Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \; 2\, a_2\, x + 3\, a_3\, x^2 + 4\, a_4\, x^3 + \cdots$$

$$f''(x) = 2\, a_2 + \; (3\times 2)a_3\, x + (4\times 3)a_4\, x^2 + \cdots$$

$$f'''(x) = (3\times 2)a_3 \; + (4\times 3\times 2)a_4\, x + \cdots$$

$$f'^{v}(x) = (4\times 3\times 2)a_4 + \cdots$$

$f(0) = a_0$

$f'(0) = a_1$

$f''(0) = 2a_2$

$f'''(0) = 3\times 2\, a_3$

$f^{iv}(0) = 4\times 3\times 2 a_4$

...

$f^{(i)}(0) = i!\, a_i$

$$\boxed{a_i = \frac{f^{(i)}(0)}{i!}}$$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i \longrightarrow \boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i\,!}\, x^i}$$

→ approximate function values

→ approximate derivatives

→ estimate erros of these approximations

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(0)}{3!}(x - x_o)^3 + \cdots$$

$$\boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i}$$

Use Taylor to approximate functions at given point

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4;\ f^{(n)}(1) = 0\ \forall\, n\ \geq 3$$

Evaluate $f(4)$

$$f(4) = \sum_{i=0}^{\infty} \frac{f^{(i)}(\overset{1}{x_0})}{i!}(\underset{4}{x} - \overset{1}{x_0})^i = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

other terms are zero!

$$= f(1) + f'(1)(4-1) + \frac{f''(1)}{2}(4-1)^2 = 2 - 3(3) + \frac{4}{2}(3)^2 = 11 \,//$$

Example:

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

$$f'(x) = \frac{-1(20)}{(20x-10)^2} \Rightarrow f'(0) = -\frac{1}{5}$$

$$f''(x) = \frac{+20(2)(20x-10)20}{(20x-10)^4} => f''(0) = -\frac{4}{5}$$

$$\boxed{t_2(x) = -\frac{1}{10} - \frac{1}{5}x - \frac{4}{10}(x)^2}$$

$$\text{error} = O(x^3)$$

Taylor Series – what is the error?

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $h = x - x_o$

$$f(x_0+h) = f(x_0) + f'(x_0)\underbrace{(x-x_0)}_{h} + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \dots$$

$$\underbrace{f(x_0+h)}_{\substack{\text{exact}\\ f(x)}} = \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}h^i}_{\substack{\text{taylor approx.}\\ t_n(x)\\ \text{truncated part}}} + \underbrace{\sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!}h^i}_{\substack{\text{error}\\ \text{what we are ignoring}}}$$

$$|\text{error}| = |f(x) - t_n(x)|$$

$$= \left| \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)\, h^i}{i!} \right|$$

$$= \left| \underbrace{\frac{f^{(n+1)}(x_0)\, h^{n+1}}{(n+1)!}}_{\text{dominant term when } h \to 0 \text{ (or } x \to x_0)} + \frac{f^{(n+2)}(x_0)\, h^{n+2}}{(n+2)!} + \cdots \right|$$

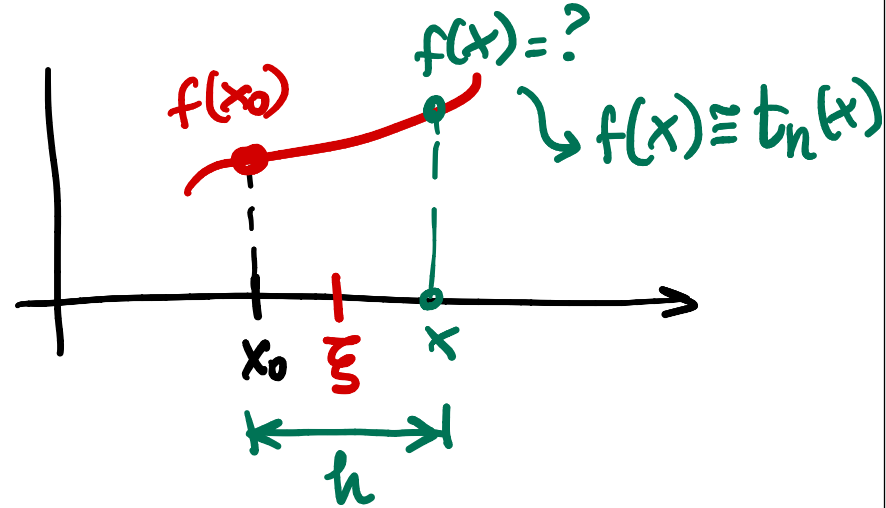
$$|\text{error}| \le |M h^{n+1}| \Rightarrow \boxed{\text{error} = O(h^{n+1})}$$

what is M?

Remainder Theorem

$f^{(n+1)}$ exists, $f^{(n)}$ continuous on $[x_0, x]$,

$h = x - x_0$



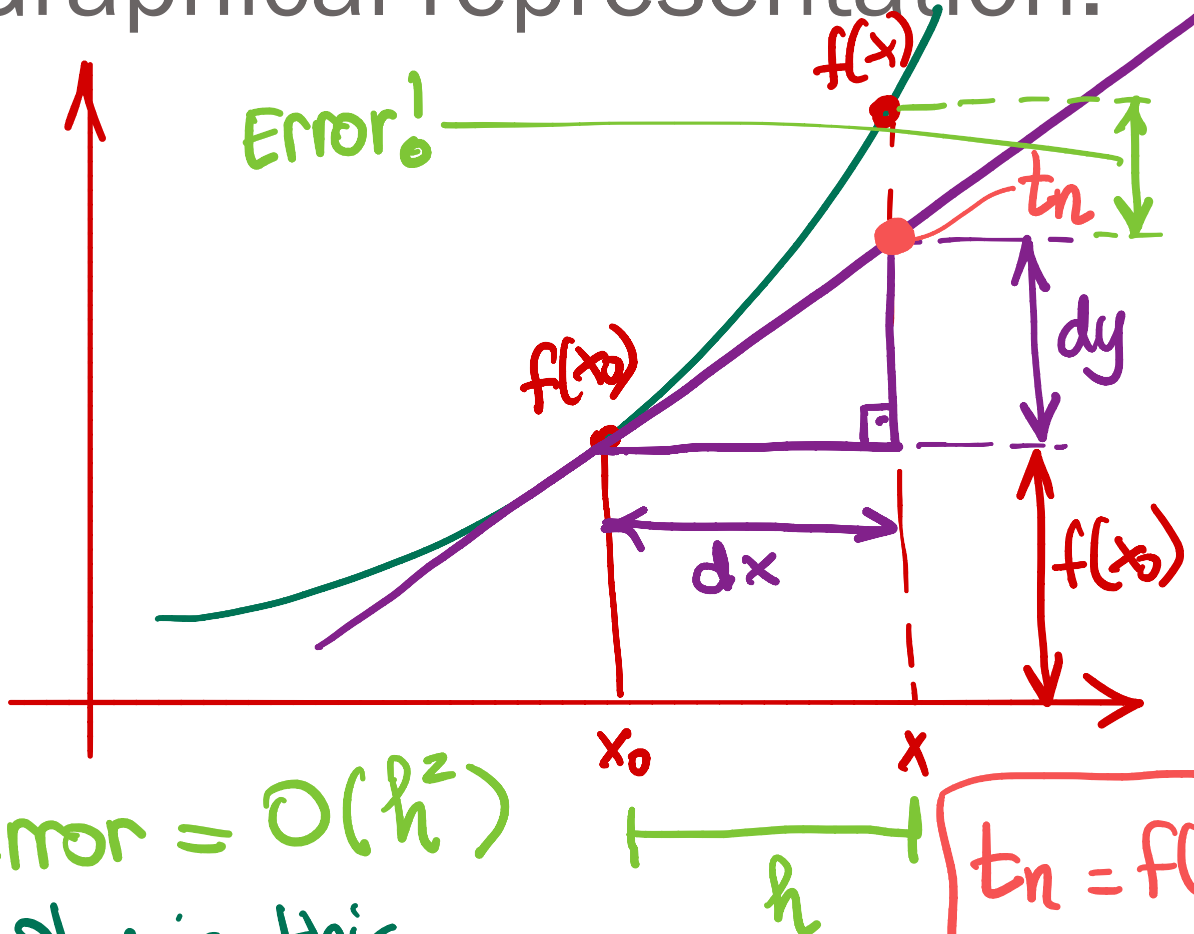
$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(\xi - x_0)^{n+1}$$

since $|\xi - x_0| \le |h|$

$$|R| \le \underbrace{\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\, h^{n+1}\right|}_{M}$$

Graphical representation:

$f'(x_0)$ → tangent at x_0



$$\text{slope} = \frac{dy}{dx}$$

$$f'(x_0) = \frac{dy}{dx}$$

$$dy = f'(x_0)\,h$$

$$t_n = dy + f(x_0)$$

$$\boxed{t_n = f(x_0) + f'(x_0)\,h}$$

Taylor approximation of degree 1.

$$\text{Error} = O(h^2)$$

Why is this information useful?

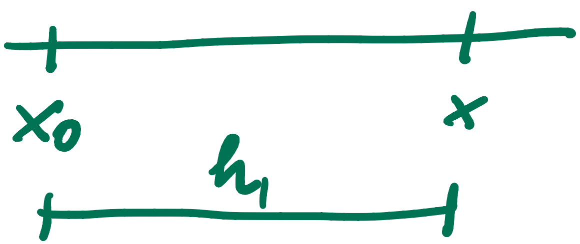
How can we use the known asymptotic behavior of the error?

$$f(x) \longrightarrow t_n(x)$$

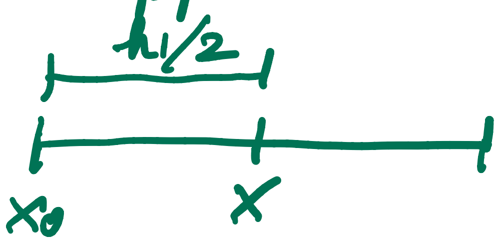
we want to approximate $f(x)$ with $t_n(x)$

We know that $h = h_1$

$\Longrightarrow$ error $= e_1$



what happens to error when $h = h_1/2$



$$e = O(h^{n+1})$$

$$e_1 \propto h_1^{n+1}$$

$$e_2 \propto h_2^{n+1}$$

$$\frac{e_1}{e_2} = \left(\frac{h_1}{h_2}\right)^{n+1}$$

$$e_2 = \left(\frac{h_2}{h_1}\right)^{n+1} e_1$$

Making error predictions

Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

What is the Taylor truncation error for $h_2 = 0.25$?

$f(x) = \sqrt{x-10} \longrightarrow t_3$ about $x_0 = 12$

$\Downarrow$

error $= O(h^4)$

where $h = x - 12$

$h_1 = 0.5 \longrightarrow e_1 = 10^{-4}$

$h_2 = 0.25 \longrightarrow e_2 = ?$

$$e_2 = \left(\frac{h_2}{h_1}\right)^4 10^{-4} =$$

Using Taylor approximations to obtain derivatives

Let's say a function has the following Taylor series expansion about $x = 2$.

$$f(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4 - \frac{5}{4}(x-2)^6 + \frac{25}{32}(x-2)^8 + O((x-2)^9)$$

Suppose you have the Taylor approximation of degree $n=4$. $\rightarrow t_4 = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4$

Take derivative:

$$t' = -5(x-2) + \frac{15}{2}(x-2)^3$$

at $x=2.3 \longrightarrow t'(2.3) = -1.2975$

at $x=3.0 \rightarrow \underbrace{t'(3.0) = }_{\text{no longer good approximation!}}\ ?$