

AI1103 : Assignment 5

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<https://github.com/cs20btech11007/Assignment-5/blob/main/Assignment%205.tex>

PROBLEM-(CSIR UGC NET EXAM) (DEC-2014),Q-116

Let Y follows multivariate normal distribution $N_n(0, I)$ and let A and B be a $n \times n$ symmetric, idempotent matrices. Then which of the following statements are true?

1. if $AB=0$, then $Y'AY$ and $Y'BY$ are independently distributed.
2. if $Y'(A+B)Y$ has chi square distribution then $Y'AY$, $Y'BY$ are independently distributed.
3. $Y'(A-B)Y$ has chi square distribution.
4. $Y'AY$, $Y'BY$ has chi square distribution.

SOLUTION

Lemma 0.1. $Y = (Y_1, Y_2, Y_3, \dots, Y_n)$ has multivariate normal distribution with mean $\mu = 0$ and variance-covariance matrix $\Sigma = I$.

The distributions $Y'AY$ and $Y'BY$ are independently distributed if and only if $A \Sigma B = 0$.

Lemma 0.2. Let $Y \sim N_k(\mu, \Sigma)$, $\Sigma > 0$ and A be a $n \times n$ real symmetric matrix.

Then $Y'AY \sim \chi_m^2(\lambda)$ with $\lambda = \frac{1}{2}\mu' A \mu$ if and only if $A \Sigma$ is idempotent of rank m .

We are proving this by proof by using moment-generating function of $\chi_m^2(\lambda)$

$$M_{\chi_m^2(\lambda)} = (1 - 2t)^{-\frac{m}{2}} e^{\frac{t\lambda}{1-2t}}, t < \frac{1}{2} \quad (0.0.1)$$

Proof. Sufficiency. Suppose $(A \Sigma)^2 = A \Sigma$ and $r(A \Sigma) = m$, where $\Sigma = KK'$ and $A_0 = K'AK$.

Then $A_0^2 = A_0$ and $r(A_0) = m$. Thus, there exists an orthogonal matrix P such that

$$A_0 = P \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} P' \quad (0.0.2)$$

where $P = (P_1, P_2)$, $P_1 P_1' = I_m$ and $z = P_1' K^{-1} Y$. It follows that

$$Y'AY = z'z \sim \chi_m^2(\lambda), \quad (0.0.3)$$

where $\lambda = \frac{1}{2}(P_1' K^{-1} Y)'(P_1' K^{-1} Y) = \frac{1}{2}\mu\mu'$

Necessity. Suppose $Y'AY \sim \chi_m^2(\lambda)$. Let $P = (P_1, \dots, P_k)$ be an orthogonal matrix such that $P'A_0P = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \dots \geq \lambda_k$ are eigenvalues of A_0 . Then

$$Y'AY = z'\Lambda z = \sum_{i=1}^k \lambda_i z_i^2 \quad (0.0.4)$$

$$M_{Y'AY}(t) = \prod_{i=1}^k M_{z_i^2}(t\lambda_i) \quad (0.0.5)$$

where $z = P'K^{-1}Y \sim N_k(P'K^{-1}\mu, I)$ and z_1, \dots, z_k are independent. Hence,

$$(1 - 2t)^{-\frac{m}{2}} e^{\frac{t\lambda}{1-2t}} = \prod_{i=1}^k (1 - 2t\lambda_i)^{-\frac{1}{2}} e^{\frac{t\lambda_i}{1-2t\lambda_i} (P_i' K^{-1} \mu)} \quad (0.0.6)$$

for $t < 1/2$ and $t\lambda_i < \frac{1}{2}$ ($i = 1, \dots, k$). Comparing the discontinuous points of the two functions on both sides results in

$$(1 - 2t)^{-\frac{m}{2}} = \prod_{i=1}^k (1 - 2t\lambda_i)^{-\frac{1}{2}} = |I_k - 2tA_0|^{-\frac{1}{2}} \quad (0.0.7)$$

which implies that $\lambda_1 = \dots = \lambda_m = 1$ and $\lambda_{m+1} = \dots = \lambda_k = 0$.

Thus, A_0 or $A \sum$ is idempotent of rank m .

The proof is completed.

□

Examining each option :

- 1) If $Y'AY$ and $Y'BY$ are independently distributed then $A \sum B = 0$.

we know that Y is a multi normalvariet distribution so the \sum is equal to I then, AB must be equal to 0.

so that the distributions $Y'AY$ and $Y'BY$ are independently distributed.

Hence, **Option 1 is correct.**

- 2) Given that $Y'(A+B)Y$ has chi square distribution then the matrix $\sum(A+B)$ must be an idempotent matrix.

$$(\sum(A+B))^2 = \sum(A+B) \quad (0.0.8)$$

We know that $\sum = I$ (identity matrix)

$$A^2 + B^2 + AB + BA = A + B \quad (0.0.9)$$

Given that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.10)$$

$$A + B + AB + BA = A + B \quad (0.0.11)$$

$$AB + BA = 0 \quad (0.0.12)$$

Then AB, BA must be equal to 0,

The distributions $Y'A_1Y$ and $Y'BY$ are independently distributed if $A \sum B = 0$.

We know that $\sum = I$ (identity matrix)

$$A \sum B = AB = 0 \quad (0.0.13)$$

clearly we can say that the distributions $Y'AY$ and $Y'BY$ are independently distributed .

Hence, **Option 2 is correct.**

- 3) For having chi square distribution for a normally distributed $Y'(A-B)Y$ distribution . Then the matrix $(\sum(A-B))$ must be an idempotent matrix.

$$(\sum(A-B))^2 = \sum(A-B) \quad (0.0.14)$$

We know that $\sum = I$ (identity matrix)

$$A^2 + B^2 - AB - BA = A - B \quad (0.0.15)$$

we know that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.16)$$

$$A + B - AB - BA = A - B \quad (0.0.17)$$

$$AB + BA = 2B \quad (0.0.18)$$

for having chi square distribution the above equation(11) need to be satisfied.

but it is not mentioned in the option 3.

so, we cannot say that the distribution $Y'(A-B)Y$ has chi square distribution.

Hence, **Option 3 is incorrect.**

4.) $Y = (Y_1, Y_2, Y_3, \dots, Y_n)$ has multivariate normal distribution with mean $\mu = 0$ and variance-covariance matrix $\sum = I$.

Then $Y'AY$ and $Y'BY$ has chi square distribution(χ^2) if and only if $\sum A$ and $\sum B$ are idempotent matrix.

we know that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.19)$$

We know that $\sum = I$ (identity matrix)

$$(\sum A)^2 = A, (\sum B)^2 = B \quad (0.0.20)$$

Then $Y'AY$ and $Y'BY$ has chi square distribution(χ^2).

Hence, **Option 4 is correct.**