Assignment 4

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***Abstract*— This document presents the tracing of conic sections.**

Download all python codes from https://github.com/cs19resch11004/hari
Download all Latex-tikz codes from https://github.com/cs19resch11004/hari

I. Problem

Tracing of following equation

$$x^2 + xy + y^2 + x + y - 1 = 0 \tag{1}$$

II. solution

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2}$$

and can be expressed as

$$\vec{x}^T V \vec{x} + 2\vec{u}^T \vec{x} + f = 0 \tag{3}$$

where

$$V = V^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{4}$$

$$\vec{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{5}$$

Comparing equations (1) and (3) we get

$$V = V^T = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{6}$$

$$\vec{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{7}$$

$$f = -1 \tag{8}$$

Expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} > 0 \tag{9}$$

Hence from above result, given equation represents the Ellipse.

The characteristic equation of V is obtained by evaluating the determinant

$$\left| V - \lambda I \right| = 0 \tag{10}$$

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \tag{11}$$

$$(1-\lambda)(1-\lambda) - 1/4 = 0 \tag{12}$$

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \tag{13}$$

The eigenvector $\vec{p}$ is defined as

$$V\vec{p} = \lambda\vec{p} \tag{14}$$

$$\implies (V - \lambda I)\vec{p} = 0 \tag{15}$$

For $\lambda_1 = \frac{1}{2}$,

$$(V - \lambda_1 I) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{16}$$

By row reduction,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow 2R_2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{17}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{18}$$

Subsituting equation (18) in an equation (15) we get

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

Where, $\vec{p} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ Let $u_1 = t$

$$u_2 = -t \tag{20}$$

Eigen vector $\vec{p_1}$ is given by

$$\vec{p_1} = \begin{pmatrix} t \\ -t \end{pmatrix} \tag{21}$$

Let $t = 1$, we get

$$\vec{p_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{22}$$

For $\lambda_2 = \frac{3}{2}$,

$$(V - \lambda_2 I) = \begin{pmatrix} 1 - \frac{3}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & 1 - \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \tag{23}$$

By row reduction,

$$\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow 2R_2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{24}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \tag{25}$$

Subsituting equation (25) in an equation (15) we get

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

Where, $\vec{p} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ Let $u_1 = t$

$$u_2 = t \tag{27}$$

Eigen vector $\vec{p_2}$ is given by

$$\vec{p_2} = \begin{pmatrix} t \\ t \end{pmatrix} \tag{28}$$

Let $t = 1$, we get

$$\vec{p_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{29}$$

By eigen decompostion V can be represented by

$$V = PDP^T \tag{30}$$

where

$$P = \begin{pmatrix} \vec{p_1} & \vec{p_2} \end{pmatrix} \tag{31}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{32}$$

Substituting equations (22), (29) in equation (31) we get

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{33}$$

Substituting equation (13) in (32) we get

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \tag{34}$$

Centre of the ellipse is given by

$$\vec{C} = -V^{-1}\vec{u} \tag{35}$$

$$\implies \vec{C} = -\begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \\ \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{pmatrix} \tag{36}$$

$$\implies \vec{C} = \begin{pmatrix} \frac{-1}{3} \\ \\ \frac{-1}{3} \end{pmatrix} \tag{37}$$

Calculating the axes, we get

$$a = \sqrt{\frac{\vec{u}^T V^{-1} \vec{u} - f}{\lambda_1}} = 1.6329931618 \tag{38}$$

$$b = \sqrt{\frac{\vec{u}^T V^{-1} \vec{u} - f}{\lambda_2}} = 0.9428090416 \tag{39}$$

Standard ellipse can be written in the form:

$$\vec{y}^T D \vec{y} = \vec{u}^T V^{-1} \vec{u} - f \tag{40}$$

Simplifying we get:

$$\vec{u}^T V^{-1} \vec{u} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{3} \tag{41}$$

Substituting equation (41) in equation (40), then we have :

$$\vec{y}^T D \vec{y} = \frac{4}{3} \tag{42}$$

Substituting equation (34), in equation (42)

$$\vec{y}^T \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \vec{y} = \frac{4}{3} \tag{43}$$

$$\vec{y}^T \begin{pmatrix} \frac{8}{3} & 0 \\ 0 & \frac{8}{9} \end{pmatrix} \vec{y} = 1 \tag{44}$$

The following figure verifies the given equation (3) as ellipse with centre $\begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix}$

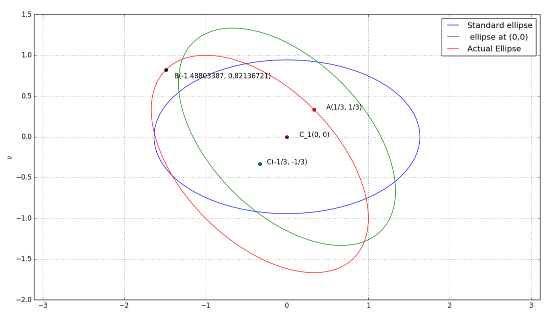

Fig. 1. Ellipse at C($\frac{-1}{3}$, $\frac{-1}{3}$), and Standard Ellipse