Amortized Analysis

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Today's music: "Money, Money, Money" by ABBA

CLICKER QUESTION 1

Review

Current topic: Efficiency

- Big Oh
- Hash tables (and mutability)

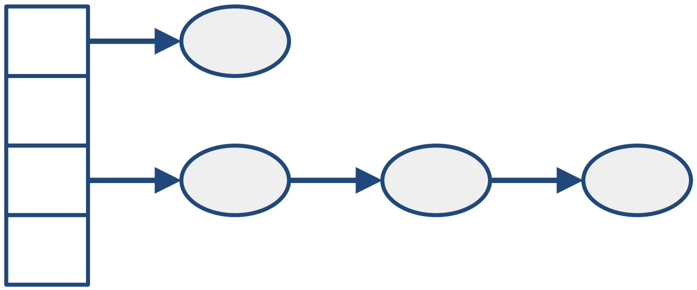
Today:

- Amortized analysis

REVIEW OF HASH TABLES

Hash table: chaining

```
type ('k, 'v) t = {
  mutable buckets
    : ('k * 'v) list array
}
```



Implementation of operations

- Insert (k, v):
  - Hash k to find bucket b
  - Search through b to delete any previous binding of k (to maintain RI)
  - Mutate bucket to add new binding of k
- Find k:
  - Hash k to find bucket b
  - Search through b to find binding of k
- Remove k:
  - Hash k to find bucket b
  - Search through b to delete any binding of k

…every operation requires search through bucket

…efficiency depends on bucket length

Load factor

Load factor = average bucket length = α
(# bindings in hash table) / (# buckets in array)

- # bindings not under implementer's control
- # buckets is
- When load factor gets above some constant, make array bigger
  - Which makes load factor smaller
  - Then redistribute keys across bigger array

CLICKER QUESTION 2

Rehashing

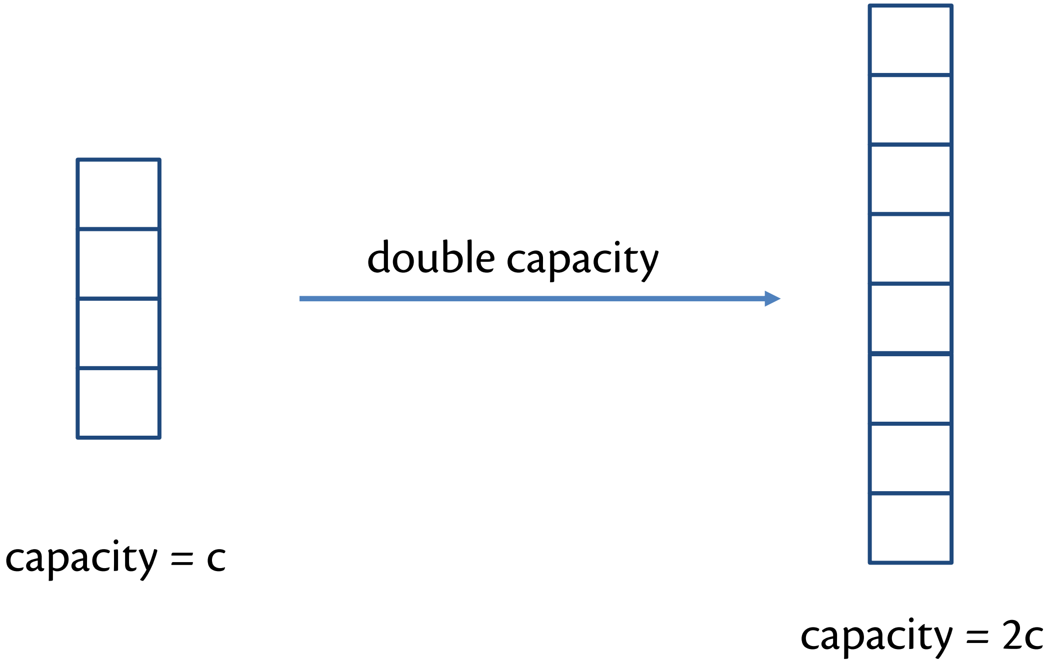
- If load factor $\geq$ 2.0 then:
  - double array size
  - rehash elements into new buckets
  - thus bringing load factor back to around 1.0

- Both OCaml **`Hashtbl`** and **`java.util.HashMap`** do this

- Efficiency:
  - find, and remove: expected O(1)
  - insert: O(n), because it can require rehashing all elements
  - but we wanted O(1)…

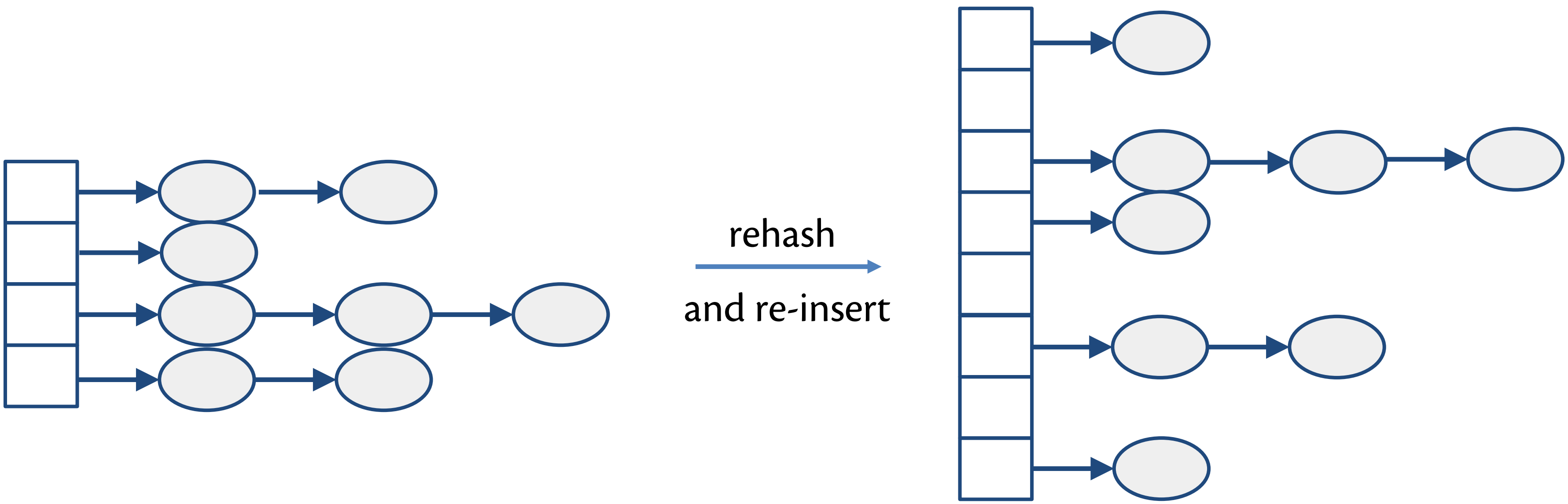
AMORTIZED ANALYSIS

Hash table resize



Cost = O(n) = O(2c)

Hash table resize



Expected cost = 2n O(1)= O(n)

Total cost to resize

Expected cost = O(n) + O(n) = O(n)

Suppose the hidden constant is r

- r = x + y + z
- x is cost to allocate
- y is cost to hash
- z is cost to insert

Let's call that \$r

Saving money

on insert

save \$r

on resize

spend \$r·n

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	$0

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	\$0
Insert 16 bindings			
16	32	2	\$16r

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	\$0
	Insert 16 bindings		
16	32	2	\$16r
	Resize and rehash		
32	32	1	-\$16r

Let’s double the amount we save: \$2rn

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	$0
Insert 16 bindings			
16	32	2	$32r

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	\$0
	Insert 16 bindings		
16	32	2	\$32r
	Resize and rehash		
32	32	1	\$0

Bank account balance

Capacity	Bindings	Load factor α	Balance
16	16	1	\$0
	Insert 16 bindings		
16	32	2	\$32r
	Resize and rehash		
32	32	1	\$0
	Insert 32 bindings		
32	64	2	\$64r
	Resize and rehash		
64	64	1	\$0

Budgeting

Mon

Tue

Wed

Thur

Fri

Budgeting

- Key idea is to analyze worst-case efficiency of
  - sequence of operations
  - not individual operations
- Rare expensive operations paid for by common inexpensive operations

Hash table efficiency

- find, and remove: expected O(1)
- insert: expected O(1), because rehashing can be paid for with amortization

TWO-LIST QUEUES

Two-list queues [lec 7]

abstract:

1	2	3	4	5	6	7	8

concrete:

front: (enqueued since front last emptied)

1	2	3

back: (recently enqueued)

8	7	6	5	4

Two-list queues: AF+RI

- Rep type:
  - front of queue: list, stored in order
  - back of queue: list, stored in reverse order
- RI: if front is empty then back is empty

Two-list queues: efficiency

- Peek: head of front O(1)
- Enqueue: cons onto back O(1)
  - But if completely empty, cons onto front instead to maintain RI O(1)
- Dequeue: tail of front O(1)
  - If front becomes empty, reverse back and make it the front to maintain RI O(n)

Amortized analysis

on enqueue (back)

save $1

on reverse

spend $n

Bank account balance

Front length	Back length	Balance
0	0	$0
Enqueue 1 element		
1	0	$0
Enqueue 9 elements		
1	9	$9
Dequeue 1 element		
0	9	$9
Reverse back and make it front		
9	0	$0
Dequeue 9 elements		
0	0	$0

KEY IDEAS OF AMORTIZED ANALYSIS

Amortized analysis

- *Amortize:* put aside money at intervals for gradual payment of debt [Webster's 1964]
- In efficiency analysis:
  - Pay extra "money" for some operations as a credit
  - Use that credit to pay higher cost of some later operations
  - a.k.a. *banker's method* and *accounting method*
- Invented by Sleator and Tarjan (1985)

Robert Tarjan

b. 1948

**Turing Award Winner (1986)
with Prof. John Hopcroft**

For fundamental achievements in the design and analysis of algorithms and data structures.

Cornell CS faculty 1972-1973

Upcoming events

- [last night] R6 due
- [Thur] A5 released
- [Fri] MS0 due – no late submissions

This is money.

THIS IS 3110