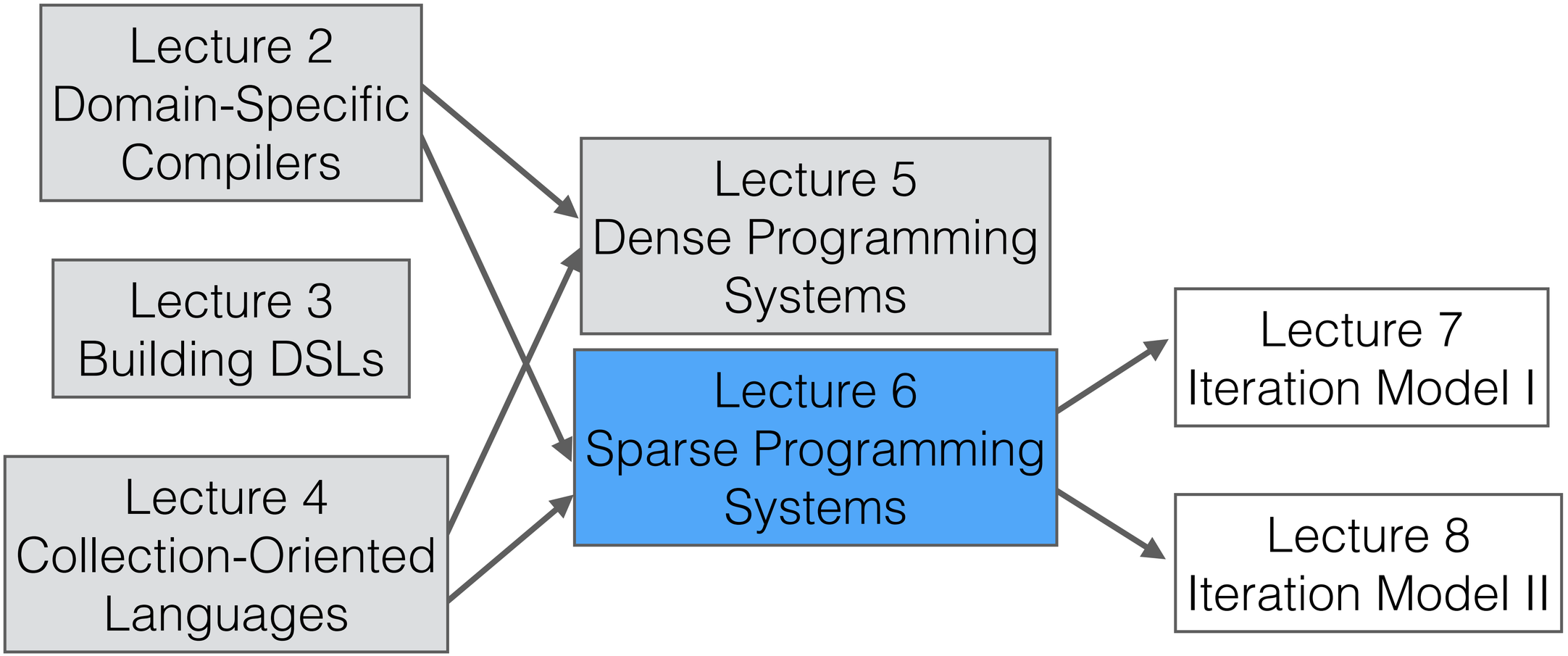
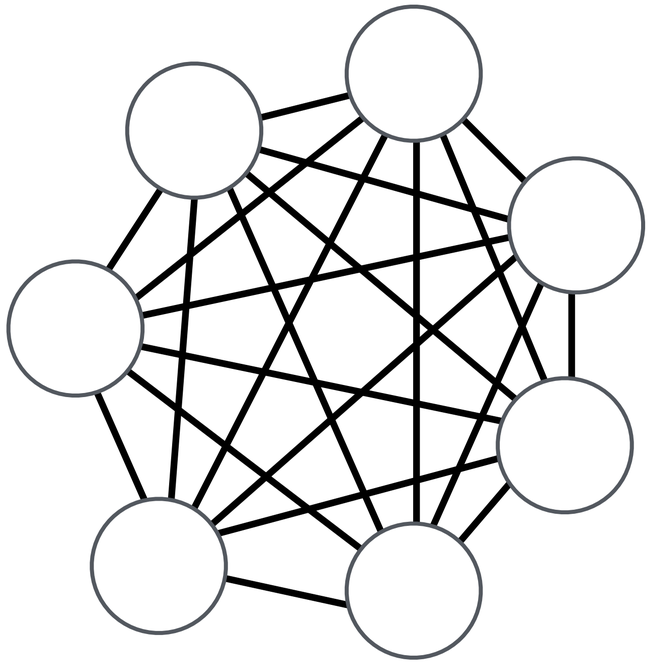
Lecture 6 — Sparse Programming Systems

Stanford CS343D (Winter 2024)
Fred Kjolstad

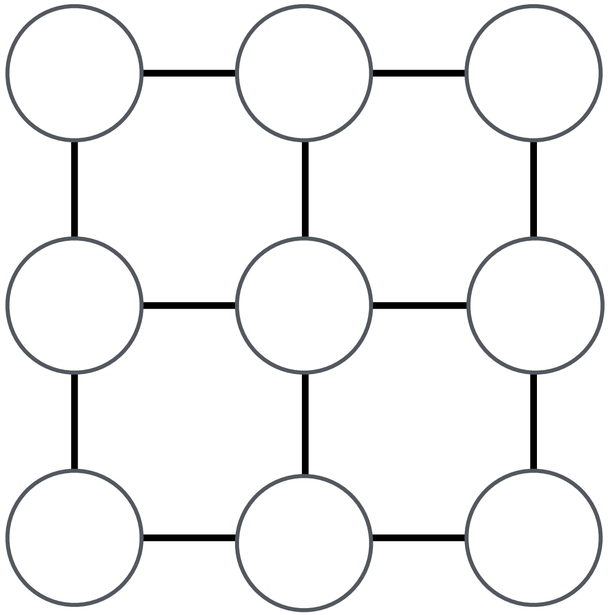

Lecture 2
Domain-Specific
Compilers
Lecture 3
Building DSLs
Lecture 4
Collection-Oriented
Languages
Lecture 5
Dense Programming
Systems
Lecture 6
Sparse Programming
Systems
Lecture 7
Iteration Model I
Lecture 8
Iteration Model II

Terminology: Regular and Irregular

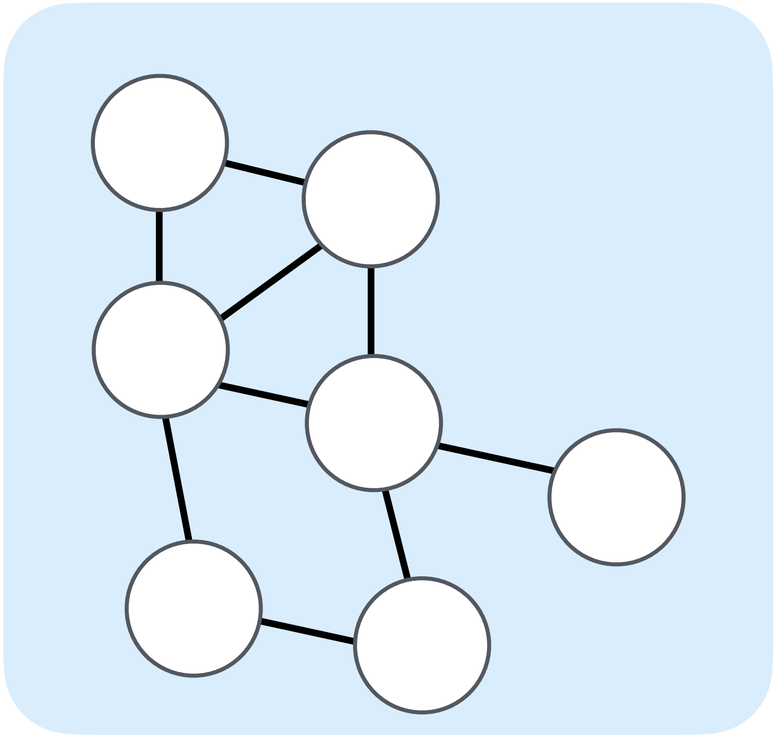
Fully Connected System



Regular System

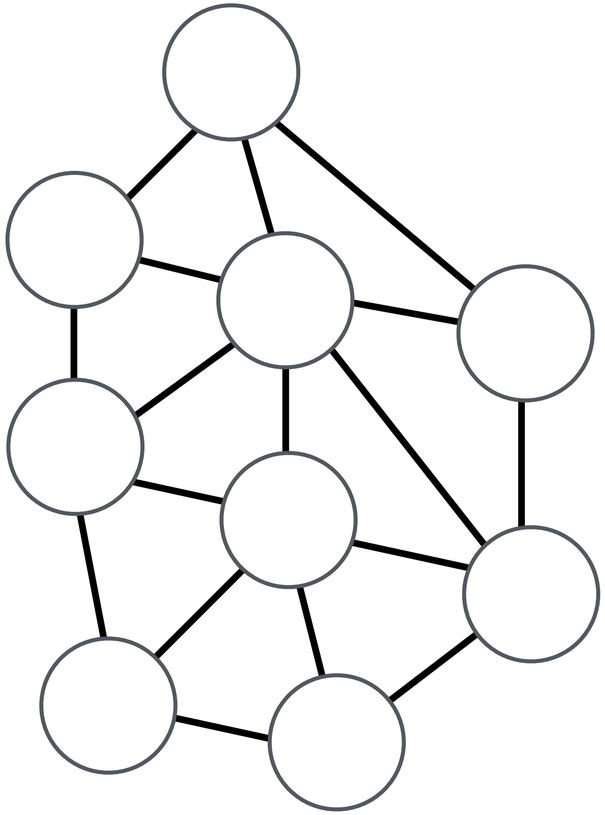


Irregular System

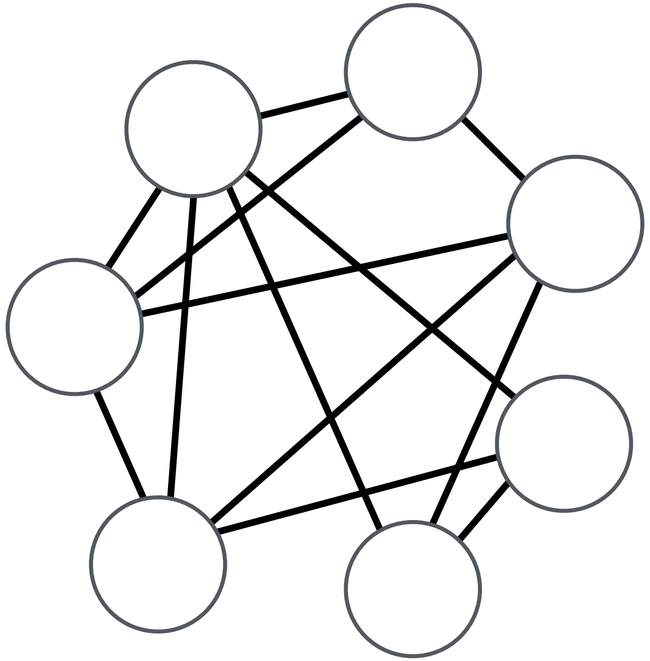


Three classes of irregular systems

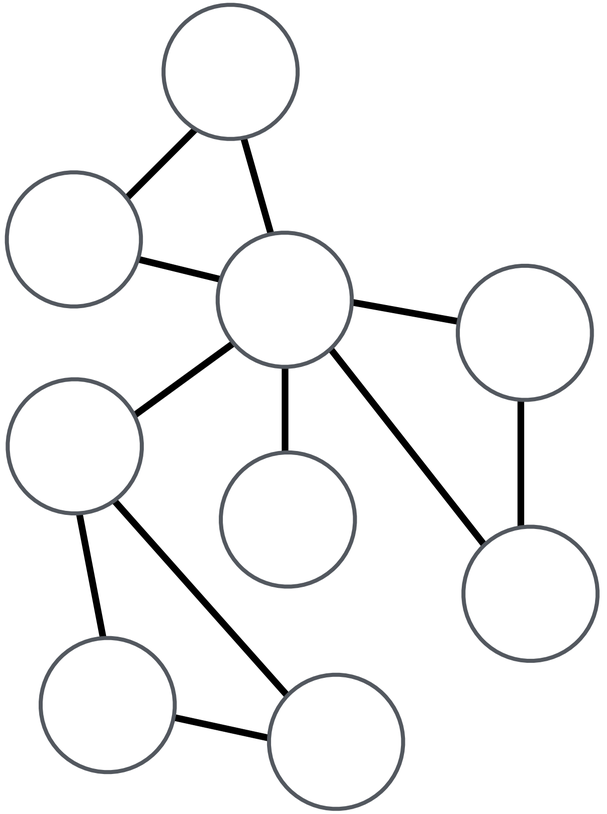
Road Networks



Fractional Sparsity



Power Law Graphs



How sparse is graph/relational data? Often asymptotically sparse.

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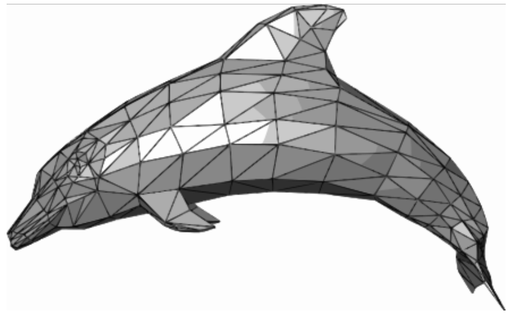
Conditioned Meshes



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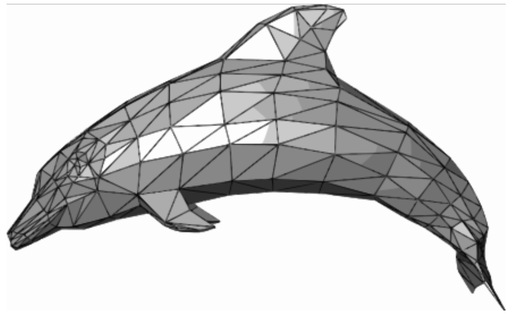

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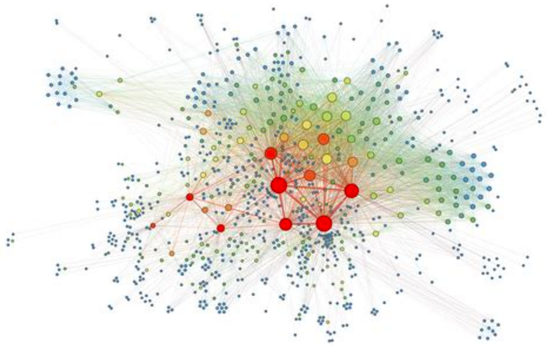

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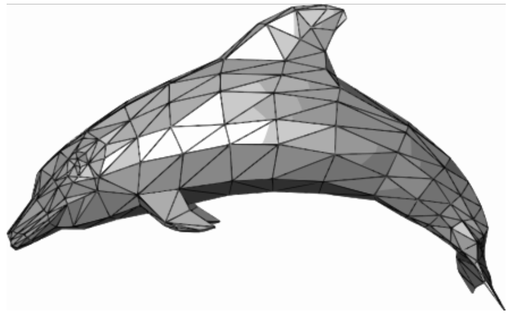

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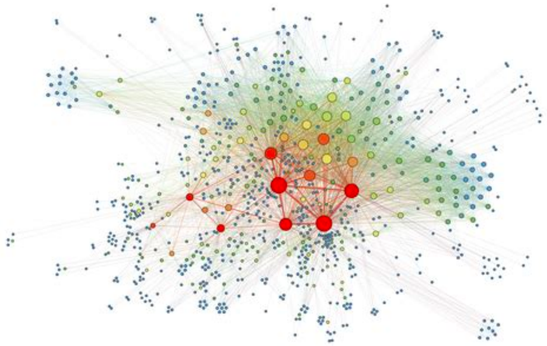

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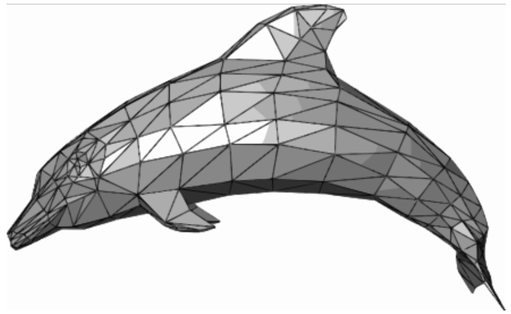

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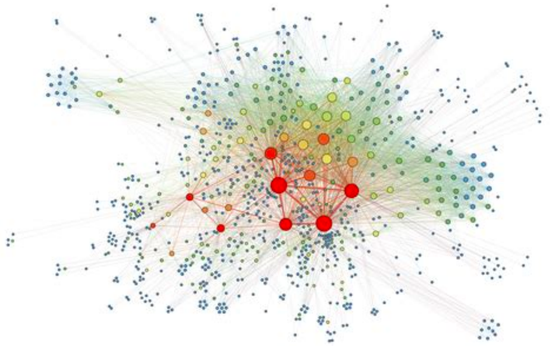

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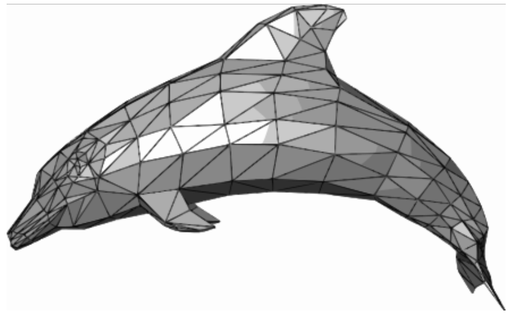

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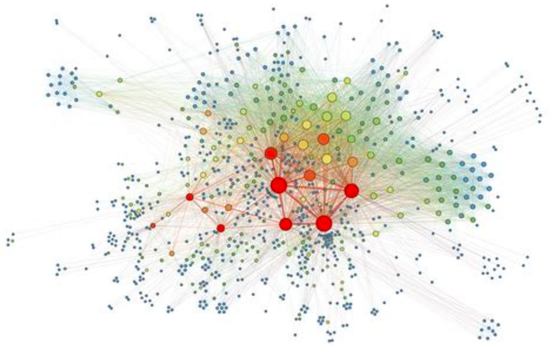

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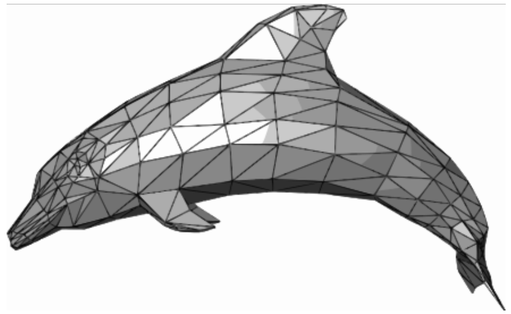

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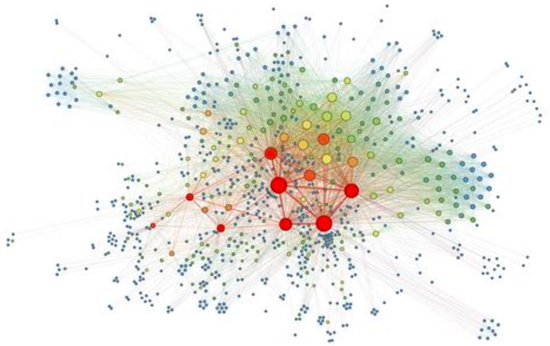

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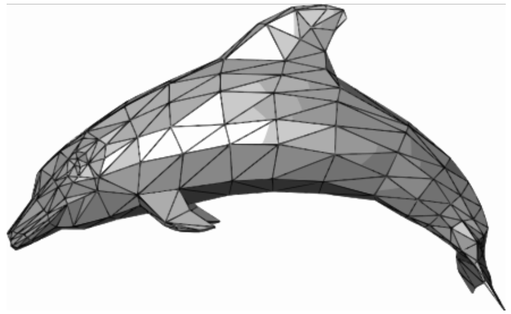

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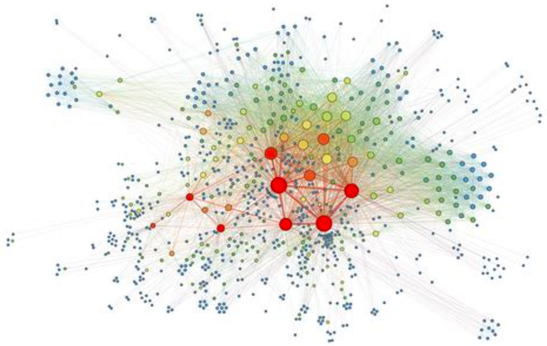

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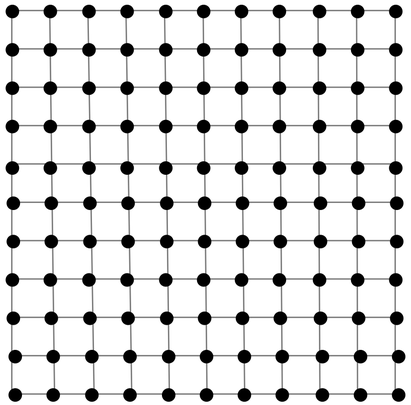
Matrix components: $O(n^2)$

Nonzeros: $O(n)$

Fraction of nonzeros: $O(1/n)$

Terminology: Dense and Sparse

Dense loop iteration space

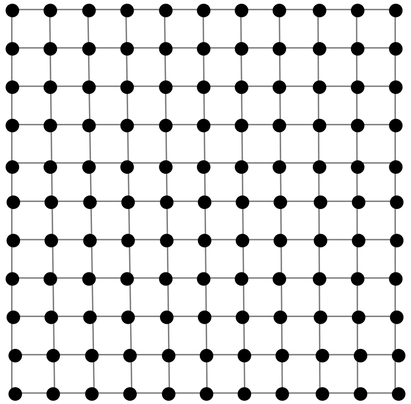


```
for (int i = 0; i < m; i++) {
  for (int j = 0; j < n; j++) {
    y[i] += A[i*n+j] * x[j];
  }
}
```

$$y = Ax$$

Terminology: Dense and Sparse

Dense loop iteration space



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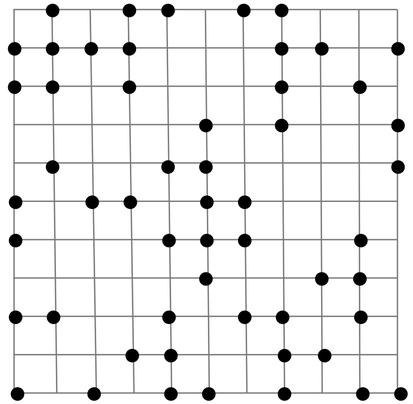
Sparse loop iteration space

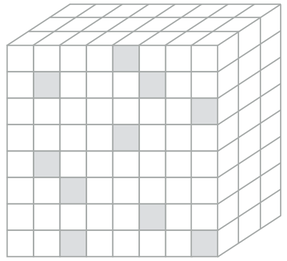


```
for (int i = 0; i < m; i++) {
 for (int pA = A2_pos[i]; pA < A_pos[i+1]; pA++) {
    int j = A_crd[pA];
    y[i] += A[pA] * x[j];
  }
}
```

$$y = Ax$$

Three sparse applications areas

Tensors



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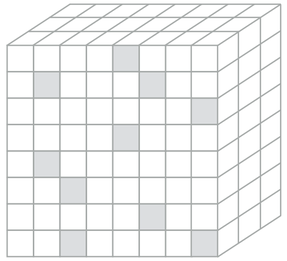

Relations

Names	City	Age
Peter	Boston	54
Mary	San Fransisco	35
Paul	New York	23
Adam	Seattle	84
Hilde	Boston	19
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Three sparse applications areas

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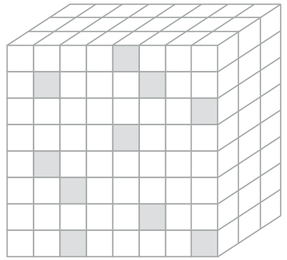

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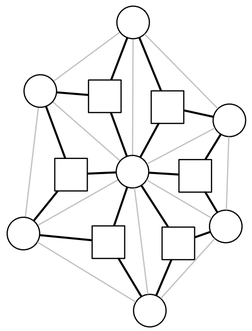
Graphs



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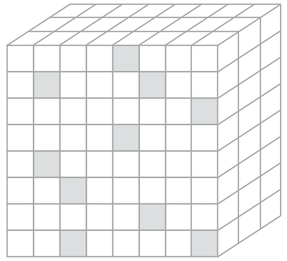

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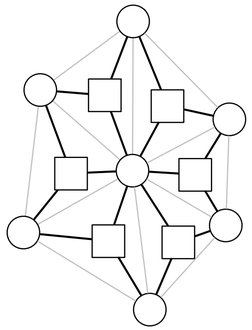
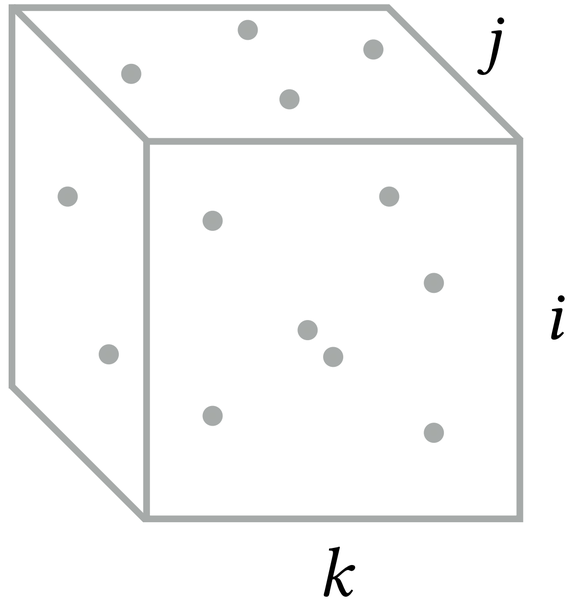


Nonzeros are a subset of the cartesian combination of sets

Sparse Iteration Spaces



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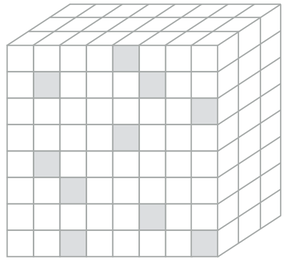

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A relation is a subset of the cartesian combination of sets

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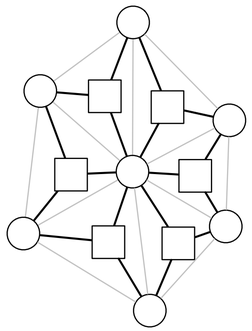

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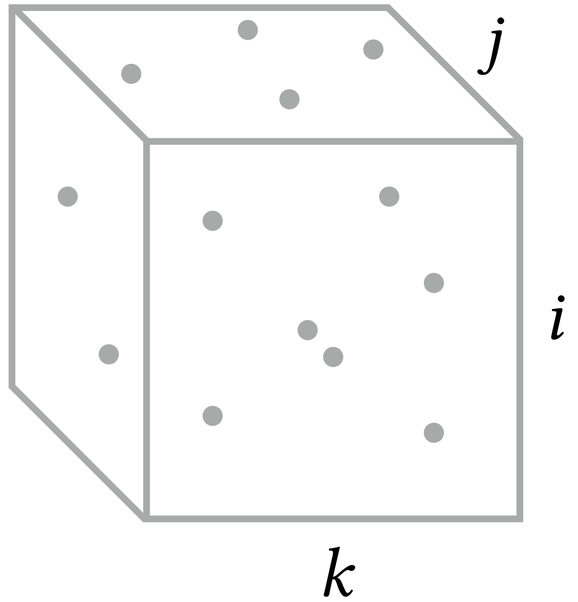

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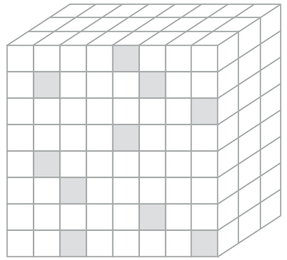

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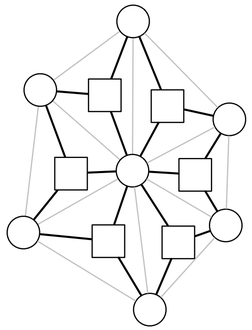
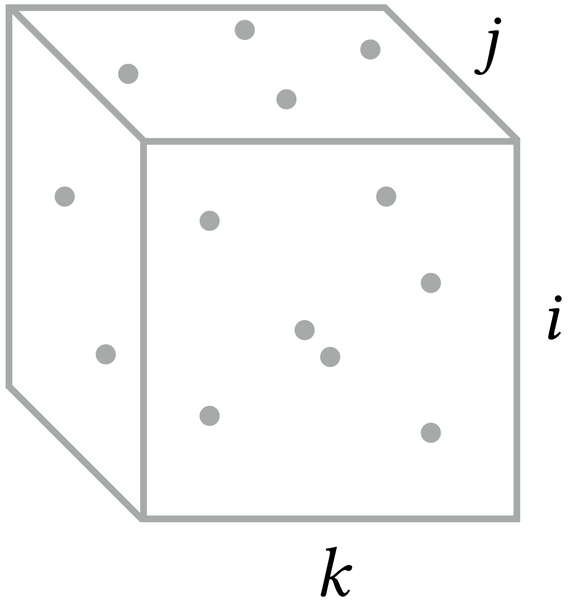


Graph edges are a subset of the cartesian combination of sets

Sparse Iteration Spaces



Relations, graphs, and tensors share a lot of structure but are specialized for different purposes

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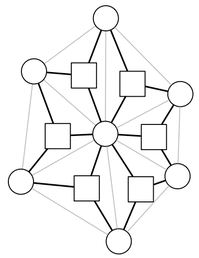

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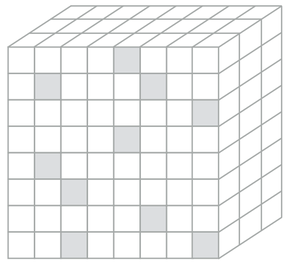

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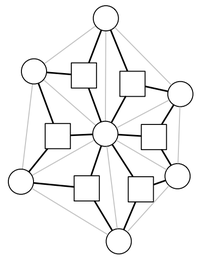
Combine data to form systems

Graphs



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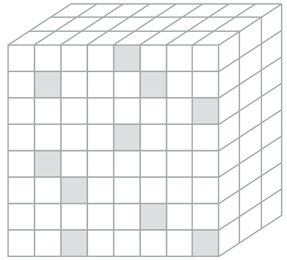

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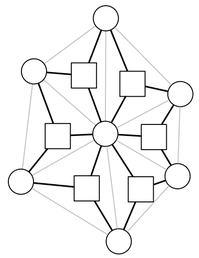
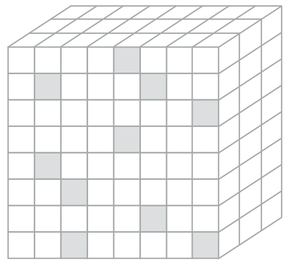


Local operations on systems

Tensors



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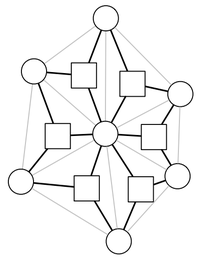
Paul	New York	23
Adam	Seattle	84
Hilde	Boston	19
Bob	Chicago	76
Sam	Portland	32
Angela	Los Angeles	62

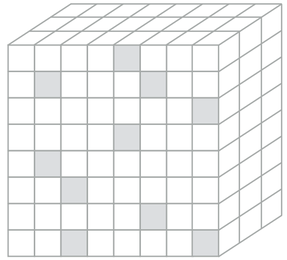
Combine data to form systems

Graphs



Local operations on systems

Tensors



Global operations on systems

Relations, graphs, and tensors share a lot of structure but are specialized for different purposes

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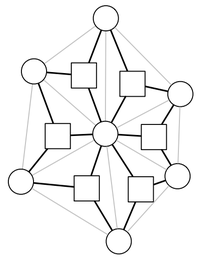
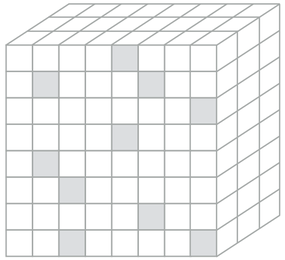
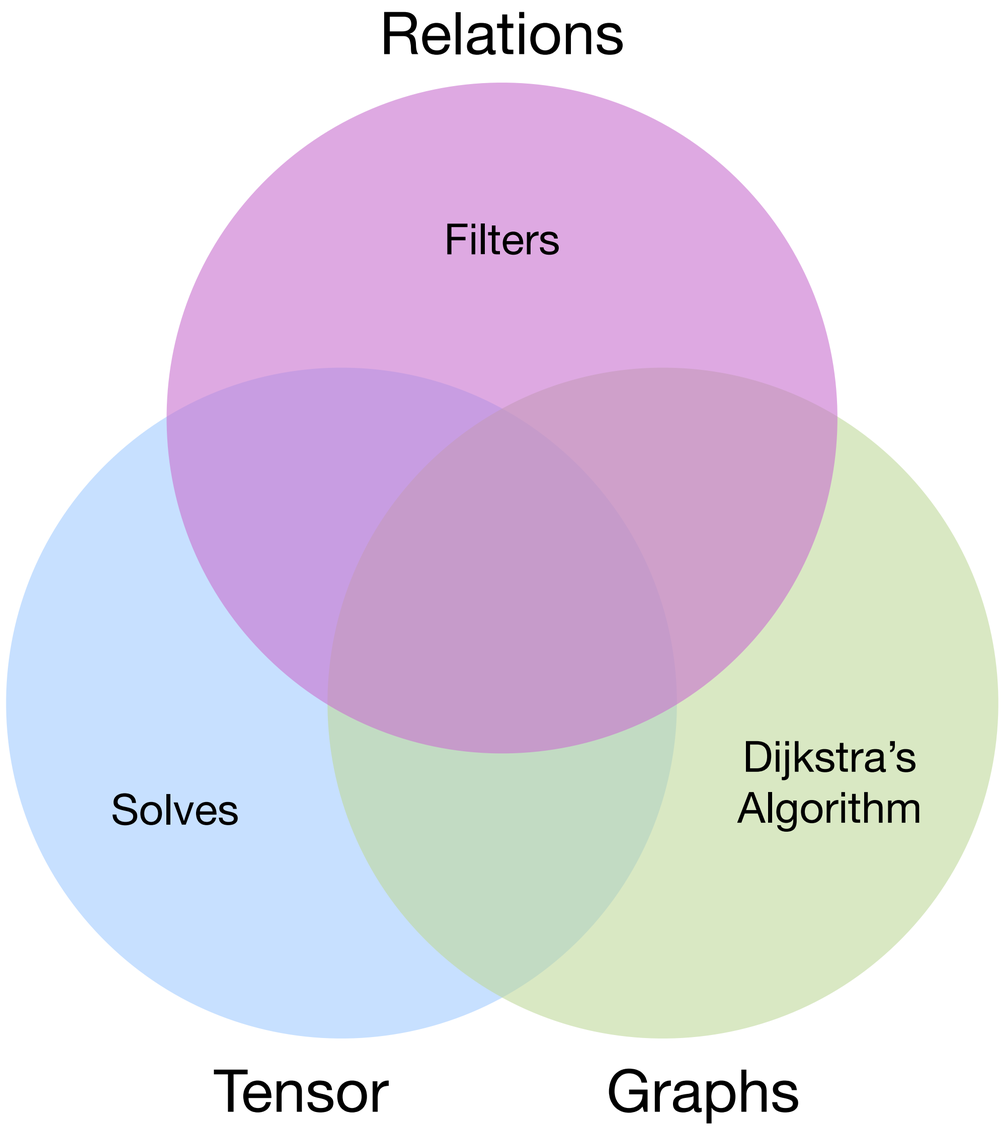


Local operations on systems

Tensors



Global operations on systems



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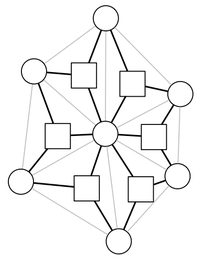
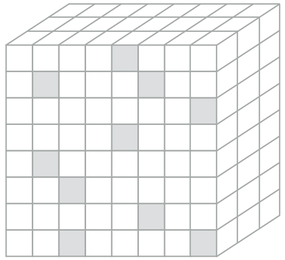
Combine data to form systems

Graphs



Local operations on systems

Tensors



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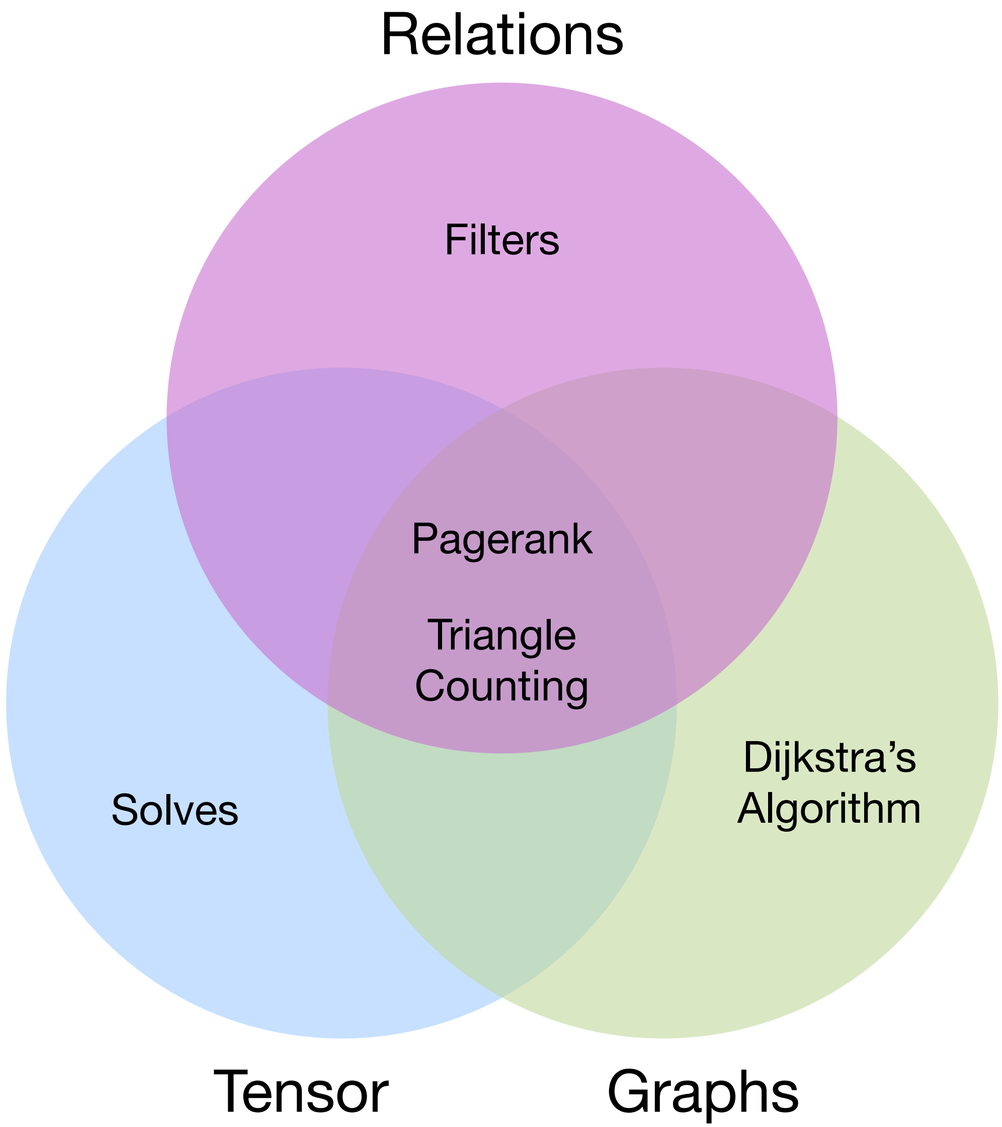
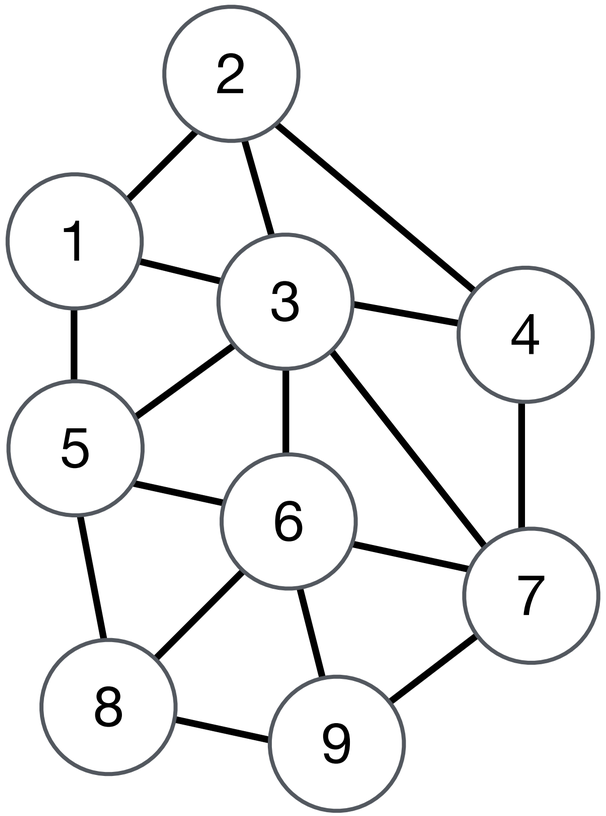


Triangle counting on graphs, relations, and tensors

On graphs



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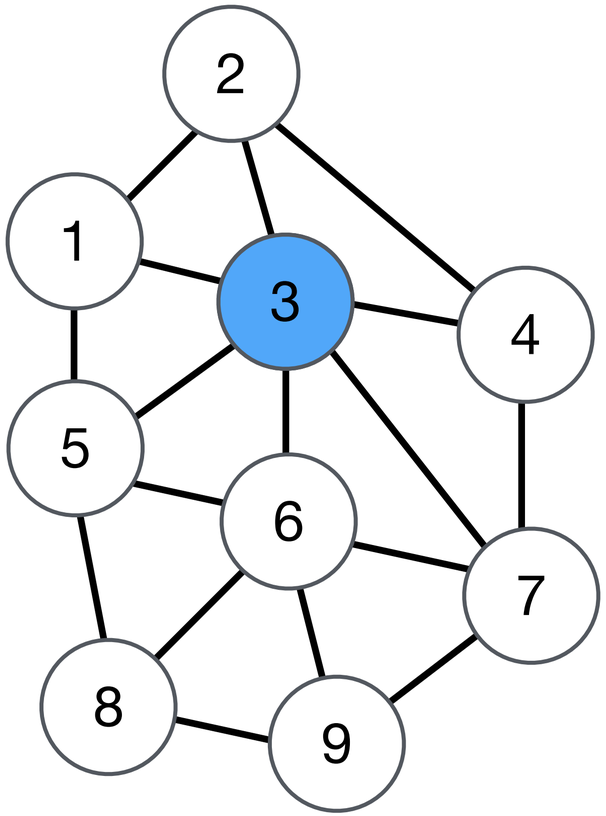

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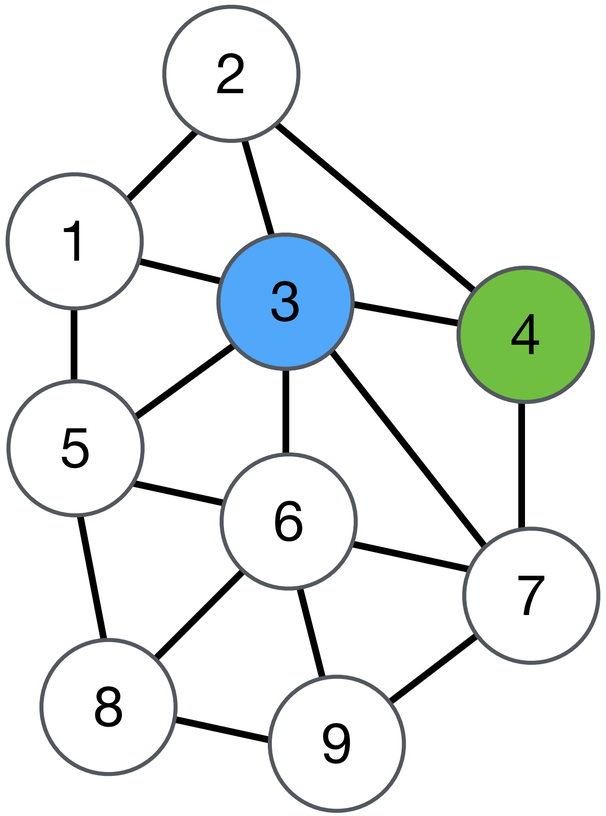

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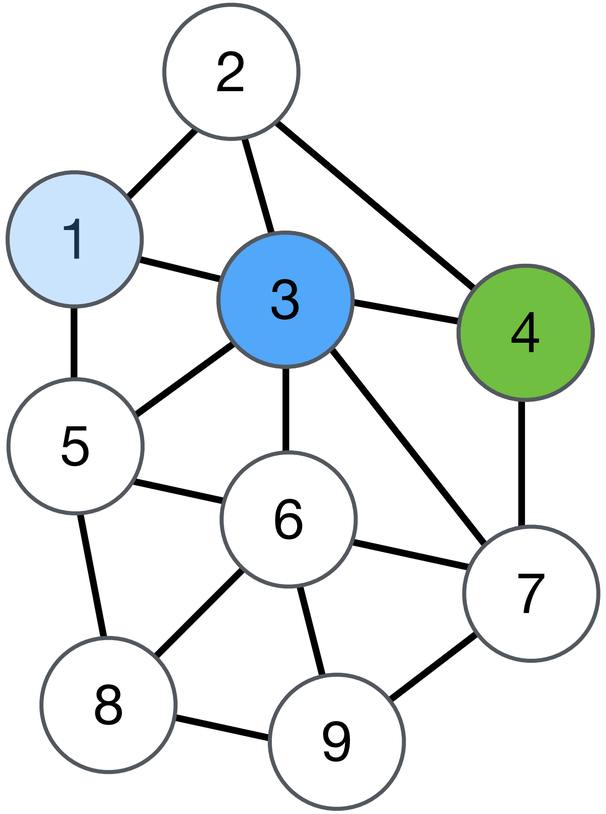
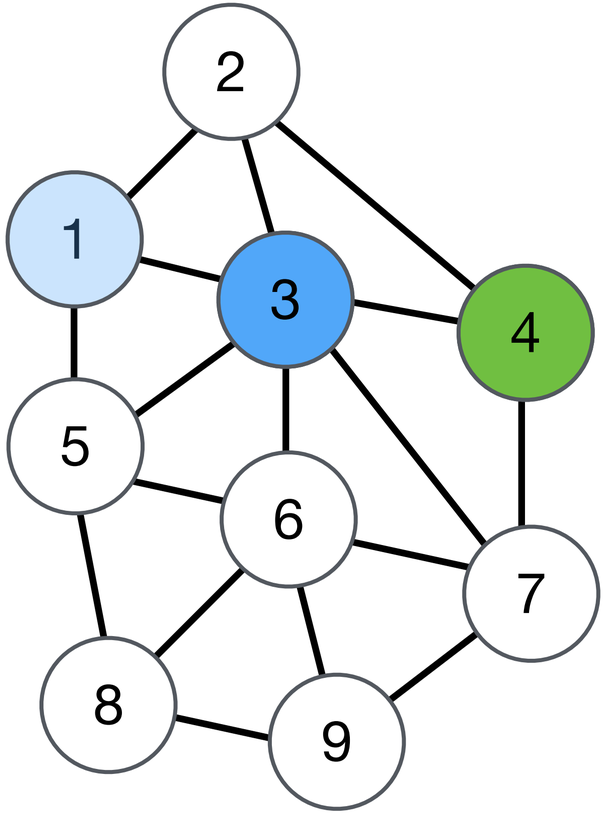


Triangle counting on graphs, relations, and tensors

On graphs



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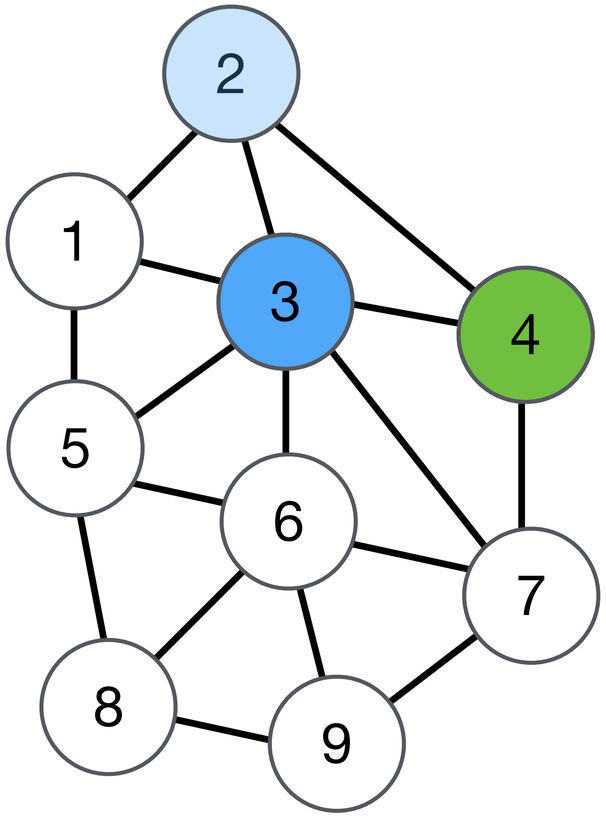

Triangle counting on graphs, relations, and tensors

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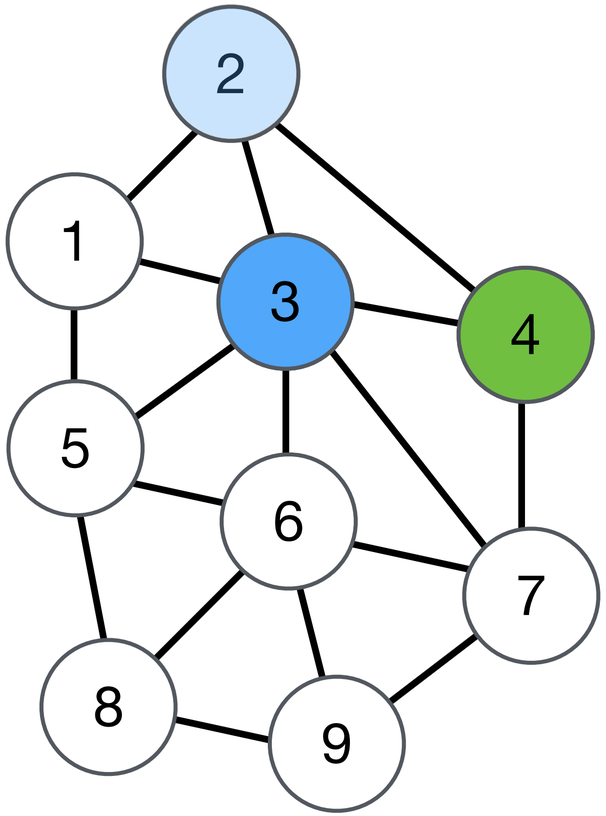

Triangle counting on graphs, relations, and tensors

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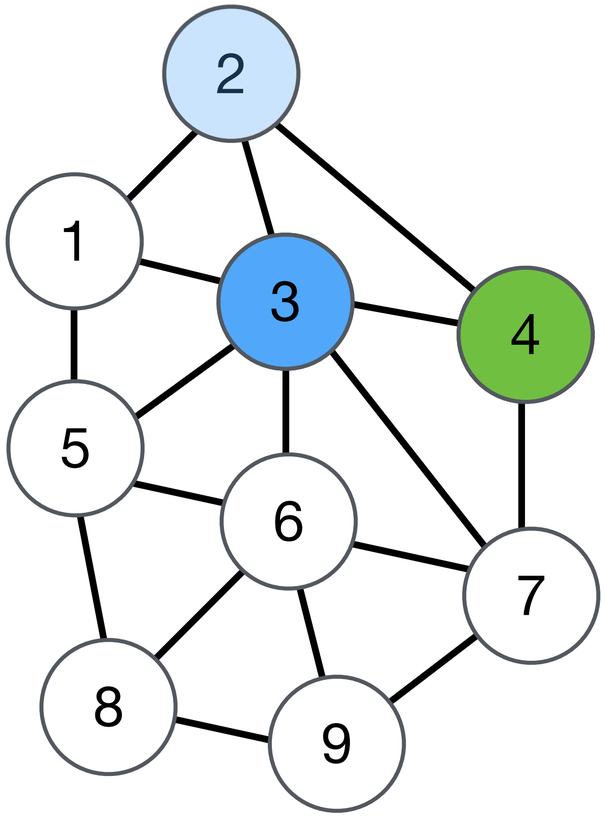
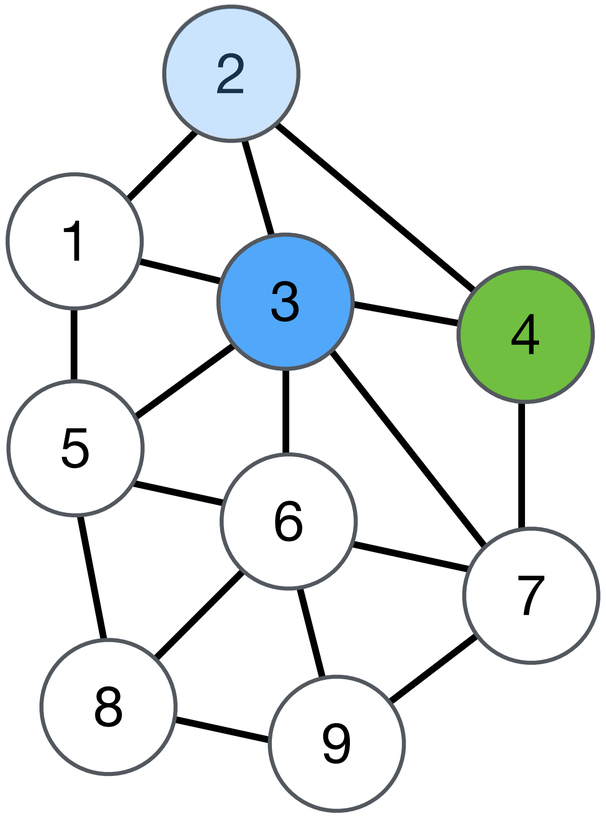


On relations

$$Q_{\triangle} = E(A, B) \bowtie E(B, C) \bowtie E(C, A)$$

Triangle counting on graphs, relations, and tensors

On graphs



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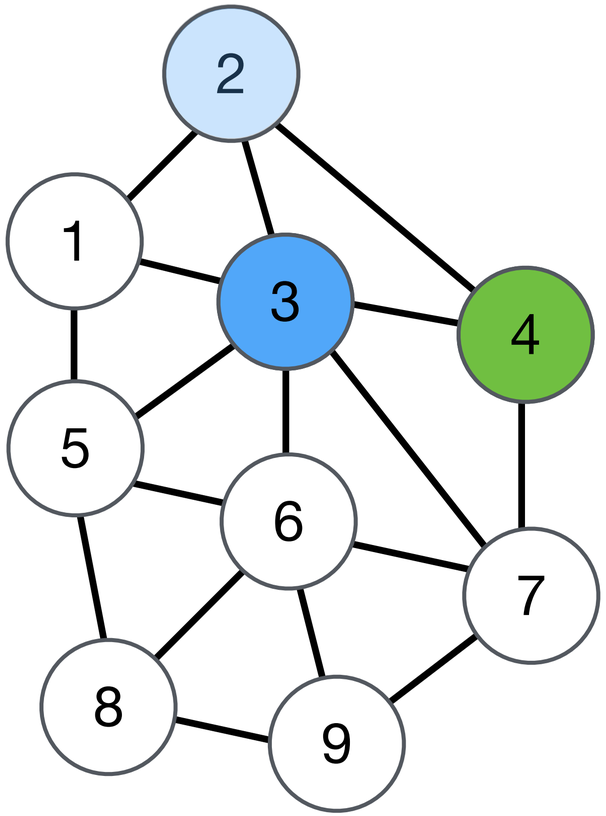

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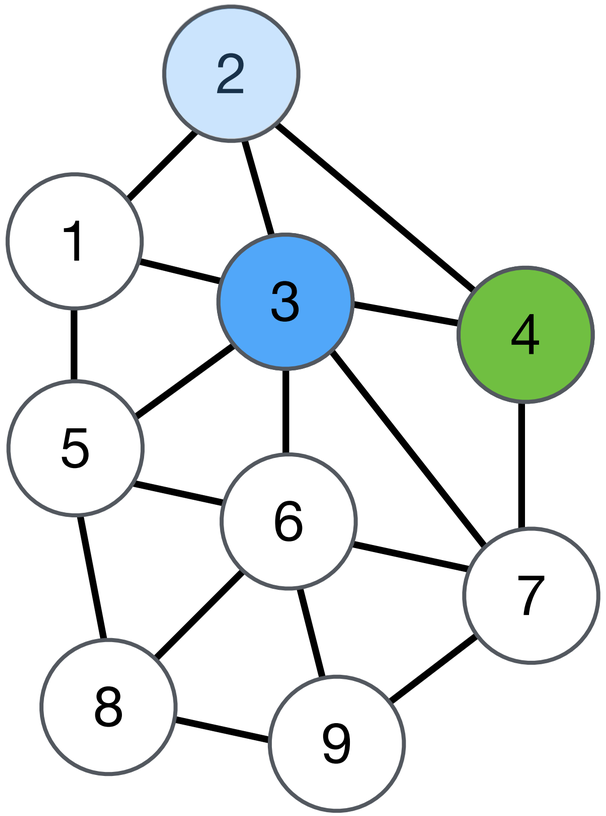

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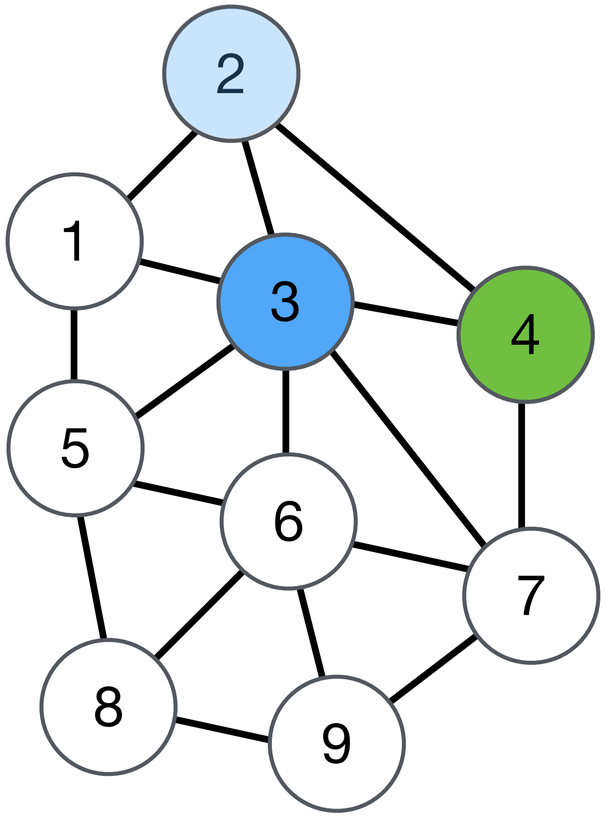


On relations

$$Q_{\triangle} = E(A,B) \bowtie E(B,C) \bowtie E(C,A)$$

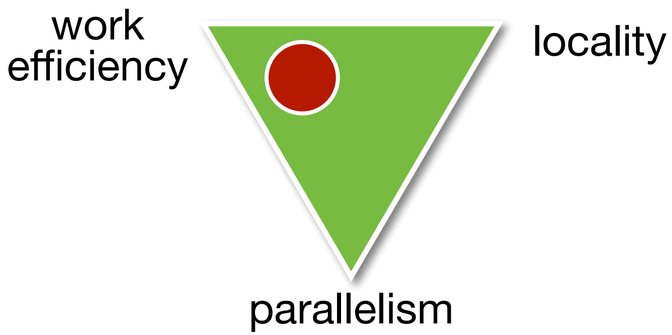
On tensors

$$\frac{1}{6}\text{trace}(A^3).$$

Some important developments in compilers and programming languages for sparse compilers

- 1960s: Development of libraries for sparse linear algebra
- 1970s: Relational algebra and the first relational database management systems: System R and INGRES
- 1980s: SQL is developed and has commercial success
- 1990s: Matlab gets sparse matrices and some dense to sparse linear algebra compilers are developed
- 2000s: Sparse linear algebra libraries for supercomputers and GPUs
- 2010s: Graph processing libraries become popular, compilers for databases, and compilers for sparse tensor algebra

Parallelism, locality, work efficiency still matters, but the key is choosing efficient data structures



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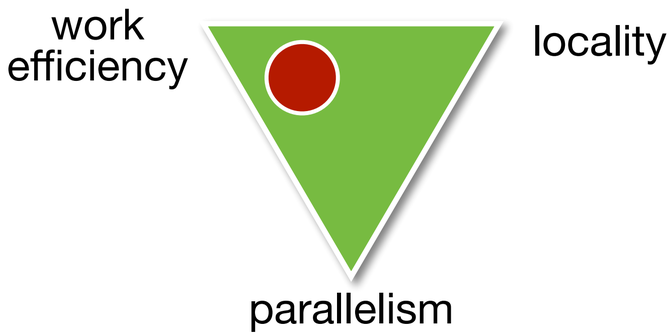

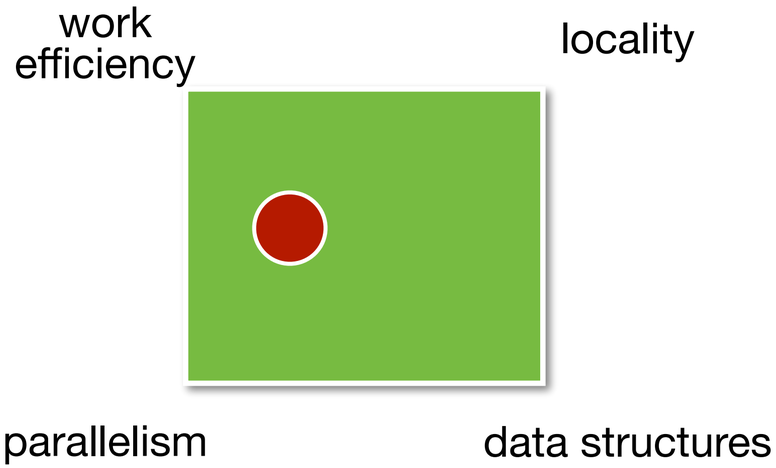

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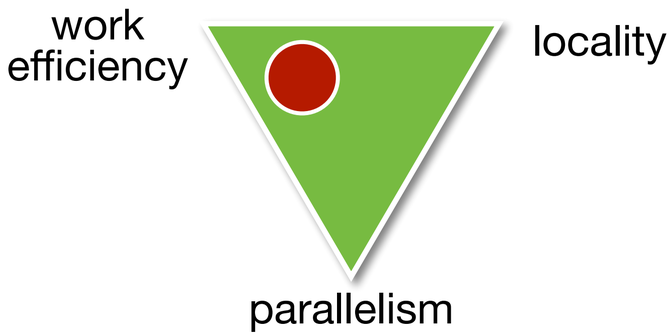

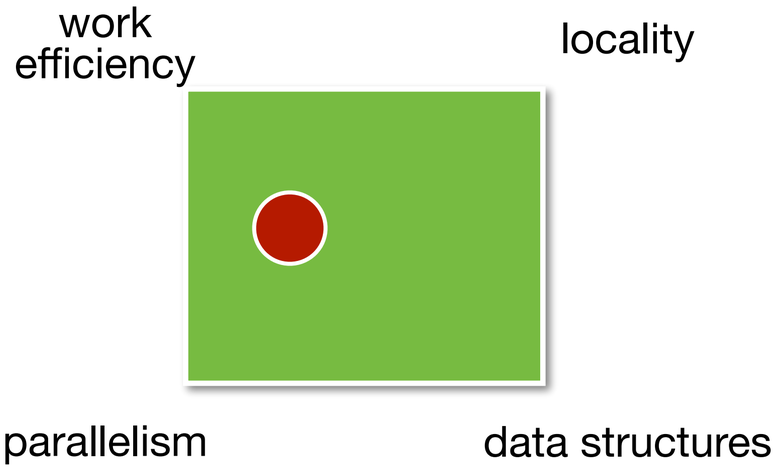

Harry	CS
Sally	EE
George	CS
Mary	ME
Rita	CS

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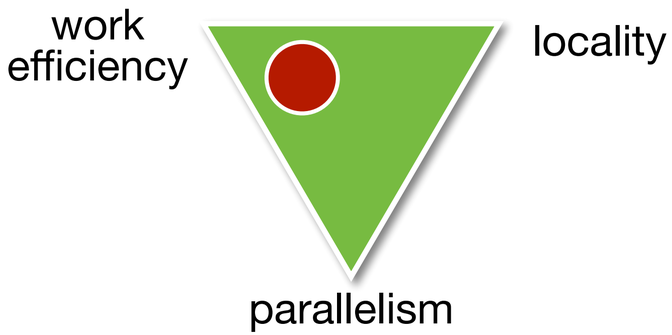

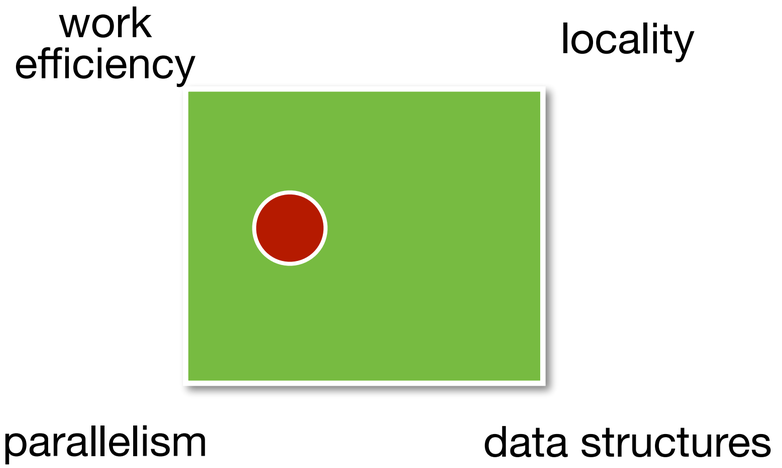

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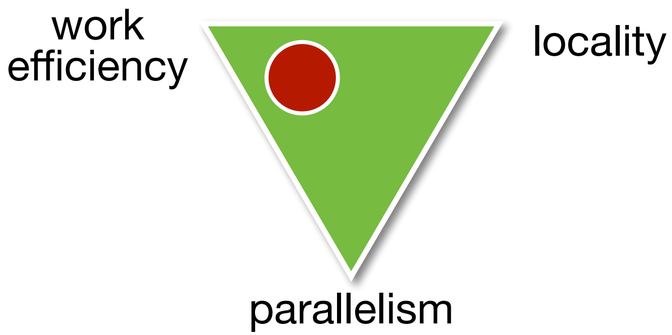

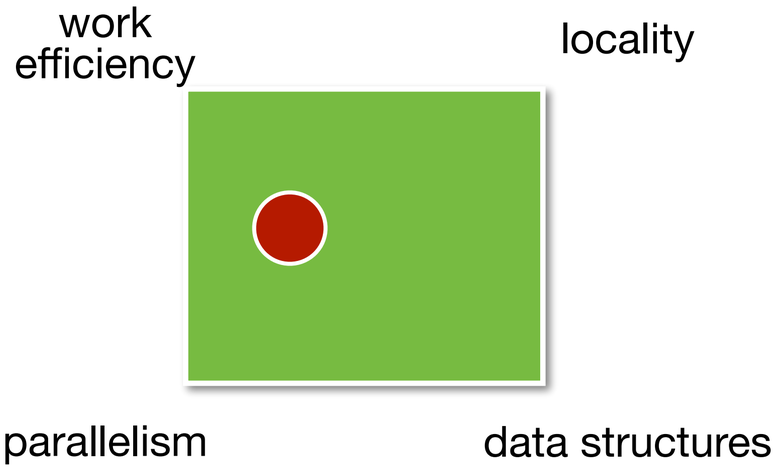

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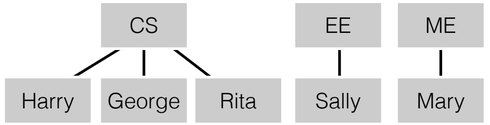


Sparse data structures in graphs, tensors, and relations encode coordinates in a sparse iteration space

Tensor (nonzeros)

(0,1)

(2,3)

(0,5)

(5,5)

(7,5)

Relation (rows)

(Harry,CS)

(Sally,EE)

(George,CS)

(Mary,ME)

(Rita,CS)

Graph (edges)

(v_1,v_5)

(v_4,v_3)

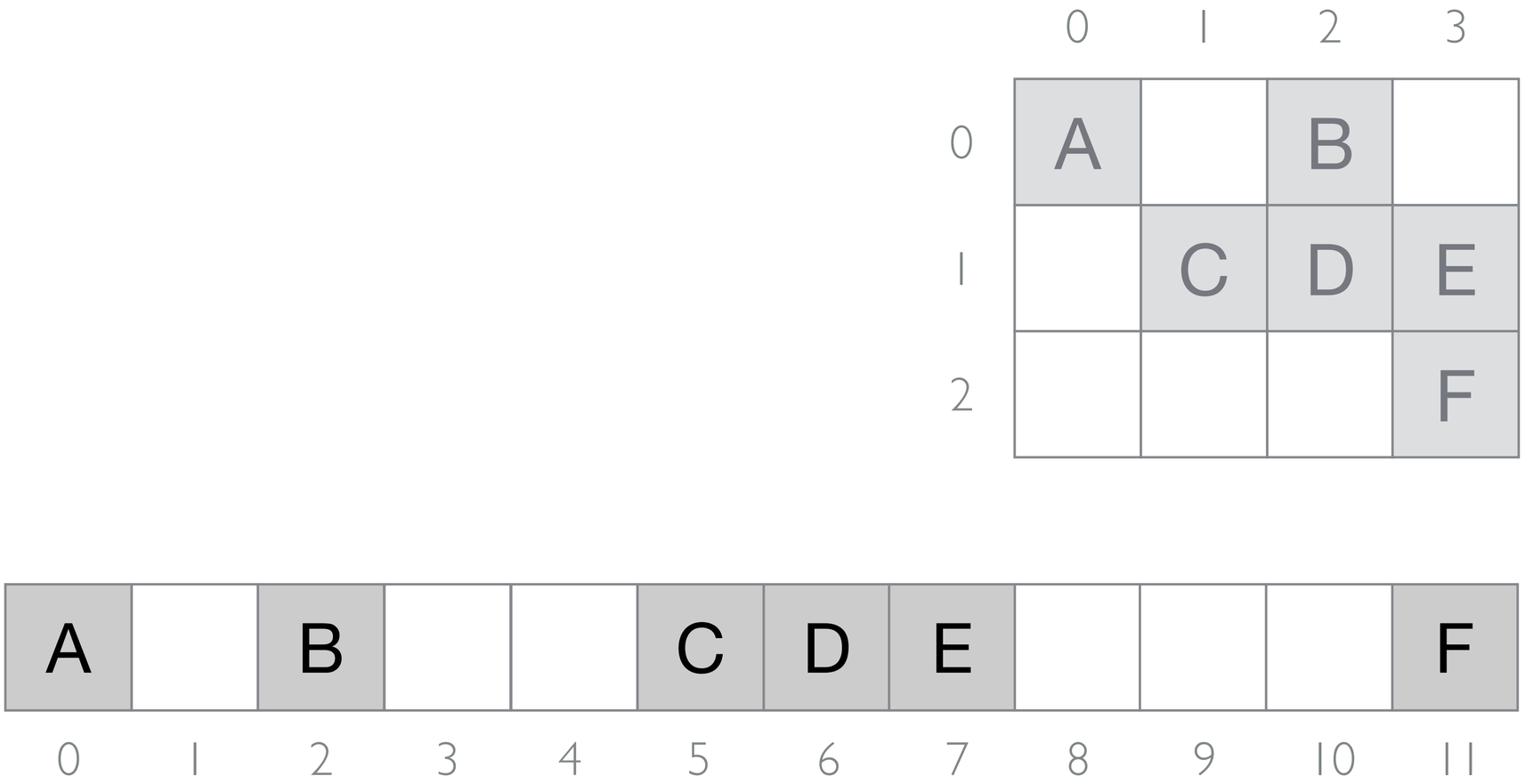
(v_5,v_3)

(v_3,v_1)

(v_3,v_5)

Values may be attached to these coordinates: e.g., nonzero values, edge attributes

Hierarchically compressed data structures (tries) reduce the number of values that need to be stored



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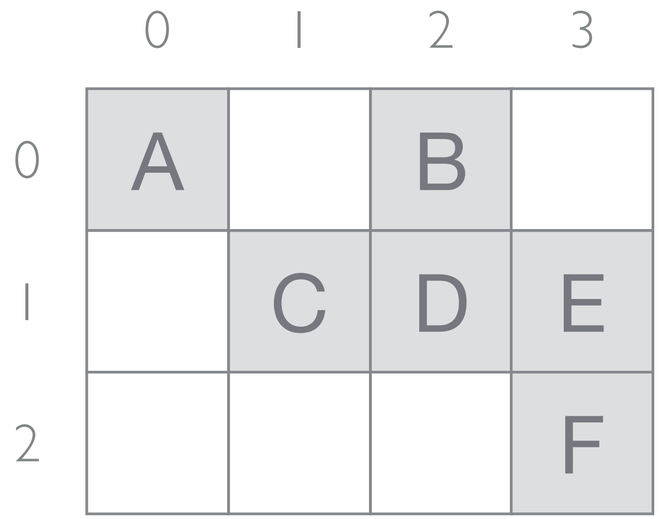
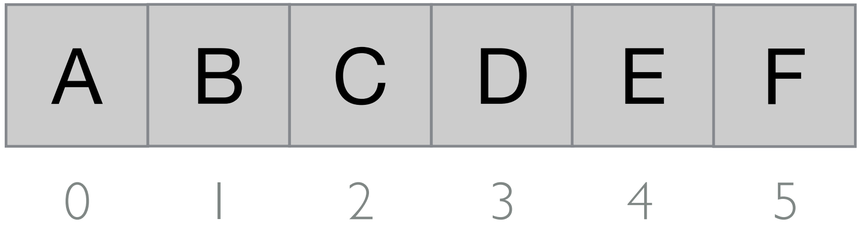
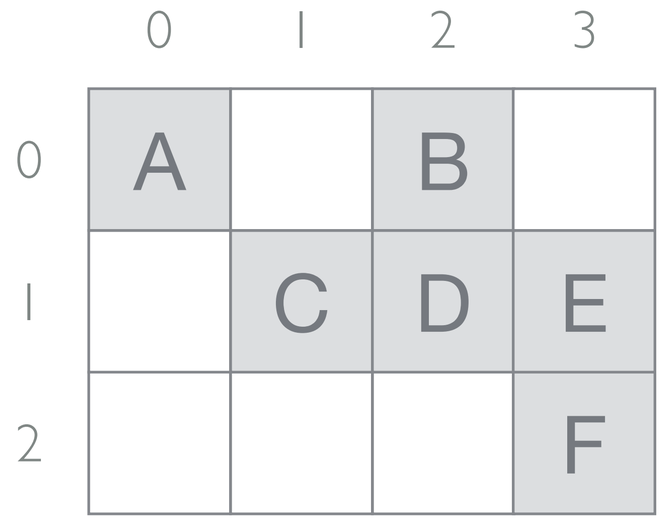
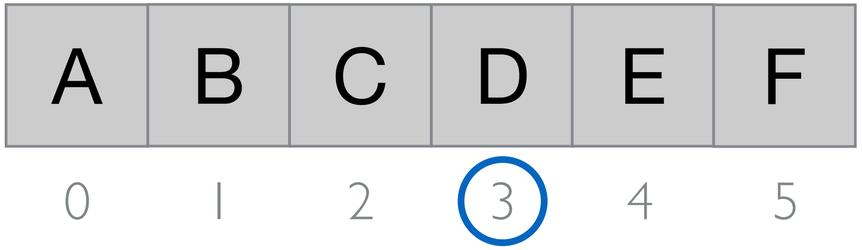




Hierarchically compressed data structures (tries) reduce the number of values that need to be stored

```
row(3) = ???
col(3) = ???
```





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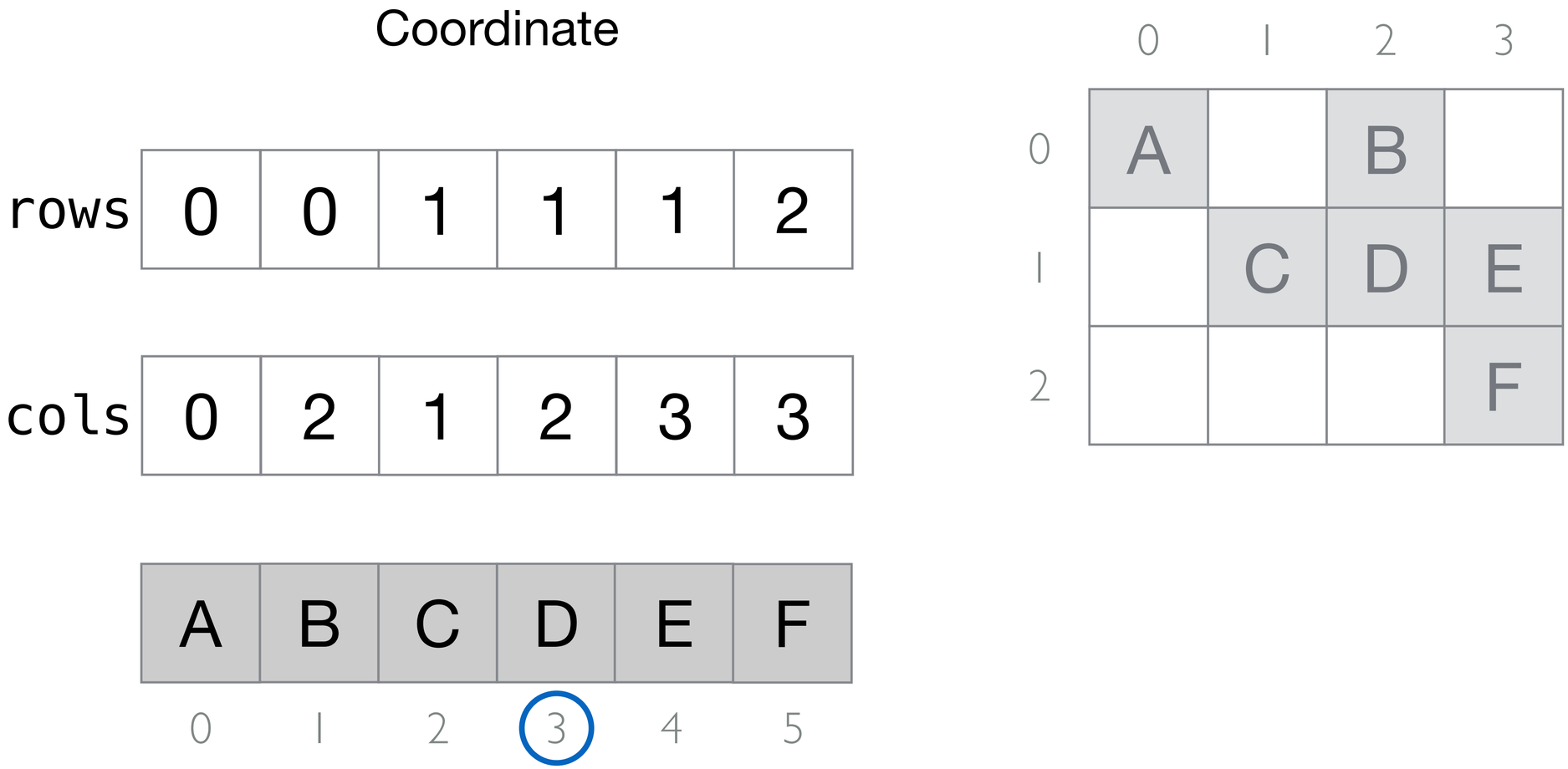

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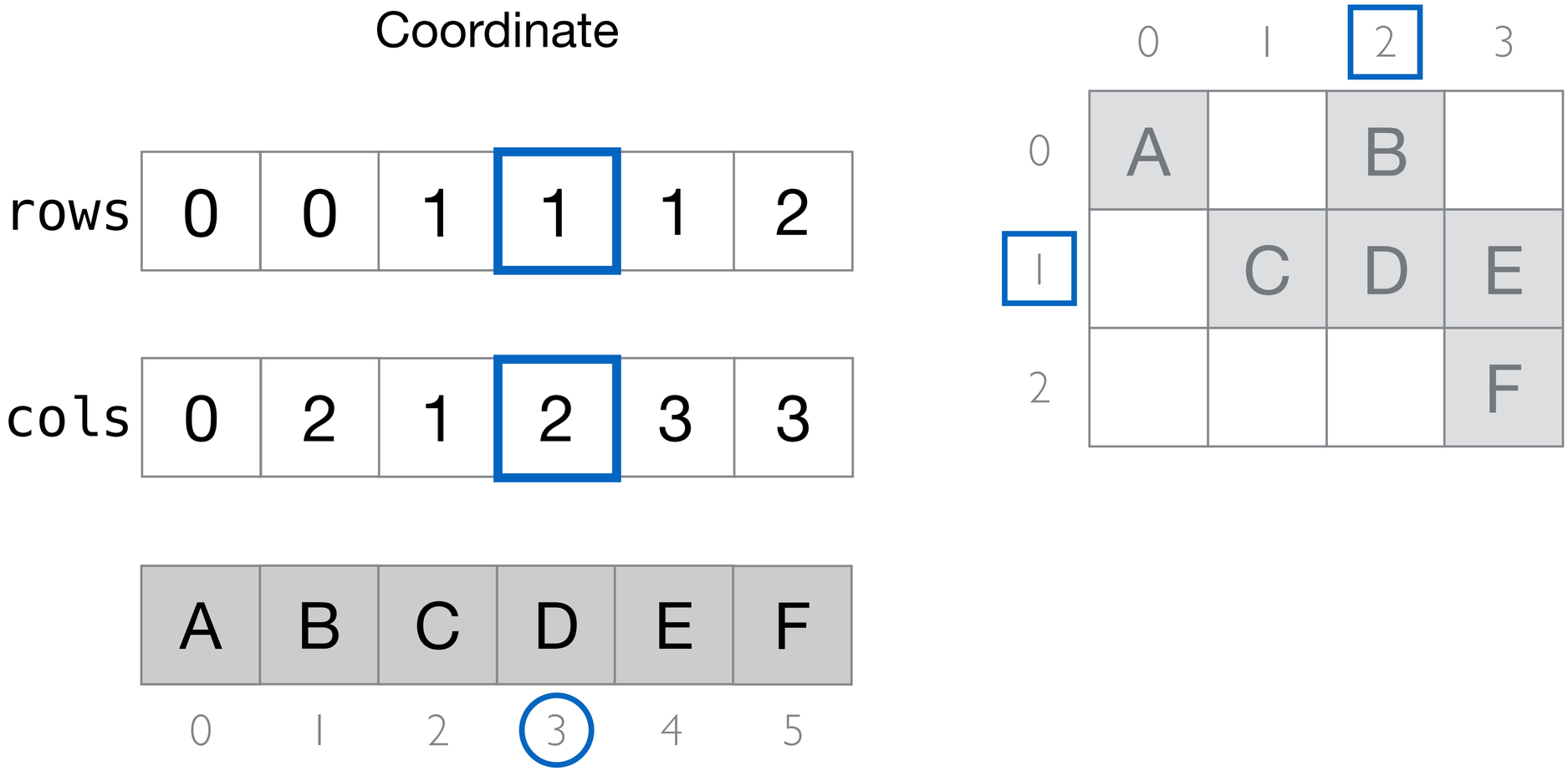

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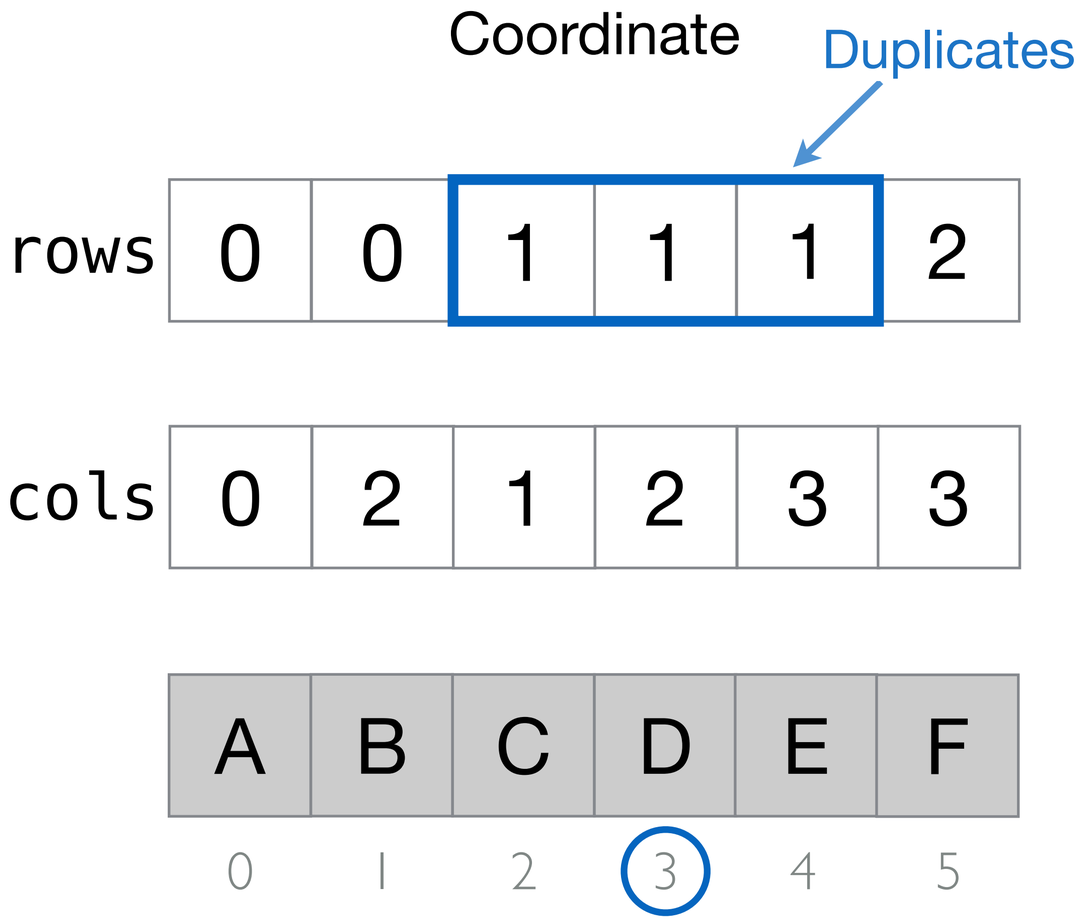

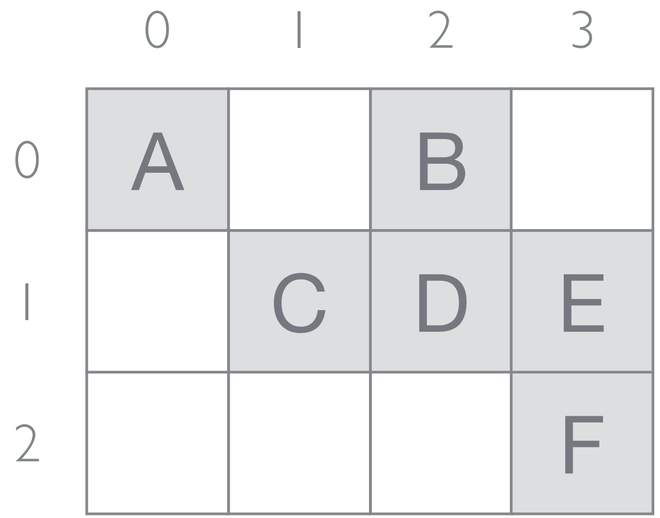

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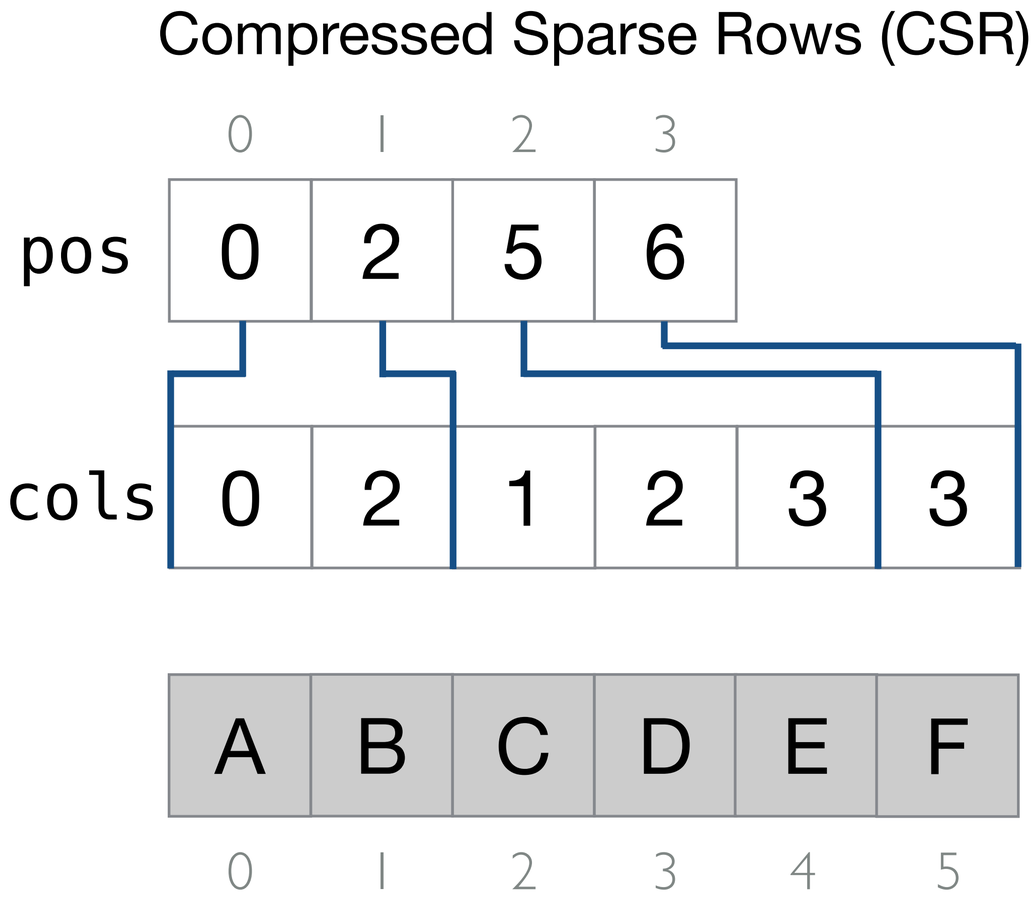

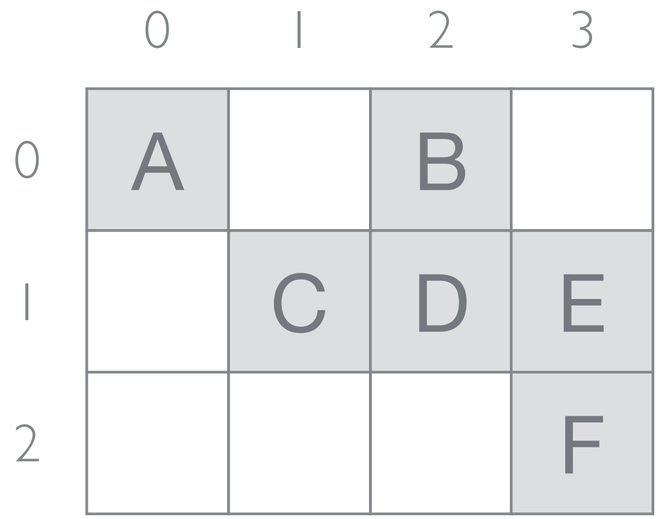

Iteration over sparse iteration spaces imply coiteration over sparse data structures

Linear Algebra: $A = B + C$

Iteration over sparse iteration spaces imply coiteration over sparse data structures

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Tensor Index Notation: $A_{ij} = B_{ij} + C_{ij}$

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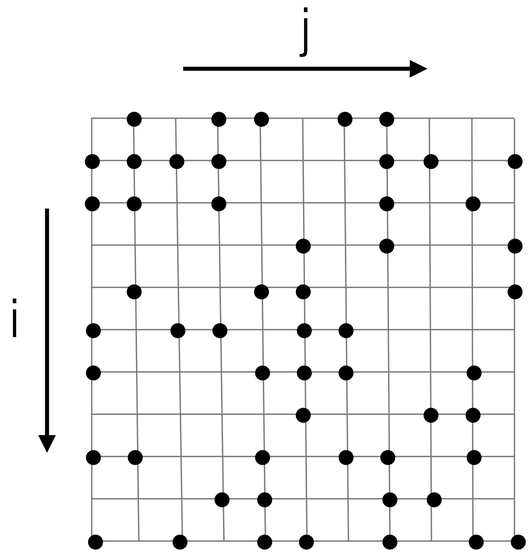

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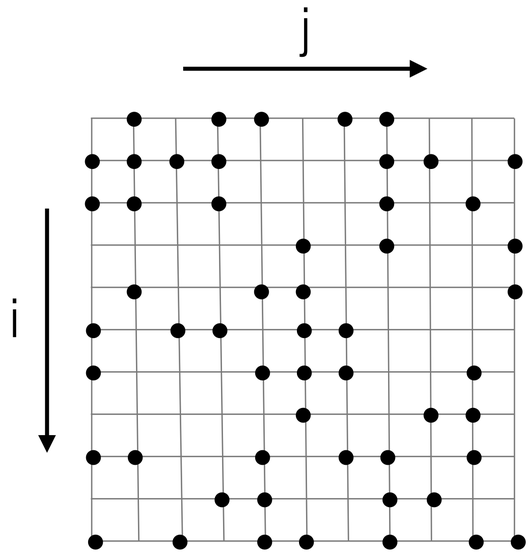


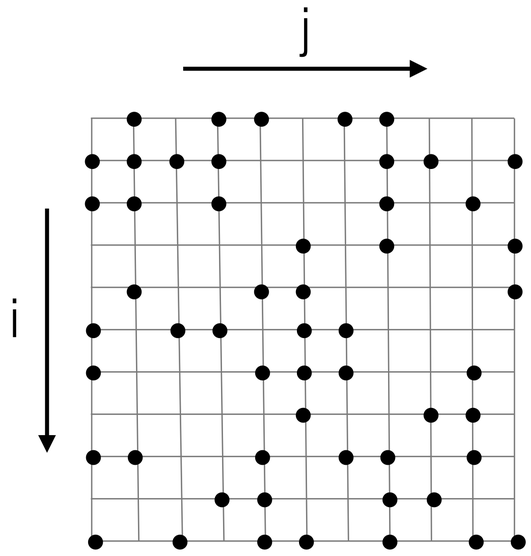
B

pos	0	3	5	8				
cols	0	2	3	0	2	1	2	3
vals	30	40	50	10	70	80	20	60

Iteration over sparse iteration spaces imply coiteration over sparse data structures

Linear Algebra: $A = B + C$

Tensor Index Notation: $A_{ij} = B_{ij} + C_{ij}$



B

pos	0	3	5	8				
cols	0	2	3	0	2	1	2	3
vals	30	40	50	10	70	80	20	60

C

pos	0	1	3
cols	0	2	3
vals	2	1	9

Iteration over sparse iteration spaces imply coiteration over sparse data structures

Linear Algebra: $A = B + C$

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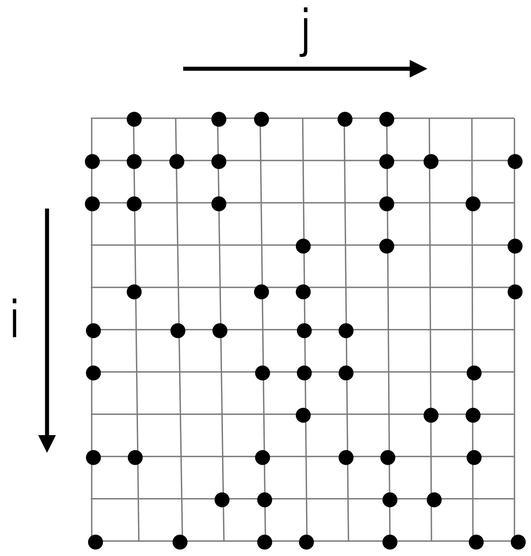

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coiterate

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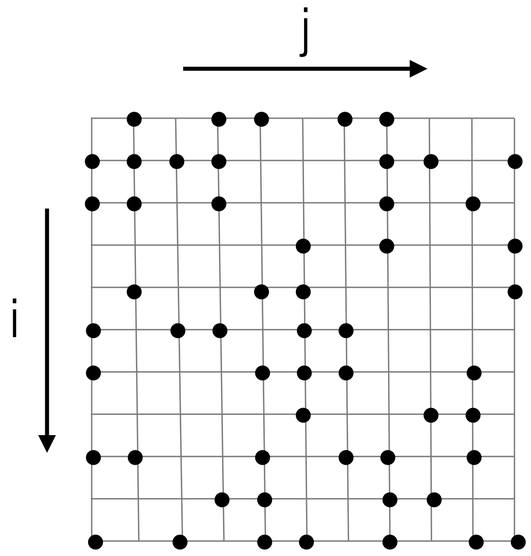

union because $x + 0 = x$

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cols	0	2	3	0	2	1	2	3
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∪

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pos	0	1	3
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coiterate

Iteration over sparse iteration spaces imply coiteration over sparse data structures

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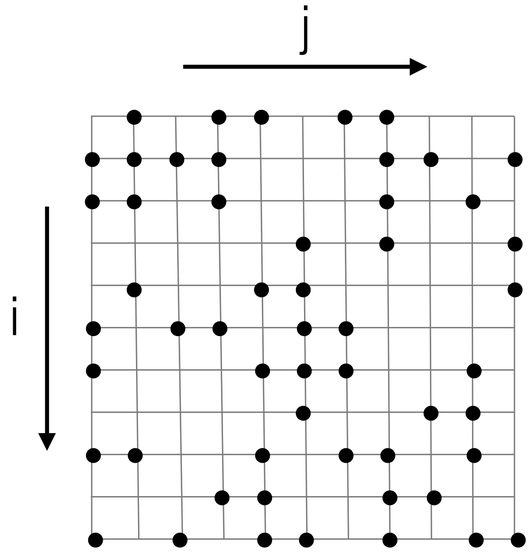
Iteration Space: $B_{ij} \cup C_{ij}$



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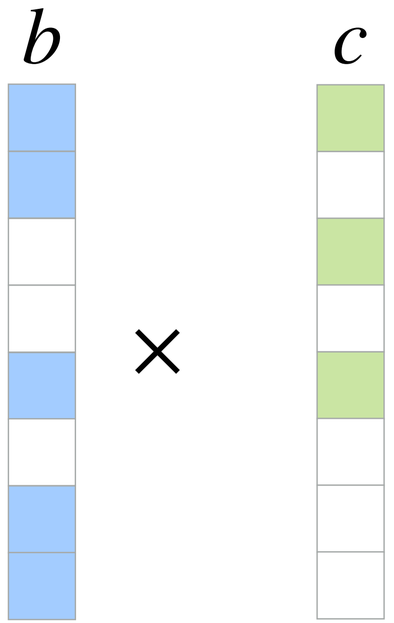
coiterate

Merged coiteration

Coordinate Space

$$a_i = b_i \, c_i$$



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Coordinate Space

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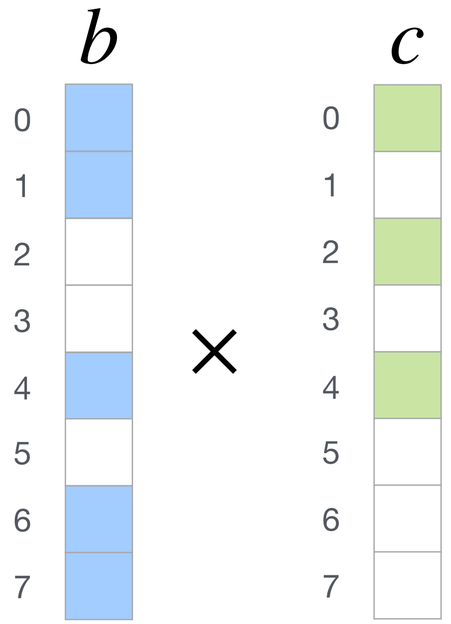

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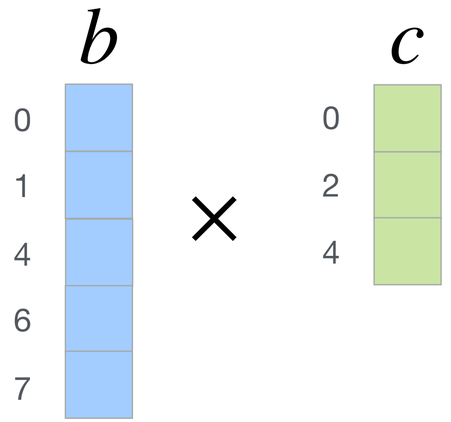

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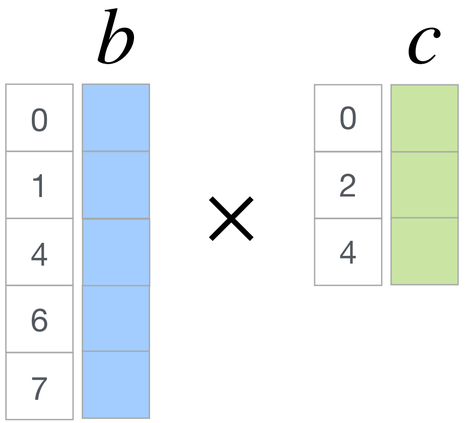
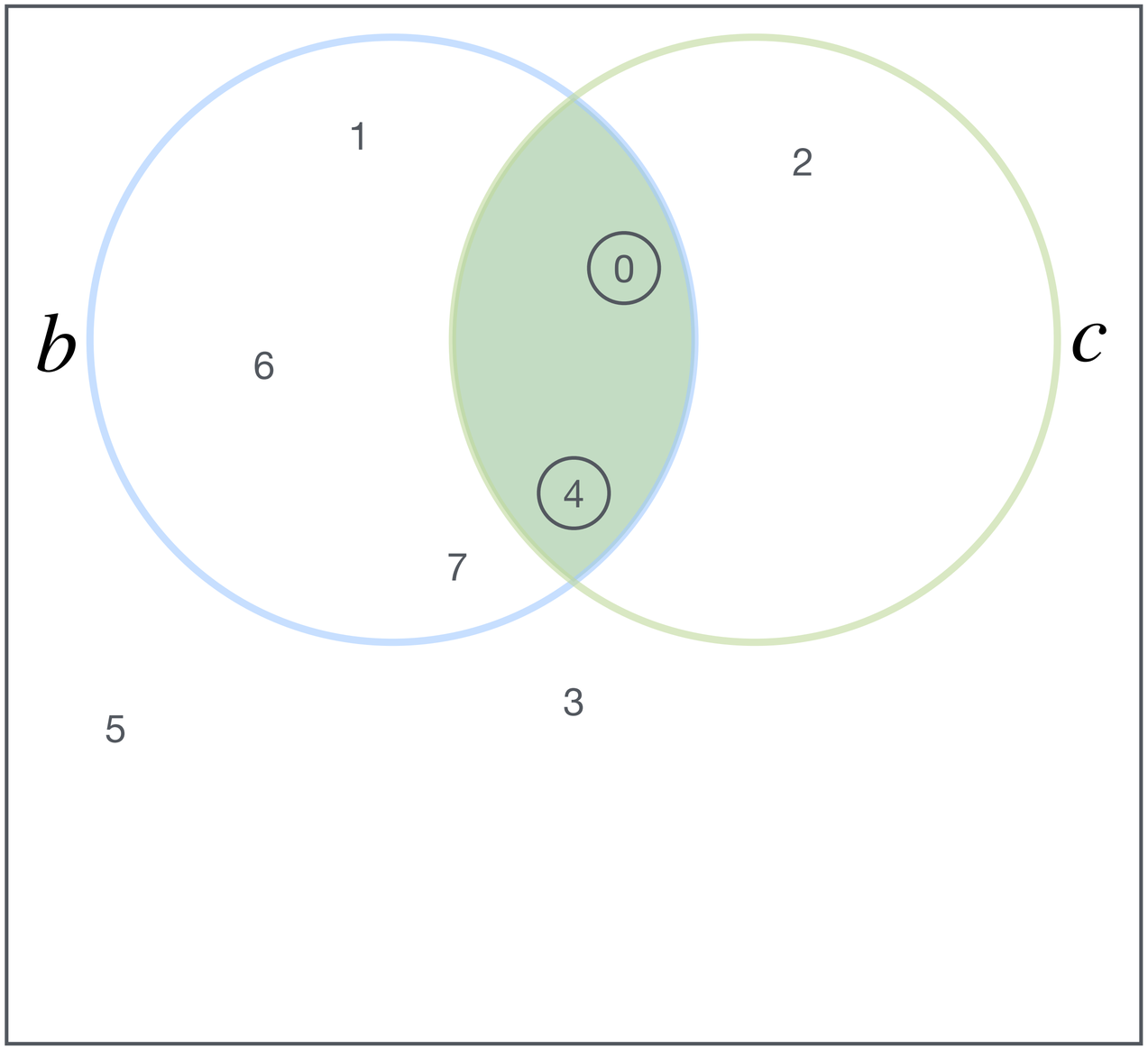


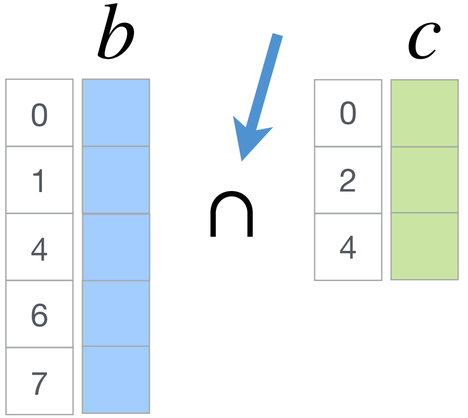
Merged coiteration

Coordinate Space

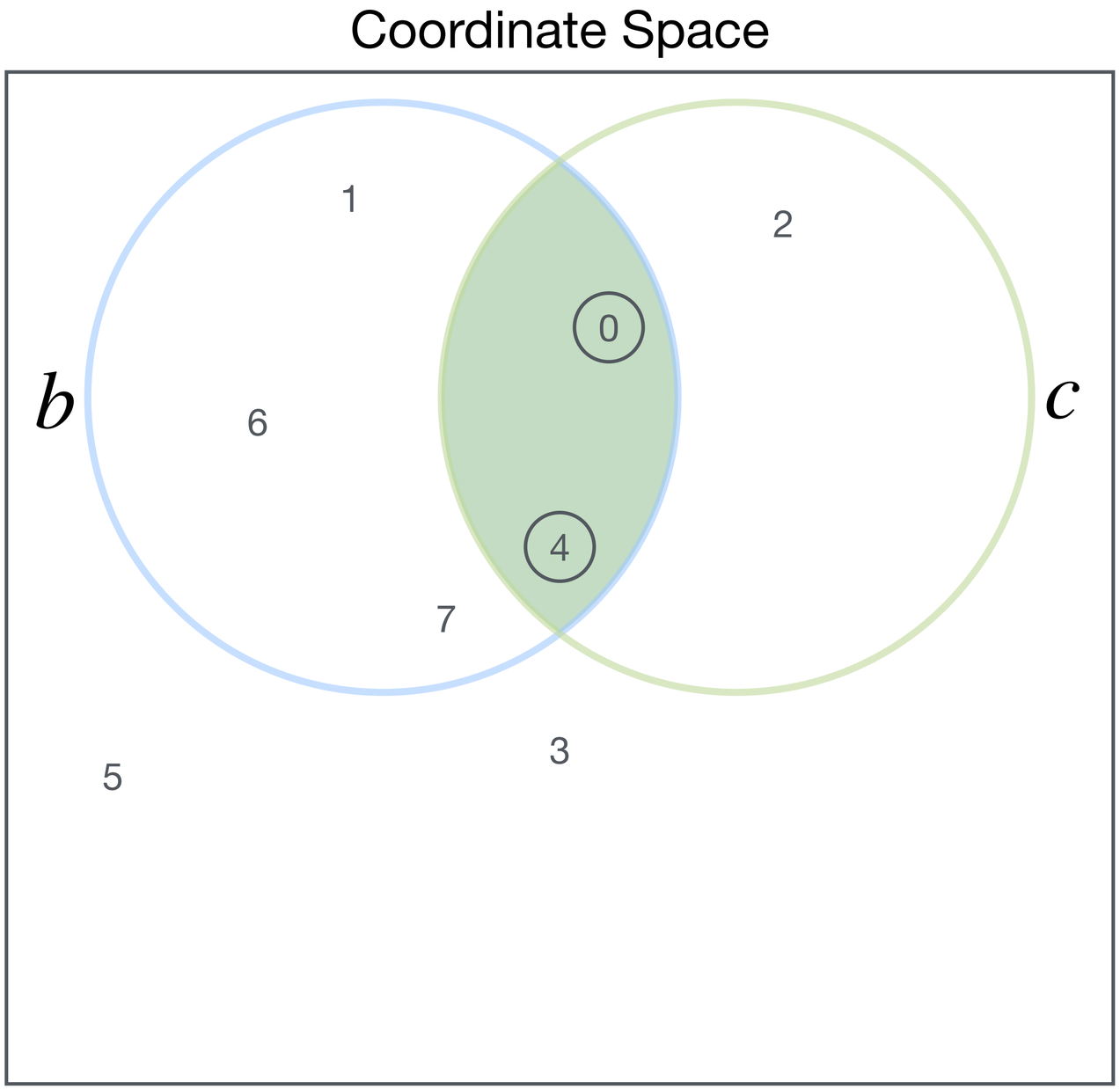


$$a_i = b_i \, c_i$$

$x \cdot 0 = 0$



Merged coiteration



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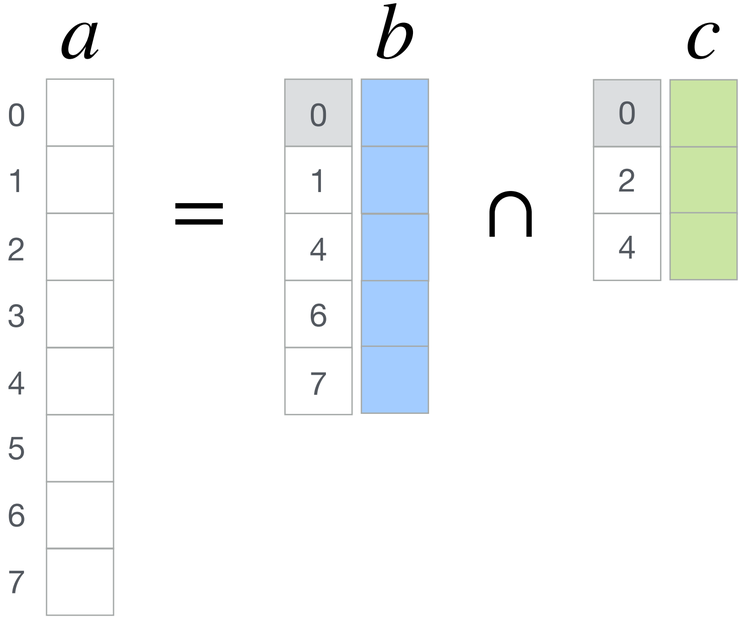
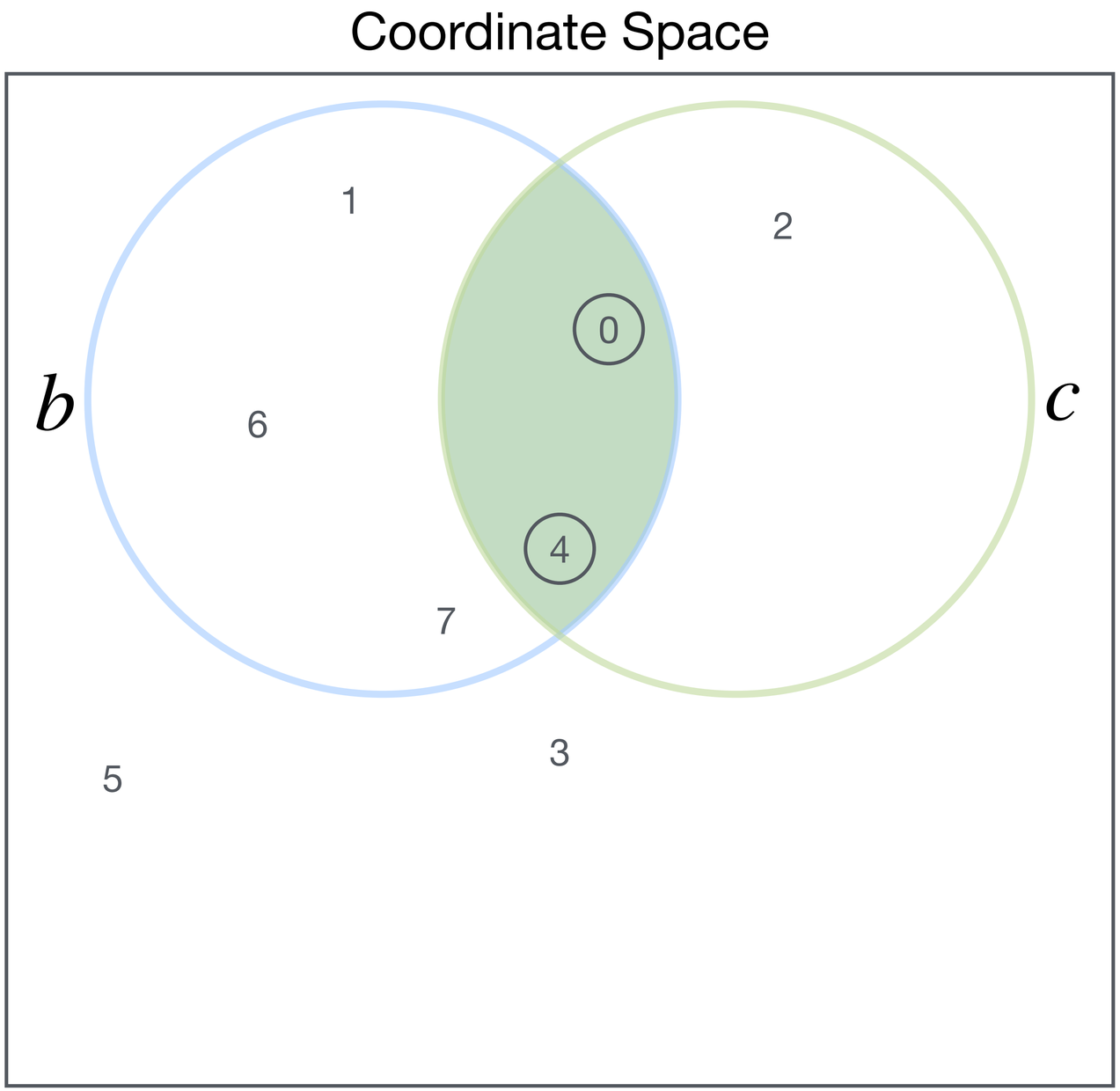


Merged coiteration



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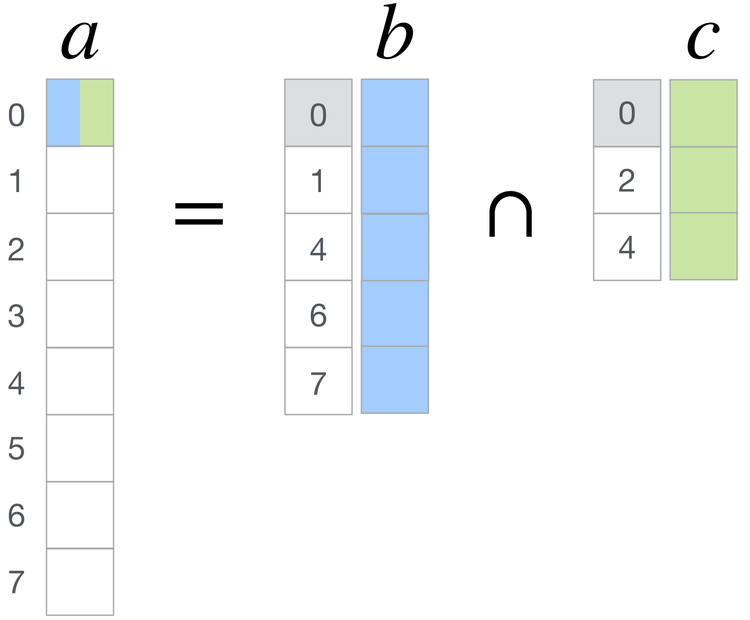
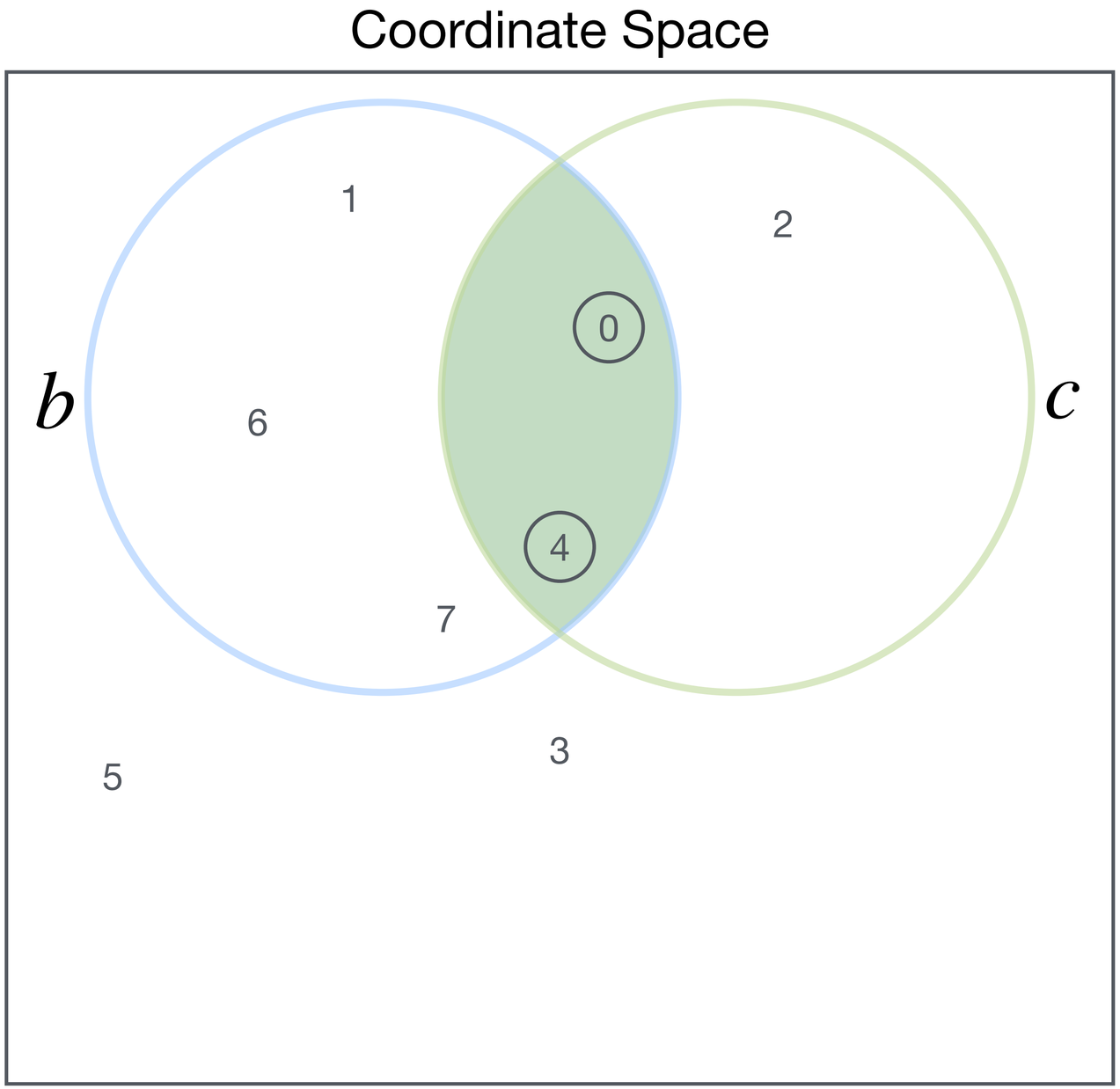


Merged coiteration



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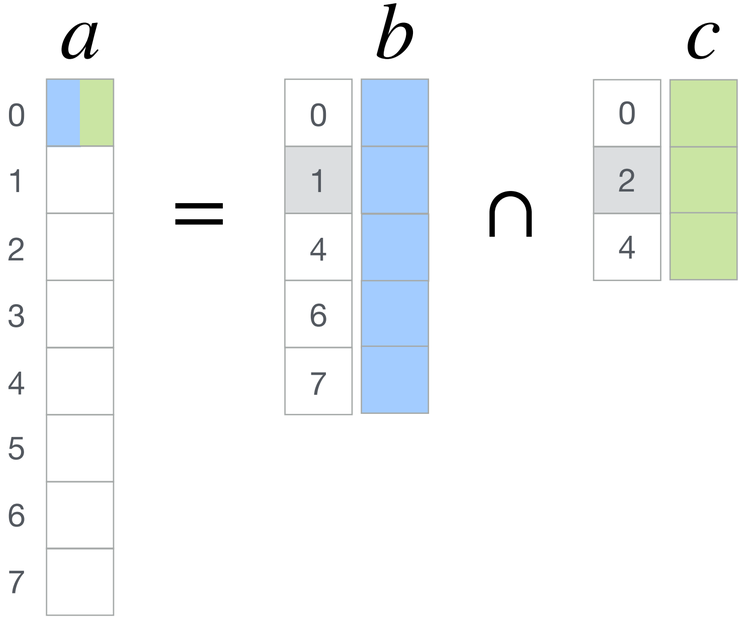
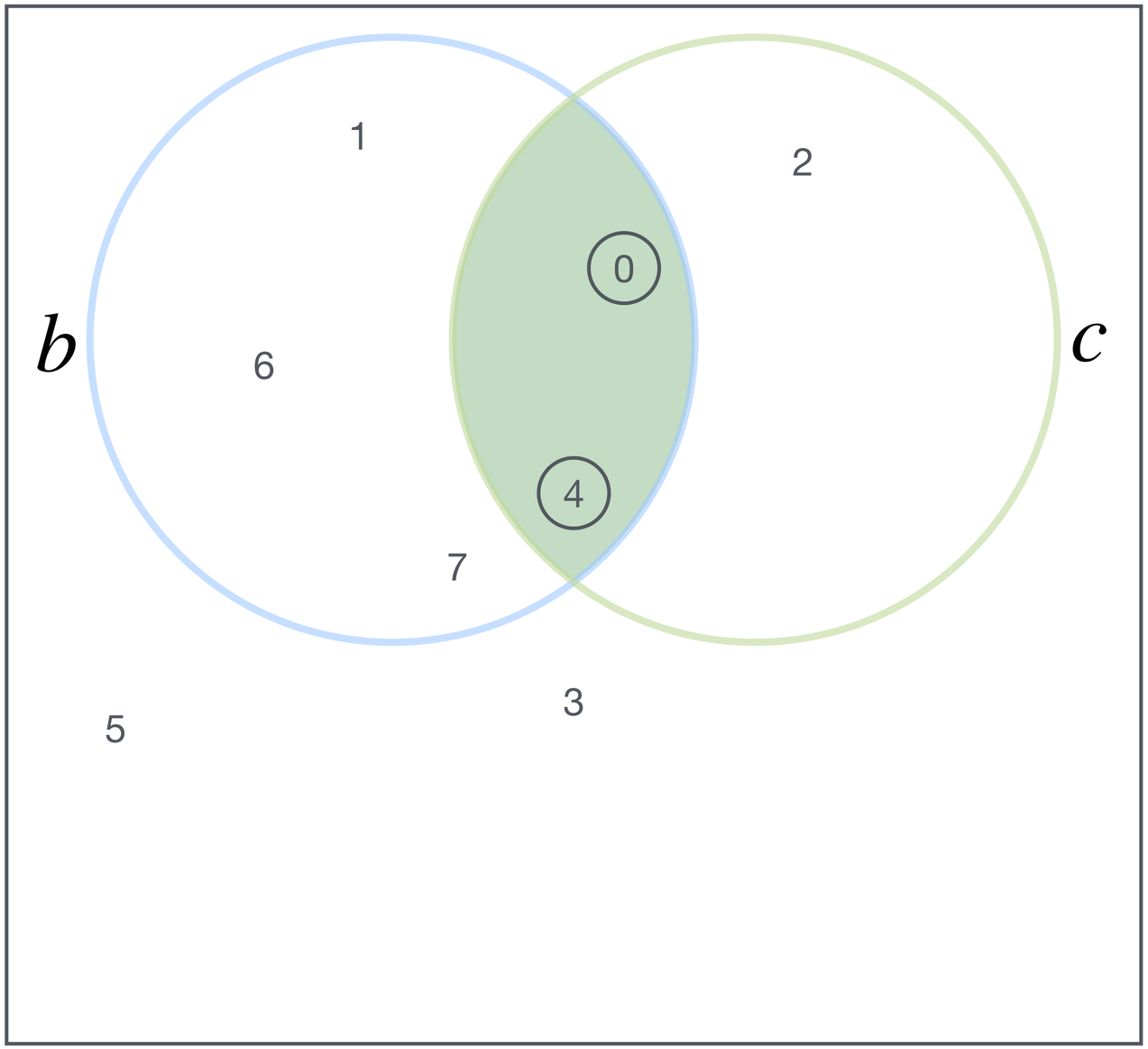


Merged coiteration



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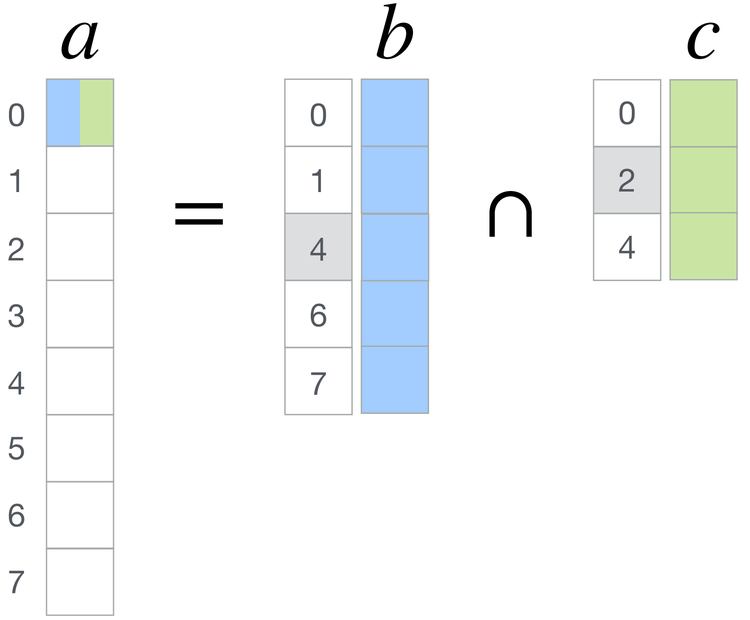
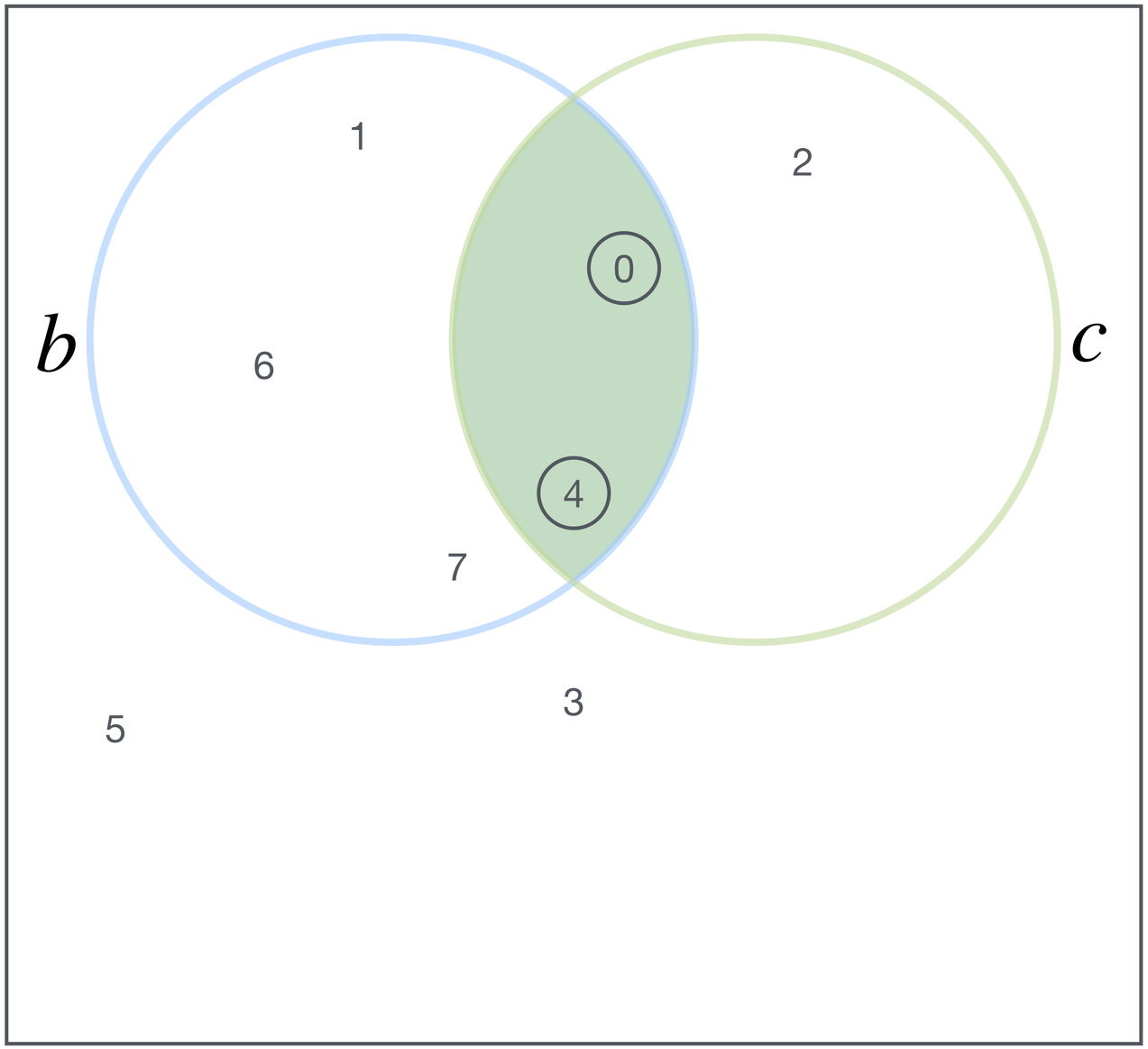


Merged coiteration

Coordinate Space



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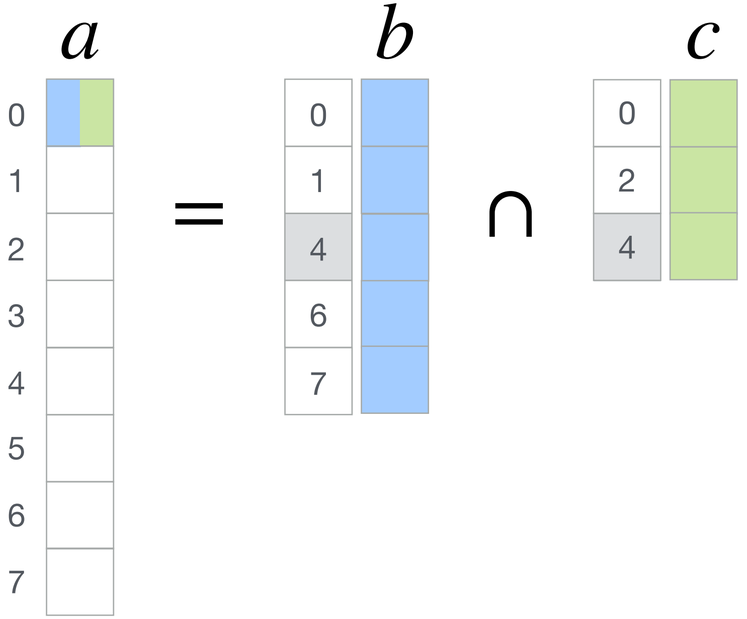

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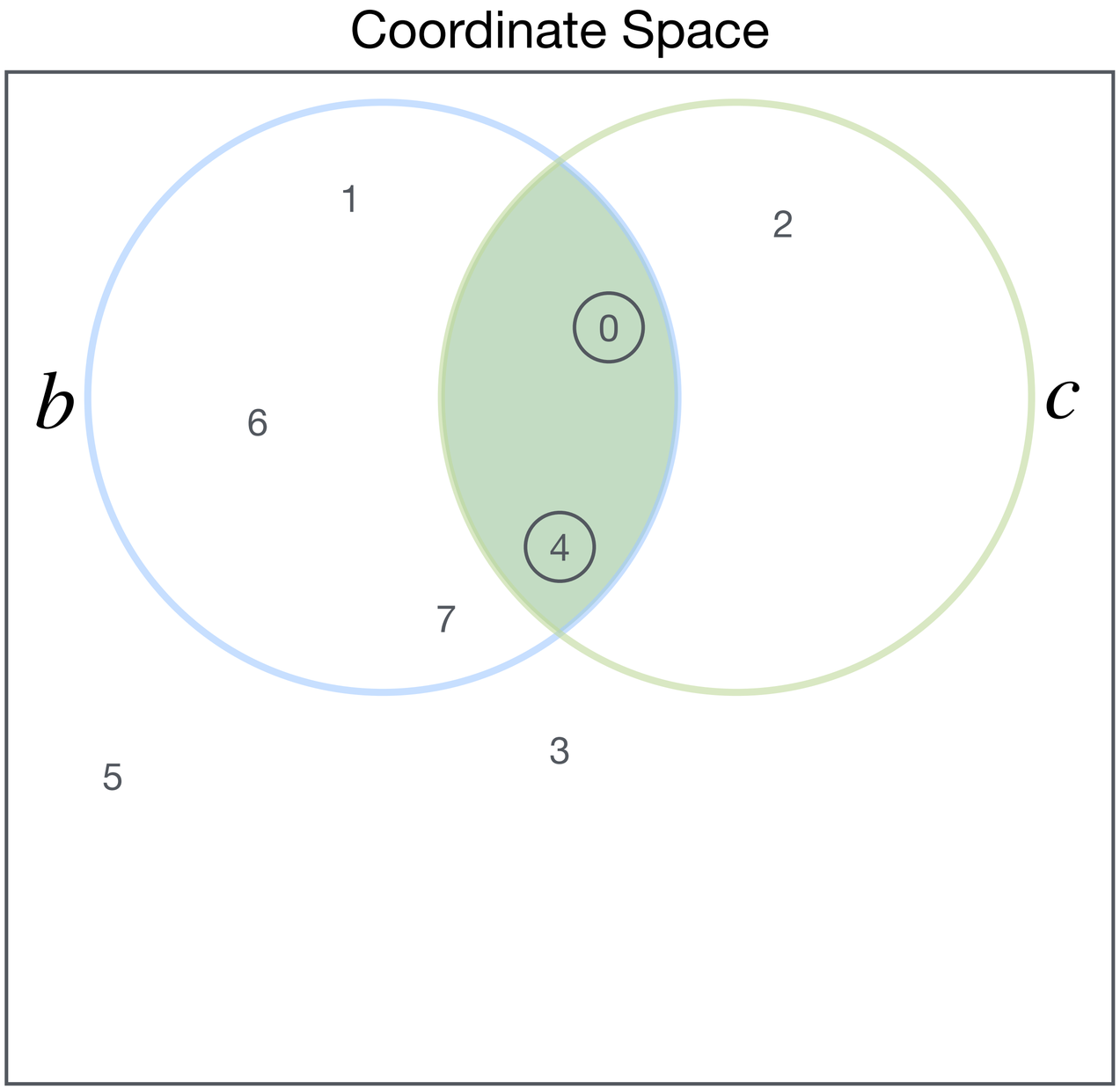

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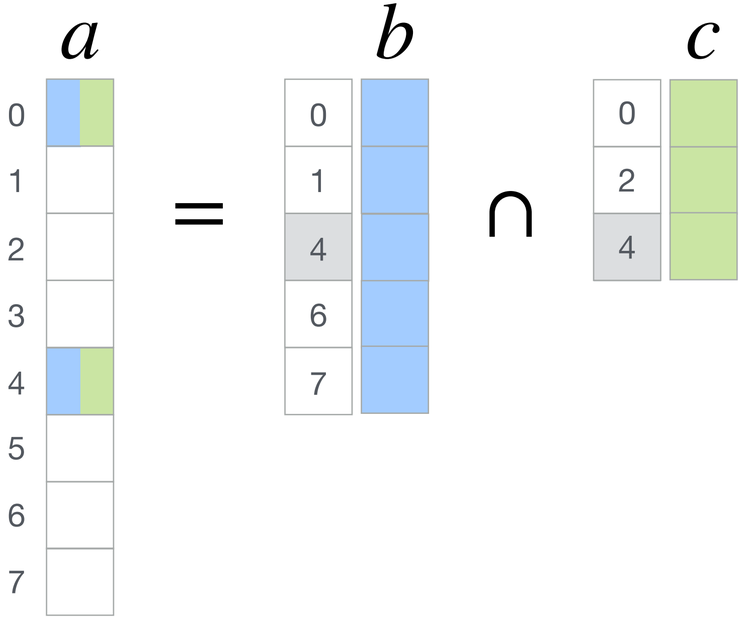
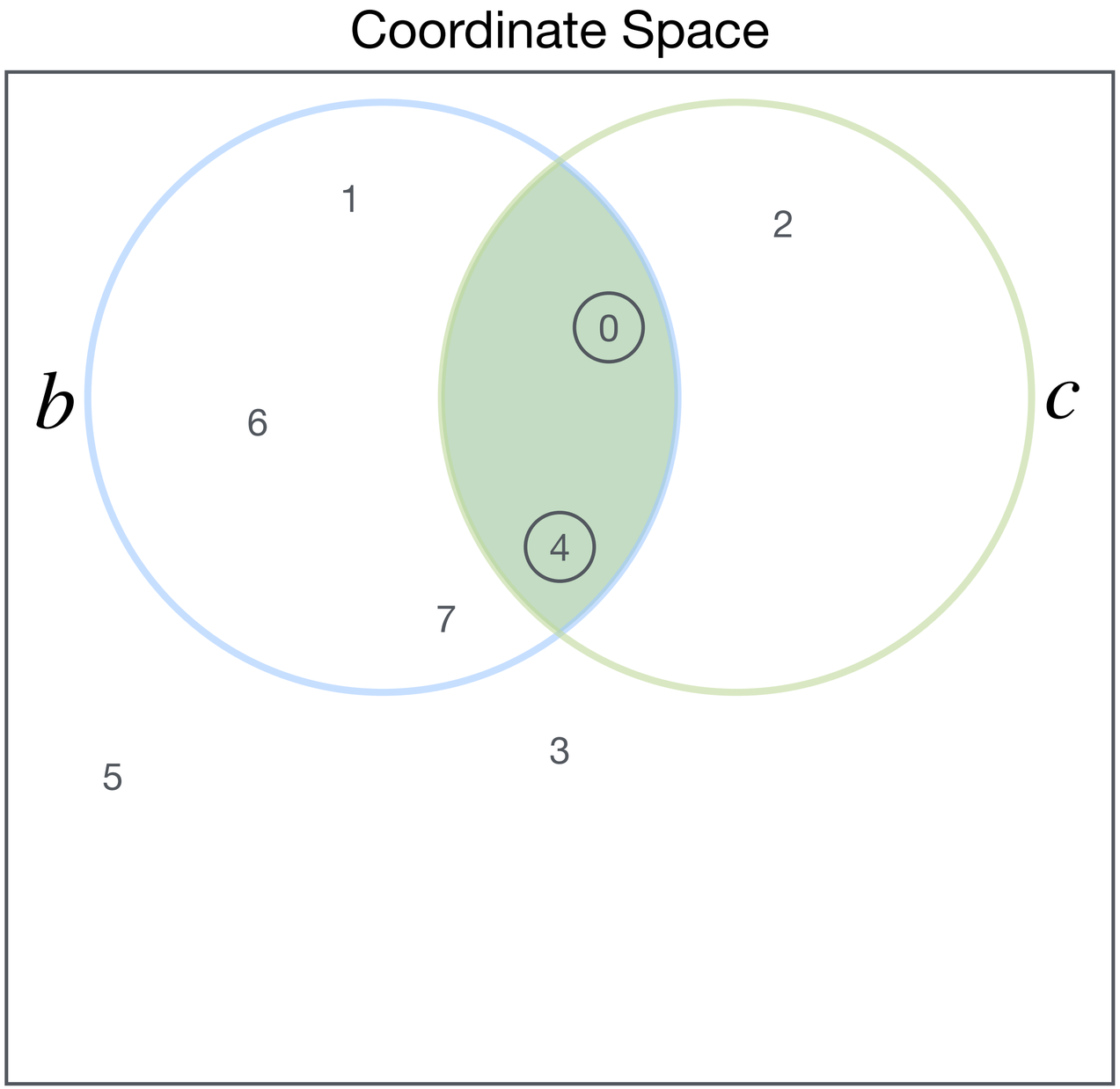


Merged coiteration



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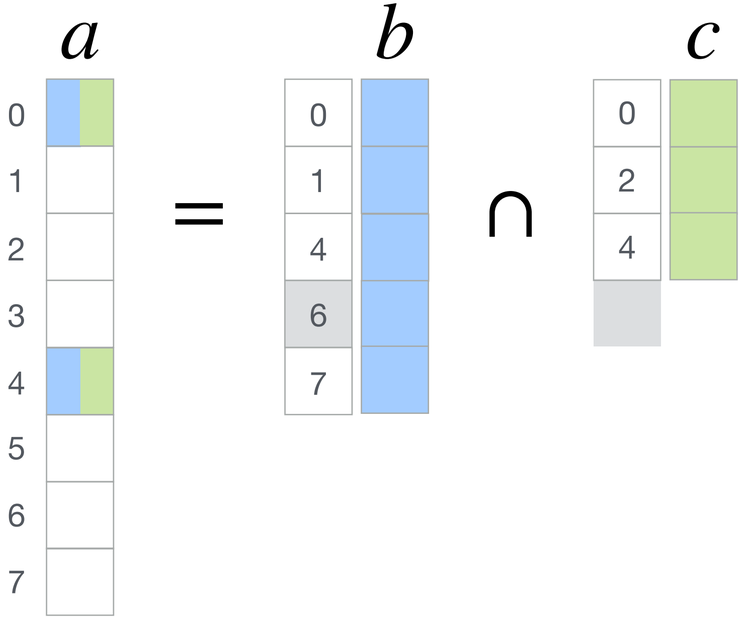

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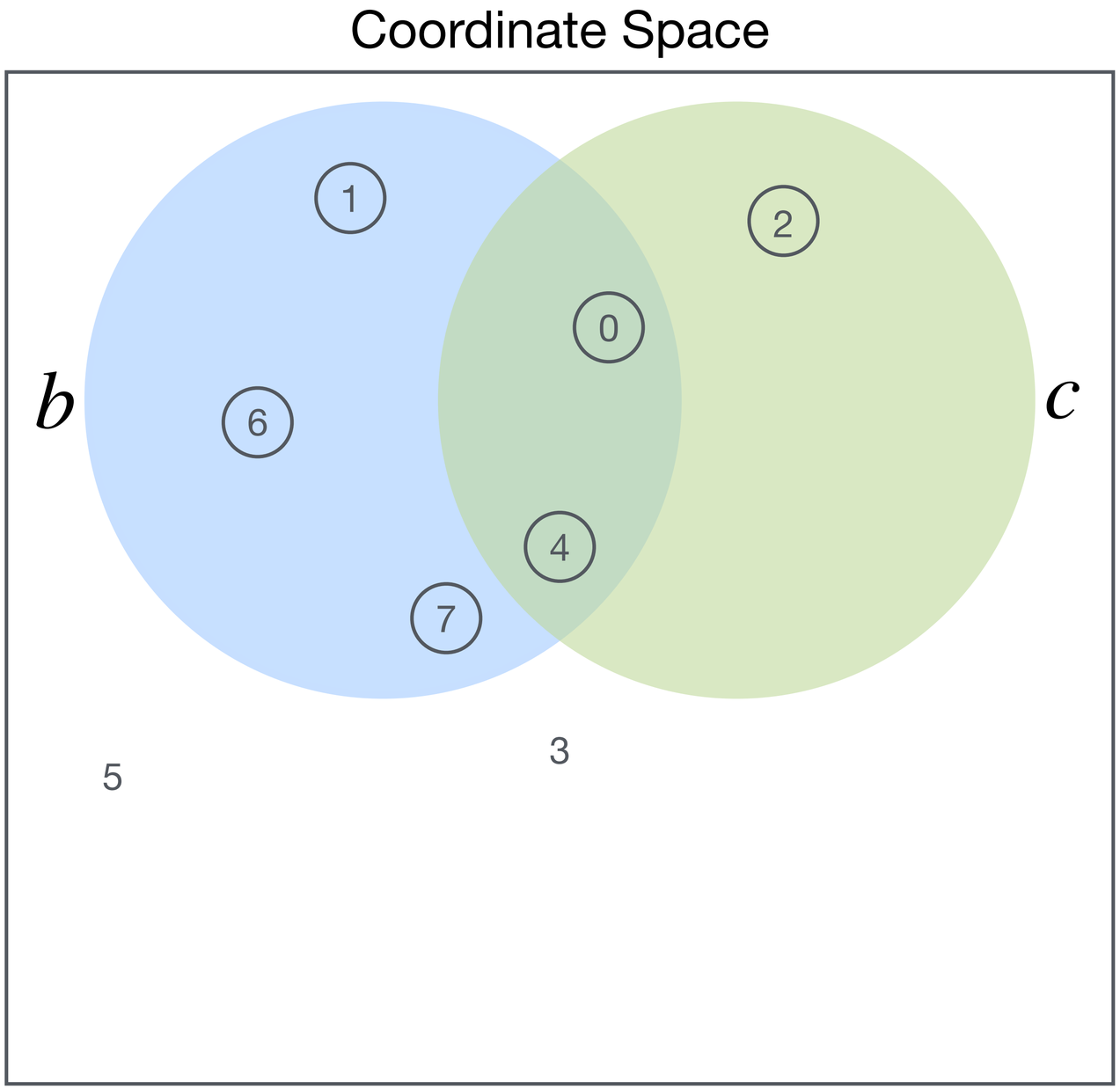
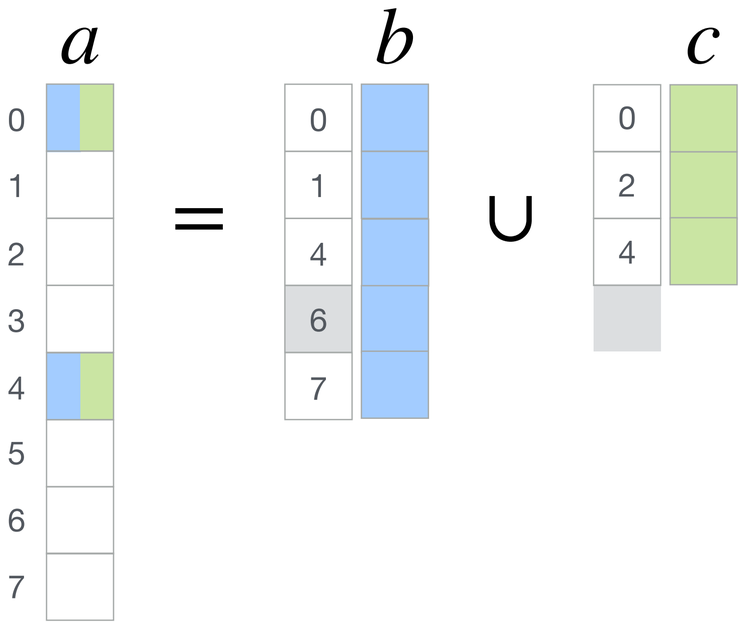
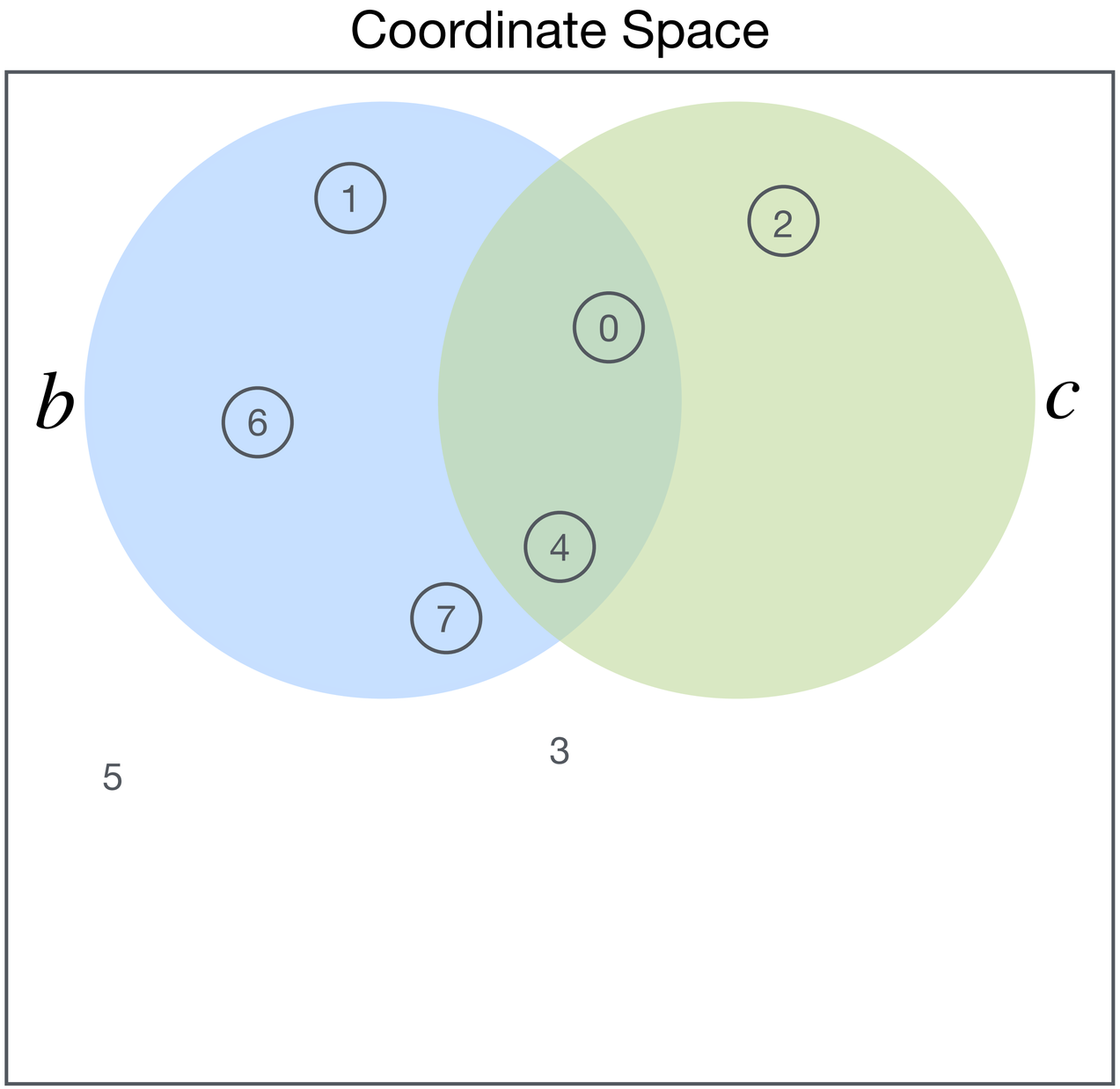


$$a_i = b_i + c_i$$



Merged coiteration



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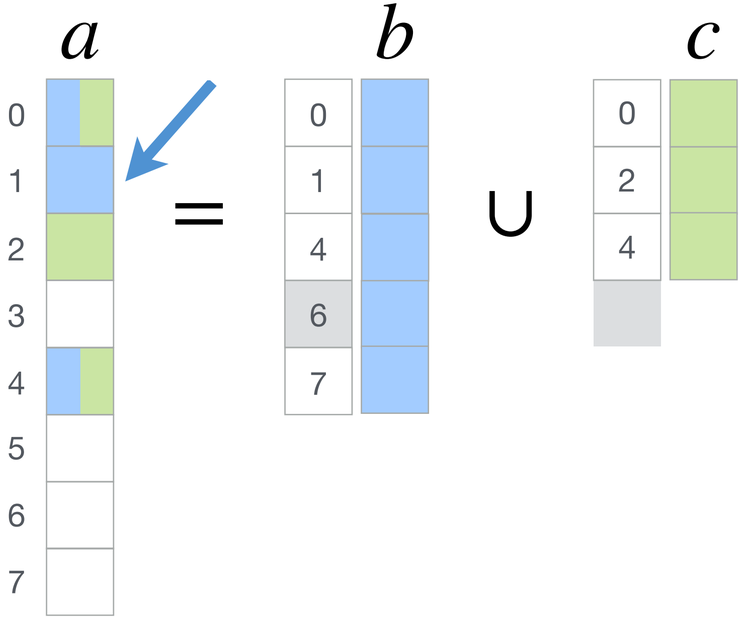

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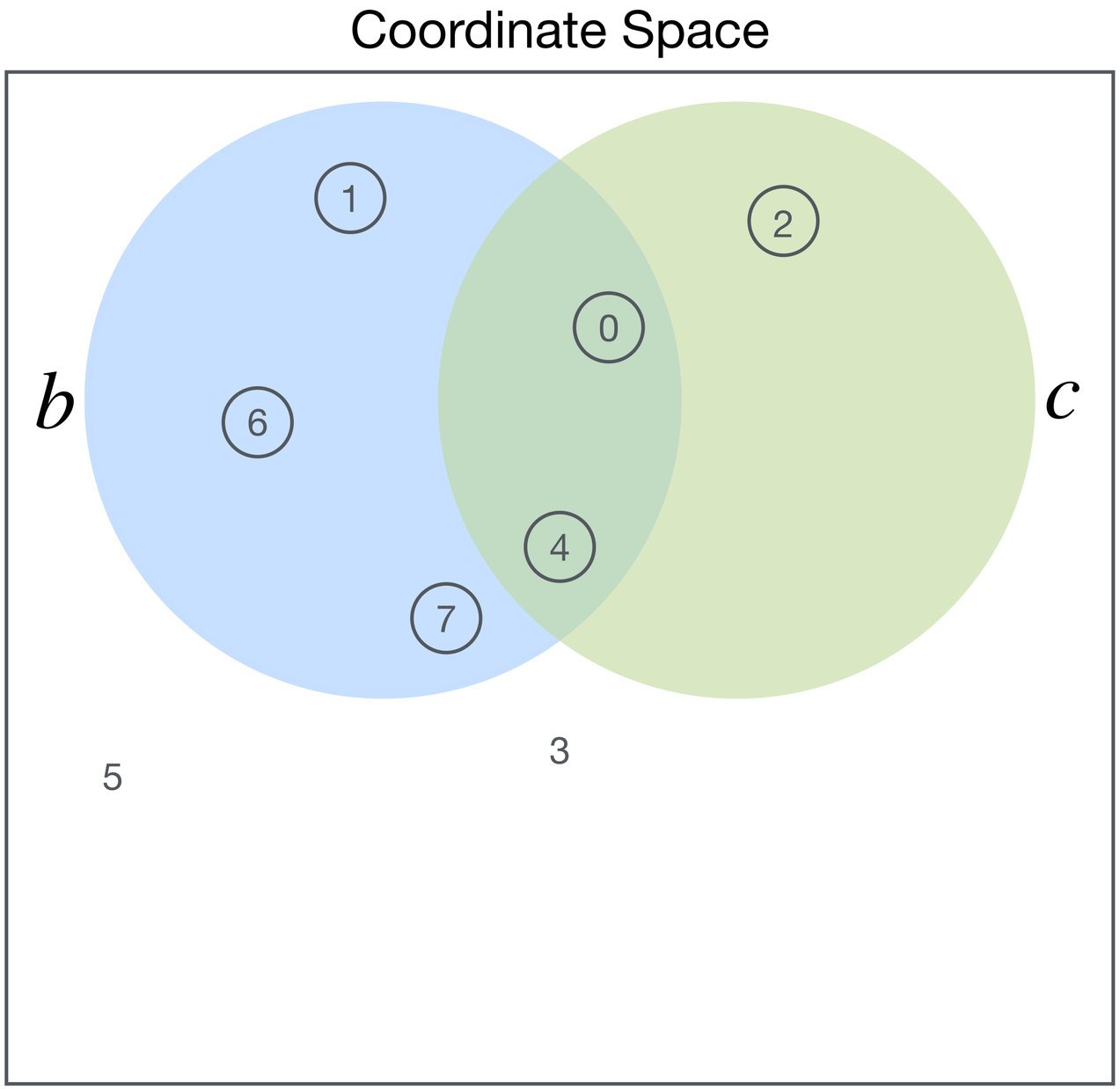

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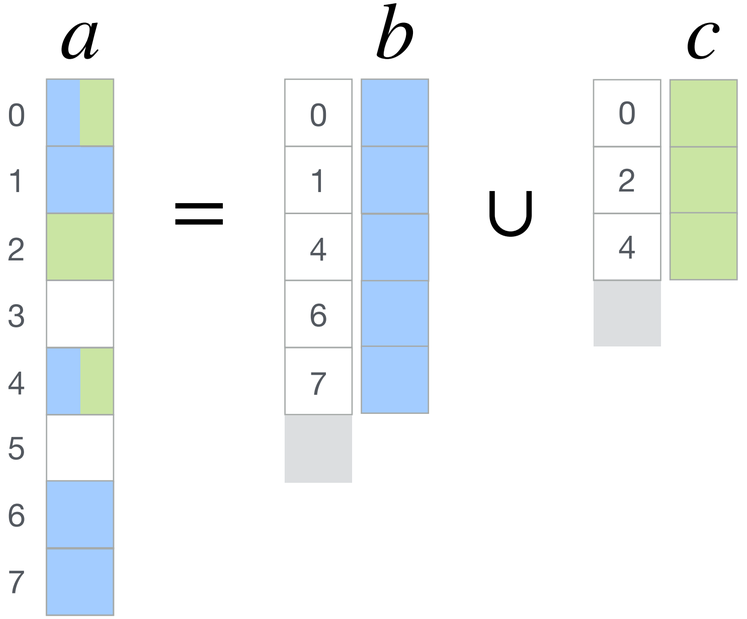
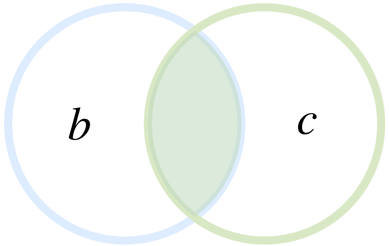


Merged coiteration code

Intersection $b \cap c$

```
int pb = b_pos[0];
int pc = c_pos[0];
while (pb < b_pos[1] && pc < c_pos[1]) {
  int ib = b_crd[pb];
  int ic = c_crd[pc];
  int i = min(ib, ic);
  if (ib == i && ic == i) {
    a[i] = b[pb] * c[pc];
  }
  if (ib == i) pb++;
  if (ic == i) pc++;
}
```



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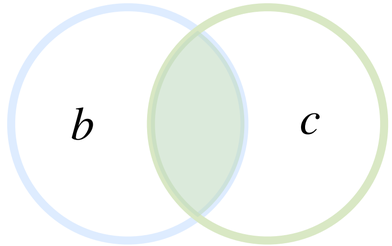
```

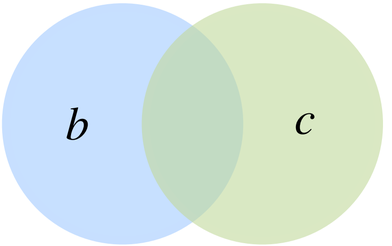


Union $b \cup c$

```
int pb = b_pos[0];
int pc = c_pos[0];
while (pb < b_pos[1] && pc < c_pos[1]) {
  int ib = b_crd[pb];
  int ic = c_crd[pc];
  int i = min(ib, ic);
  if (ib == i && ic == i) {
    a[i] = b[pb] + c[pc];
  }
  else if (ib == i) {
    a[i] = b[pb];
  }
  else {
    a[i] = c[pc];
  }
  if (ib == i) pb++;
  if (ic == i) pc++;
}

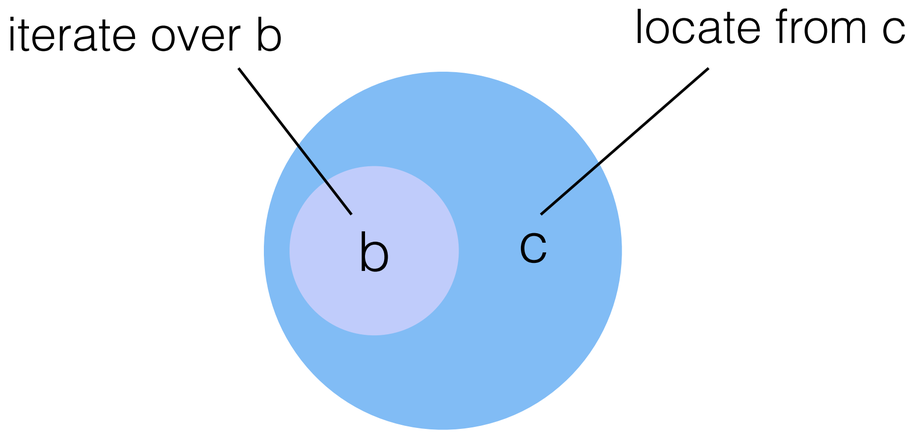
while (pb < b_pos[1]) {
  int i = b_crd[pb];
  a[i] = b[pb++];
}

while (pc < c_pos[1]) {
  int i = c_crd[pc];
  a[i] = c[pc++];
}
```



Iterate-and-locate examples (intersection)

$$a = \sum_i b_i c_i$$



Iterate-and-locate examples (intersection)

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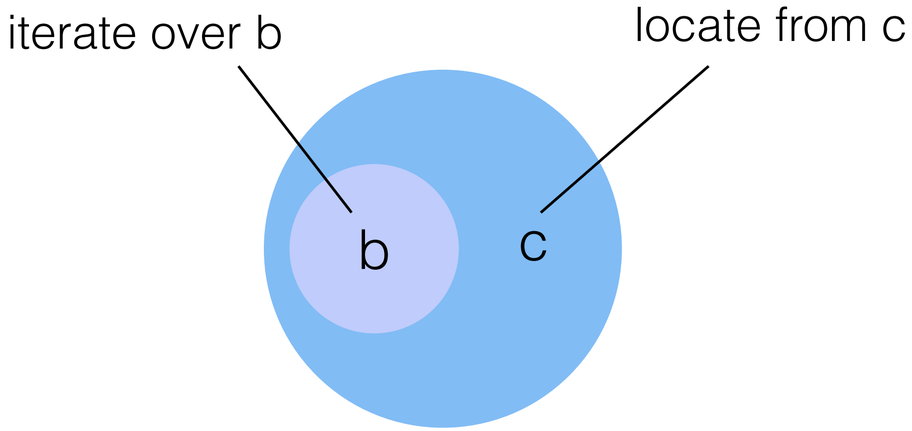
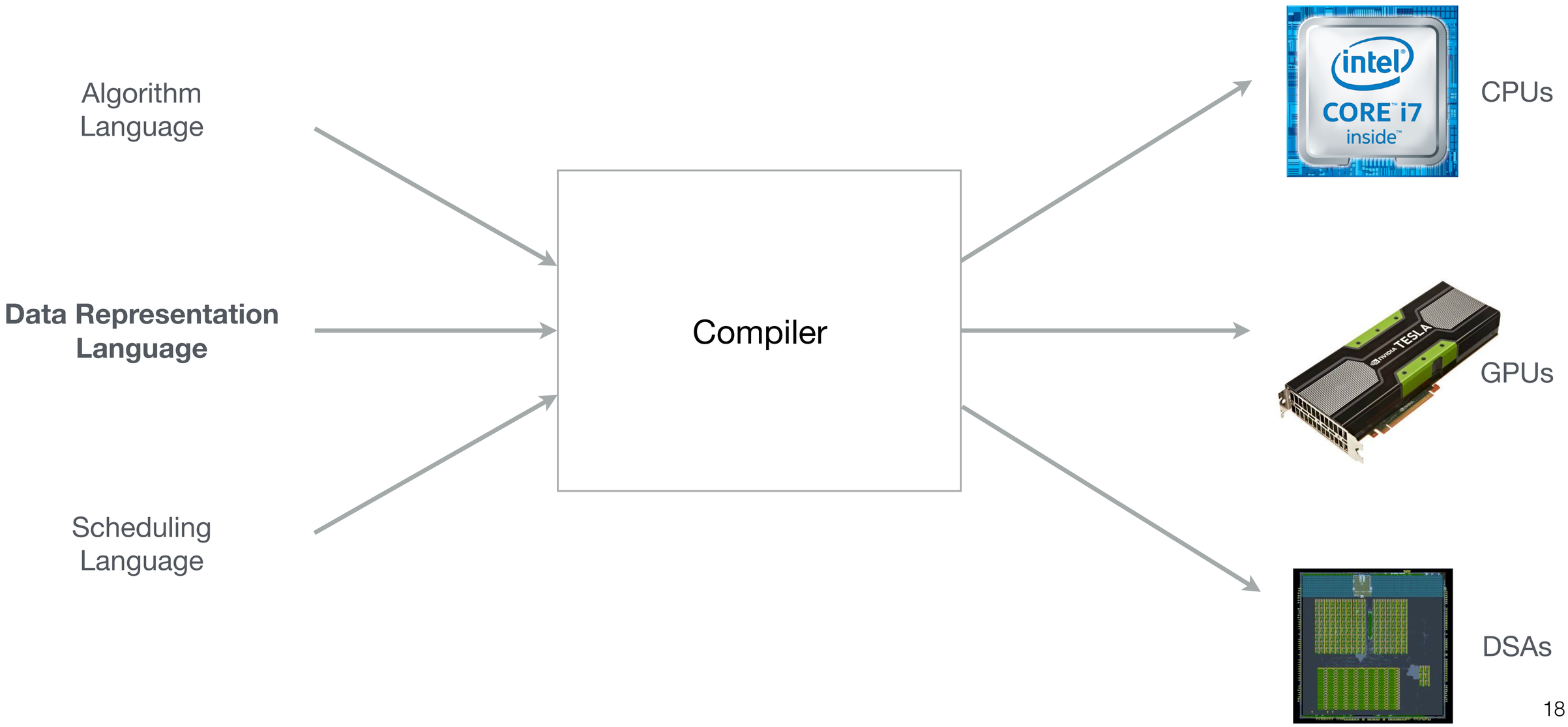



```
for (int pb = b_pos[0]; pb < b_pos[1]; pb ++) {
  int i = b_crd[pb];
  a += b[pb] * c[i];
}
```

Separation of Algorithm, Data Representation, and Schedule



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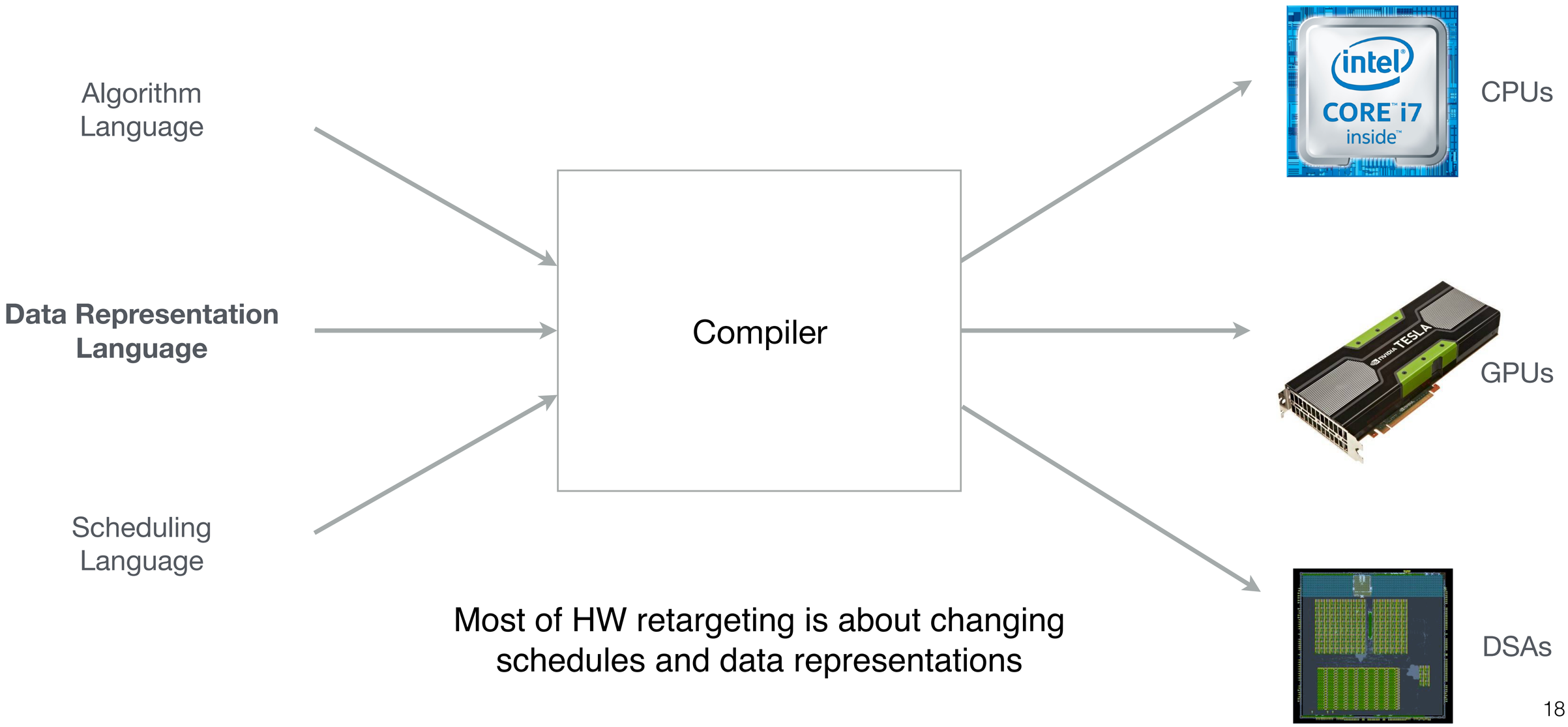

Most of HW retargeting is about changing schedules and data representations