CHEME 5820: Machine Learning for Engineers **Spring 2025**

Lecture 2c: Singular Value Decomposition (SVD of) Data and Systems

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

1 Introduction

In this lecture, we will discuss the singular value decomposition (SVD) of data and systems. The SVD is a fundamental matrix decomposition that is used in many areas of science and engineering. The SVD is a generalization of the eigenvalue decomposition and is used to analyze the structure of a matrix. The SVD is used in many applications, including data compression, image processing, and control theory.

What is the SVD?

Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$. The SVD of $\mathbf{A}$ is a factorization of the form: $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{n \times n}$ and $\mathbf{V} \in \mathbb{R}^{m \times m}$ are orthogonal matrices, i.e., $\mathbf{U} \cdot \mathbf{U}^T = \mathbf{I}$ and $\mathbf{\Sigma} \in \mathbb{R}^{n \times m}$ is a diagonal matrix containing the singular values $\sigma_i = \Sigma_{ii}$. The matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ can be decomposed as:

$$\mathbf{A} = \sum_{i=1}^{r_{\mathbf{A}}} \sigma_i \cdot (\mathbf{u}_i \otimes \mathbf{v}_i)$$

where $r_{\mathbf{A}}$ is the rank of matrix $\mathbf{A}$ and $\otimes$ denotes the outer product. The outer product $\hat{\mathbf{A}}_i = \mathbf{u}_i \otimes \mathbf{v}_i$ is a rank-1 matrix with elements:

$$\hat{a}_{jk} = u_j v_k \qquad j = 1, 2, \ldots, n \text{ and } k = 1, 2, \ldots, m$$

The vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are the left (right) singular vectors, and σ_i are the singular values (ordered).

Singular value decomposuition is a special sort of eigendecomposition, thus, we could use QR-itereation to compute the SVD. The columns of $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^T$, the columns of $\mathbf{V}$ are eigenvectors of $\mathbf{A}^T\mathbf{A}$ and the singular values σ_i are the square roots of the eigenvalues of the matrix products $\mathbf{A}\mathbf{A}^T$ or $\mathbf{A}^T\mathbf{A}$.

References