

Assignment 4

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Abstract— This document presents the tracing of conic sections.

Download all python codes from
<https://github.com/cs19resch11004/hari>
 Download all Latex-tikz codes from
<https://github.com/cs19resch11004/hari>

I. PROBLEM

Tracing of following equation

$$x^2 + xy + y^2 + x + y - 1 = 0 \quad (1)$$

II. SOLUTION

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2)$$

and can be expressed as

$$\vec{x}^T V \vec{x} + 2\vec{u}^T \vec{x} + f = 0 \quad (3)$$

where

$$V = V^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (4)$$

$$\vec{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (5)$$

Comparing equations 1 and 3 we get

$$V = V^T = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (6)$$

$$\vec{u} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (7)$$

$$f = -1 \quad (8)$$

Expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} > 0 \quad (9)$$

Hence from above result, given equation represents the Ellipse.

The characteristic equation of V is obtained by evaluating the determinant

$$|V - \lambda I| = 0 \quad (10)$$

$$\begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (11)$$

$$(1 - \lambda)(1 - \lambda) - 1/4 = 0 \quad (12)$$

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (13)$$

The eigenvector \vec{p} is defined as

$$V\vec{p} = \lambda\vec{p} \quad (14)$$

$$\Rightarrow (V - \lambda I)\vec{p} = 0 \quad (15)$$

For $\lambda_1 = \frac{1}{2}$,

$$(V - \lambda_1 I) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (16)$$

By row reduction,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow 2R_2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (17)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (18)$$

Substituting equation 18 in an equation 15 we get

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (19)$$

Where, $\vec{p} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ Let $u_1 = t$

$$u_2 = -t \quad (20)$$

Eigen vector \vec{p}_1 is given by

$$\vec{p}_1 = \begin{pmatrix} t \\ -t \end{pmatrix} \quad (21)$$

Let $t = 1$, we get

$$\vec{p}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (22)$$

For $\lambda_2 = \frac{3}{2}$,

$$(V - \lambda_2 I) = \begin{pmatrix} 1 - \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 - \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (23)$$

By row reduction,

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow 2R_2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad (24)$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad (25)$$

Substituting equation 25 in an equation 15 we get

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (26)$$

Where, $\vec{p} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ Let $u_1 = t$

$$u_2 = t \quad (27)$$

Eigen vector \vec{p}_2 is given by

$$\vec{p}_2 = \begin{pmatrix} t \\ t \end{pmatrix} \quad (28)$$

Let $t = 1$, we get

$$\vec{p}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (29)$$

By eigen decomposition V can be represented by

$$V = PDP^T \quad (30)$$

where

$$P = (\vec{p}_1 \quad \vec{p}_2) \quad (31)$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (32)$$

Substituting equations 22, 29 in equation 31 we get

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad (33)$$

Substituting equation 13 in 32 we get

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \quad (34)$$

Centre of the ellipse is given by

$$\vec{C} = -V^{-1}\vec{u} \quad (35)$$

$$\Rightarrow \vec{C} = - \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (36)$$

$$\Rightarrow \vec{C} = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix} \quad (37)$$

Calculating the axes, we get

$$a = \sqrt{\frac{\vec{u}^T V^{-1} \vec{u} - f}{\lambda_1}} = 1.6329931618 \quad (38)$$

$$b = \sqrt{\frac{\vec{u}^T V^{-1} \vec{u} - f}{\lambda_2}} = 0.9428090416 \quad (39)$$

Standard ellipse can be written in the form:

$$\vec{y}^T D \vec{y} = \vec{u}^T V^{-1} \vec{u} - f \quad (40)$$

Simplifying we get:

$$\vec{u}^T V^{-1} \vec{u} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{3} \quad (41)$$

Substituting equation 41 in equation 40, then we have :

$$\vec{y}^T D \vec{y} = \frac{4}{3} \quad (42)$$

Substituting equation 34, in equation (42)

$$\vec{y}^T \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \vec{y} = \frac{4}{3} \quad (43)$$

$$\vec{y}^T \begin{pmatrix} \frac{8}{3} & 0 \\ 0 & \frac{8}{3} \end{pmatrix} \vec{y} = 1 \quad (44)$$

The following figure verifies the given equation 3 as ellipse with centre $\begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix}$

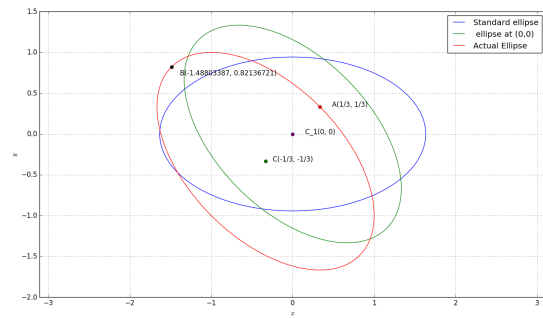


Fig. 1. Ellipse at $C(-\frac{1}{3}, -\frac{1}{3})$, and Standard Ellipse