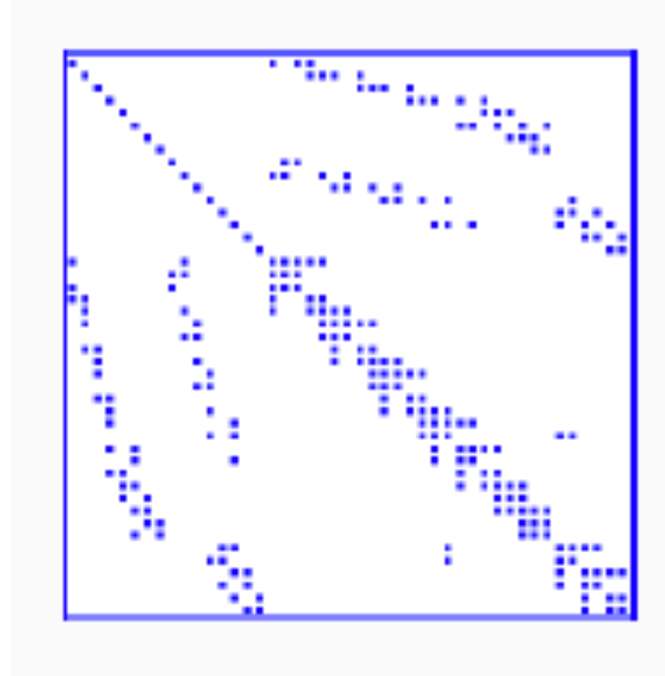
Sparse Systems

Sparse Matrices



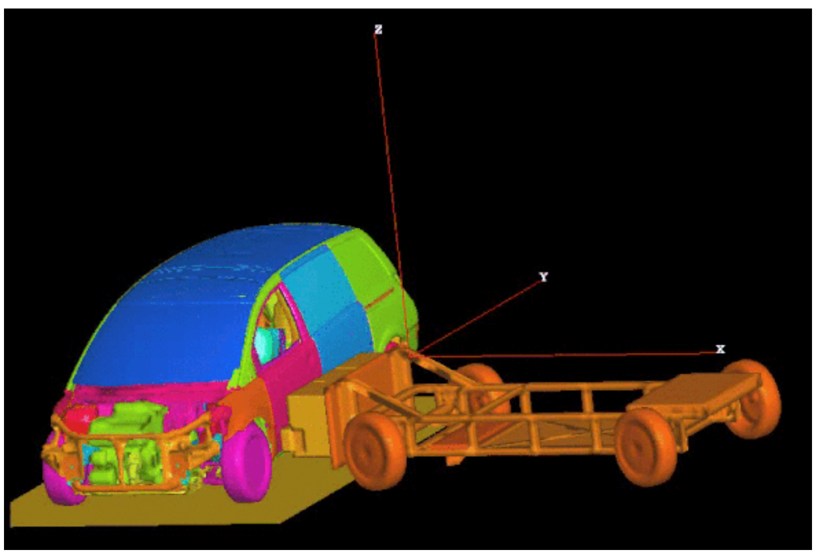
Some type of matrices contain many zeros.
Storing all those zero entries is wasteful!

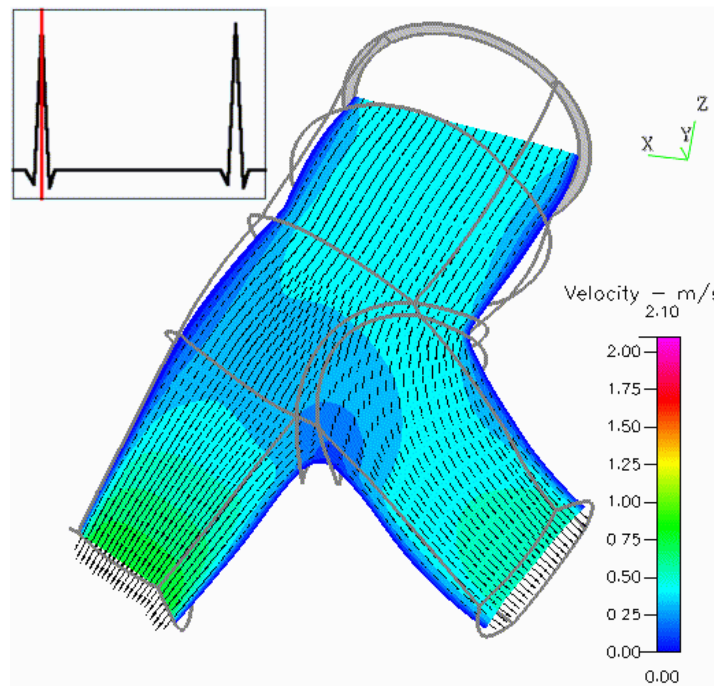
How can we efficiently store large matrices without storing tons of zeros?

- **Sparse matrices** (vague definition): matrix with few non-zero entries.
- For practical purposes: an $m \times n$ matrix is sparse if it has $O(\min(m, n))$ non-zero entries.
- This means roughly a constant number of non-zero entries per row and column.
- Another definition: "matrices that allow special techniques to take advantage of the large number of zero elements" (J. Wilkinson)

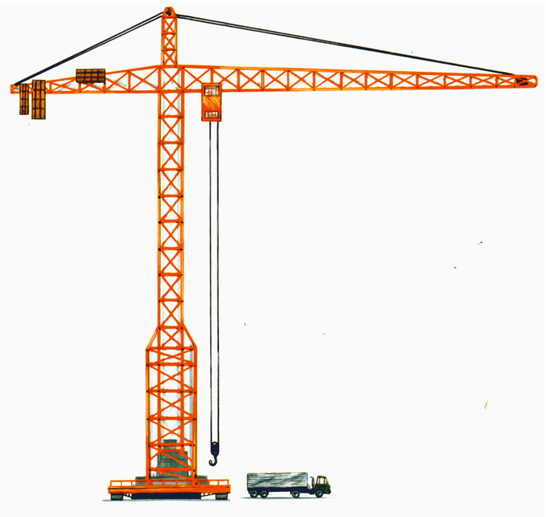
Sparse Matrices: Goals

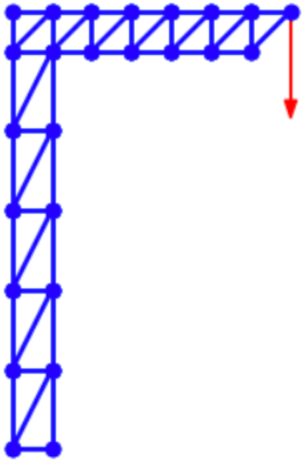
- **Perform standard matrix computations economically, i.e., without storing the zeros of the matrix.**
- For typical Finite Element and Finite Difference matrices, the number of non-zero entries is $O(n)$

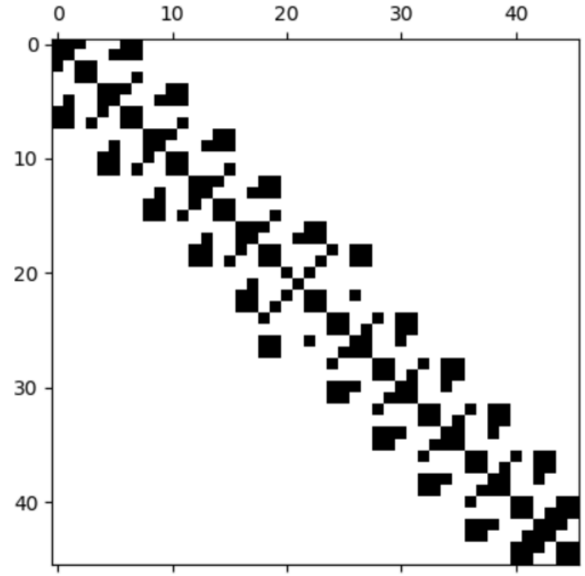




Sparse Matrices: MP example







Sparse Matrices

EXAMPLE:

Number of operations required to add two square dense matrices:

$$O(n^2)$$

Number of operations required to add two sparse matrices $\mathbf{A}$ and $\mathbf{B}$:

$$O(\mathrm{nnz}(\mathbf{A}) + \mathrm{nnz}(\mathbf{B}))$$

where $\mathrm{nnz}(\mathbf{X}) =$ number of non-zero elements of a matrix $\mathbf{X}$

Popular Storage Structures

DNS	Dense	**ELL**	Ellpack-Itpack
BND	Linpack Banded	**DIA**	Diagonal
COO	Coordinate	**BSR**	Block Sparse Row
CSR	Compressed Sparse Row	**SSK**	Symmetric Skyline
CSC	Compressed Sparse Column	**BSR**	Nonsymmetric Skyline
MSR	Modified CSR	**JAD**	Jagged Diagonal
LIL	Linked List		

note: CSR = CRS, CCS = CSC, SSK = SKS in some references

We will focus on COO and CSR!

Dense (DNS)

$$A = \begin{bmatrix} 0. & 1.9 & 0. & -5.2 \\ 0.3 & 0. & 9.1 & 0. \\ 4.4 & 5.8 & 3.6 & 0. \\ 0. & 0. & 7.2 & 2.7 \end{bmatrix}$$

$$Ashape = (nrow, ncol)$$

$$A_{dense} = [\underbrace{0. \quad 1.9 \quad 0. \quad -5.2}_{\text{Row 0}} \quad \underbrace{0.3 \quad 0. \quad 9.1 \quad 0.}_{\text{Row 1}} \quad \underbrace{4.4 \quad 5.8 \quad 3.6 \quad 0.}_{\text{Row 2}} \quad \underbrace{0. \quad 0. \quad 7.2 \quad 2.7}_{\text{Row 3}}]$$

- Simple
- Row-wise
- Easy blocked formats
- Stores all the zeros

Coordinate Form (COO)

$$A = \begin{bmatrix} 0. & 1.9 & 0. & -5.2 \\ 0.3 & 0. & 9.1 & 0. \\ 4.4 & 5.8 & 3.6 & 0. \\ 0. & 0. & 7.2 & 2.7 \end{bmatrix}$$

- Simple
- Does not store the zero elements
- Not sorted
- *row* and *col*: array of integers
- *data*: array of doubles

Representing a Sparse Matrix in Coordinate (COO) Form

1 point

Consider the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 1.3 \\ -1.5 & 0.2 & 0 \\ 5 & 0 & 0 \\ 0 & 0.3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Suppose we store one row index (a 32-bit integer), one column index (a 32-bit integer), and one data value (a 64-bit float) for each non-zero entry in A. How many bytes in total are stored? Please note that 1 byte is equal to 8 bits.

Compressed Sparse Row (CSR) format

$$A = \begin{bmatrix} 0. & 1.9 & 0. & -5.2 \\ 0. & 0. & 0. & 0. \\ 4.4 & 5.8 & 3.6 & 0. \\ 0. & 0. & 7.2 & 2.7 \end{bmatrix}$$

Compressed Sparse Row (CSR)

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 3 & 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

$$\begin{array}{lll} data & = & [\ 1.0 \quad 2.0 \quad 3.0 \quad 4.0 \quad 5.0 \quad 6.0 \quad 7.0 \quad 8.0 \quad 9.0 \quad 10.0 \quad 11.0 \quad 12.0\ ] \\ col & = & [\ 0 \quad 3 \quad 0 \quad 1 \quad 3 \quad 0 \quad 2 \quad 3 \quad 4 \quad 2 \quad 3 \quad 4\ ] \\ rowptr & = & [\ 0 \quad 2 \quad 5 \quad 9 \quad 11 \quad 12\ ] \end{array}$$

- Does not store the zero elements
- Fast arithmetic operations between sparse matrices, and fast matrix-vector product
- *col*: contain the column indices (array of nnz integers)
- *data*: contain the non-zero elements (array of nnz doubles)
- *rowptr*: contain the row offset (array of $n + 1$ integers)