Assignment 5

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***Abstract*—This assignment deals with QR decomposition .**

Download all python codes from https://github.com/cs19resch11004/hari
Download all Latex-tikz codes from https://github.com/cs19resch11004/hari

I. Problem

Perform QR decomposition on matrix V

$$V = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{1}$$

II. Solution

The columns of matrix V can be represented in $\vec{\alpha}$ and $\vec{\beta}$ as

$$\implies \vec{\alpha} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{2}$$

$$\vec{\beta} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{3}$$

For QR decomposition, matrix V can be expressed as

$$V = QR \tag{4}$$

where, Q and R are expressed as

$$Q = \begin{pmatrix} \vec{u_1} & \vec{u_2} \end{pmatrix} \tag{5}$$

$$R = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{6}$$

Note that R is an upper triangular matrix.

Now,we calculate

$$k_1 = \|\tilde{\alpha}\| = \frac{\sqrt{5}}{2} \tag{7}$$

$$\vec{u_1} = \frac{\vec{\alpha}}{k_1} = \frac{2}{\sqrt{5}} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{8}$$

$$r_1 = \frac{\vec{u_1}^T \vec{\beta}}{\|\tilde{\mathbf{u_1}}\|^2} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \frac{2}{\sqrt{5}} \tag{9}$$

$$\vec{u_2} = \frac{\vec{\beta} - r_1\vec{u_1}}{\|\vec{\beta} - r_1\vec{u_1}\|} \tag{10}$$

Consider

$$\vec{\beta} - r_1\vec{u_1} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} - \frac{2}{\sqrt{5}} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \tag{11}$$

$$\beta - r_1\vec{u_1} = \begin{pmatrix} \frac{-3}{10} \\ \frac{3}{5} \end{pmatrix} \tag{12}$$

$$\|\beta - r_1\vec{u_1}\| = \frac{2\sqrt{5}}{3} \tag{13}$$

Substitute equation (12), equation (13) in an equation (10), we get

$$\vec{u_2} = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{14}$$

$$k_2 = \vec{u_2}^T \beta = \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{15}$$

$$\implies k_2 = \frac{3}{2\sqrt{5}} \tag{16}$$

Therefore, from (5) and (6)

$$Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{17}$$

$$R = \begin{pmatrix} \frac{\sqrt{5}}{2} & \frac{2}{\sqrt{5}} \\ 0 & \frac{3}{2\sqrt{5}} \end{pmatrix} \tag{18}$$

Note that,

$$Q^TQ = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{19}$$

$$Q^TQ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{20}$$

Now matrix V can be written as (4)

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{2} & \frac{2}{\sqrt{5}} \\ 0 & \frac{3}{2\sqrt{5}} \end{pmatrix} \tag{21}$$