies \lambda^2 - \frac{9}{2}\lambda + \frac{7}{2} = 0 \tag{36}$$

Hence eigen values of M^TM are,

$$\lambda_4 = \frac{7}{2} \tag{37}$$

$$\lambda_5 = 1 \tag{38}$$

Hence the eigen vectors of M^TM are,

$$v_1 = \begin{pmatrix}-3\\1\end{pmatrix}, v_2 = \begin{pmatrix}\frac{1}{3}\\1\end{pmatrix} \tag{39}$$

Normalizing the eigen vectors we get,

$$v_1 = \begin{pmatrix}-\frac{3}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\end{pmatrix}, v_2 = \begin{pmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{pmatrix} \tag{40}$$

Hence we obtain V of (19) as follows,

$$V = \begin{pmatrix}-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix} \tag{41}$$

Finally from (19) we get the Singualr Value Decomposition of M as follows,

$$M = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T \tag{42}$$

Now, Moore-Penrose Pseudo inverse of S is given by,

$$S_+ = \begin{pmatrix} \sqrt{\frac{2}{7}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{43}$$

From (23) we get,

$$x = VS_+U^Tb = \begin{pmatrix} \frac{29}{14} \\ \frac{-33}{14} \end{pmatrix} \tag{44}$$

Verifying the solution of (44) using,

$$M^TMx = M^Tb \tag{45}$$

Evaluating the R.H.S in (45) we get,

$$M^TMx = \begin{pmatrix} \frac{17}{2} \\ \frac{-9}{2} \end{pmatrix} \tag{46}$$

$$\implies \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} x = \begin{pmatrix} \frac{17}{2} \\ \frac{-9}{2} \end{pmatrix} \tag{47}$$

Solving the augmented matrix of (47) we get,

$$\begin{pmatrix} \frac{13}{4} & -\frac{3}{4} & \frac{17}{2} \\ -\frac{3}{4} & \frac{5}{4} & \frac{-9}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 4R_1]{R_2 \leftarrow 4R_2} \begin{pmatrix} 13 & -3 & 34 \\ -3 & 5 & -18 \end{pmatrix} \tag{48}$$

$$\xleftrightarrow{R_2 \leftarrow 13R_2} \begin{pmatrix} 13 & -3 & 34 \\ -39 & 65 & -234 \end{pmatrix} \tag{49}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 13 & -3 & 34 \\ 0 & 56 & -132 \end{pmatrix} \tag{50}$$

Hence, Solution of (45) is given by,

$$x = \begin{pmatrix} \frac{29}{14} \\ \frac{-33}{14} \end{pmatrix} \tag{51}$$

Comparing results of x from (44) and (51) we conclude that the solution is verified.

Thus, the point $\vec{Q}$ (foot of the perpendicular) can be obtained by putting values of α_1 and α_2 in (14):

$$Q = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} + \frac{29}{14} \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} + \frac{-33}{14} \begin{pmatrix} 0 \\ 1 \\ \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} \frac{-13}{14} \\ \frac{23}{14} \\ \frac{46}{14} \end{pmatrix} \tag{52}$$

$$\boxed{Q = \begin{pmatrix} \frac{-13}{14} \\ \frac{23}{14} \\ \frac{46}{14} \end{pmatrix}} \tag{53}$$

Foot of the perpenicular:

$$\boxed{Q = \begin{pmatrix} \frac{-13}{14} \\ \frac{23}{14} \\ \frac{46}{14} \end{pmatrix}} \tag{54}$$