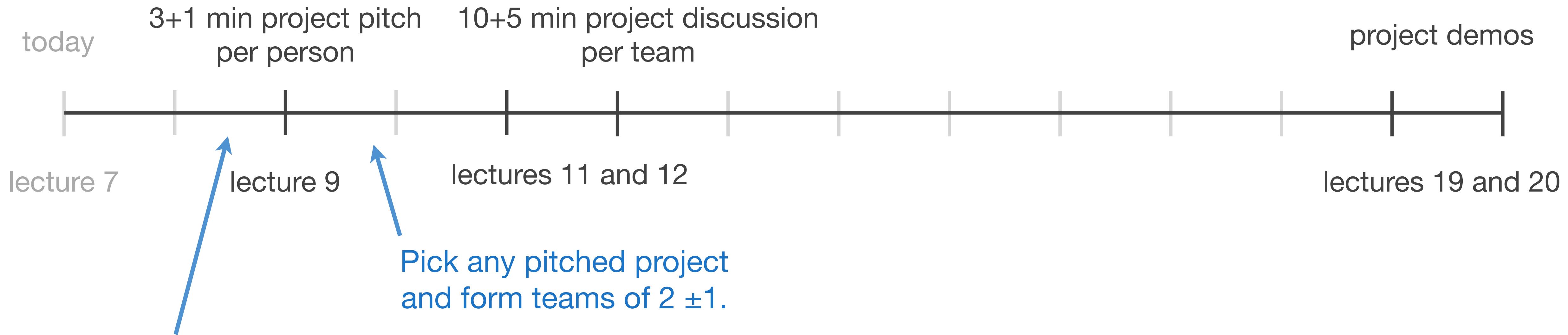


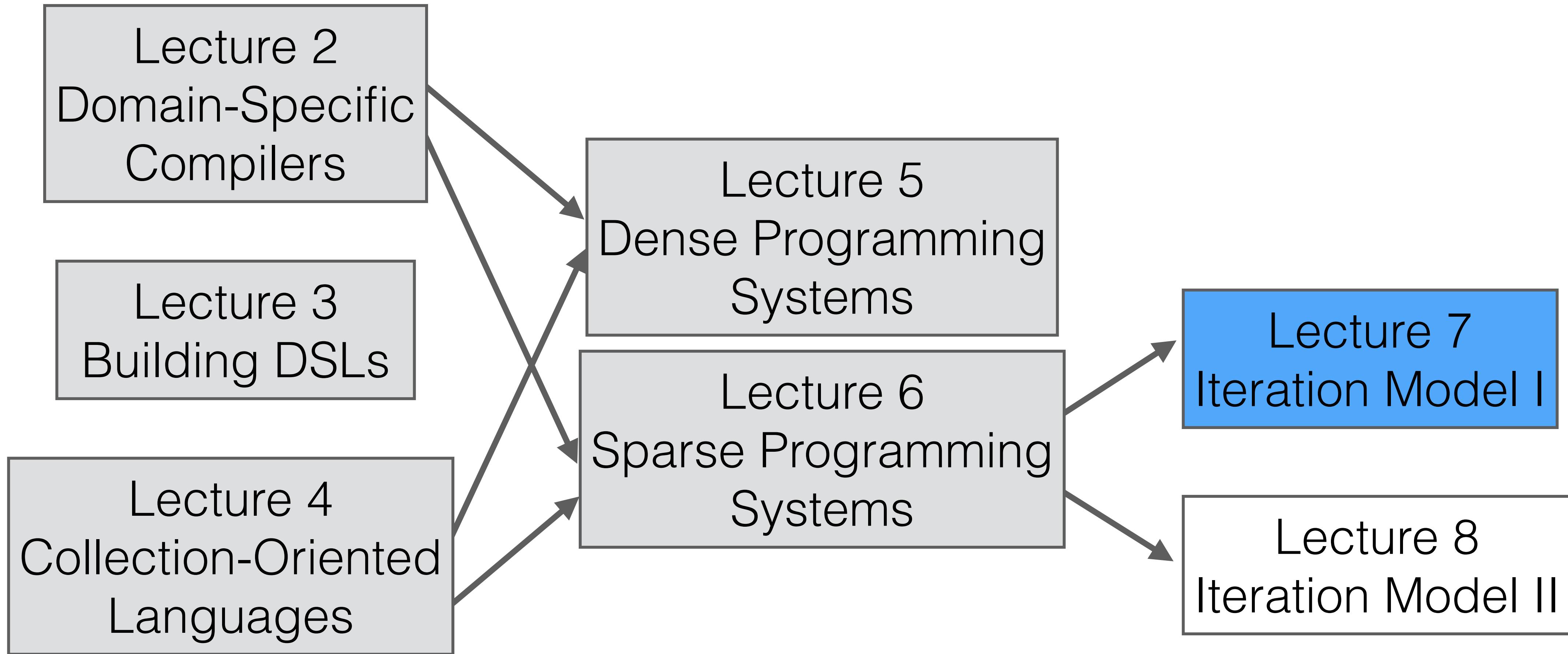
# Lecture 7 – Sparse Iteration Model I

Stanford CS343D (Winter 2025)  
Fred Kjolstad

# Course Project



Each person contributes one  
pitch slide to a google slide deck.  
These pitches are not binding.



# Overview of topics

## Lecture 7

- Data representation
- Iteration spaces
- Iteration graph IR
- Iteration lattices to represent coiteration

## Lecture 8

- Concrete index notation IR
- Code generation algorithm
- Derived iteration spaces
- Optimizing transformations

# Sparse Tensor Algebra Compilation

## Tensor Index Notation Expression

$$\begin{array}{lll} A = Bc + a & a = Bc \\ A = B \odot C & A = B + C & a = \alpha Bc + \beta a \\ A = BCd & A = \alpha B & A = 0 \quad A = BC \\ A_{ij} = \sum_{kl} B_{ikl}C_{lj}D_{kj} & A = B^T & a = B^T Bc \\ A_{ijk} = \sum_l B_{ikl}C_{lj} & A_{ik} = \sum_j B_{ijk}c_j & A_{kj} = \sum_{il} B_{ikl}C_{lj}D_{ij} \\ C = \sum_{ijkl} M_{ij}P_{jk}\overline{M_{lk}}\overline{P_{il}} & A_{ij} = (\sum_k B_{ijk}C_{ijk}) + D_{ij} \\ a = \sum_{ijklmnp} M_{ij}P_{jk}M_{kl}P_{lm}\overline{M_{nm}}P_{no}\overline{M_{po}}\overline{P_{ip}} \end{array}$$

Formats

Schedule

Sparse Tensor Algebra  
Compiler (taco)

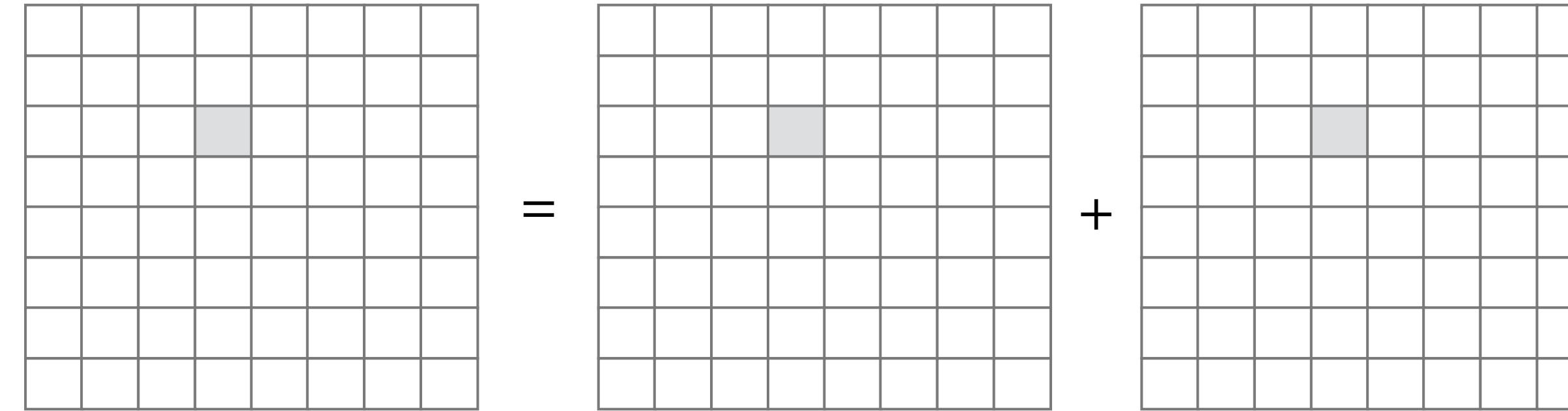


DSAs



# Tensor index notation for expressing functionality

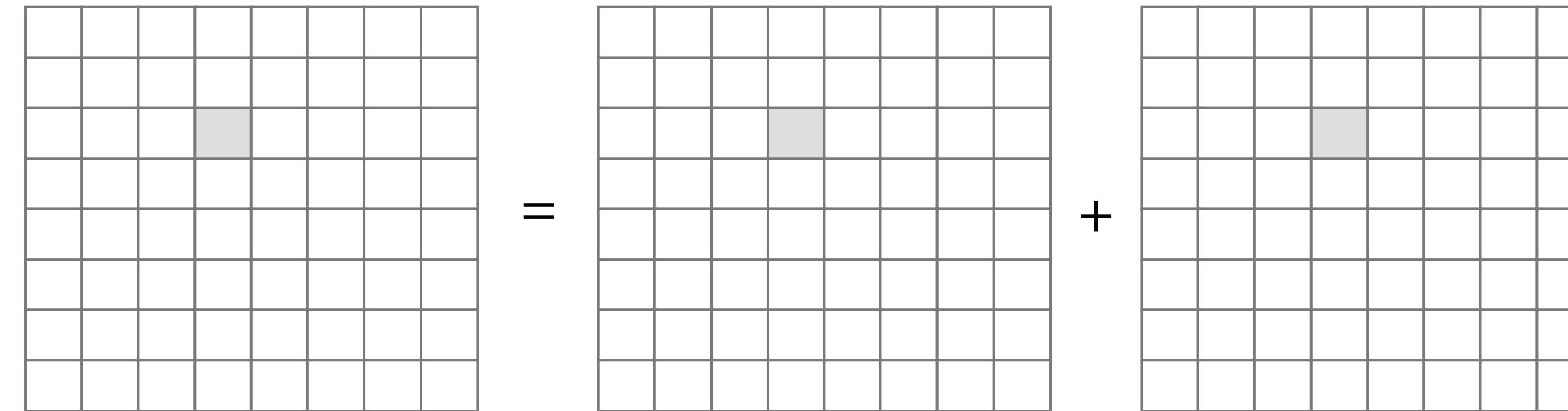
$$A_{ij} = B_{ij} + C_{ij}$$



element-wise

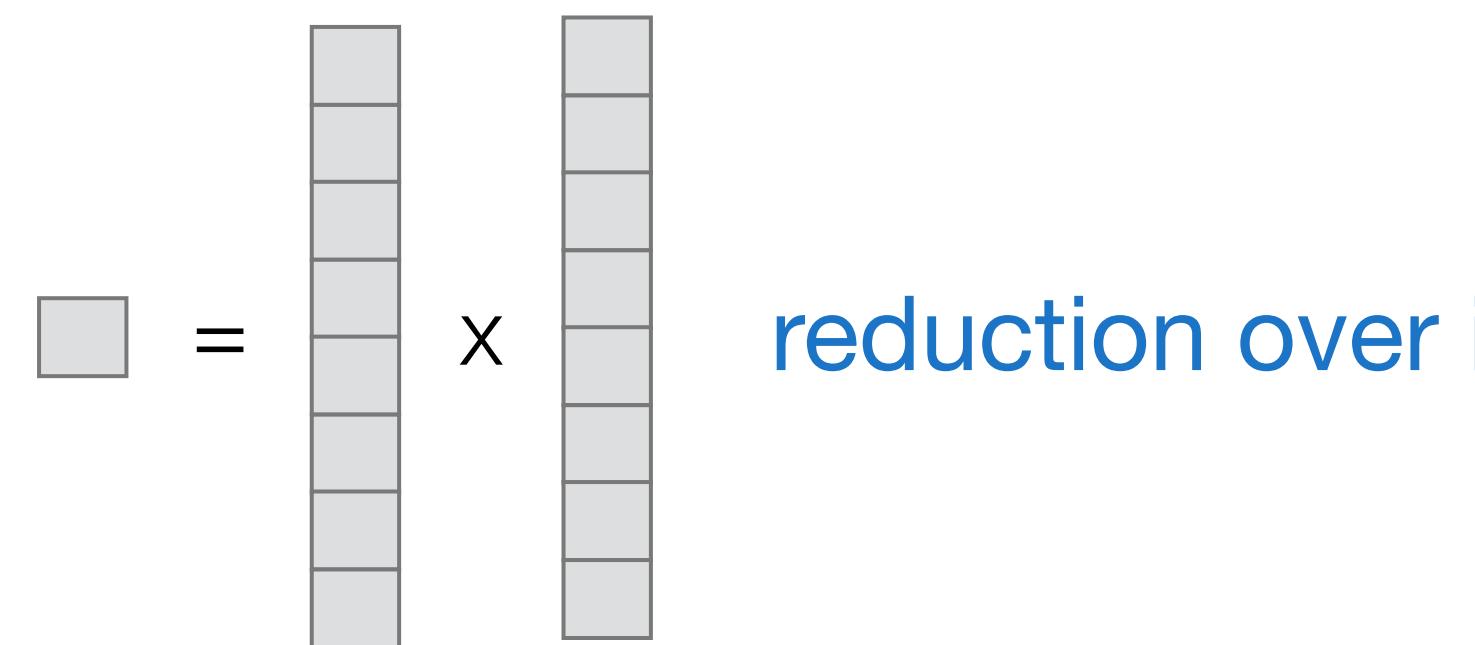
# Tensor index notation for expressing functionality

$$A_{ij} = B_{ij} + C_{ij}$$



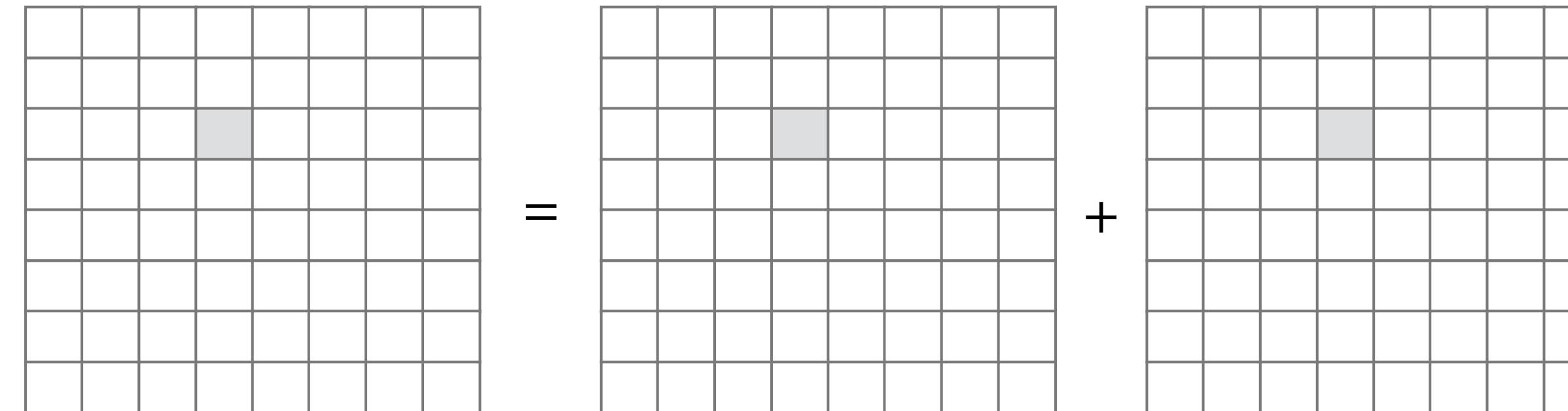
element-wise

$$\alpha = \sum_i b_i c_i$$



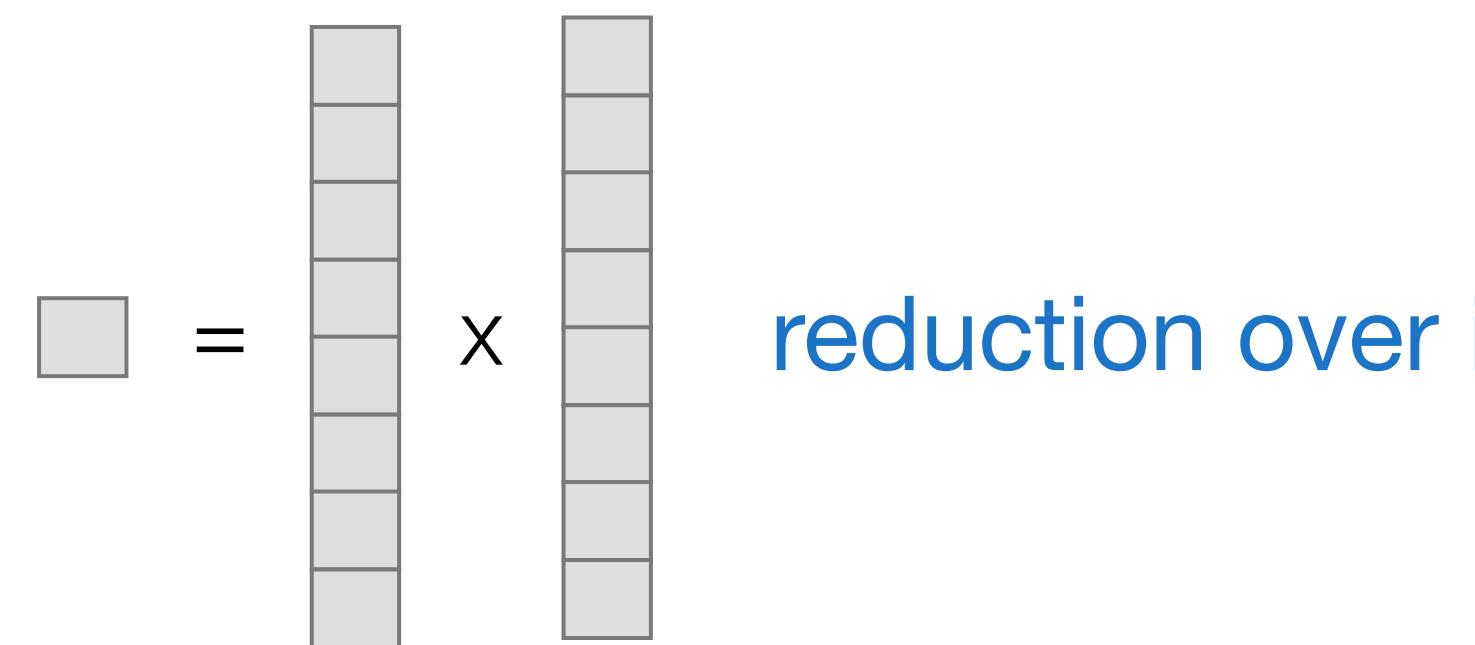
# Tensor index notation for expressing functionality

$$A_{ij} = B_{ij} + C_{ij}$$

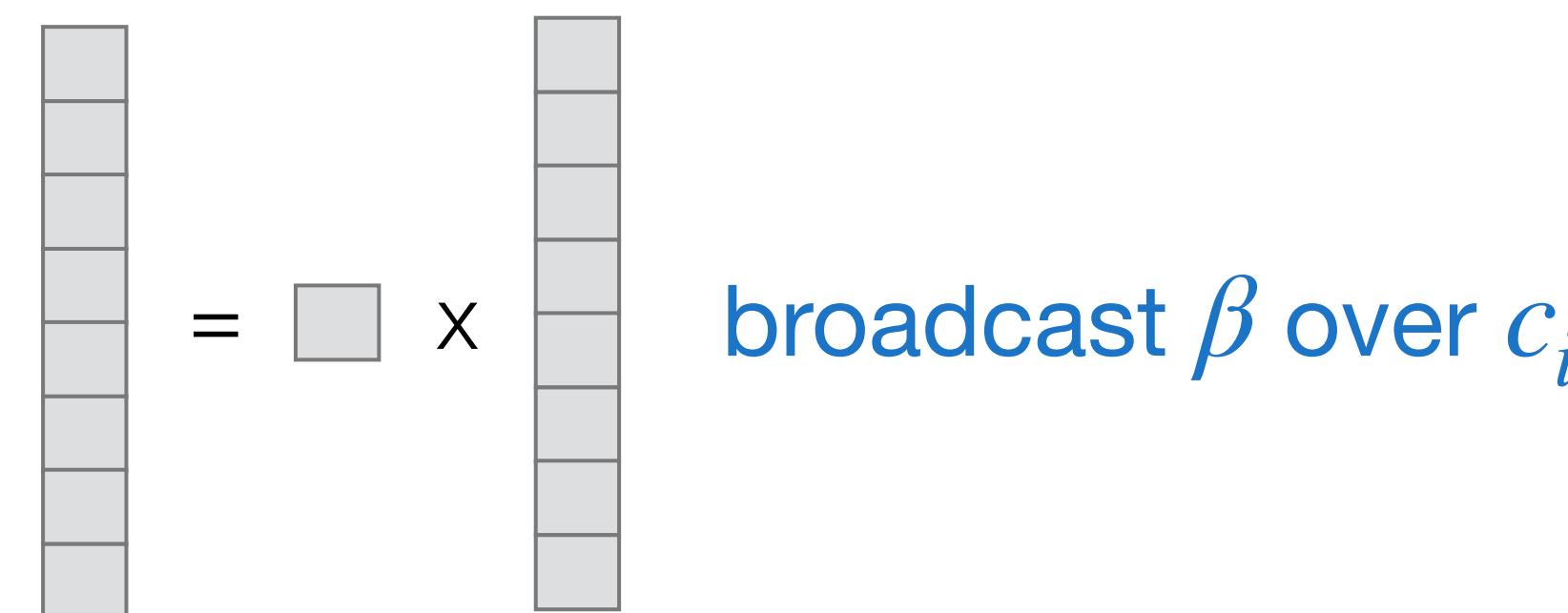


element-wise

$$\alpha = \sum_i b_i c_i$$

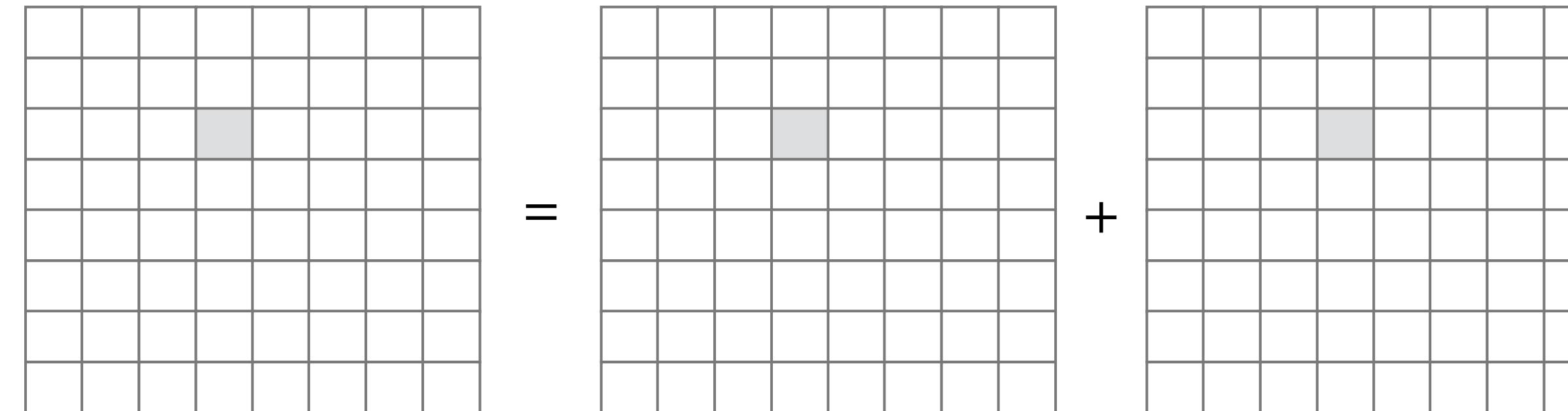


$$a_i = \beta c_i$$



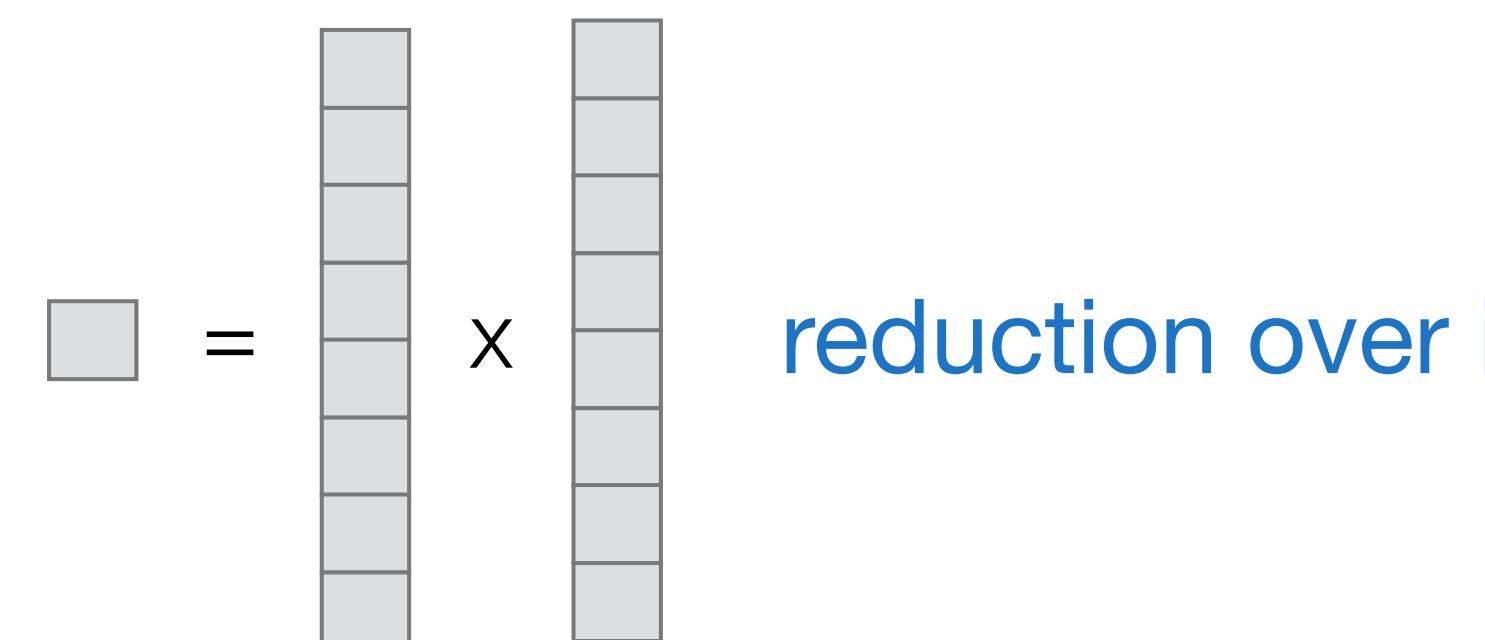
# Tensor index notation for expressing functionality

$$A_{ij} = B_{ij} + C_{ij}$$



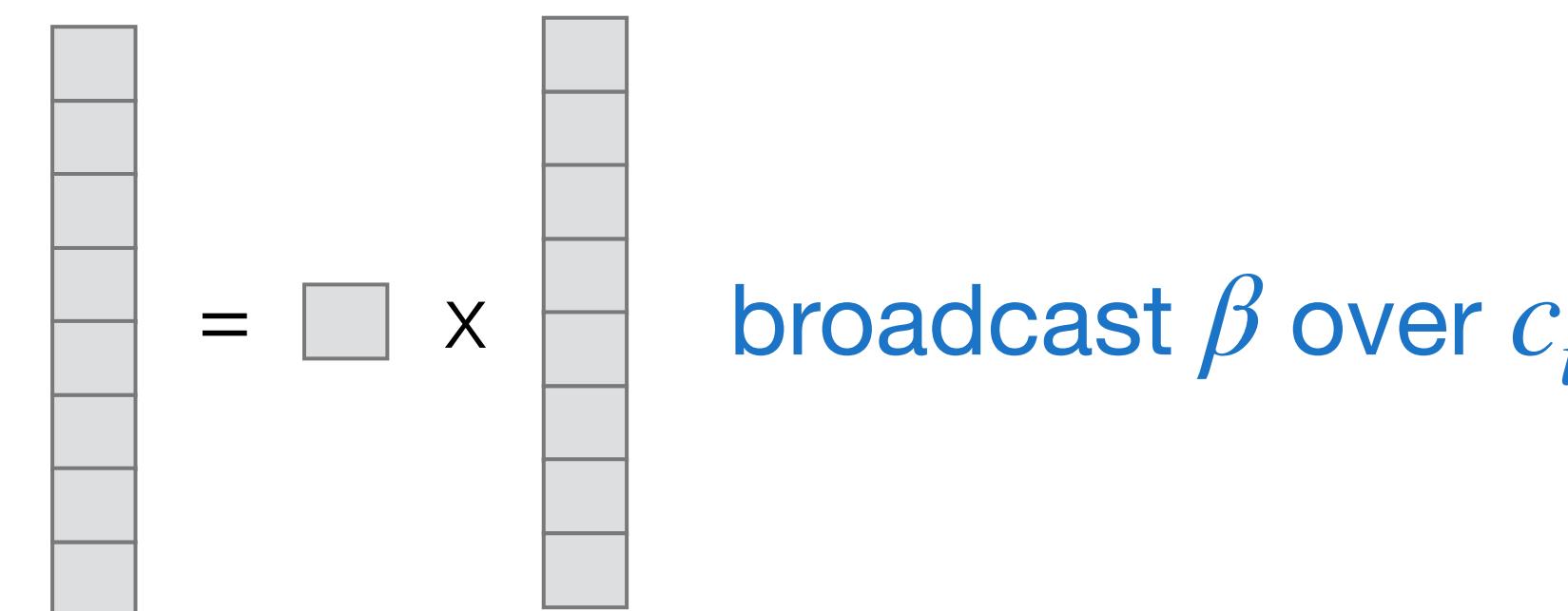
element-wise

$$\alpha = \sum_i b_i c_i$$



reduction over i

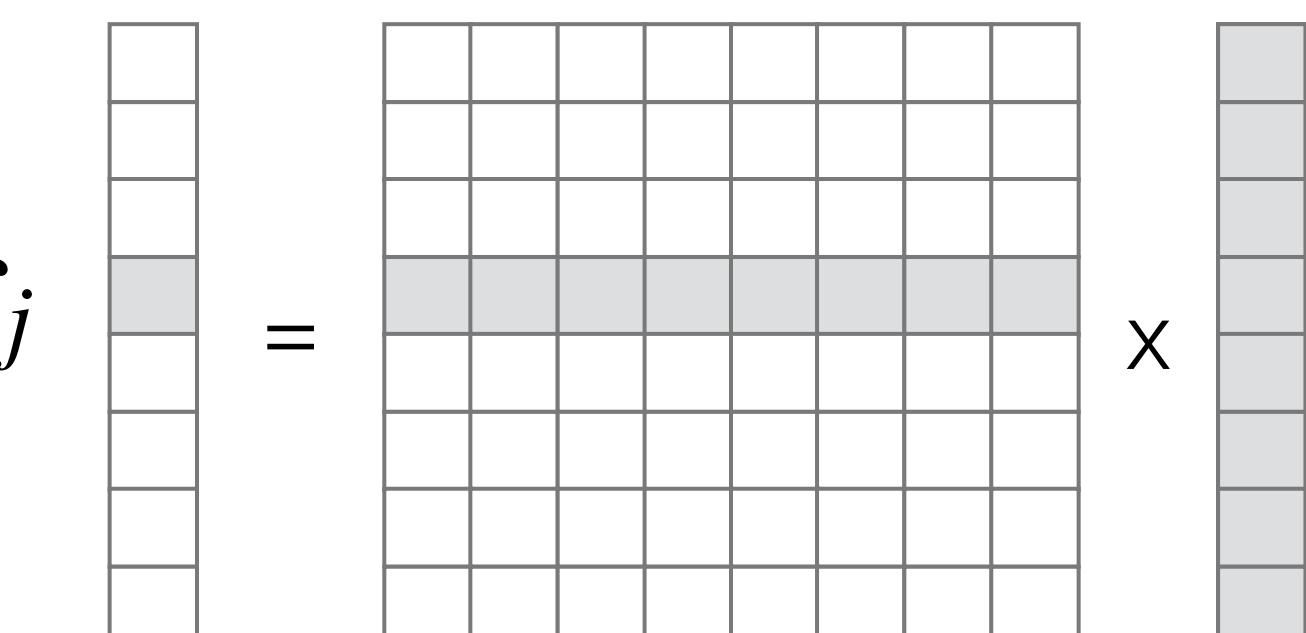
$$a_i = \beta c_i$$



broadcast  $\beta$  over  $c_i$

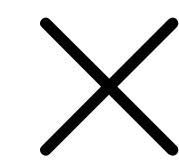
$$a_i = \sum_j B_{ij} c_j$$

all three patterns



# Generates fast code for any tensor index notation expression with the given formats and schedule

$$\begin{aligned}
 & a = Bc \\
 & a = Bc + a \\
 & a = Bc + b \quad A = B + C \quad a = \alpha Bc + \beta a \\
 & \quad a = B^T c \quad A = \alpha B \quad a = B(c + d) \\
 & a = B^T c + d \quad A = B + C + D \quad A = BC \\
 & A = B \odot C \quad a = b \odot c \quad A = 0 \quad A = B \odot (CD) \\
 & \quad A = BCd \quad A = B^T \quad a = B^T Bc \\
 & a = b + c \quad A = B \quad K = A^T CA \\
 & A_{ij} = \sum_{kl} B_{ikl} C_{lj} D_{kj} \quad A_{kj} = \sum_{il} B_{ikl} C_{lj} D_{ij} \\
 & \quad A_{lj} = \sum_{ik} B_{ikl} C_{ij} D_{kj} \quad A_{ij} = \sum_k B_{ijk} c_k \\
 & A_{ijk} = \sum_l B_{ikl} C_{lj} \quad A_{ik} = \sum_j B_{ijk} c_j \\
 & \quad A_{jk} = \sum_i B_{ijk} c_i \quad A_{ijl} = \sum_k B_{ikl} C_{kj} \\
 & C = \sum_{ijkl} M_{ij} P_{jk} \overline{M_{lk}} \overline{P_{il}} \quad \tau = \sum_i z_i (\sum_j z_j \theta_{ij}) (\sum_k z_k \theta_{ik}) \\
 & \quad a = \sum_{ijklmnop} M_{ij} P_{jk} M_{kl} P_{lm} \overline{M_{nm}} P_{no} \overline{M_{po}} \overline{P_{ip}}
 \end{aligned}$$



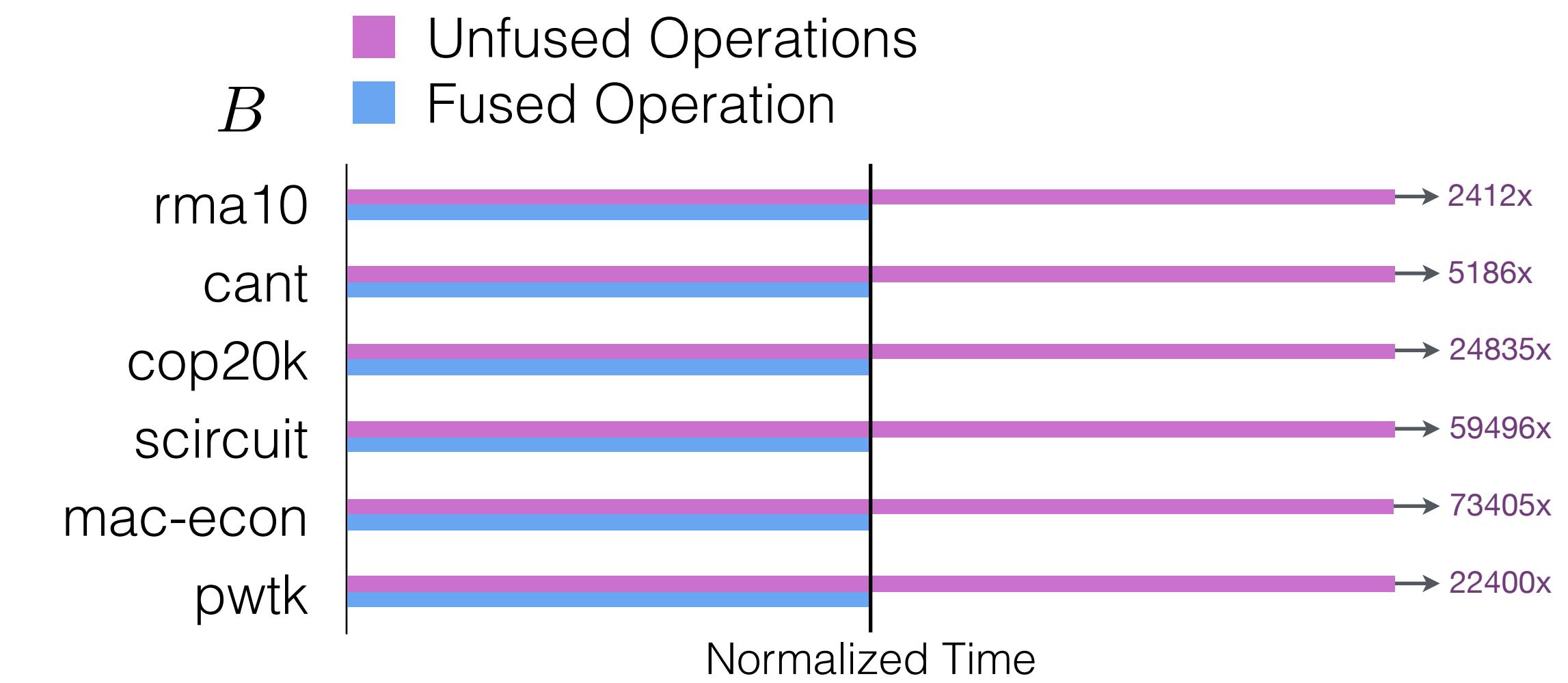
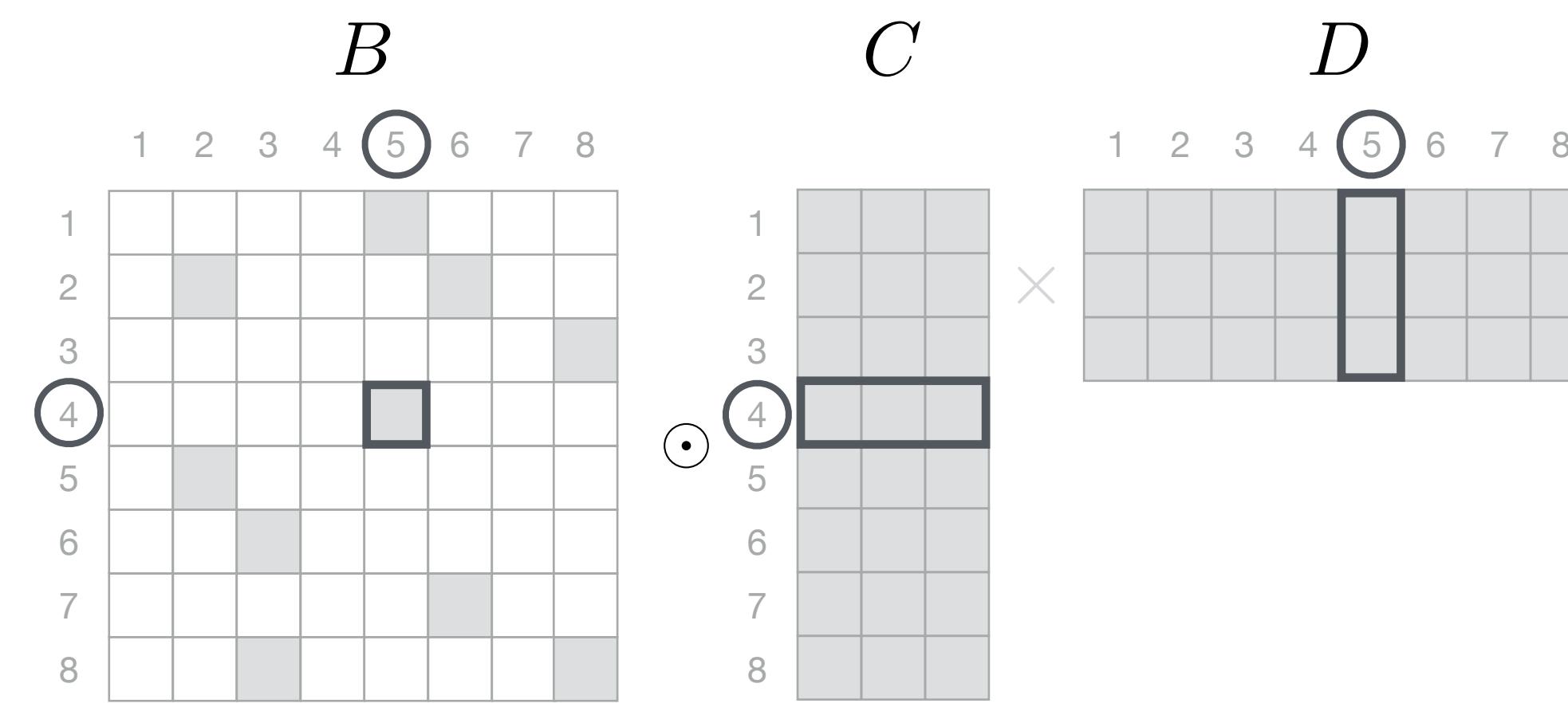
Dense Matrix				
CSR	DCSR	BCSR		
COO	ELLPACK	CSB	CPU	
Blocked COO		CSC	GPUs	TPUs
DIA	Blocked DIA	DCSC		
Sparse vector	Hash Maps			
Coordinates				
CSF	Dense Tensors			
	Blocked Tensors			

×

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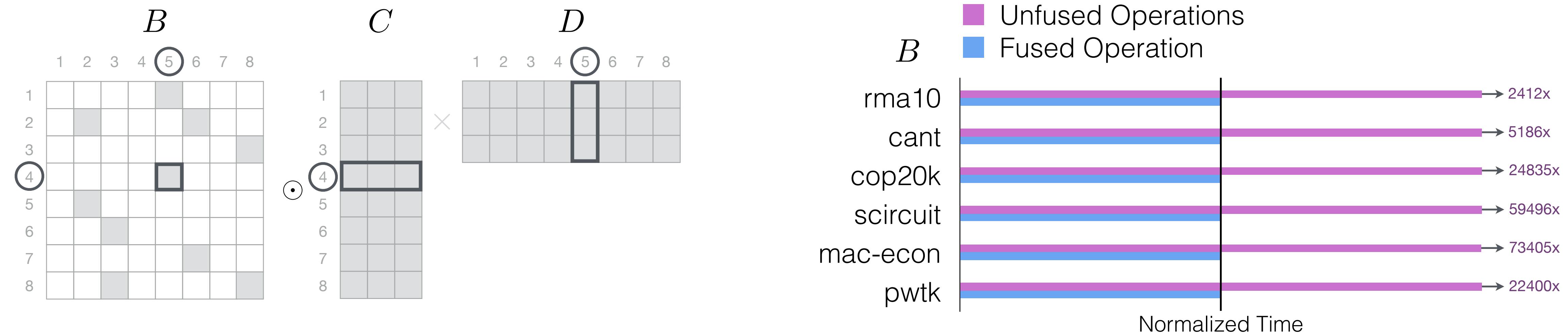
# Compound expressions matter for performance

$$A = B \odot (CD)$$



# Compound expressions matter for performance

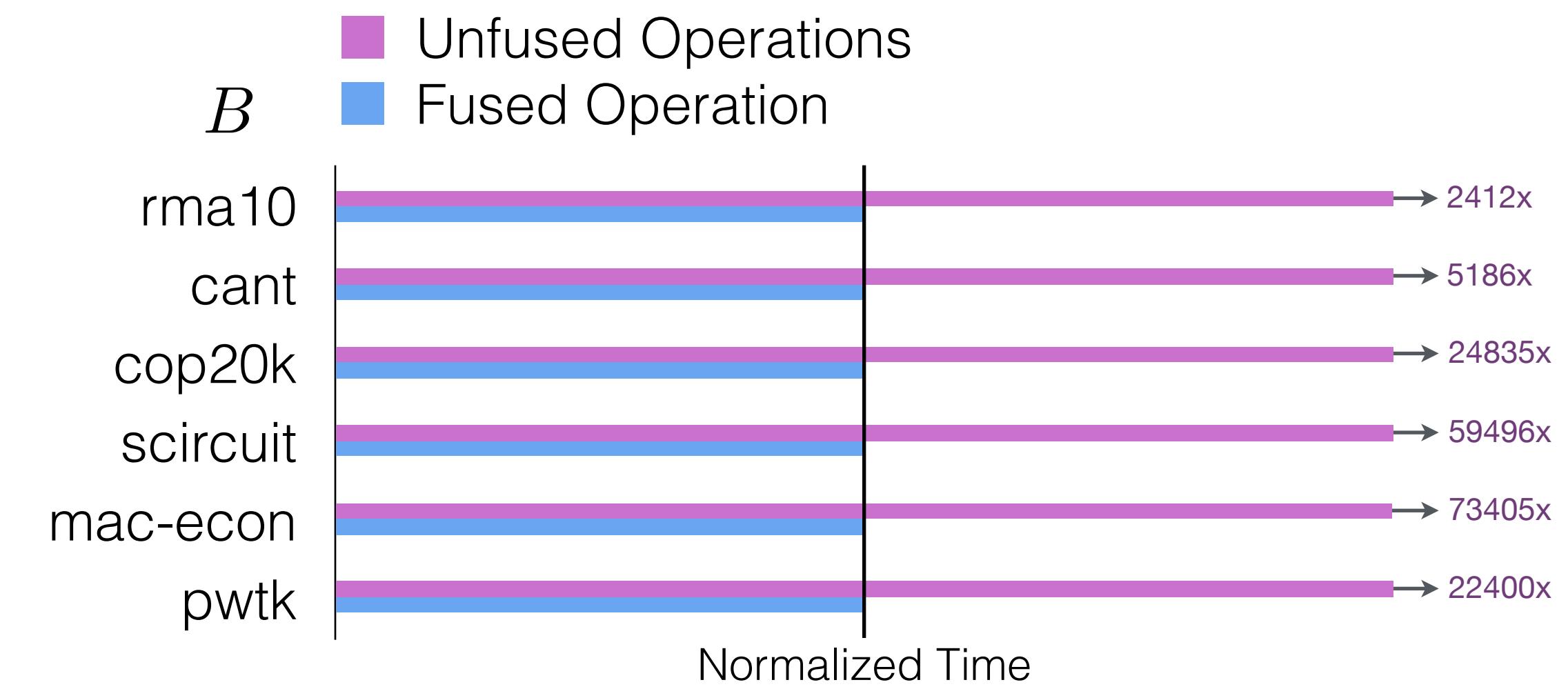
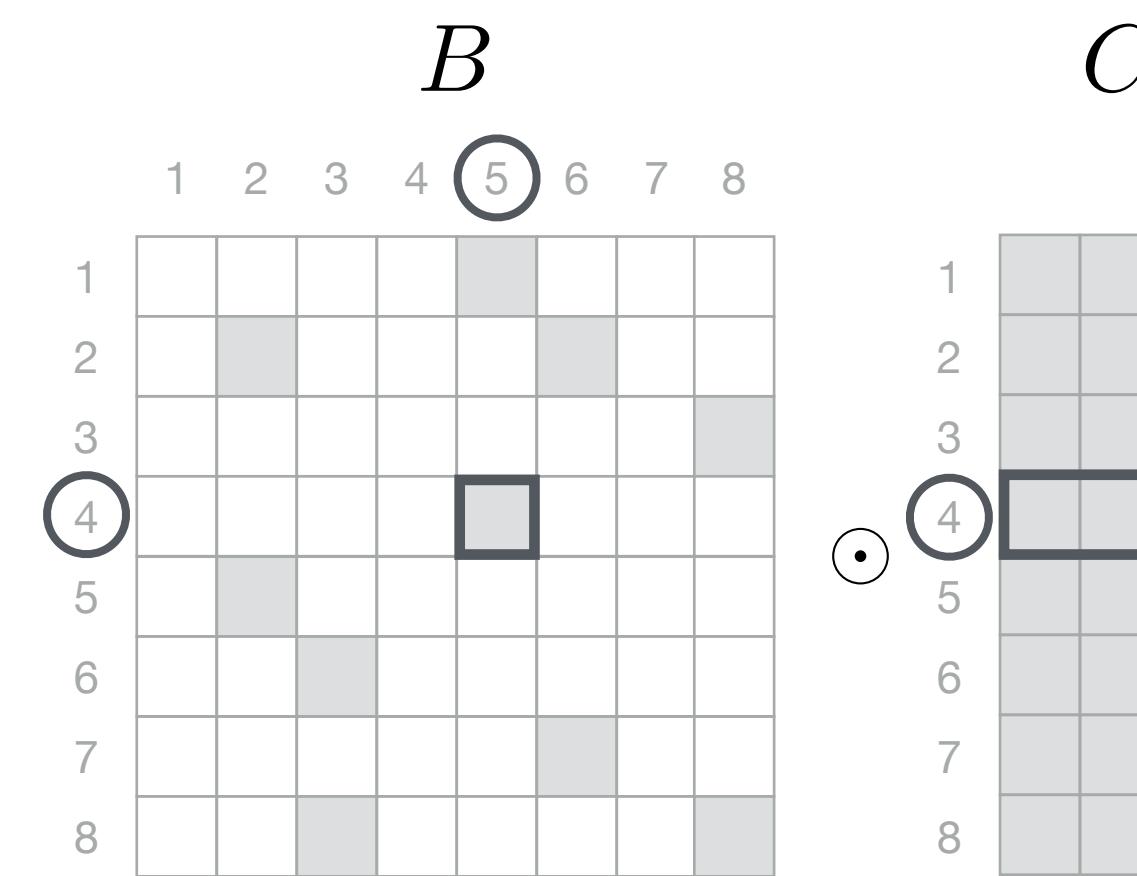
$$A = B \odot (CD)$$



Unfused:  $\Theta(n^2k)$

# Compound expressions matter for performance

$$A = B \odot (CD)$$

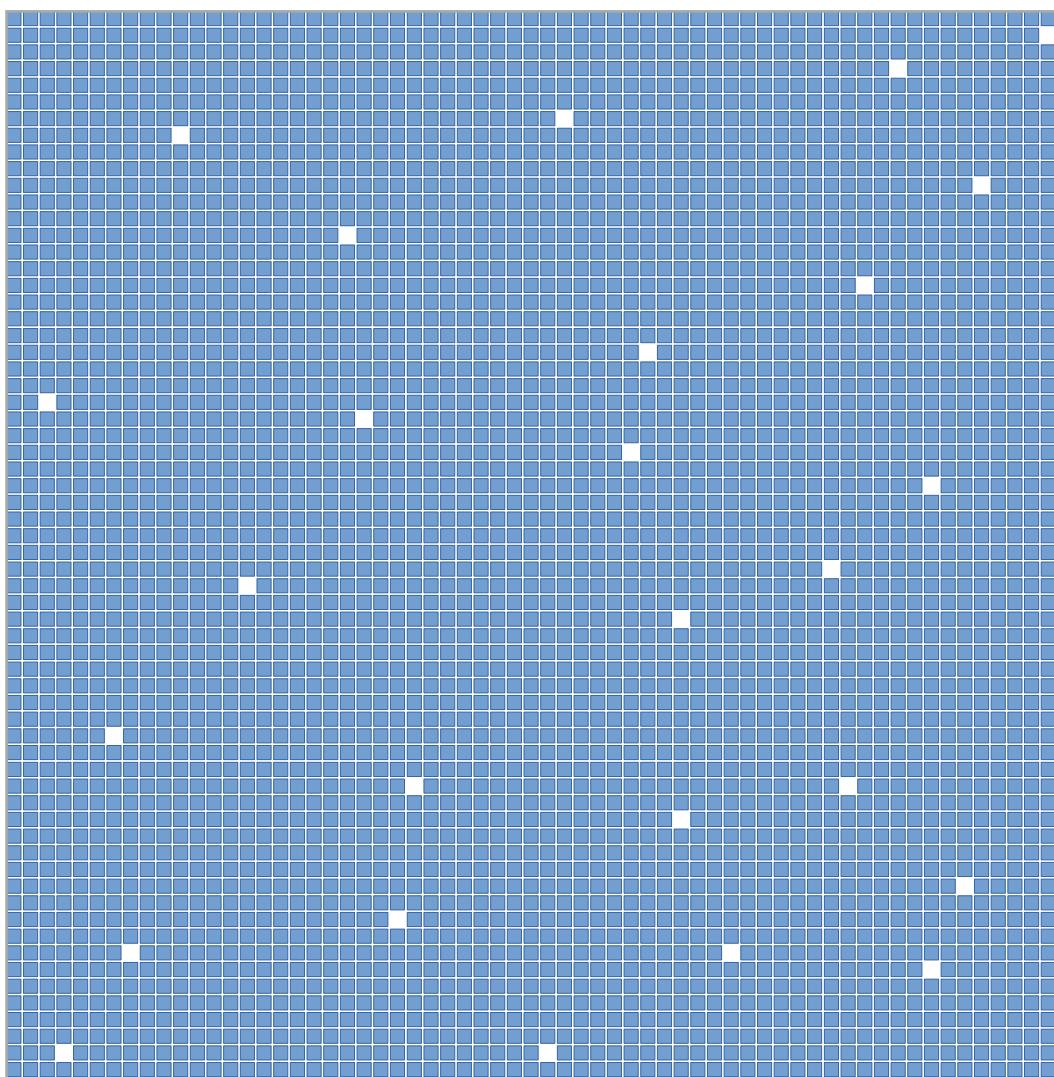


Unfused:  $\Theta(n^2k)$

Fused:  $\Theta(\text{nnz}_B \cdot k)$

# Formats matter for performance

Dense Matrix



Formats

Best performance

Dense

List of Rows

CSR

DCSR

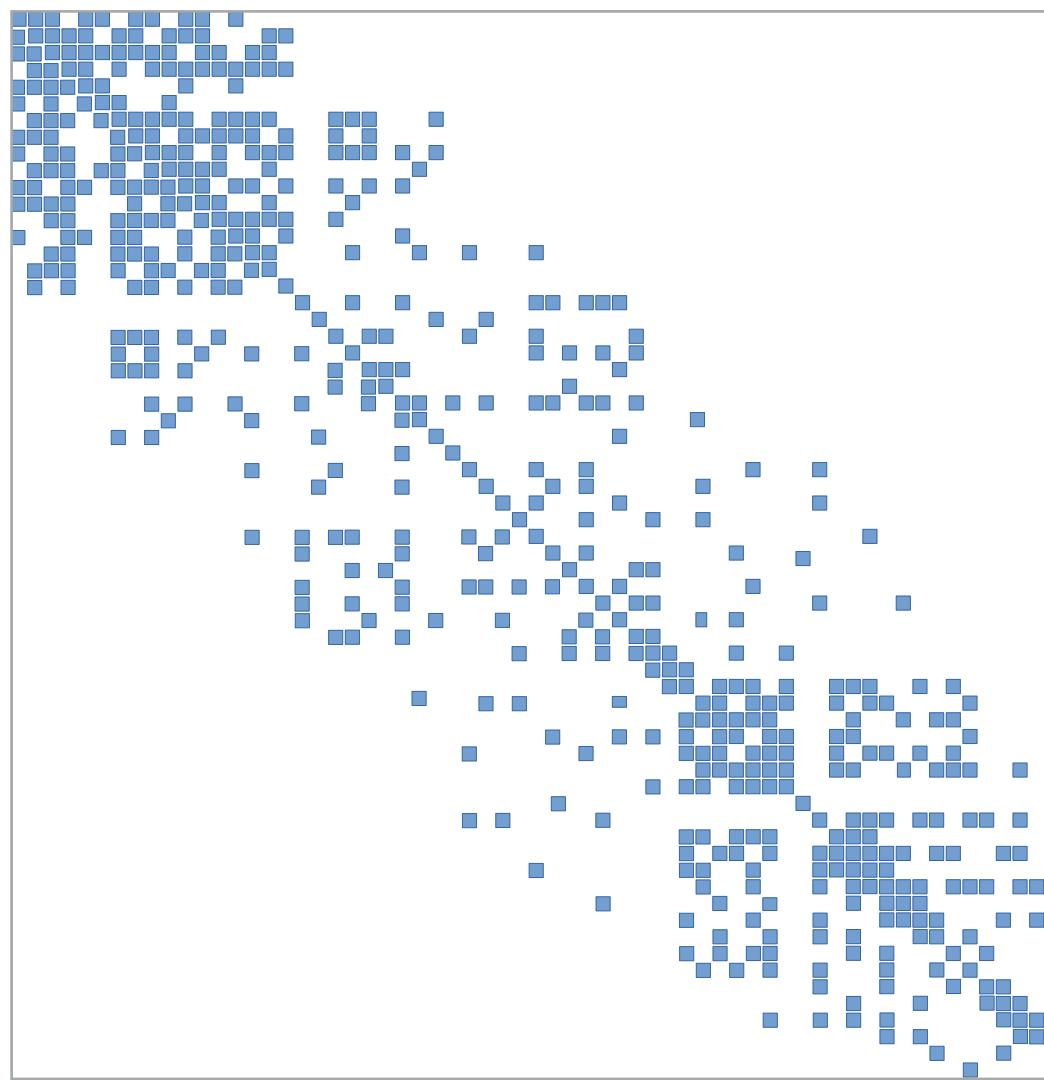
$$y = Ax$$



Normalized time

# Formats matter for performance

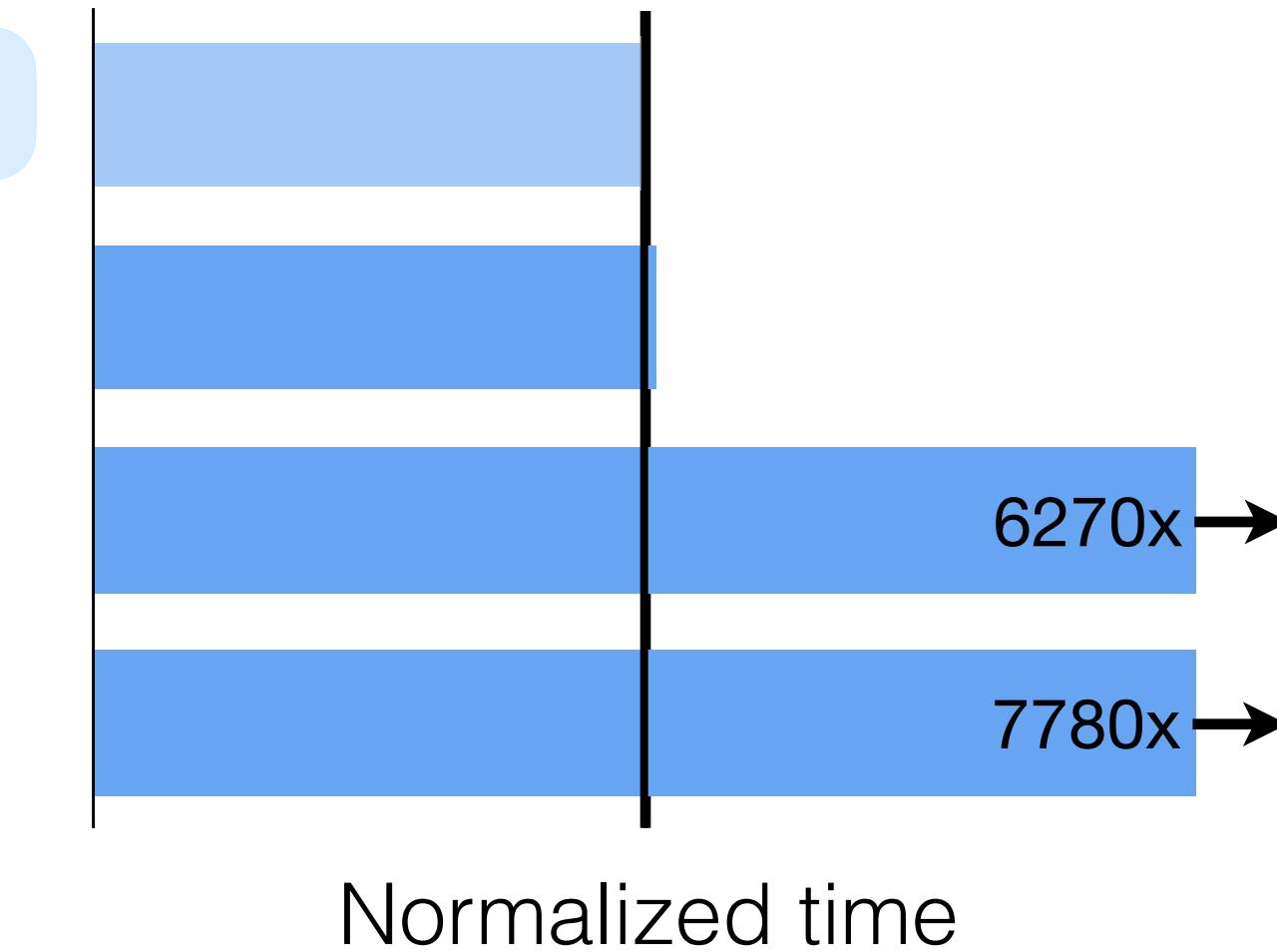
Thermal Matrix



Formats

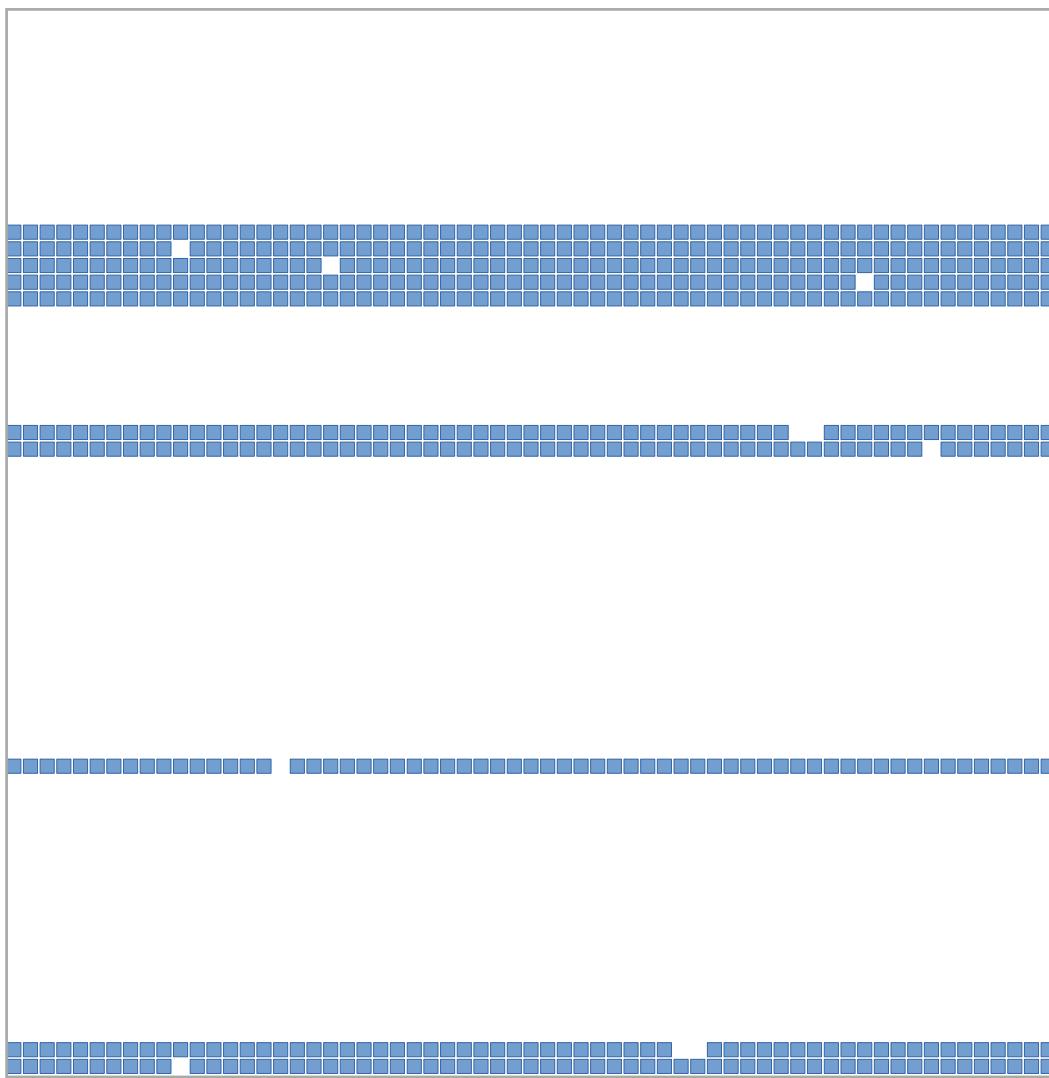
- Best performance
- CSR
  - DCSR
  - Dense
  - List of Rows

$$y = Ax$$



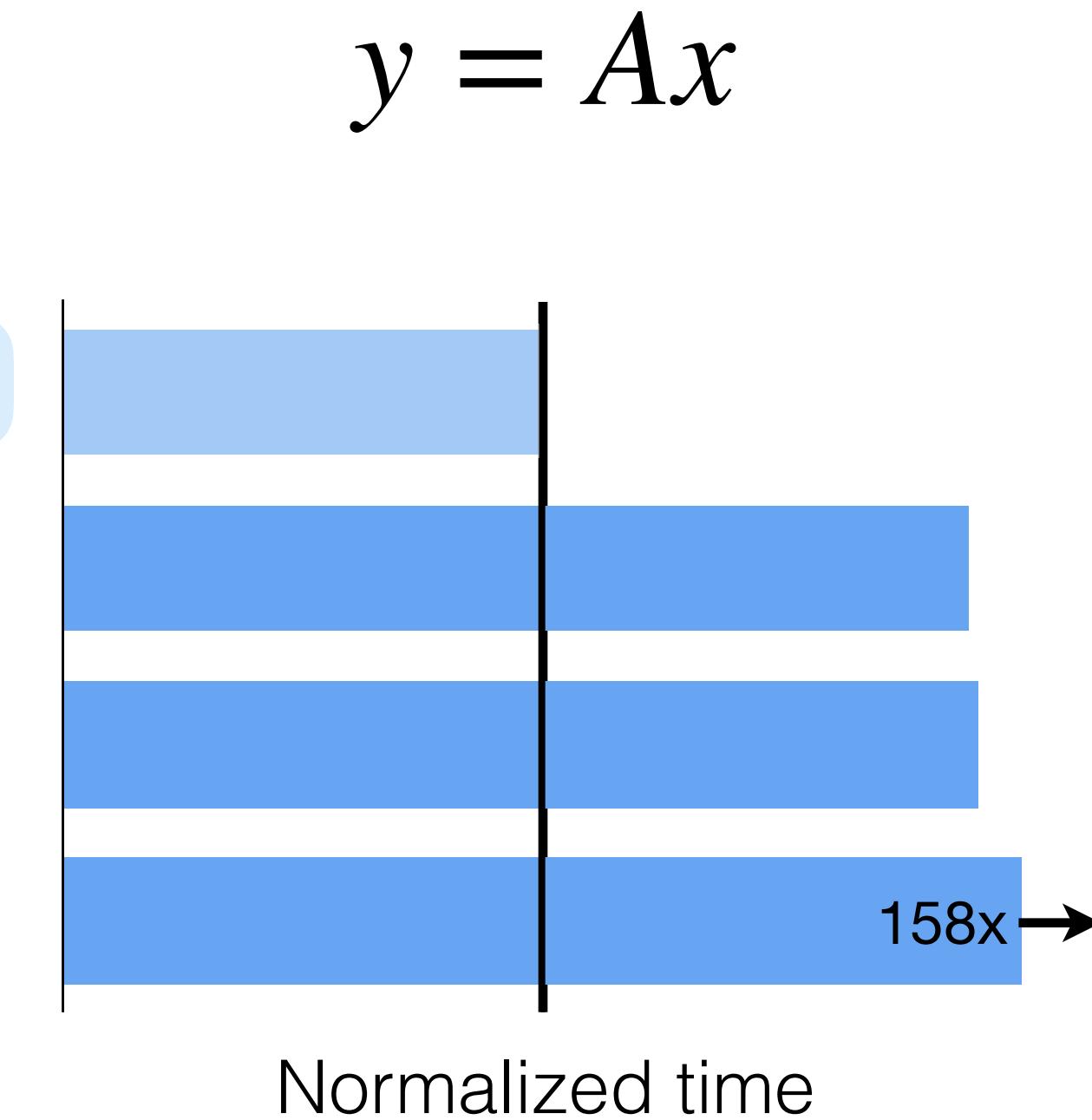
# Formats matter for performance

Row-sliced Matrix



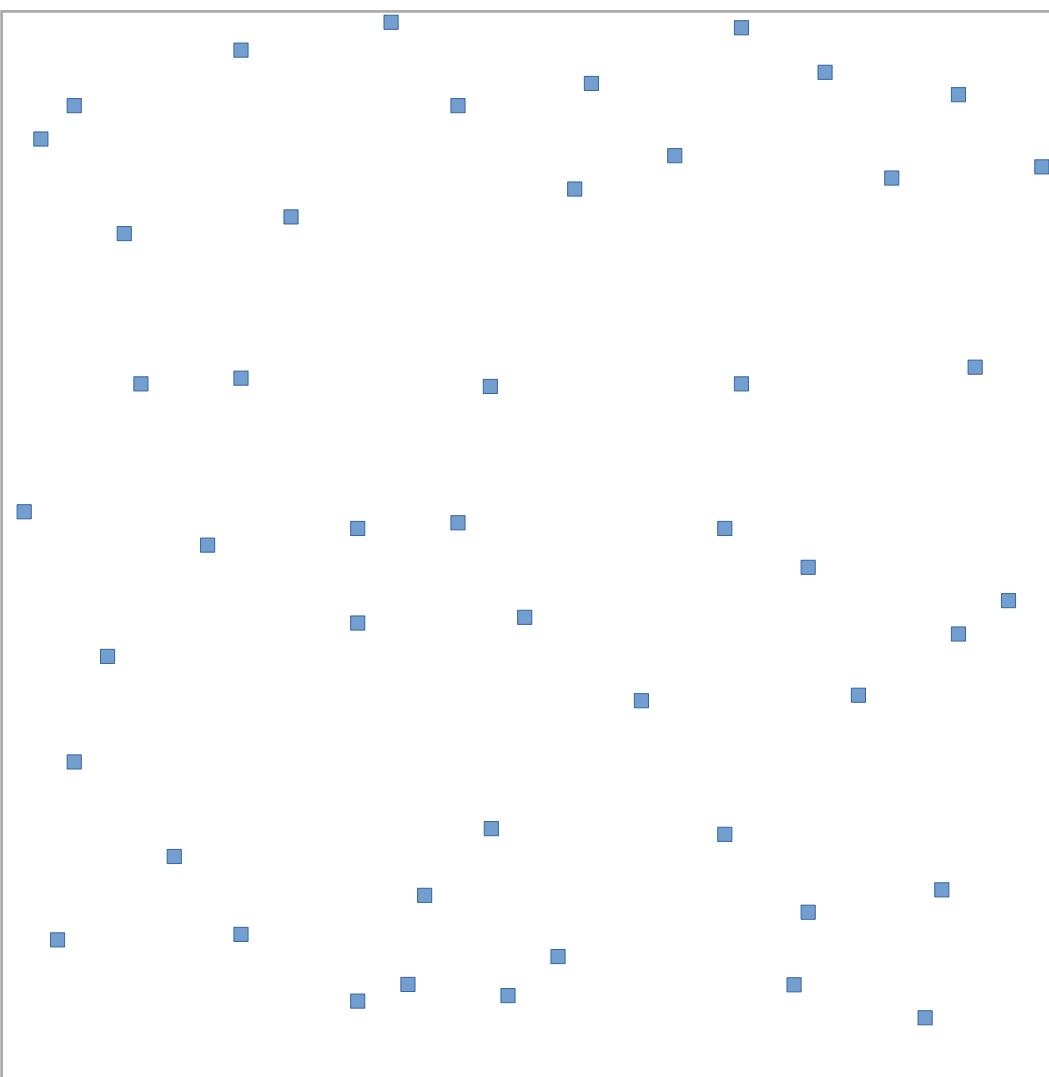
Formats

Best performance  
List of Rows  
DCSR  
CSR  
Dense



# Formats matter for performance

Hypersparse Matrix



Formats

Best performance

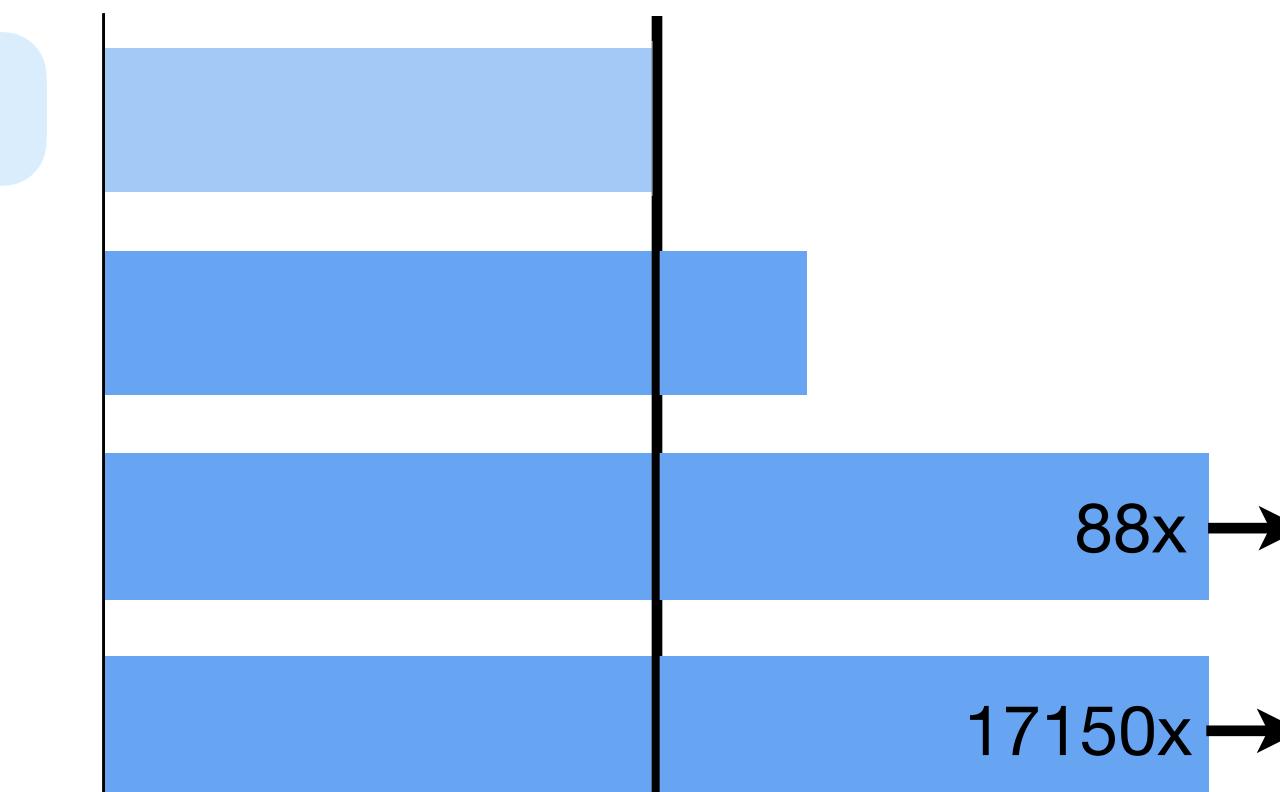
DCSR

CSR

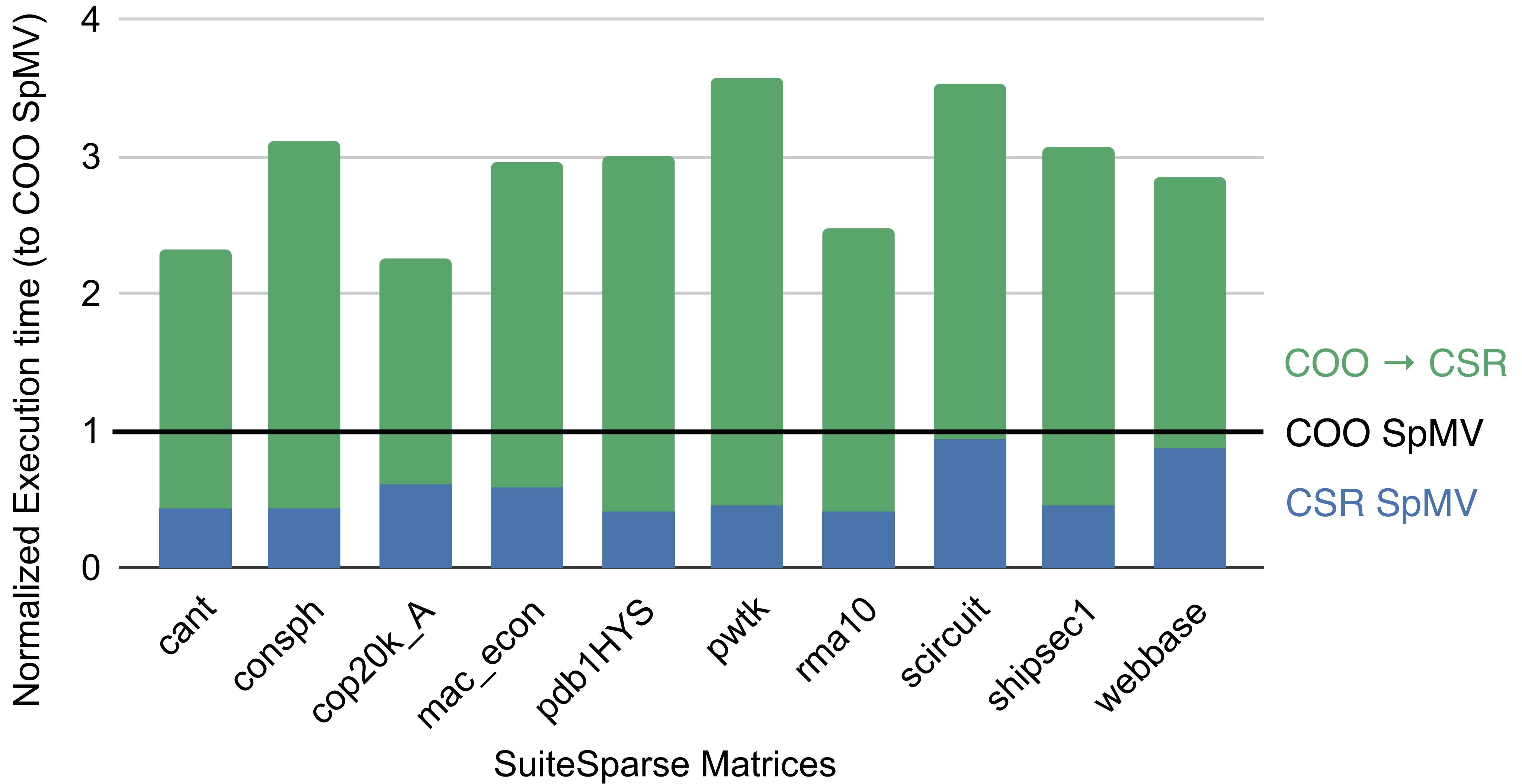
List of Rows

Dense

$$y = Ax$$



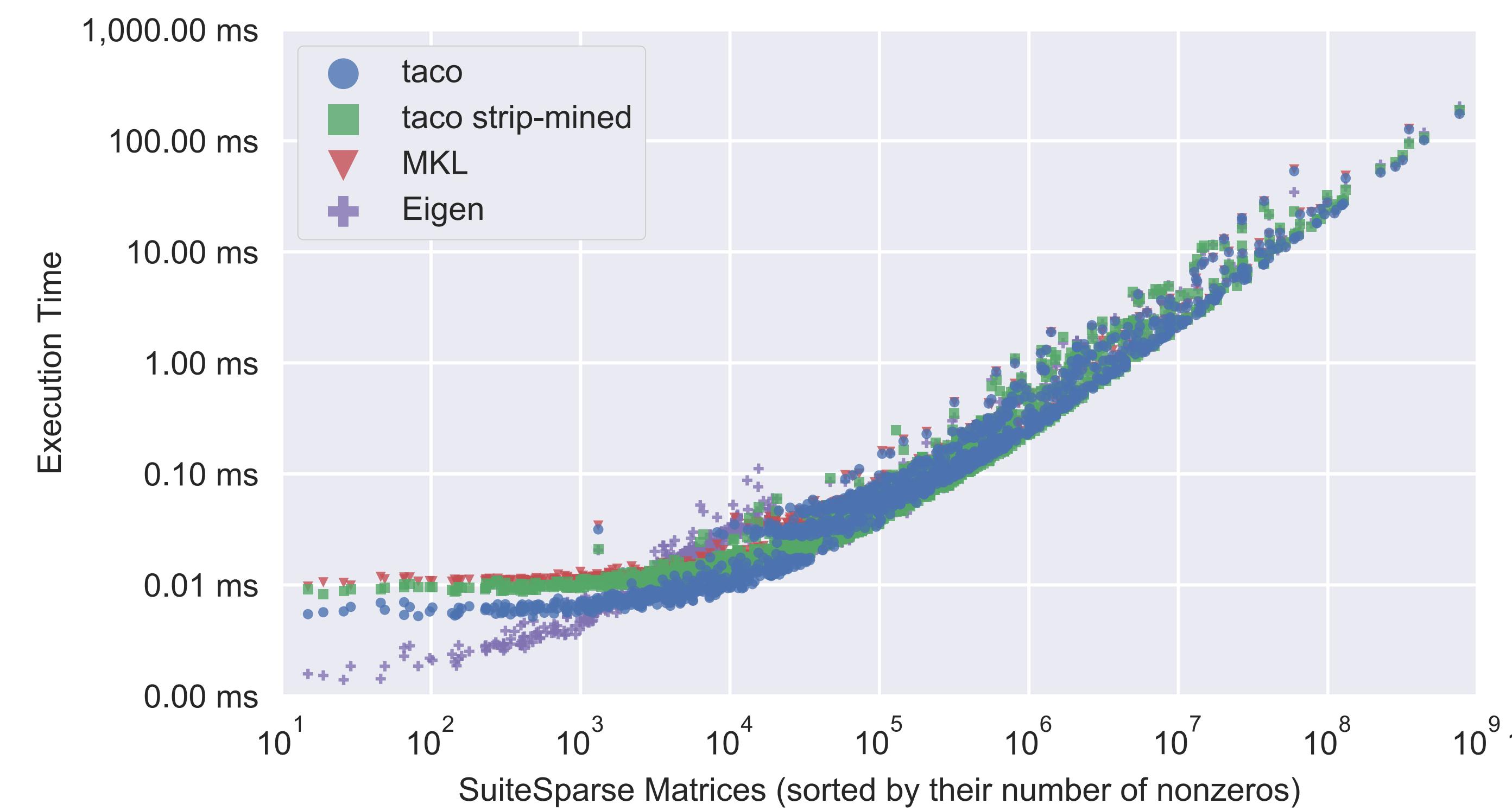
# CSR vs COO



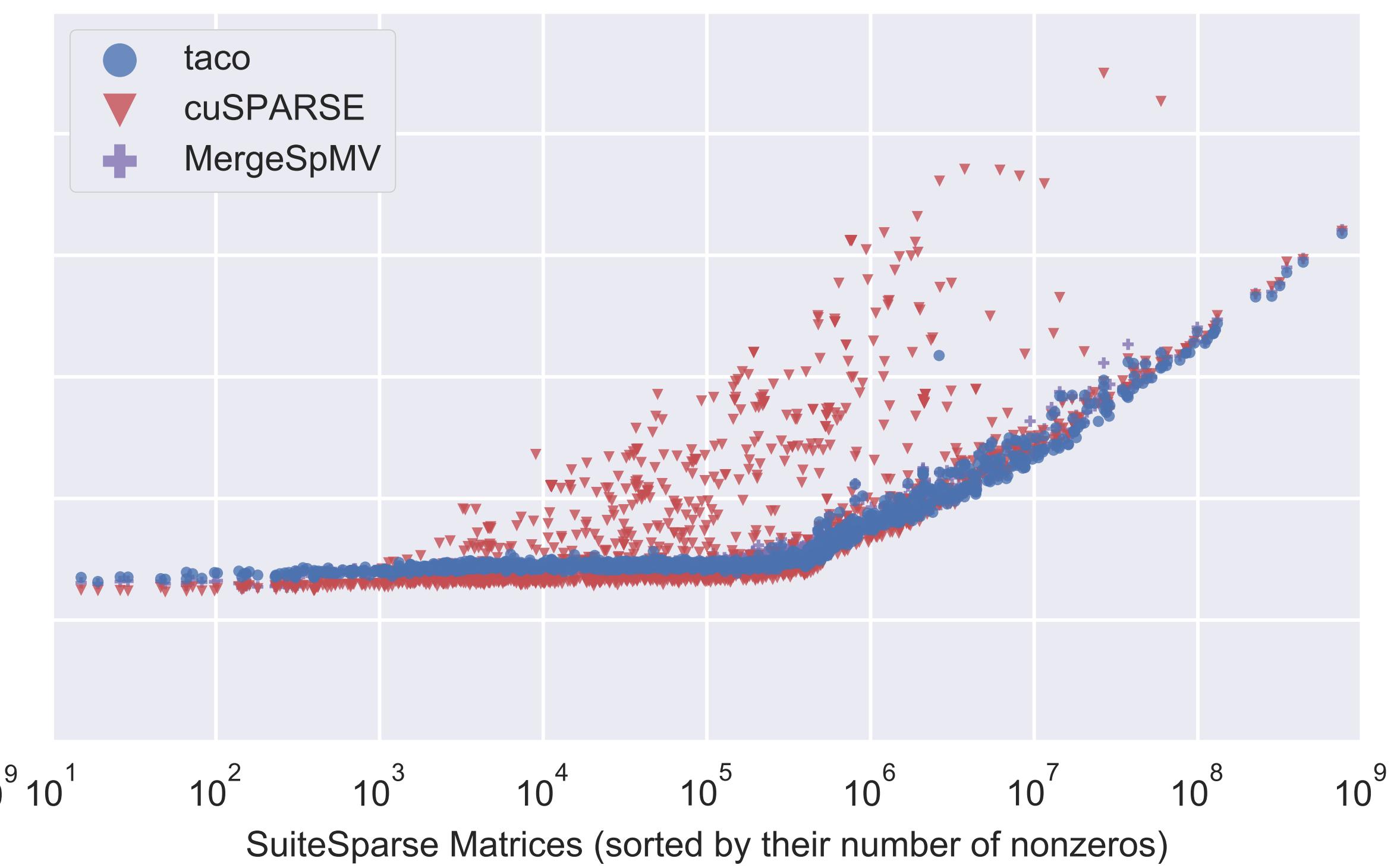
# Machines matter for performance

$$a_i = \sum_j B_{ij} c_j \quad (\text{SpMV})$$

CPU



GPU

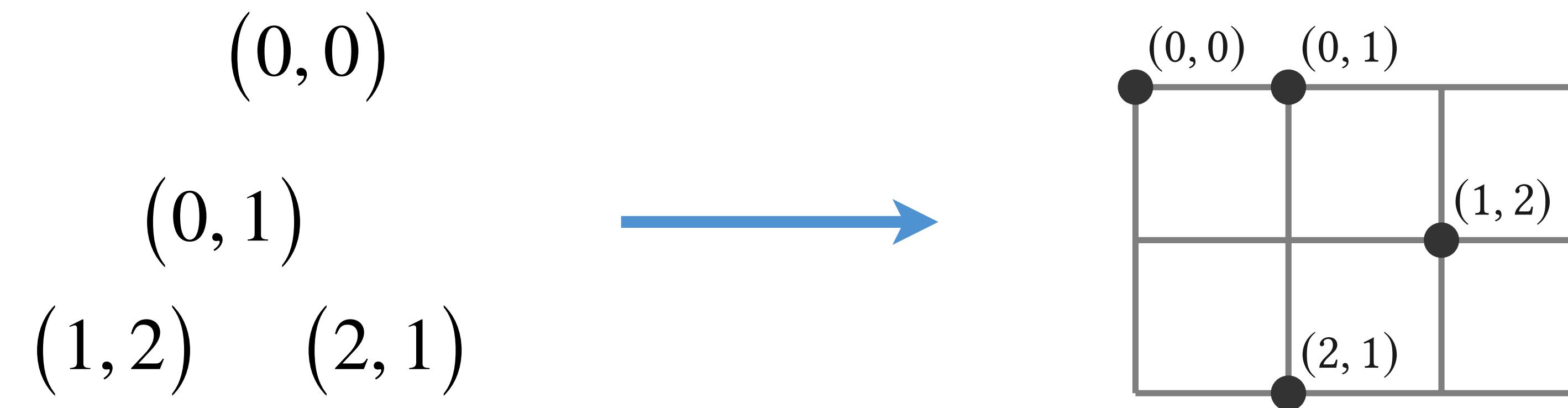


# Sparse data structures in graphs, tensors, and relations encode coordinates in a sparse iteration space

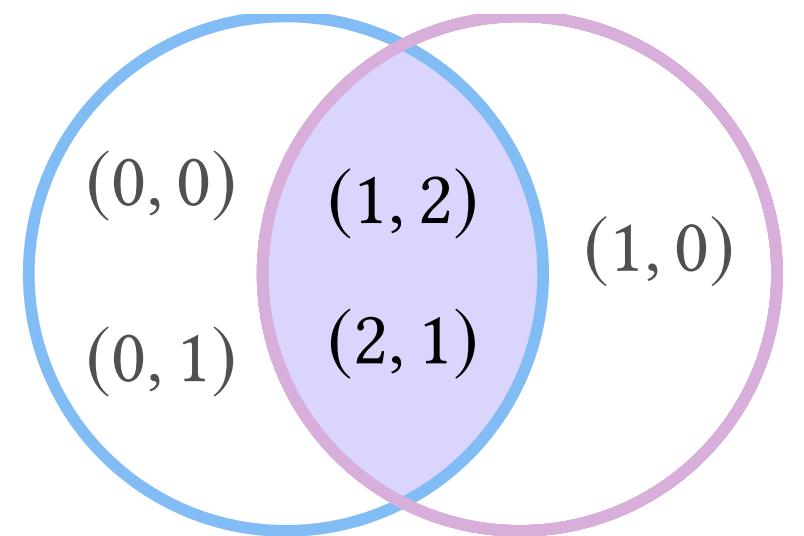
Tensor (nonzeros)		Relation (rows)	Graph (edges)
	(0,1)	(Harry,CS)	(v <sub>1</sub> ,v <sub>5</sub> )
(2,3)	(0,5)	(Sally,EE)	(v <sub>4</sub> ,v <sub>3</sub> )
(5,5)	(7,5)	(George,CS)	(v <sub>5</sub> ,v <sub>3</sub> )
		(Rita,CS)	(v <sub>3</sub> ,v <sub>5</sub> )
		(Mary,ME)	(v <sub>3</sub> ,v <sub>1</sub> )

Values may be attached to these coordinates: e.g., nonzero values, edge attributes

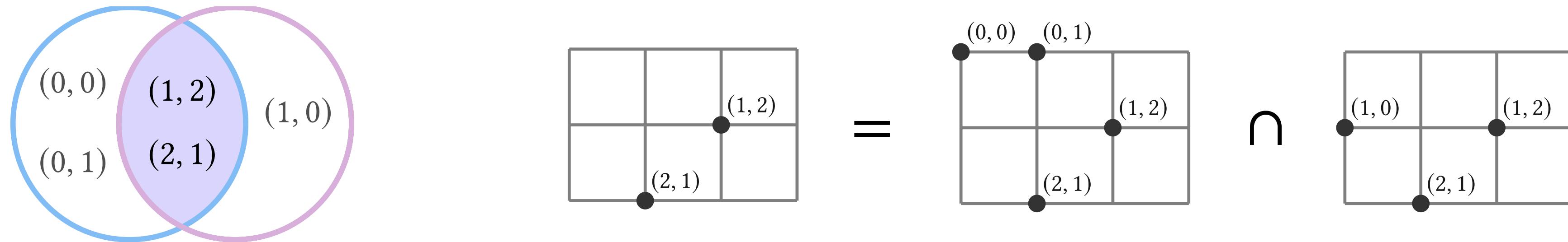
# Iteration spaces from coordinate relations



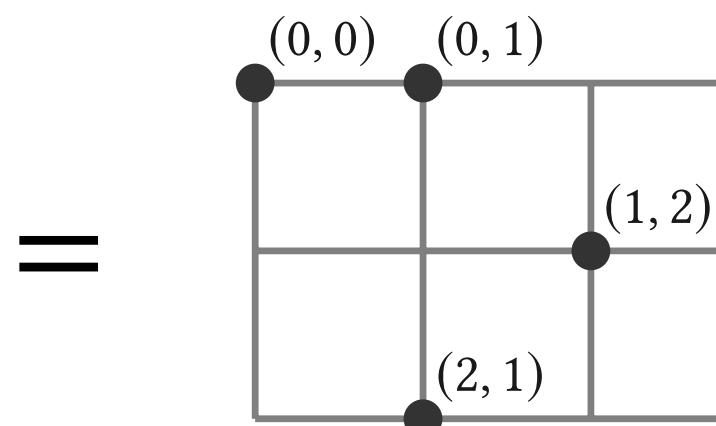
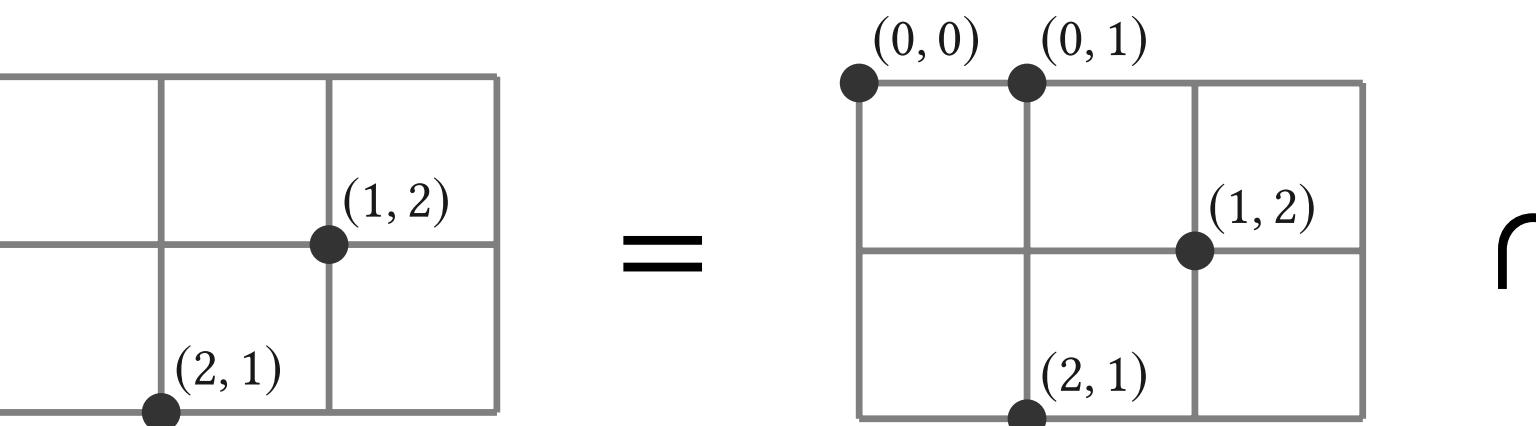
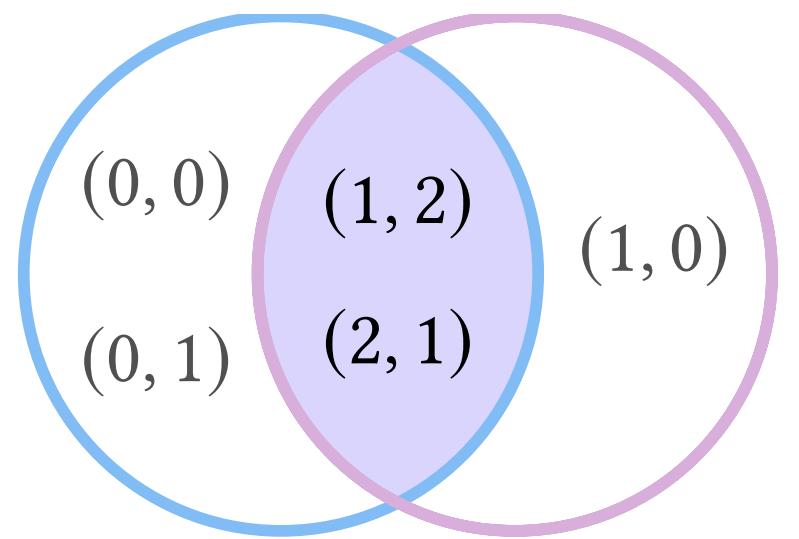
# Iteration spaces from set operations



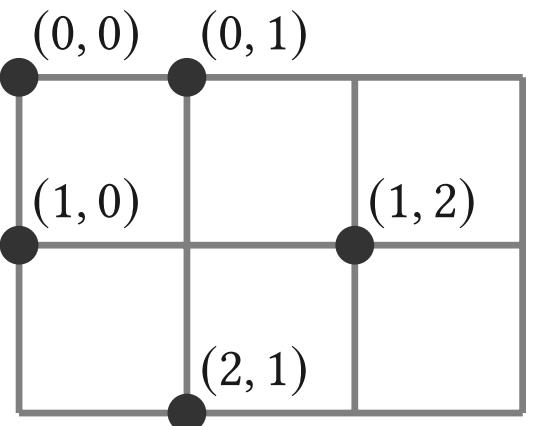
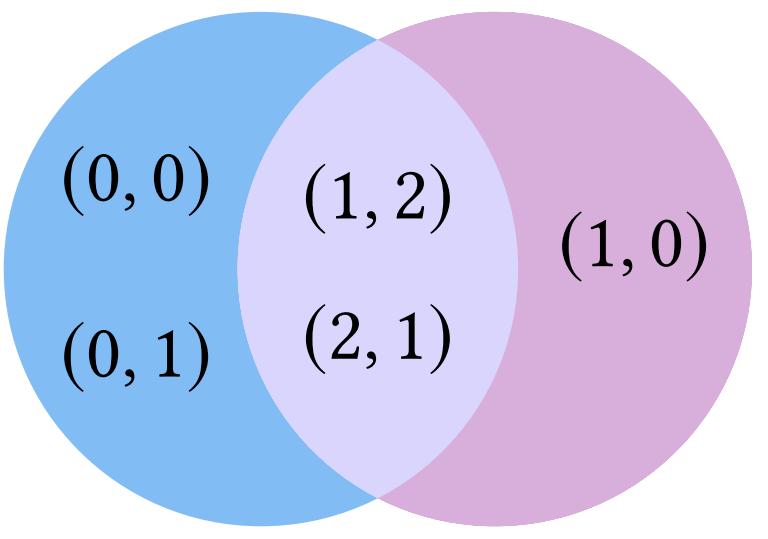
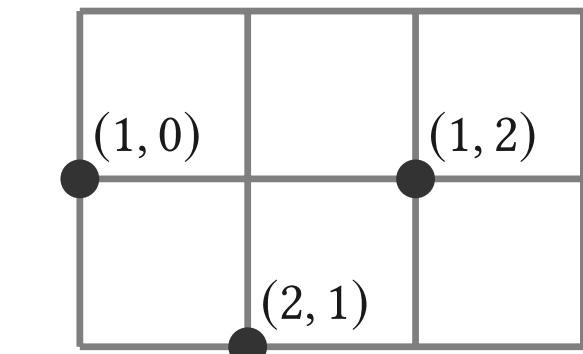
# Iteration spaces from set operations



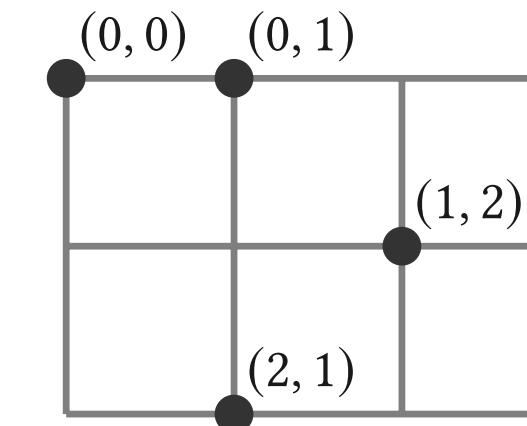
# Iteration spaces from set operations



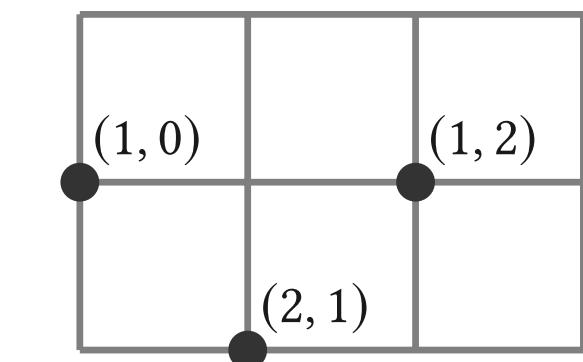
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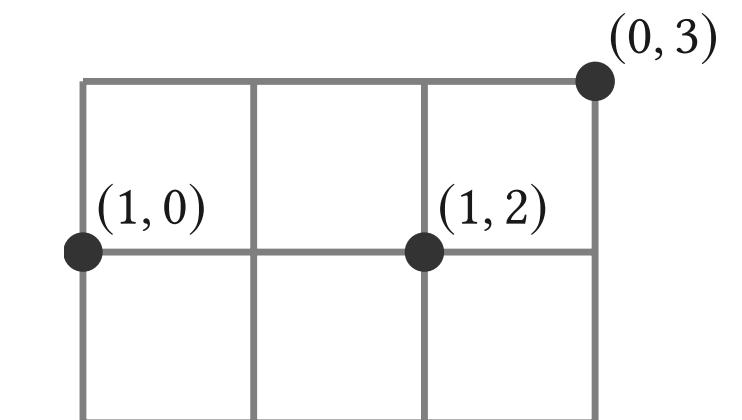
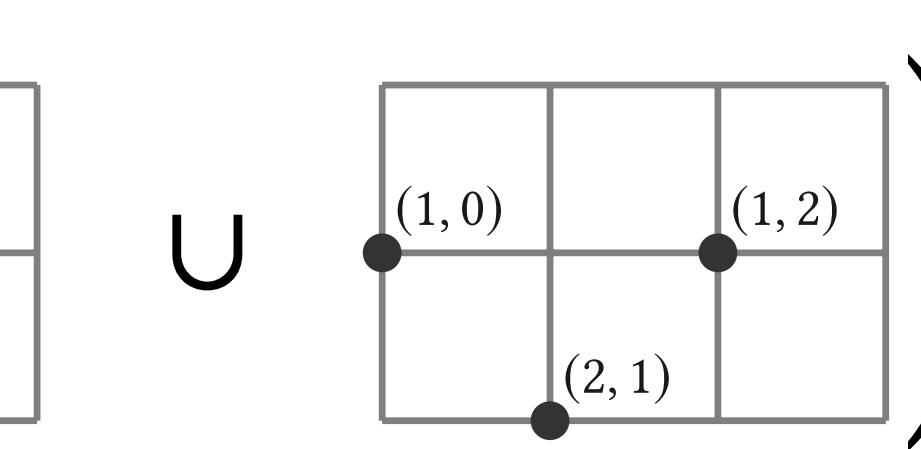
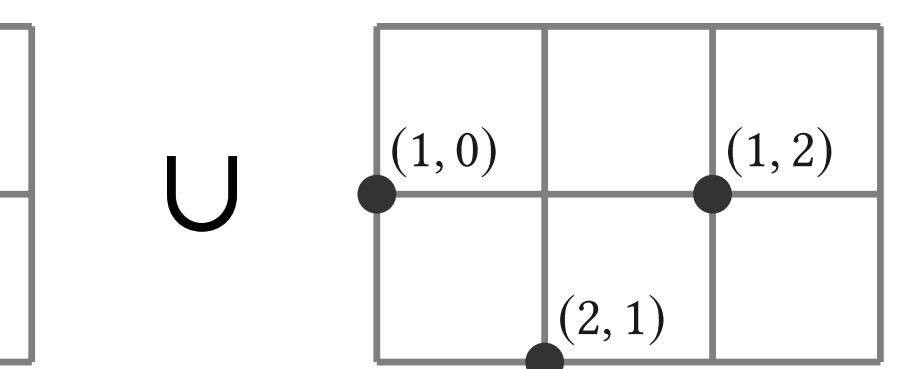
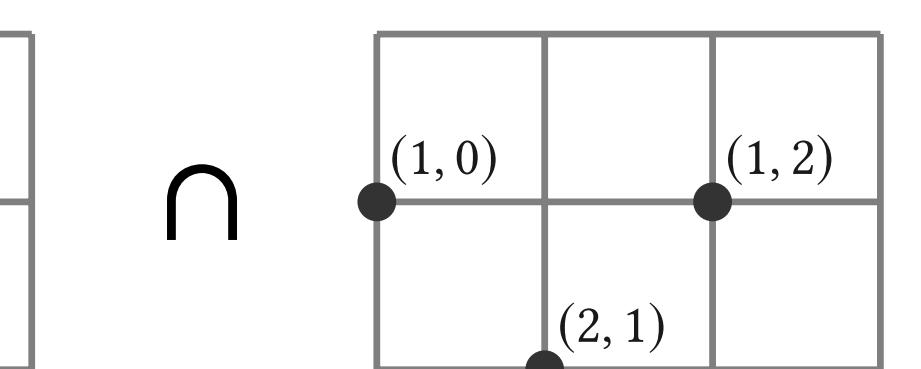
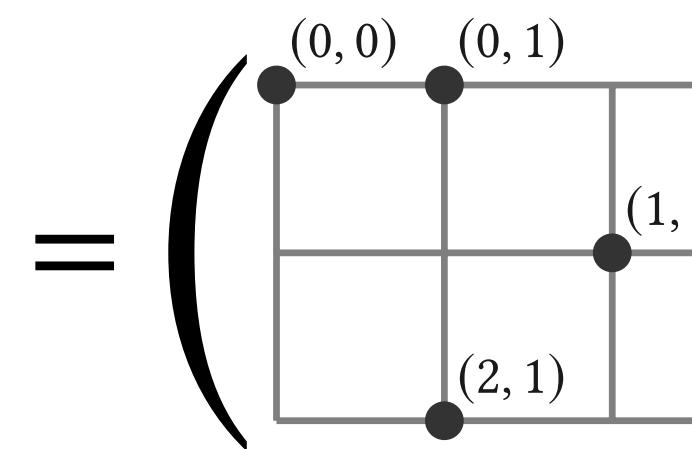
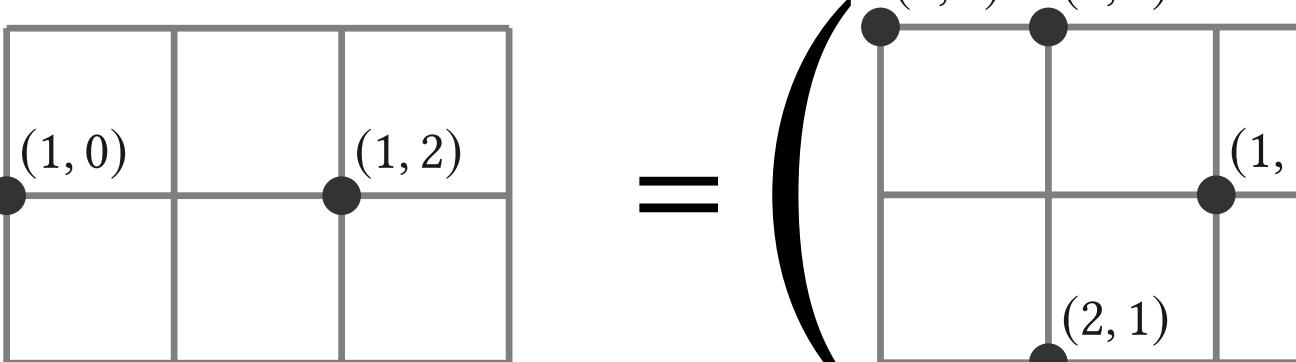
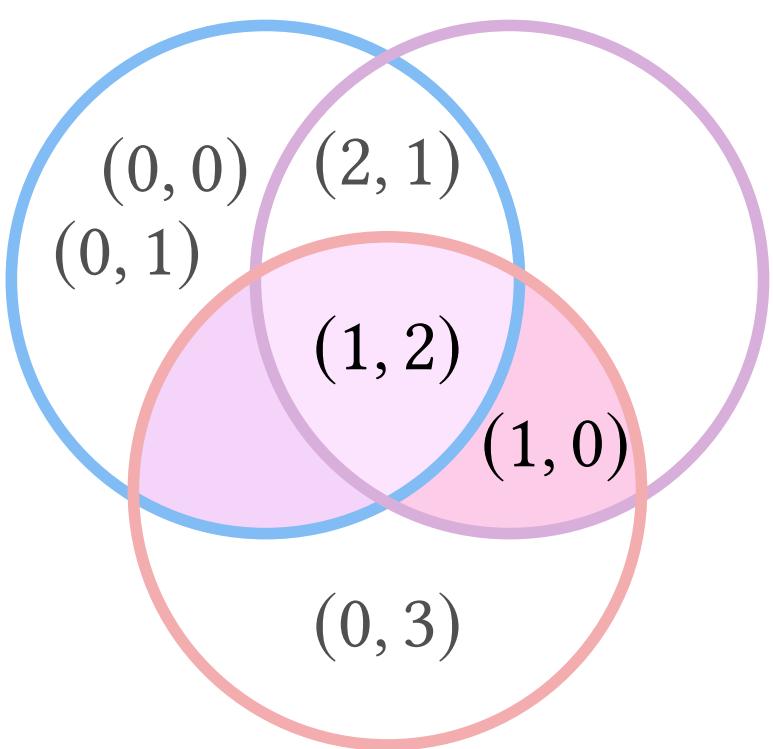
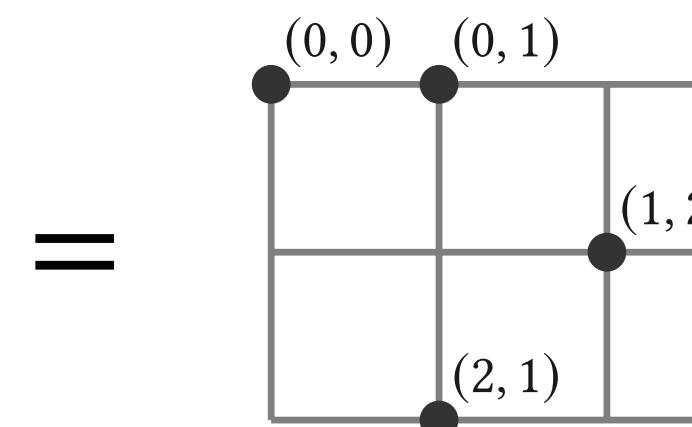
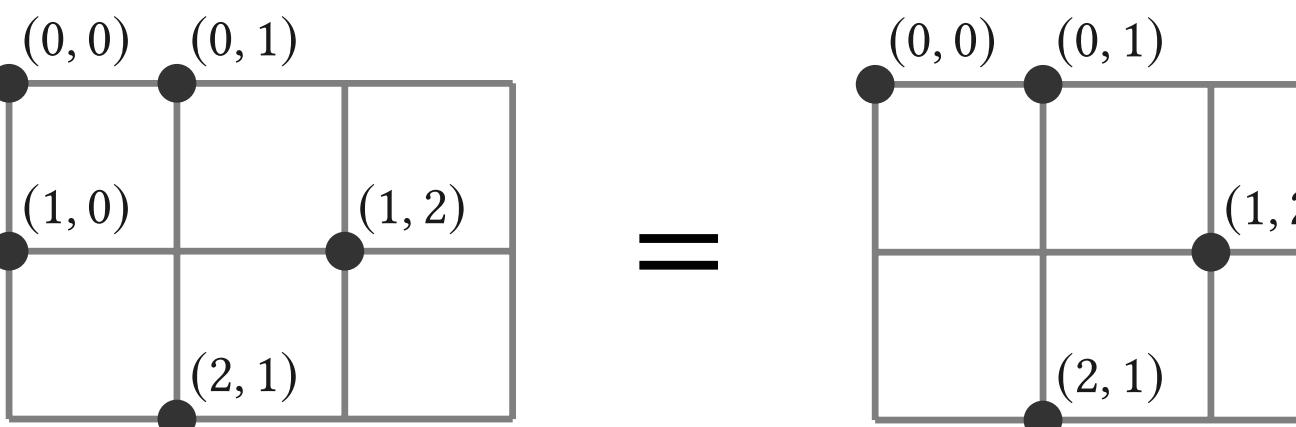
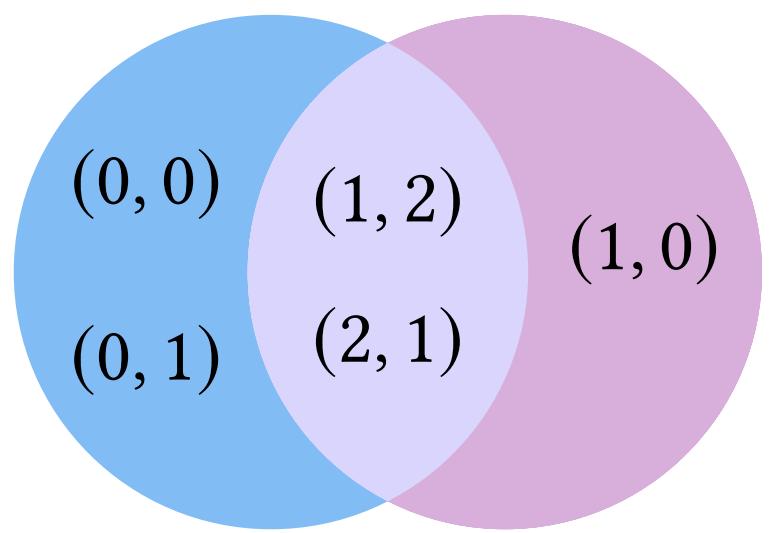
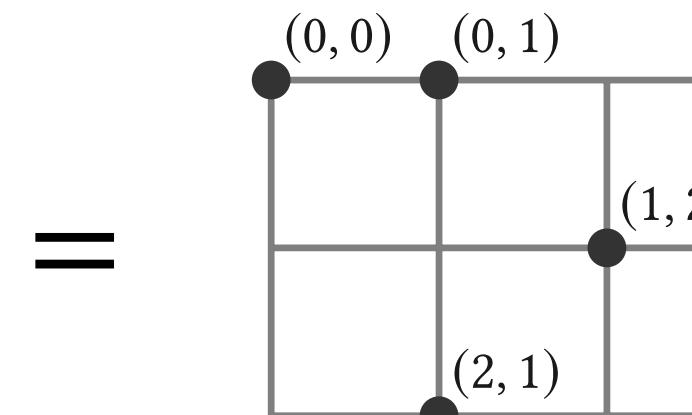
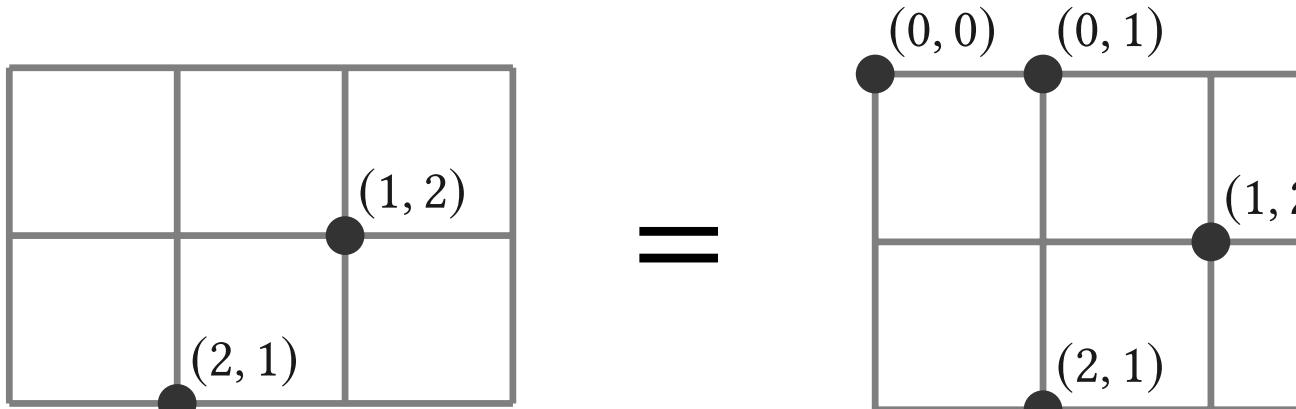
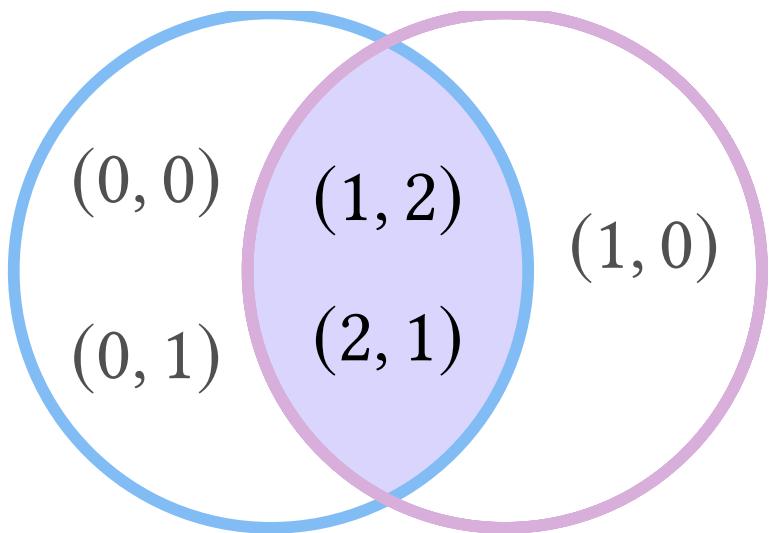
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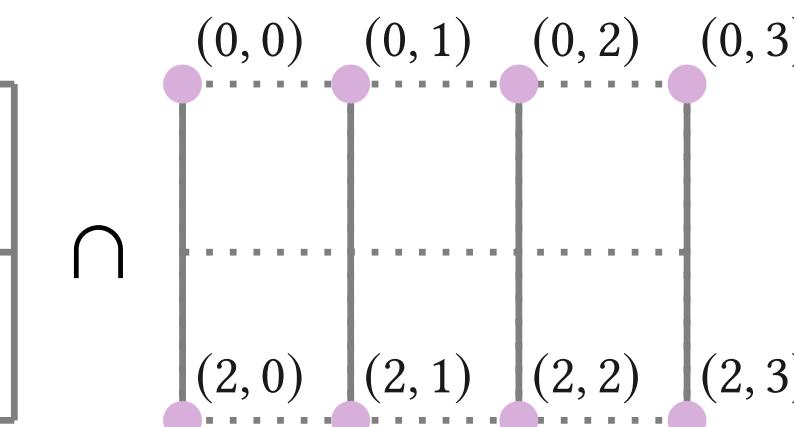
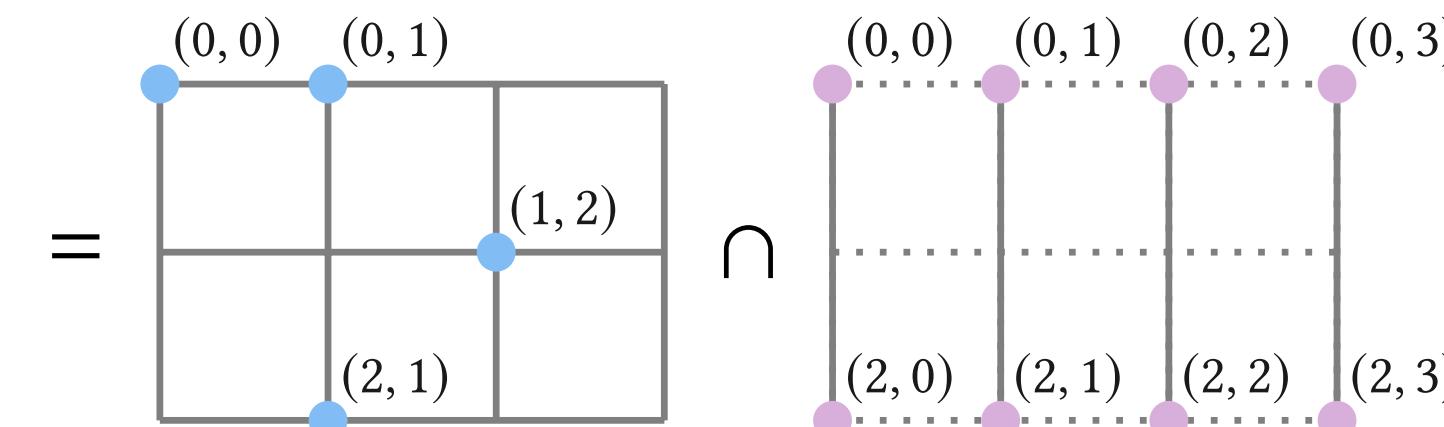
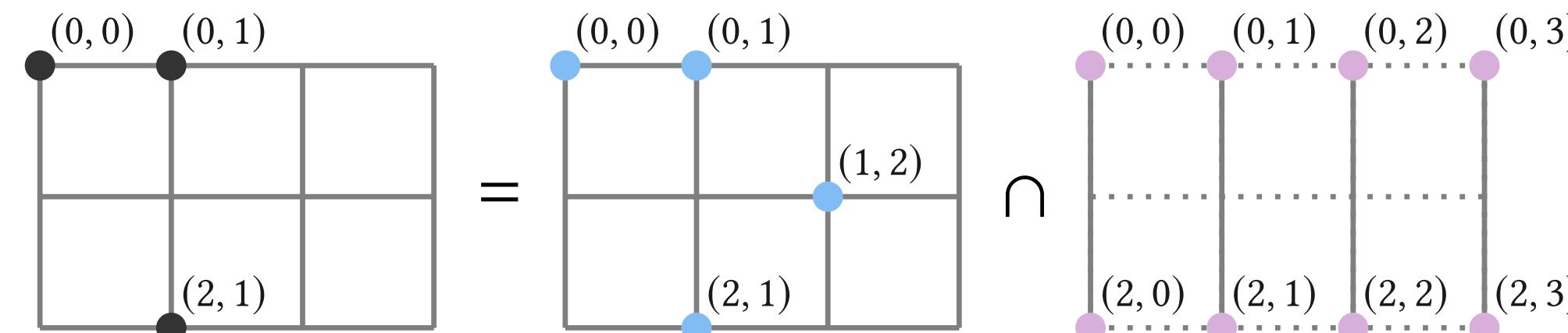
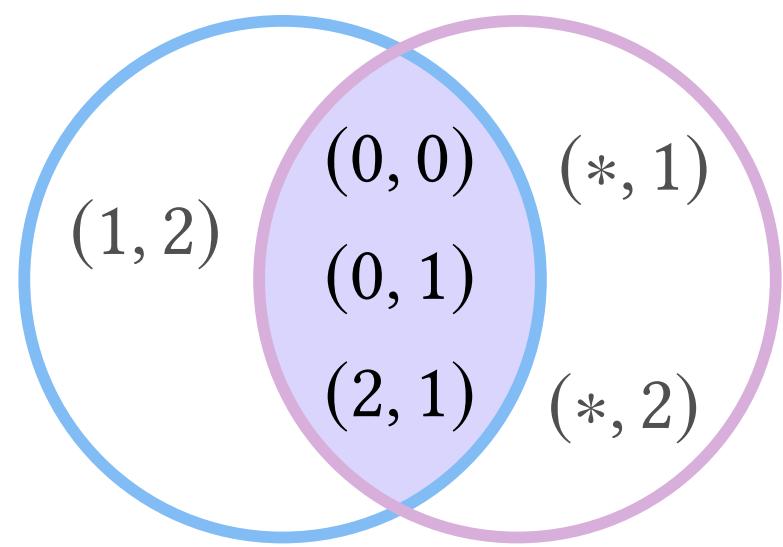
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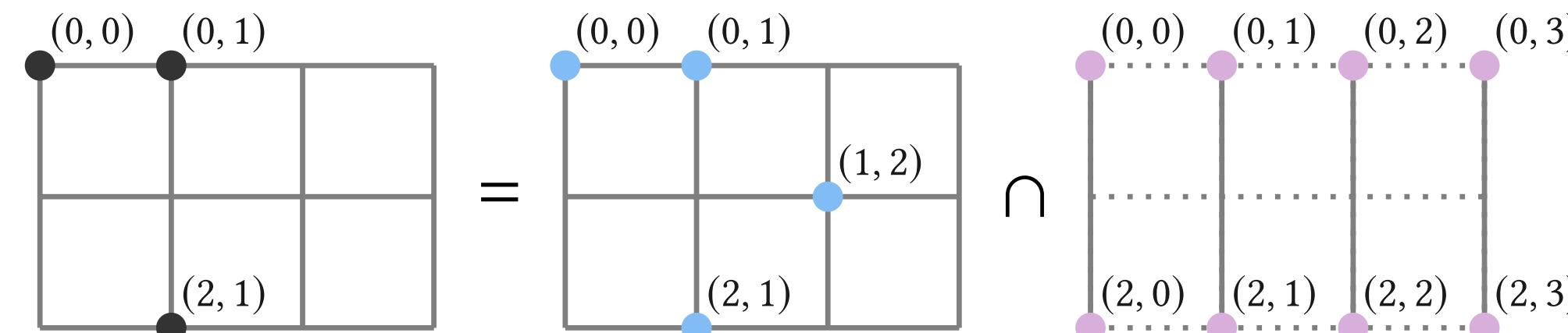
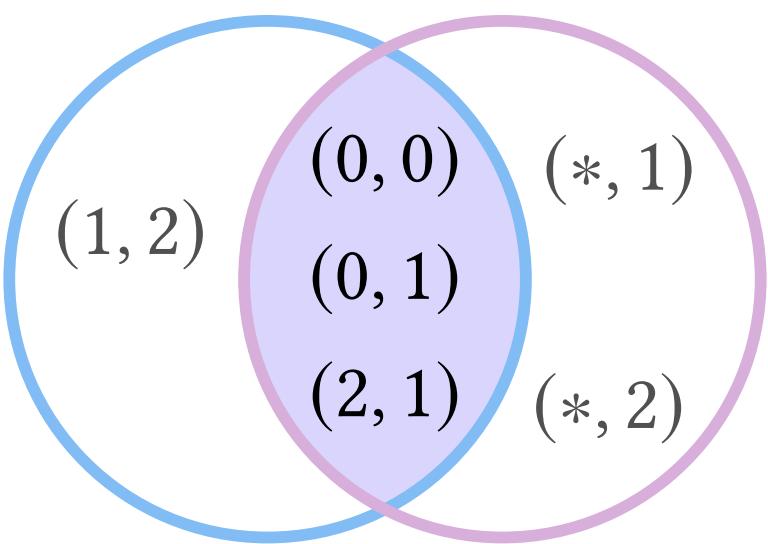
# Iteration spaces from set operations



# Iteration spaces from broadcast operations

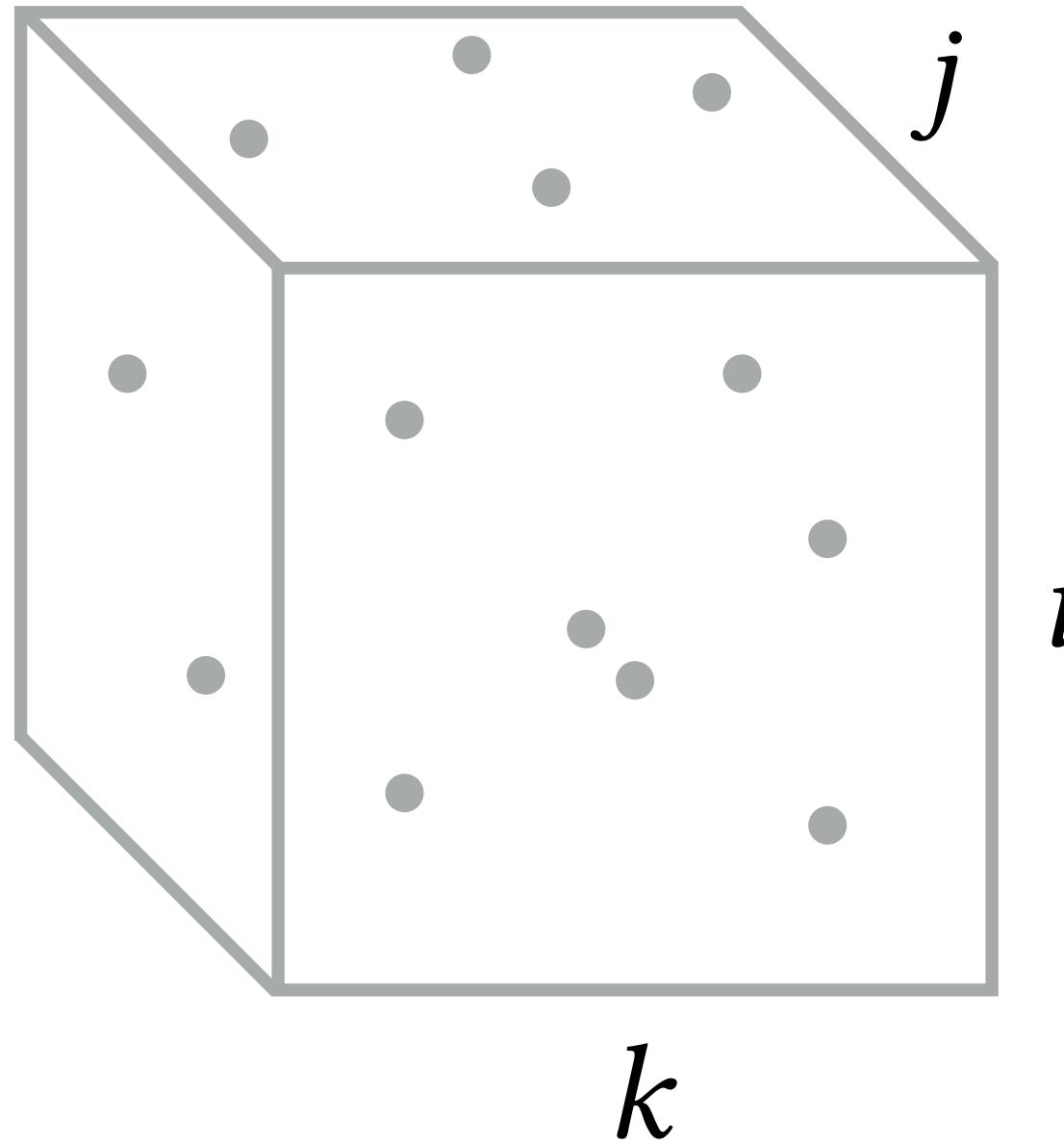


# Iteration spaces from broadcast operations

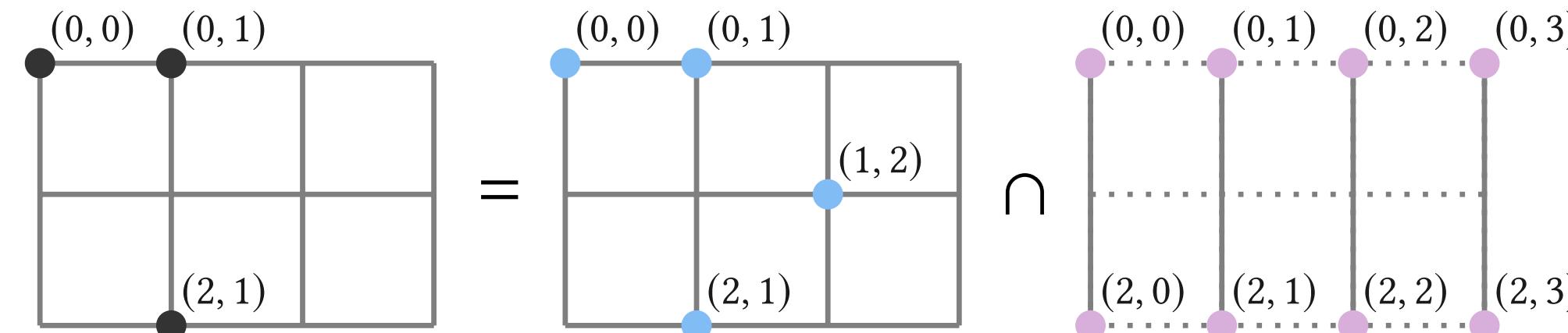
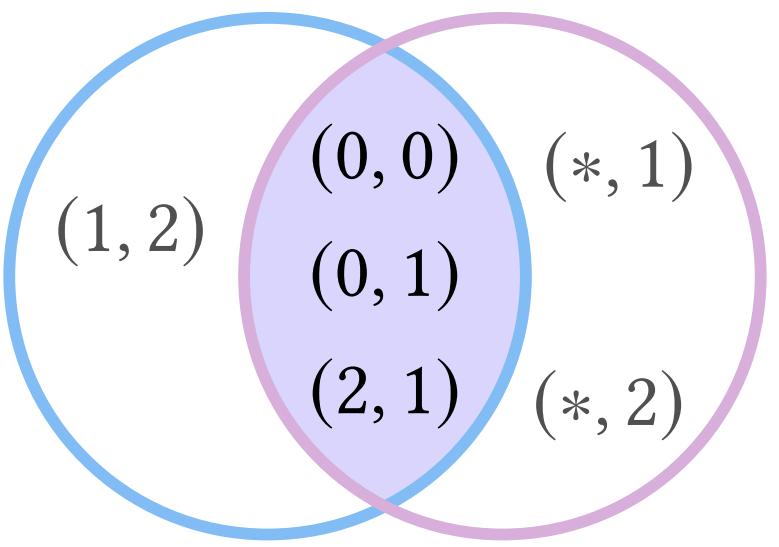


$$A_{ij} = \sum_k B_{ik} C_{kj}$$

$$B_{ik} \cap C_{kj}$$

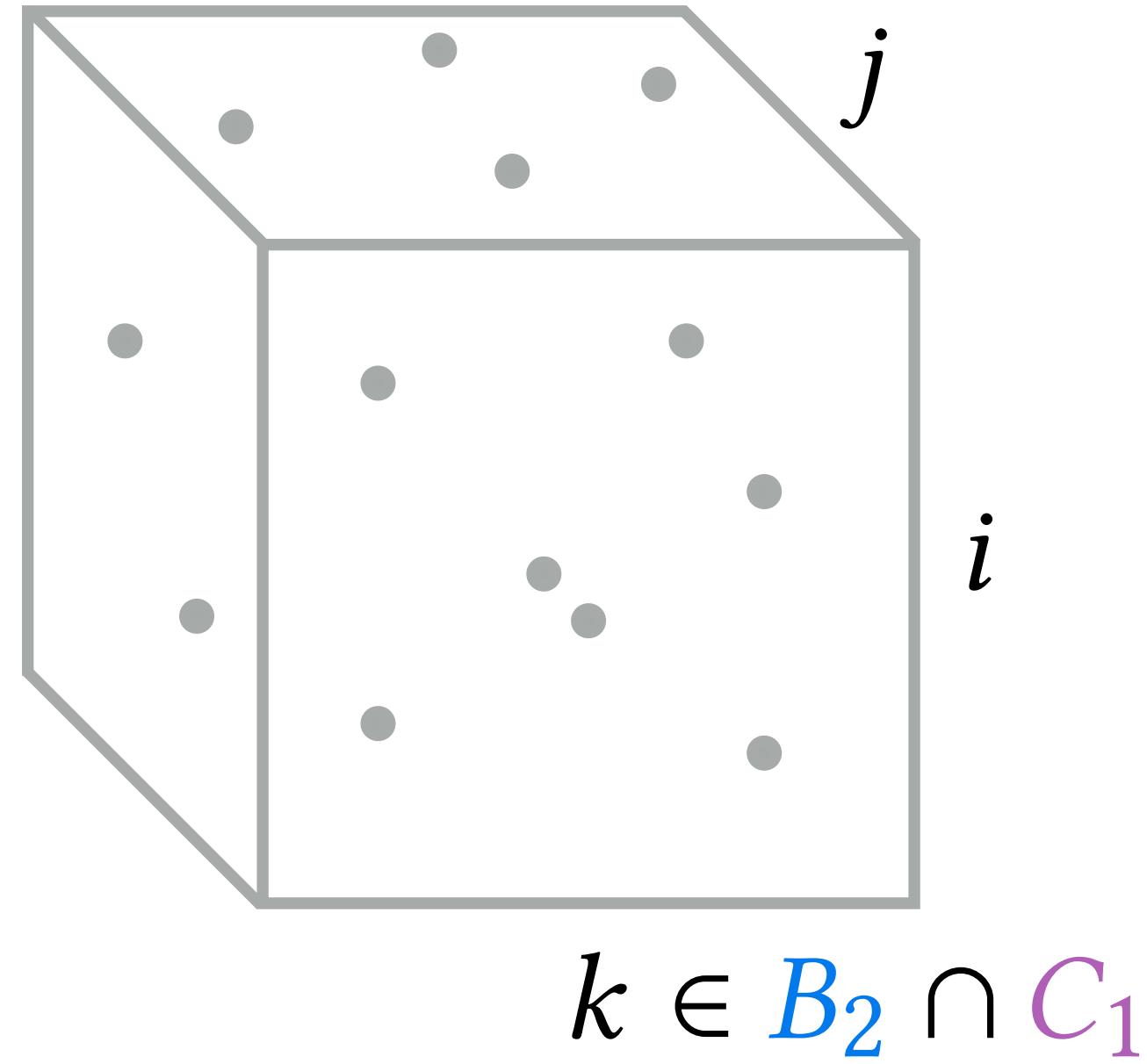


# Iteration spaces from broadcast operations

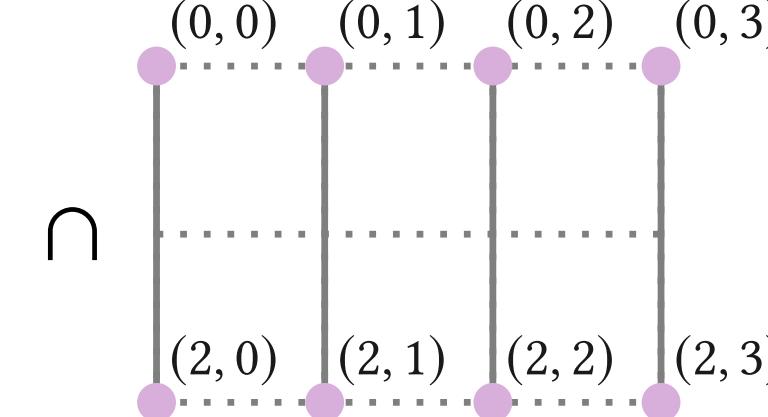
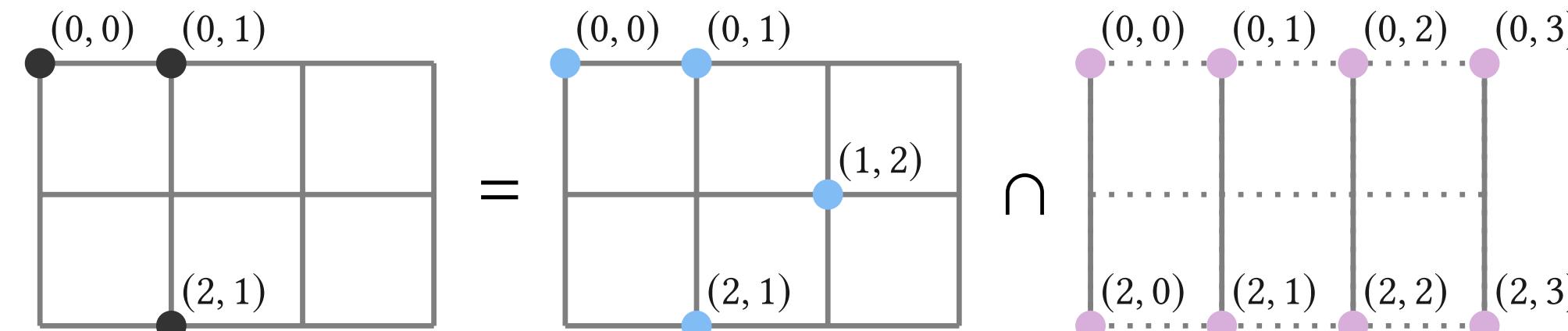
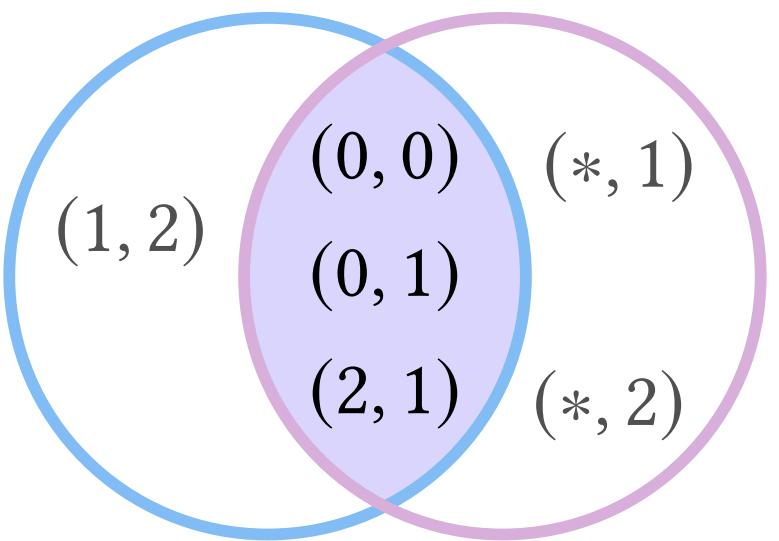


$$A_{ij} = \sum_k B_{ik} C_{kj}$$

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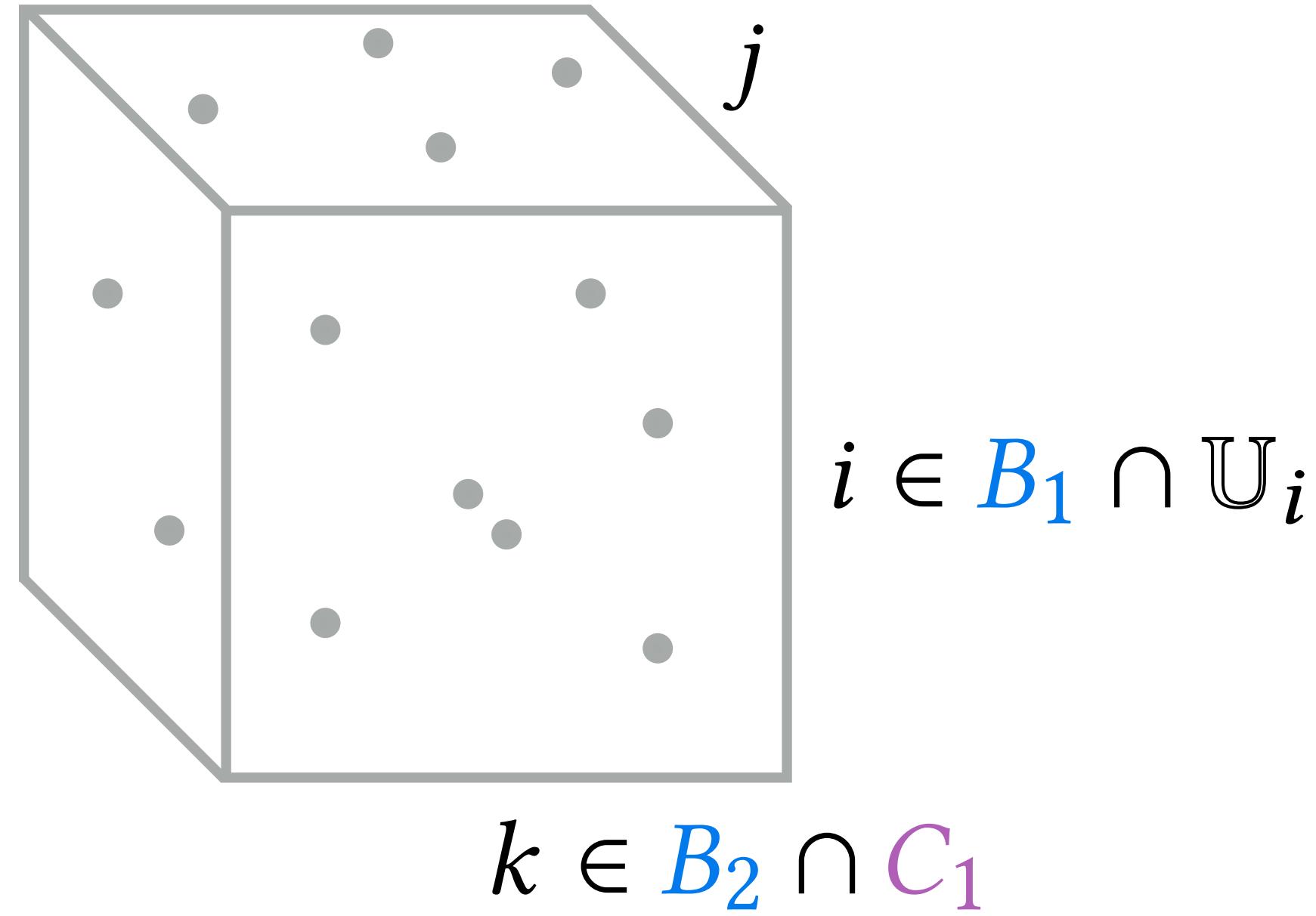


# Iteration spaces from broadcast operations

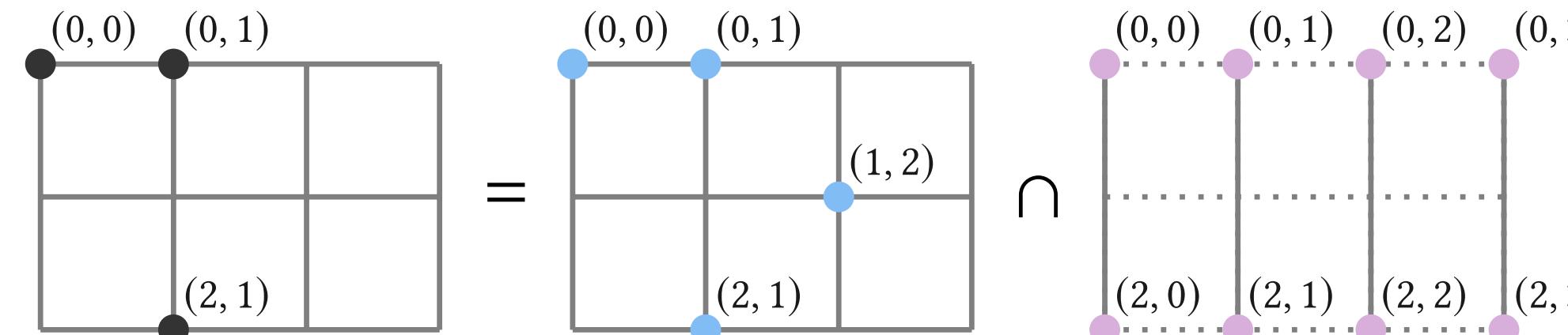
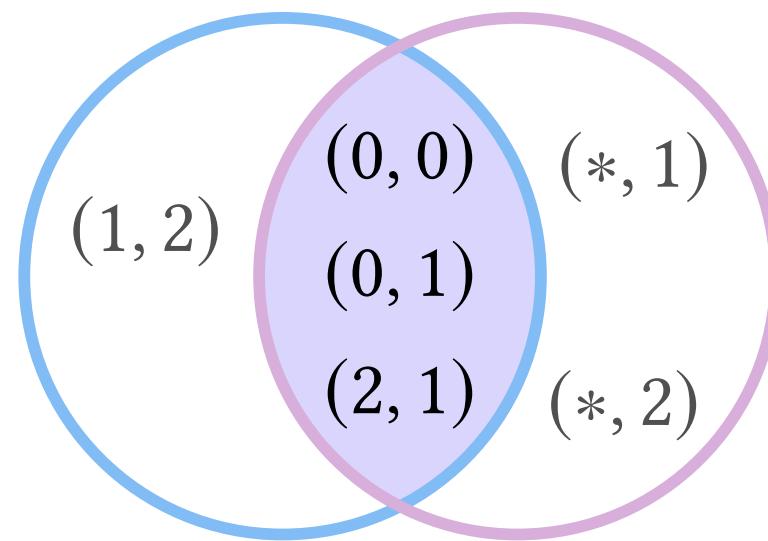


$$A_{ij} = \sum_k B_{ik} C_{kj}$$

$$B_{ik} \cap C_{kj}$$

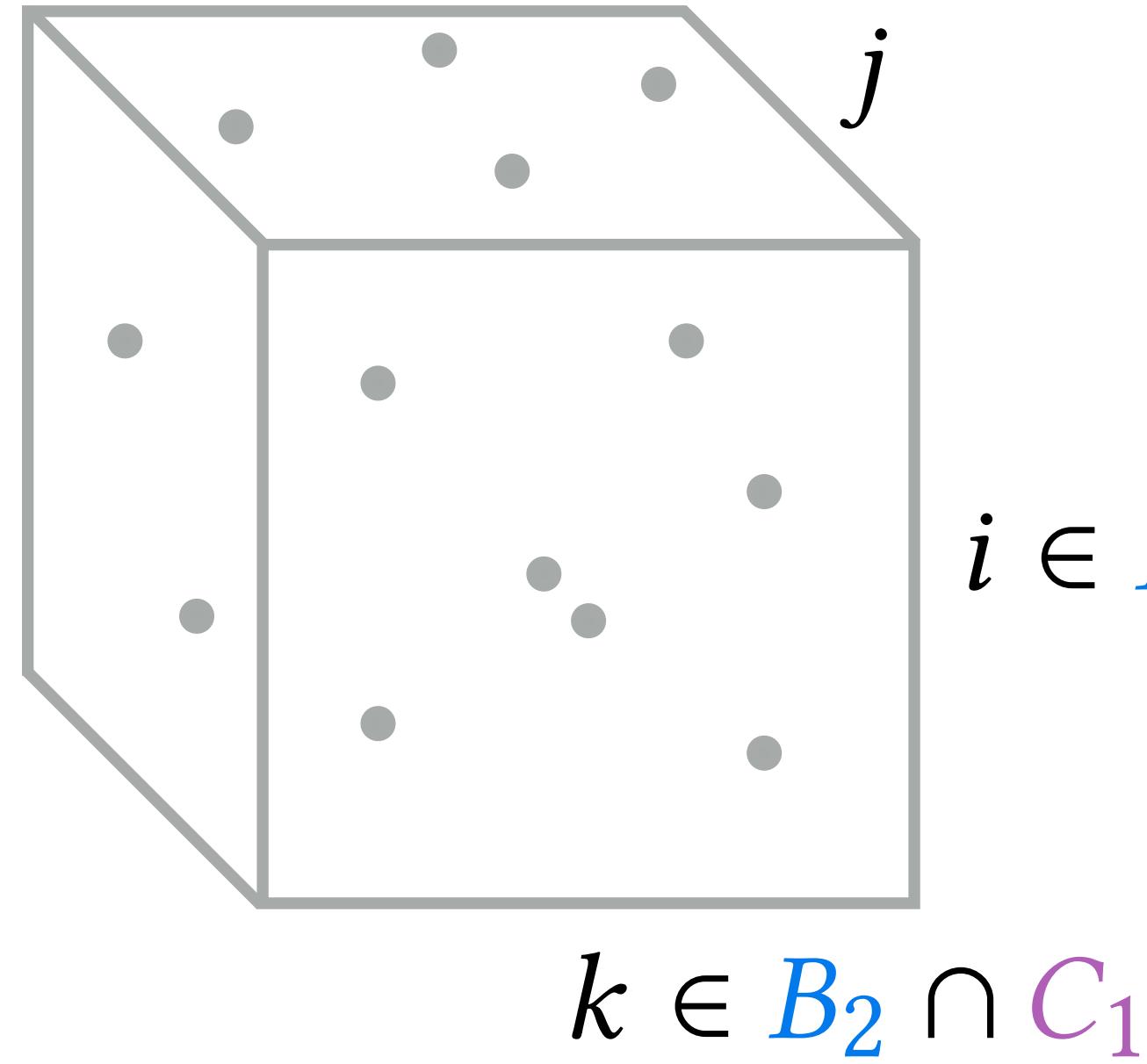


# Iteration spaces from broadcast operations



$$A_{ij} = \sum_k B_{ik} C_{kj}$$

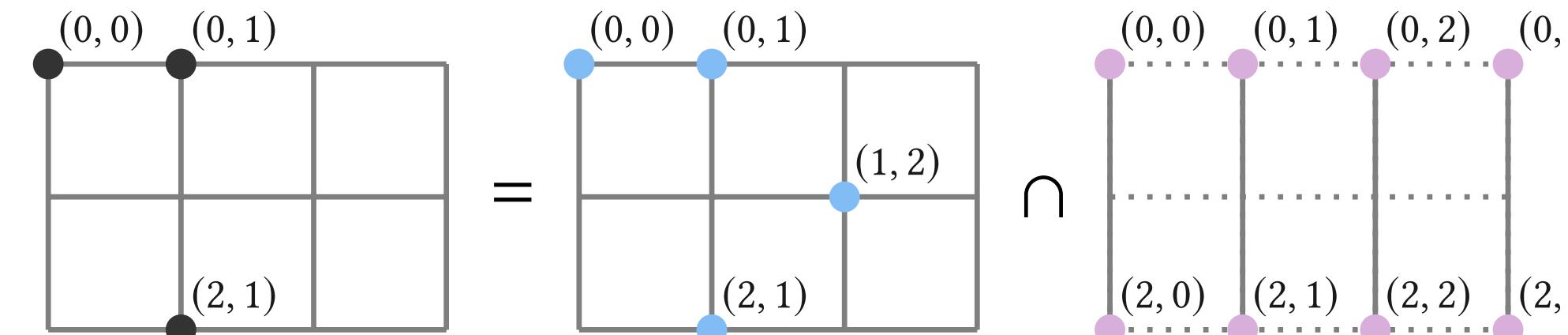
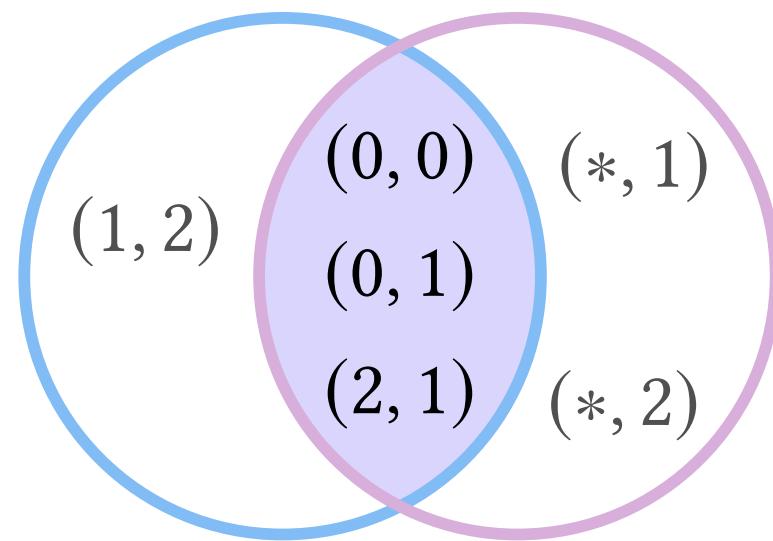
$$B_{ik} \cap C_{kj}$$



$$i \in B_1 \cap \mathbb{U}_i = B_1$$

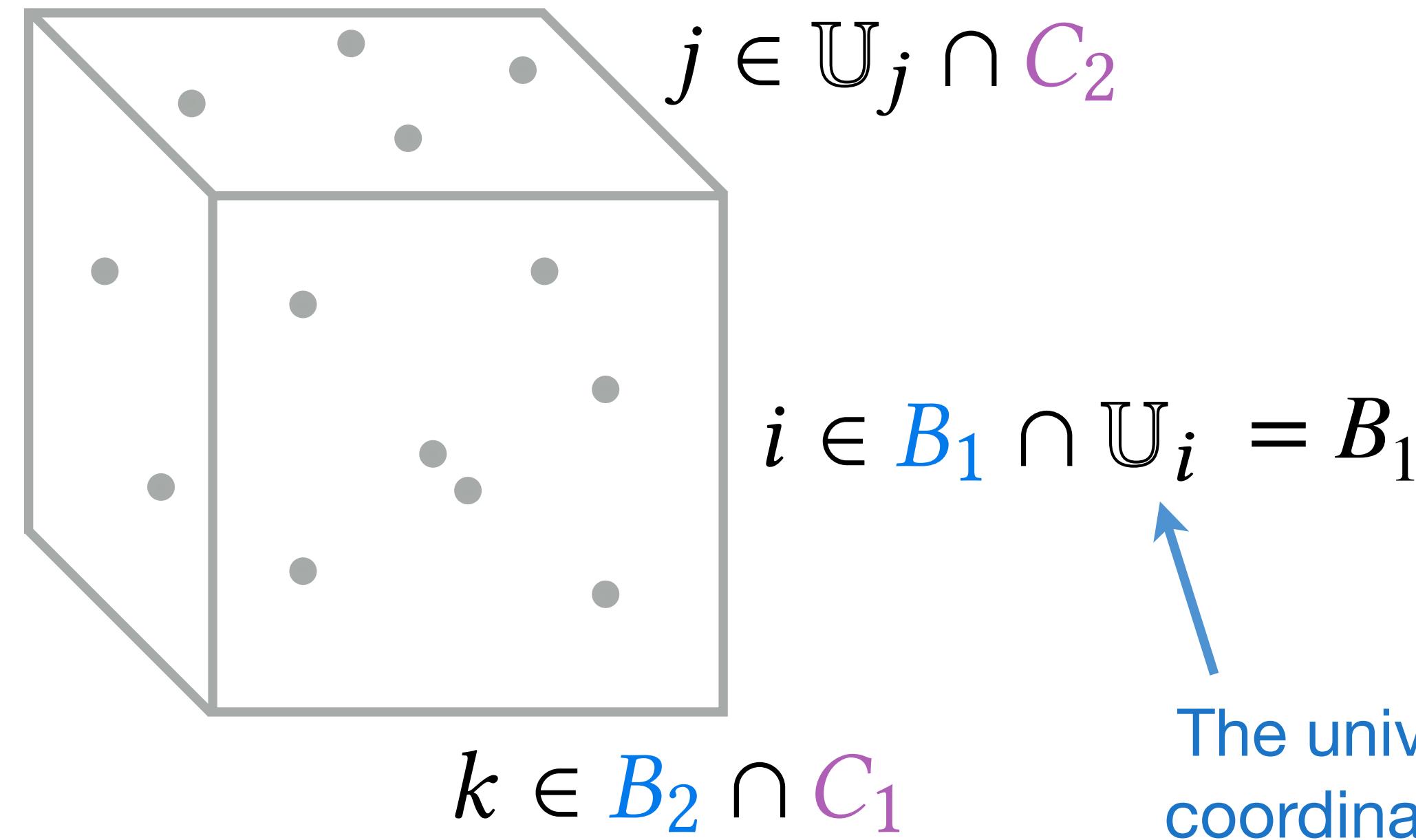
The universe of  $i$  consist of all coordinates, of which any data structure stores a subset.

# Iteration spaces from broadcast operations



$$A_{ij} = \sum_k B_{ik} C_{kj}$$

$$B_{ik} \cap C_{kj}$$



The universe of  $i$  consist of all coordinates, of which any data structure stores a subset.

# Coordinate relations → coordinate trees (abstractly)

Matrix

	$j_1$	$j_2$	$j_3$
$i_1$	a	b	
$i_2$		e	
$i_3$	g	h	i

# Coordinate relations → coordinate trees (abstractly)

Matrix

	$j_1$	$j_2$	$j_3$
$i_1$	a	b	
$i_2$		e	
$i_3$	g	h	i



Coordinate Relation

$$(i_1, j_1) \rightarrow a \quad (i_1, j_2) \rightarrow b$$

$$(i_3, j_3) \rightarrow i \quad (i_2, j_2) \rightarrow e$$

$$(i_3, j_1) \rightarrow g$$

$$(i_3, j_2) \rightarrow h$$

# Coordinate relations → coordinate trees (abstractly)

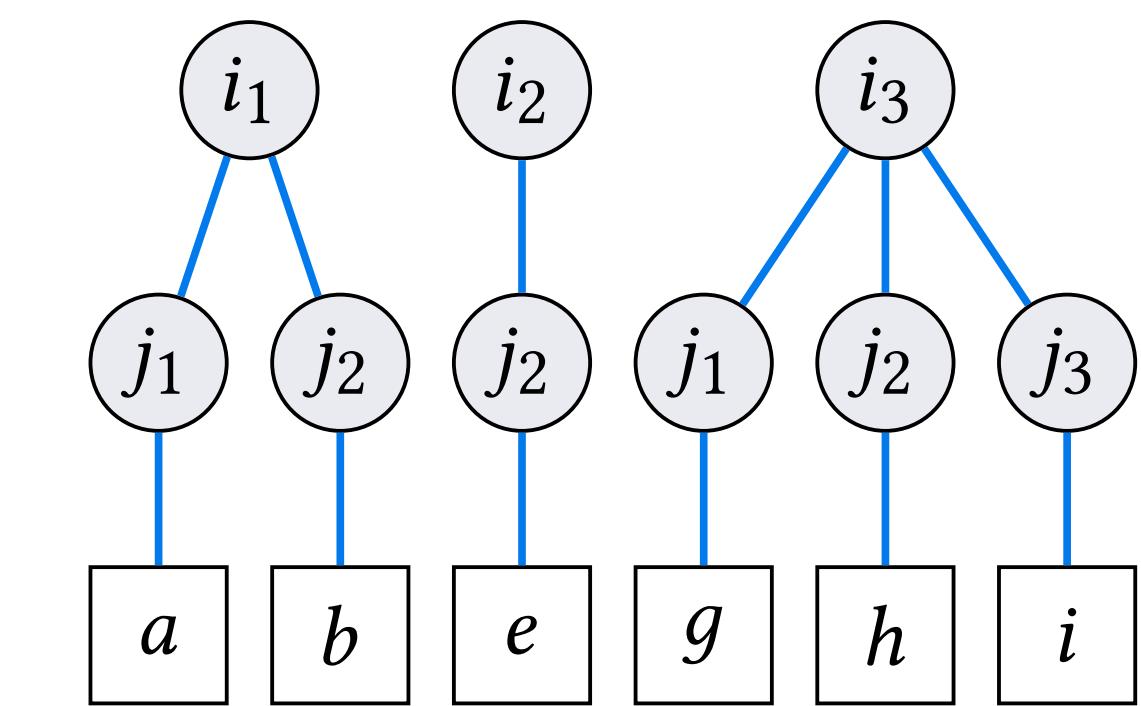
Matrix

	$j_1$	$j_2$	$j_3$
$i_1$	a	b	
$i_2$		e	
$i_3$	g	h	i

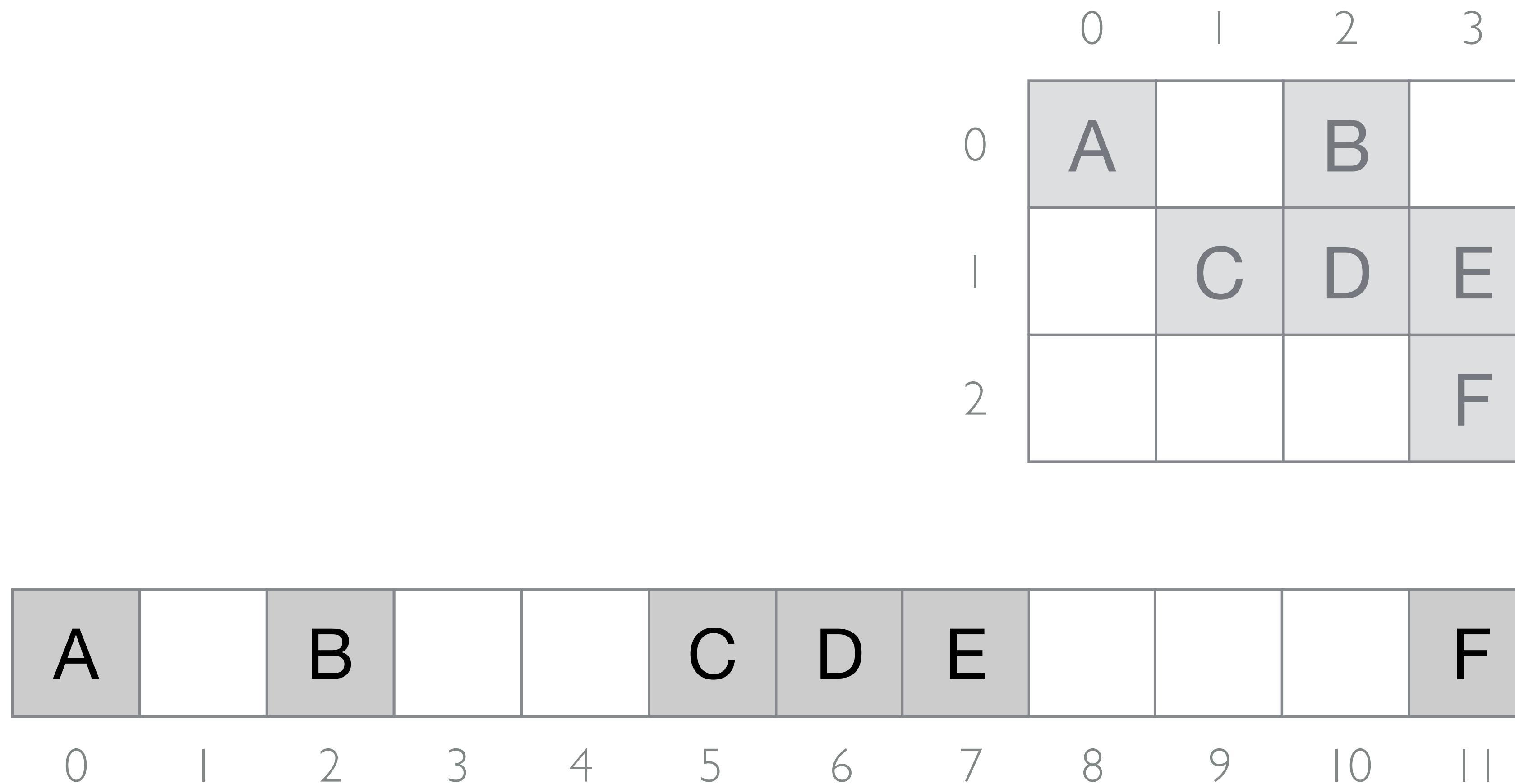
Coordinate Relation

$$\begin{array}{ll} (i_1, j_1) \rightarrow a & (i_1, j_2) \rightarrow b \\ (i_3, j_3) \rightarrow i & (i_2, j_2) \rightarrow e \\ (i_3, j_1) \rightarrow g & \\ (i_3, j_2) \rightarrow h & \end{array}$$

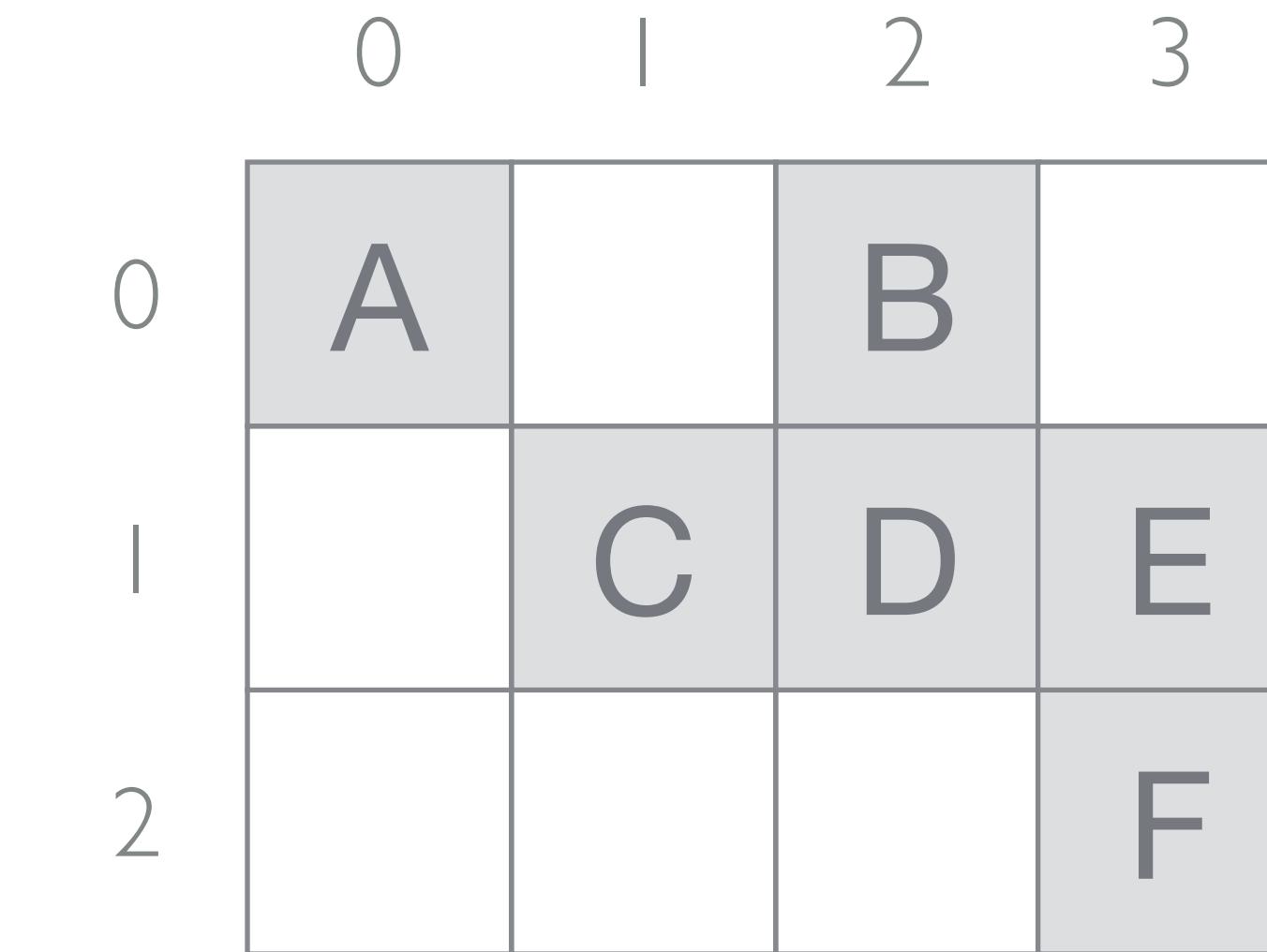
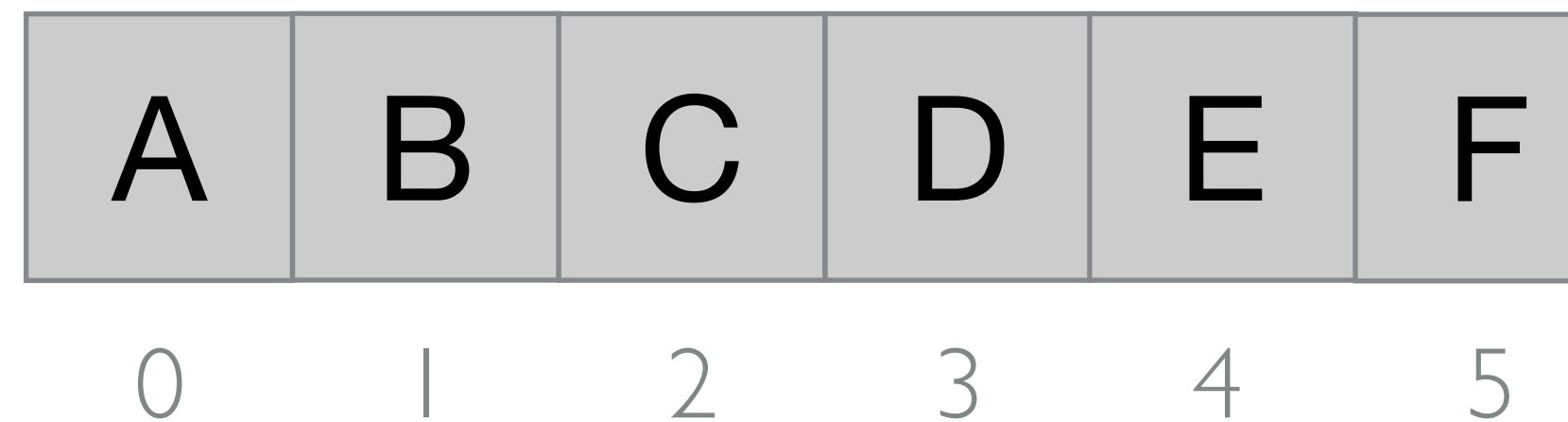
Coordinate Tree



# Coordinate relations → coordinate trees (concretely)



# Coordinate relations → coordinate trees (concretely)



# Coordinate relations → coordinate trees (concretely)

$\text{row}(3) = ???$

$\text{col}(3) = ???$

	0		2	3
0	A		B	
		C	D	E
2				F



# Coordinate relations → coordinate trees (concretely)

		Coordinate						
		rows	0	0	1	1	1	2
cols	0	0	2	1	2	3	3	3
		A	B	C	D	E	F	

0    |    2    3    4    5

0	A	B		
1		C	D	E
2				F

# Coordinate relations → coordinate trees (concretely)

Coordinate

	Coordinate					
rows	0	0	1	1	1	2
cols	0	2	1	2	3	3

0	I	2	3
A		B	
	C	D	E
2			F



# Coordinate relations → coordinate trees (concretely)

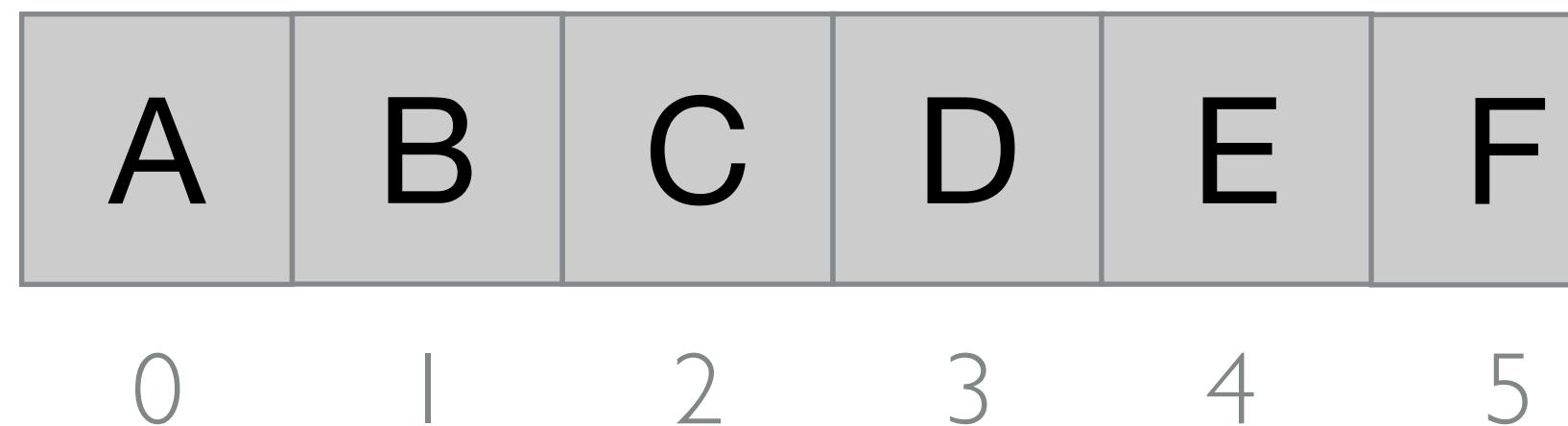
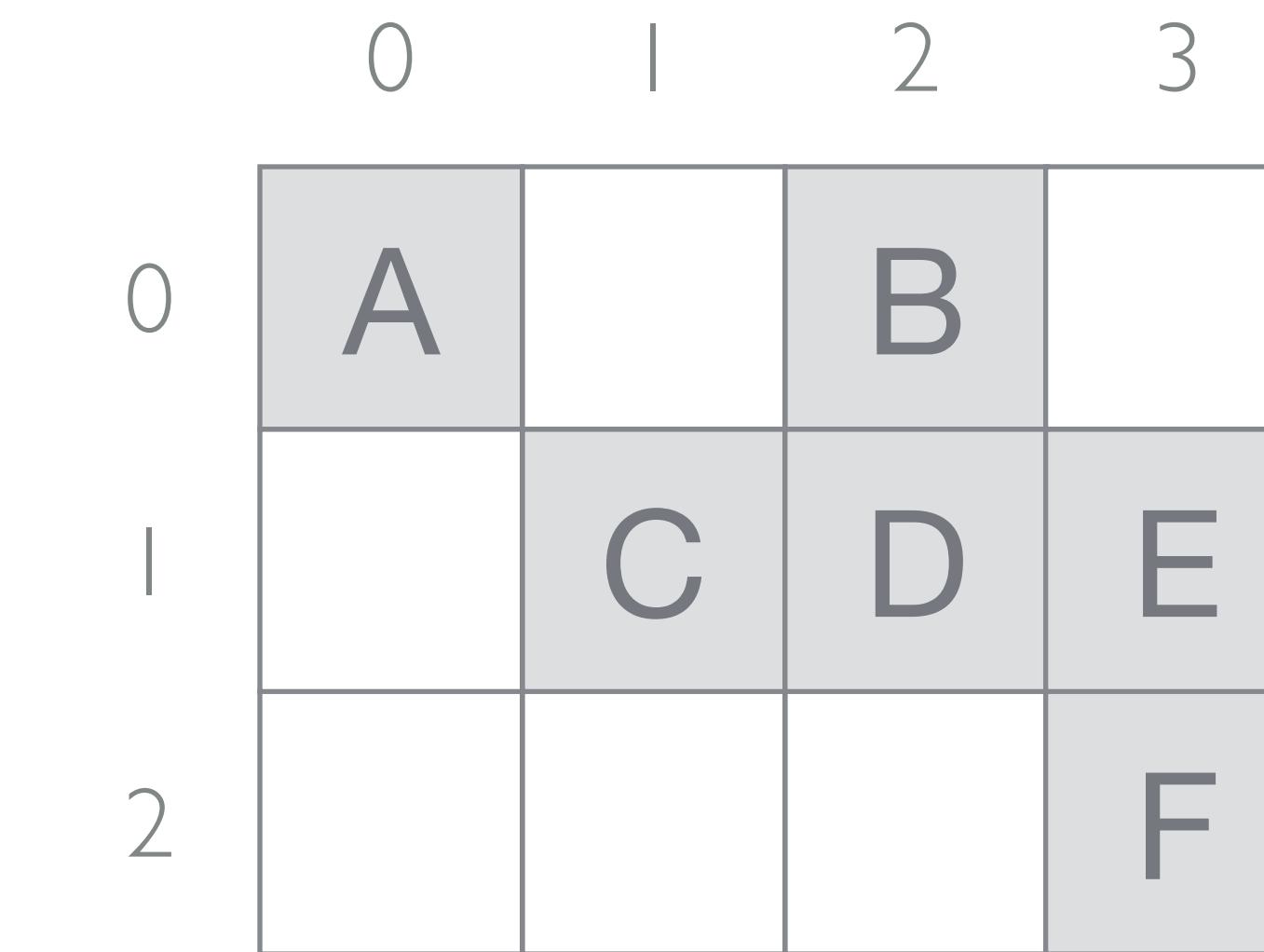
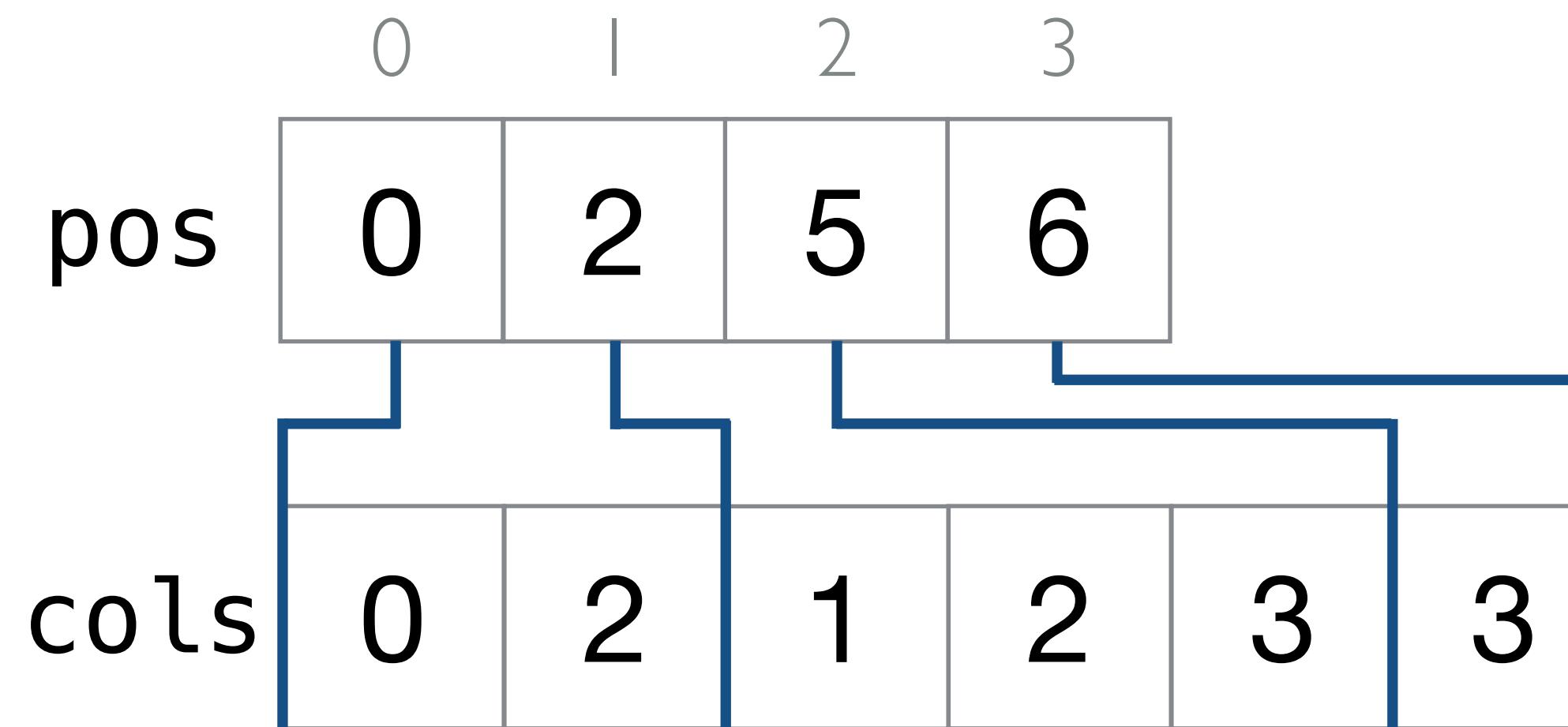
		Coordinate			Duplicates
		rows	0	1	2
cols	0	0	1	1	2

0	1	2	3
A		B	
	C	D	E
			F

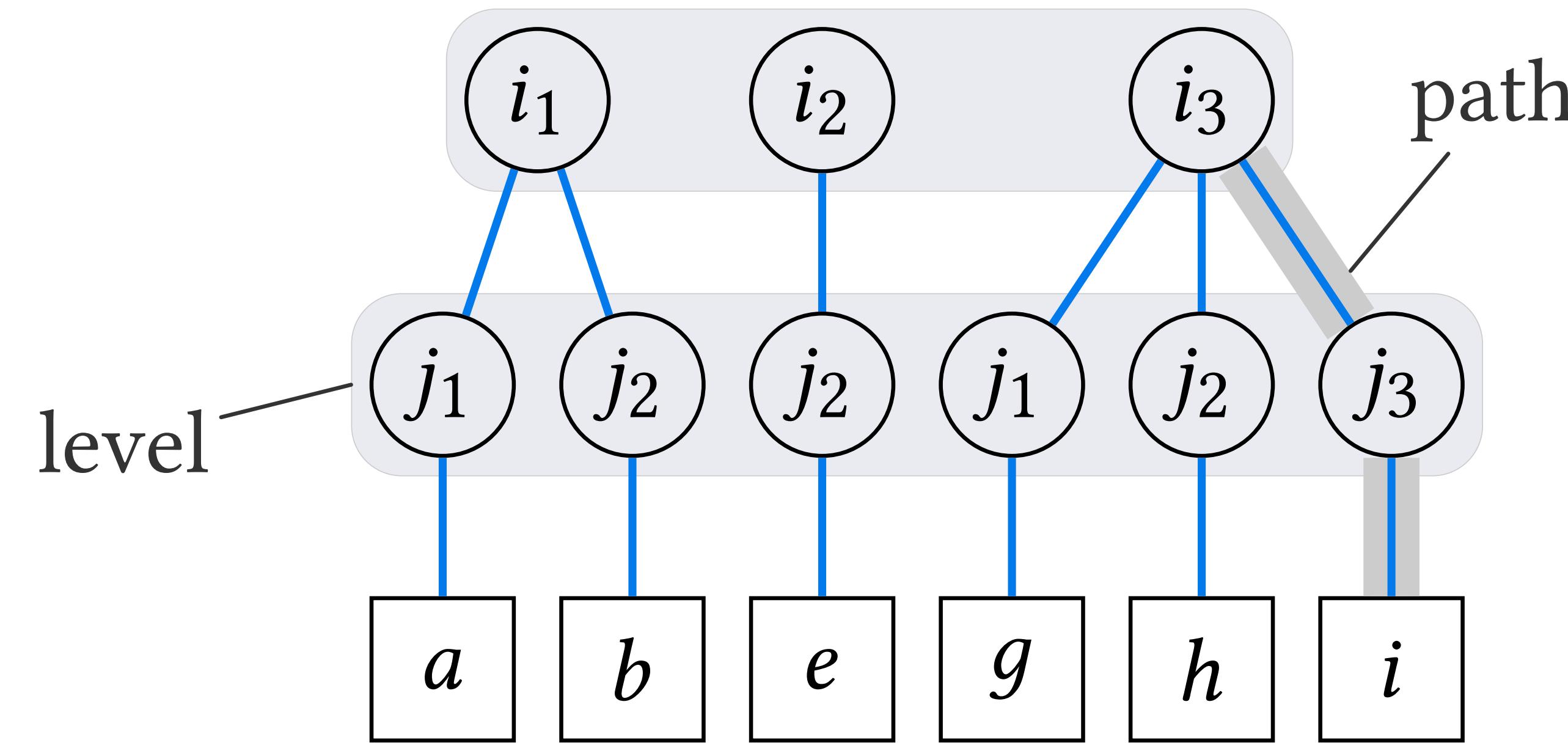


# Coordinate relations → coordinate trees (concretely)

Compressed Sparse Rows (CSR)

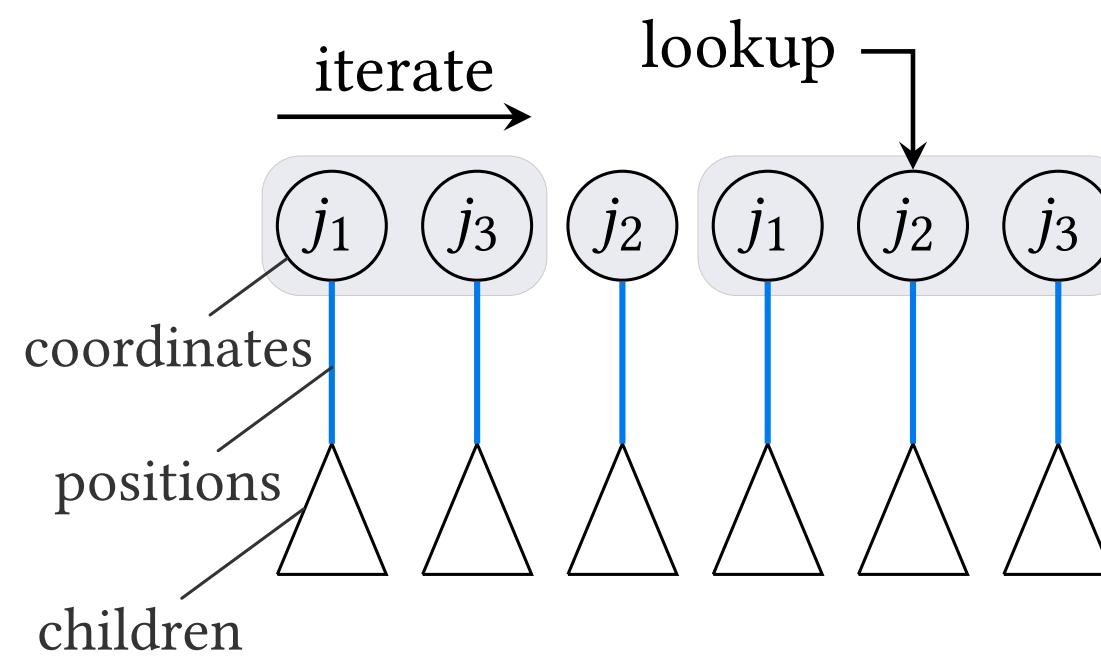


# Level-based representation: compiler abstraction

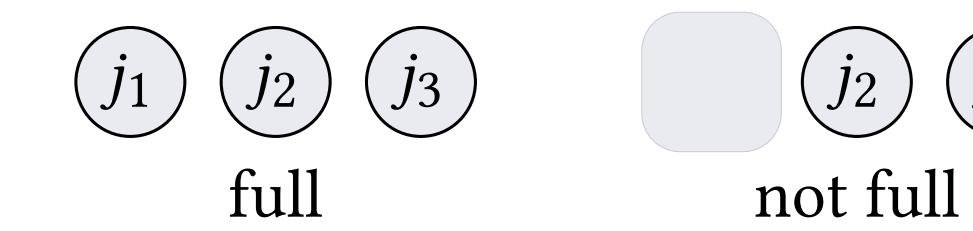


# Level abstraction: capabilities and properties

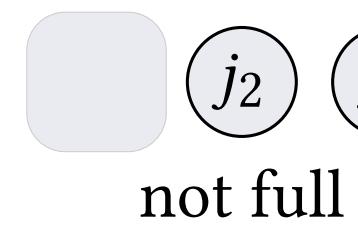
## Capabilities



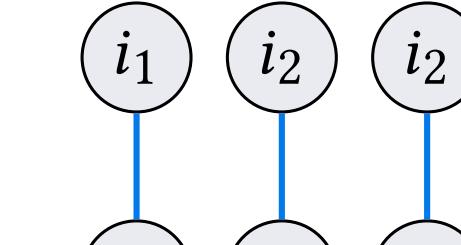
## Properties



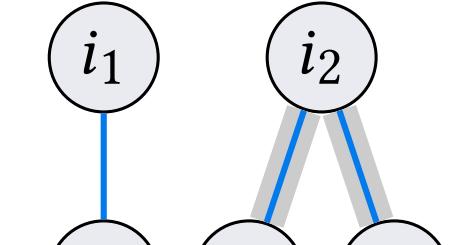
full



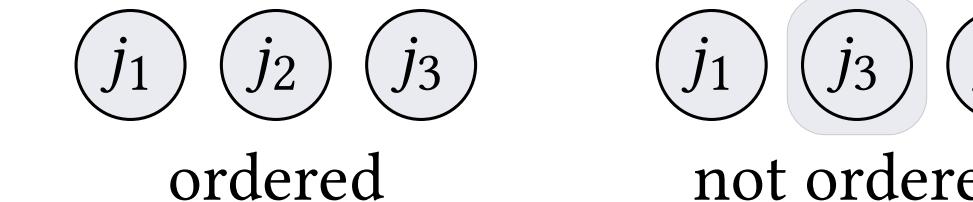
not full



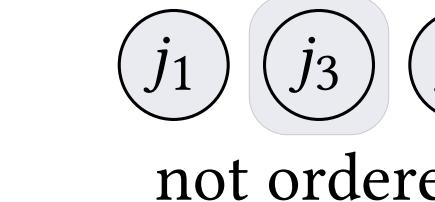
branchless



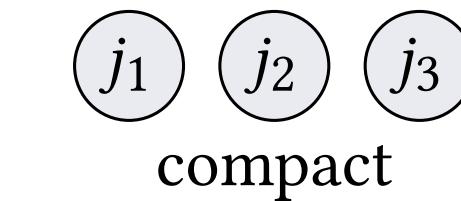
not branchless



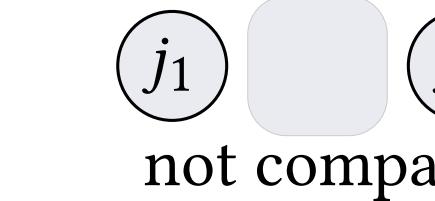
ordered



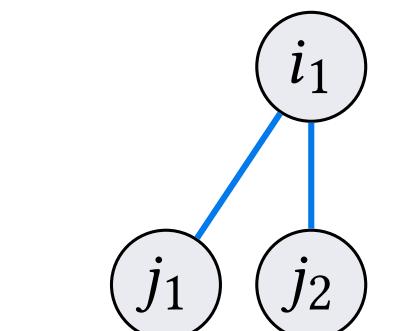
not ordered



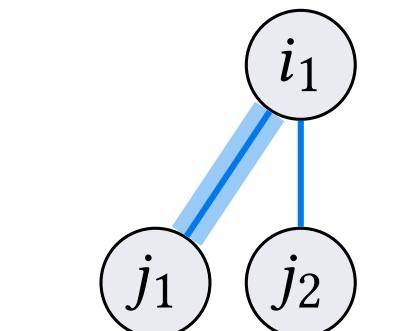
compact



not compact



unique



not unique

The code generator sees only the level abstraction and not specific level types

# Level types: dense and compressed

Dense locate capability:

```
locate(pk-1, i1, ..., ik):  
    return <pk-1 * Nk + ik, true>
```

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Compressed iterate capability

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pos_bounds(pk-1):  
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```
pos_access(pk, i1, ..., ik-1):  
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    return <crd[pk], true>
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---

$$y = Ax$$

```
for (int i = 0; i < m; i++) {  
    for (int pA = A_pos[i]; pA < A_pos[i+1]; pA++) {  
        int j = A_crd[pA];  
        y[i] += A[pA] * x[j];  
    }  
}
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    }  
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}
```

Dense locate

Compressed iterate

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Dense locate capability:

```
locate(pk-1, i1, ..., ik):  
    return <pk-1 * Nk + ik, true>
```

Compressed iterate capability

```
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        int j = A_crd[pA];  
        y[i] += A[pA] * x[j];  
    }  
}
```

Dense locate

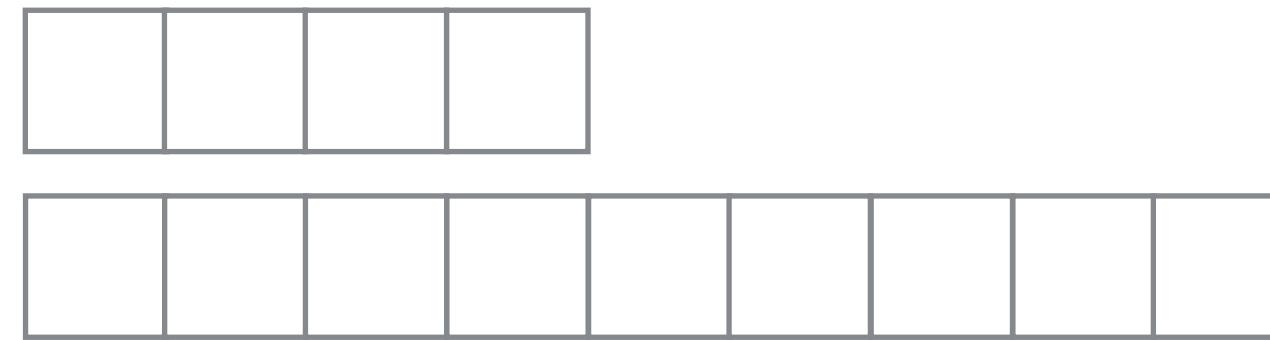
Compressed iterate

# Level types can be composed in many ways

Dense



Compressed



Singleton

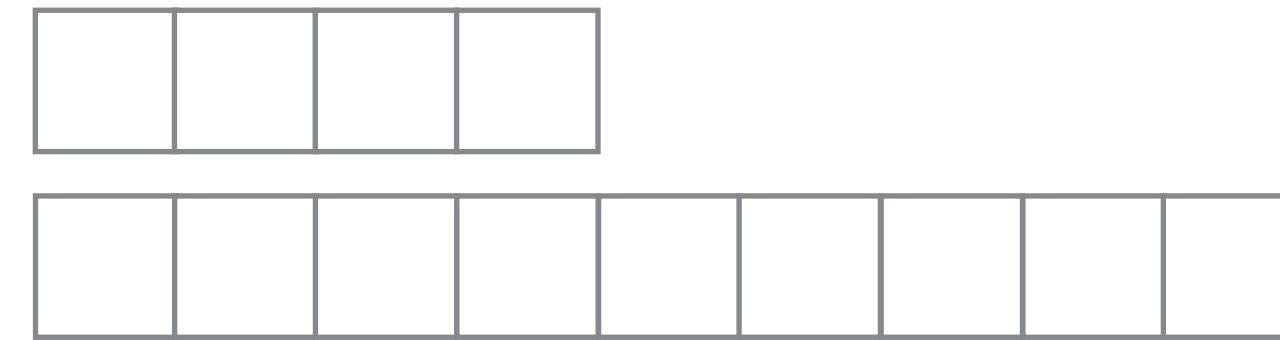


# Level types can be composed in many ways

Dense



Compressed



Singleton



0 | 1 | 2 | 3

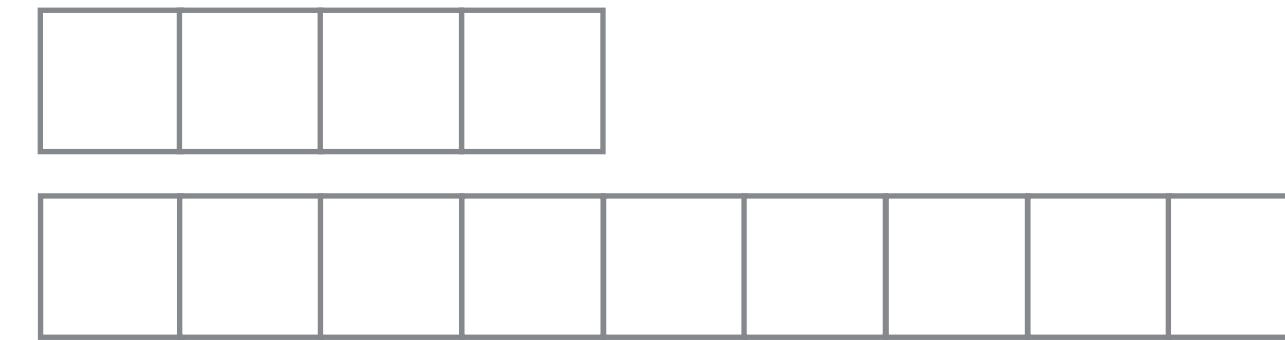
A		B	
	C	D	E
			F

# Level types can be composed in many ways

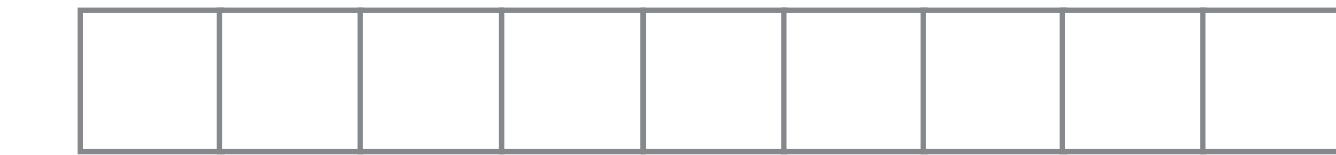
Dense



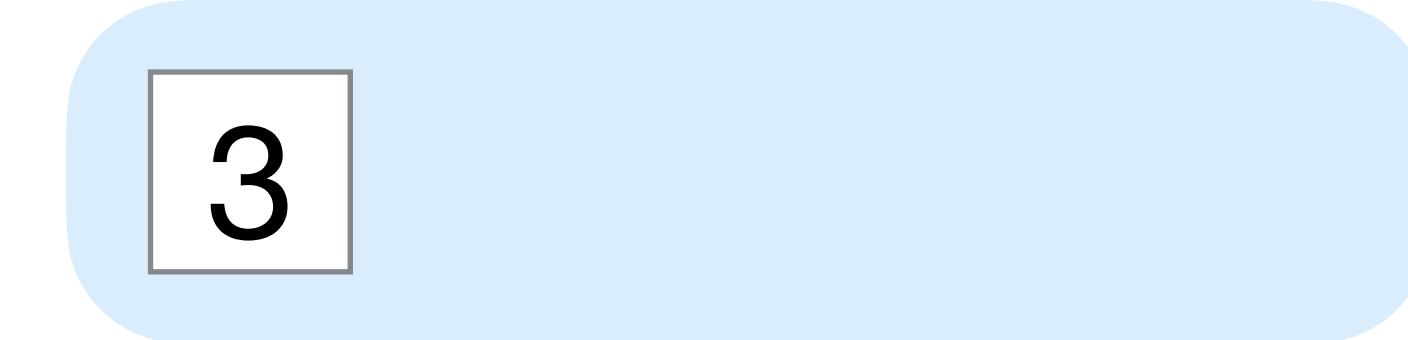
Compressed



Singleton



Dense



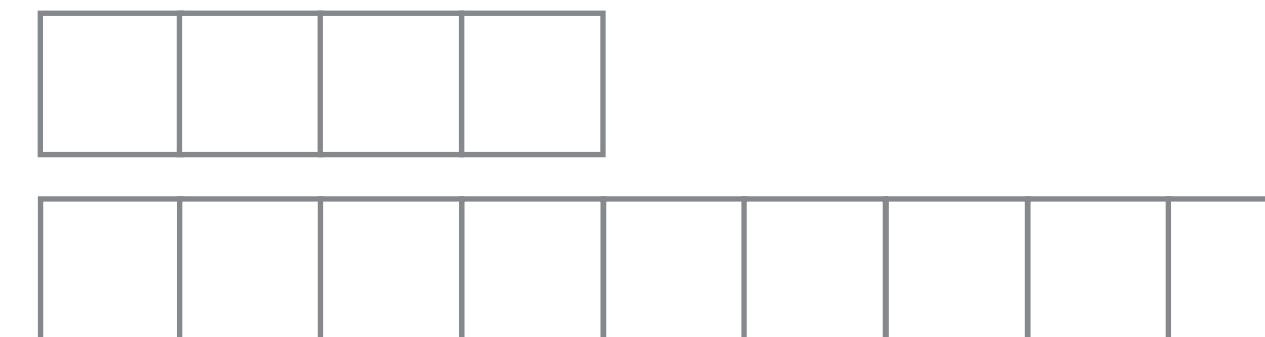
	0	1	2	3
	A		B	
	C	D	E	
				F

# Level types can be composed in many ways

Dense



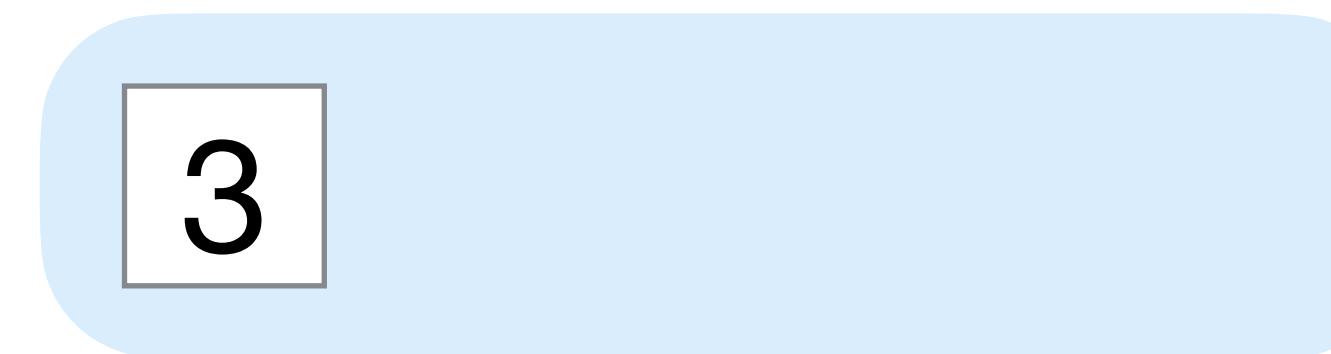
Compressed



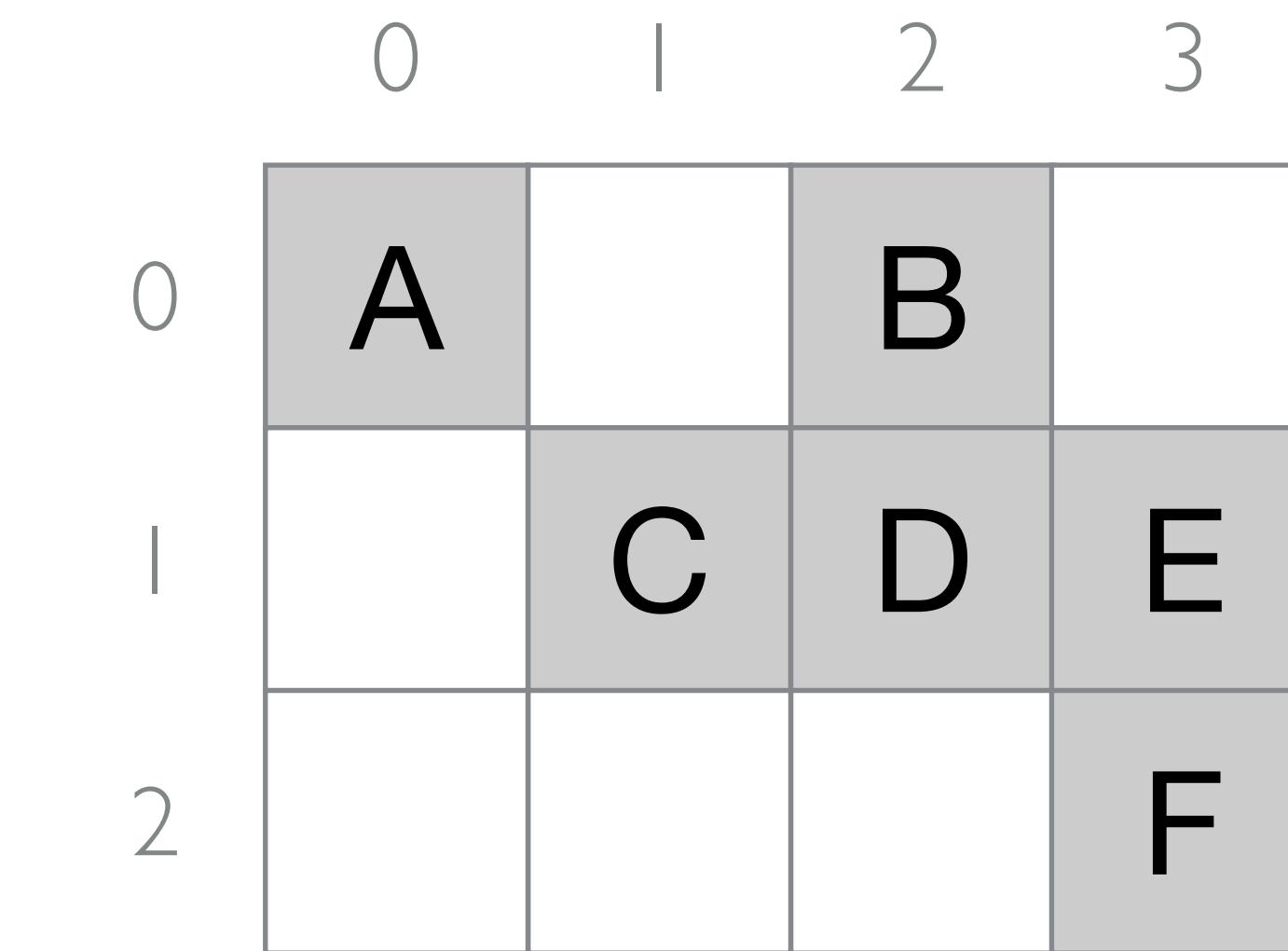
Singleton



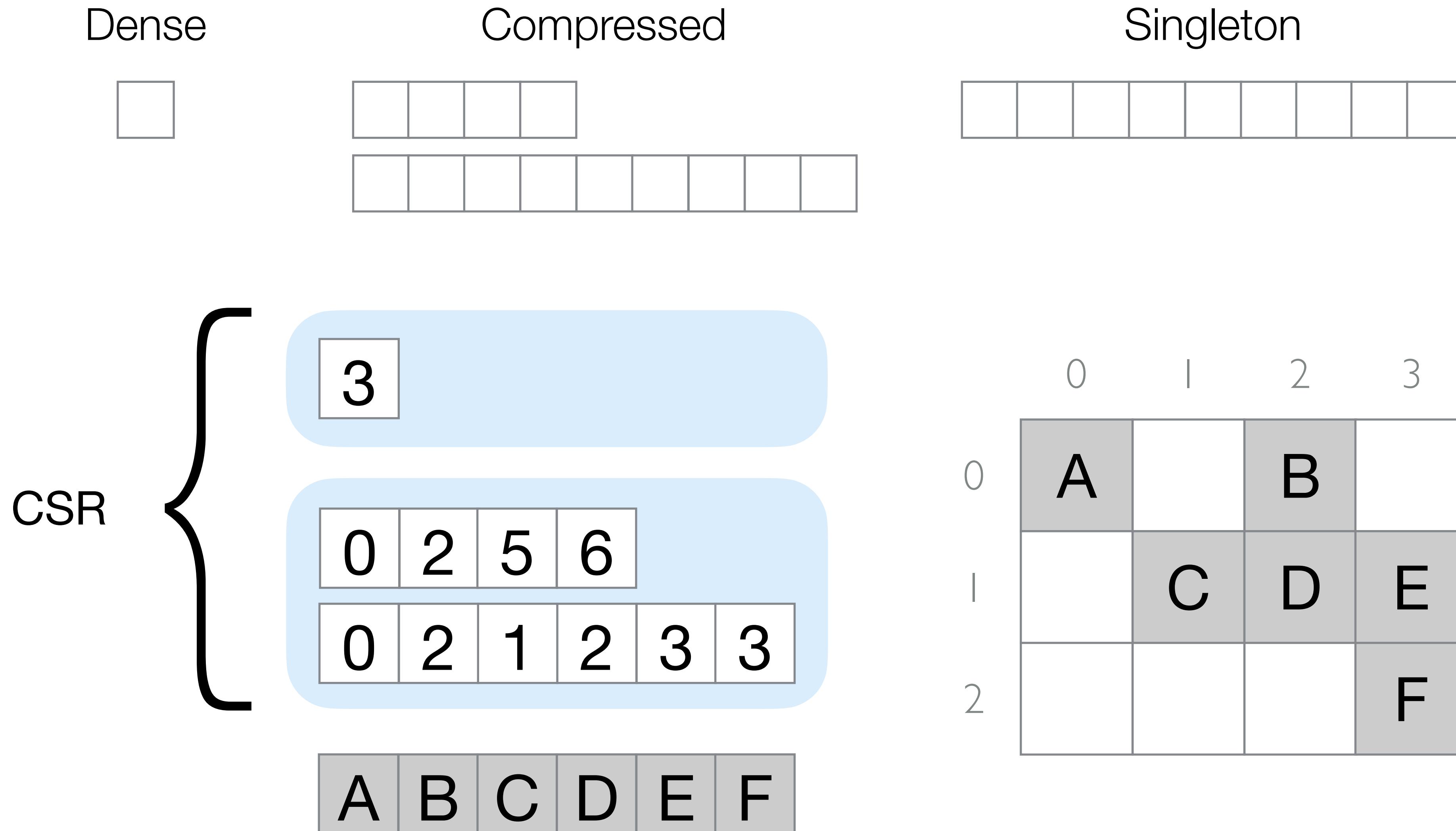
Dense



Compressed



# Level types can be composed in many ways

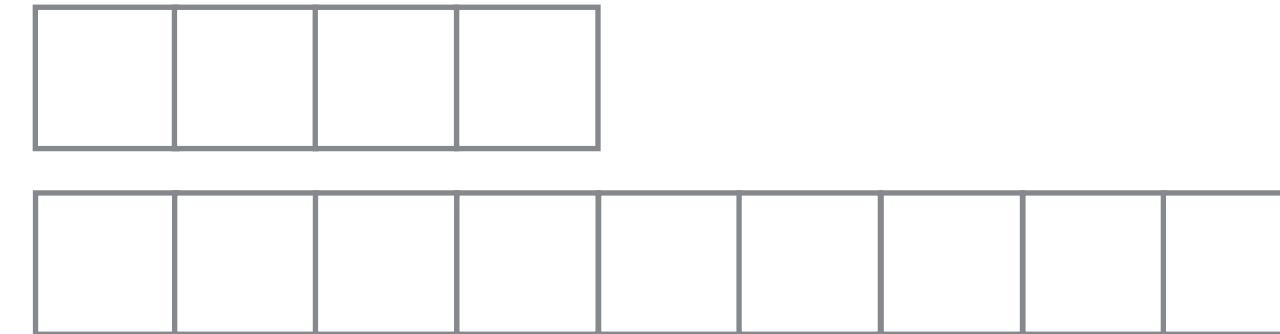


# Level types can be composed in many ways

Dense



Compressed



Singleton



0      |      2      3

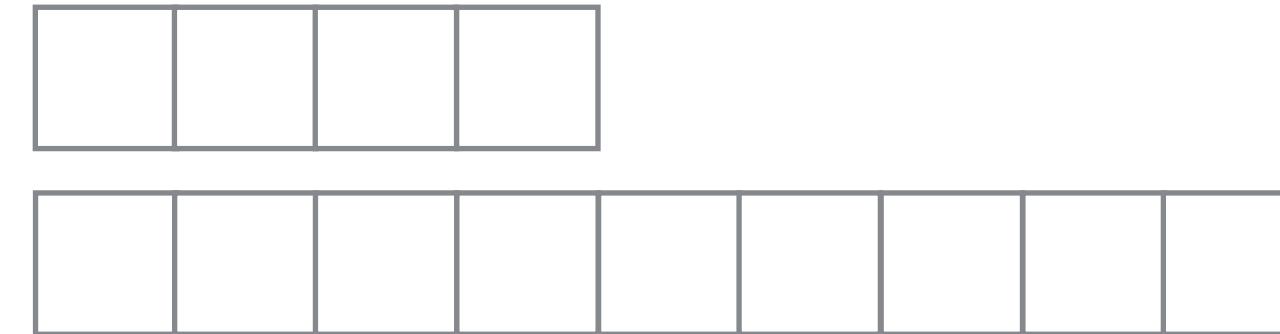
A		B	
C	D	E	
			F

# Level types can be composed in many ways

Dense



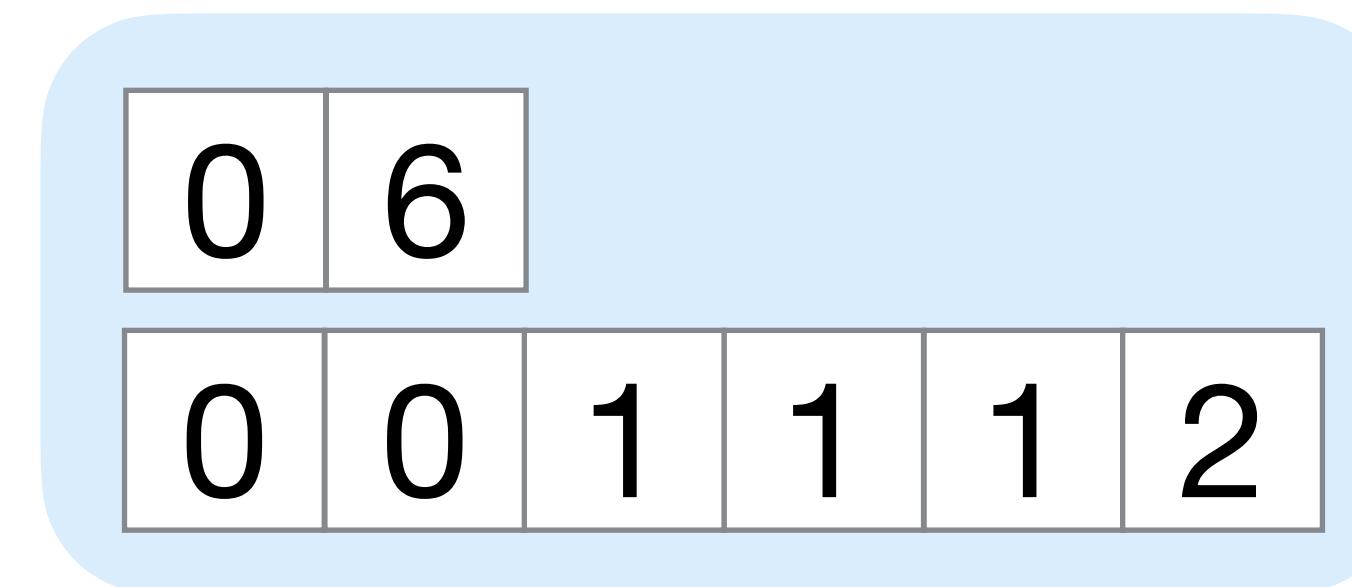
Compressed



Singleton

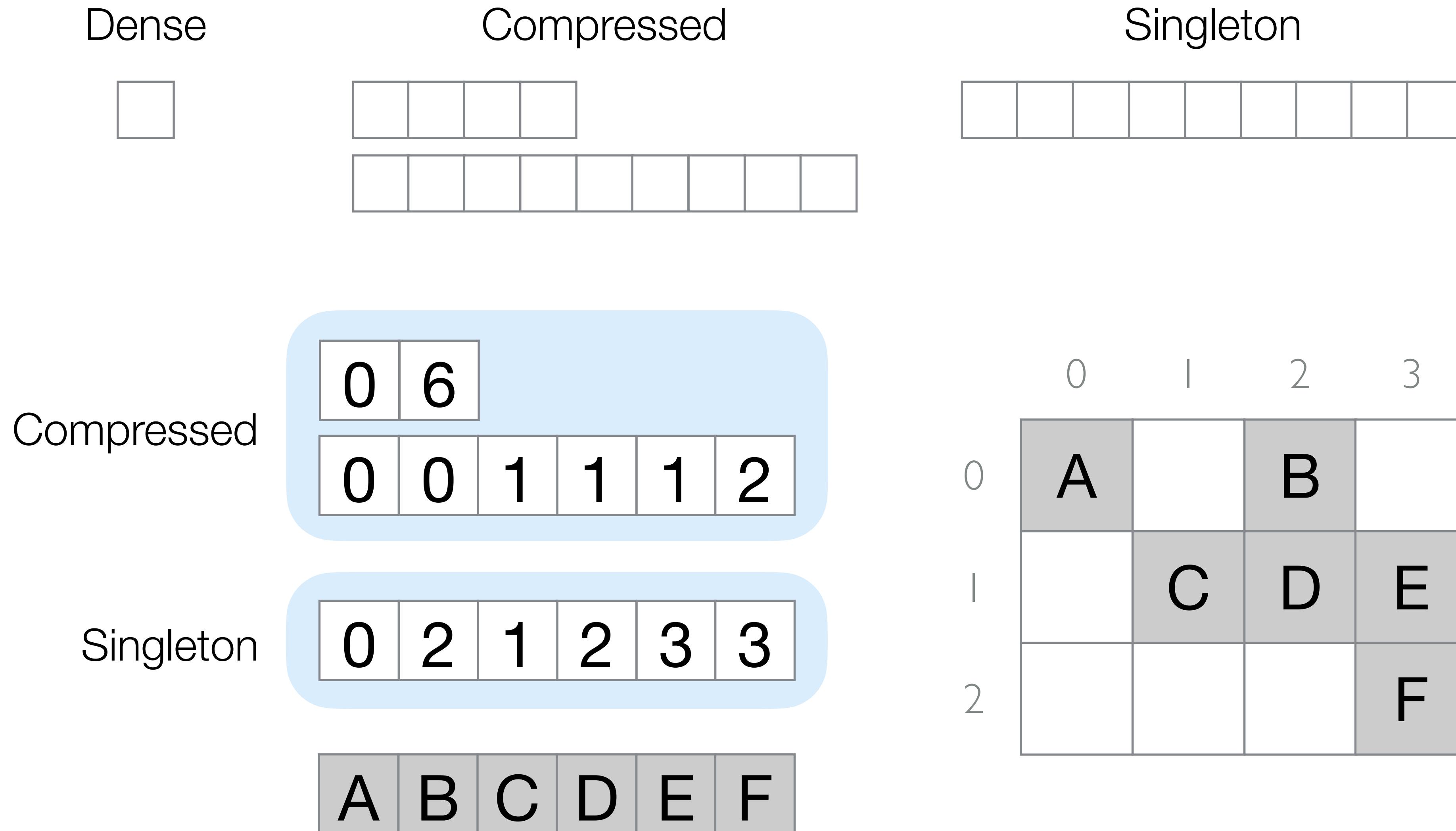


Compressed

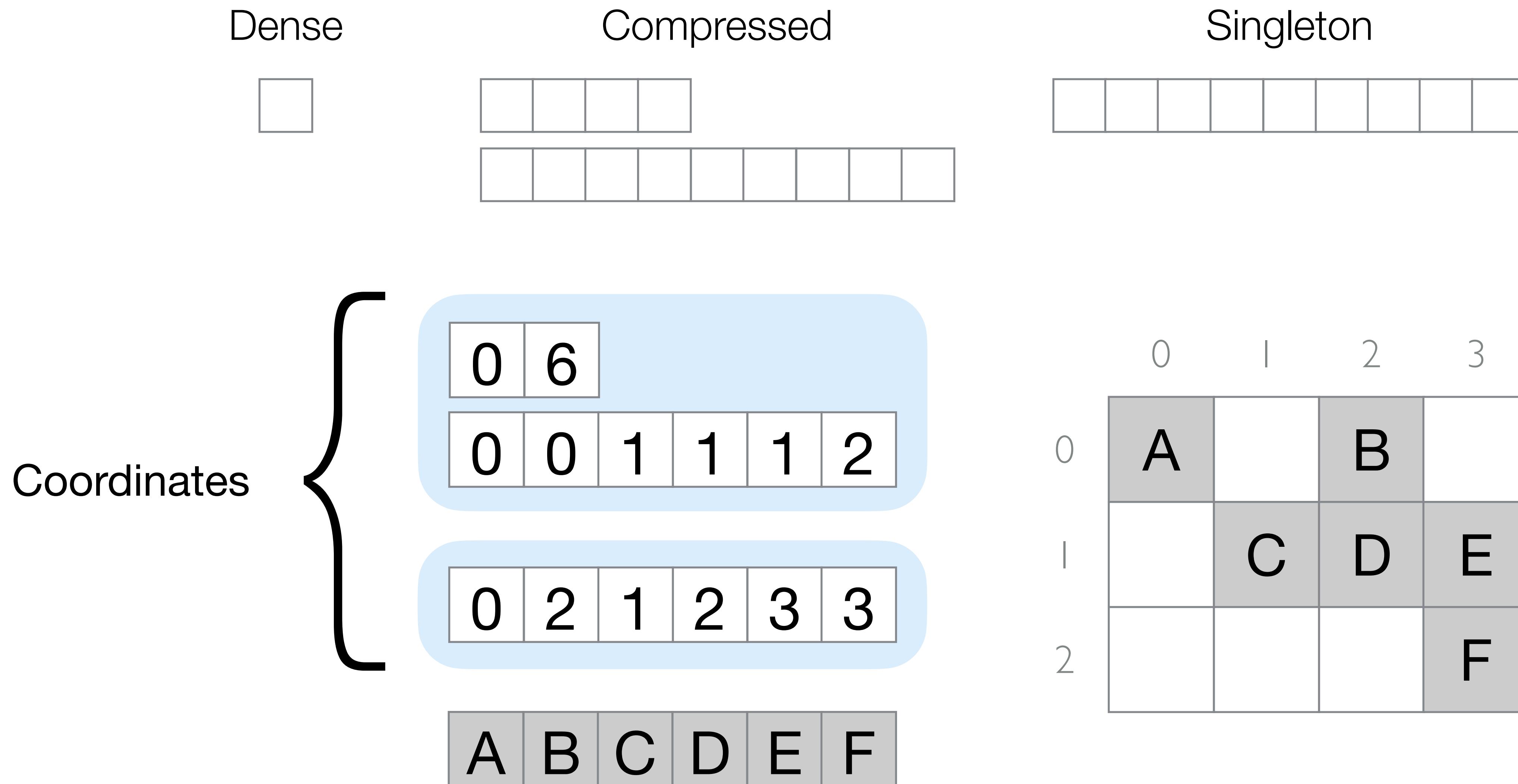


0	A		B	
1	C	D	E	
2			F	

# Level types can be composed in many ways



# Level types can be composed in many ways



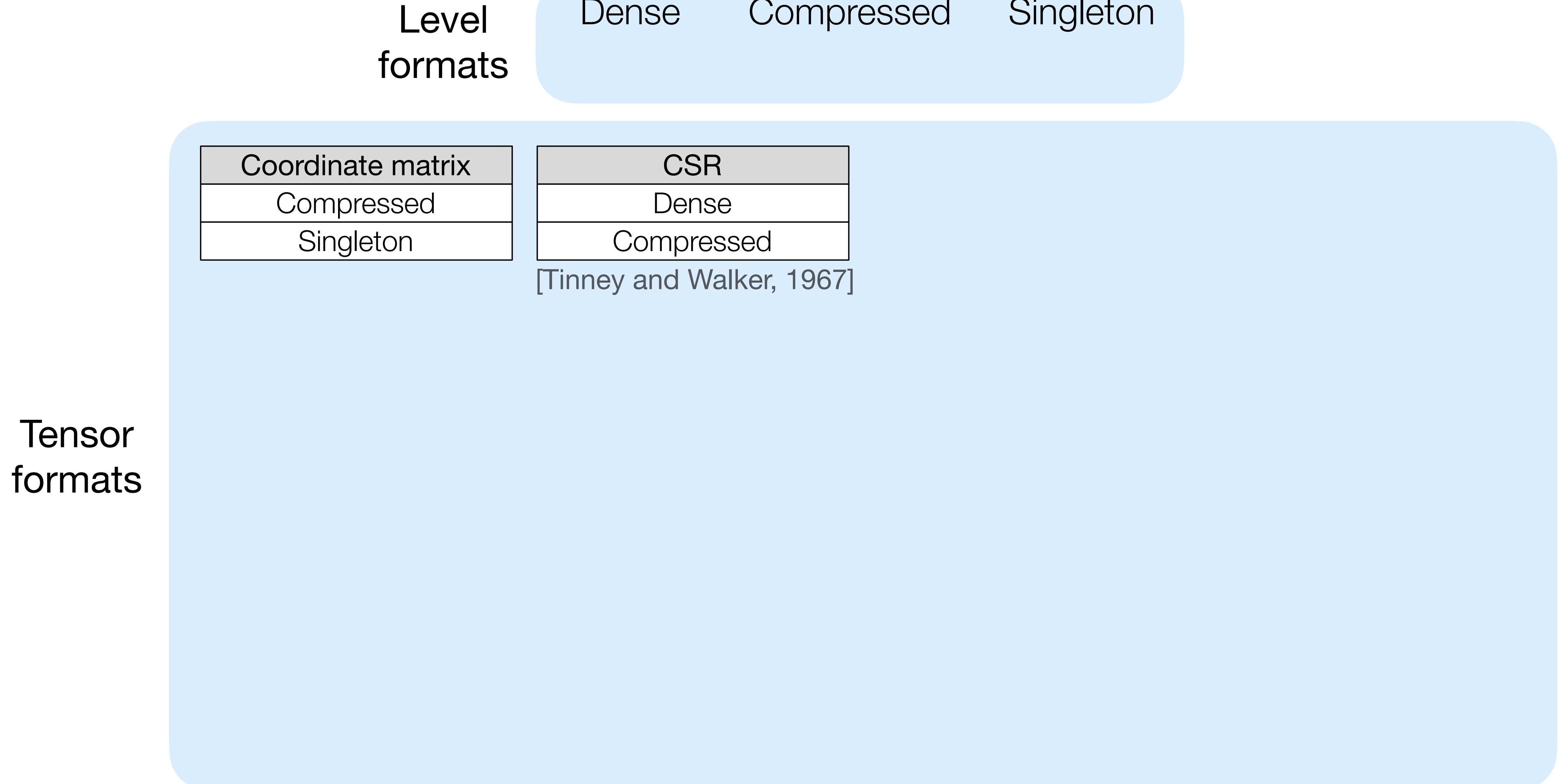
# Level types can be composed in many ways

Dense

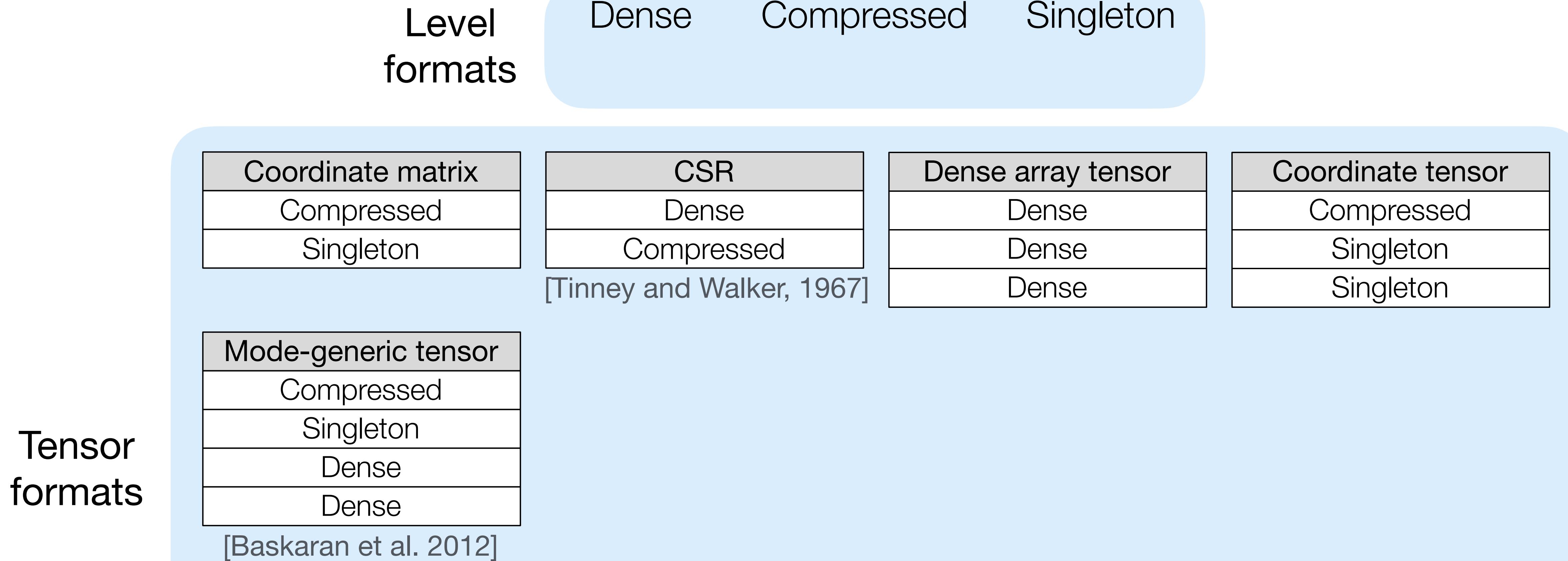
Compressed

Singleton

# Level types can be composed in many ways



# Level types can be composed in many ways



# Level types can be composed in many ways

Tensor formats	Level formats	Dense	Compressed	Singleton	
		Coordinate matrix	CSR	Dense array tensor	Coordinate tensor
	Compressed	Compressed	Dense	Dense	Compressed
	Singleton	Singleton	Compressed	Dense	Singleton
	[Tinney and Walker, 1967]				
	Mode-generic tensor	BCSR	CSB	ELLPACK	
	Compressed	Dense	Dense	Dense	
	Singleton	Compressed	Dense	Dense	
	Dense	Dense	Compressed	Singleton	
	Dense	Dense	Singleton	[Kincaid et al. 1989]	
	[Baskaran et al. 2012]	[Im and Yelick 1998]	[Buluç et al. 2009]		

# Level types can be composed in many ways

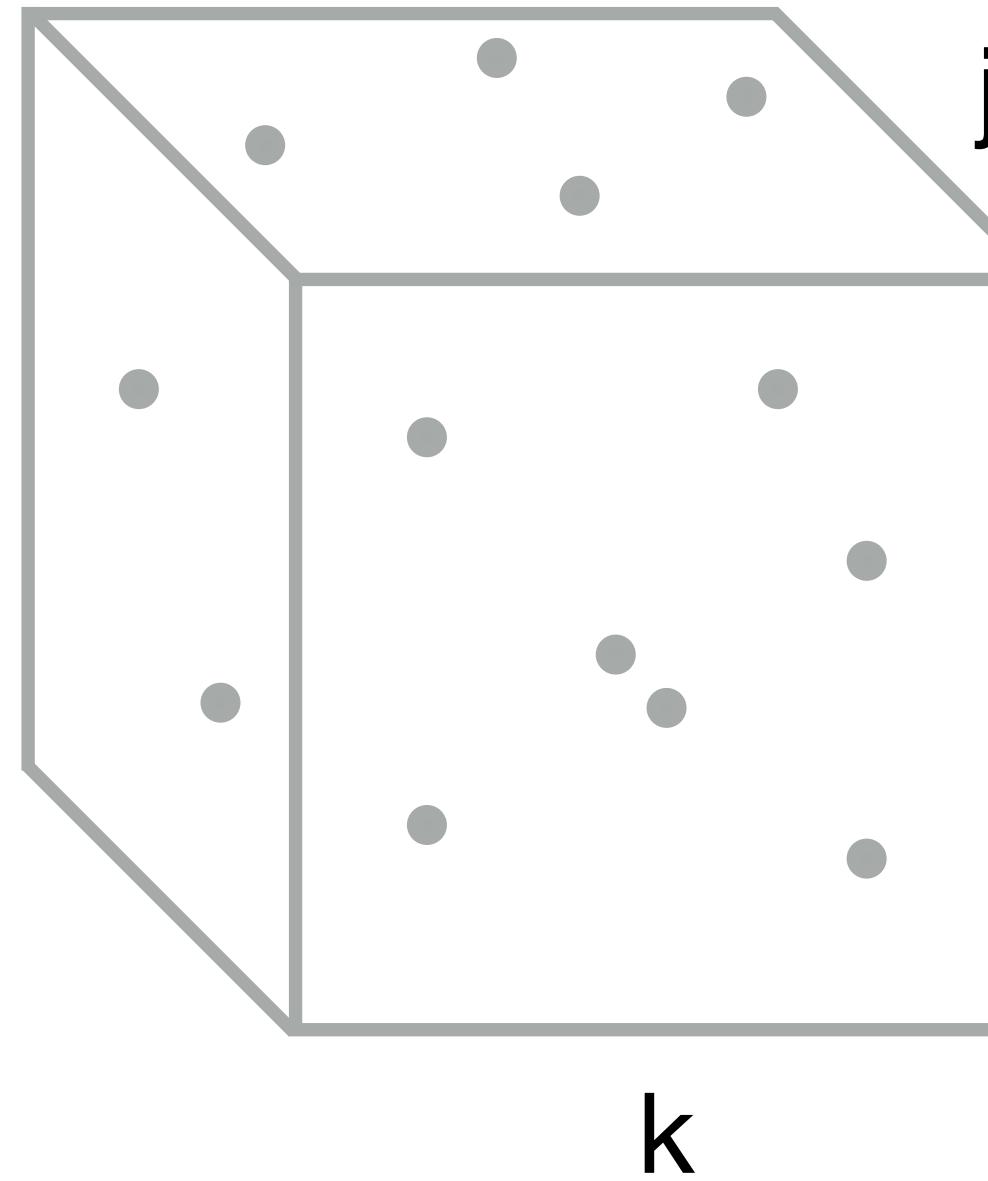
Tensor formats	Level formats	Dense	Compressed	Singleton
		Hashed	Range	Offset
Coordinate matrix	CSR	Dense	Dense array tensor	Coordinate tensor
	Dense	Compressed	Dense	Compressed
	Compressed	Compressed	Dense	Singleton
	[Tinney and Walker, 1967]		Dense	Singleton
Mode-generic tensor	BCSR	Dense	CSB	ELLPACK
	Dense	Compressed	Dense	Dense
	Compressed	Dense	Compressed	Dense
	Dense	Dense	Singletton	Singletton
	[Baskaran et al. 2012]		[Buluç et al. 2009]	[Kincaid et al. 1989]
Tensor formats	CSR	Dense	Dense array tensor	Coordinate tensor
	Dense	Compressed	Dense	Compressed
	Compressed	Compressed	Dense	Singleton
	[Tinney and Walker, 1967]		Dense	Singleton

Level types can be composed in many ways

Tensor formats	Level formats	Dense	Compressed	Singleton
		Hashed	Range	Offset
Coordinate matrix	CSR			
Compressed	Dense			
Singleton	Compressed			
	[Tinney and Walker, 1967]			
Mode-generic tensor	BCSR			
Compressed	Dense			
Singleton	Compressed			
Dense	Dense			
Dense	Dense			
	[Baskaran et al. 2012]			
Hash map vector	Hash map matrix			
Hashed	Hashed			
	[Patwary et al. 2015]			
Dense array tensor				
Dense				
Dense				
Dense				
Coordinate tensor				
Compressed				
Singleton				
Singleton				
CSB				
Dense				
Dense				
Compressed				
Singleton				
ELLPACK				
Dense				
Dense				
Singleton				
	[Kincaid et al. 1989]			
Block DIA				
Dense				
Range				
Offset				
Dense				
Dense				
DIA				
Dense				
Range				
Offset				
	[Saad 2003]			

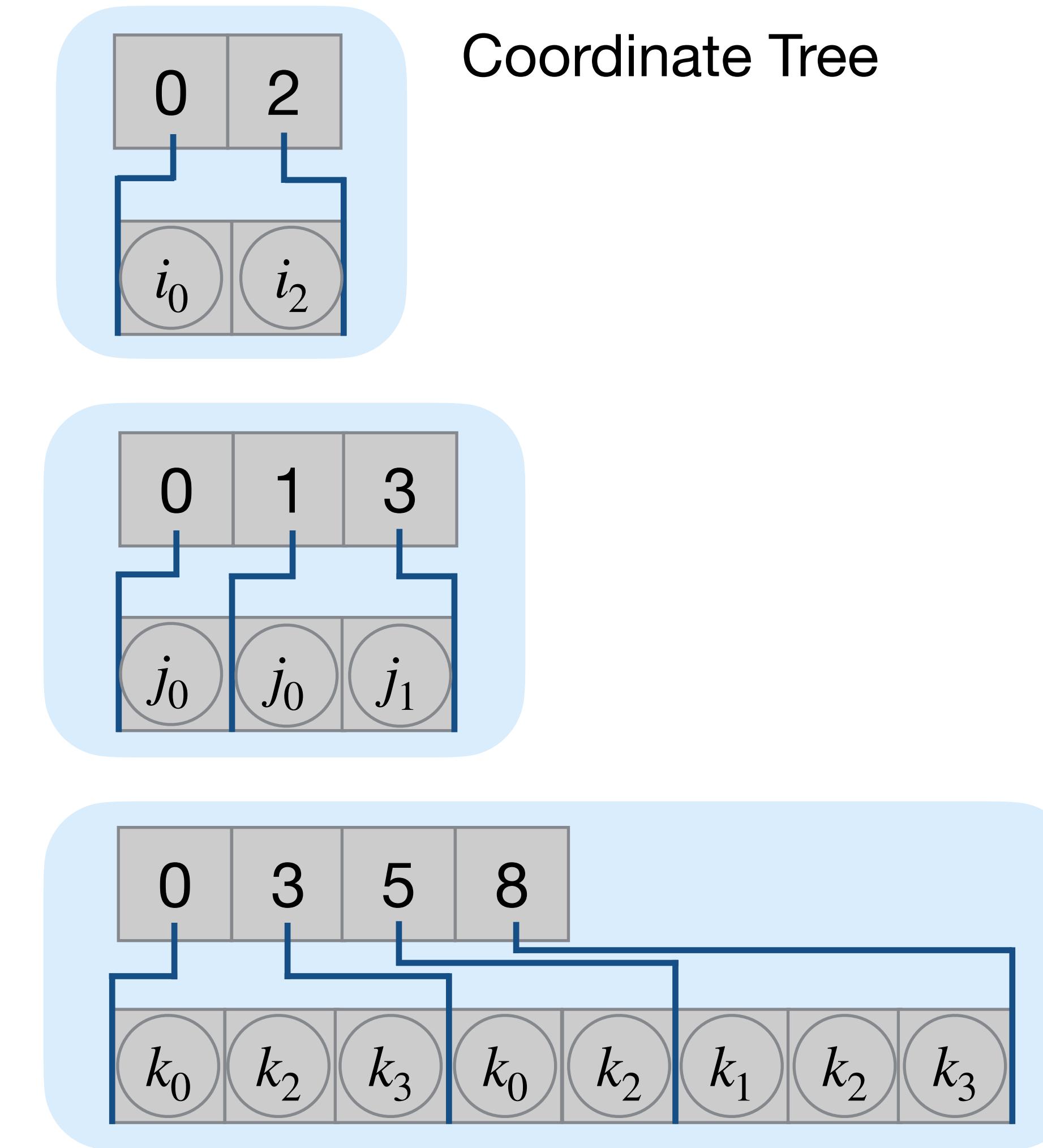
# Iteration graphs express iteration spaces and data structure ordering

Iteration Space



$B_{ijk}$

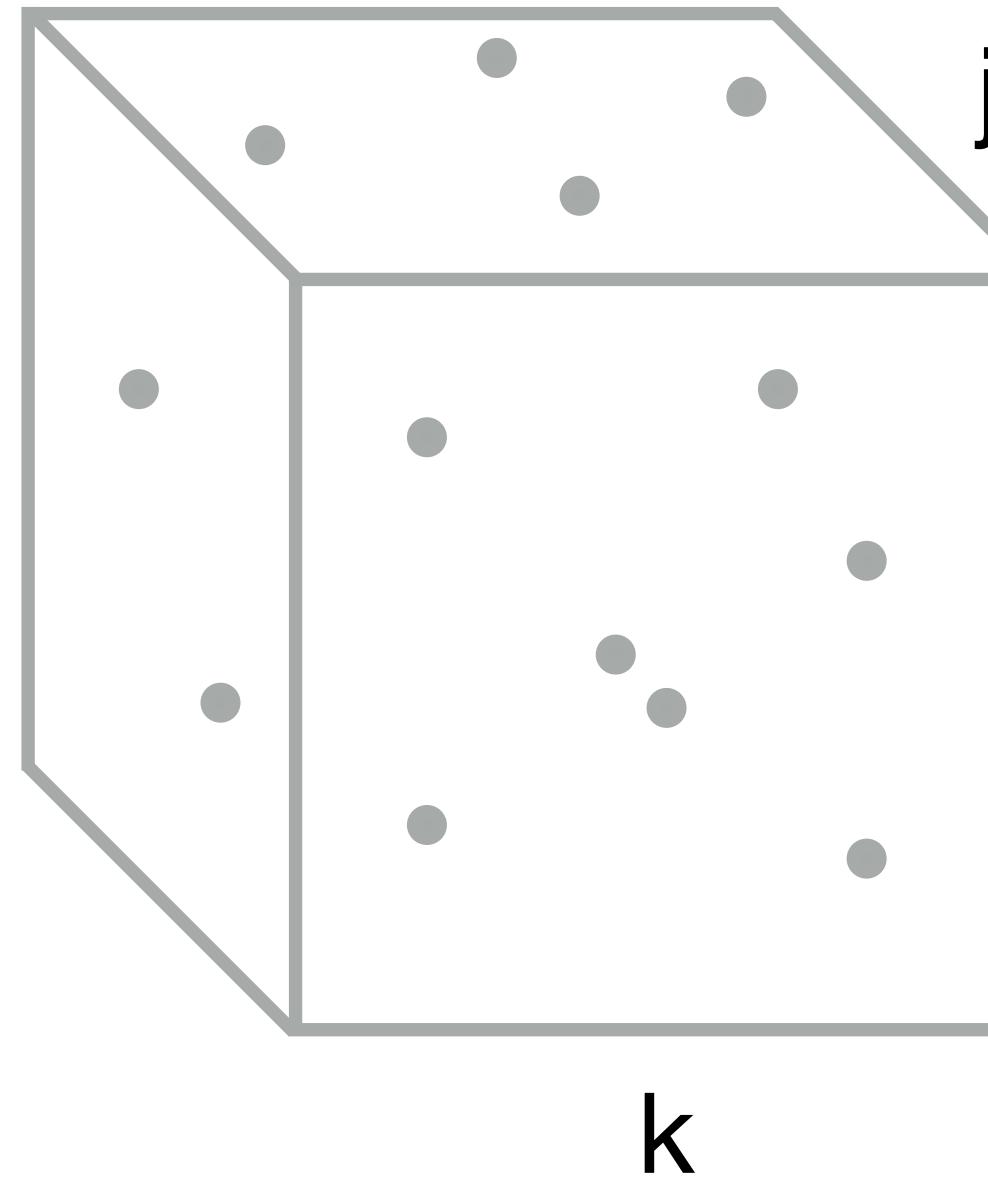
Coordinate Tree



30	40	50	10	70	80	20	60
----	----	----	----	----	----	----	----

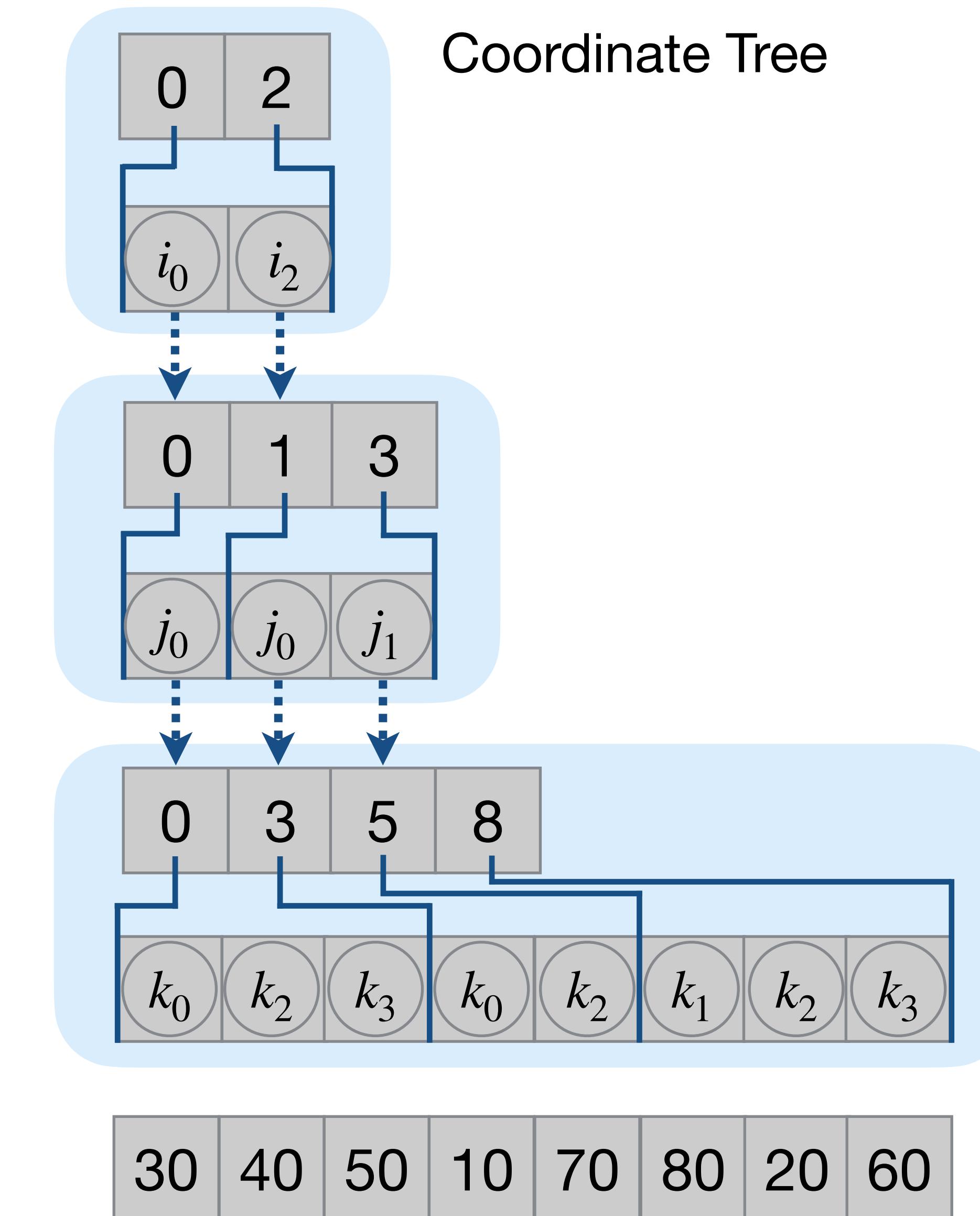
# Iteration graphs express iteration spaces and data structure ordering

Iteration Space



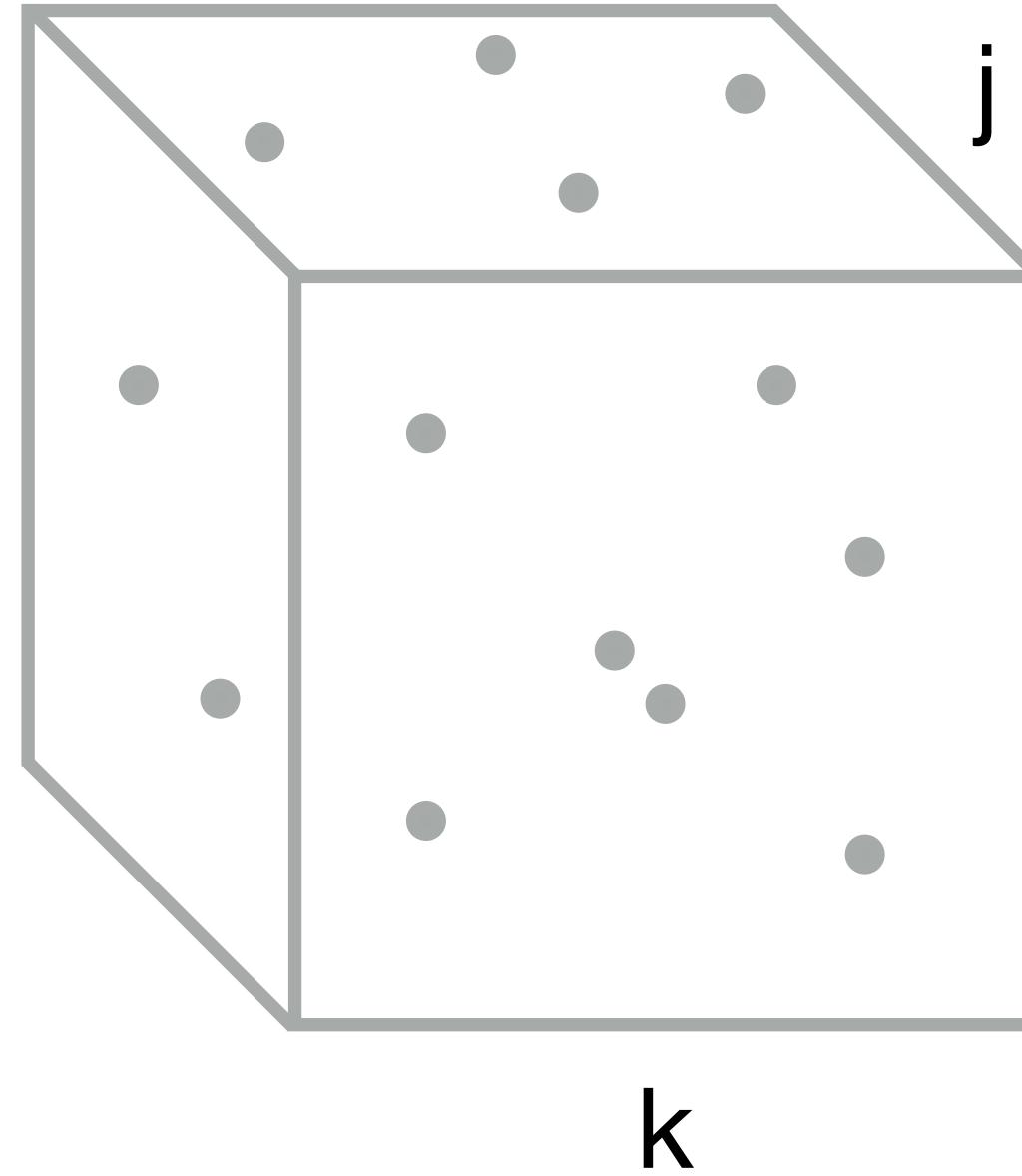
$B_{ijk}$

Coordinate Tree



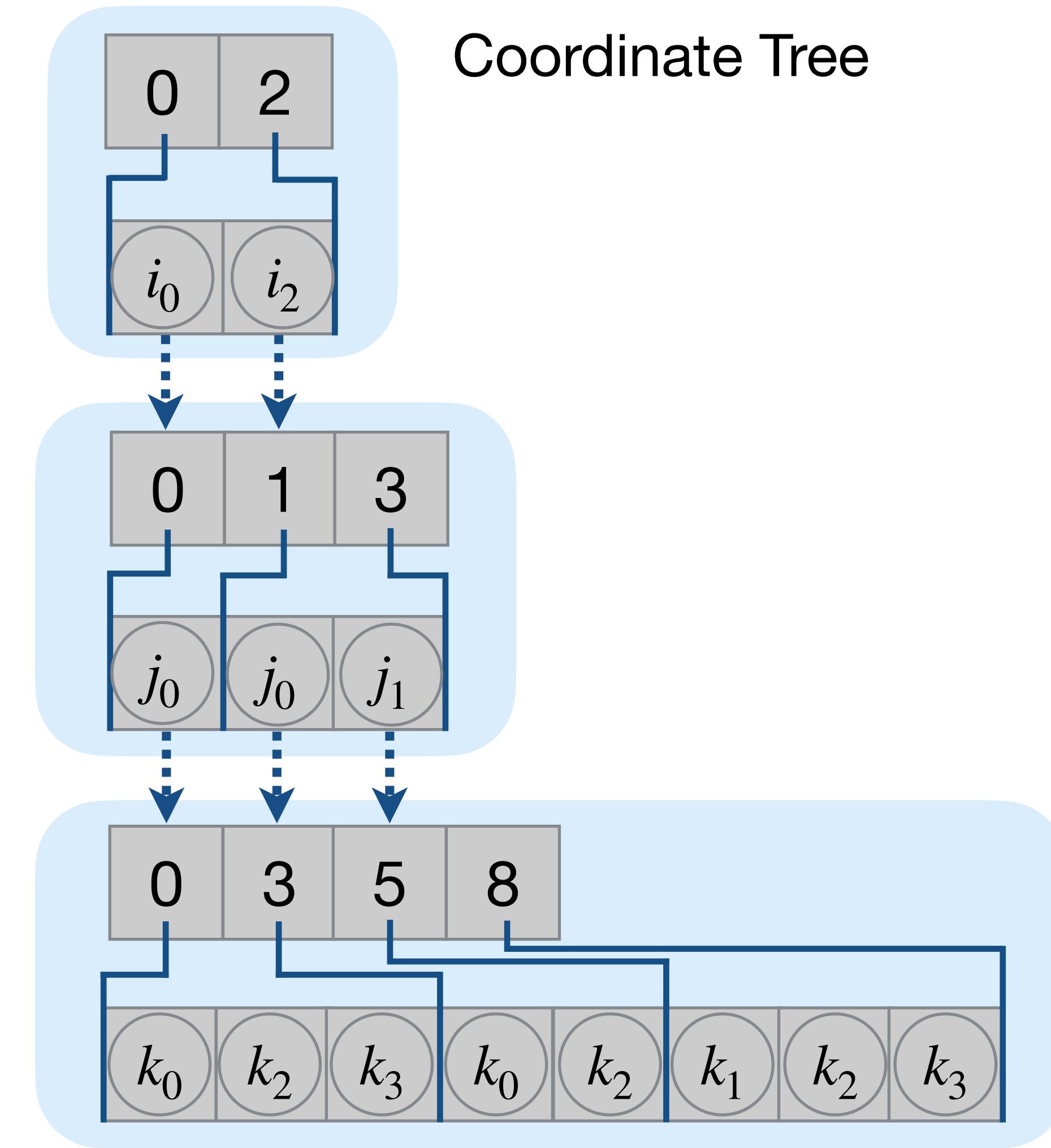
# Iteration graphs express iteration spaces and data structure ordering

Iteration Space



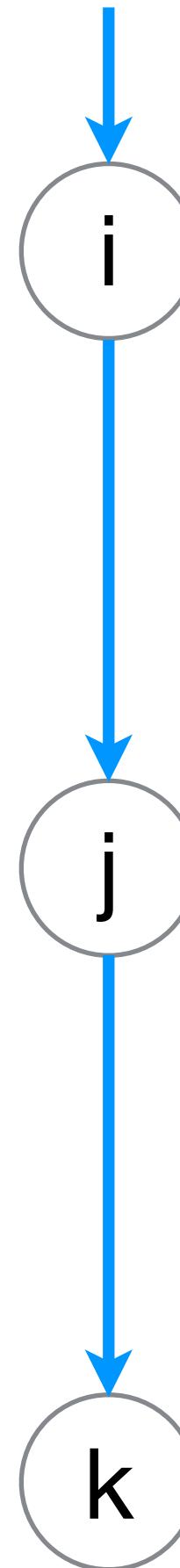
$B_{ijk}$

Coordinate Tree



30 40 50 10 70 80 20 60

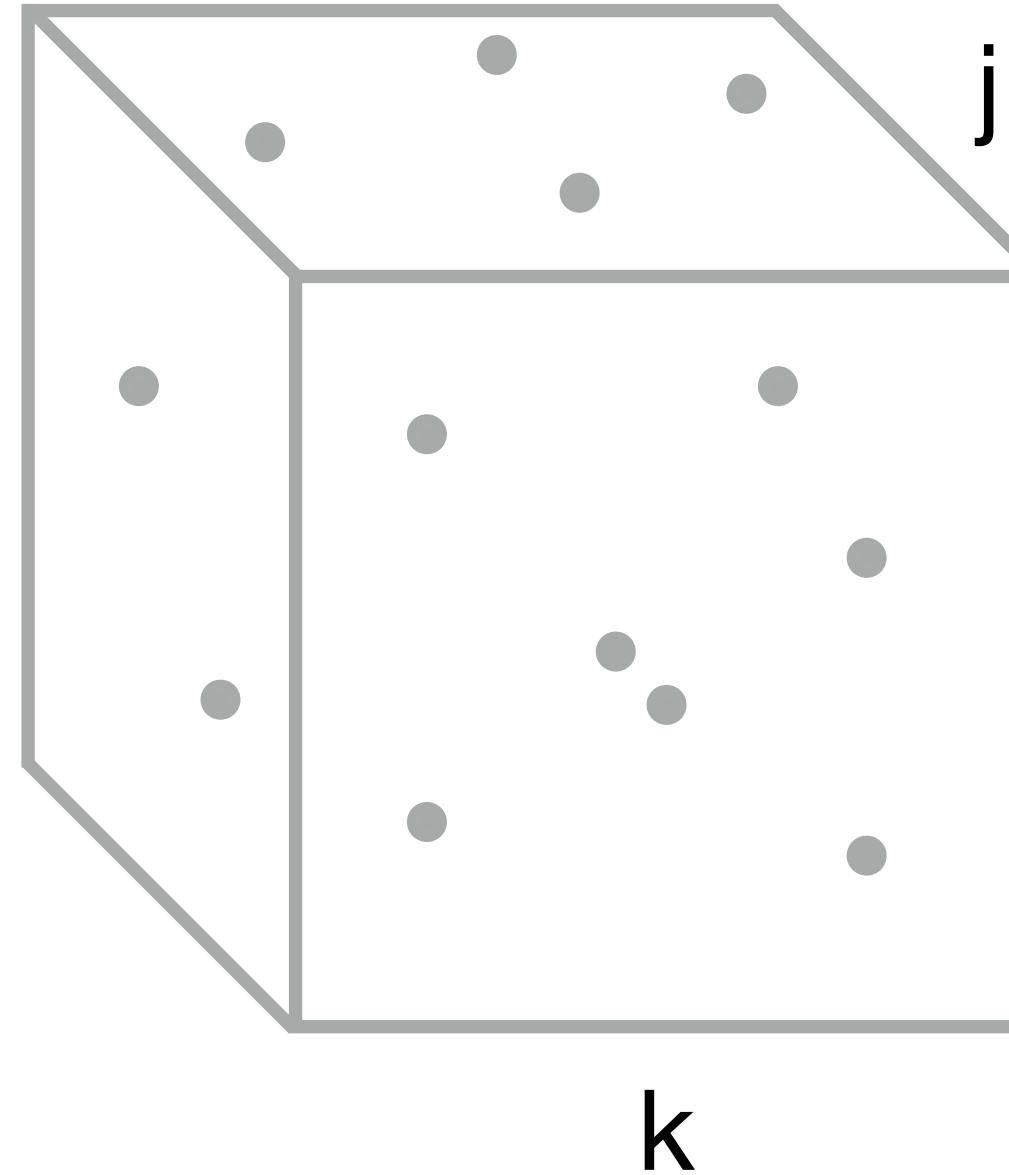
Iteration Graph



$\forall_i \forall_j \forall_k B_{ijk}$

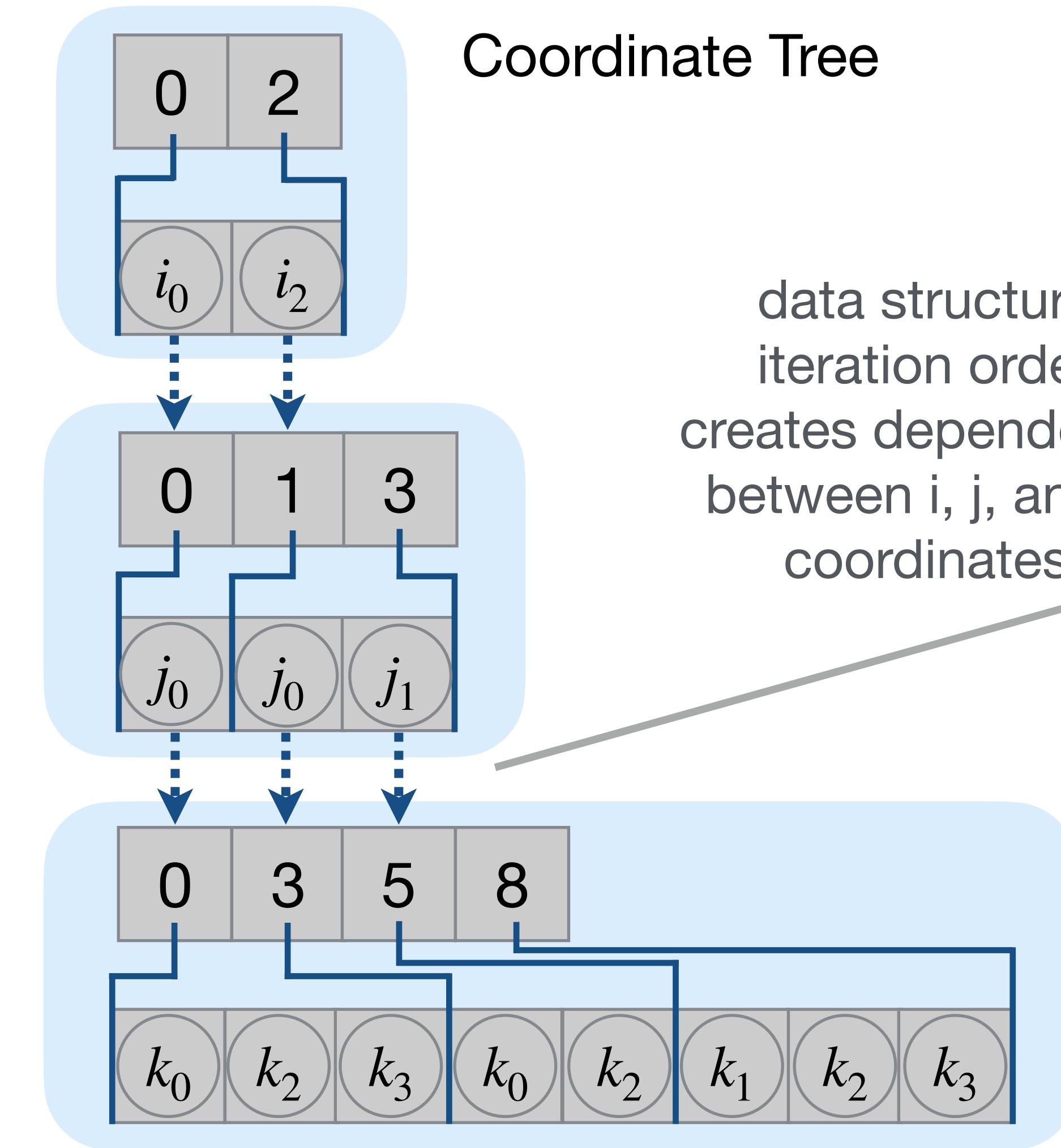
# Iteration graphs express iteration spaces and data structure ordering

Iteration Space



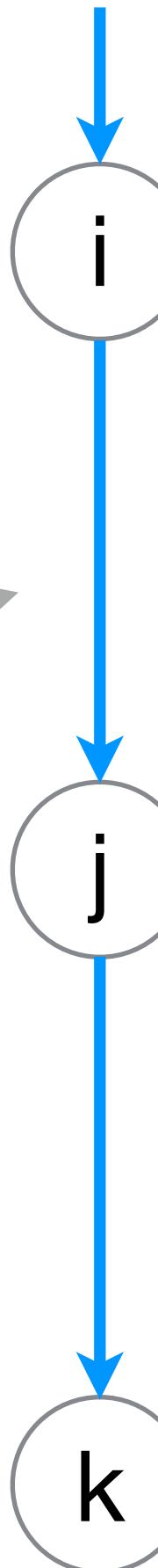
$B_{ijk}$

Coordinate Tree

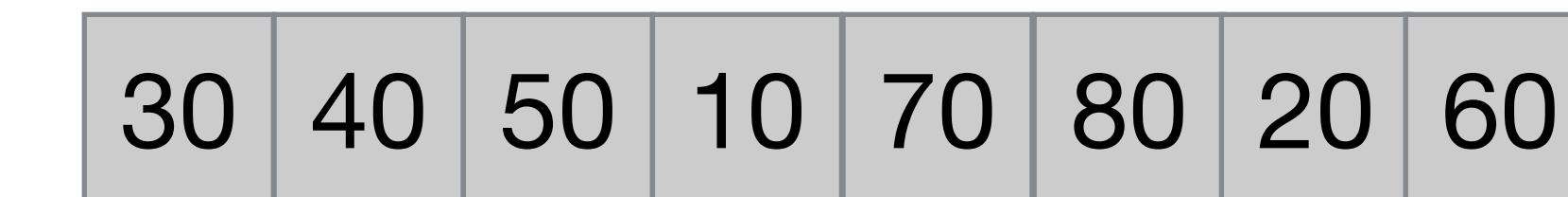


data structure  
iteration order  
creates dependency  
between *i*, *j*, and *k*  
coordinates

Iteration Graph



$\forall_i \forall_j \forall_k B_{ijk}$



# Sparse iteration spaces and Iteration Graphs

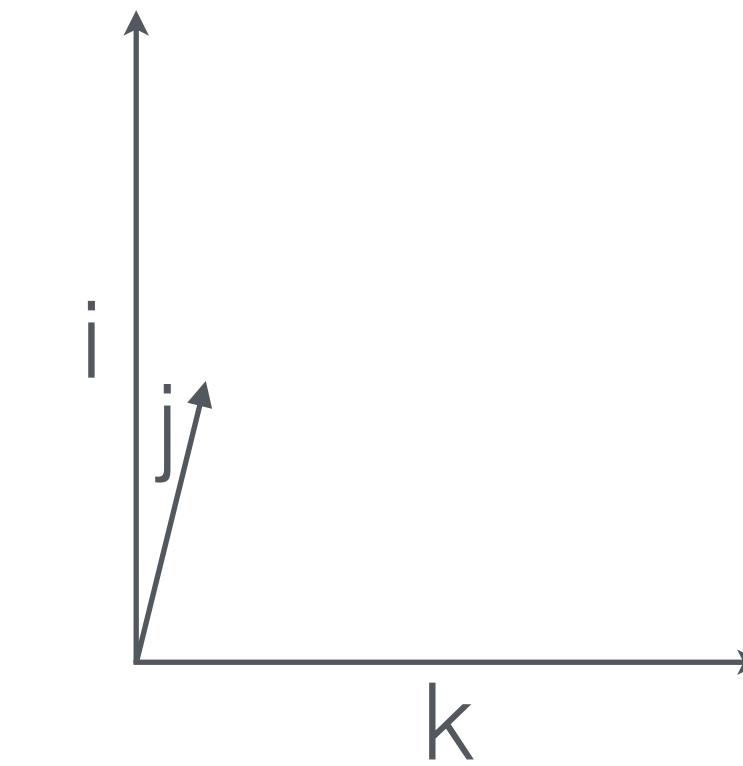
$$A_{ij} = \sum_k B_{ijk} * c_k$$

# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * c_k$$

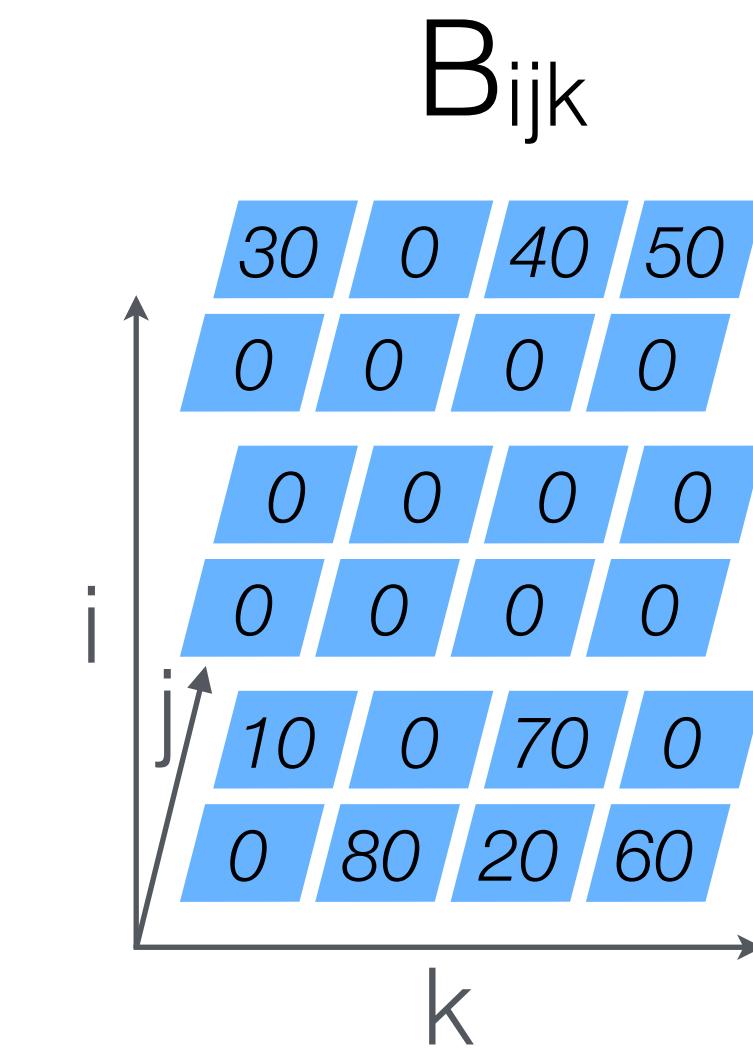

# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * c_k$$



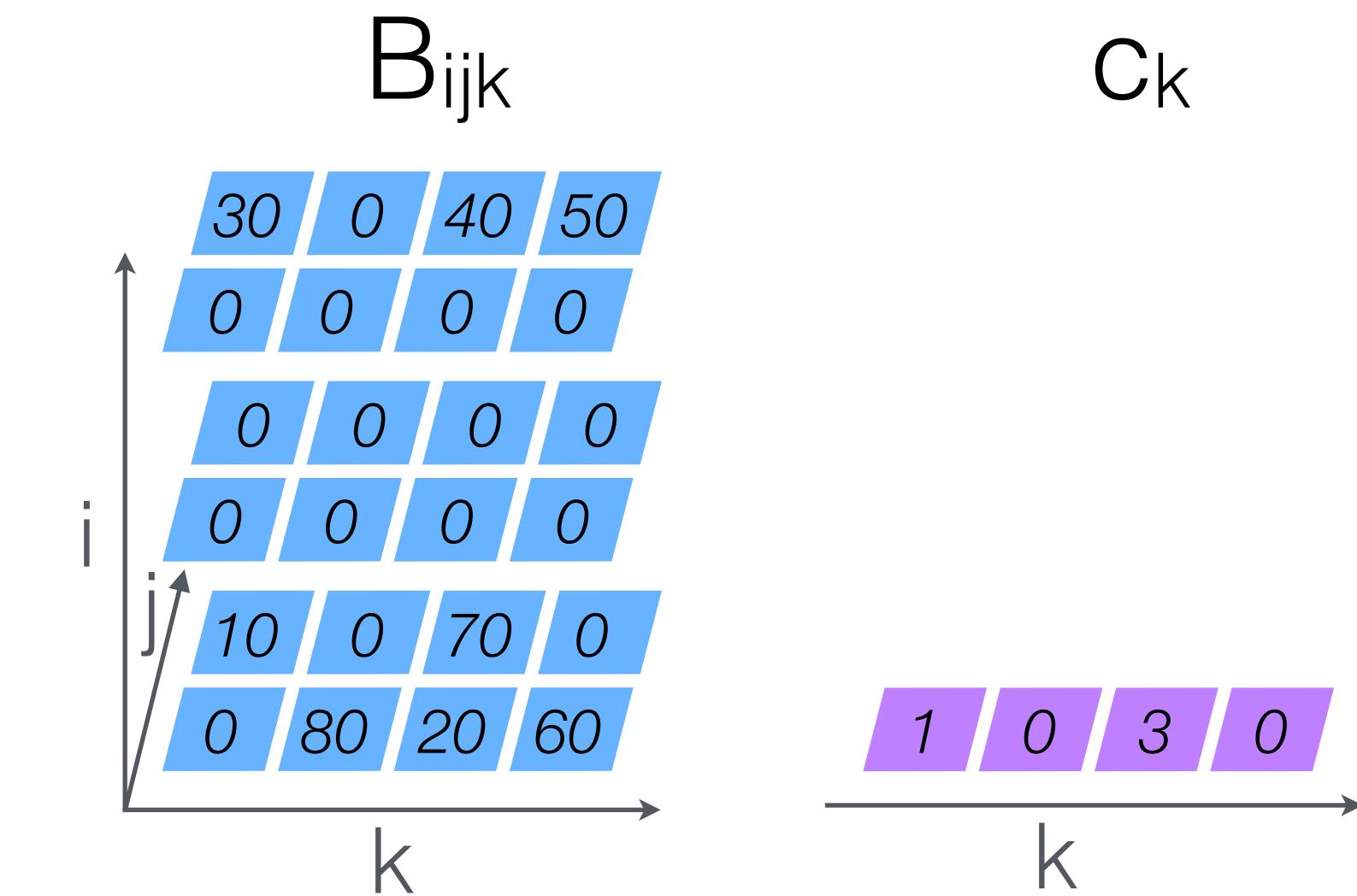
# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * c_k$$



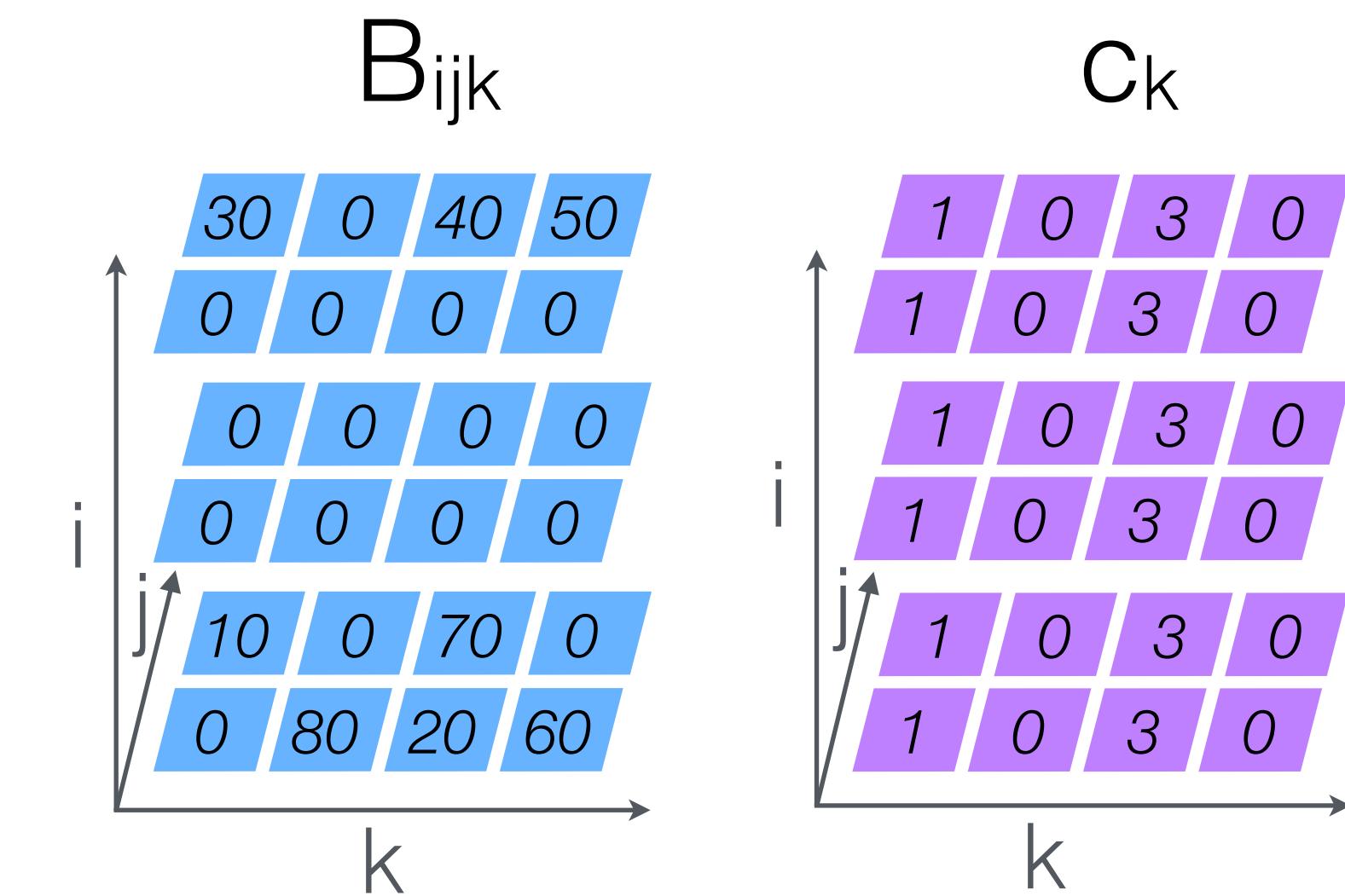
# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * c_k$$



# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$



# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$

B <sub>ijk</sub> * C <sub>k</sub>			
i	j	k	
		30	0
		0	120
		0	0
		0	0
		0	0
		10	210
		0	60

Dense

# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$

$B_{ijk}$

	30			40	50
i					
j	10			70	
		80	20	60	
k					

$C_k$

1			3	
1			3	
1			3	
1			3	
1			3	
j				
k				

$B_{ijk} * C_k$

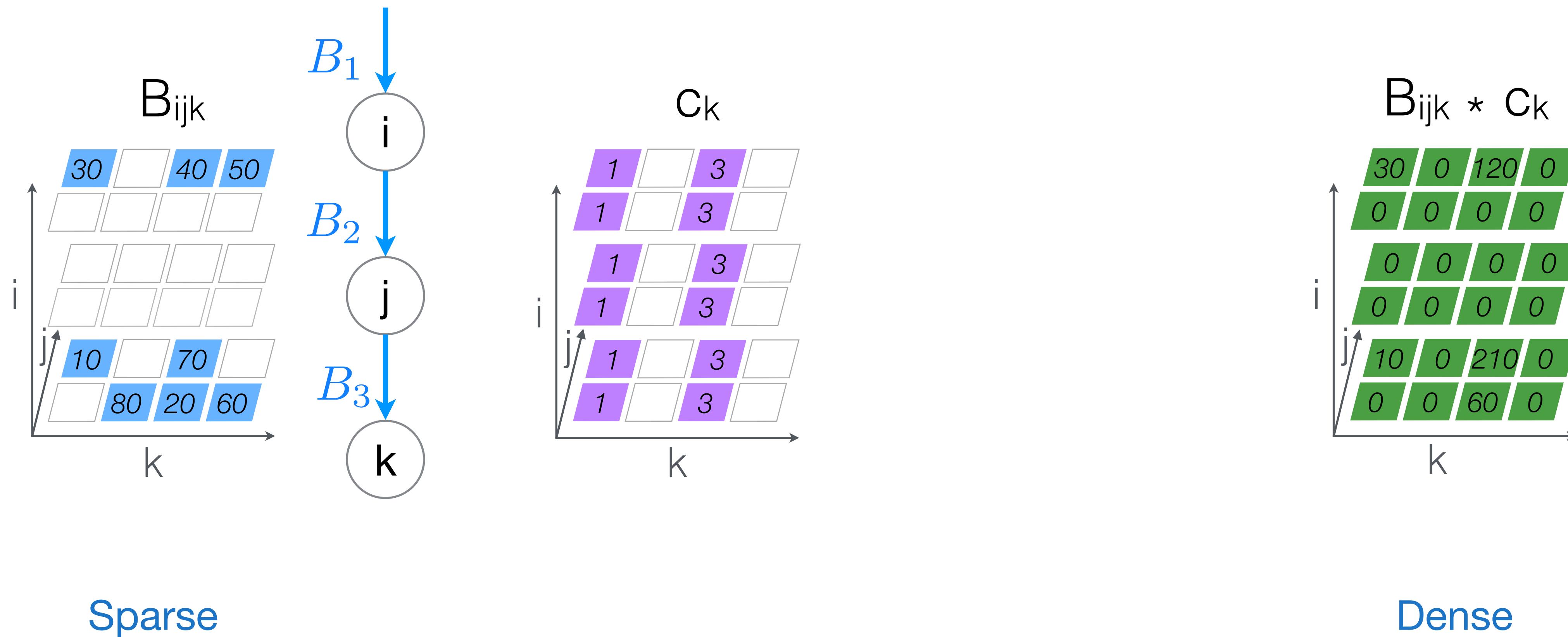
30	0	120	0
0	0	0	0
0	0	0	0
10	0	210	0
0	0	60	0
i			
j			
k			

Sparse

Dense

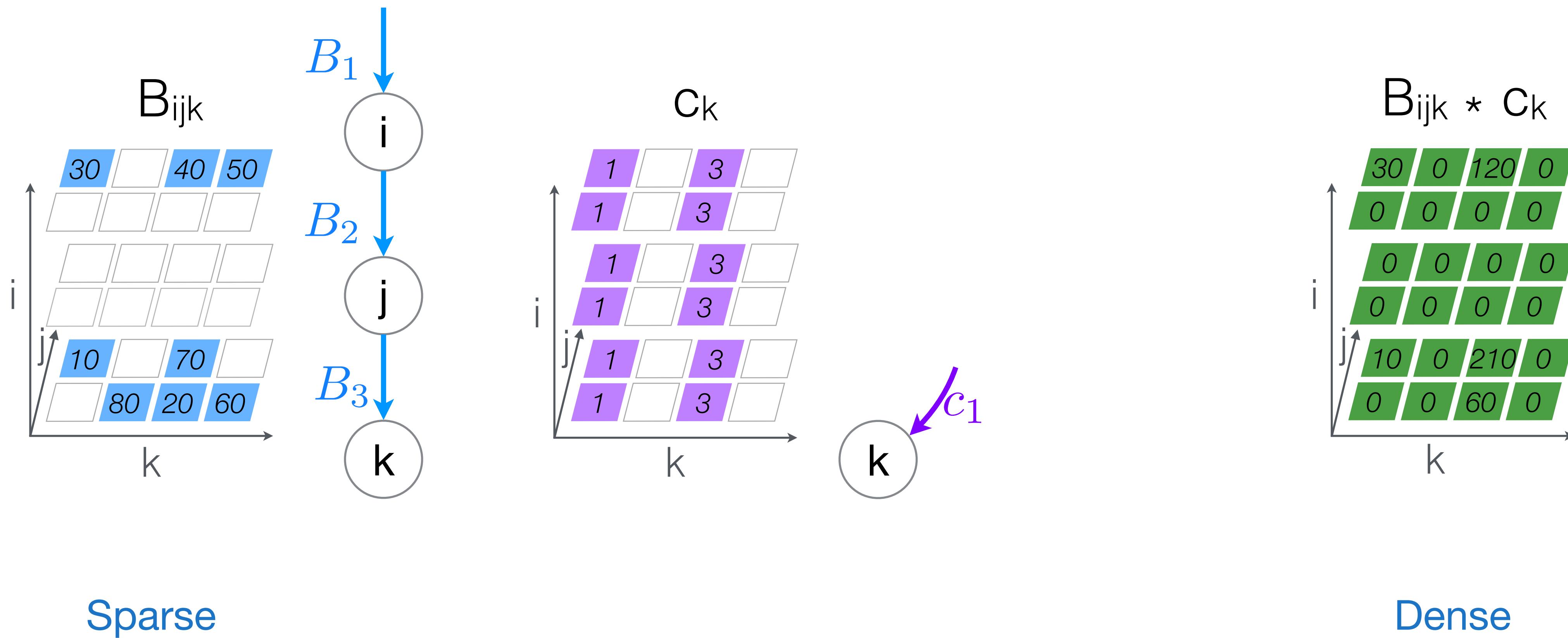
# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$



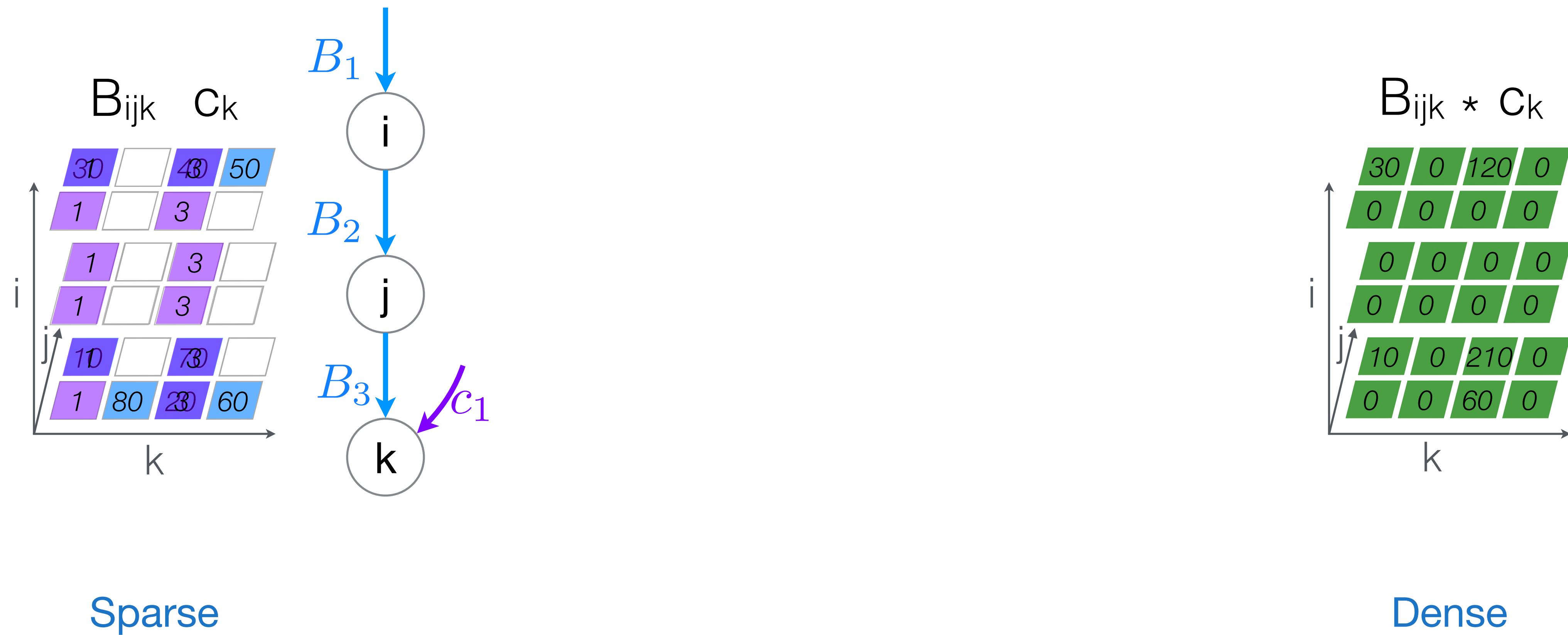
# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$



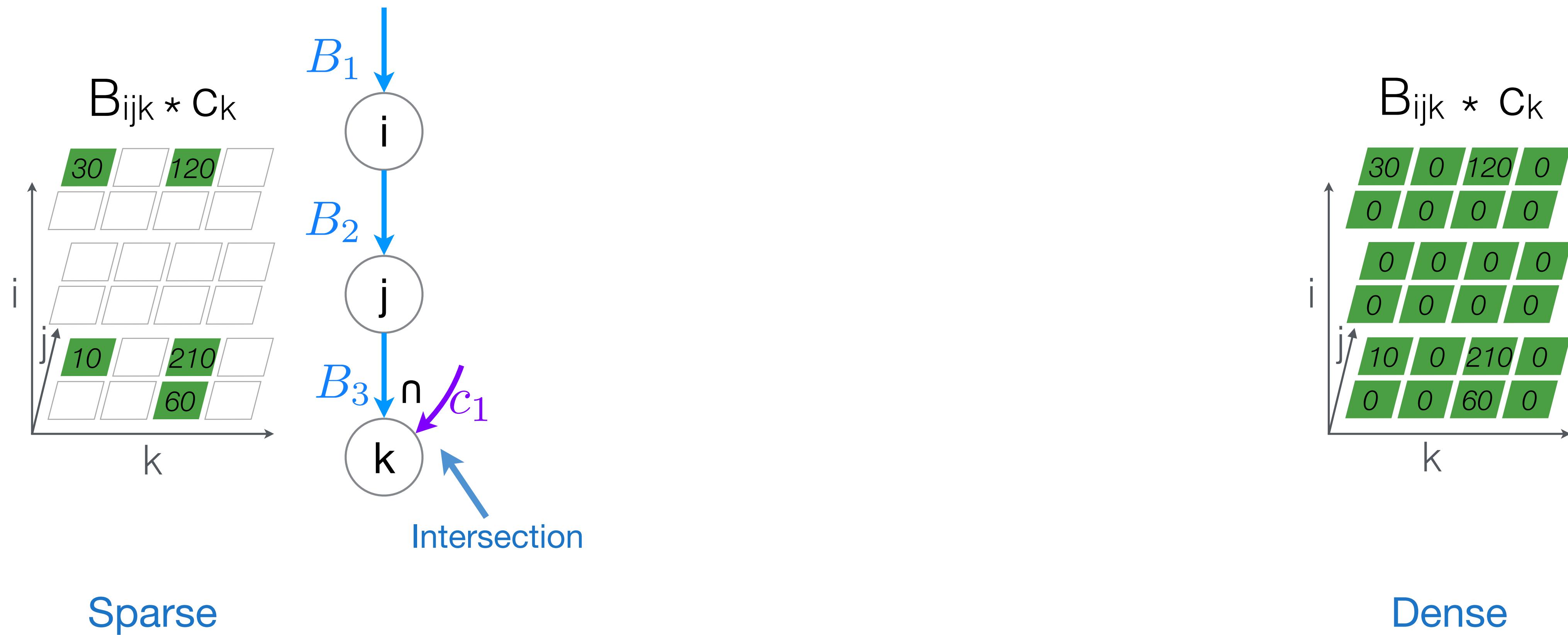
# Sparse iteration spaces and Iteration Graphs

$$A_{ij} = \sum_k B_{ijk} * C_k$$

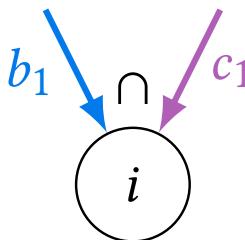


# Sparse iteration spaces and Iteration Graphs

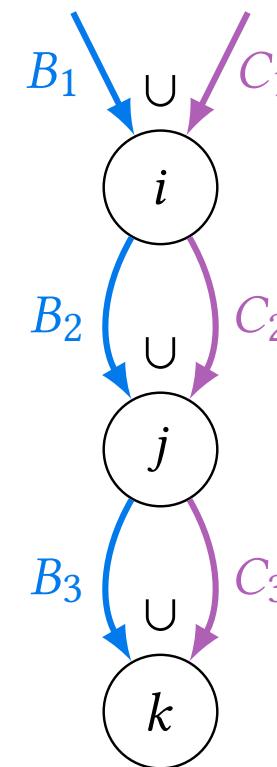
$$A_{ij} = \sum_k B_{ijk} * C_k$$



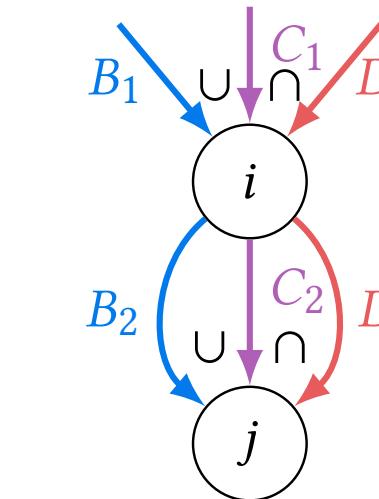
# Iteration graph examples



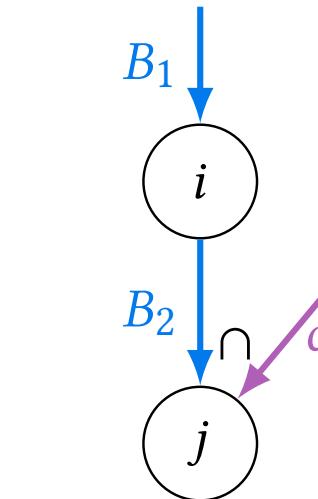
$$\frac{\forall_i \ b_i \cap c_i}{i \in b_1 \cap c_1}$$



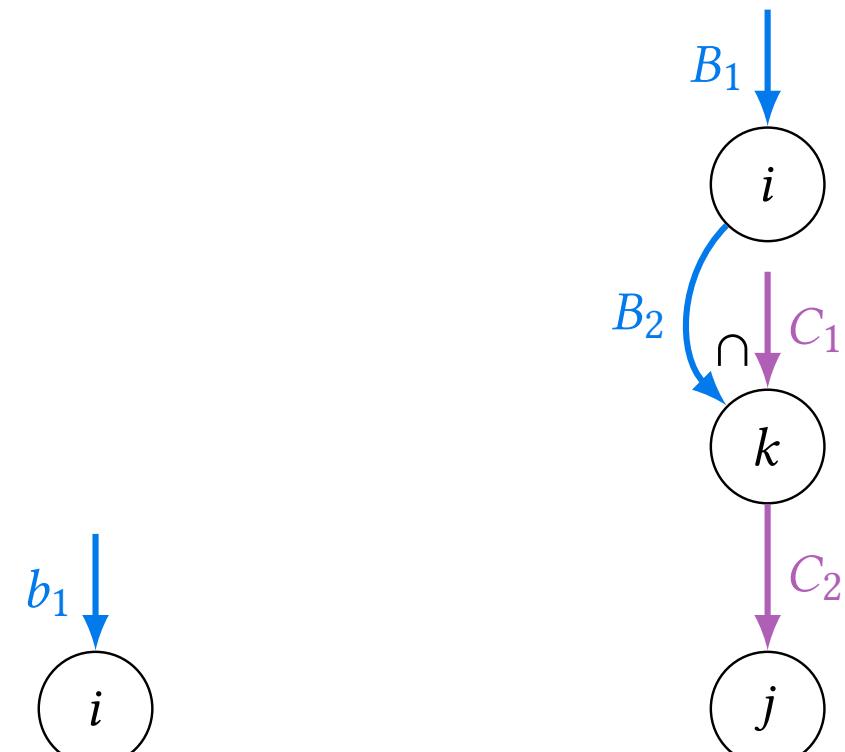
$$\frac{\forall_i \forall_j \forall_k \ B_{ijk} \cup C_{ijk}}{i \in B_1 \cup C_1 \\ j \in B_2 \cup C_2 \\ k \in B_3 \cup C_3}$$



$$\frac{\forall_i \forall_j (B_{ij} \cup C_{ij}) \cap D_{ij}}{i \in (B_1 \cup C_1) \cap D_1 \\ j \in (B_2 \cup C_2) \cap D_2}$$

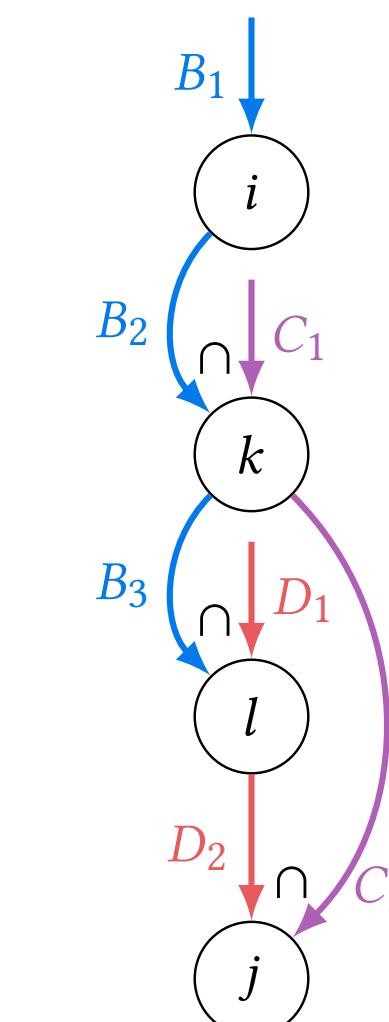


$$\frac{\forall_i \forall_j B_{ij} \cap c_j}{i \in B_1 \cap \mathbb{U}_i \\ j \in B_2 \cap c_1}$$

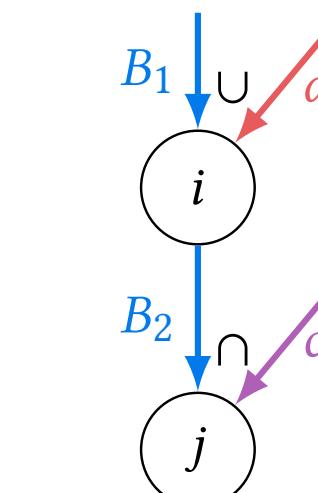


$$\frac{\forall_i \alpha \cup b_i}{i \in \mathbb{U}_i \cap b_1}$$

$$\frac{\forall_i \forall_k \forall_j B_{ik} \cap C_{kj}}{i \in B_1 \cap \mathbb{U}_i \\ k \in B_2 \cap C_1 \\ j \in \mathbb{U}_j \cap C_2}$$



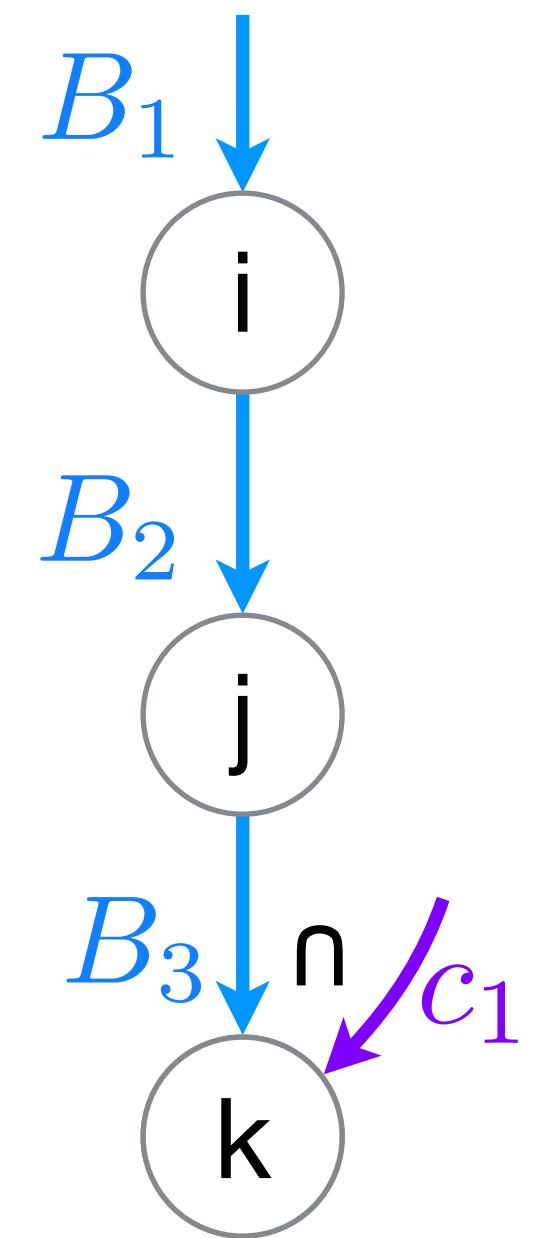
$$\frac{\forall_i \forall_k \forall_l \forall_j B_{ikl} \cap C_{kj} \cap D_{lj}}{i \in B_1 \cap \mathbb{U}_i \cap \mathbb{U}_i \\ k \in B_2 \cap C_1 \cap \mathbb{U}_k \\ l \in B_3 \cap \mathbb{U}_l \cap D_1 \\ j \in \mathbb{U}_j \cap C_2 \cap D_2}$$



$$\frac{\forall_i (\forall_j B_{ij} \cap c_j) \cup d_i}{i \in (B_1 \cap \mathbb{U}_i) \cup d_1 \\ j \in B_2 \cap c_1}$$

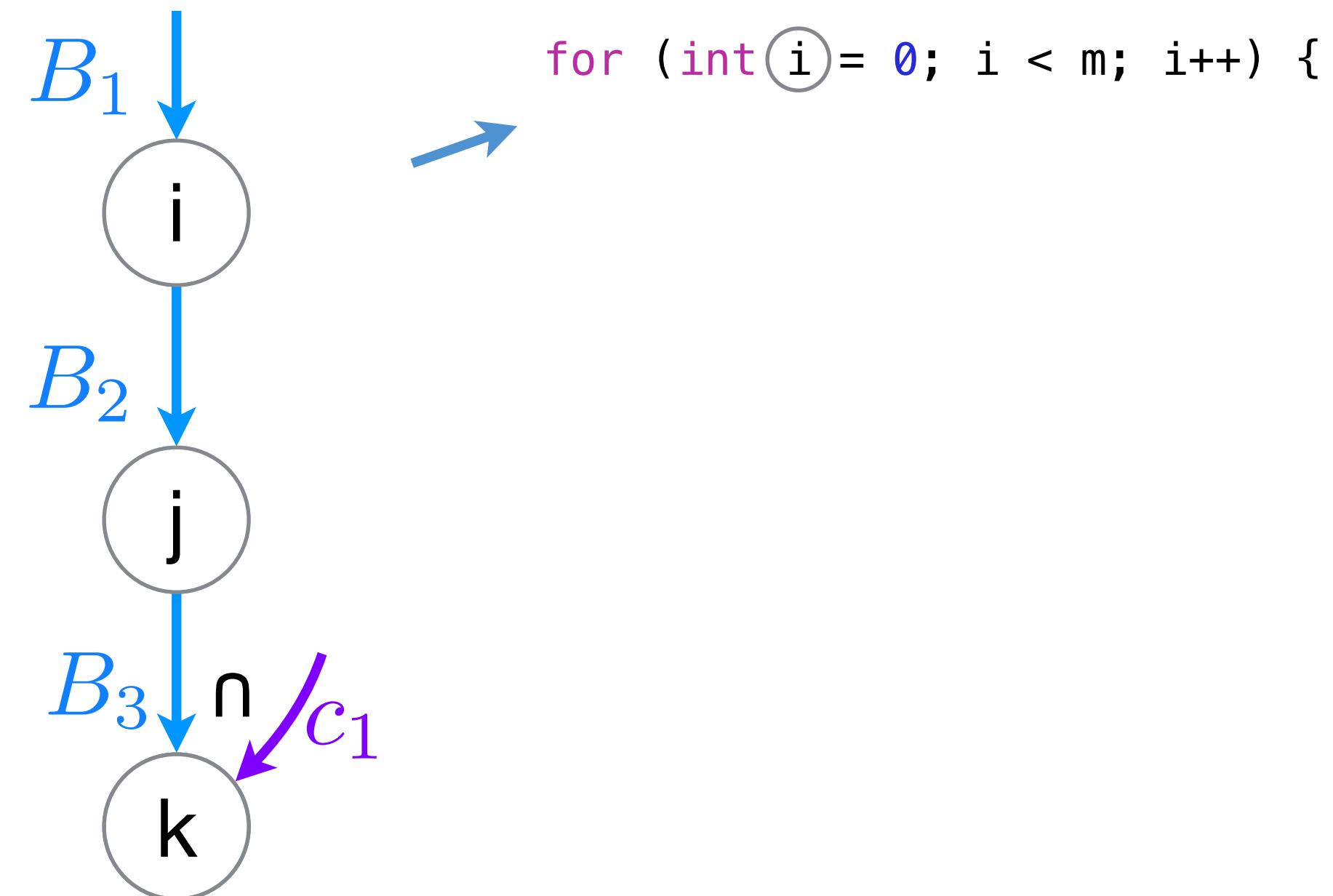
# Iteration graphs are lowered to sparse code

$$A_{ij} = \sum_k B_{ijk} c_k$$



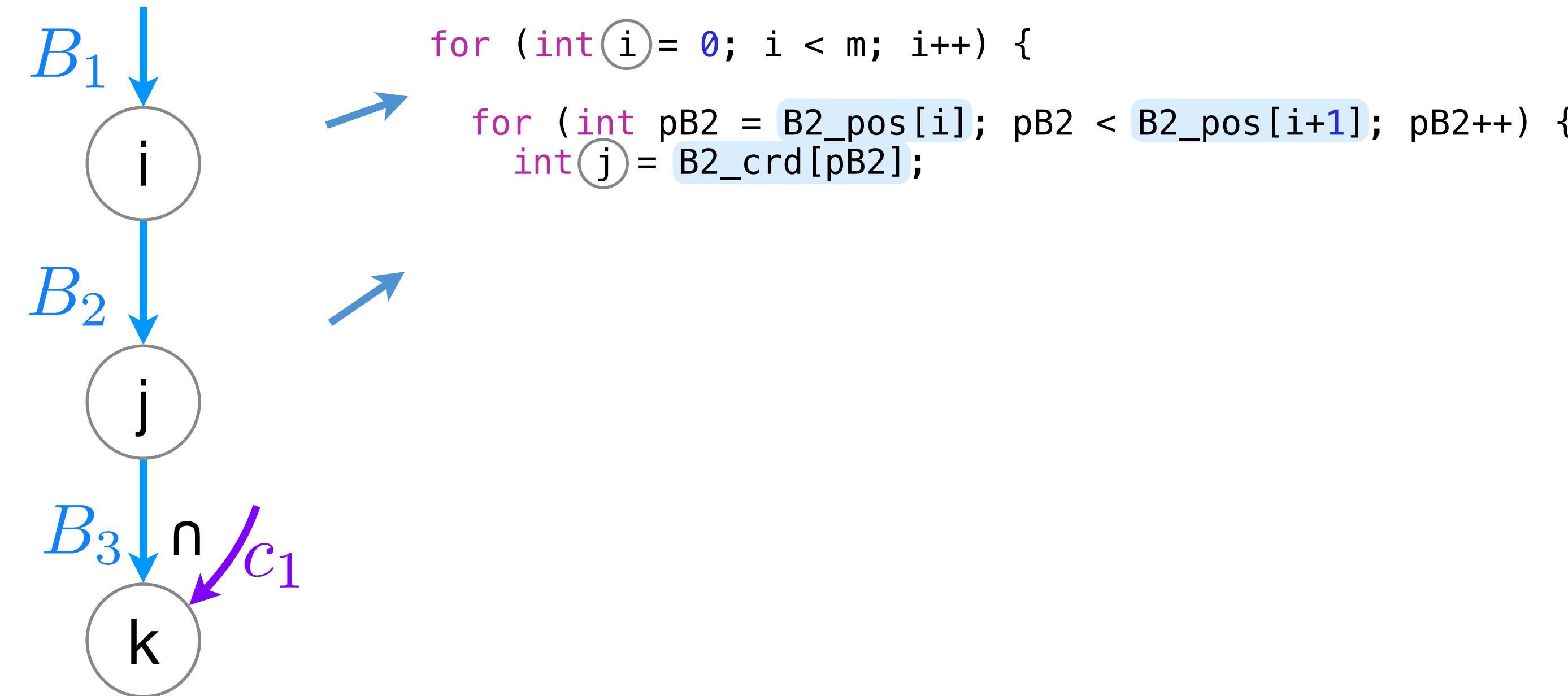
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$$A_{ij} = \sum_k B_{ijk} c_k$$



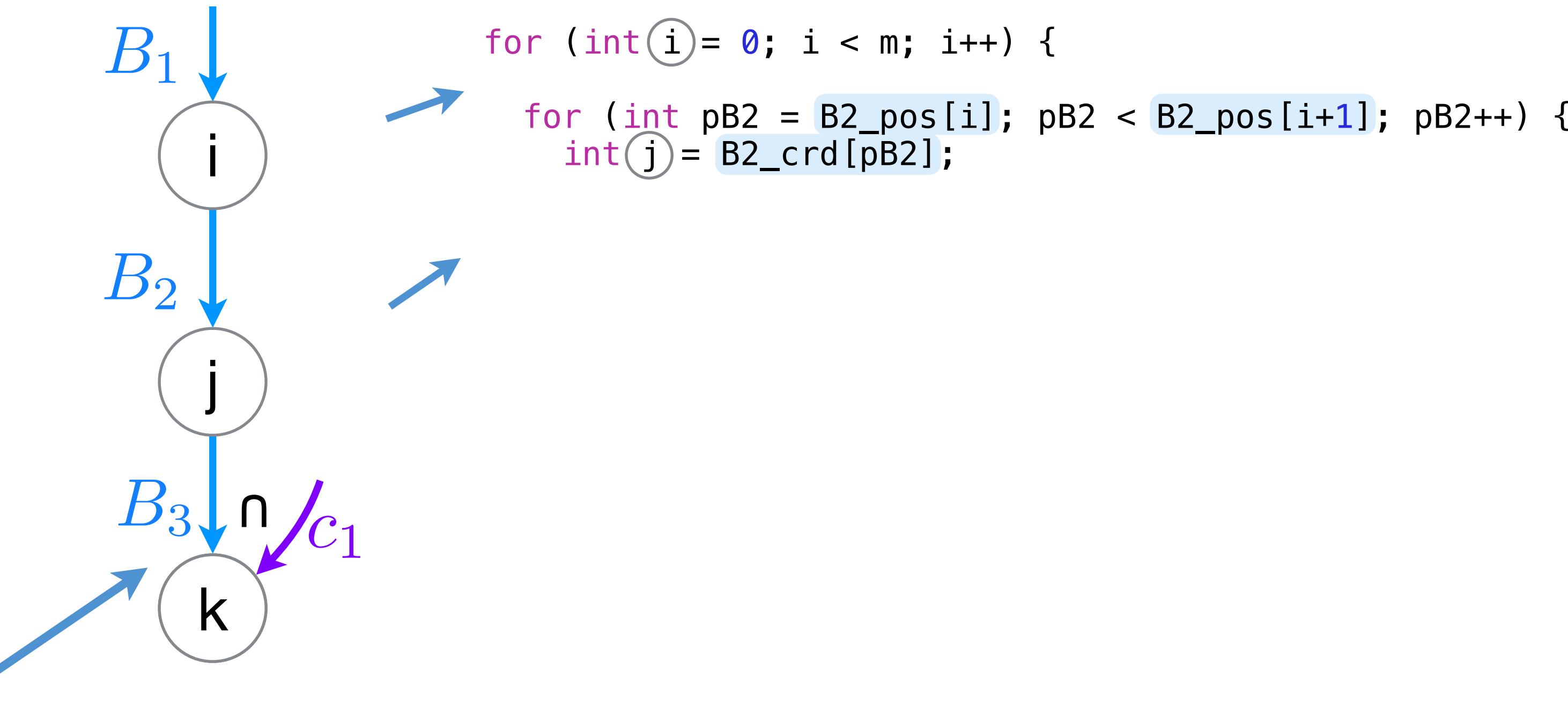
# Iteration graphs are lowered to sparse code

$$A_{ij} = \sum_k B_{ijk} c_k$$



# Iteration graphs are lowered to sparse code

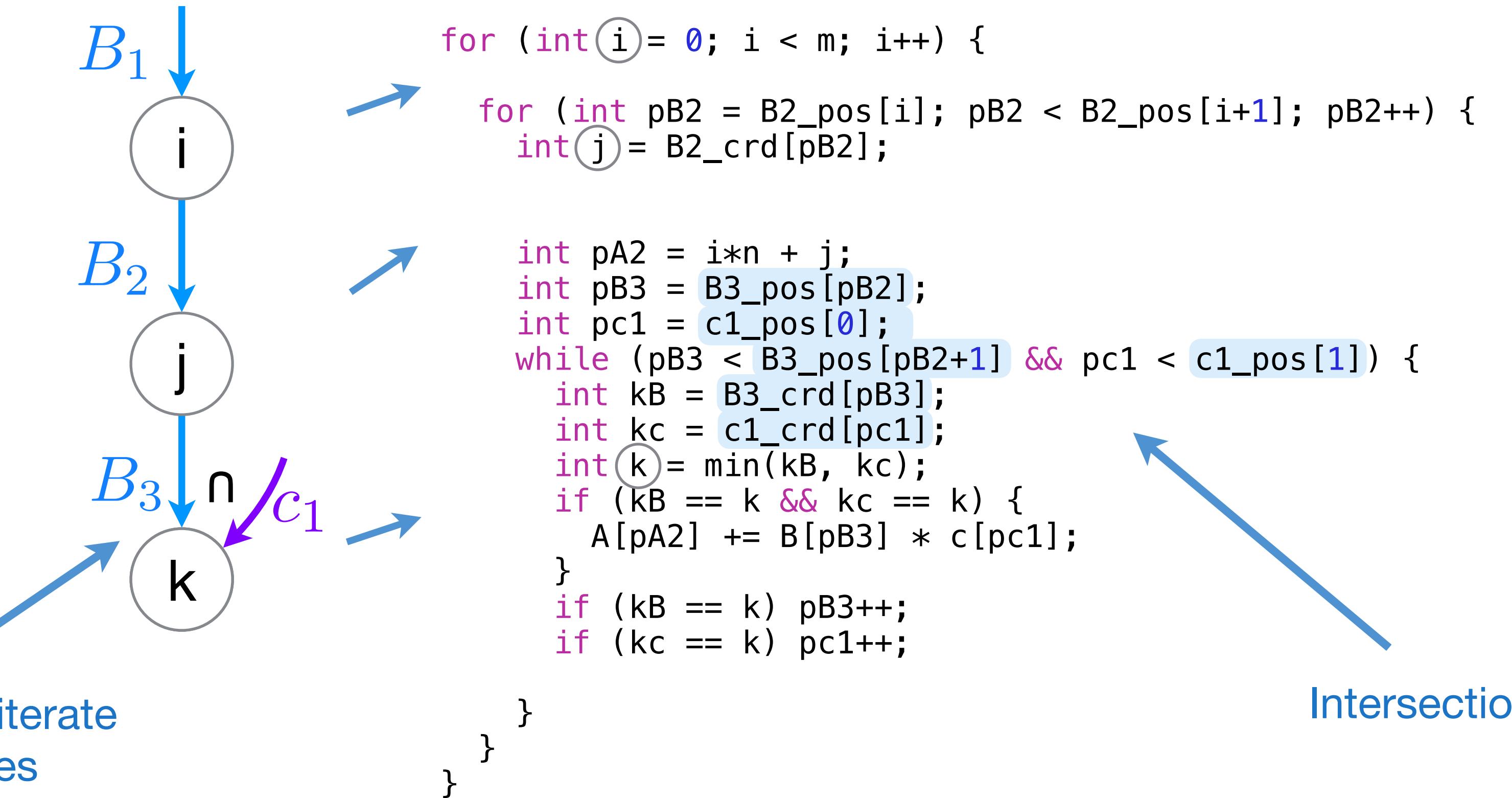
$$A_{ij} = \sum_k B_{ijk} c_k$$



Key operation is to coiterate  
over data structures

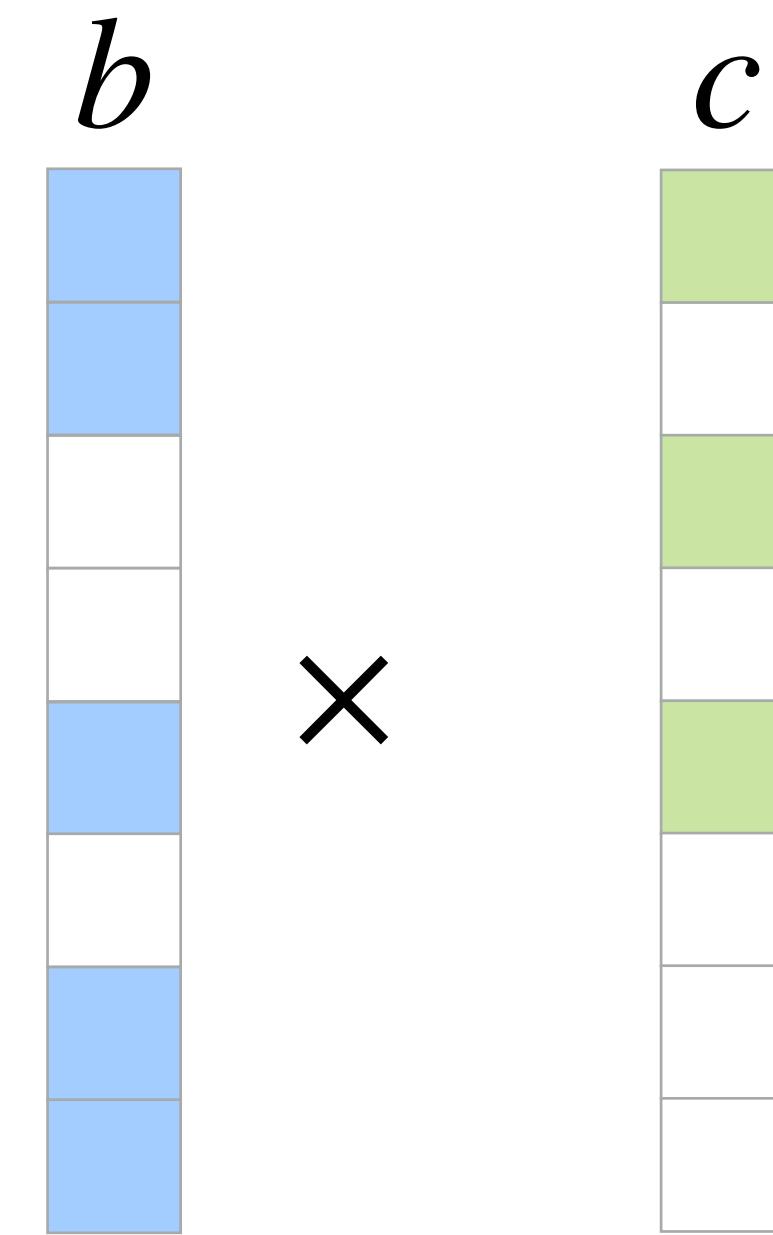
# Iteration graphs are lowered to sparse code

$$A_{ij} = \sum_k B_{ijk} c_k$$



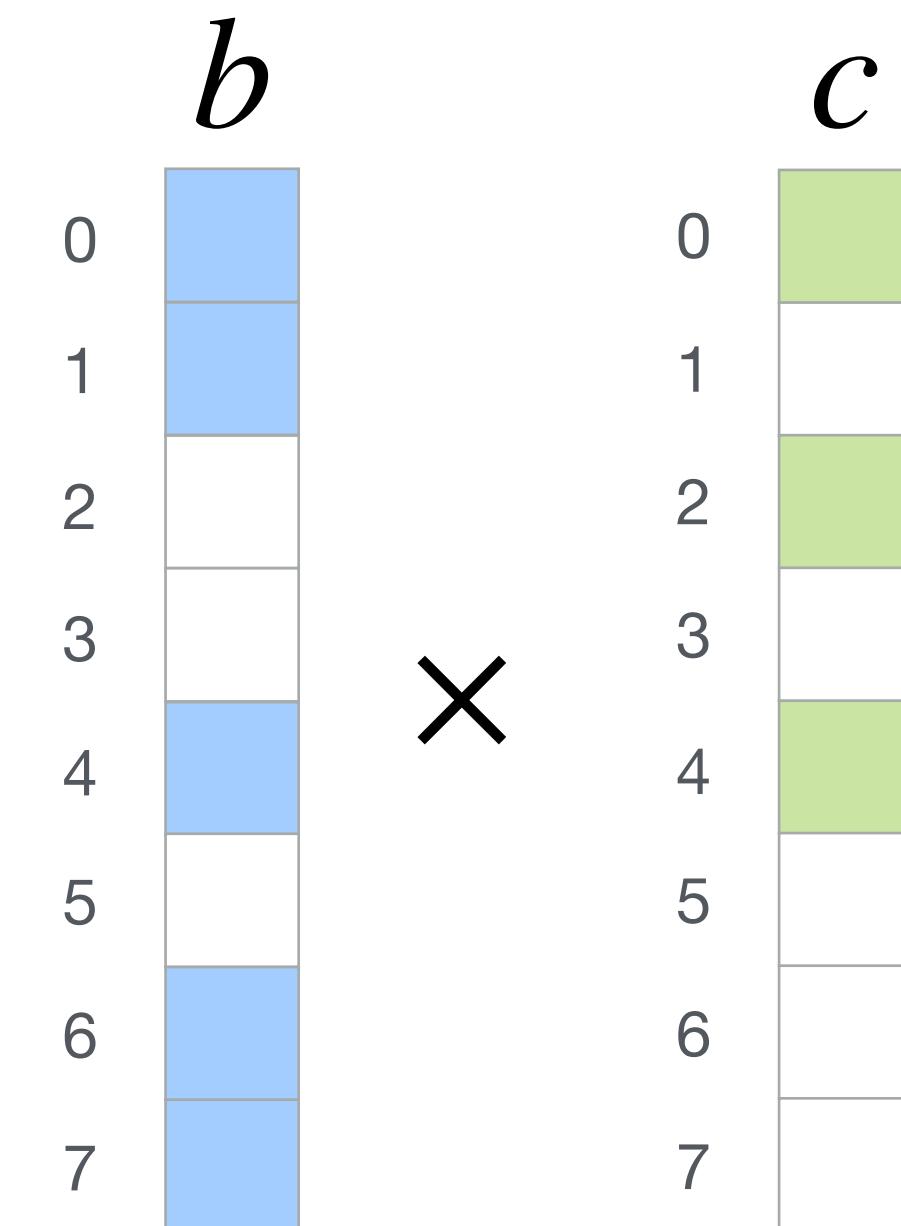
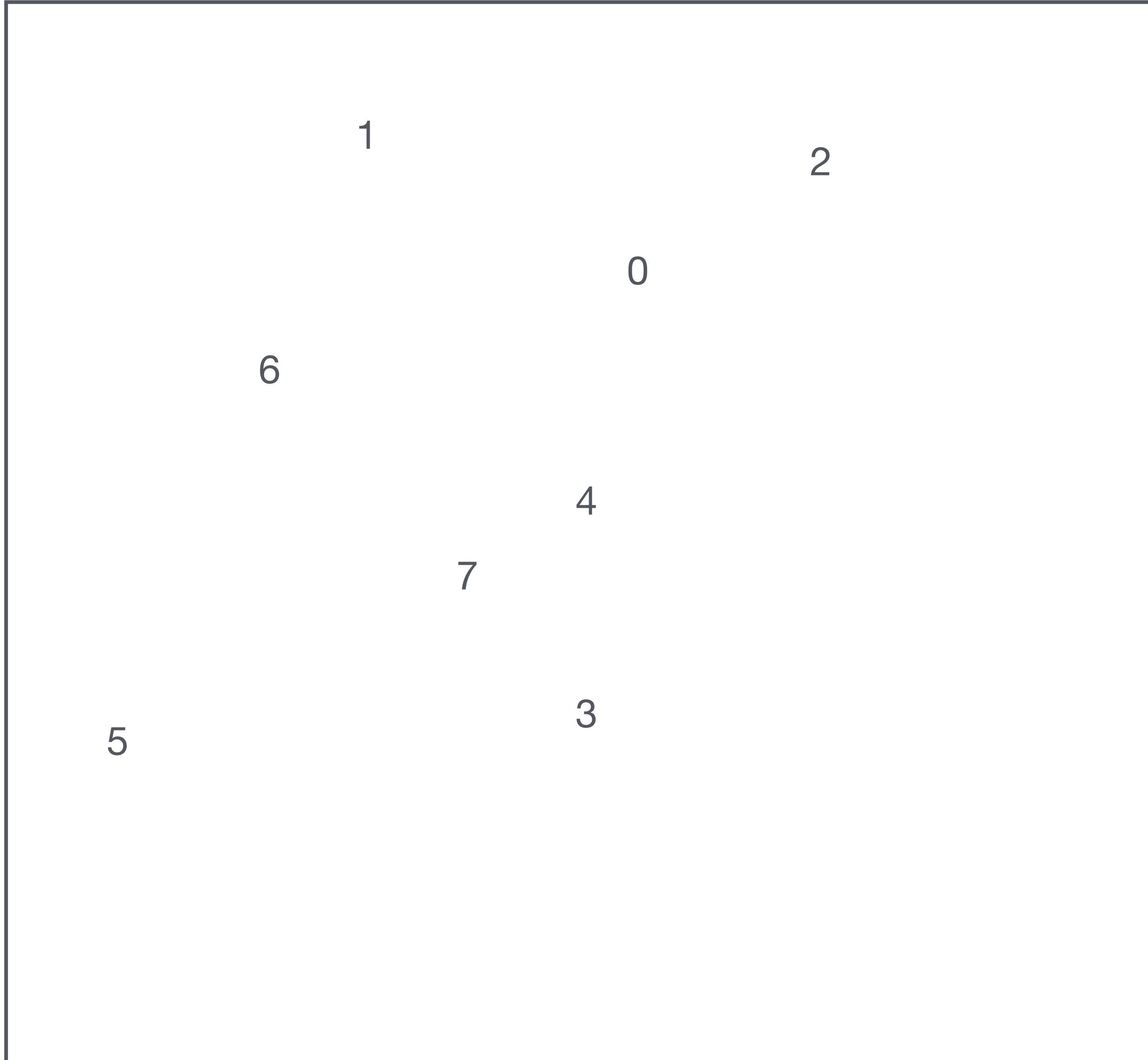
# Data structure coiteration

Coordinate Space



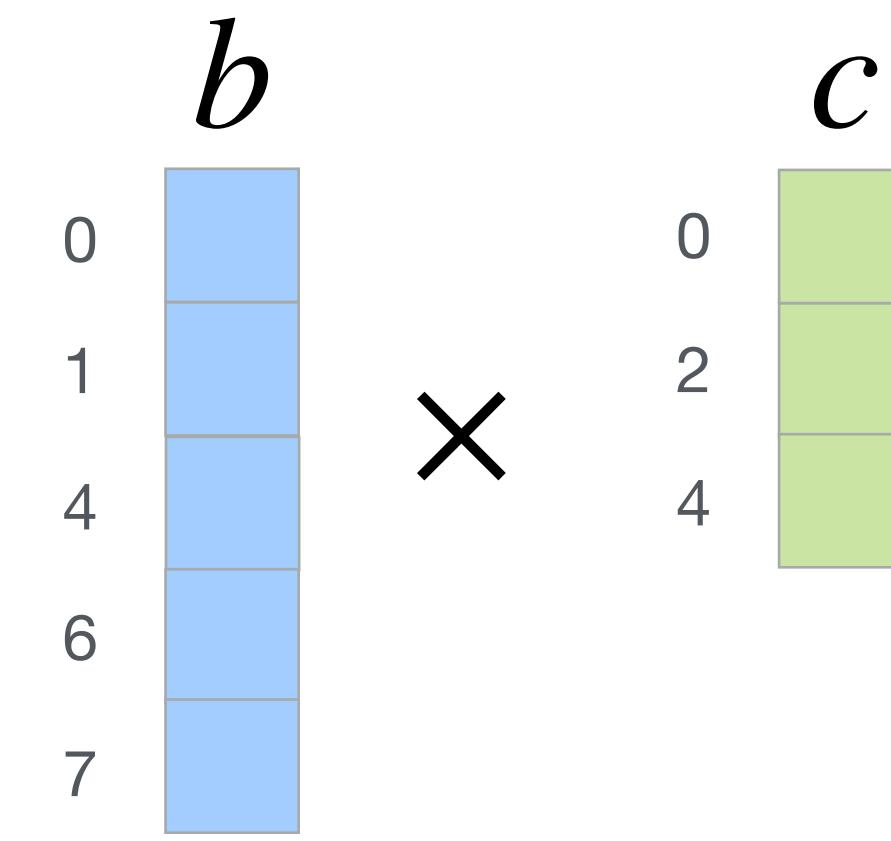
# Data structure coiteration

Coordinate Space



# Data structure coiteration

Coordinate Space



# Data structure coiteration

Coordinate Space



$$b \times c$$

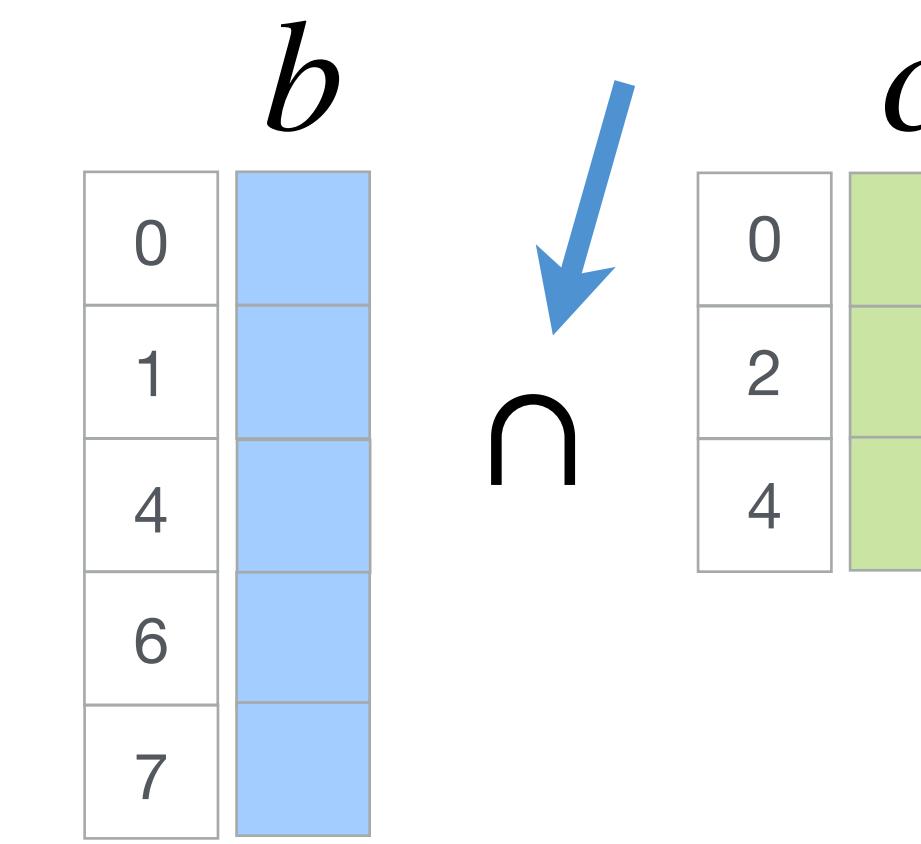
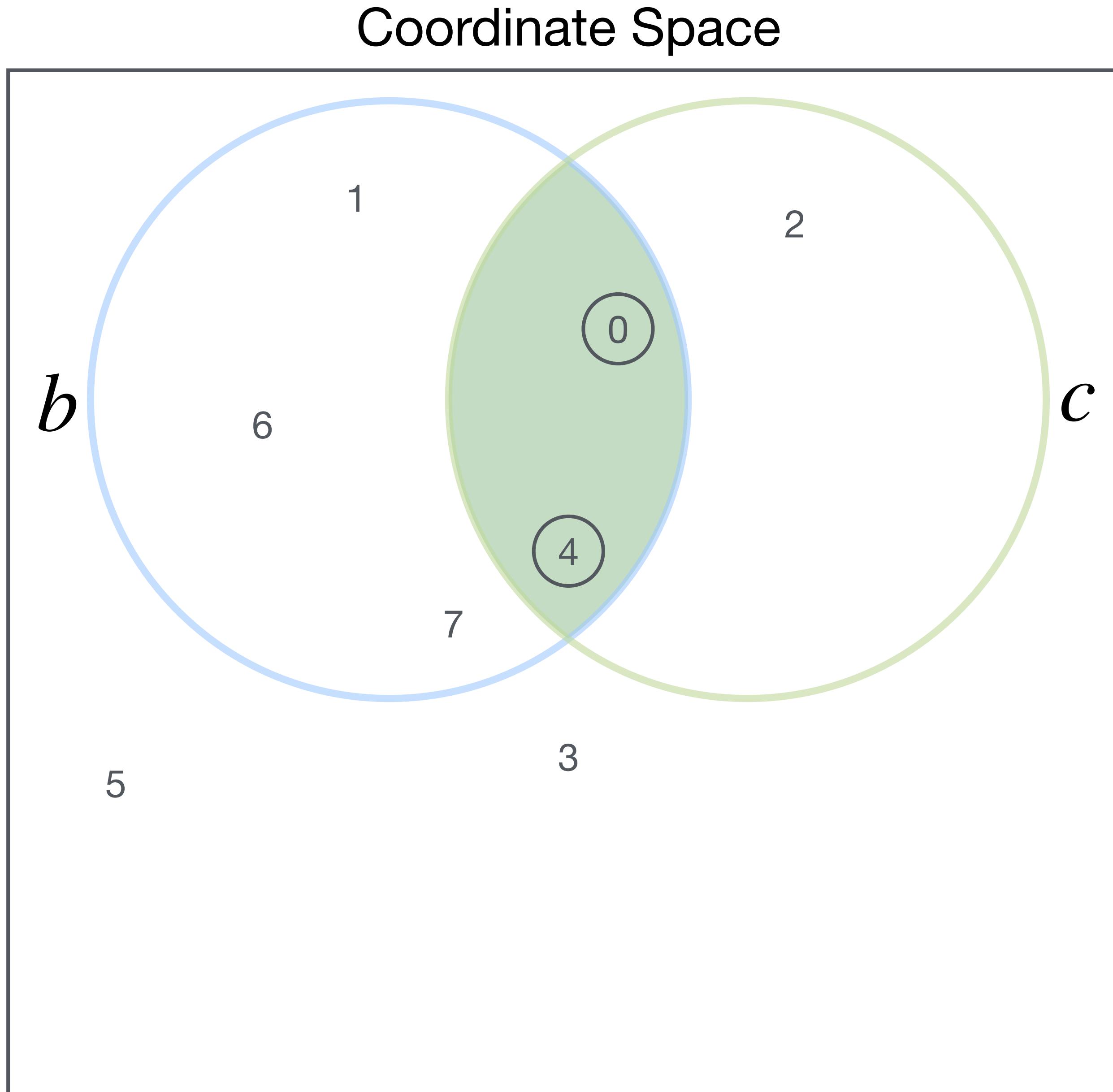
Matrix  $b$ :

0	
1	
4	
6	
7	

Matrix  $c$ :

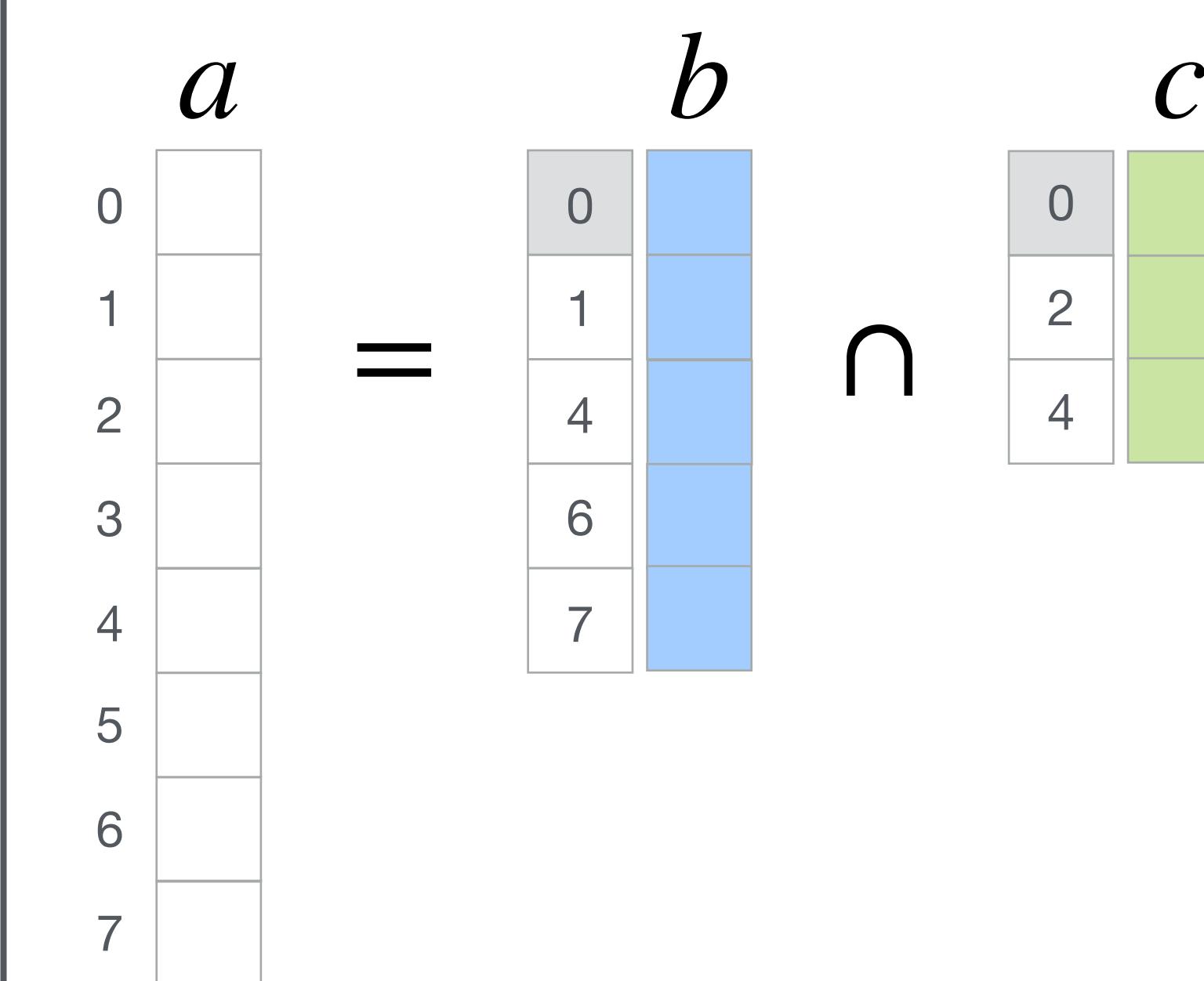
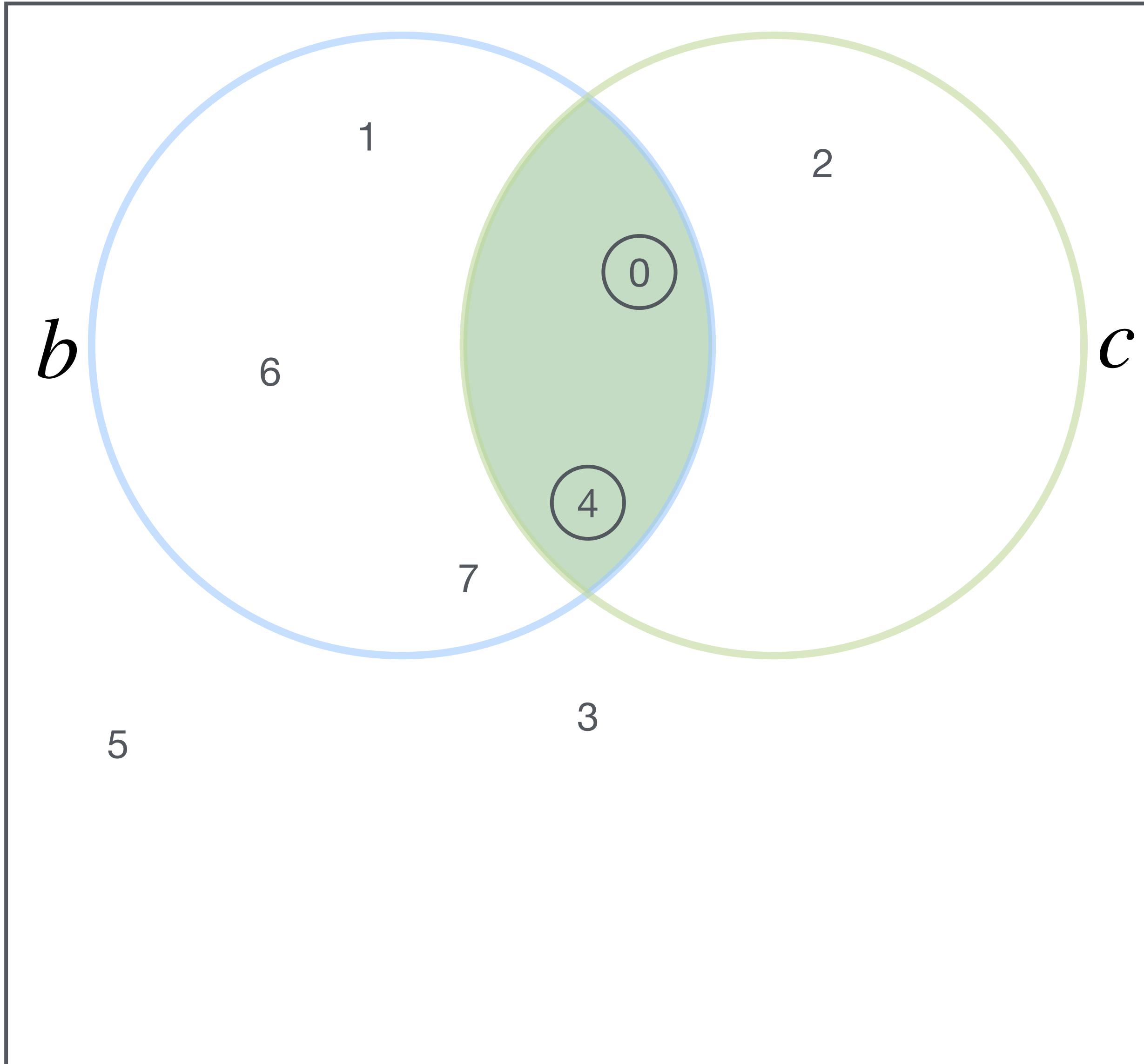
0	
2	
4	

# Data structure coiteration



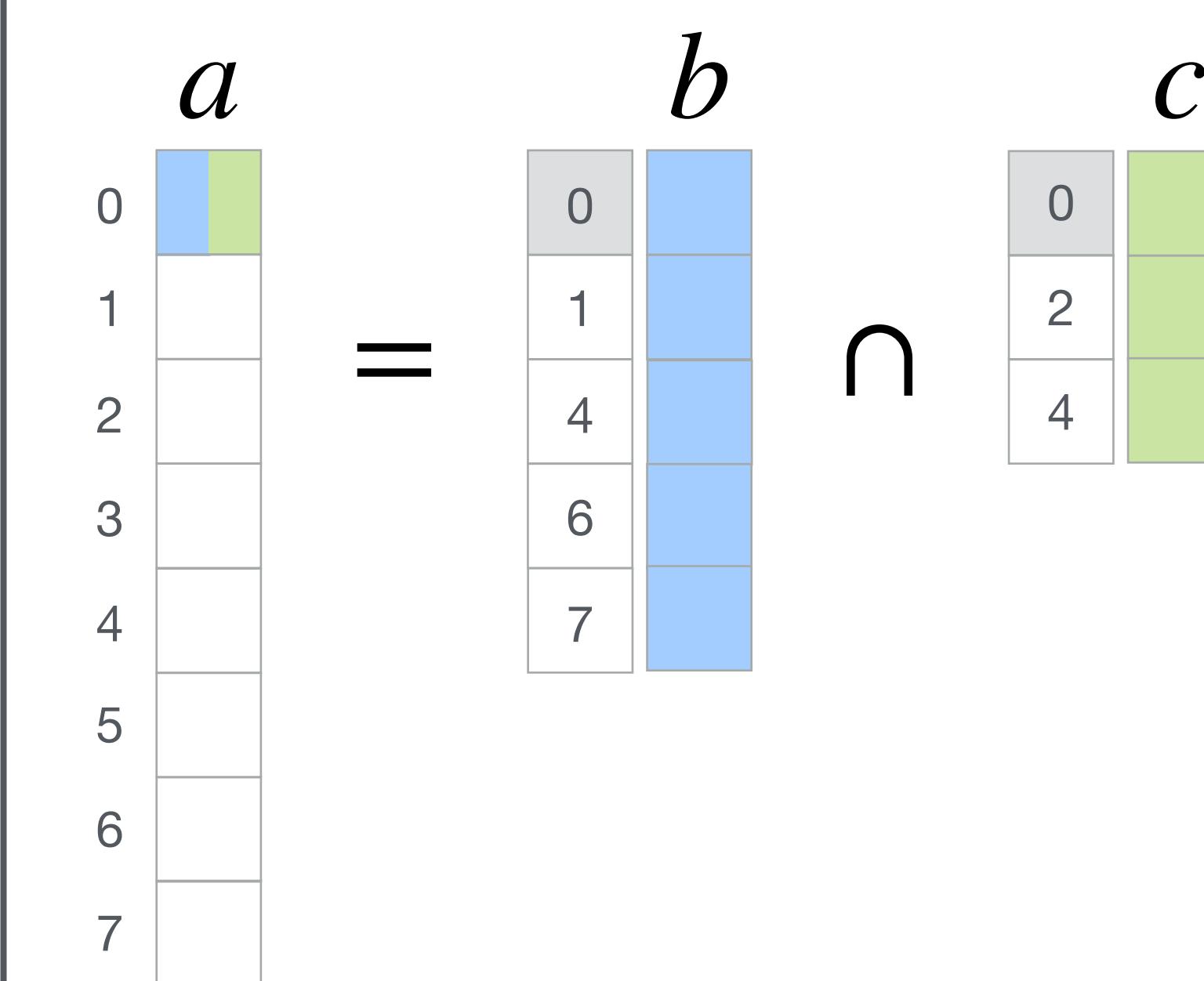
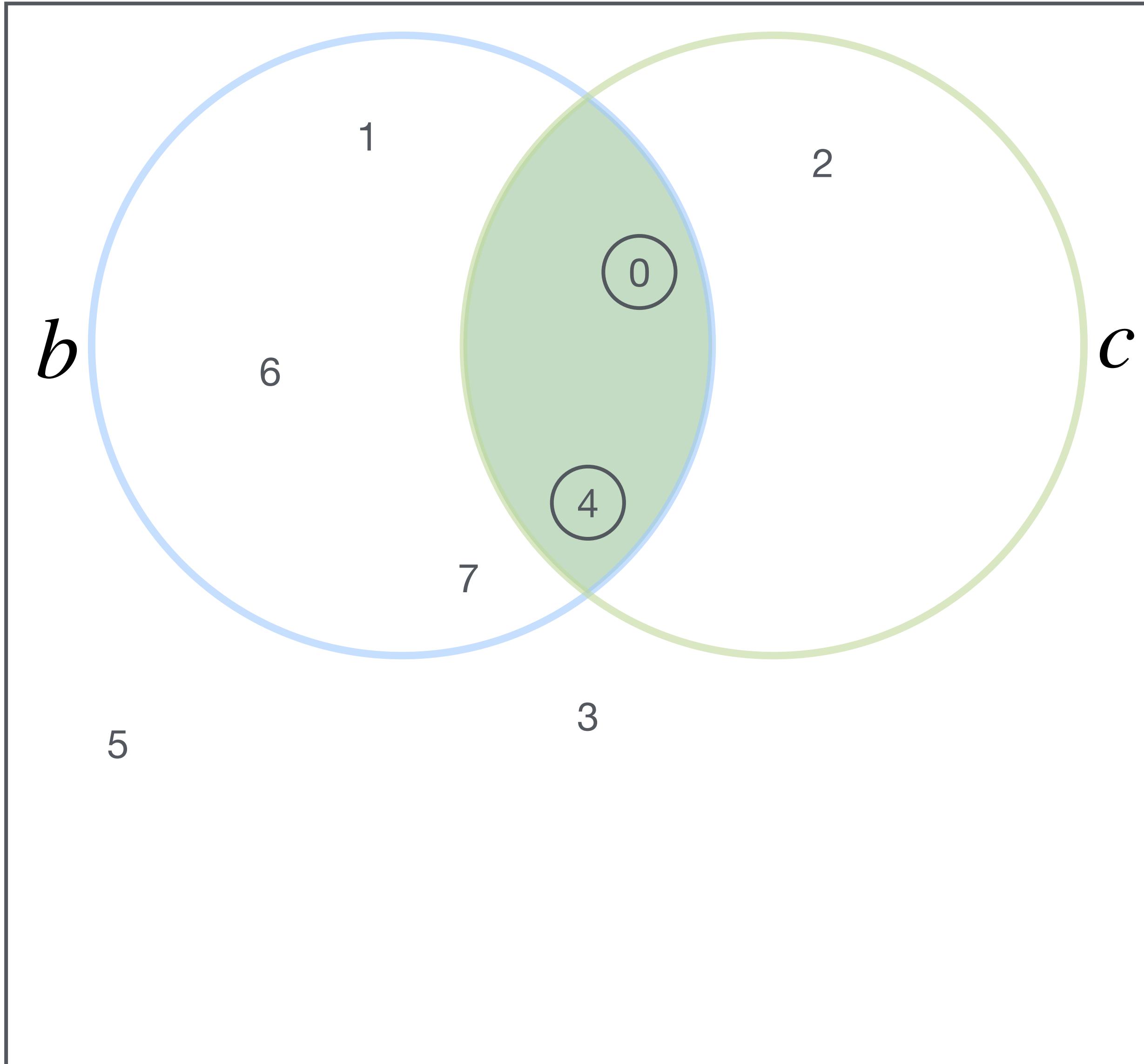
# Data structure coiteration

Coordinate Space



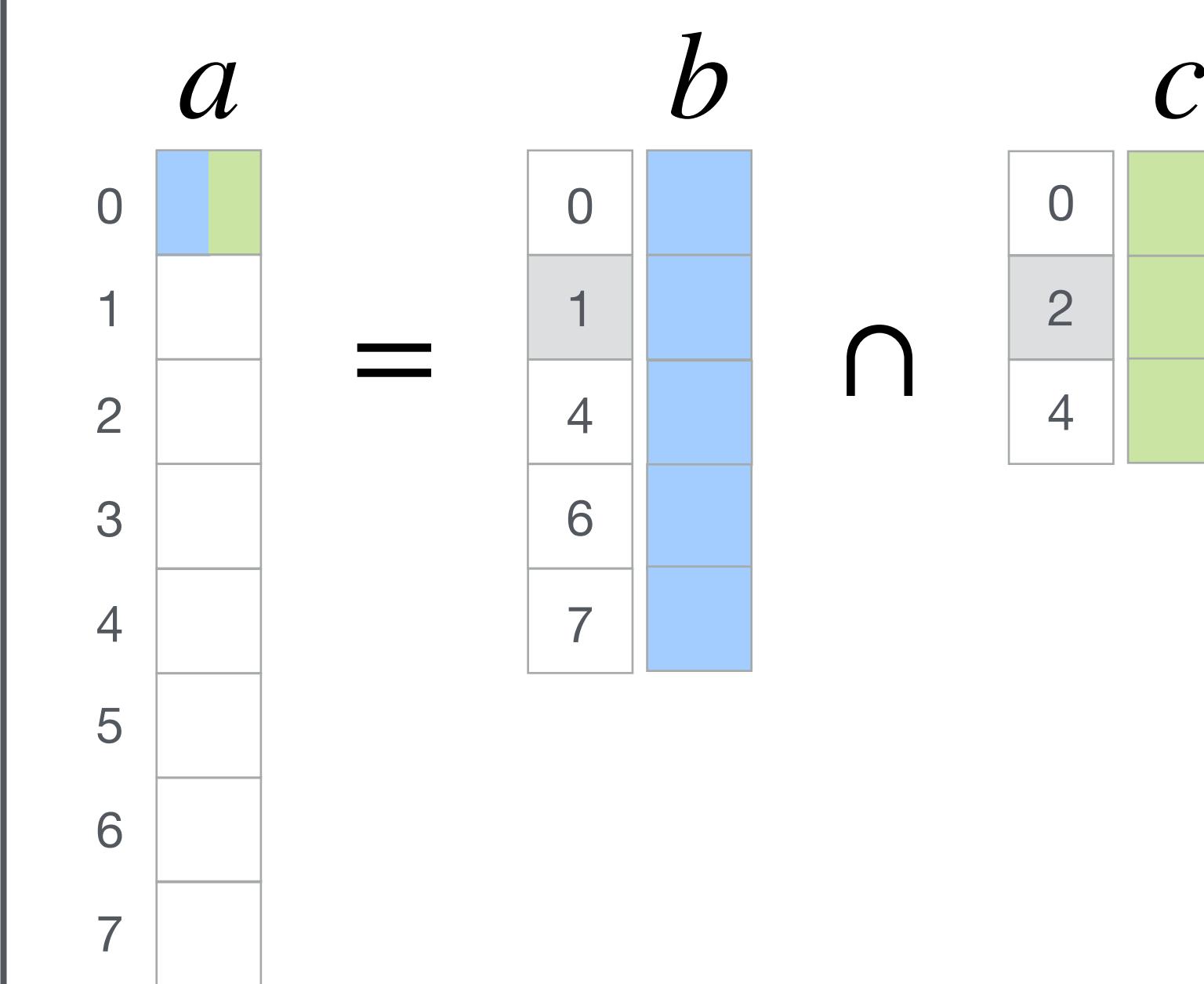
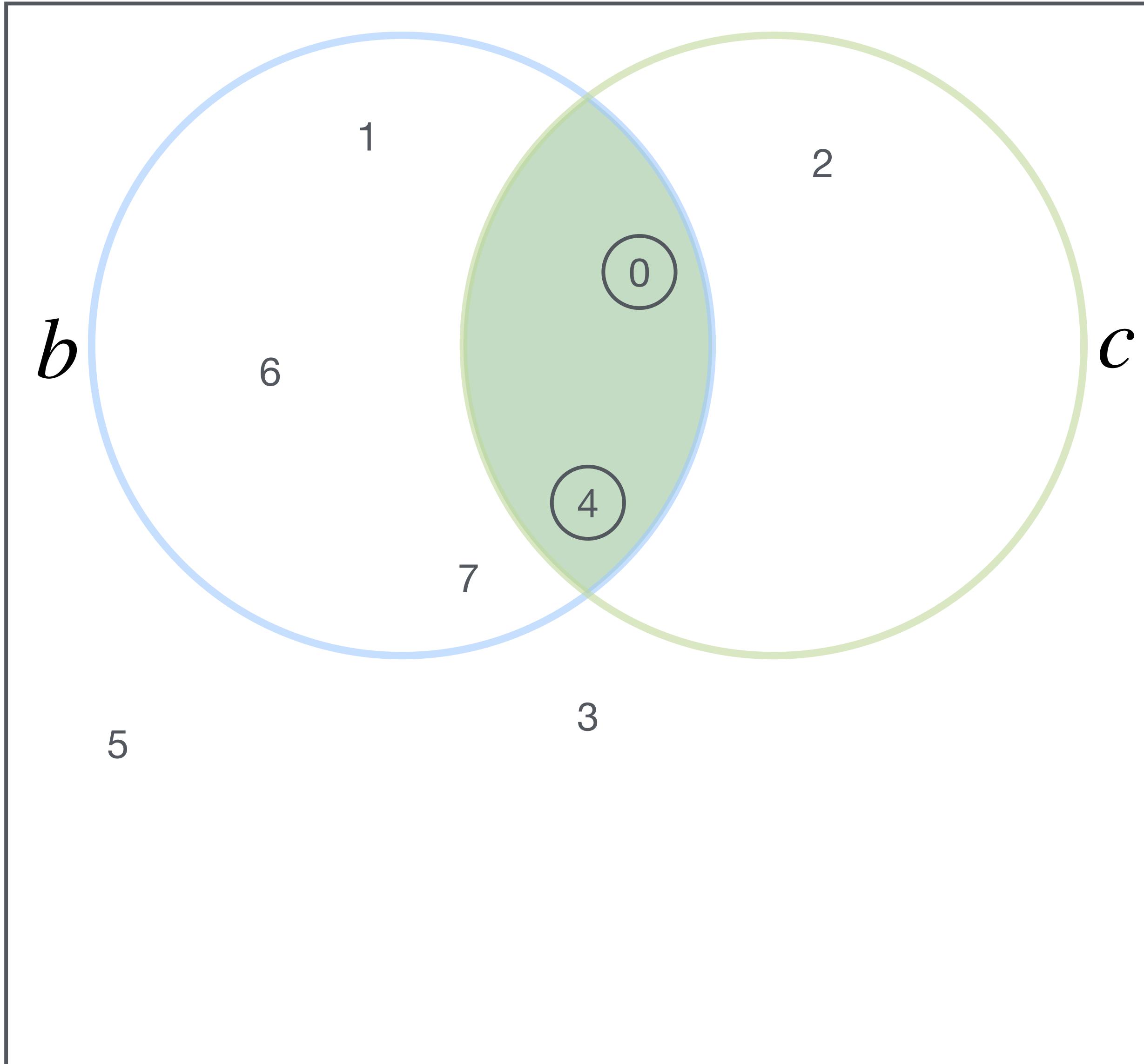
# Data structure coiteration

Coordinate Space



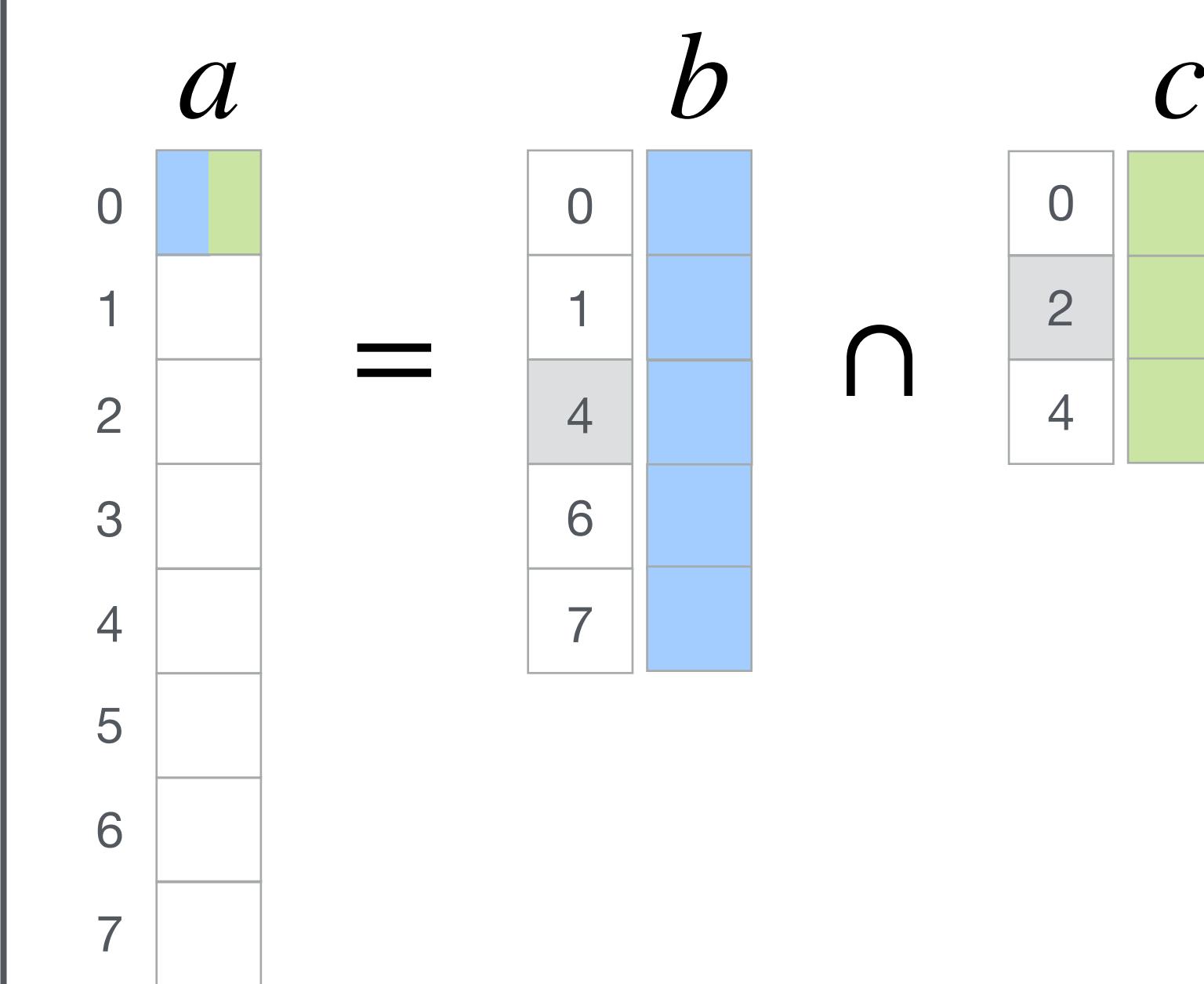
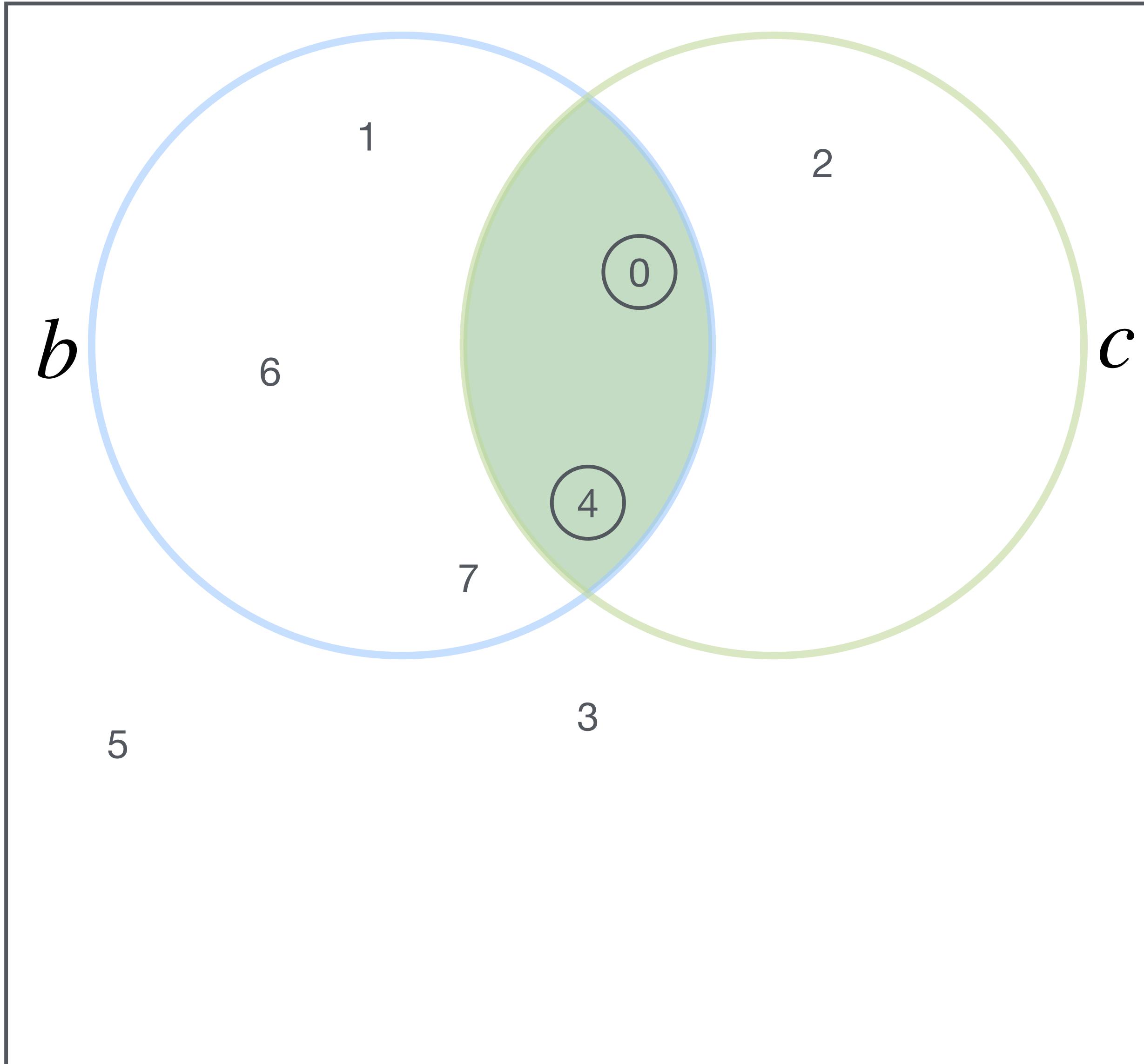
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Coordinate Space



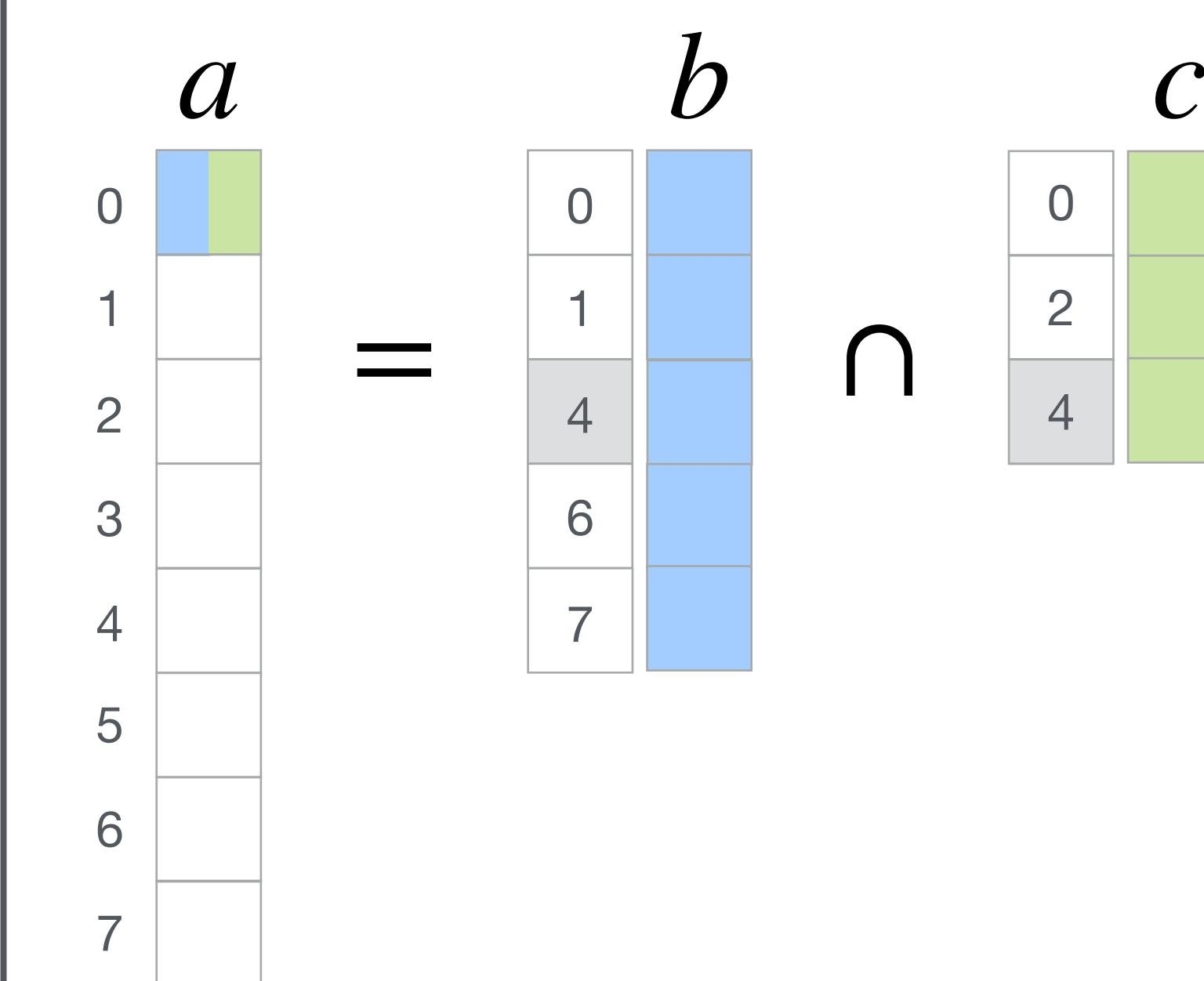
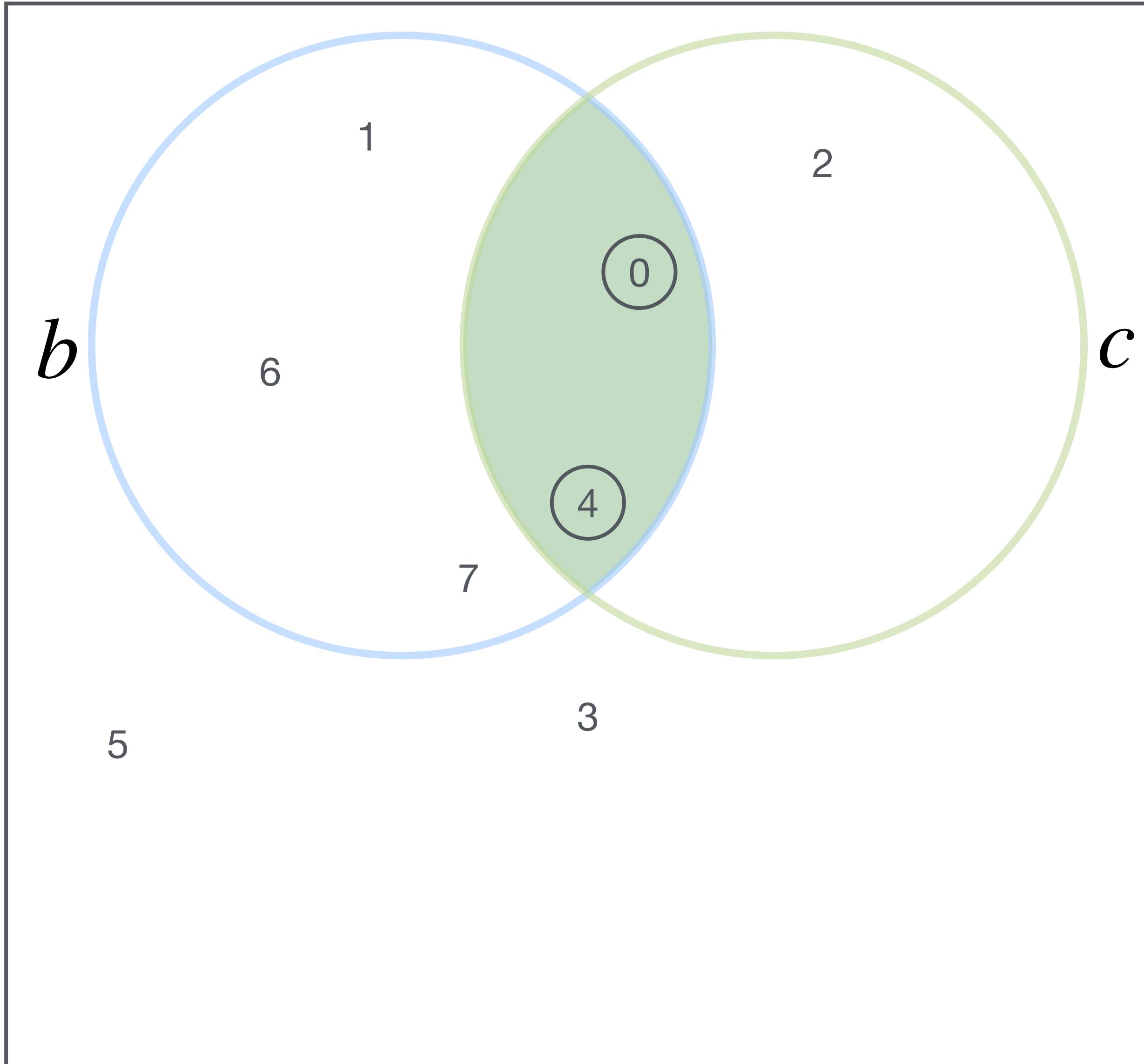
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Coordinate Space



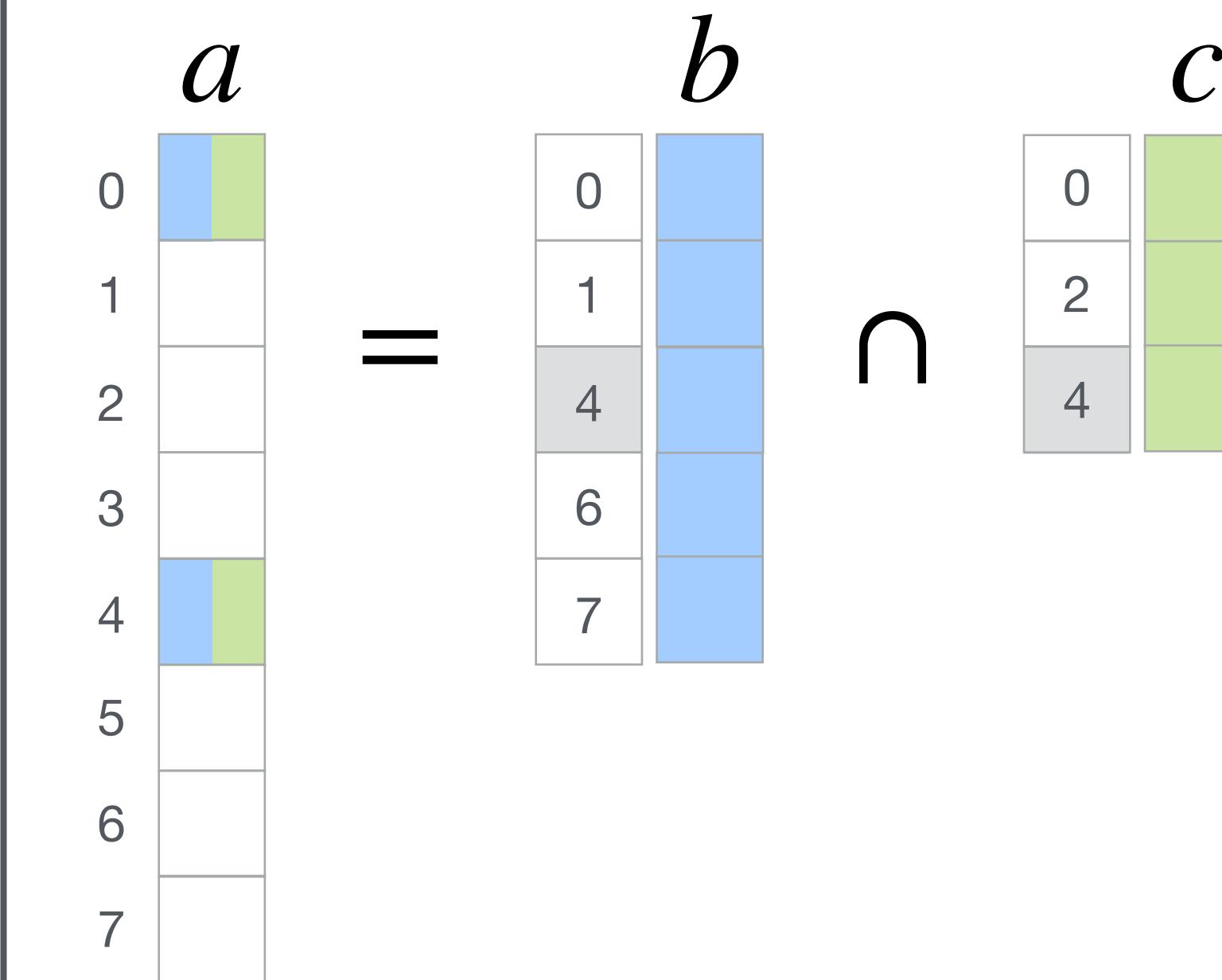
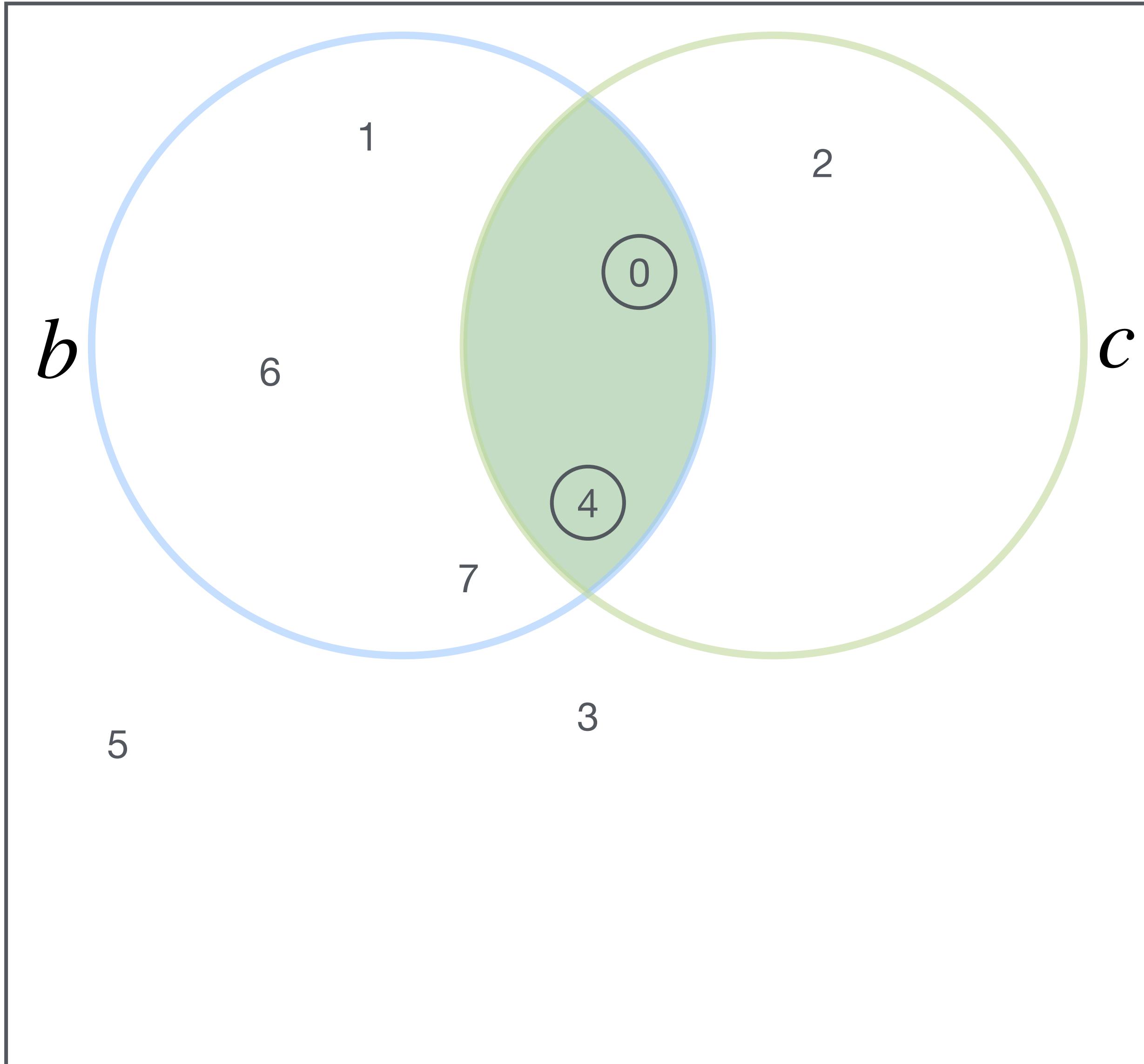
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Coordinate Space



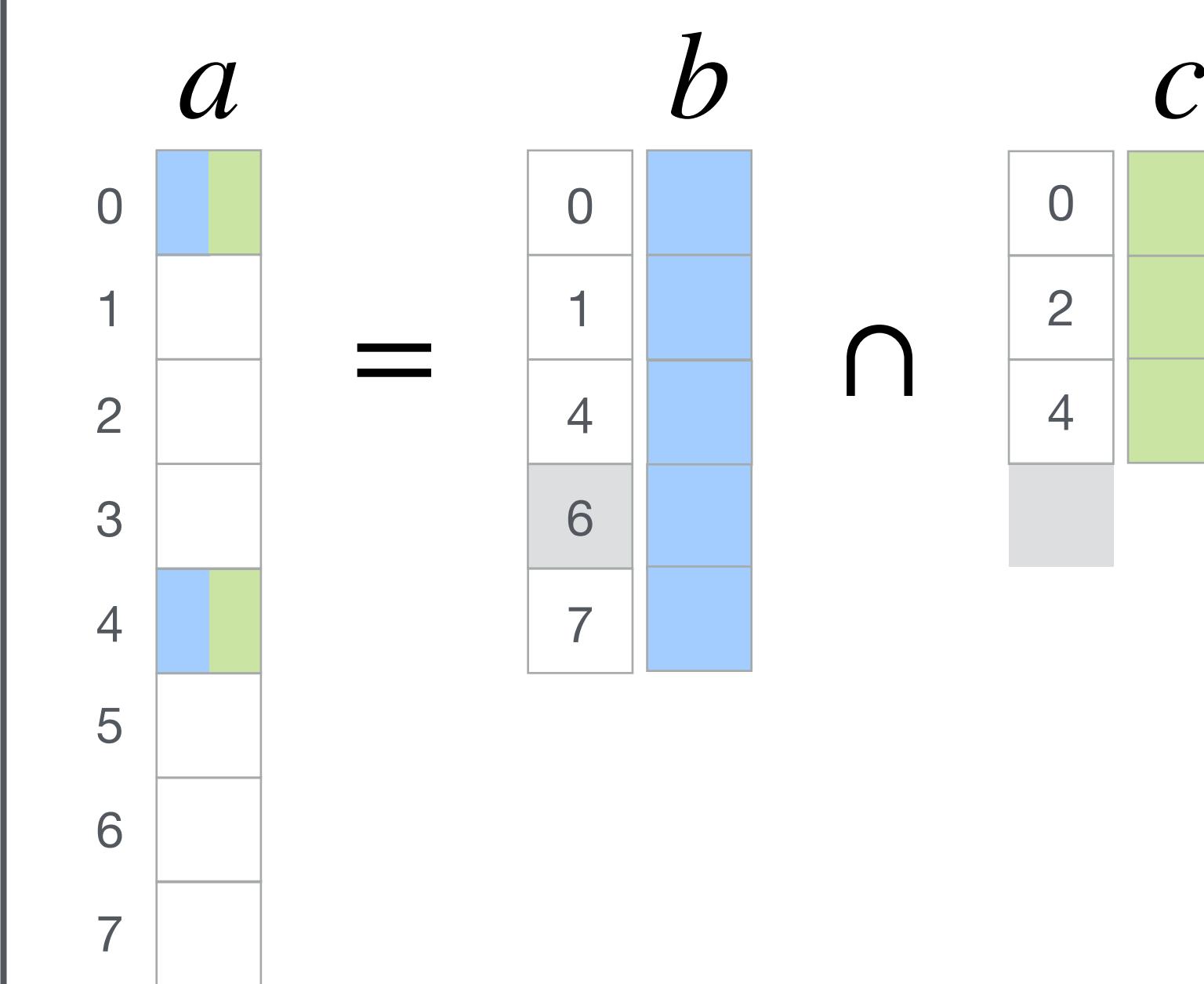
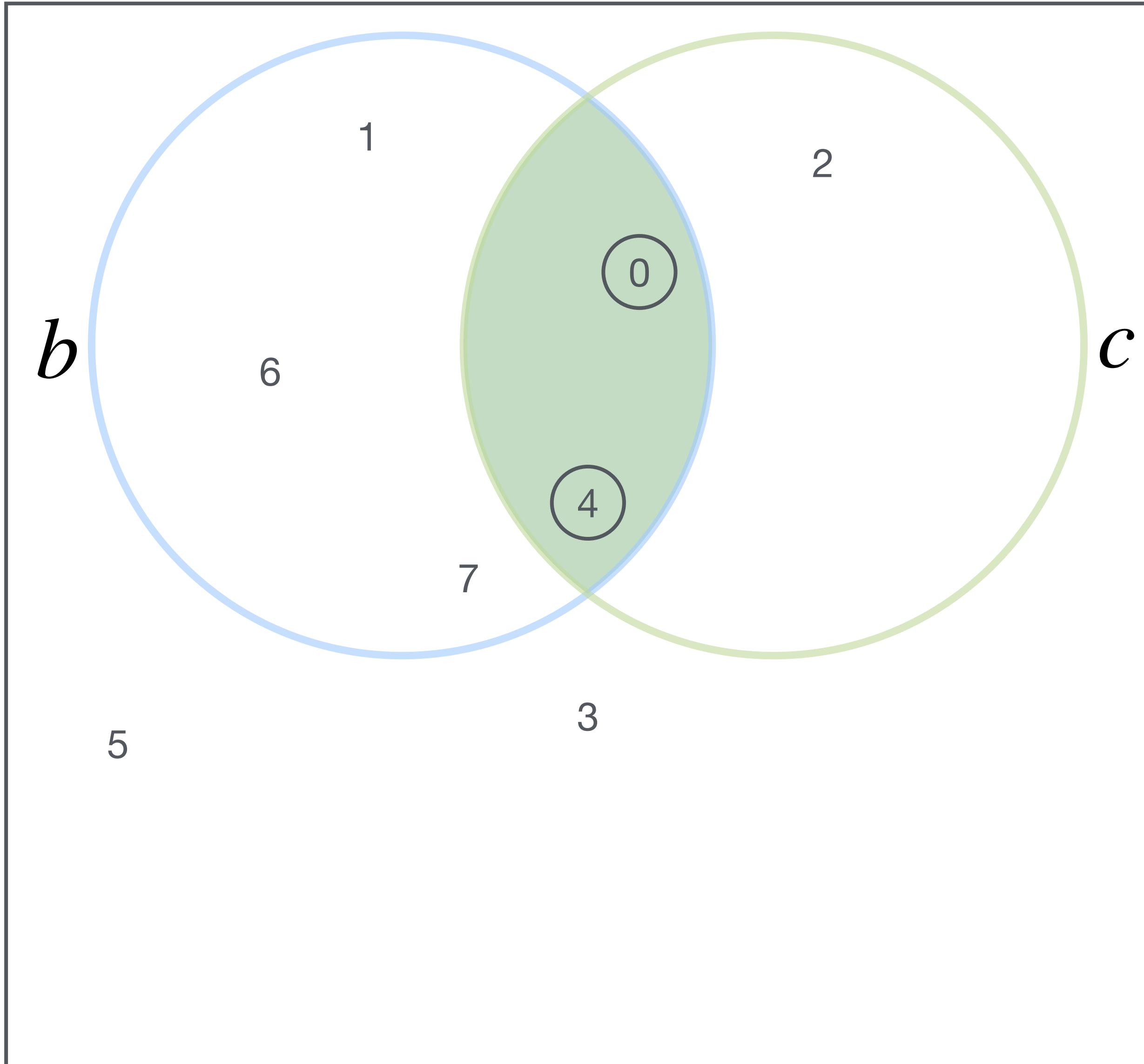
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Coordinate Space



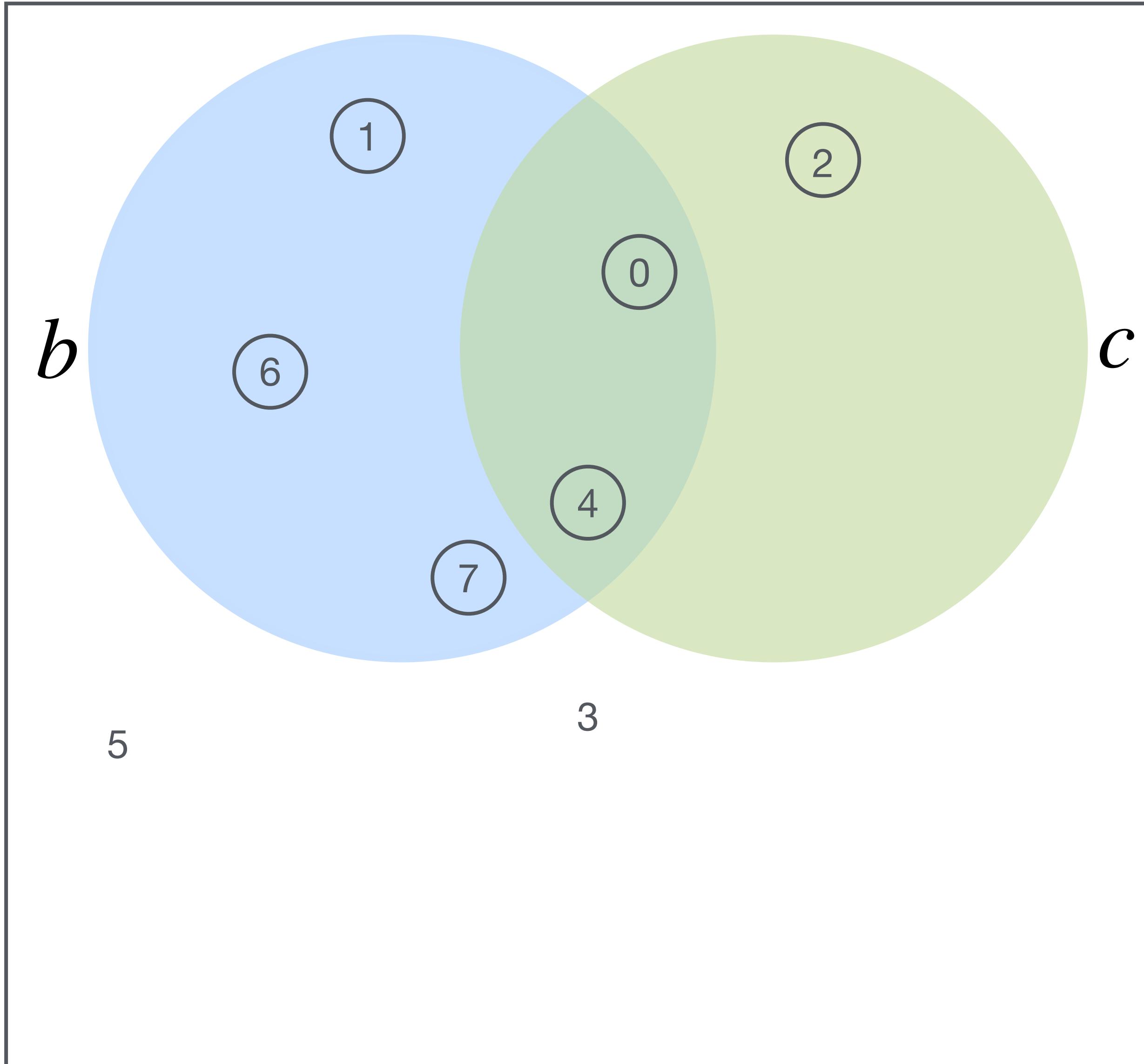
# Data structure coiteration

Coordinate Space



# Data structure coiteration

Coordinate Space

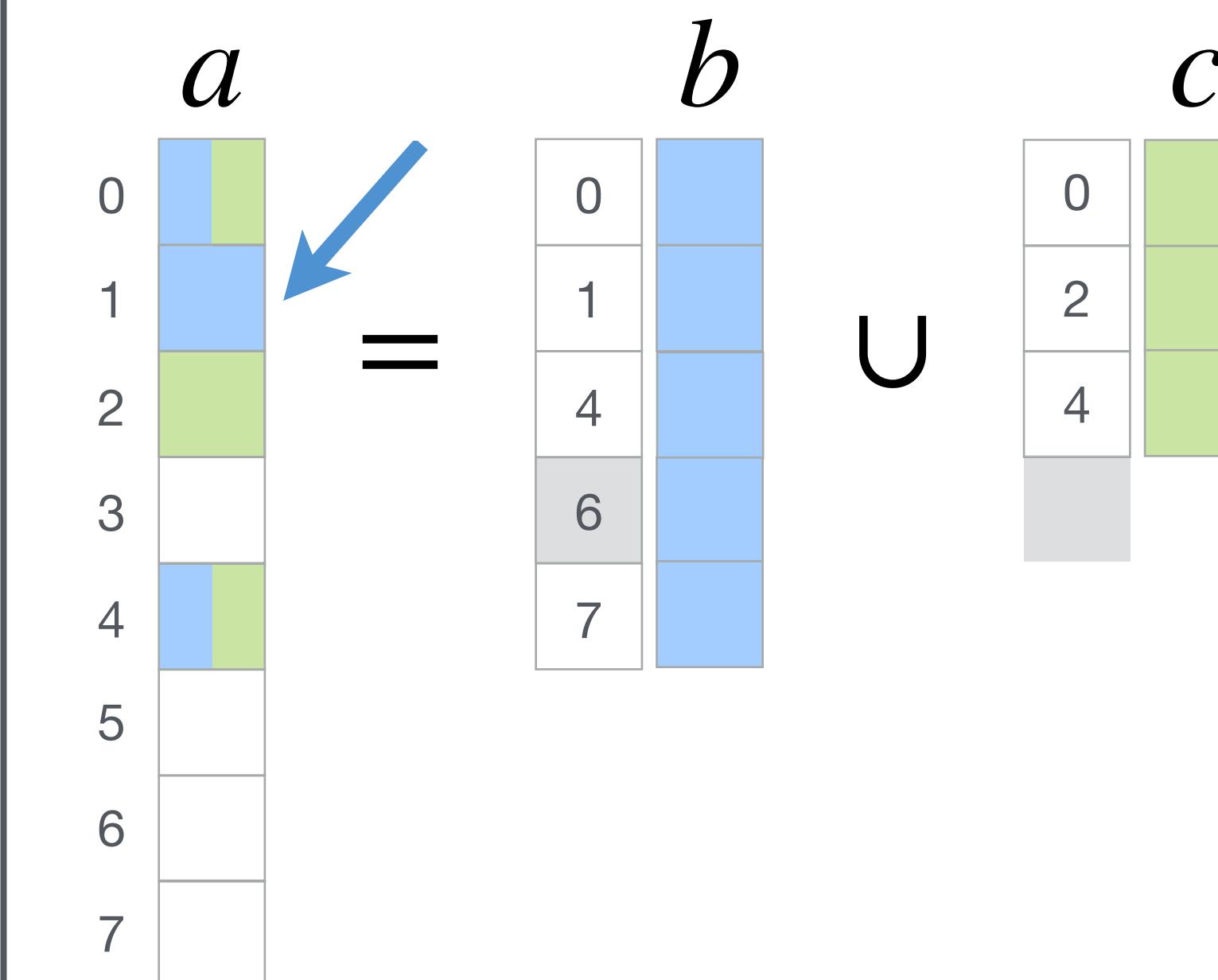
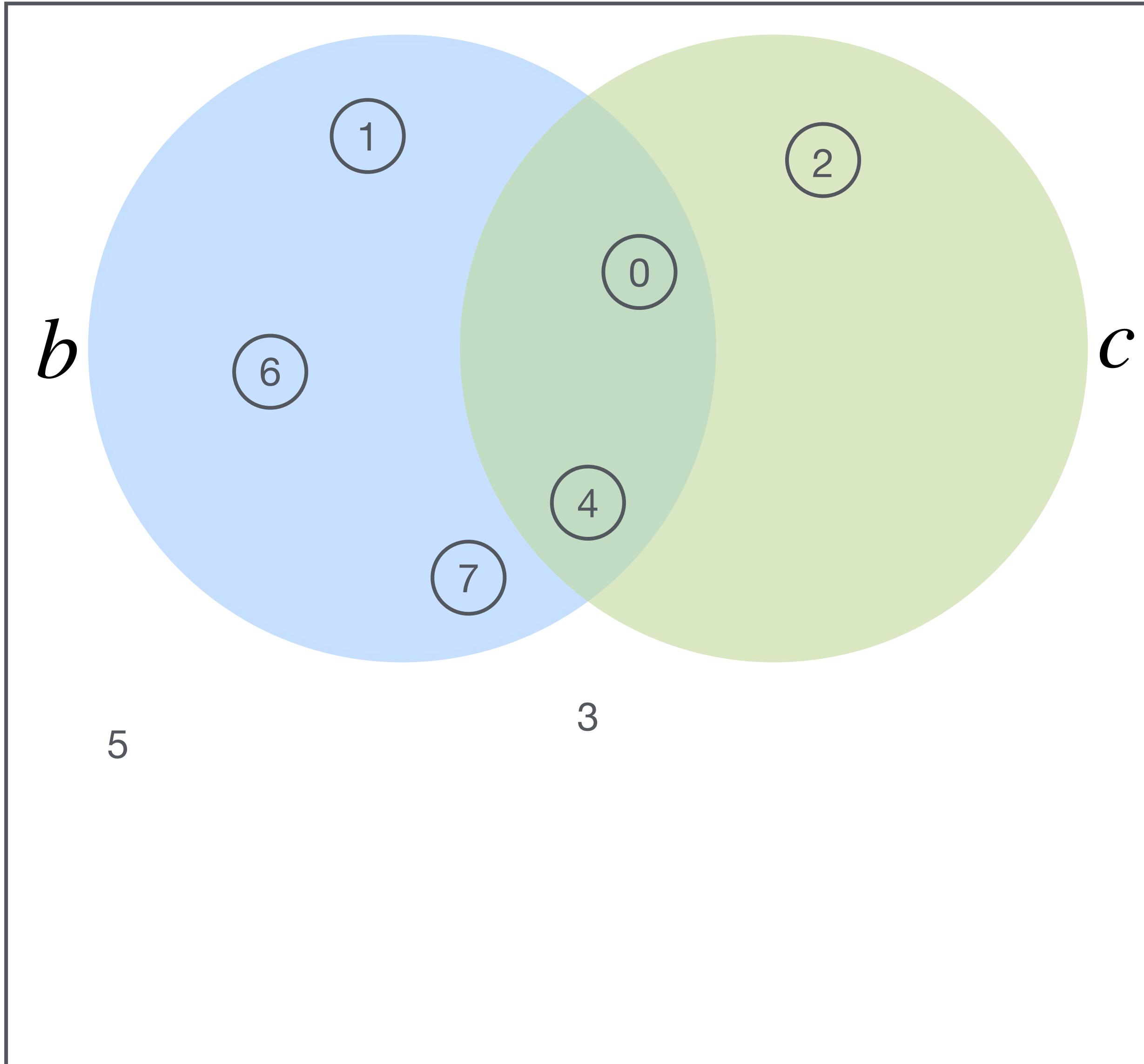


$$a = \begin{array}{c|c} 0 & \text{blue} \\ \hline 1 & \text{white} \\ \hline 2 & \text{white} \\ \hline 3 & \text{white} \\ \hline 4 & \text{blue} \\ \hline 5 & \text{white} \\ \hline 6 & \text{white} \\ \hline 7 & \text{blue} \end{array} \quad b = \begin{array}{c|c} 0 & \text{white} \\ \hline 1 & \text{white} \\ \hline 4 & \text{white} \\ \hline 6 & \text{gray} \\ \hline 7 & \text{blue} \end{array} \quad c = \begin{array}{c|c} 0 & \text{white} \\ \hline 2 & \text{white} \\ \hline 4 & \text{white} \\ \hline \end{array}$$

$a = b \cup c$

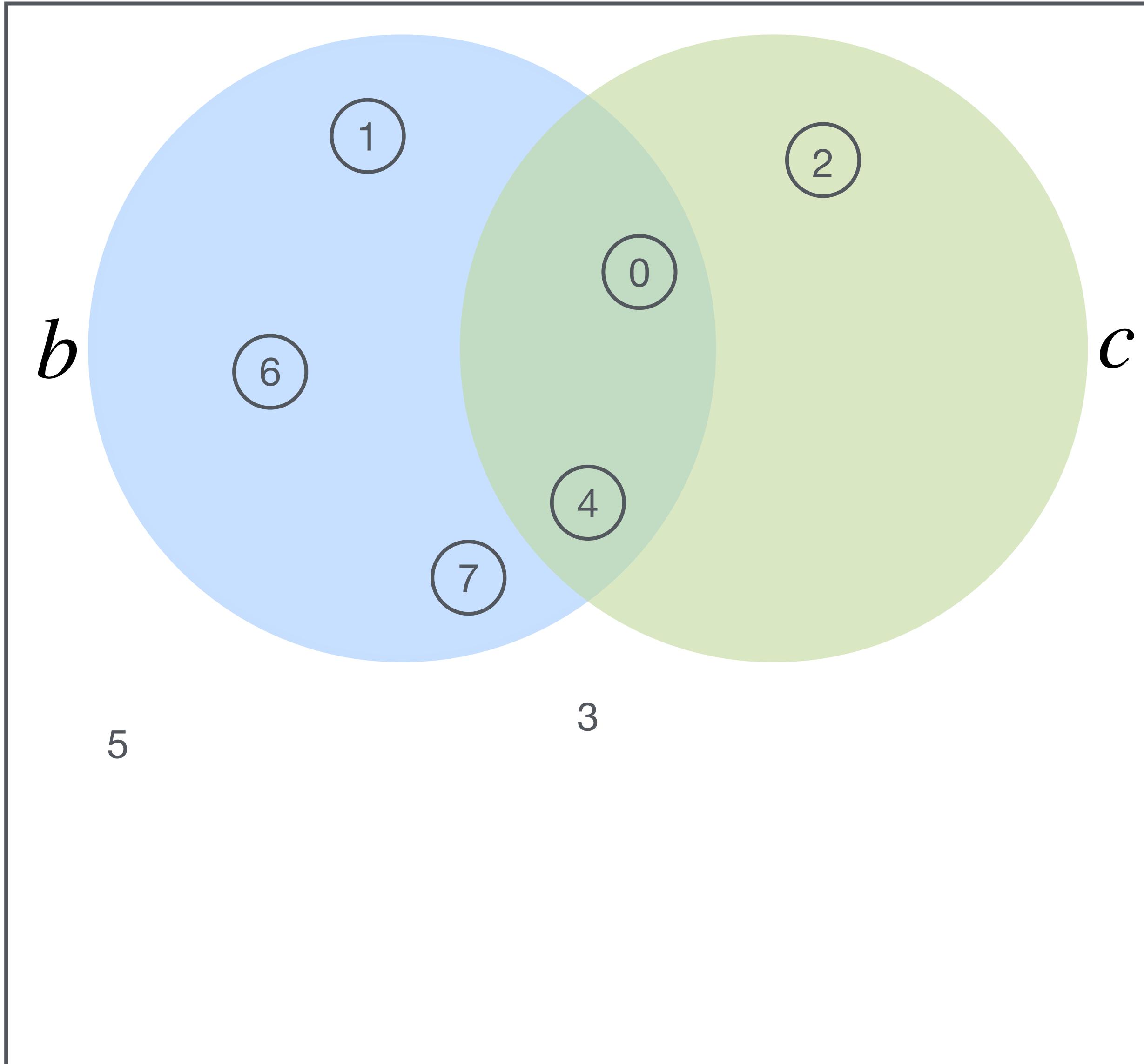
# Data structure coiteration

Coordinate Space



# Data structure coiteration

Coordinate Space

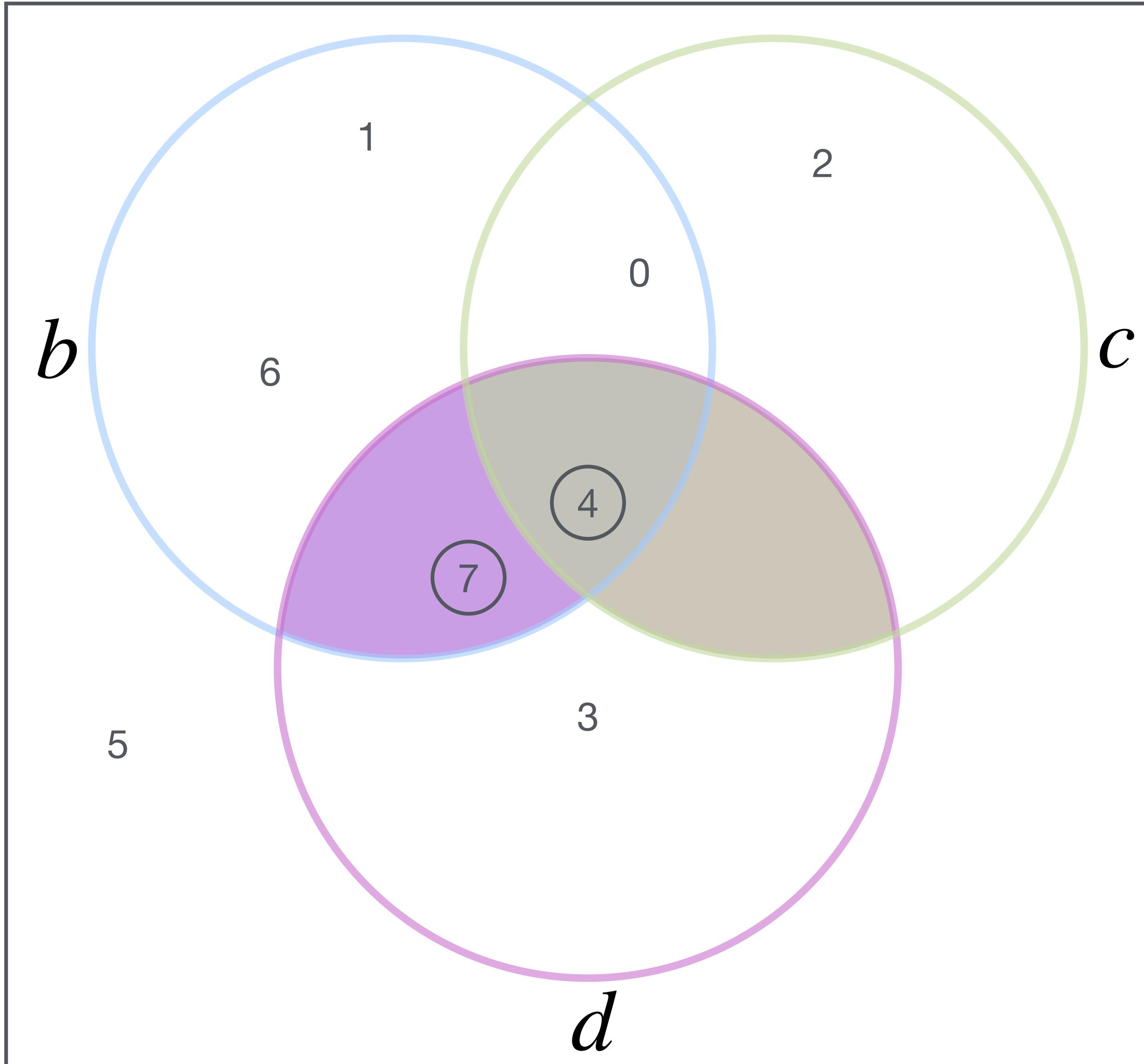


$$a = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \text{blue} & \text{green} \\ 1 & \text{blue} & \text{green} \\ 2 & \text{green} & \text{white} \\ 3 & \text{white} & \text{white} \\ 4 & \text{blue} & \text{green} \\ 5 & \text{white} & \text{white} \\ 6 & \text{blue} & \text{white} \\ 7 & \text{blue} & \text{white} \end{array} \quad b = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \text{white} & \text{white} \\ 1 & \text{white} & \text{white} \\ 2 & \text{white} & \text{white} \\ 3 & \text{white} & \text{white} \\ 4 & \text{white} & \text{white} \\ 5 & \text{white} & \text{white} \\ 6 & \text{white} & \text{white} \\ 7 & \text{white} & \text{white} \end{array} \quad c = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \text{white} & \text{white} \\ 1 & \text{white} & \text{white} \\ 2 & \text{white} & \text{white} \\ 3 & \text{white} & \text{white} \\ 4 & \text{white} & \text{white} \\ 5 & \text{white} & \text{white} \\ 6 & \text{white} & \text{white} \\ 7 & \text{white} & \text{white} \end{array}$$

$=$   $\cup$

# Data structure coiteration

Coordinate Space

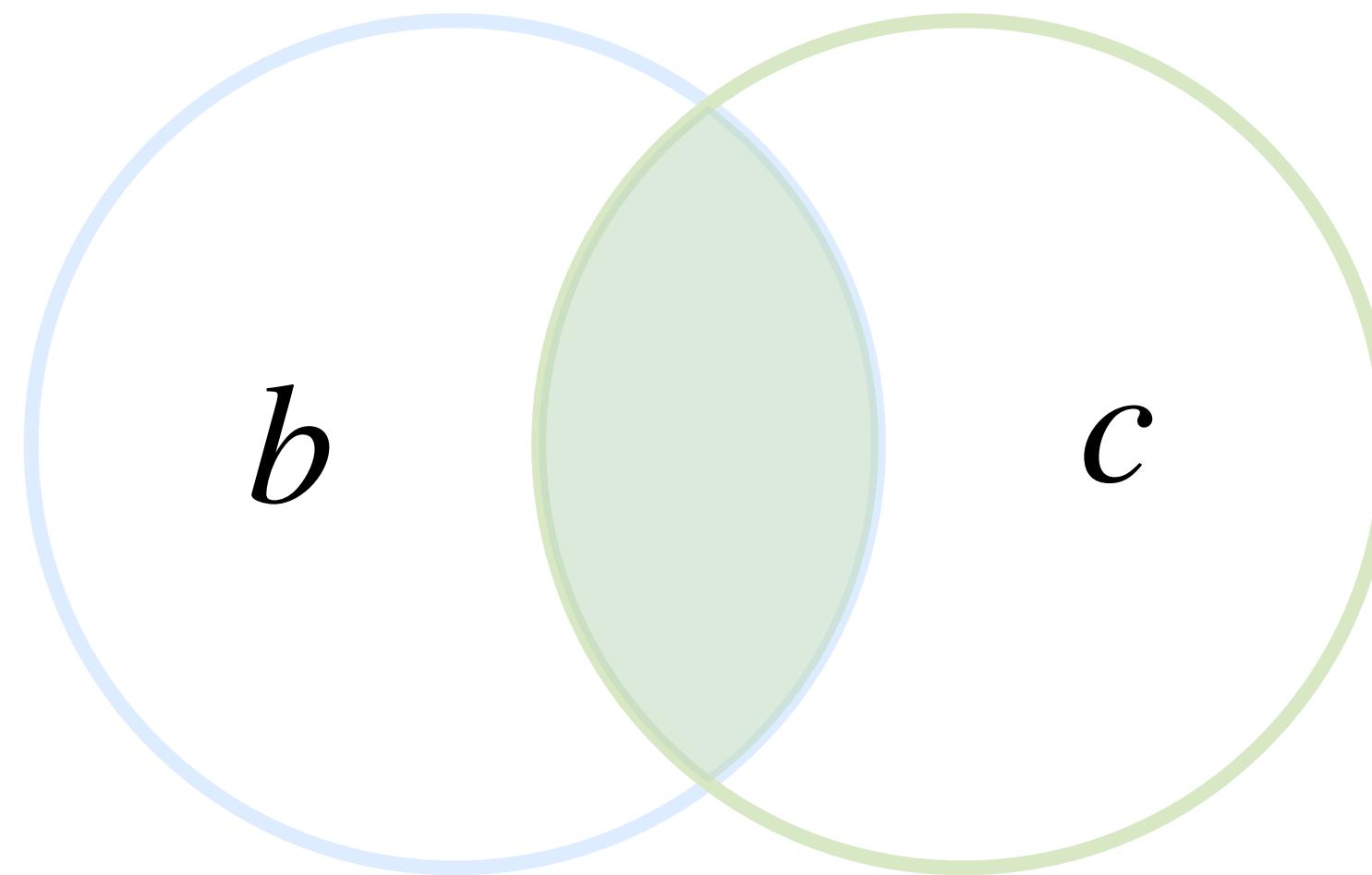


$$a = \left( \begin{array}{c|c} \text{Region } 0 & \text{Region } b \\ \hline 0 & 0 \\ 1 & 1 \\ 2 & 4 \\ 3 & 6 \\ 4 & 0, 1, 4, 7 \\ 5 & \\ 6 & \\ 7 & 0, 4, 7 \end{array} \right) \cup \left( \begin{array}{c|c} \text{Region } c & \text{Region } d \\ \hline 0 & 0 \\ 1 & 2 \\ 2 & 4 \\ 3 & \\ 4 & \\ 5 & \\ 6 & \\ 7 & 3, 4, 7 \end{array} \right) \cap$$

# Iteration lattice for multiplications

$$a_i = b_i c_i$$

# Iteration lattice for multiplications



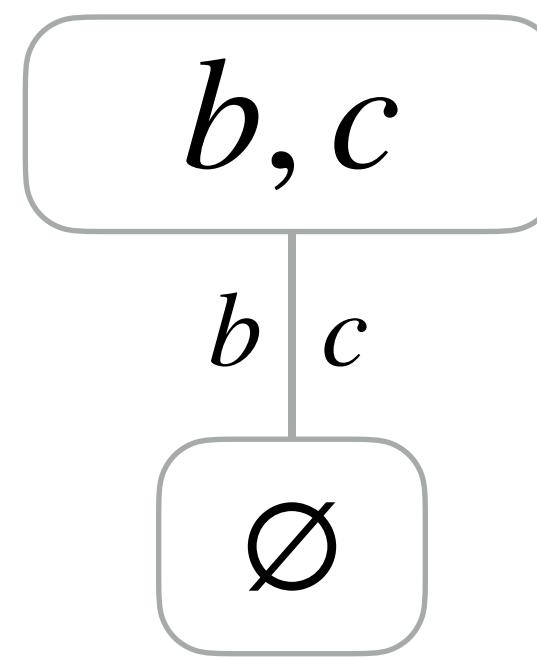
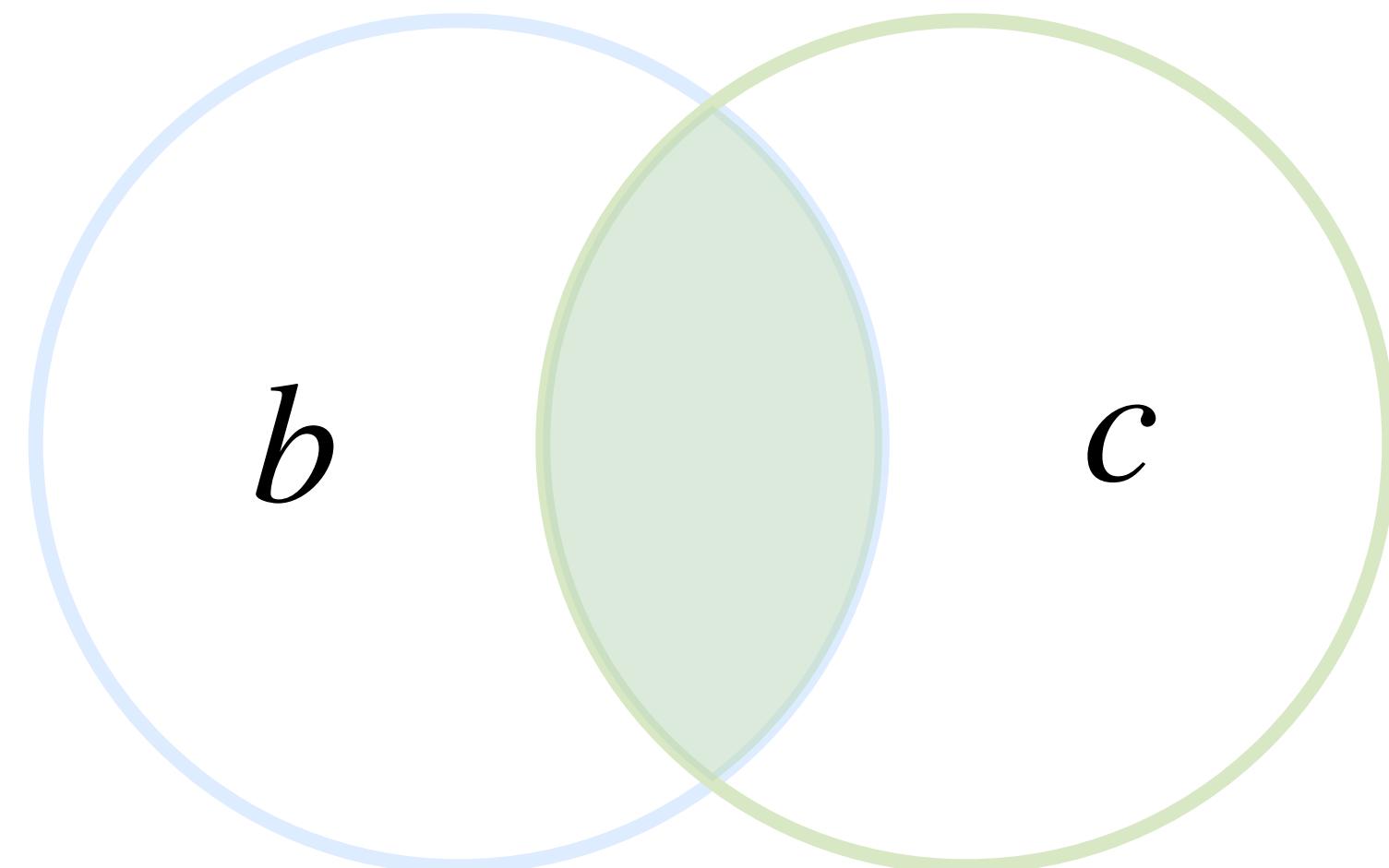
$$a_i = b_i c_i$$

Multiplication requires intersection

$$b \cap c$$

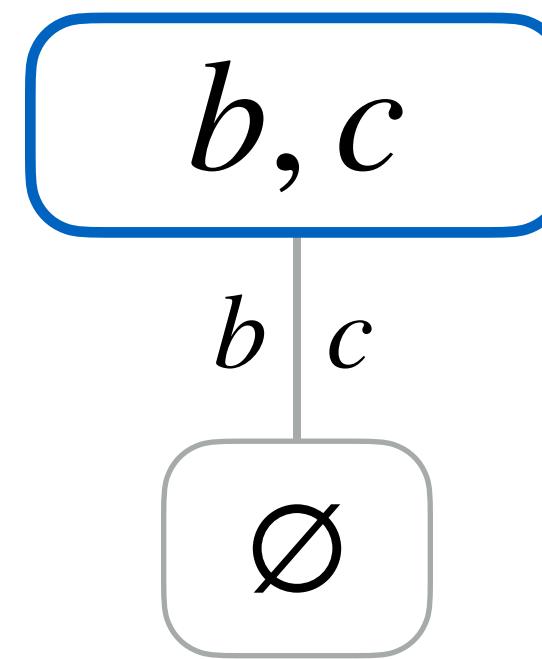
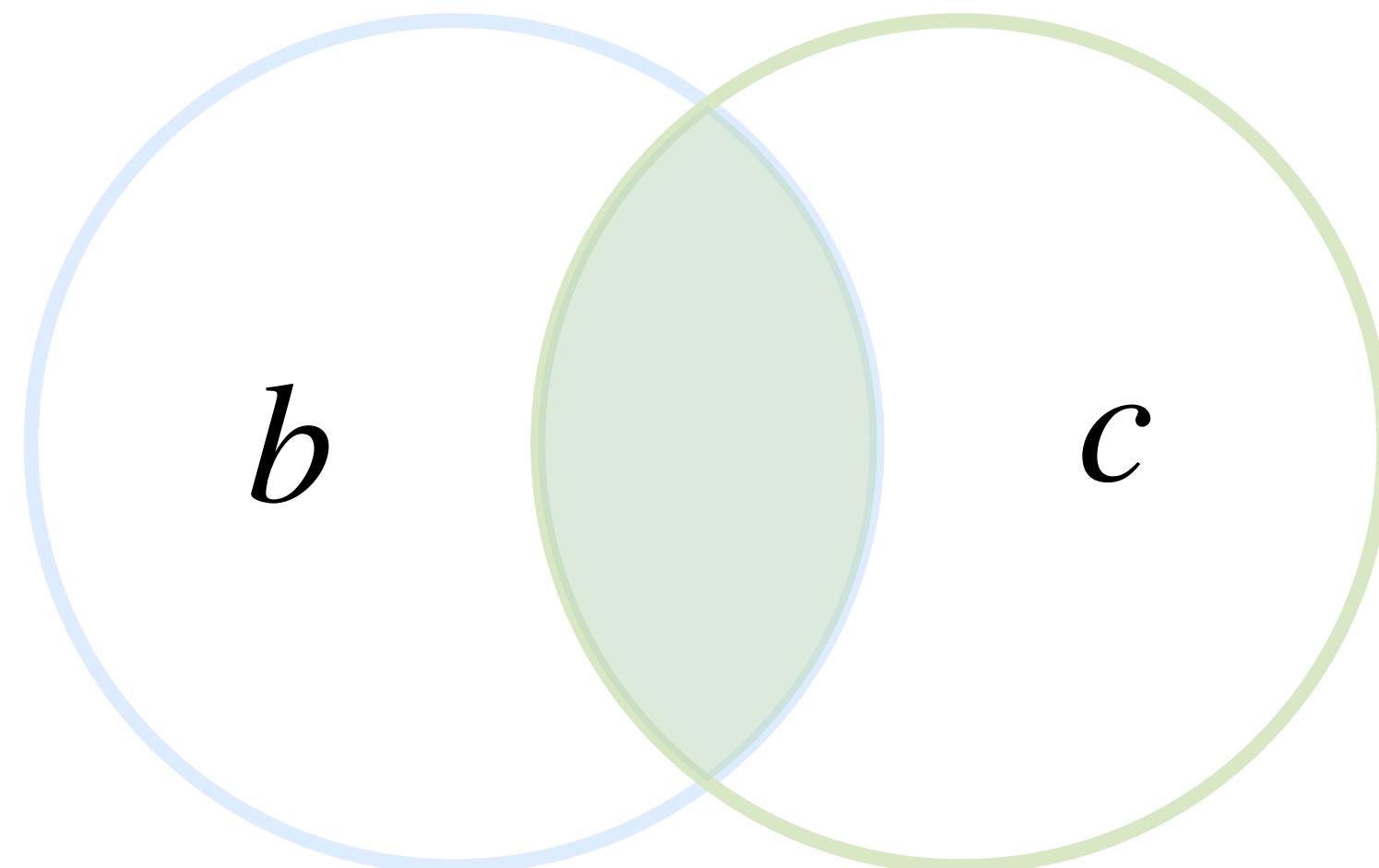
# Iteration lattice for multiplications

$$a_i = b_i c_i$$



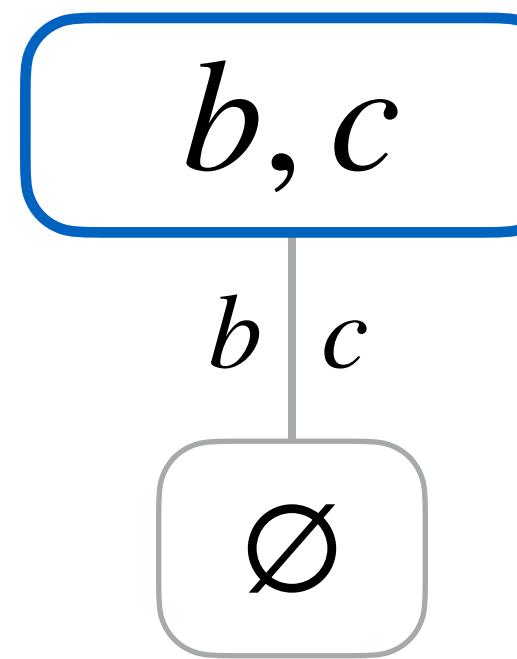
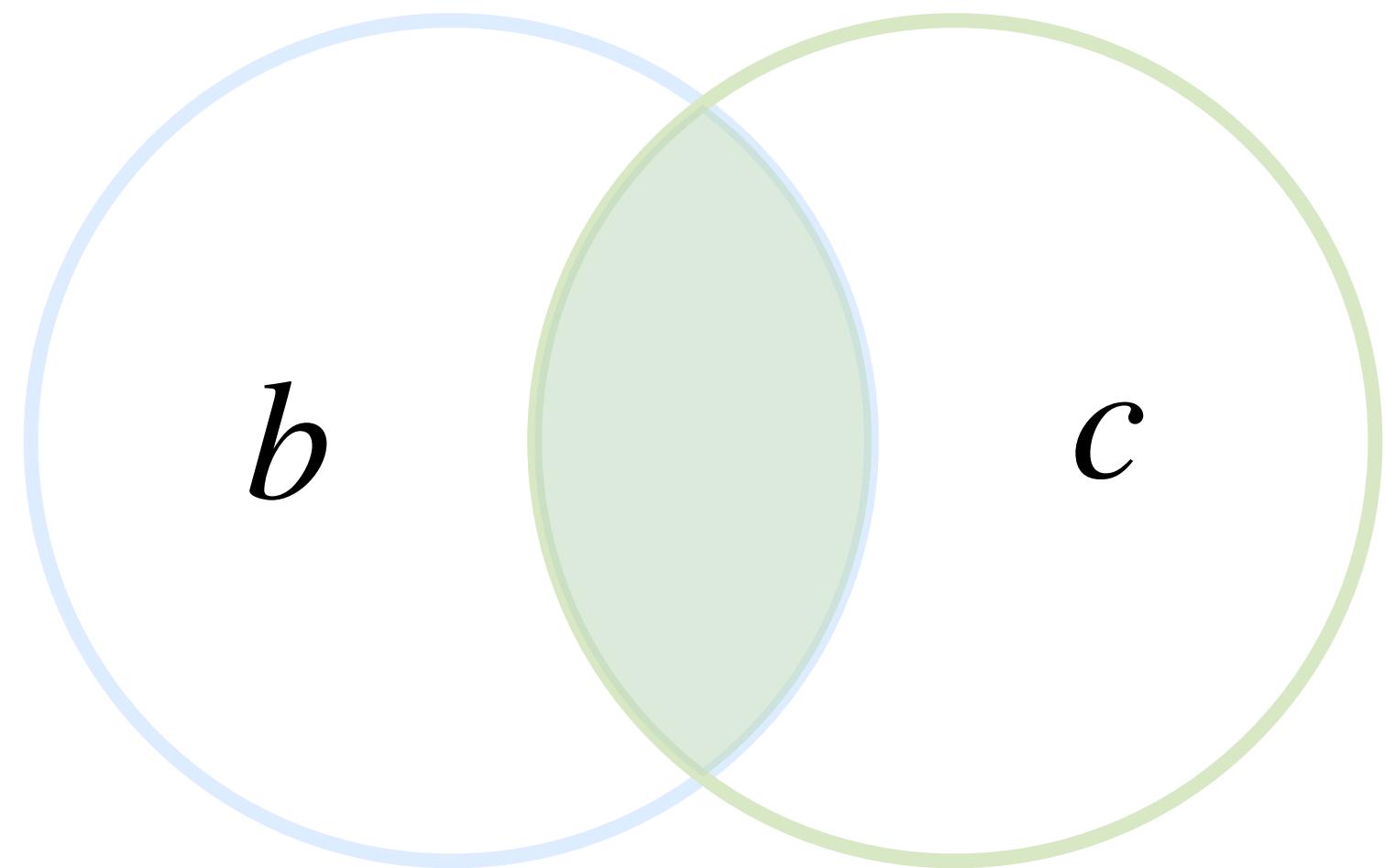
# Iteration lattice for multiplications

$$a_i = b_i c_i$$



# Iteration lattice for multiplications

$$a_i = b_i c_i$$

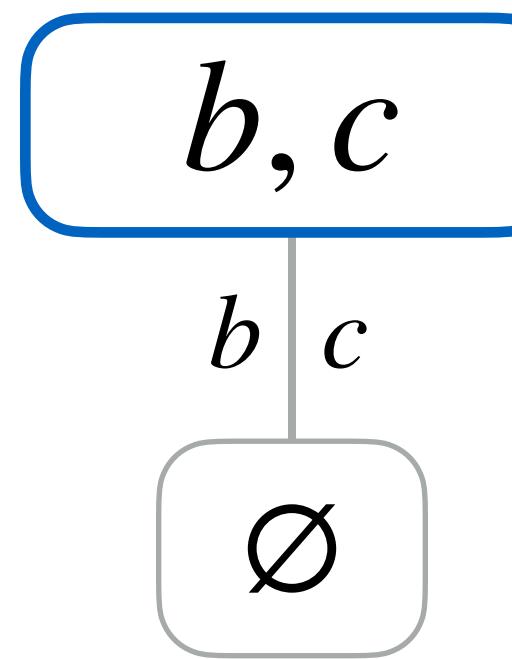
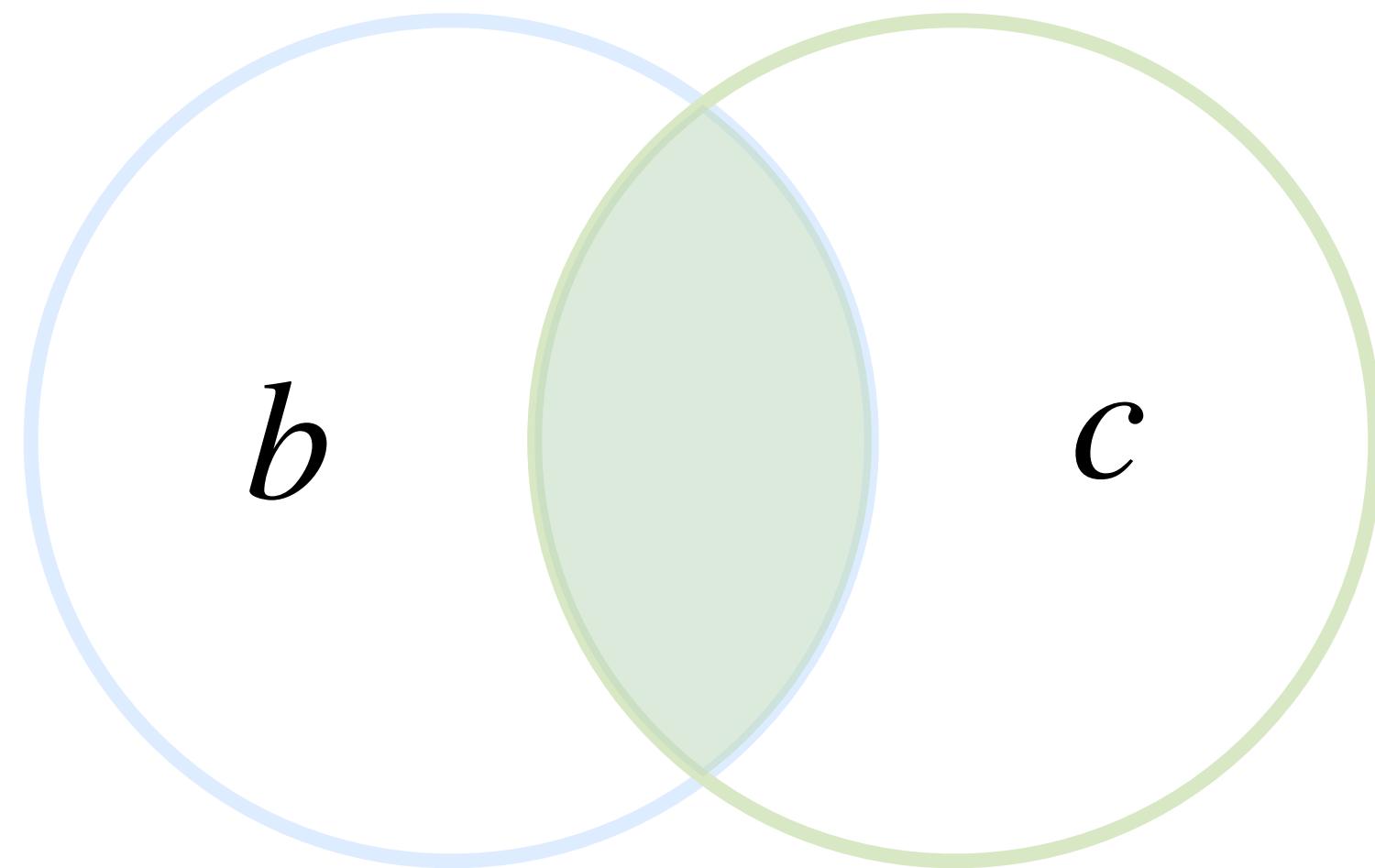


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);

    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for multiplications

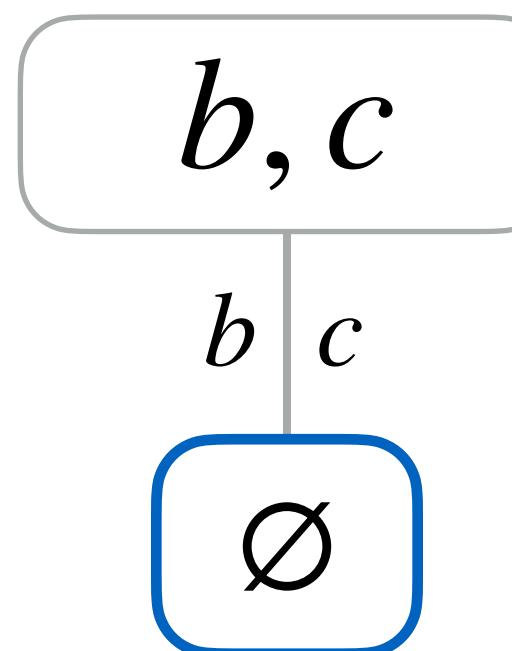
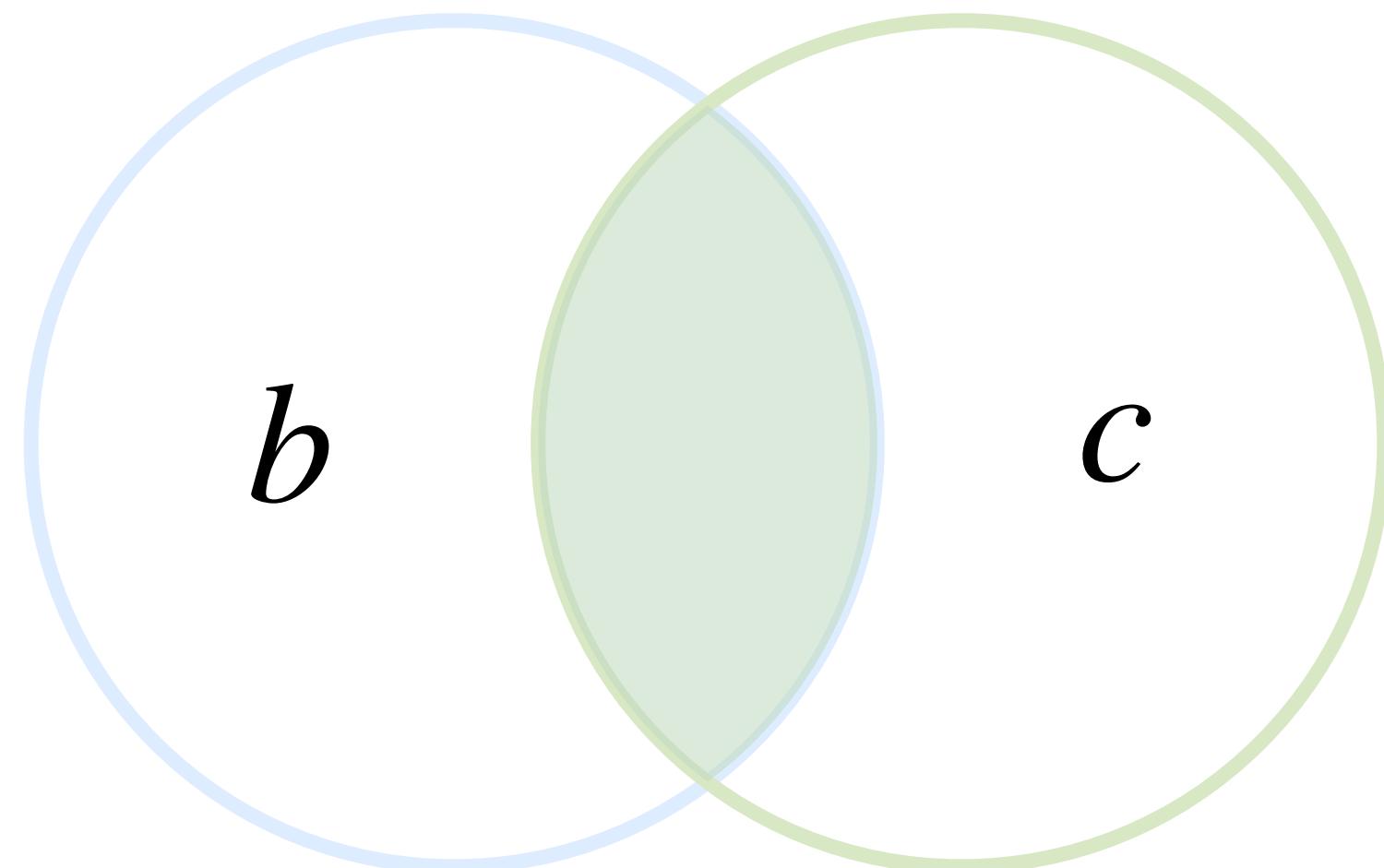
$$a_i = b_i c_i$$



```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] * c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for multiplications

$$a_i = b_i c_i$$

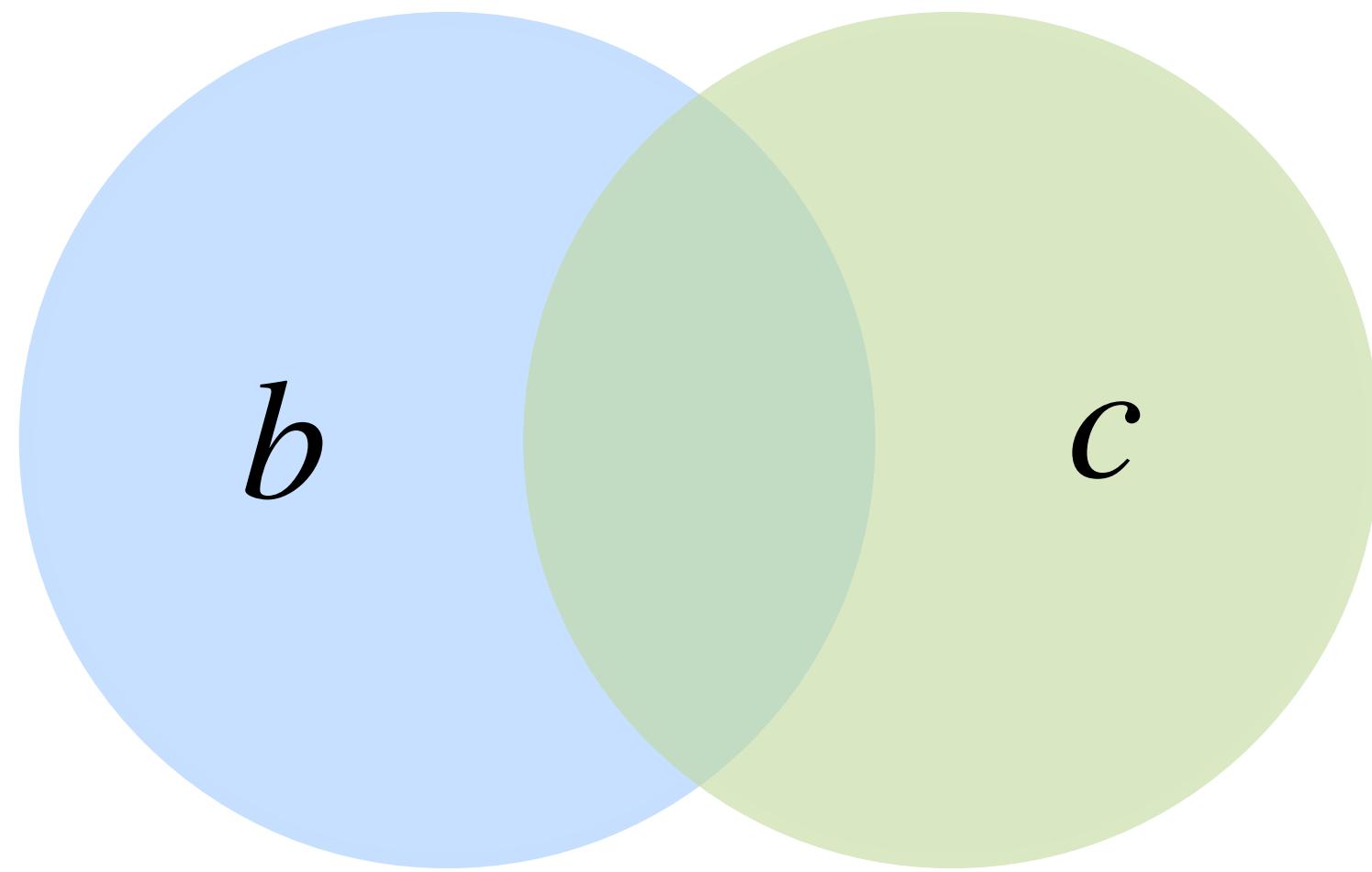


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] * c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

$$a_i = b_i + c_i$$

# Iteration lattice for additions

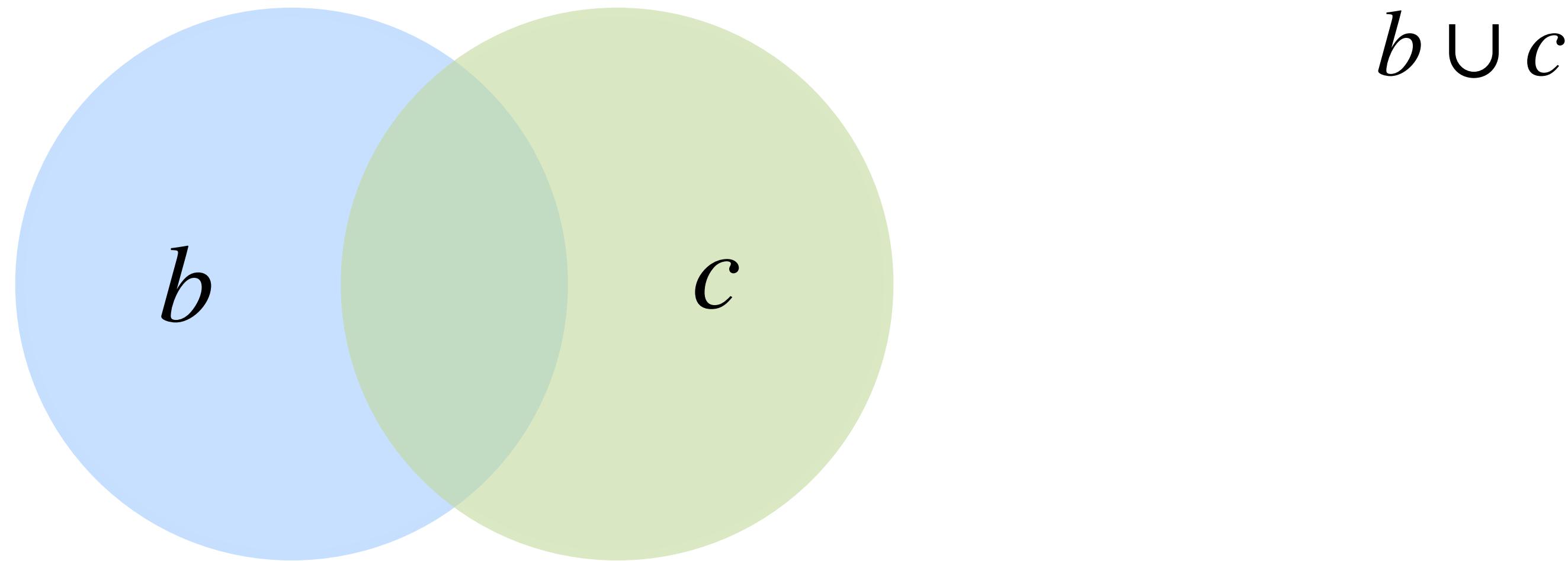


$$a_i = b_i + c_i$$

Addition requires union

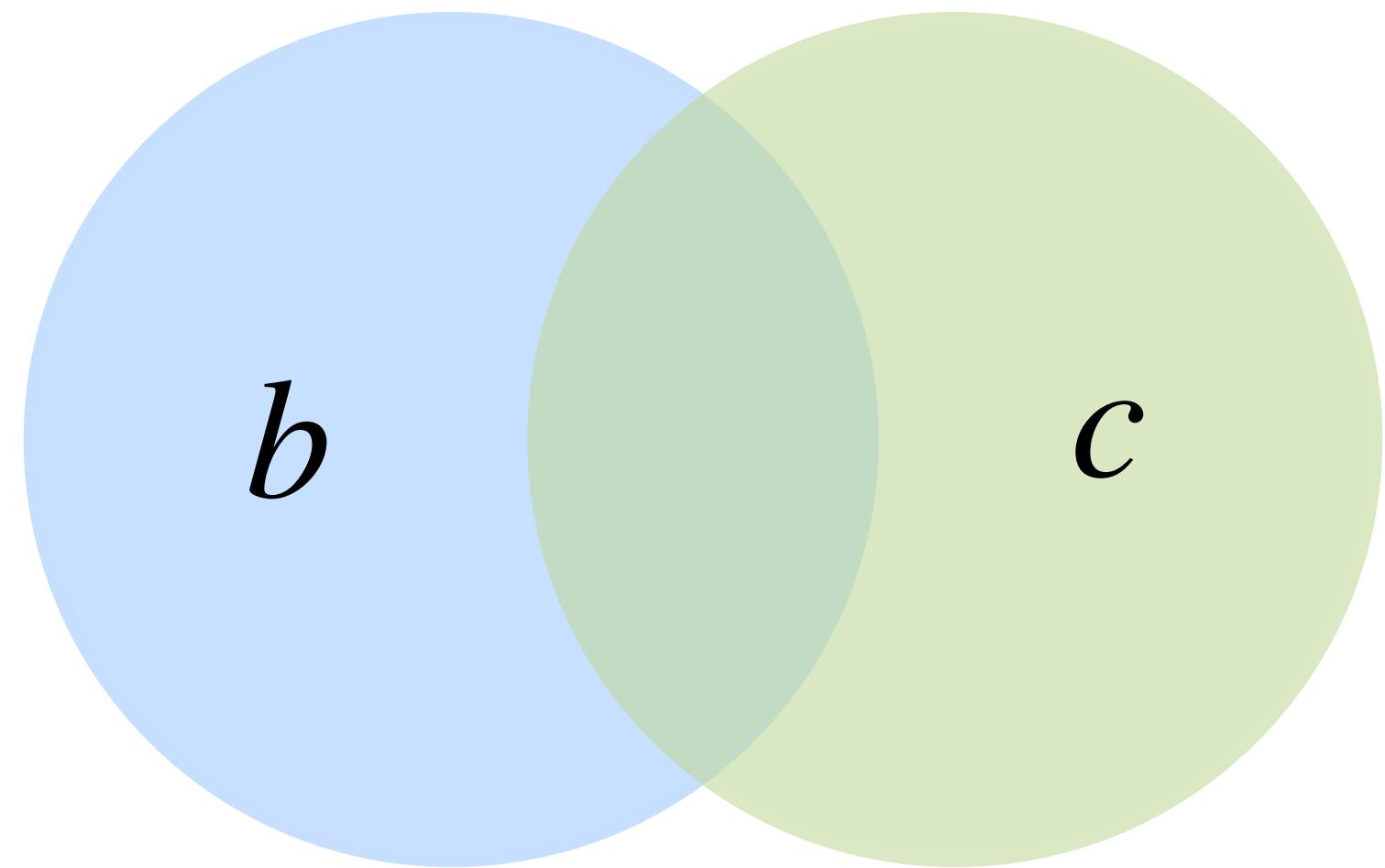
# Iteration lattice for additions

$$a_i = b_i + c_i$$



# Iteration lattice for additions

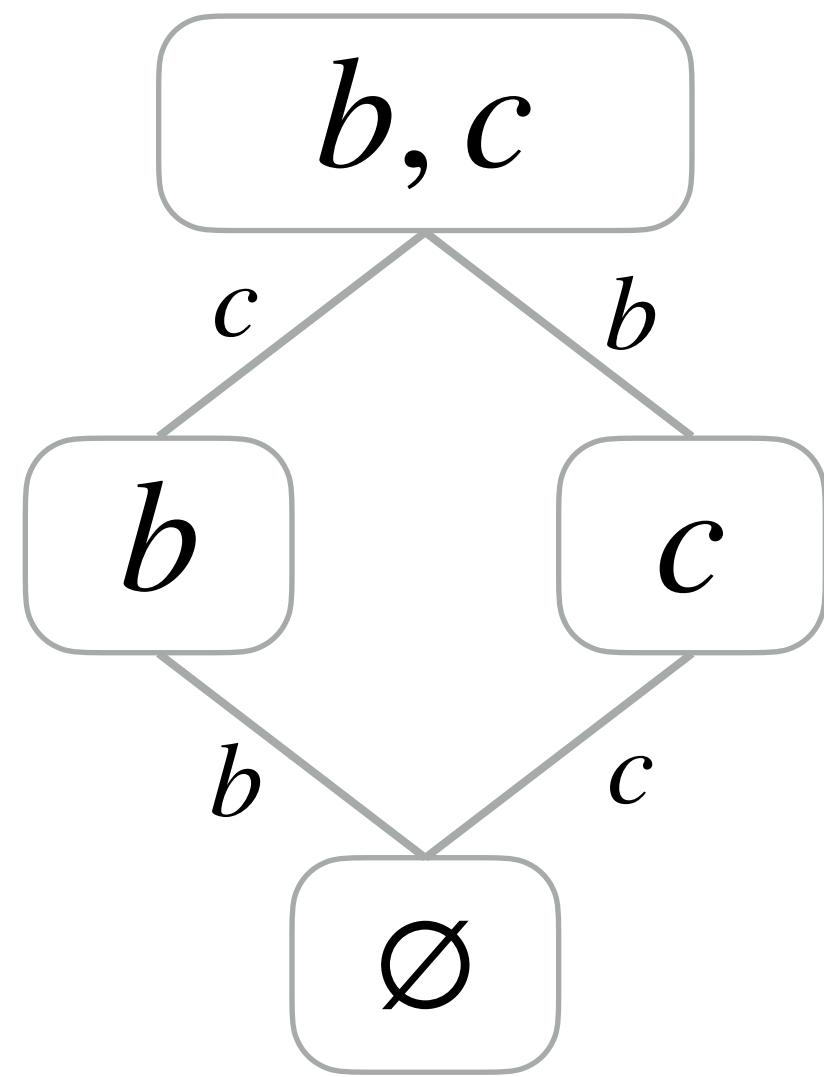
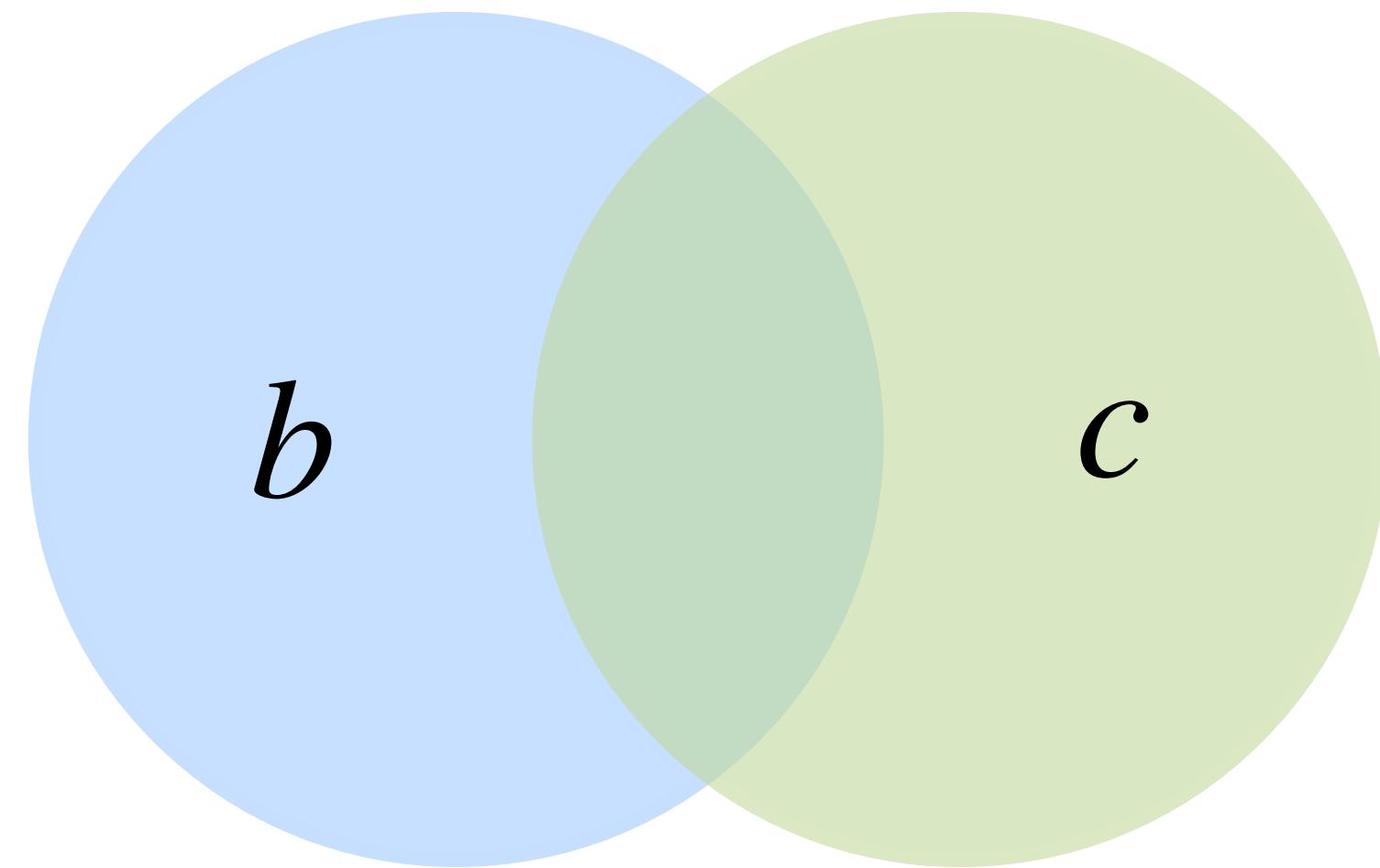
$$a_i = b_i + c_i$$



$$b \cup c = (b \cap c) \cup b \cup c$$

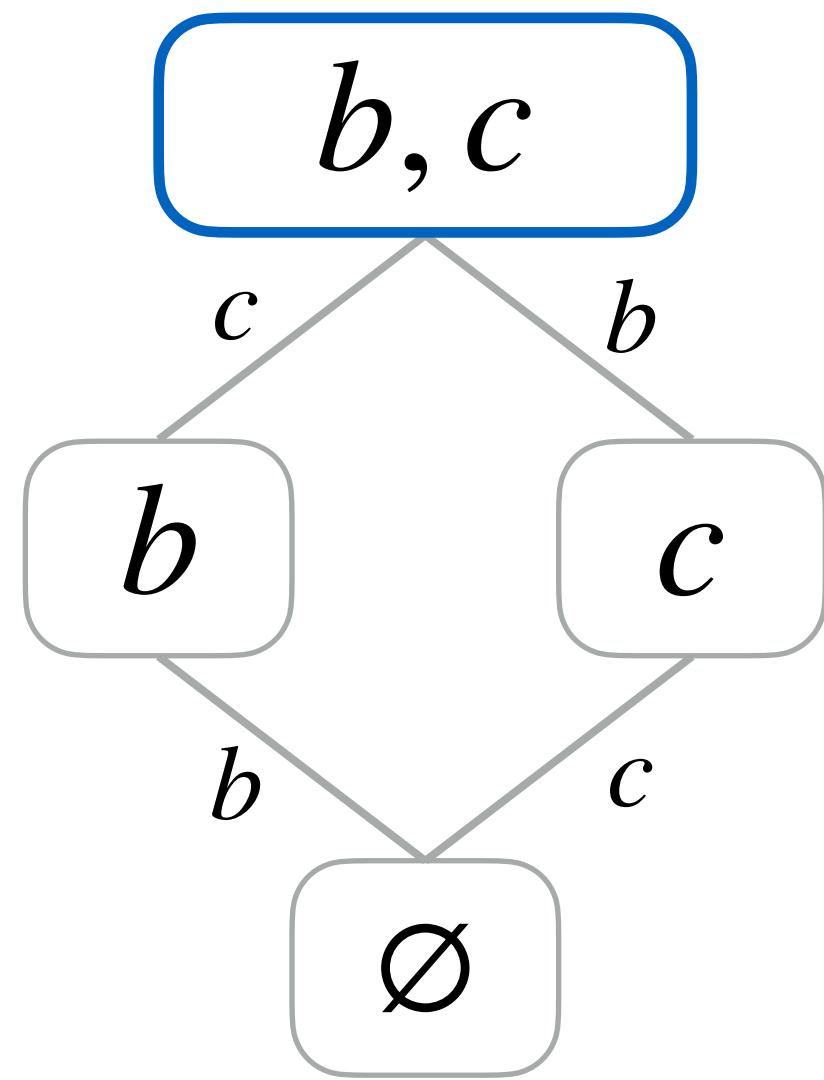
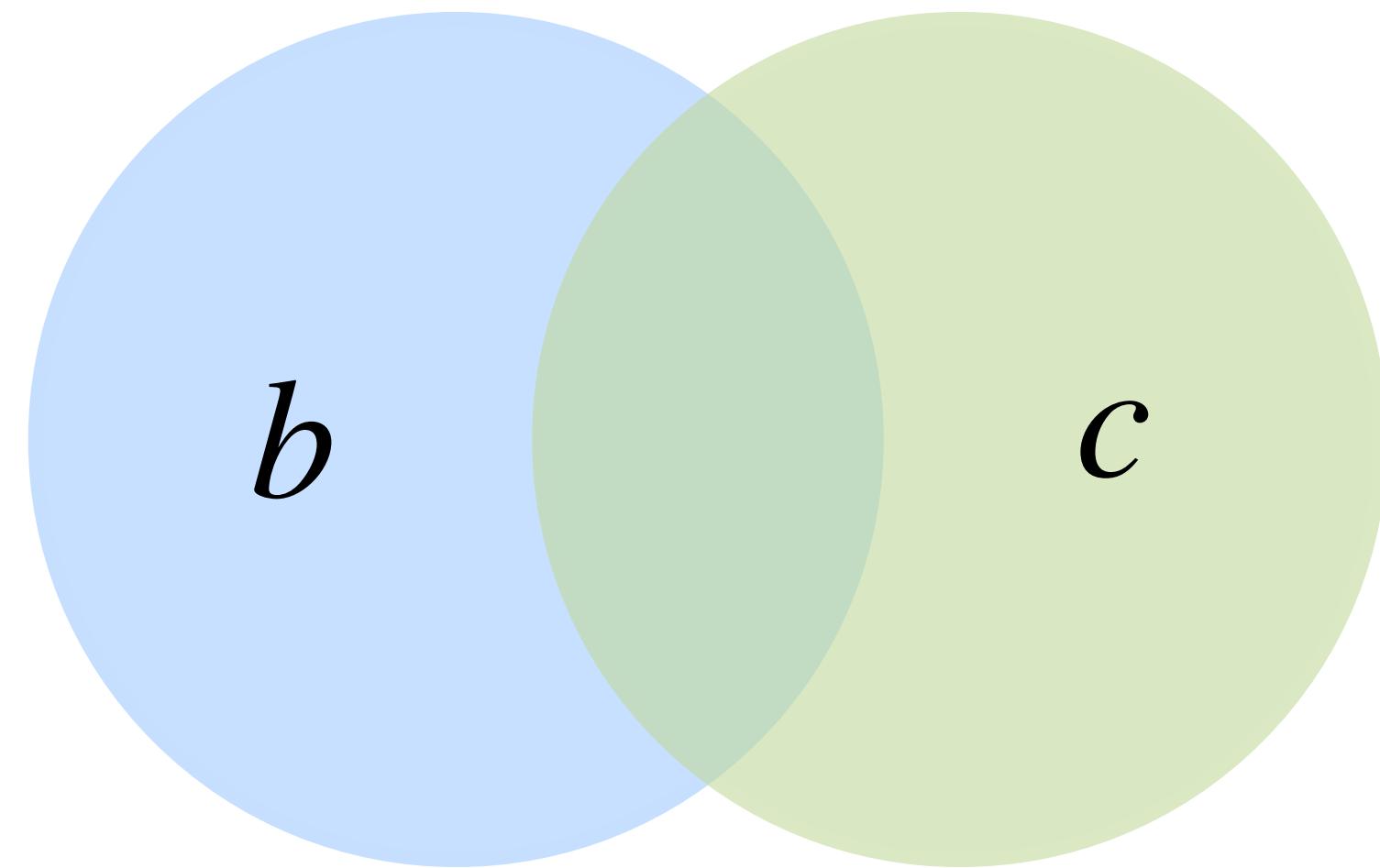
# Iteration lattice for additions

$$a_i = b_i + c_i$$



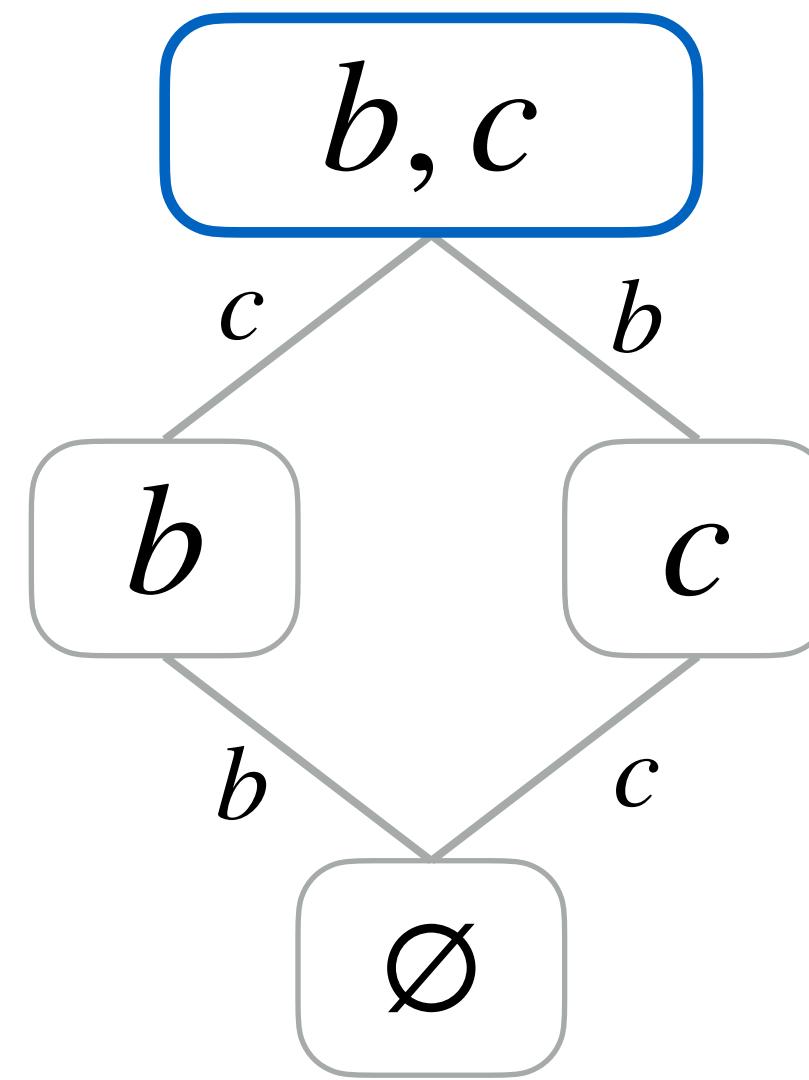
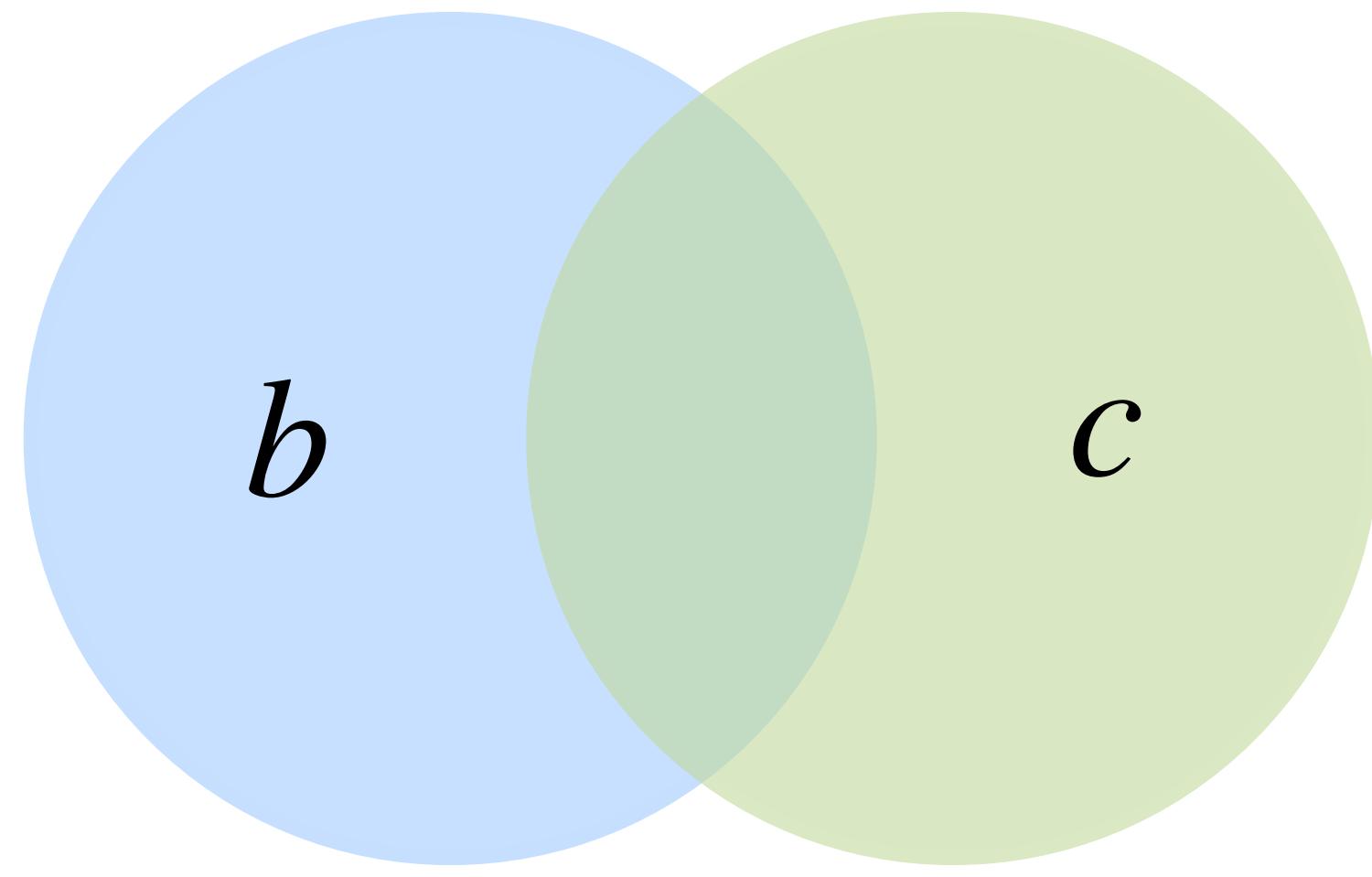
# Iteration lattice for additions

$$a_i = b_i + c_i$$



# Iteration lattice for additions

$$a_i = b_i + c_i$$

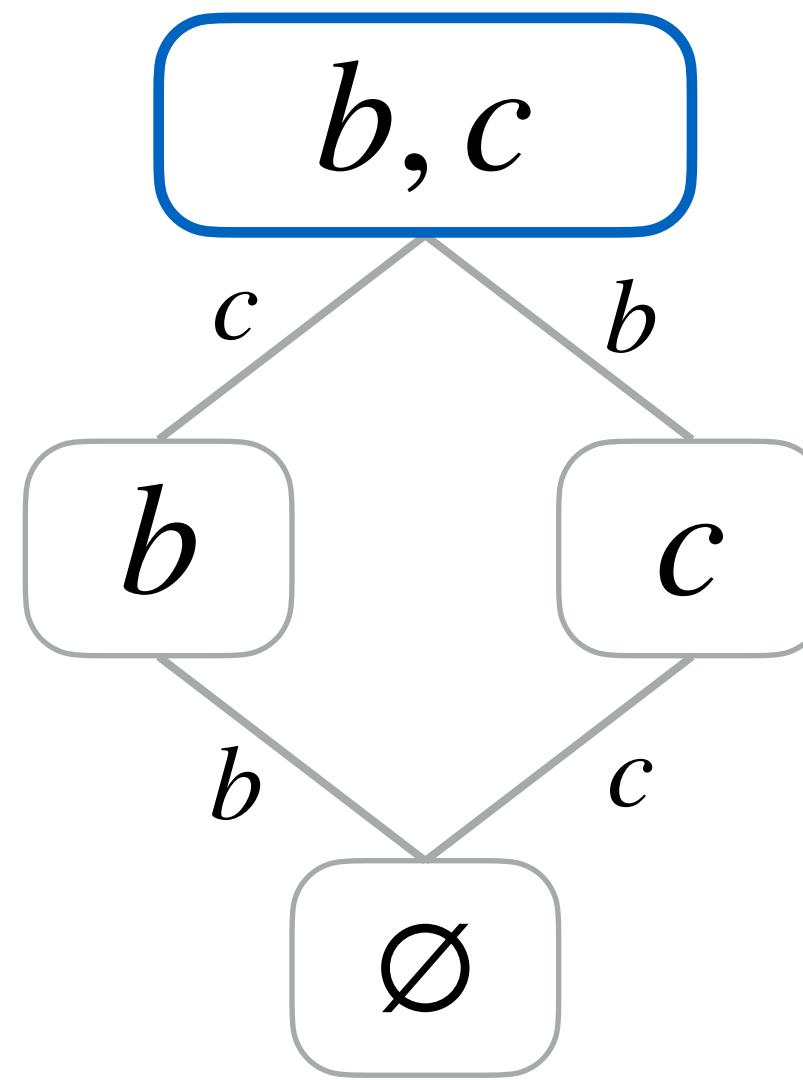
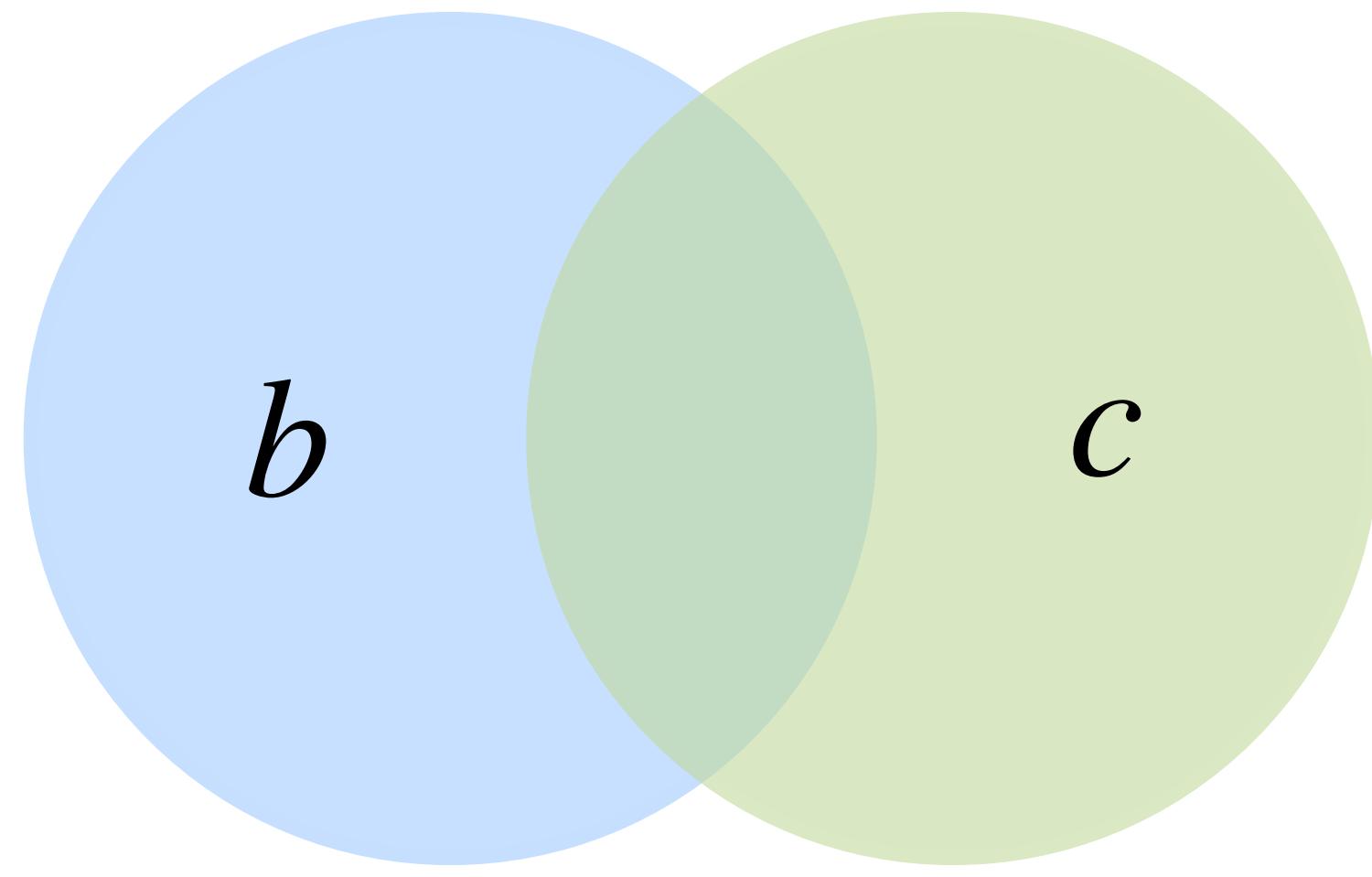


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);

    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

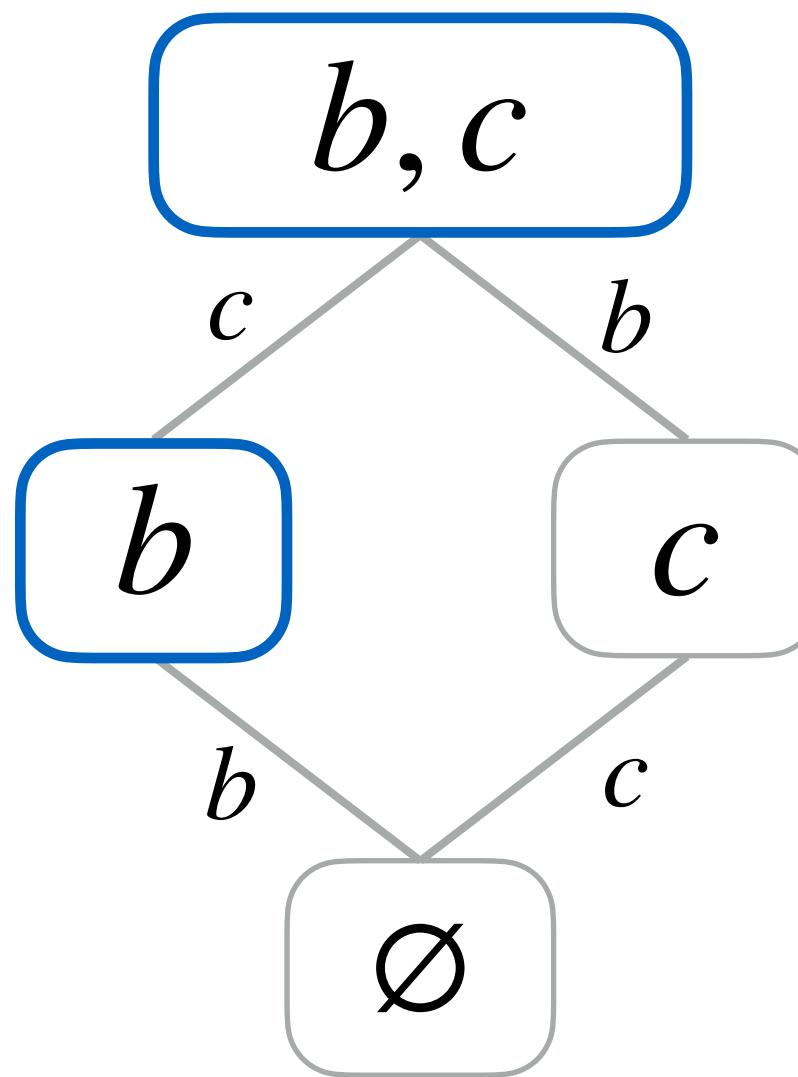
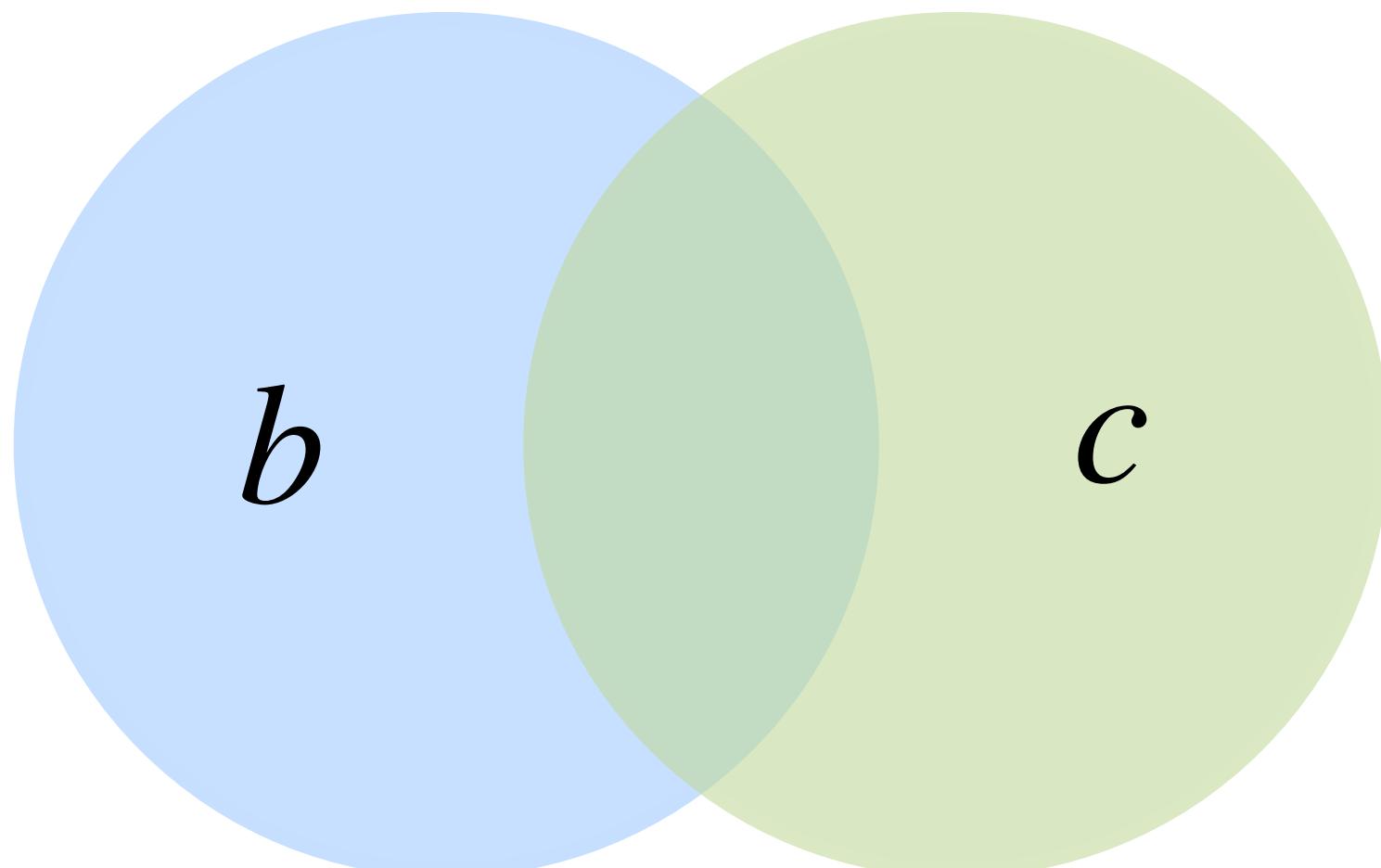
$$a_i = b_i + c_i$$



```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

$$a_i = b_i + c_i$$

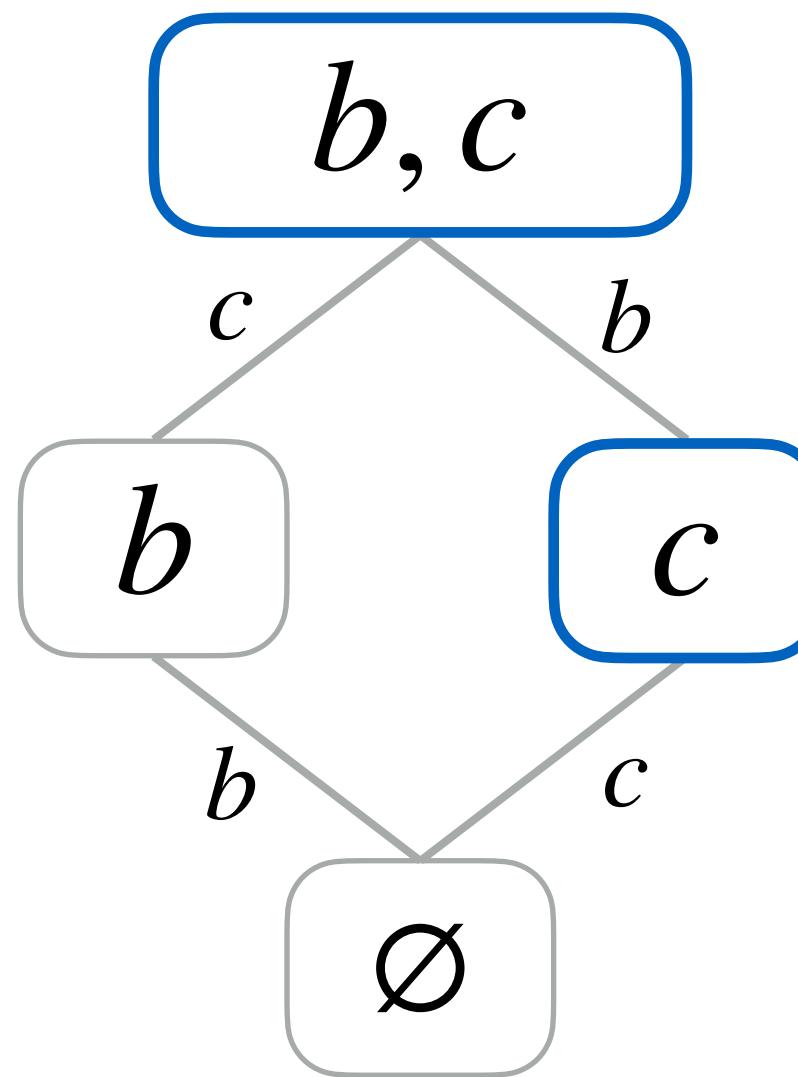
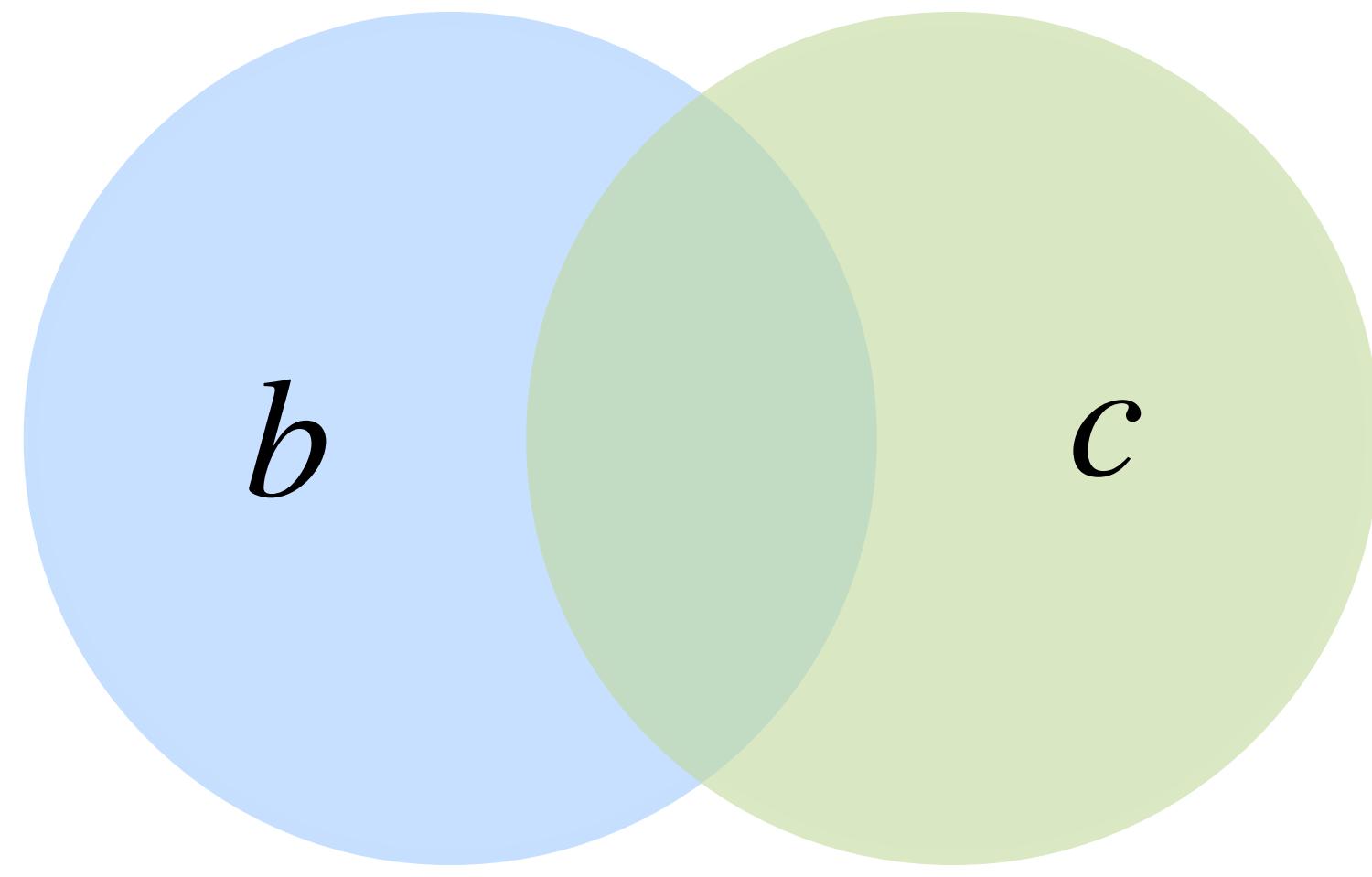


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    } else if (ib == i) {
        a[i] = b[pb1];
    }

    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

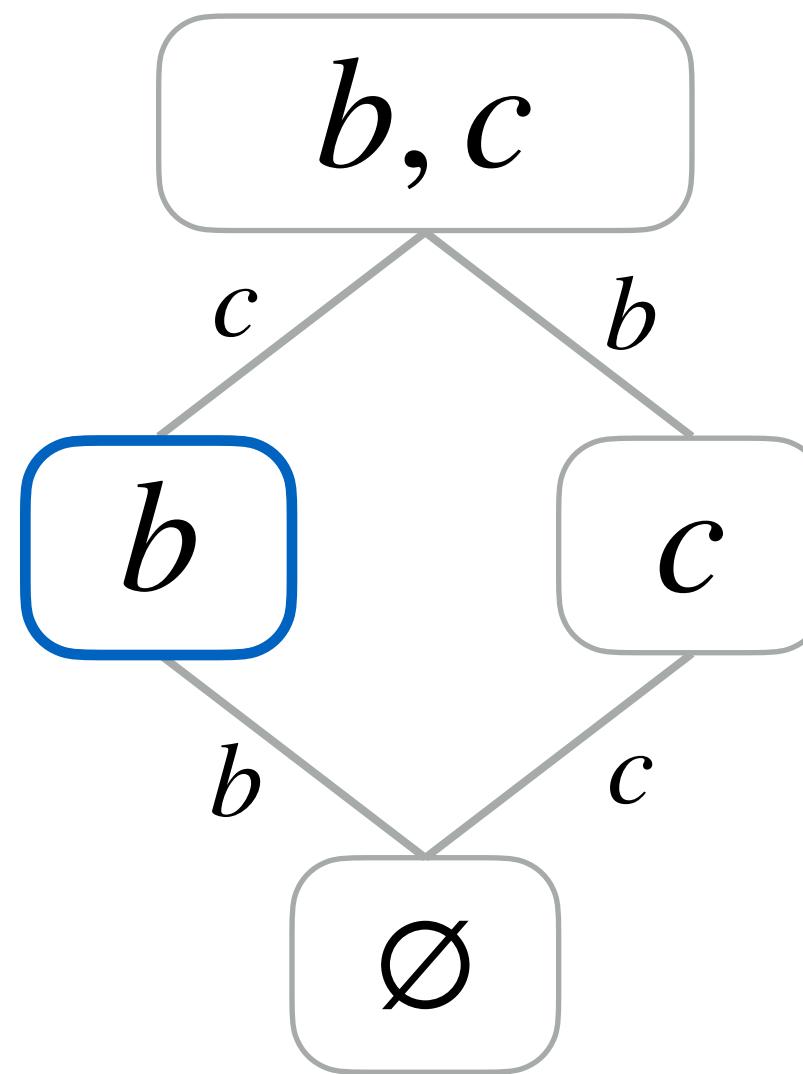
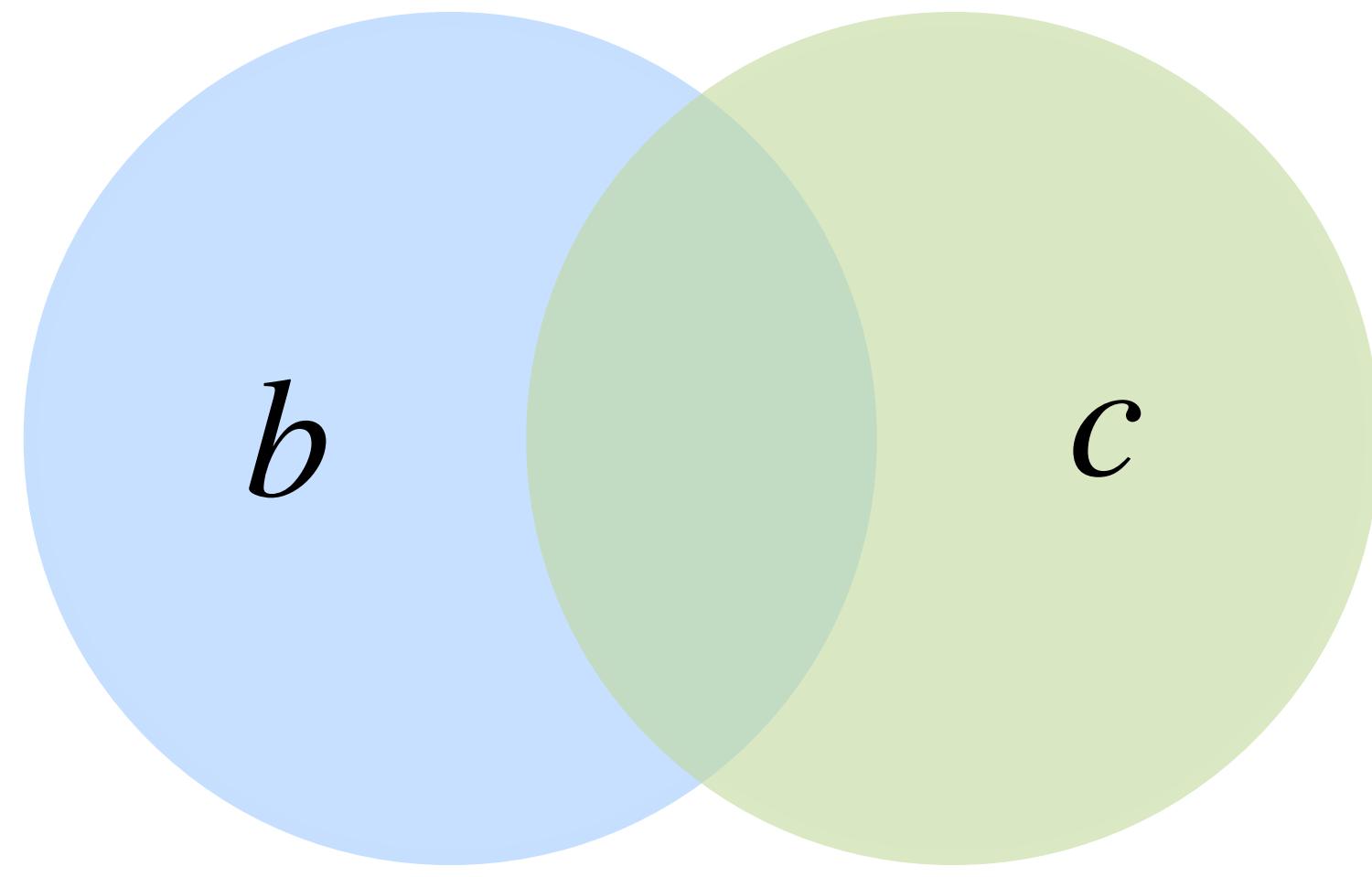
$$a_i = b_i + c_i$$



```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    }
    else if (ib == i) {
        a[i] = b[pb1];
    }
    else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

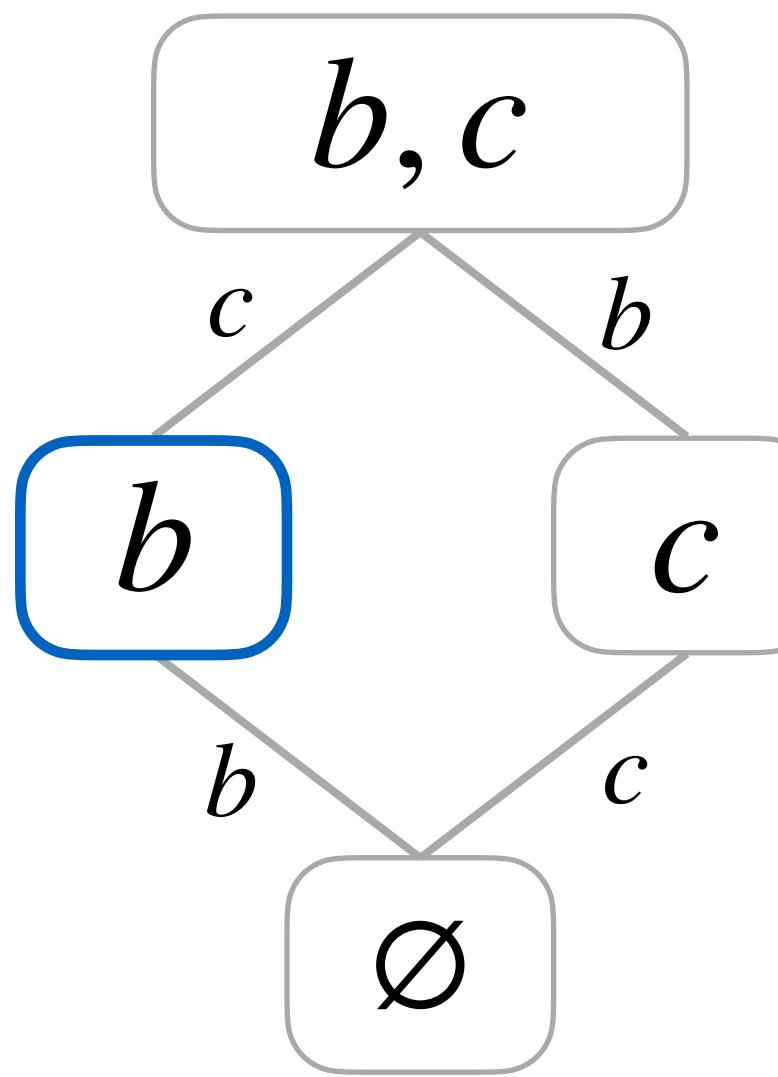
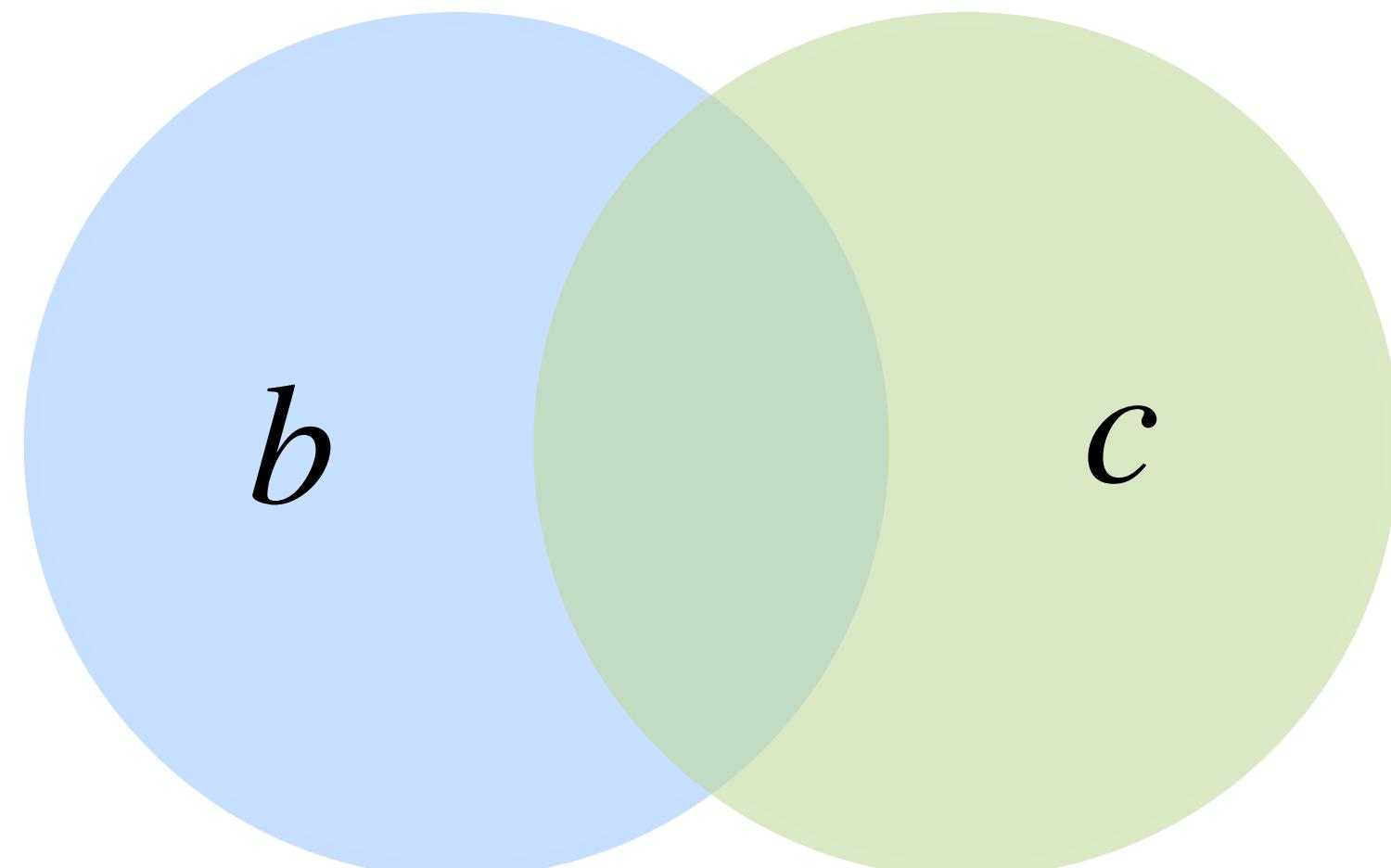
$$a_i = b_i + c_i$$



```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    }
    else if (ib == i) {
        a[i] = b[pb1];
    }
    else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}
```

# Iteration lattice for additions

$$a_i = b_i + c_i$$

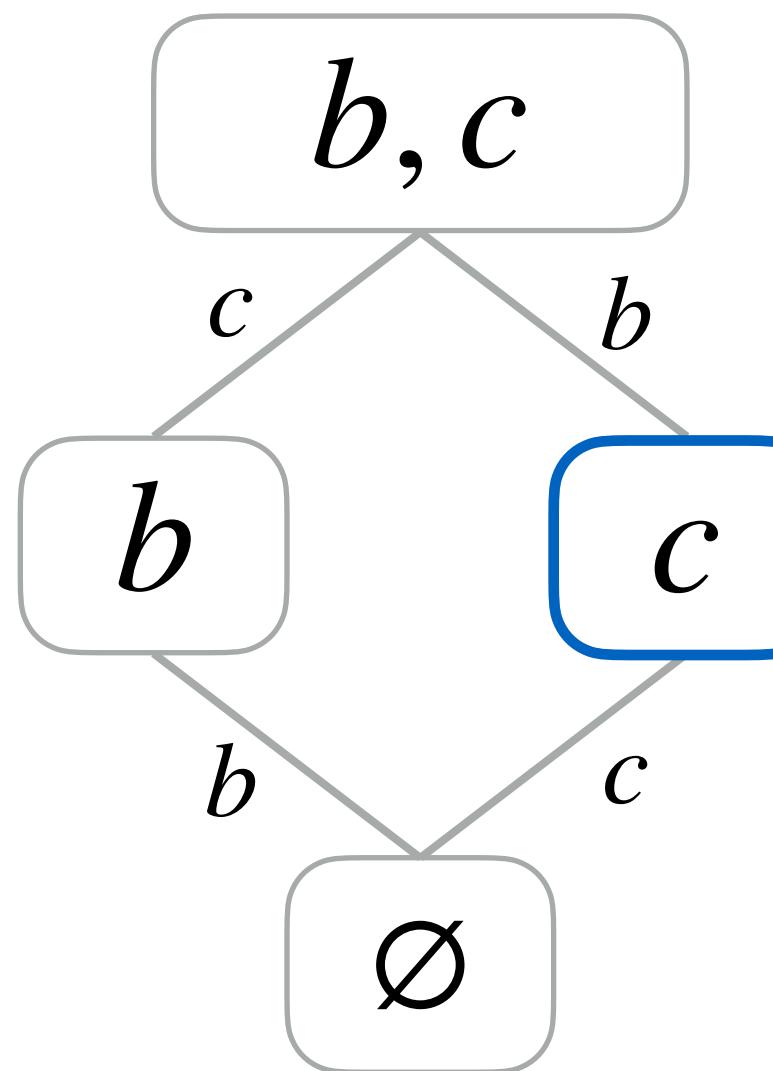
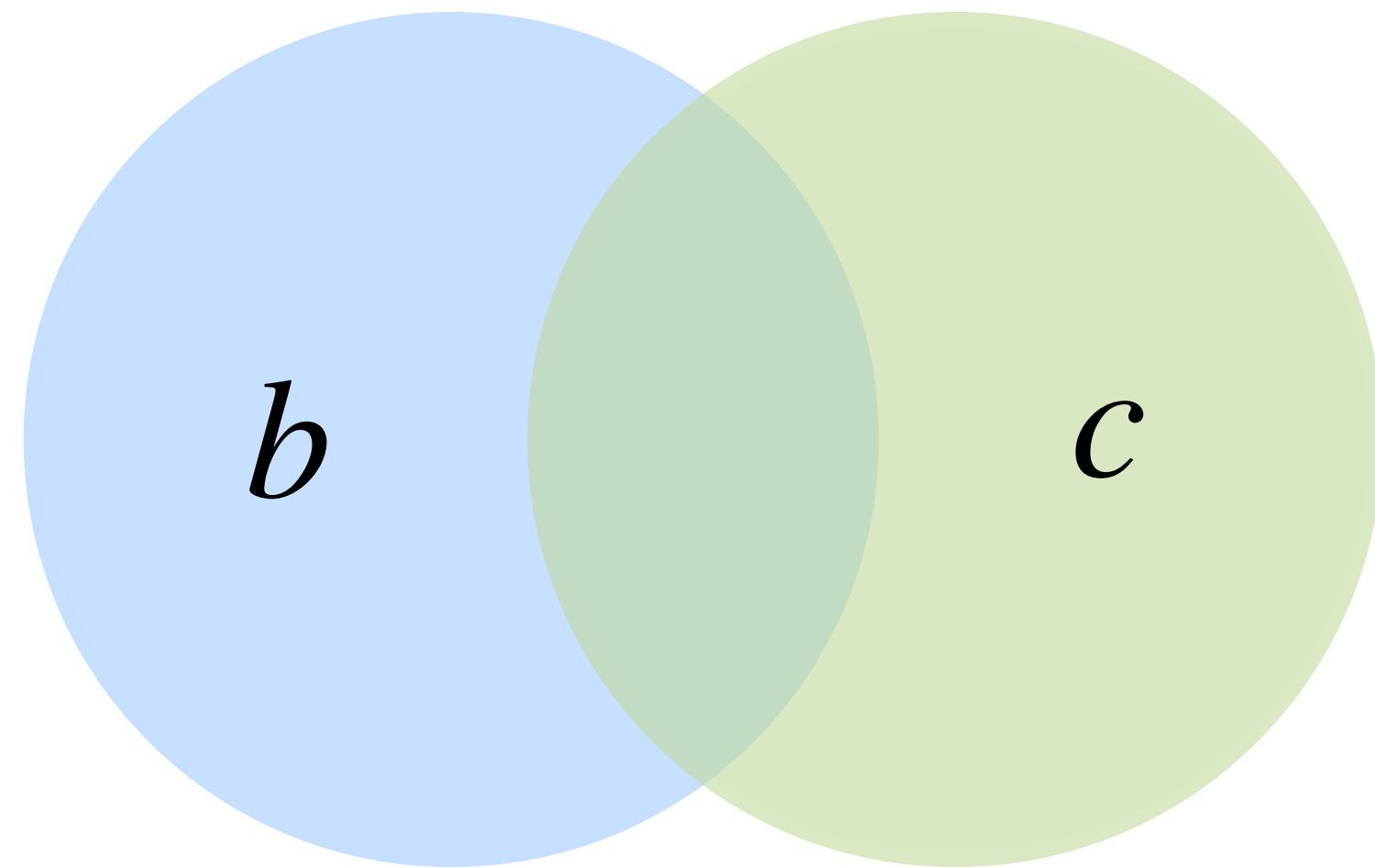


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    }
    else if (ib == i) {
        a[i] = b[pb1];
    }
    else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}

while (pb1 < b1_pos[1]) {
    int i = b1_crd[pb1];
    a[i] = b[pb1++];
}
```

# Iteration lattice for additions

$$a_i = b_i + c_i$$



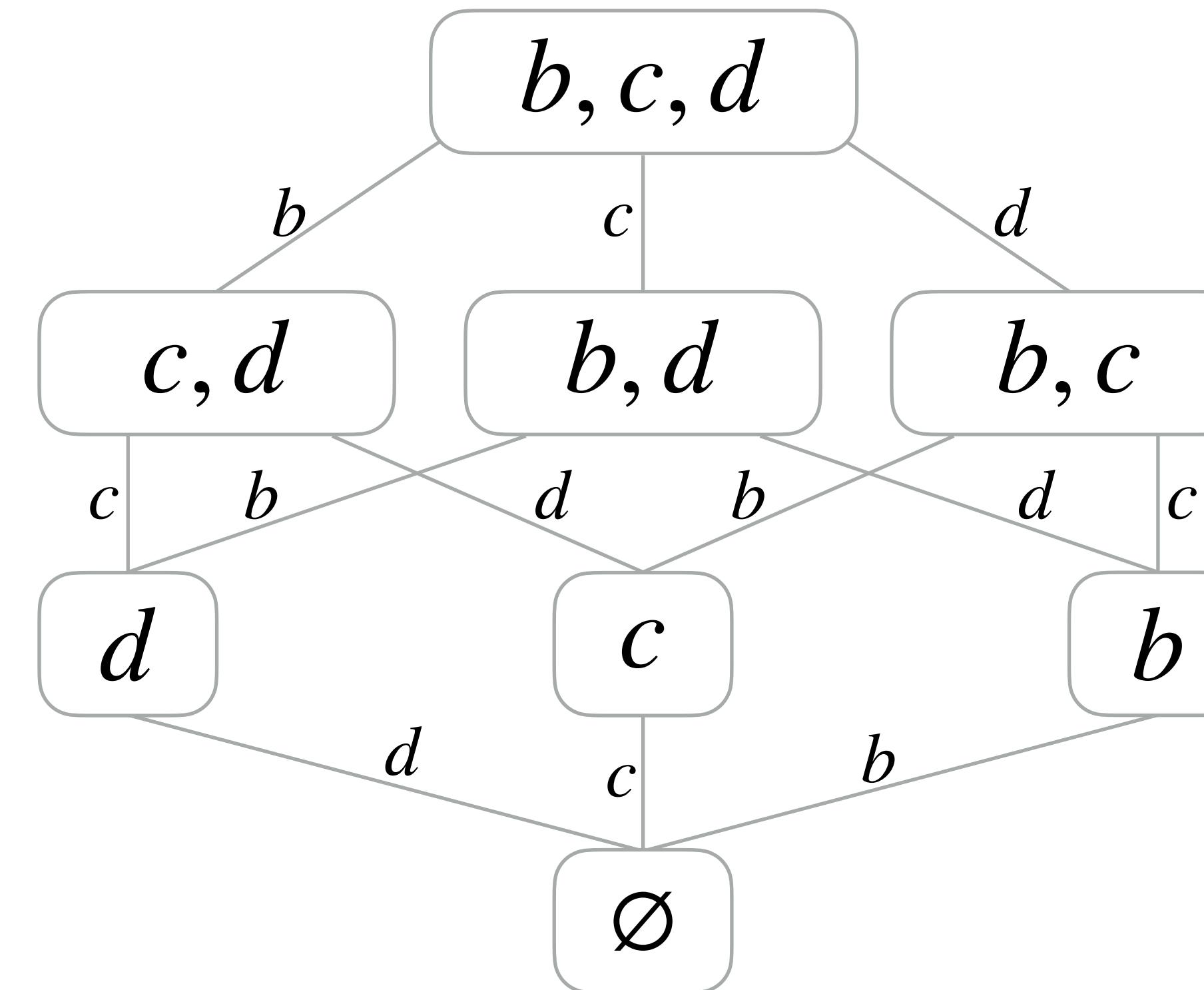
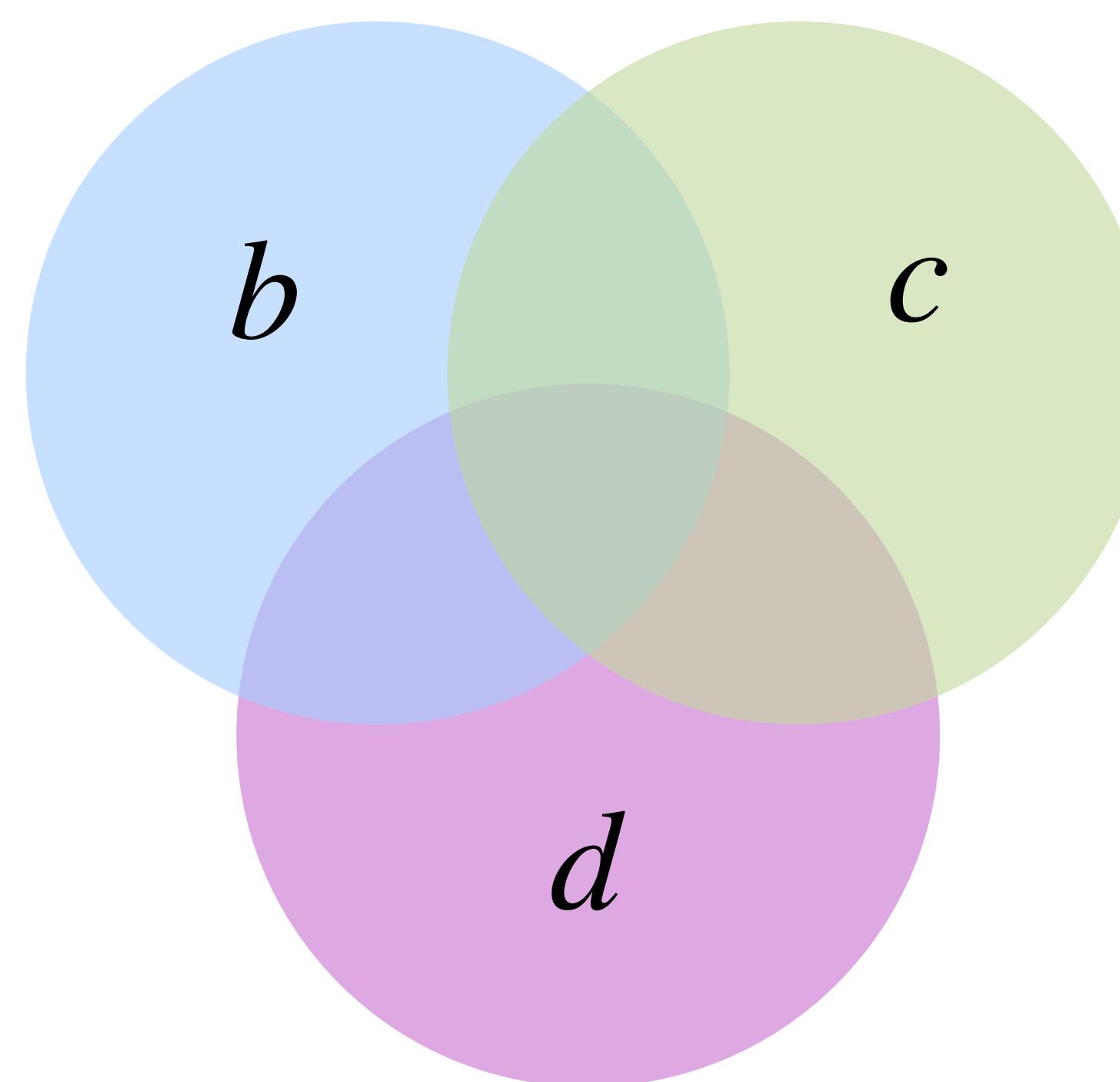
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    }
    else if (ib == i) {
        a[i] = b[pb1];
    }
    else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}

while (pb1 < b1_pos[1]) {
    int i = b1_crd[pb1];
    a[i] = b[pb1++];
}

while (pc1 < c1_pos[1]) {
    int i = c1_crd[pc1];
    a[i] = c[pc1++];
}
```

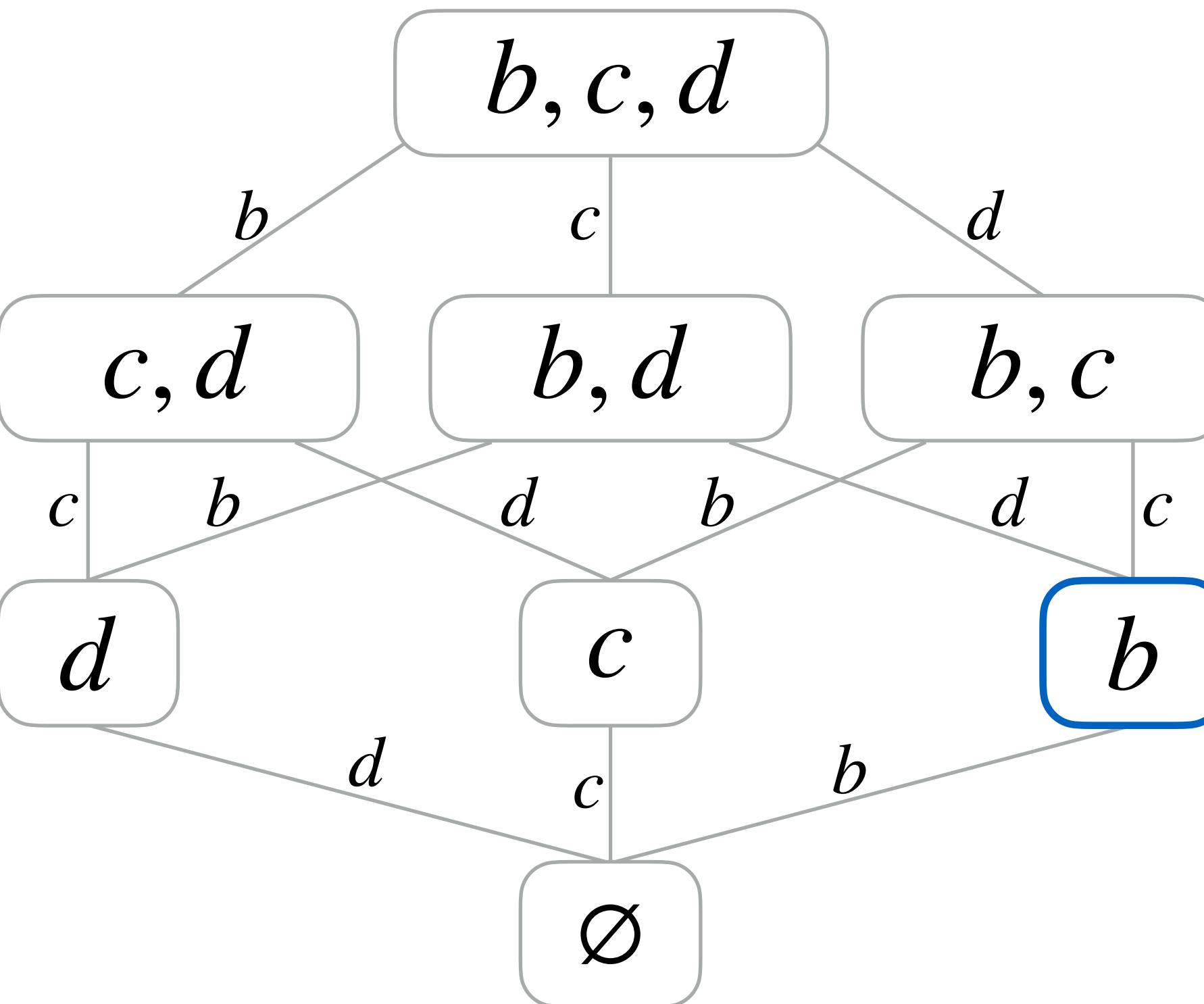
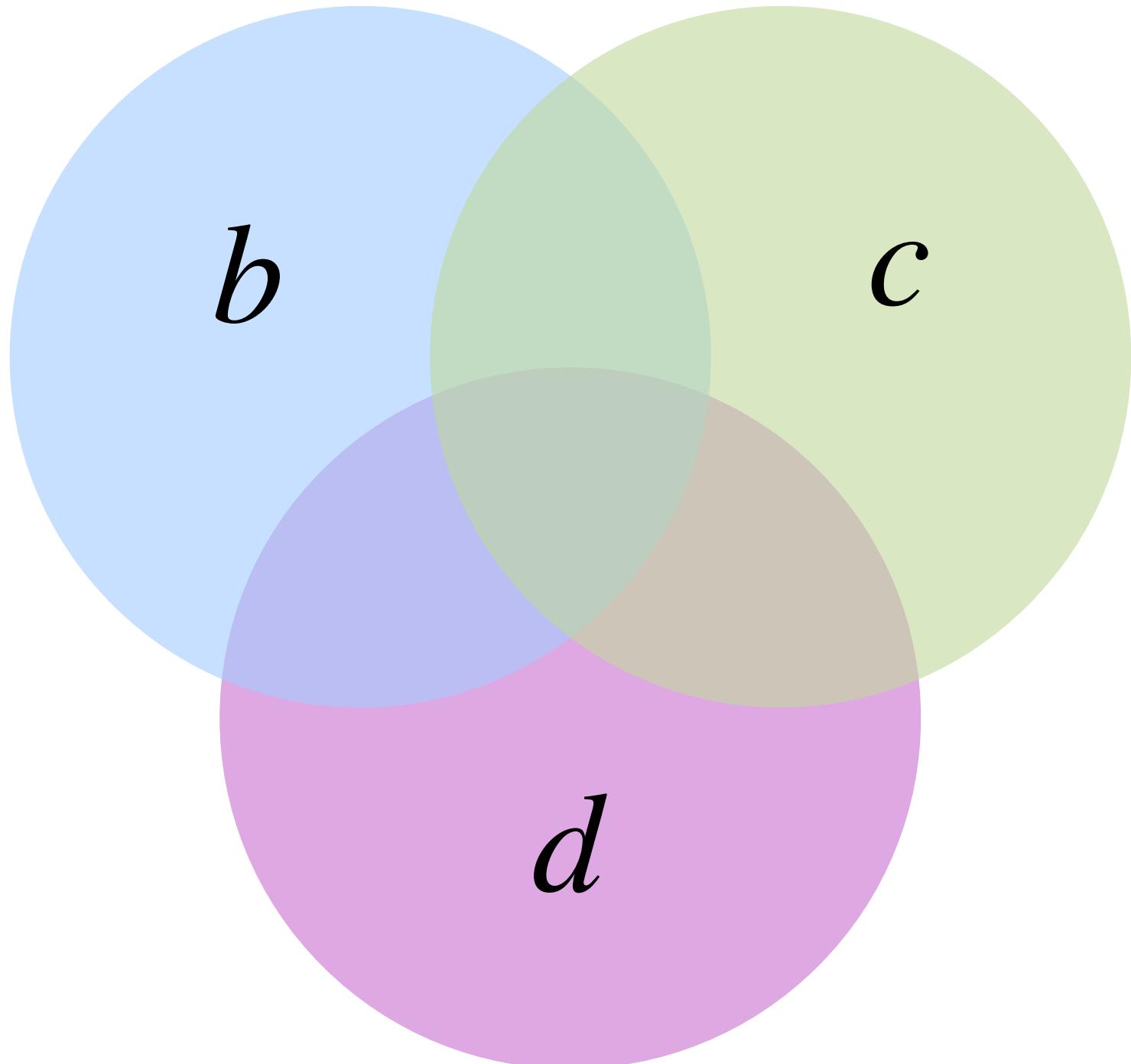
# Iteration lattice for a compound expression

$$a_i = b_i + c_i + d_i$$



# Iteration lattice for a compound expression

$$a_i = b_i + c_i + d_i$$



```

int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] + d[pd1];
    } else if (ic == i) {
        a[i] = c[pc1];
    } else {
        a[i] = d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] + d[pd1];
    } else if (ib == i) {
        a[i] = b[pb1];
    } else {
        a[i] = d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    } else if (ib == i) {
        a[i] = b[pb1];
    } else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}

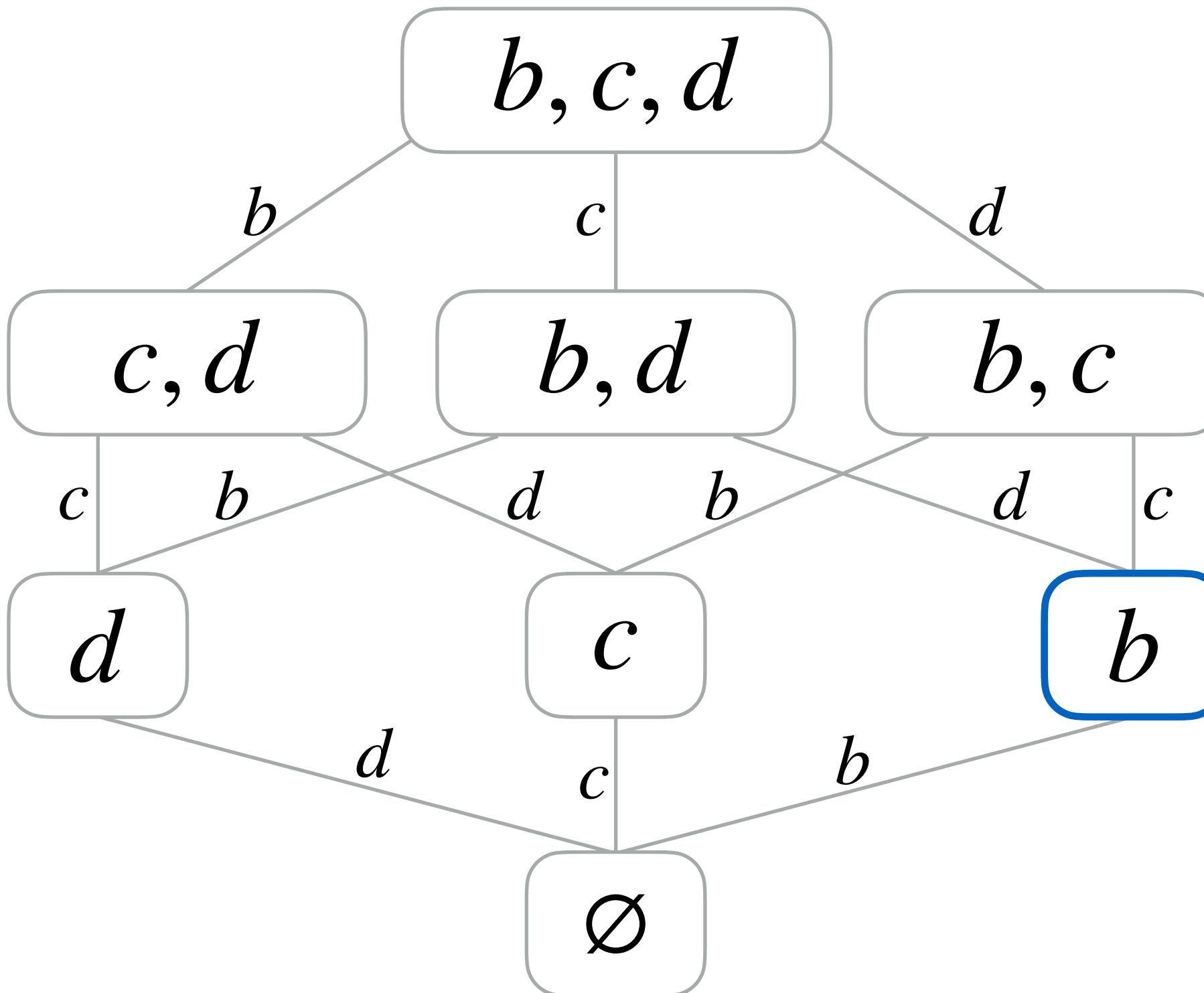
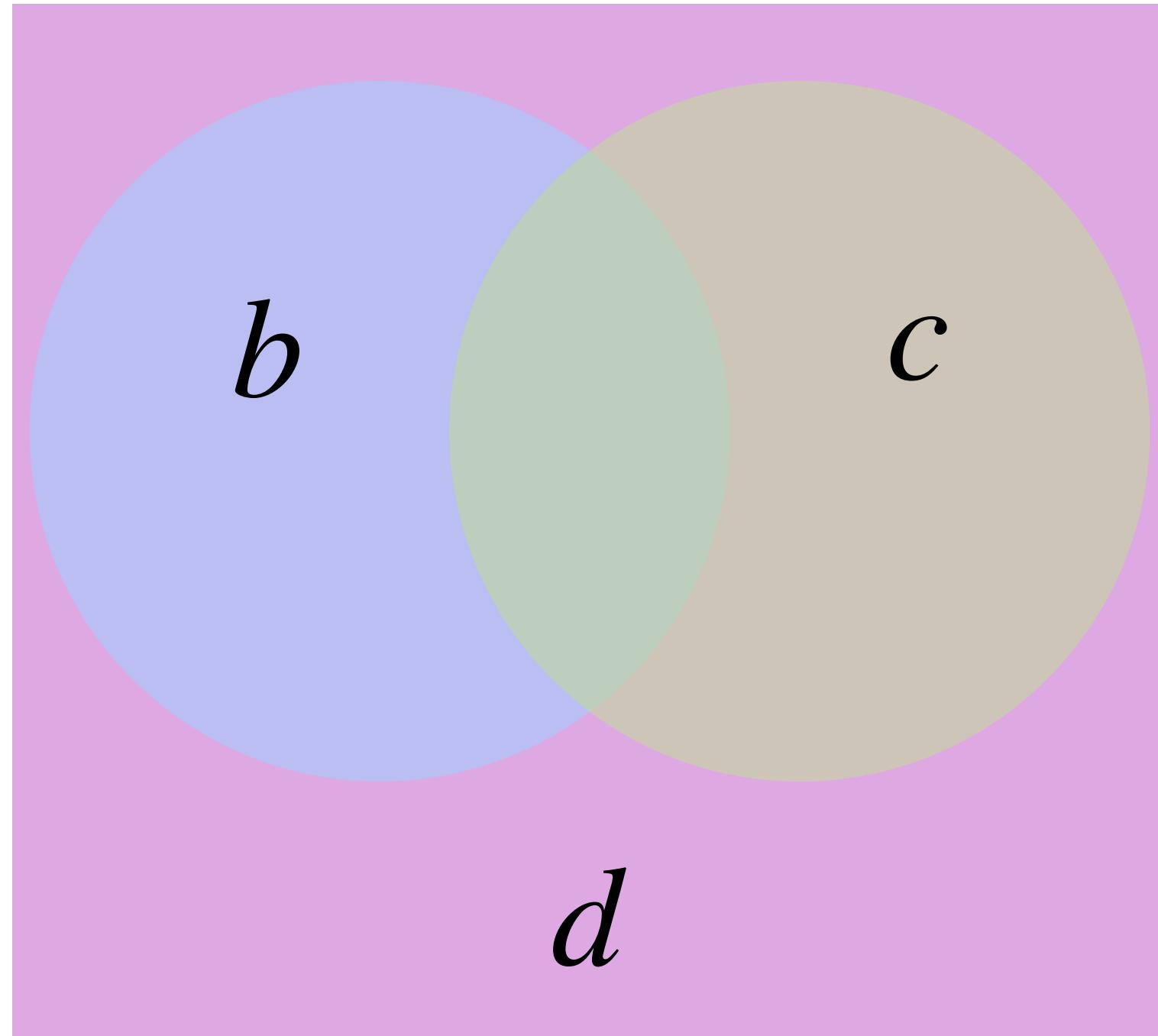
while (pd1 < d1_pos[1]) {
    int id = d1_crd[pd1];
    a[id] = d[pd1];
    pd1++;
}

while (pc1 < c1_pos[1]) {
    int ic = c1_crd[pc1];
    a[ic] = c[pc1];
    pc1++;
}

while (pb1 < b1_pos[1]) {
    int ib = b1_crd[pb1];
    a[ib] = b[pb1];
    pb1++;
}
  
```

# Iteration lattice for a compound expression

$$a_i = b_i + c_i + d_i \quad \text{Dense}$$



```

int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] + d[pd1];
    } else if (ic == i) {
        a[i] = c[pc1];
    } else {
        a[i] = d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] + d[pd1];
    } else if (ib == i) {
        a[i] = b[pb1];
    } else {
        a[i] = d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = b[pb1] + c[pc1];
    } else if (ib == i) {
        a[i] = b[pb1];
    } else {
        a[i] = c[pc1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}

while (pd1 < d1_pos[1]) {
    int id = d1_crd[pd1];
    a[id] = d[pd1];
    pd1++;
}

while (pb1 < b1_pos[1]) {
    int ib = b1_crd[pb1];
    a[ib] = b[pb1];
    pb1++;
}

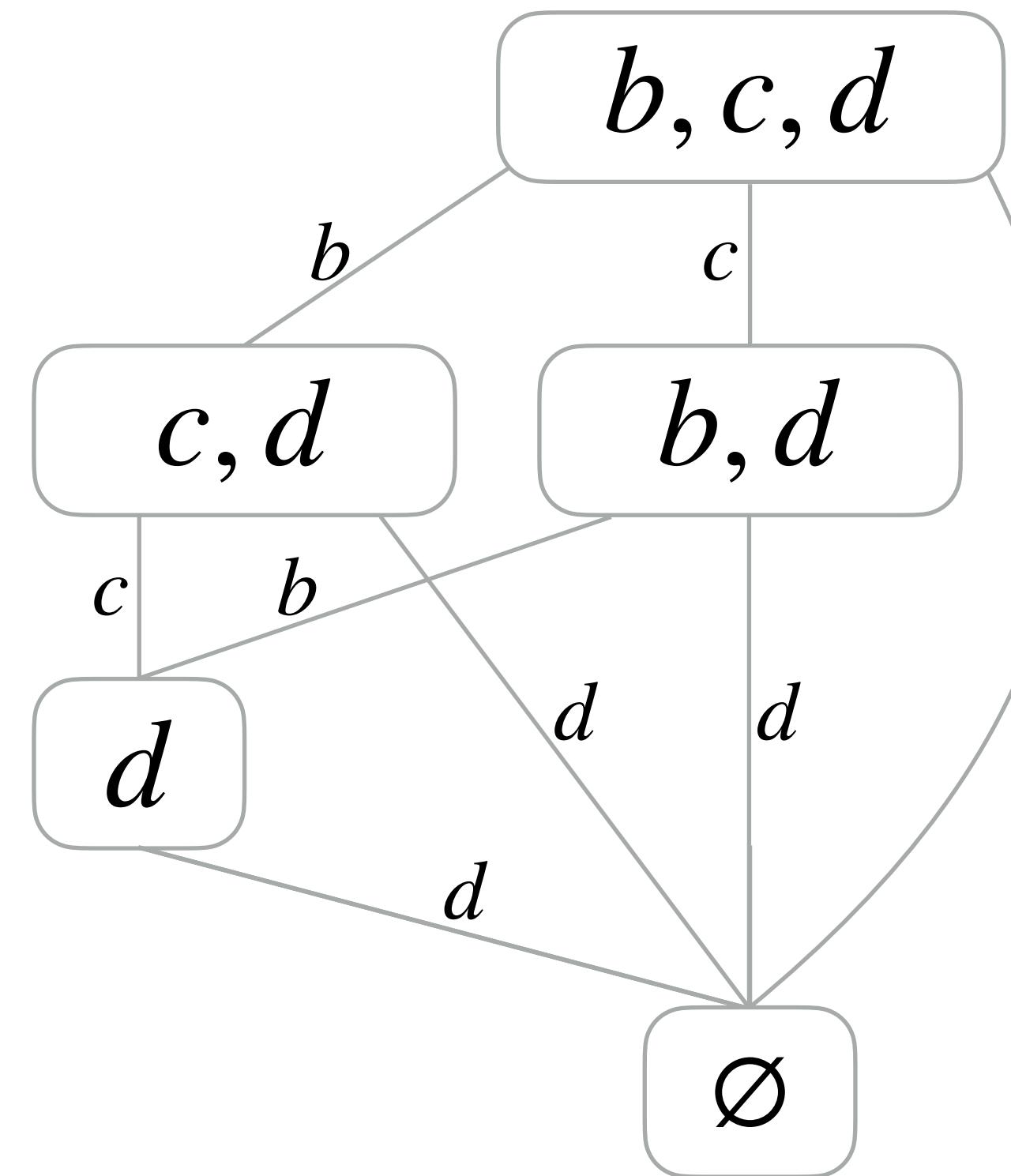
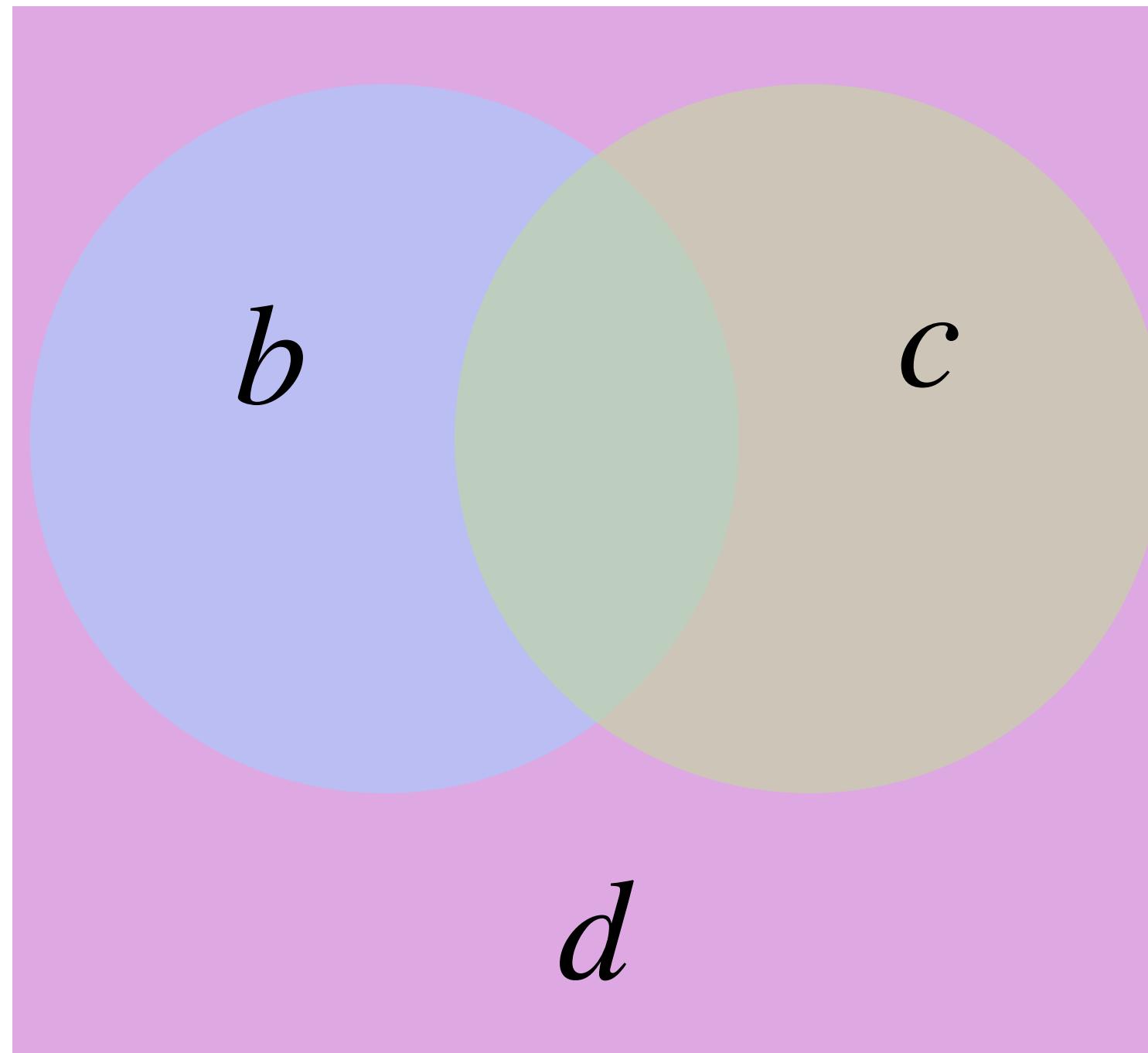
while (pc1 < c1_pos[1]) {
    int ic = c1_crd[pc1];
    a[ic] = c[pc1];
    pc1++;
}

while (pb1 < b1_pos[1]) {
    int ib = b1_crd[pb1];
    a[ib] = b[pb1];
    pb1++;
}

```

# Iteration lattice for a compound expression

$$a_i = b_i + c_i + d_i \quad \text{Dense}$$



```

int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int id = 0;
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int pd1 = id;
    int pa1 = id;
    if (ib == id) {
        a[pa1] = c[pc1] + d[pd1];
    } else {
        a[pa1] = d[pd1];
    }
    if (ib == id && ic == id) {
        a[pa1] = b[pb1] + c[pc1] + d[pd1];
    } else if (ib == id) {
        a[pa1] = b[pb1] + d[pd1];
    } else if (ic == id) {
        a[pa1] = c[pc1] + d[pd1];
    } else {
        a[pa1] = d[pd1];
    }
    if (ib == id) pb1++;
    if (ic == id) pc1++;
    id++;
}

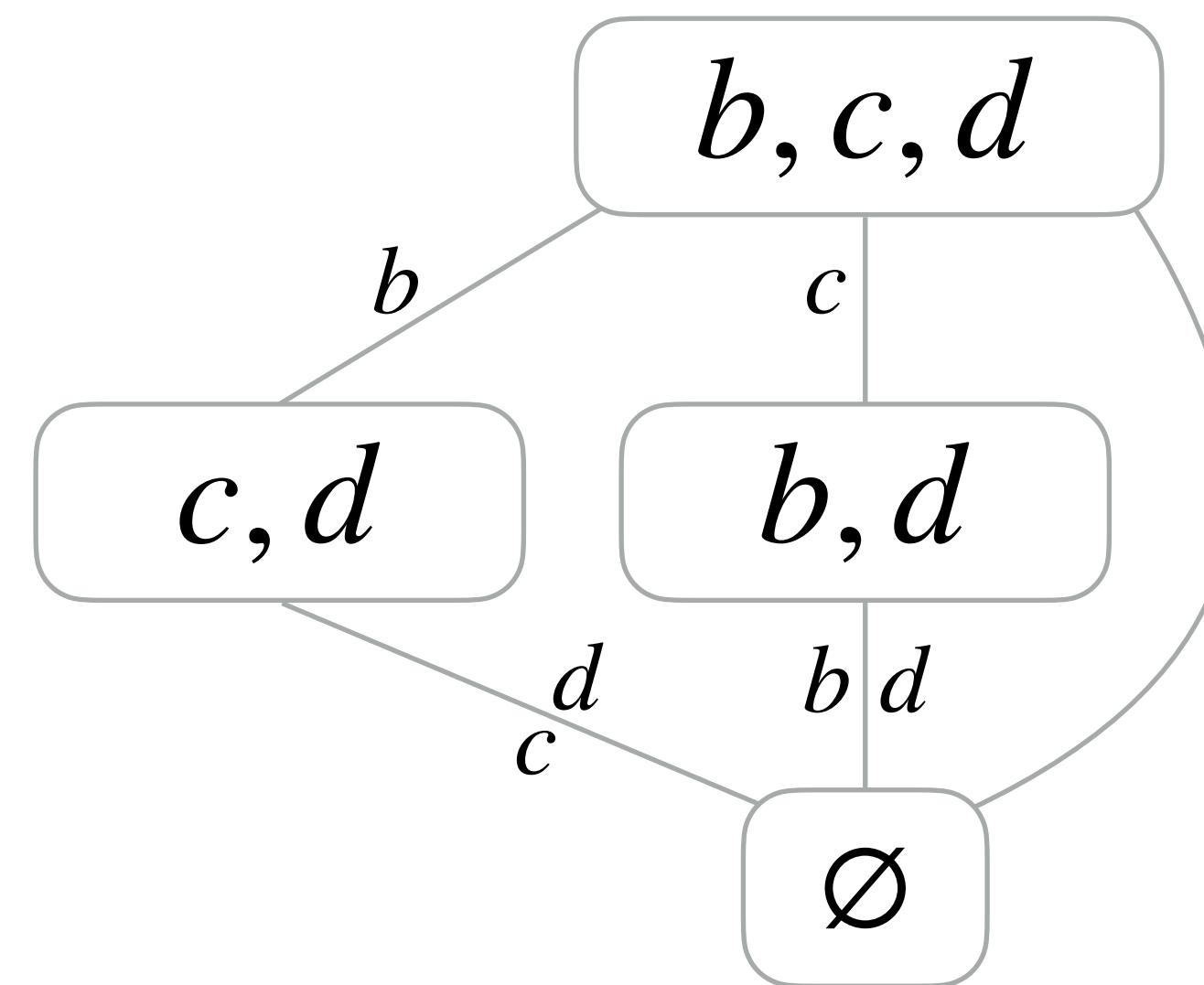
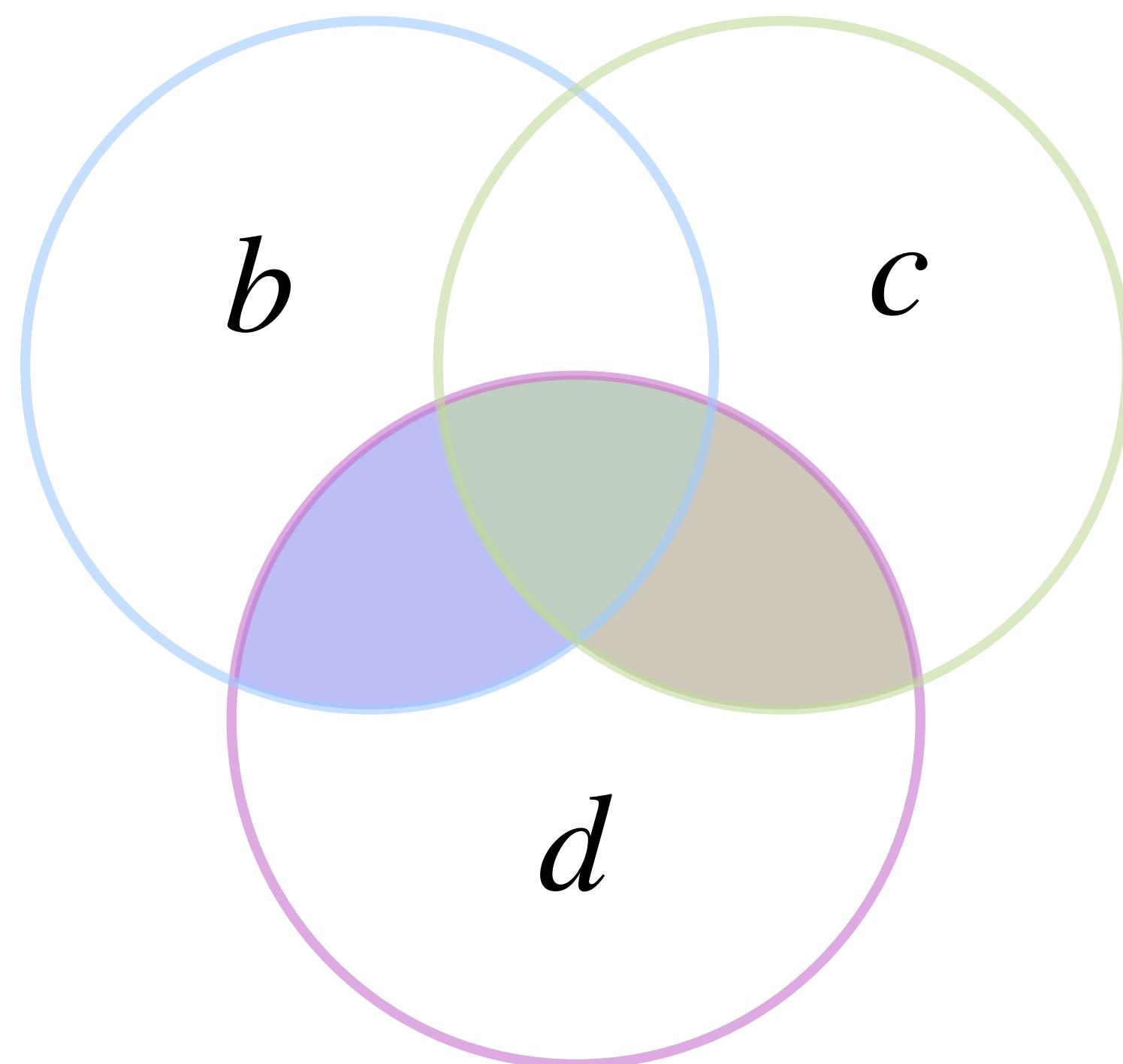
while (pb1 < b1_pos[1]) {
    int ib = b1_crd[pb1];
    int pd1 = id;
    int pa1 = id;
    if (ib == id) {
        a[pa1] = b[pb1] + d[pd1];
    } else {
        a[pa1] = d[pd1];
    }
    if (ib == id) pb1++;
    id++;
}

while (id < d1_dimension) {
    int pd1 = id;
    int pa1 = id;
    a[pa1] = d[pd1];
    id++;
}

```

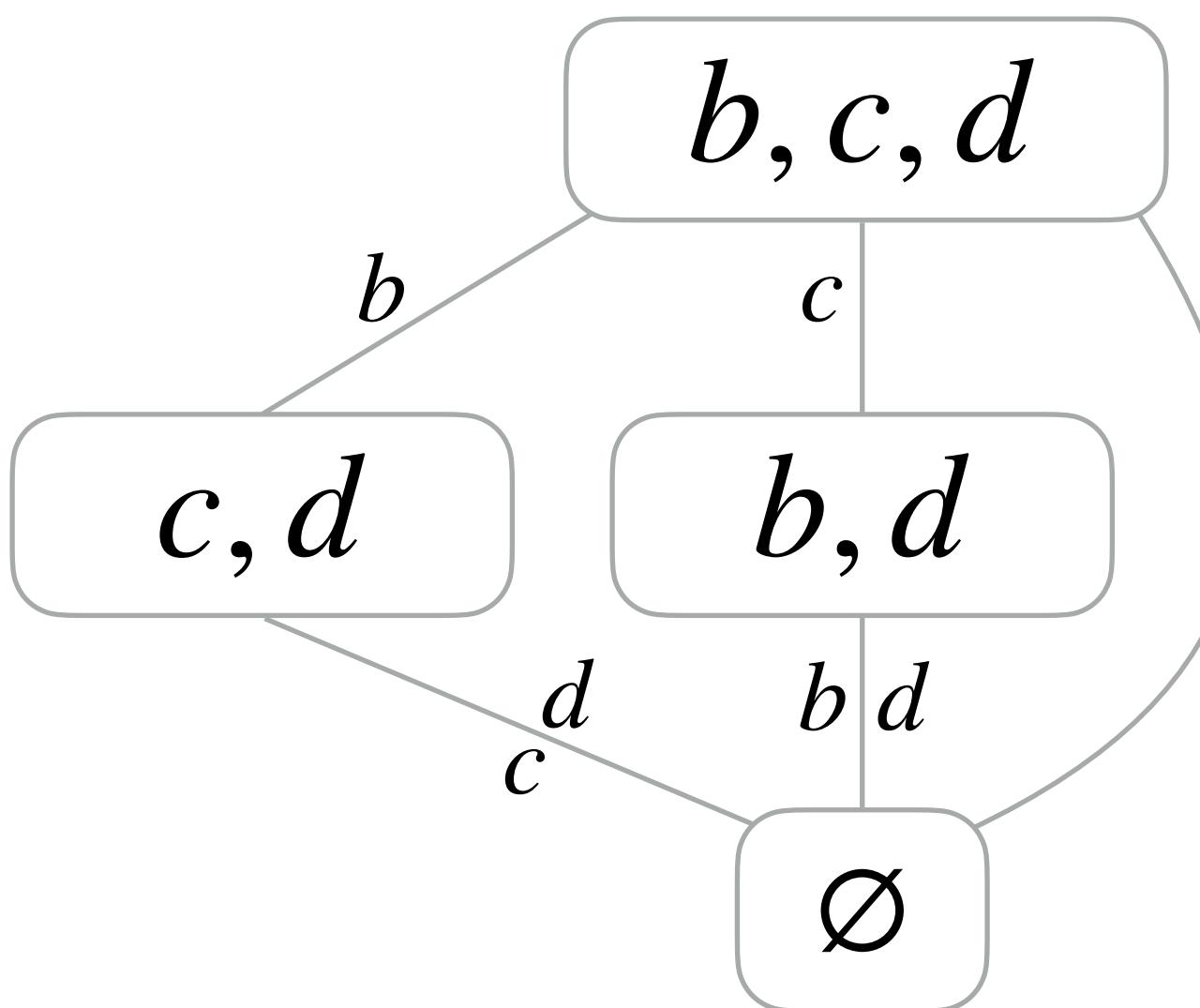
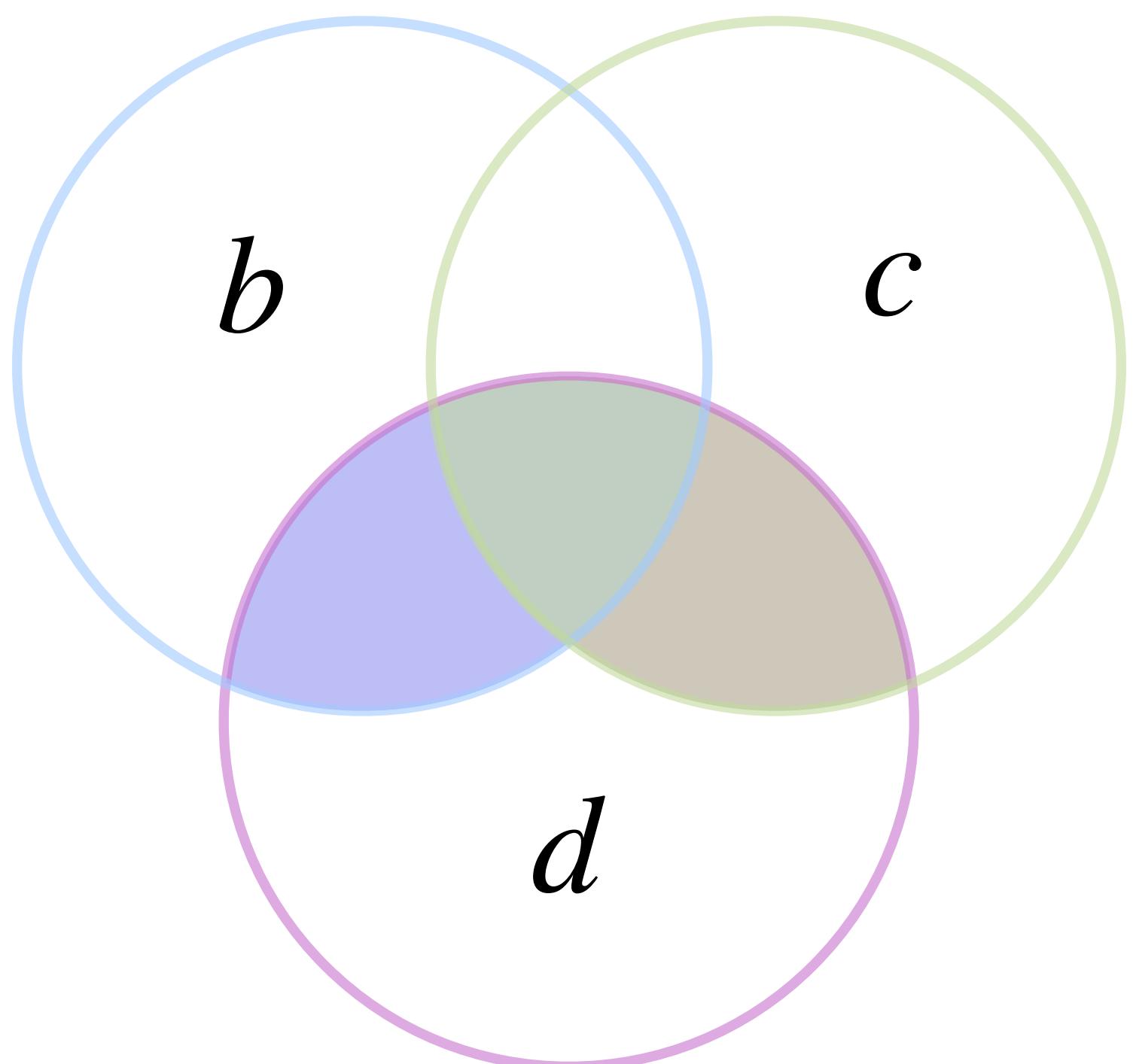
# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i$$



# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i$$



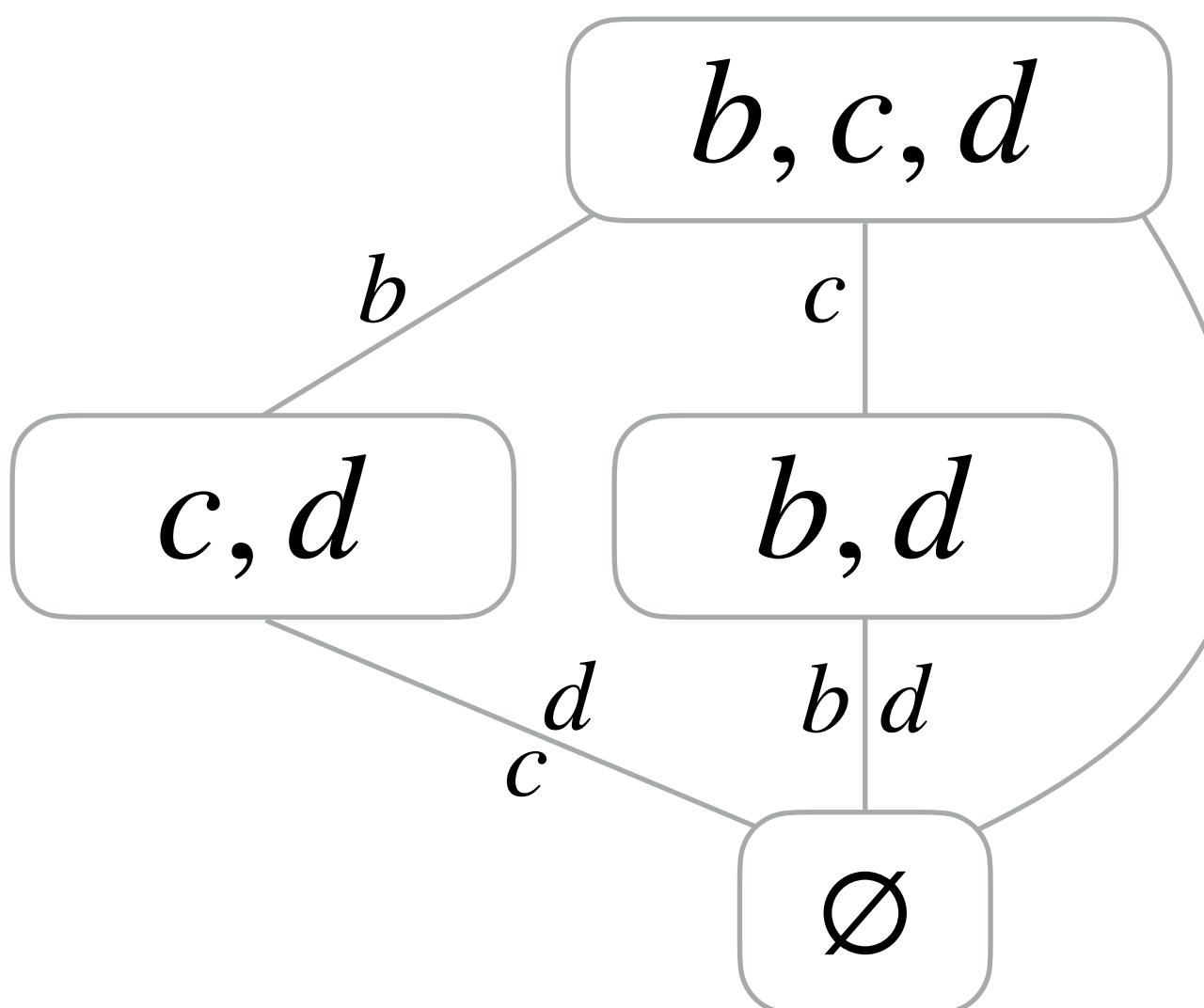
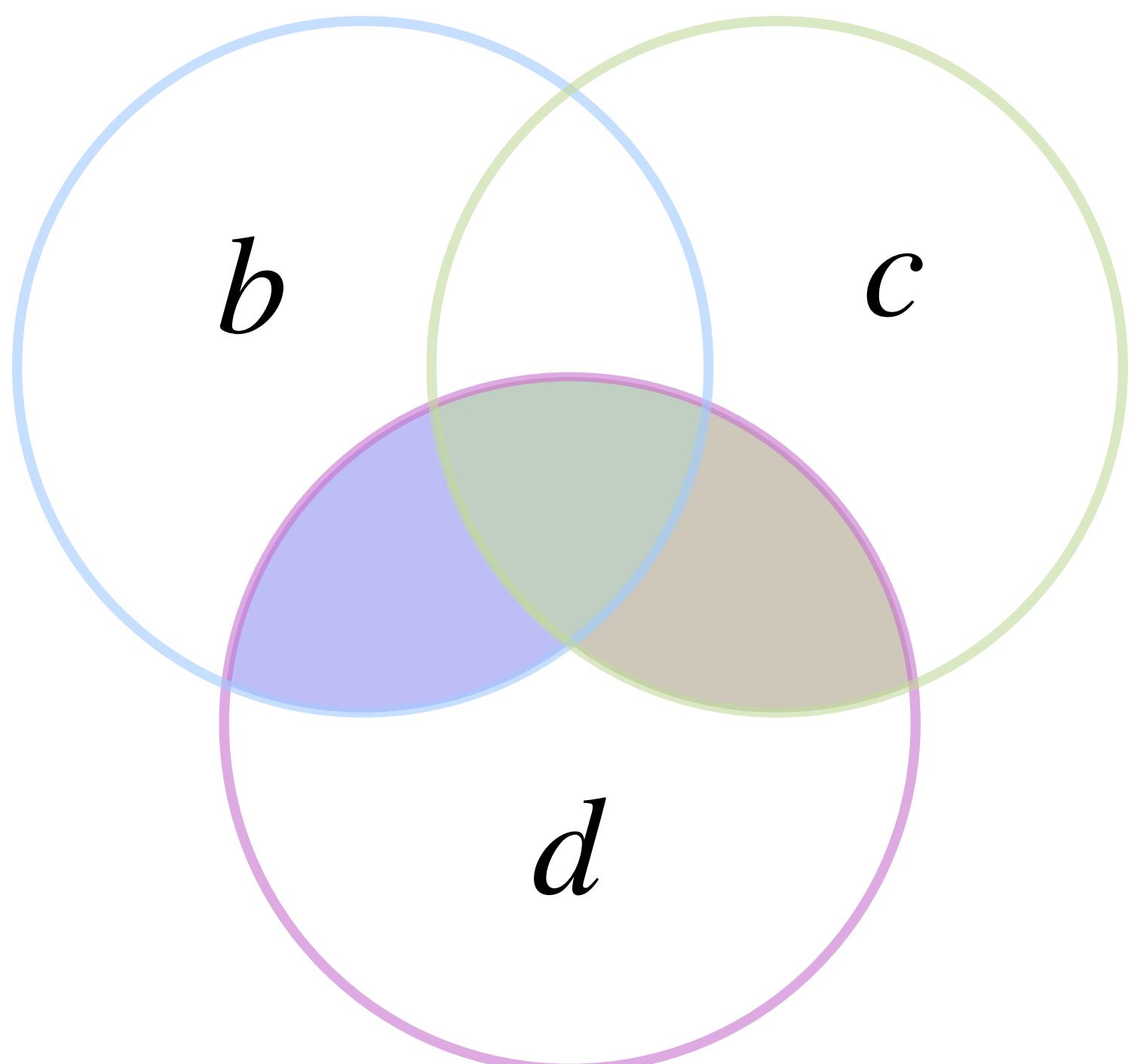
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i \quad \text{Dense}$$



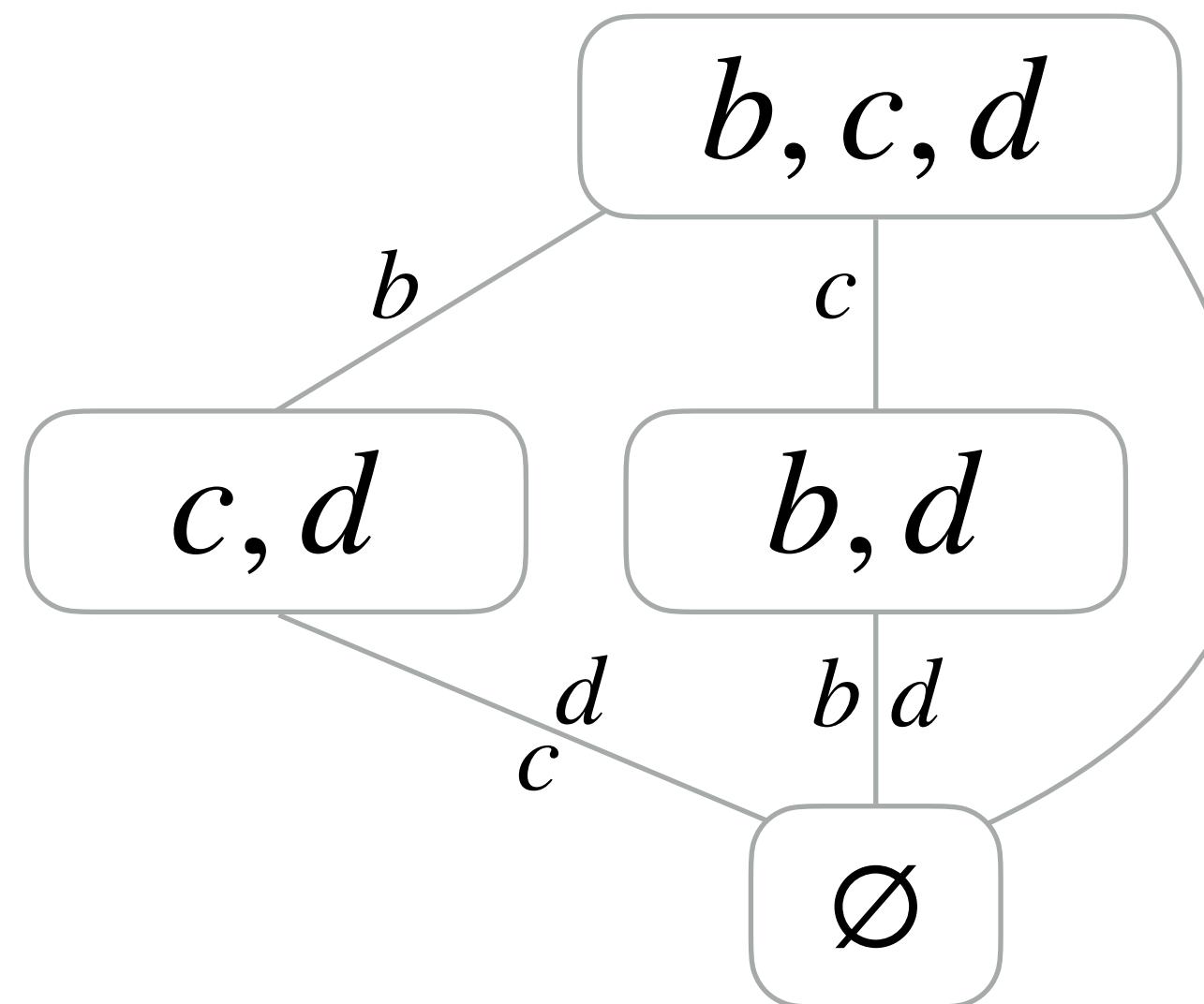
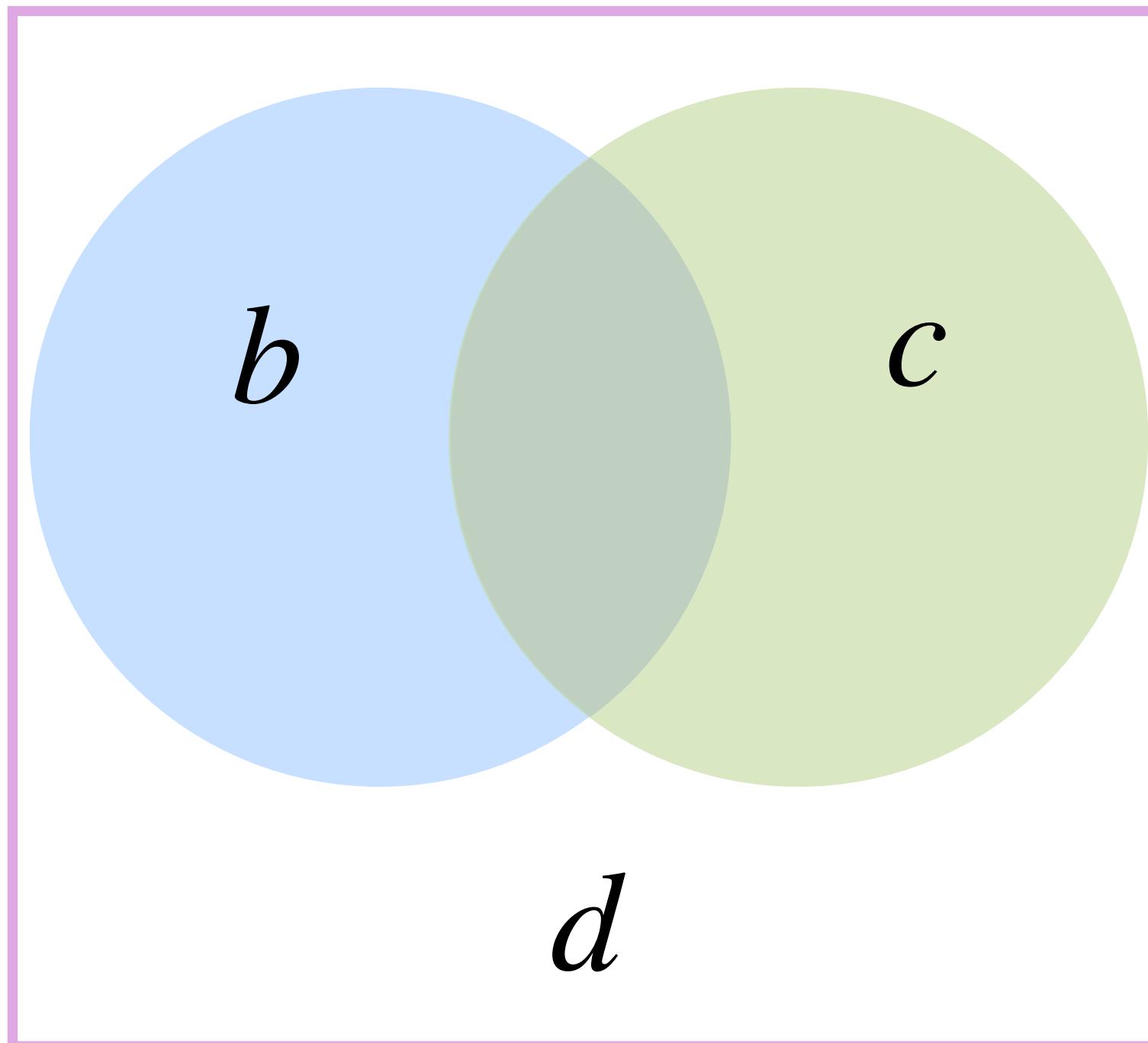
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i \quad \text{Dense}$$



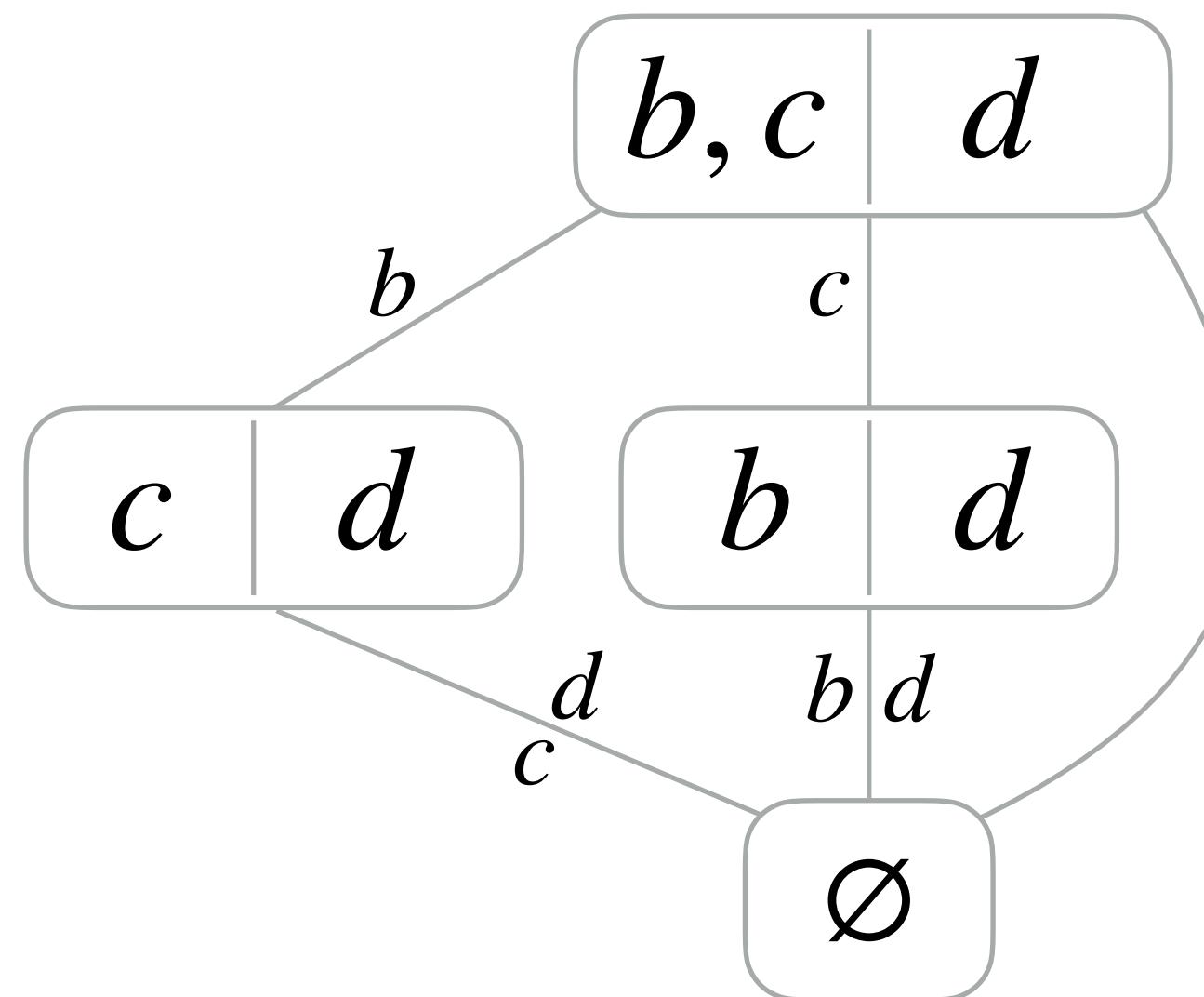
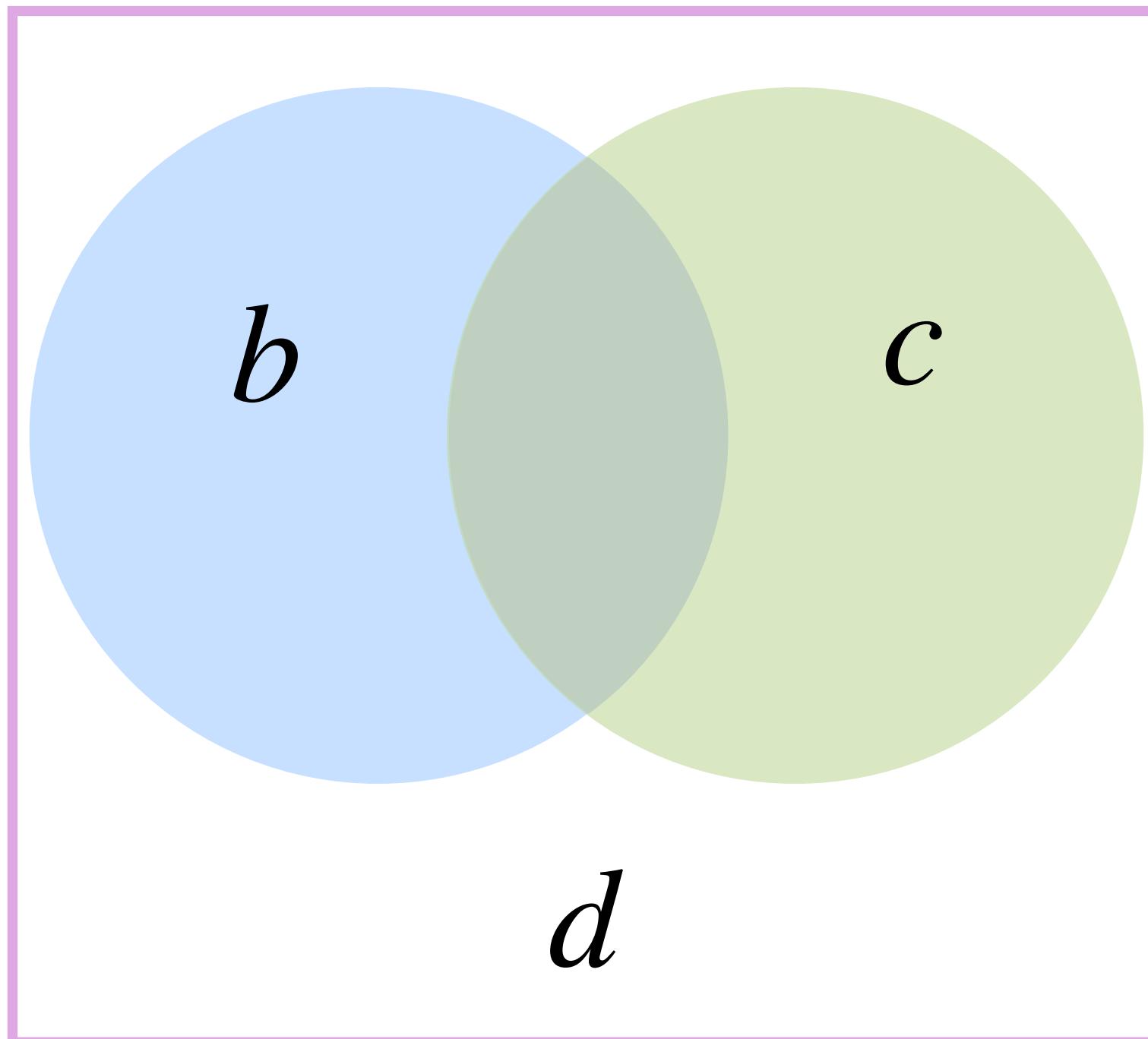
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i \quad \text{Dense}$$

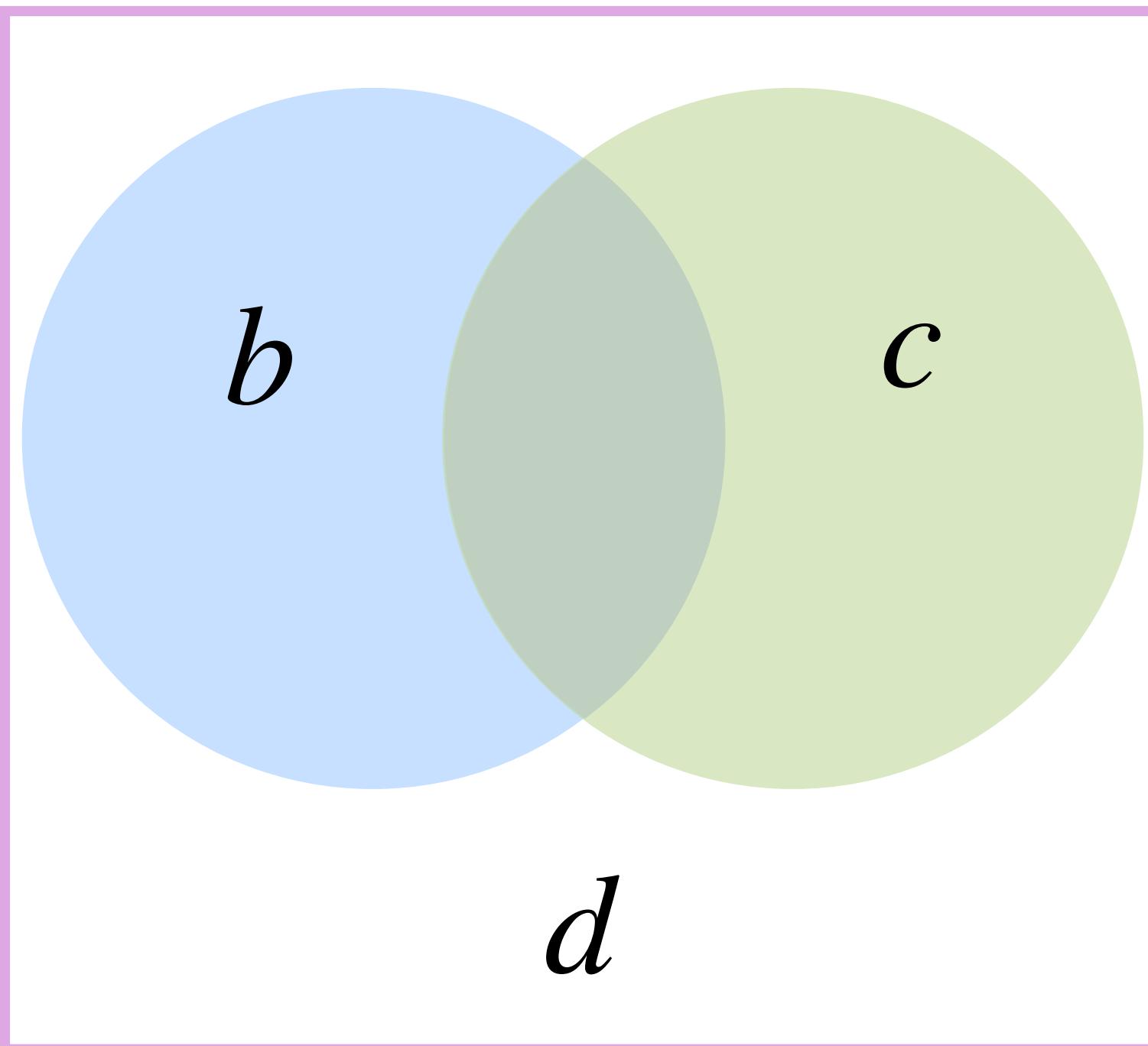


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

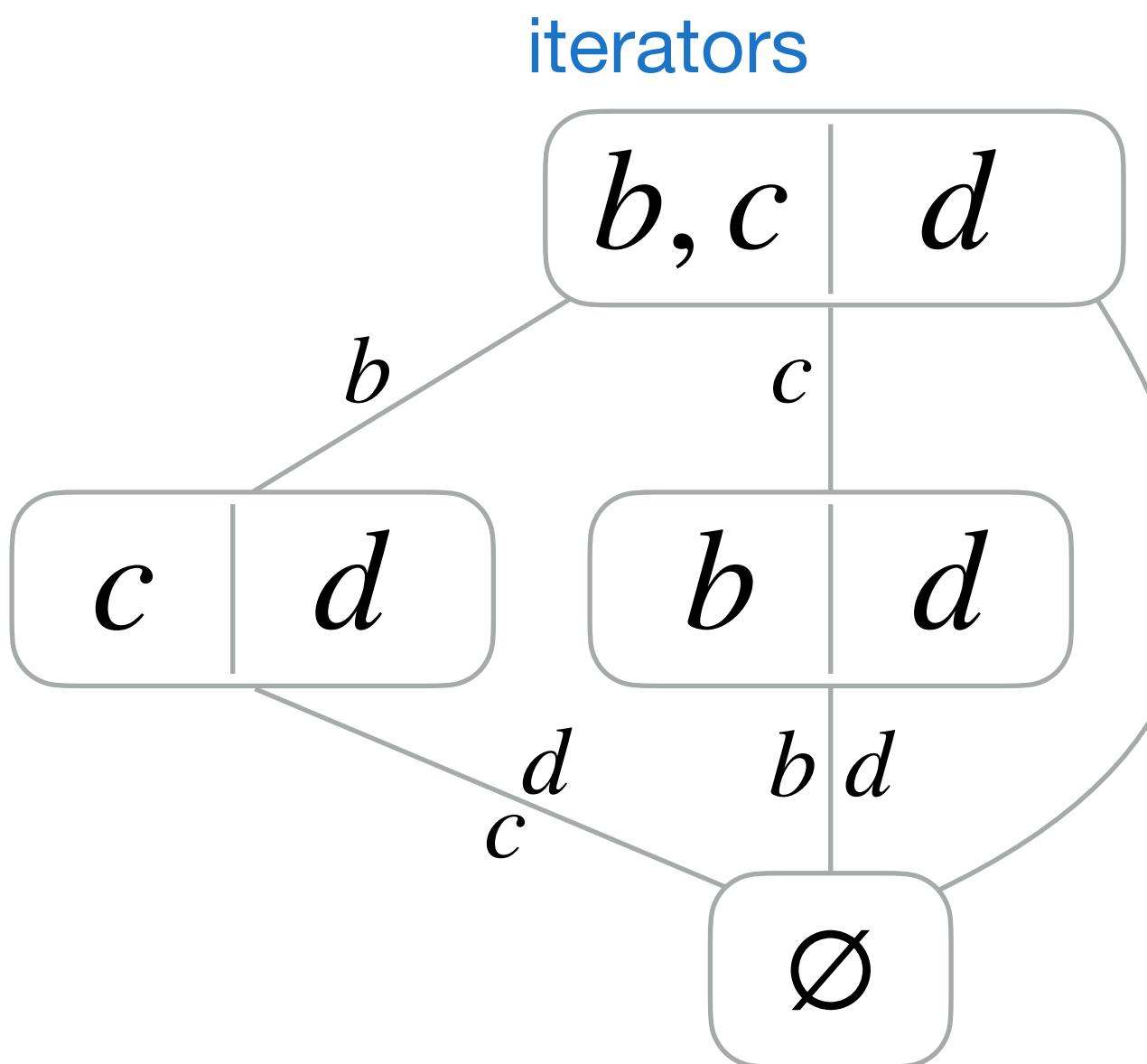
while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression



$$a_i = (b_i + c_i)d_i$$

Dense



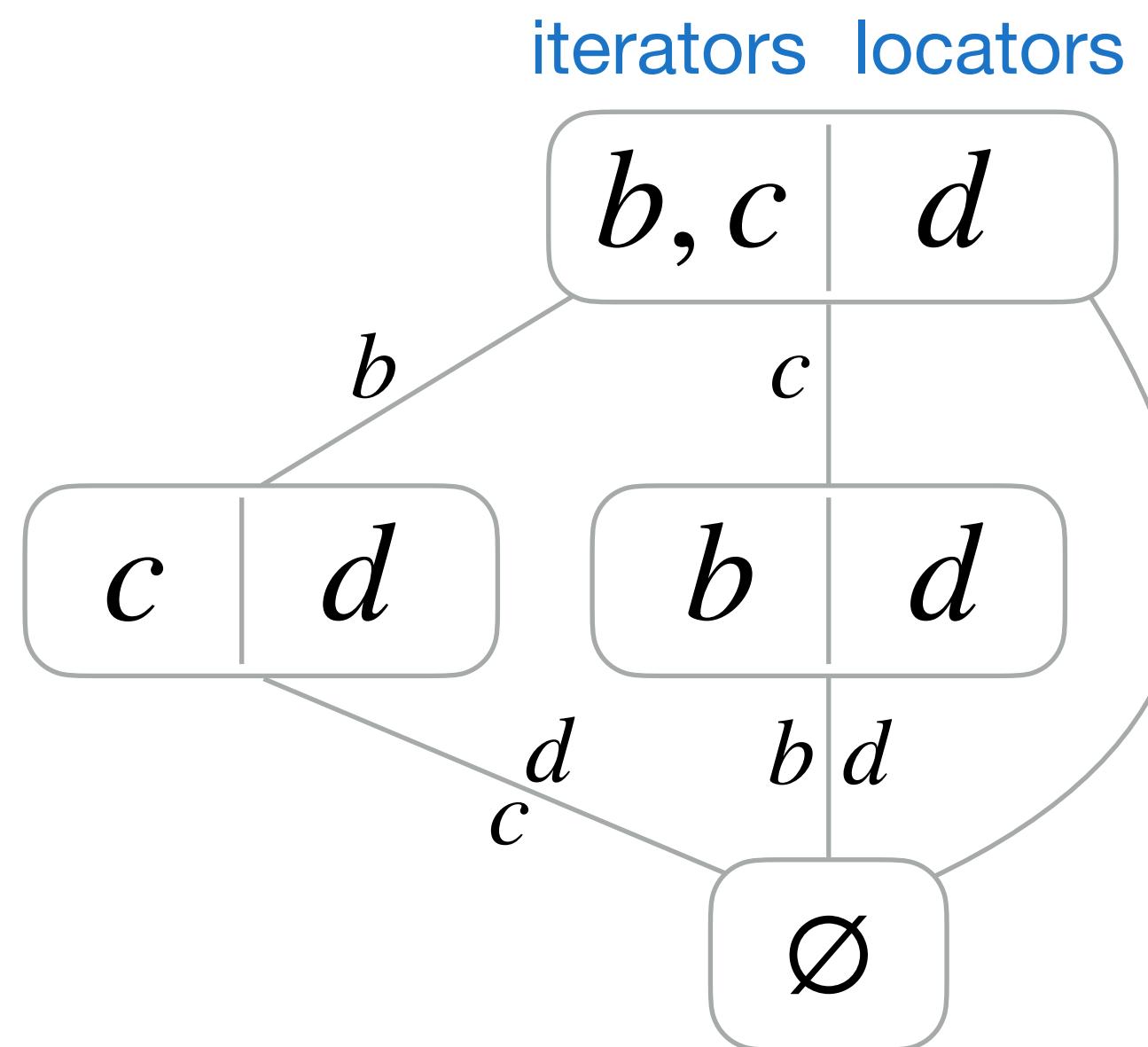
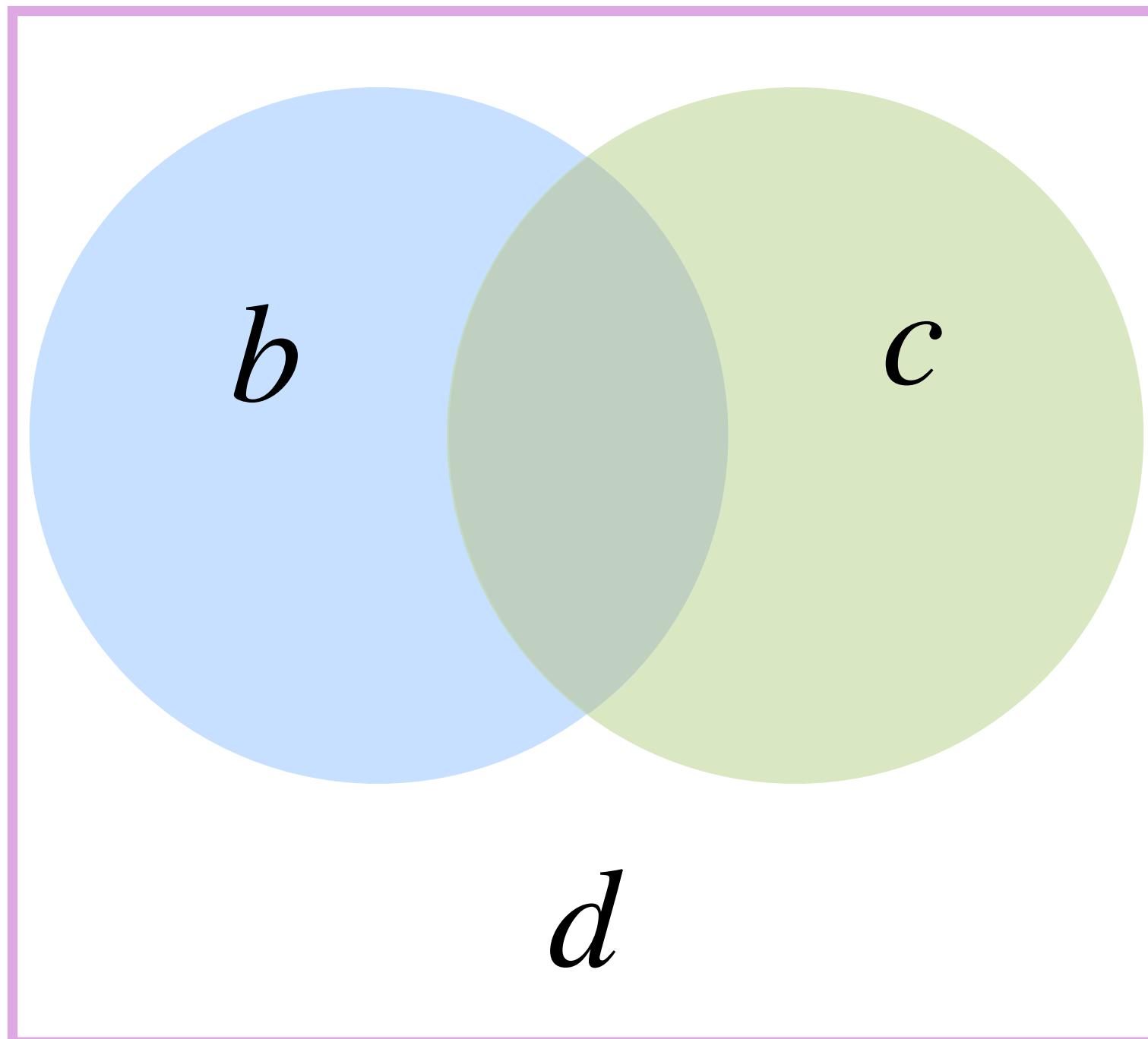
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i \quad \text{Dense}$$



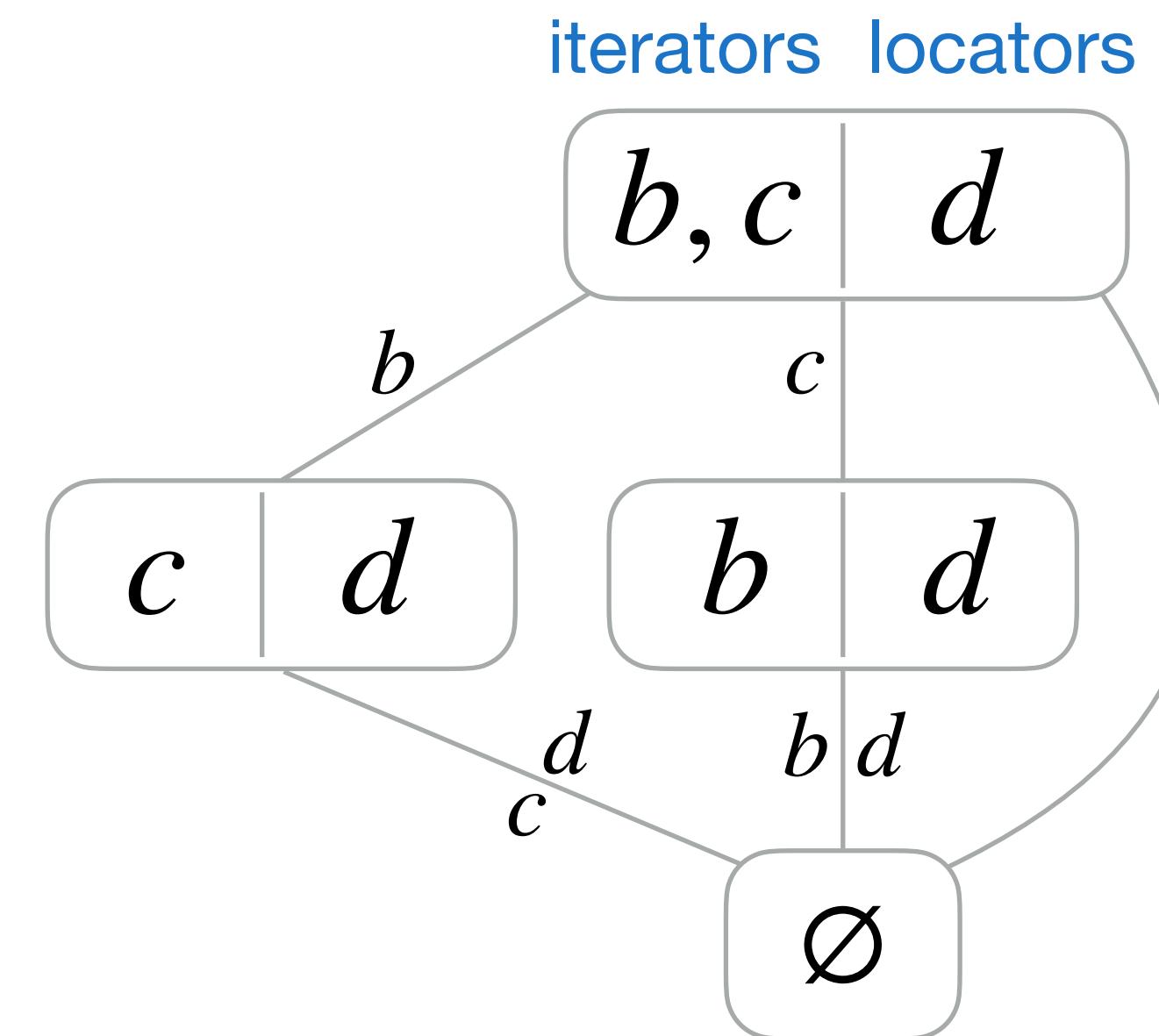
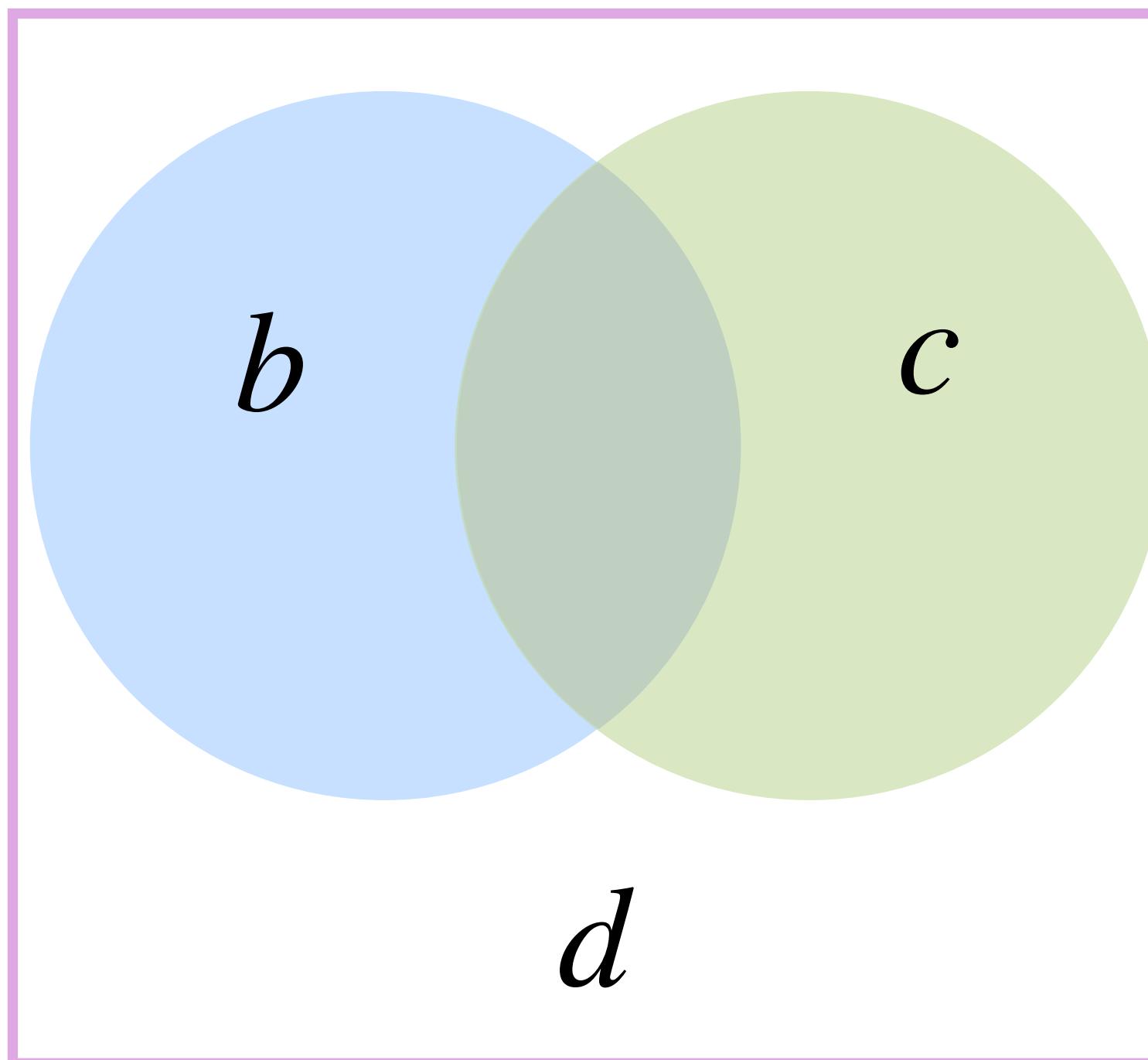
```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
int pd1 = d1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ib, ic, id);
    if (ib == i && ic == i && id == i) {
        a[i] = (b[pb1] + c[pc1]) * d[pd1];
    } else if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    } else if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pc1 < c1_pos[1] && pd1 < d1_pos[1]) {
    int ic = c1_crd[pc1];
    int id = d1_crd[pd1];
    int i = min(ic, id);
    if (ic == i && id == i) {
        a[i] = c[pc1] * d[pd1];
    }
    if (ic == i) pc1++;
    if (id == i) pd1++;
}

while (pb1 < b1_pos[1] && pd1 < d1_pos[1])
{
    int ib = b1_crd[pb1];
    int id = d1_crd[pd1];
    int i = min(ib, id);
    if (ib == i && id == i) {
        a[i] = b[pb1] * d[pd1];
    }
    if (ib == i) pb1++;
    if (id == i) pd1++;
}
```

# Iteration lattice for a compound expression

$$a_i = (b_i + c_i)d_i \quad \text{Dense}$$

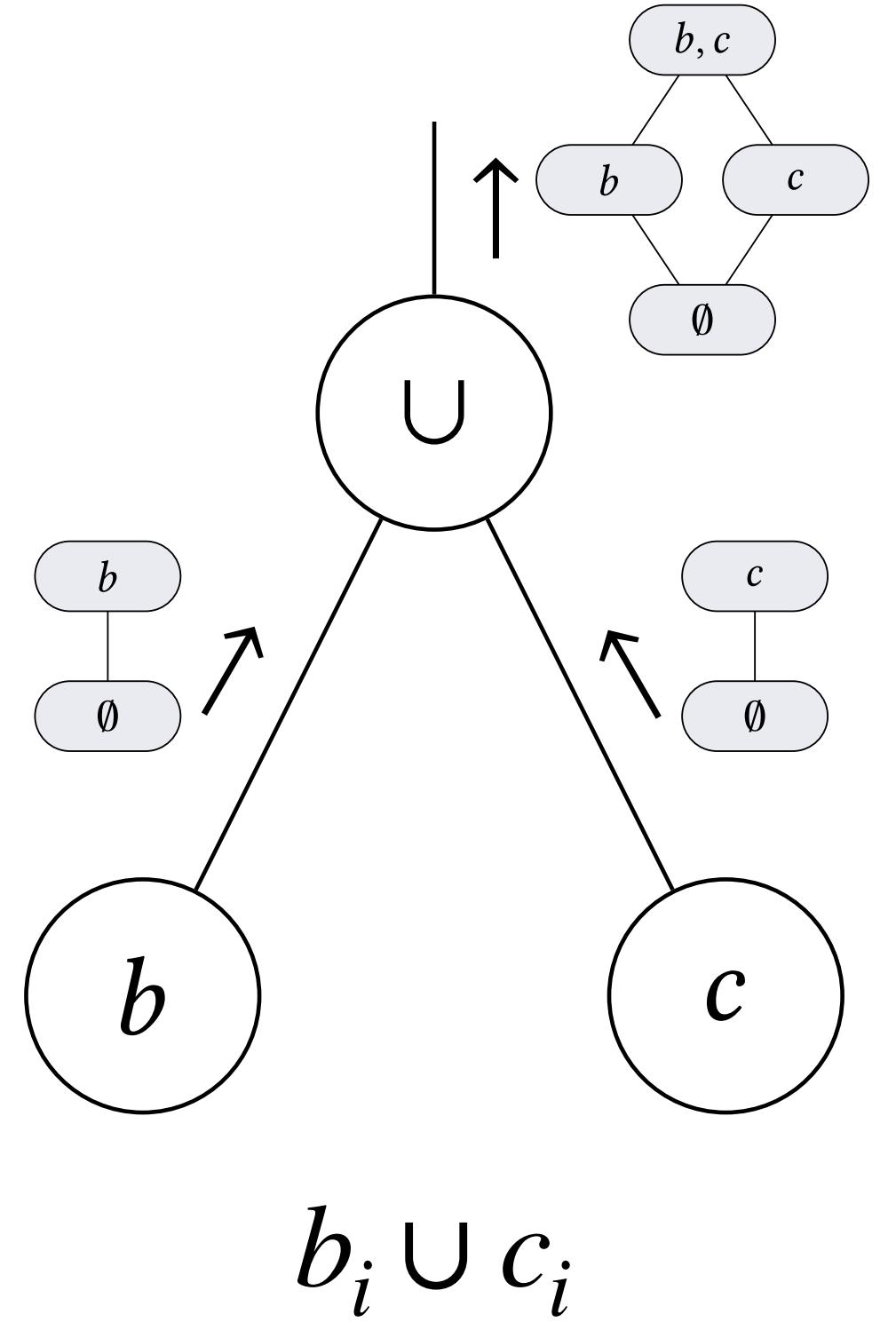


```
int pb1 = b1_pos[0];
int pc1 = c1_pos[0];
while (pb1 < b1_pos[1] && pc1 < c1_pos[1]) {
    int ib = b1_crd[pb1];
    int ic = c1_crd[pc1];
    int i = min(ib, ic);
    if (ib == i && ic == i) {
        a[i] = (b[pb1] + c[pc1]) * d[i];
    } else if (ib == i) {
        a[i] = b[pb1] * d[i];
    } else if (ic == i) {
        a[i] = c[pc1] * d[i];
    }
    if (ib == i) pb1++;
    if (ic == i) pc1++;
}

while (pb1 < b1_pos[1]) {
    int i = b1_crd[pb1];
    a[i] = b[pb1] * d[i];
    pb1++;
}

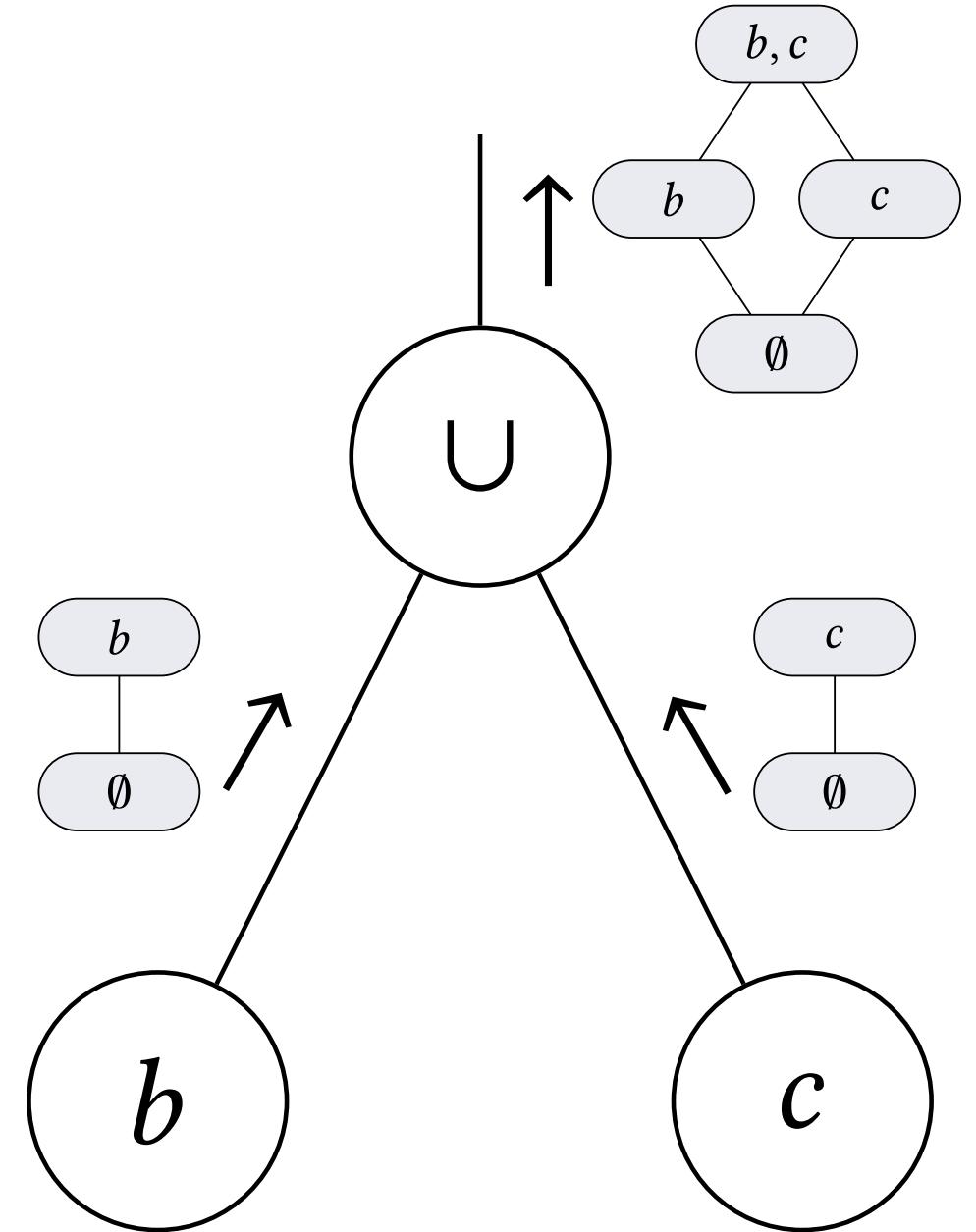
while (pc1 < c1_pos[1]) {
    int i = c1_crd[pc1];
    a[i] = c[pc1] * d[i];
    pc1++;
}
```

# Iteration lattice construction



Bottom-up construction from set expression:  
create and merge iteration lattices

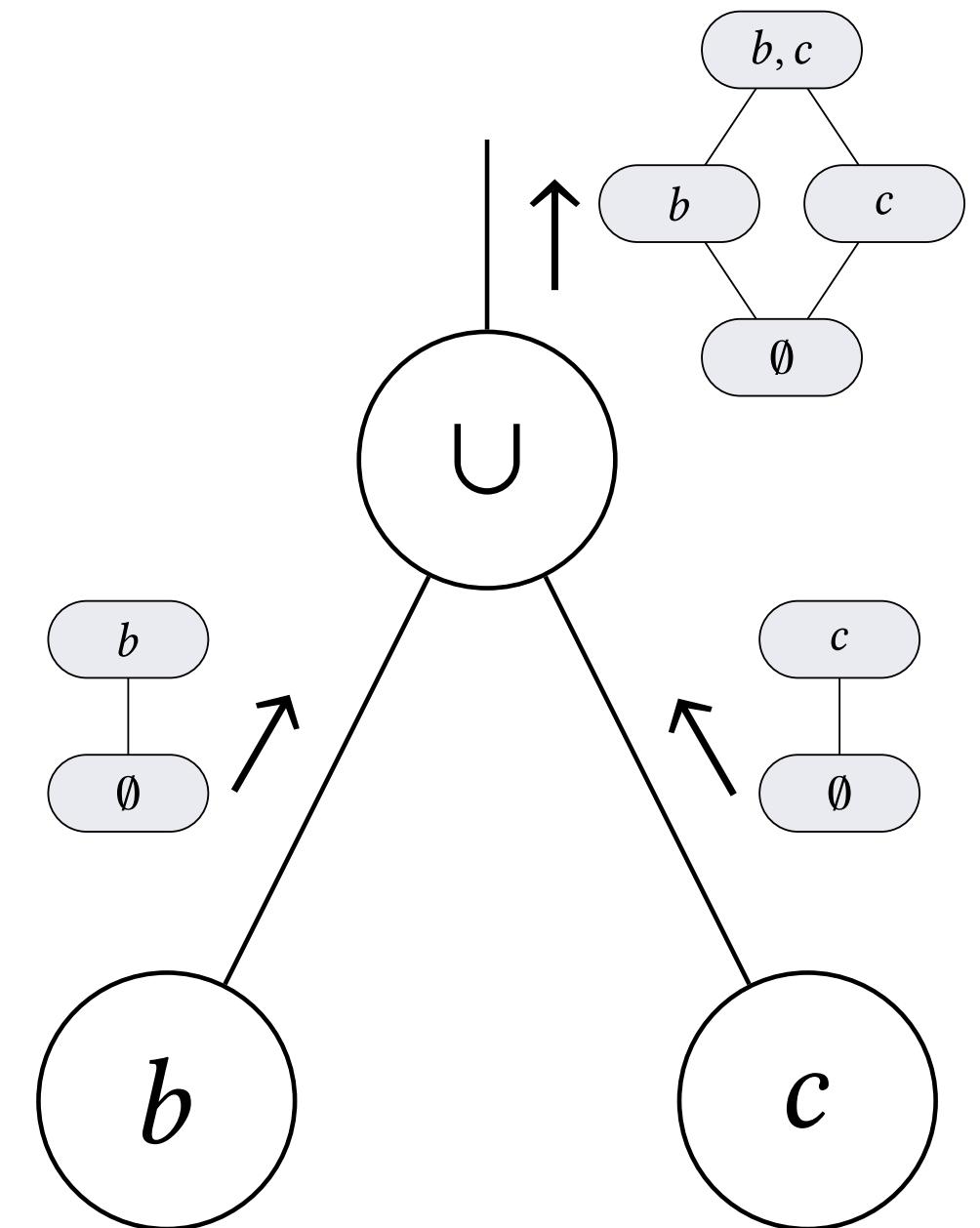
# Iteration lattice construction



Bottom-up construction from set expression:  
create and merge iteration lattices

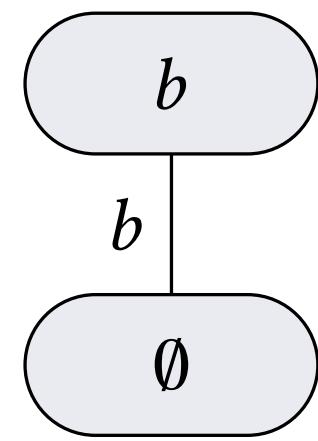
Base cases:

# Iteration lattice construction



$$b_i \cup c_i$$

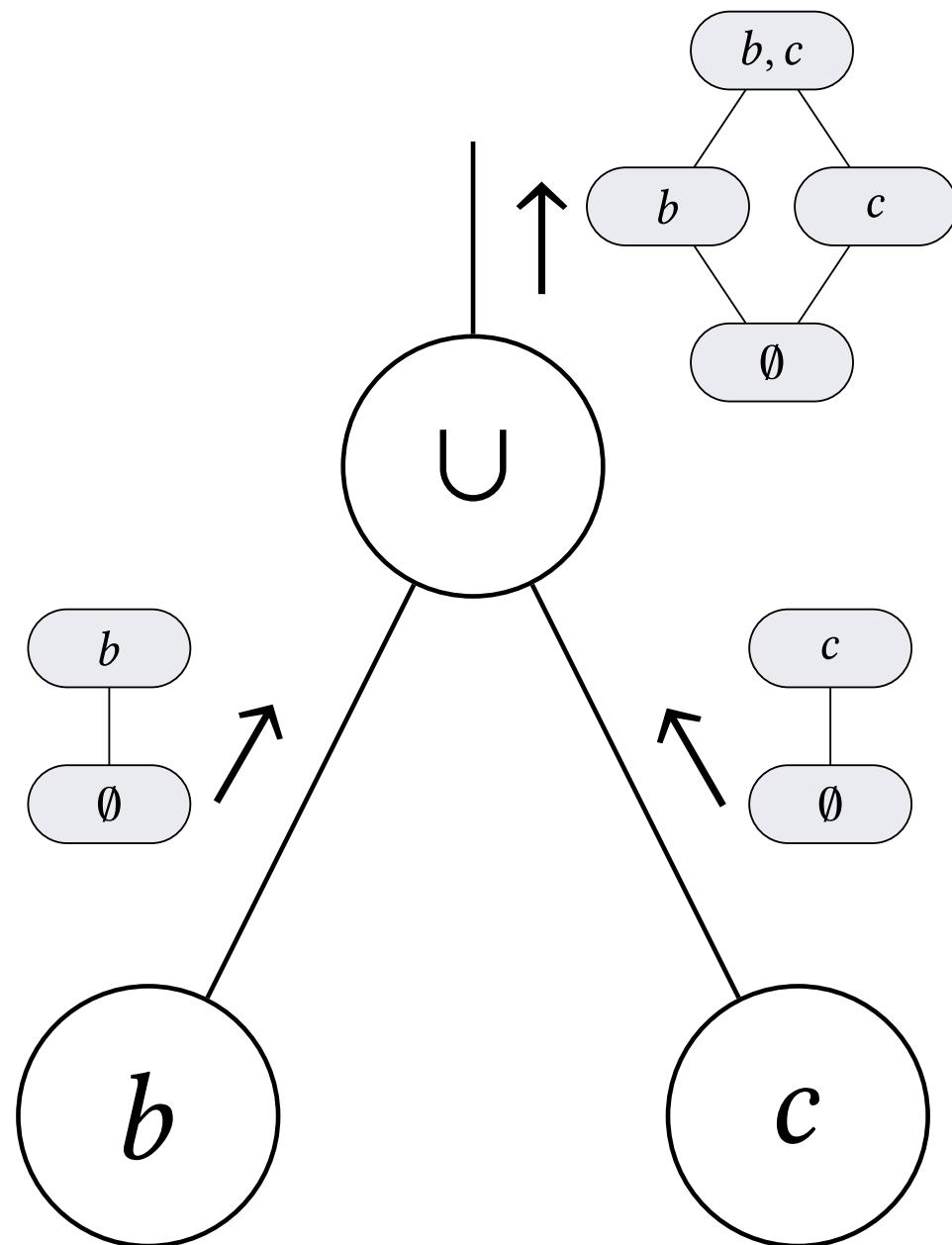
Bottom-up construction from set expression:  
create and merge iteration lattices



*b* has an iterator

Base cases:

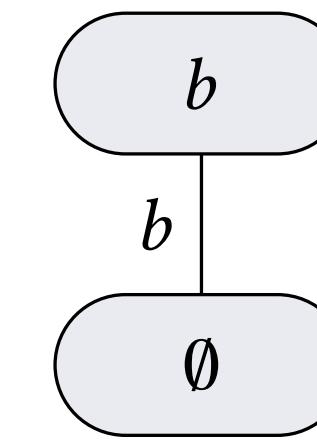
# Iteration lattice construction



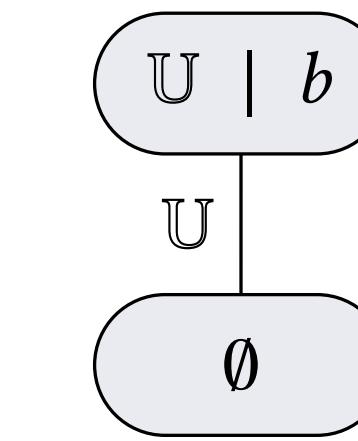
$$b_i \cup c_i$$

Bottom-up construction from set expression:  
create and merge iteration lattices

Base cases:

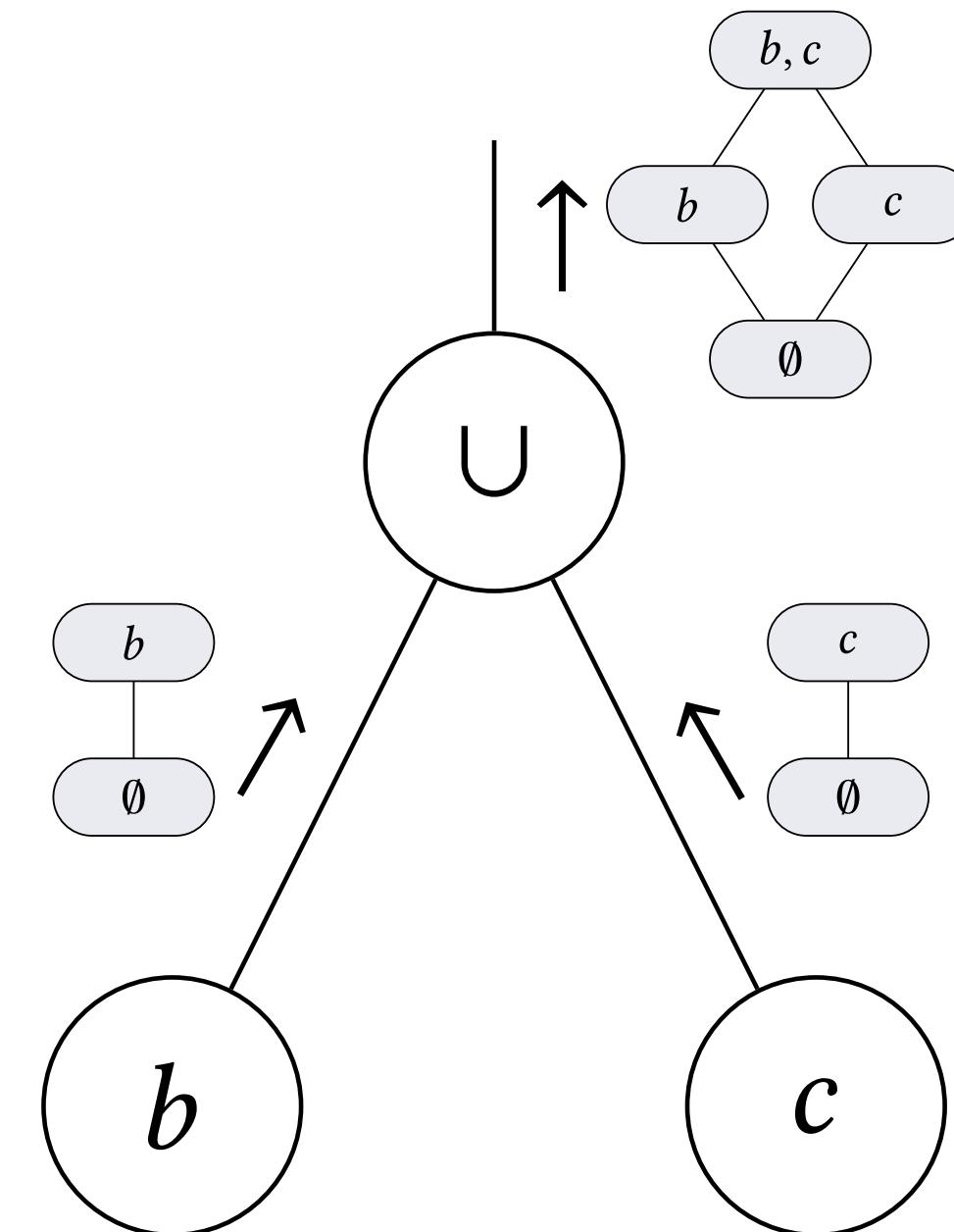


*b* has an iterator



*b* does not have an iterator,  
but supports locate

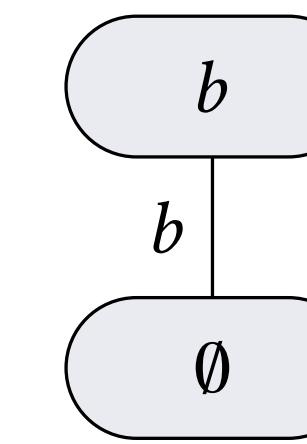
# Iteration lattice construction



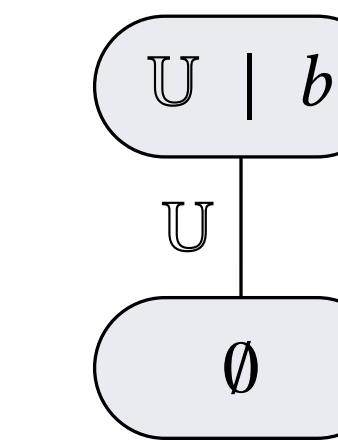
$$b_i \cup c_i$$

Bottom-up construction from set expression:  
create and merge iteration lattices

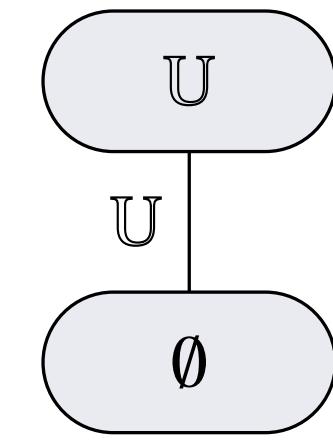
Base cases:



$b$  has an iterator

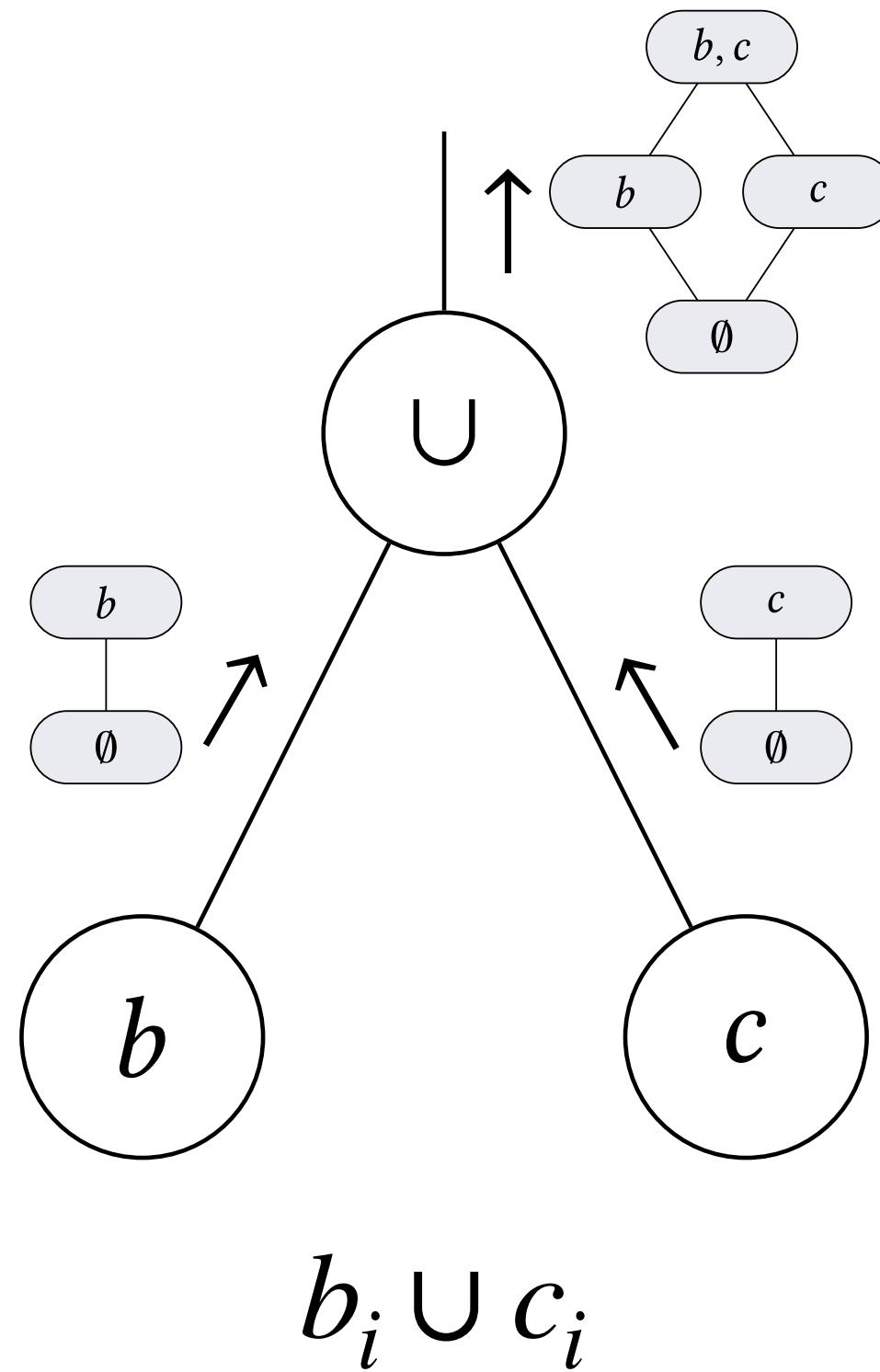


$b$  does not have an iterator,  
but supports locate

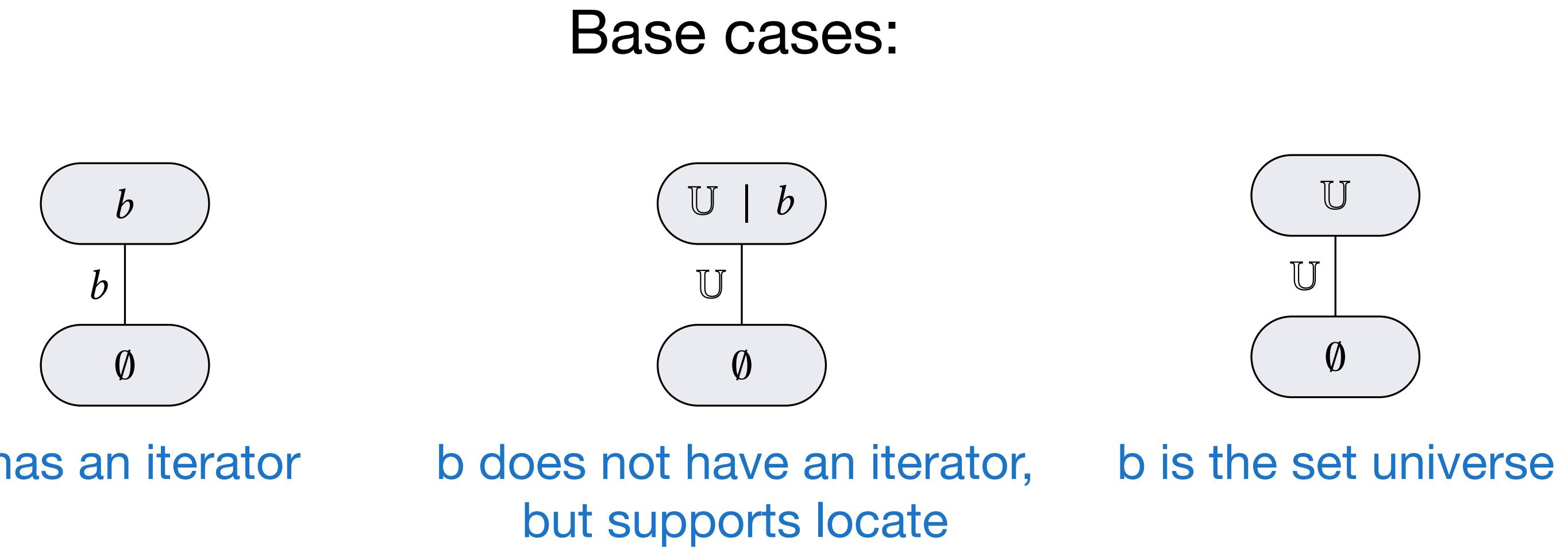


$b$  is the set universe

# Iteration lattice construction



Bottom-up construction from set expression:  
create and merge iteration lattices



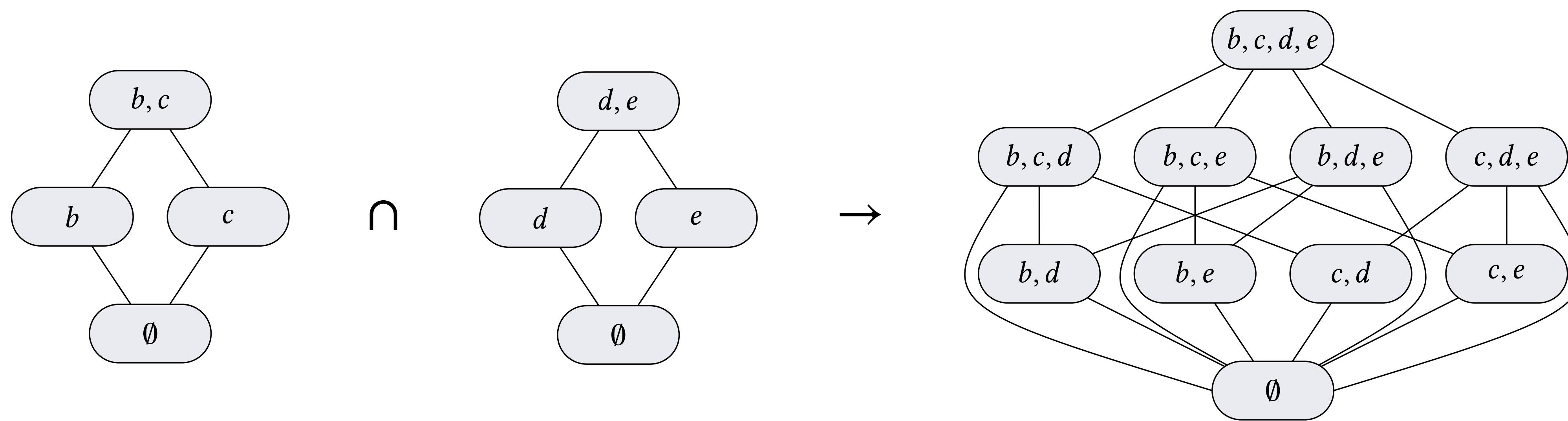
Lattice point merging:

$$((b, c), (d, e)) \rightarrow (b, c, d, e)$$

Lattice points are merged by  
taking the union of their iterator  
and locator sets respectively

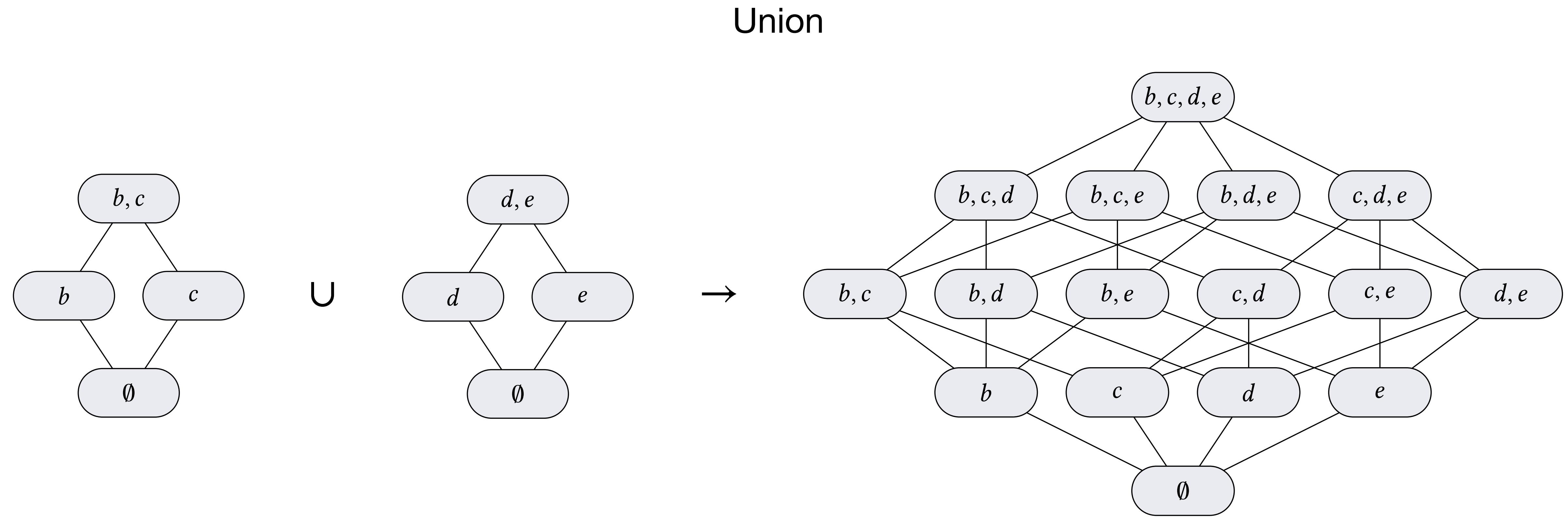
# Iteration lattice construction

Intersection



The intersection of two lattices is computed by merging the lattice point pairs in the Cartesian combination of their lattice points.

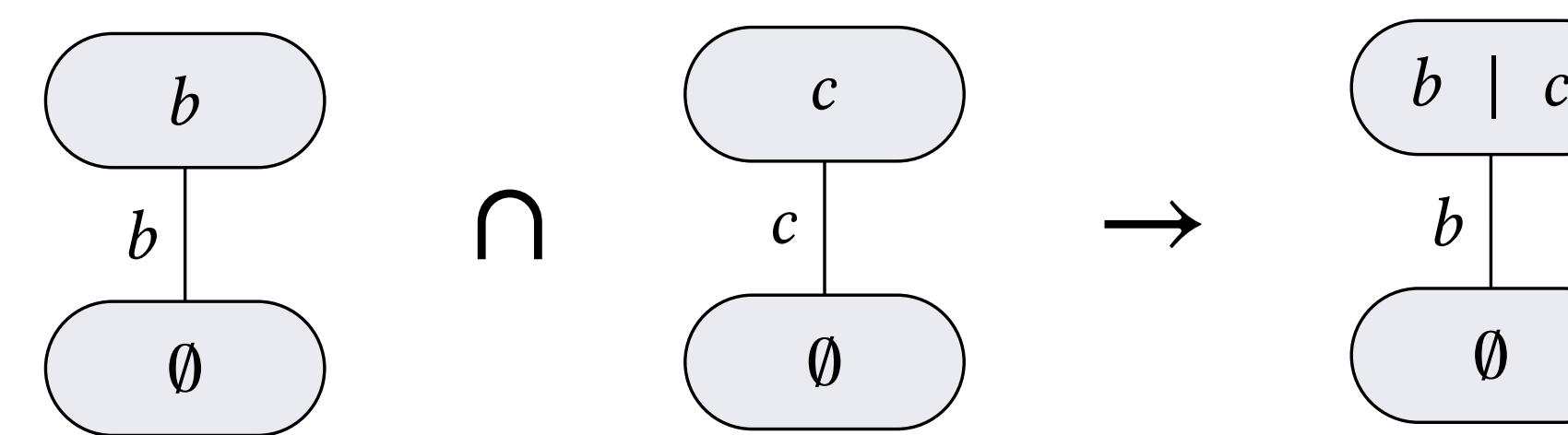
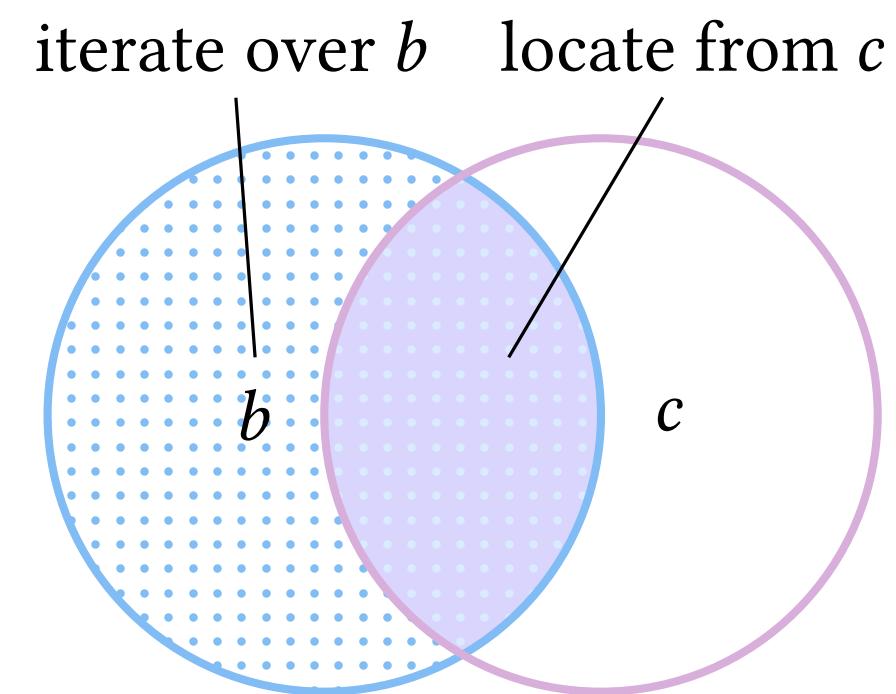
# Iteration lattice construction



The union of two lattices is computed by first merging the lattice point pairs in the Cartesian combination of their lattice points. The union of the lattices is then the union of the result and the two initial lattices.

# Iteration lattice optimization example

## Intersection Optimization



When intersecting two lattices, move the operands with the locate capability from one side of the intersection from the iterators to the locators set.