# AI1103 : Assignment 5

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https://github.com/cs20btech11007/Assignment−5/blob/main/Assignment%205.tex

### Problem-(CSIR UGC NET EXAM) (Dec-2014),Q-116

Let Y follows multivariate normal distribution $N_n(0, I)$ and let A and B be a $n \times n$ symmetric,idempotent matrices. Then which of the following statements are true?

1.if AB=0 ,then Y'AY and Y'BY are independently distributed.

2.if Y'(A+B)Y has chi square distribution then Y'AY, Y'BY are independently distributed.

3.Y'(A-B)Y has chi square distribution.

4.Y'AY, Y'BY has chi square distribution.

### Solution

**Lemma 0.1.** *$Y = (Y_1, Y_2, Y_3....., Y_n)$ has multivariate normal distribution with mean $\mu = 0$ and variance-covariance matrix $\sum = I$.*
*The distributions Y'AY and Y'BY are independently distributed if and only $A \sum B$=0.*

**Lemma 0.2.** *Let $Y \sim N_k(\mu, \sum)$, $\sum$ ¿ 0 and A be a $n \times n$ real symmetric matrix.*
*Then $Y'AY \sim \chi^2_m(\lambda)$ with $\lambda = \frac{1}{2}\mu' A\mu$ if and only if $A \sum$ is idempotent of rank m.*

We are proving this by proof by using moment-generating function of $\chi^2_m(\lambda)$

$$M_{\chi^2_m(\lambda)} = (1 - 2t)^{\frac{-m}{2}} e^{\frac{2t\lambda}{1-2t}}, t < \frac{1}{2} \tag{0.0.1}$$

*Proof.* Suppose $(A \sum)^2 = A \sum$ and $r(A \sum) = m$, where $\sum = KK'$ and $A_0 = K'AK$. Then $A_0^2 = A_0$ and $r(A_0) = m$. Thus, there exists an orthogonal matrix P such that

$$A_0 = P \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} P' \tag{0.0.2}$$

where P = $(P_1, P_2)$, $P_1 P_1 = I_m$ and z = $P_1' K^{-1} Y$. It follows that

$$Y'AY = z'z \sim \chi^2_m(\lambda), \tag{0.0.3}$$

where $\lambda = \frac{1}{2}(P_1' K^{-1} Y)'(P_1' K^{-1} Y) = \frac{1}{2}\mu\mu'$

**Necessity**. Suppose $Y'AY \sim \chi^2_m(\lambda)$. Let $P = (P_1, ....., P_k)$ be an orthogonal matrix such that $P' A_0 P = \Lambda = diag(\lambda_1, ..., \lambda_k)$, where $\lambda_1 \geq ... \geq \lambda_k$ are eigenvalues of $A_0$. Then

$$Y'AY = z'\Lambda z = \sum_{i=1}^{k} \lambda_i z_i^2 \tag{0.0.4}$$

$$M_{Y'AY}(t) = \prod_{i=1}^{k} M_{z_i^2}(t\lambda_i) \tag{0.0.5}$$

where $z = P'K^{-1}Y \sim N_k(P'K^{-1}\mu, I)$ and $z_1, ..., z_k$ are independent. Hence,

$$(1 - 2t)^{\frac{-m}{2}} e^{\frac{2t\lambda}{1-2t}} = \prod_{i=1}^{k} (1 - 2t\lambda_i)^{\frac{-1}{2}} e^{\frac{t\lambda_i}{1-2t\lambda_i}(p_i' K^{-1}\mu)} \tag{0.0.6}$$

for t ¡ 1/2 and $t\lambda_i < \frac{1}{2}(i = 1, ..., k)$. Comparing the discontinuous points of the two functions on both sides results in

$$(1 - 2t)^{\frac{-m}{2}} = \prod_{i=1}^{k} (1 - 2t\lambda_i)^{\frac{-1}{2}} = |I_k - 2tA_0|^{\frac{-1}{2}} \tag{0.0.7}$$

which implies that $\lambda_1 = ... = \lambda_m = 1$ and $\lambda_{m+1} = .... = \lambda_k = 0$.

Thus, $A_0$ or $A\sum$ is idempotent of rank m.
**The proof is completed.**

□

**Examining each option :**

1) If Y'AY and Y'BY are independently distributed then $A\sum B = 0$.
we know that Y is a multi normalvariet distribution so the $\sum$ is equal to I then, $AB$ must be equal to 0.
so that the distributions Y'AY and Y'BY are independently distributed.

Hence, **Option 1 is correct.**

2) Given that Y'(A+B)Y has chi square distribution then the matrix $\sum(A+B)$ must be an idempotent matrix.

$$(\sum(A+B))^2 = \sum(A+B) \quad (0.0.8)$$

We know that $\sum = I$ (identity matrix)

$$A^2 + B^2 + AB + BA = A + B \quad (0.0.9)$$

Given that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.10)$$

$$\cancel{A} + \cancel{B} + AB + BA = \cancel{A} + \cancel{B} \quad (0.0.11)$$

$$AB + BA = 0 \quad (0.0.12)$$

Then AB,BA must be equal to 0,
The distributions Y'$A_1$Y and Y'$B$Y are independently distributed if $A\sum B$=0.

We know that $\sum = I$ (identity matrix)

$$A\sum B = AB = 0 \quad (0.0.13)$$

clearly we can say that the distributions Y'$A$Y and Y'$B$Y are independently distributed .

Hence, **Option 2 is correct.**

3) For having chi square distribution for a normally distributed Y'(A-B)Y distribution .
Then the matrix $(\sum(A - B)$ must be an idempotent matrix.

$$(\sum(A-B))^2 = \sum(A-B) \quad (0.0.14)$$

We know that $\sum = I$ (identity matrix)

$$A^2 + B^2 - AB - BA = A - B \quad (0.0.15)$$

we know that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.16)$$

$$\cancel{A} + B - AB - BA = \cancel{A} + -B \quad (0.0.17)$$

$$AB + BA = 2B \quad (0.0.18)$$

for having chi square distribution the above equation(11) need to be satisfied.
but it is not mentioned in the option 3.
so,we cannot say that the distribution $Y'(A-B)Y$ has chi square distribution.

Hence, **Option 3 is incorrect.**

4) $Y = (Y_1, Y_2, Y_3....., Y_n)$ has multivariate normal distribution with mean $\mu = 0$ and variance-covariance matrix $\sum = I$.
Then Y'AY and Y'BY has chi square distribution($\chi^2$) if and only if $\sum A$ and $\sum B$ are idempotent matrix.

we know that A and B are idempotent matrices.

$$A^2 = A, B^2 = B \quad (0.0.19)$$

We know that $\sum = I$ (identity matrix)

$$(\sum A)^2 = A, (\sum B)^2 = B \quad (0.0.20)$$

Then $Y'AY$ and $Y'BY$ has chi square distribution($\chi^2$).

Hence, **Option 4 is correct.**