Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\,x + \ a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \ 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + \cdots$$

$$f''(x) = 2\,a_2 + \ (3{\times}2)a_3\,x + (4{\times}3)a_4\,x^2 + \cdots$$

$$f'''(x) = (3{\times}2)a_3 \ + (4{\times}3{\times}2)a_4\,x + \cdots$$

$$f'^{v}(x) = (4{\times}3{\times}2)a_4 + \cdots$$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(x_o)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i$$

Use Taylor to approximate functions at given point

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4;\ f^{(n)}(1) = 0\ \forall\ n\ \geq 3$$

Evaluate $f(4)$

Example:

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

Taylor Series – what is the error?

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $h = x - x_o$

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

$$f(x) = t_n(x) + R(x) \qquad R(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!}(h)^i$$

Then there exists a $\xi \in (x_o, x)$ so that

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(\xi - x_o)^{n+1}$$

Graphical representation:

How can we use the known asymptotic behavior of the error?

Making error predictions

Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

What is the Taylor truncation error for $h_2 = 0.25$?

Using Taylor approximations to obtain derivatives

Let's say a function has the following Taylor series expansion about $x = 2$.

$$f(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4 - \frac{5}{4}(x-2)^6 + \frac{25}{32}(x-2)^8 + O((x-2)^9)$$