Sparse Recovery

Given linear sketch Ax

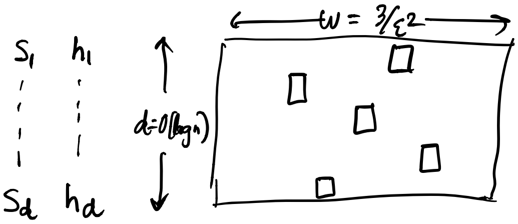
Find z s.t. $\|z\|_0 \le k$, $\|x - z\|_2$ is minimized

minimum error $err_2^k := \left(\sum_{i \notin S} x_i^2\right)^{1/2}$

S = largest k coordinates in x

Compressed Sensing

Count Sketch



$\forall i \quad |\hat{x}_i - x_i| \le \varepsilon \sqrt{F_2}$

$F_2 = err_2^0(x)$

Now: $w = \dfrac{3k}{\varepsilon^2}$, $d = O(\log n)$

Lemma: w.h.p. $\forall i \quad |\hat{x}_i - x_i| \le \dfrac{\varepsilon}{\sqrt{k}} err_2^k(x)$

Proof: $\tilde{x}_{ij}$: estimate of x_i from j^{th} row $\{C_{ij}\}$

$C_{j h_j(i)} \equiv C[j, h_j(i)]$

$\sigma_j(i) \equiv s_j(i)$

$\tilde{x}_{ij} = C_{j h_j(i)} \cdot \sigma_j(i)$

S : largest k coordinates of x

A_i : event that $\exists\, i' \in S \setminus i$ s.t. $h_j(i) = h_j(i')$

$$\Pr\left[|x_i - \tilde{x}_{ij}| \ge \frac{\varepsilon}{\sqrt{k}} err_2^k\right] = \Pr[A_i] \cdot \Pr\left[|x_i - \tilde{x}_{ij}| \ge \frac{\varepsilon}{\sqrt{k}} err_2^k \,\middle|\, A_i\right]$$
$$+ \Pr[\bar{A_i}] \cdot \Pr\left[|x_i - \tilde{x}_{ij}| \ge \frac{\varepsilon}{\sqrt{k}} err_2^k \,\middle|\, \bar{A_i}\right]$$
$$\le \Pr[A_i] + \Pr\left[|x_i - \tilde{x}_{ij}| \ge \frac{\varepsilon}{\sqrt{k}} err_2^k \,\middle|\, \bar{A_i}\right]$$

$\Pr[A_i] \le \dfrac{1}{w} \cdot k = \dfrac{\varepsilon^2}{3}$

$$\mathbb{E}\left[\,|x_i - \tilde{x}_{ij}|^2 \mid \bar{A}_i\right] = \frac{(err_2^k(x))^2}{w}$$

Warning: I lied here! The analysis of last time does not work with conditioning. Think about how to fix this issue.

$$\Pr\left[\,|x_i - \tilde{x}_{ij}| \geq \frac{\varepsilon}{\sqrt{k}}\, err_2^k(x) \,\middle|\, \bar{A}_i\right] \leq \frac{1}{w} \cdot \frac{k}{\varepsilon^2} = \frac{1}{3}$$

$$\Pr\left[\,|x_i - \tilde{x}_{ij}| \geq \frac{\varepsilon}{\sqrt{k}}\, err_2^k(x)\right] \leq \frac{1}{3} + \frac{\varepsilon^2}{3}$$

Using median $\{\tilde{x}_{ij}\}$ we get w.h.p.

$$|\hat{x}_i - x_i| \leq \frac{\varepsilon}{\sqrt{k}}\, err_2^k(x)$$

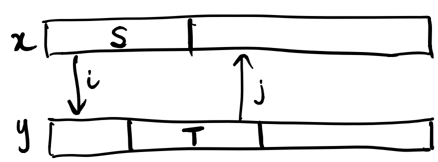
Lemma: Given $\|x-y\|_\infty \leq \frac{\varepsilon}{\sqrt{k}}\, err_2^k(x)$

x: unknown vector
y: count sketch estimates

T: largest k coordinates of y

$z = y_T \qquad \|x-z\|_2 \leq (1+5\varepsilon)\, err_2^k(x)$

Pf:



Break down error $\|x-z\|_2^2$ into 3 terms

$$\|x-z\|_2^2 = \|(x-z)_T\|^2 + \|x_{S\setminus T}\|^2 + \|x_{[n]\setminus(S\cup T)}\|^2$$

Let $E = err_2^k(x)$

$$\|x-y\|_\infty \leq \frac{\varepsilon}{\sqrt{k}}\, err_2^k(x) = \frac{\varepsilon \cdot E}{\sqrt{k}}$$

$$\|(x-z)_T\|^2 \leq k \cdot \frac{\varepsilon^2}{k} E^2 = \varepsilon^2 E^2$$

$\forall i \in S\setminus T \qquad |x_i| \leq |y_i| + \frac{\varepsilon}{\sqrt{k}} E$

$\forall j \in T\setminus S \qquad -|x_j| \leq -|y_j| + \frac{\varepsilon}{\sqrt{k}} E \Leftarrow |x_j| \geq |y_j| - \frac{\varepsilon}{\sqrt{k}} E$

$$|x_i| - |x_j| \leq |y_i| - |y_j| + \frac{2\varepsilon}{\sqrt{k}} E \leq \frac{2\varepsilon}{\sqrt{k}} E$$

$$|y_j| \geq |y_i|$$

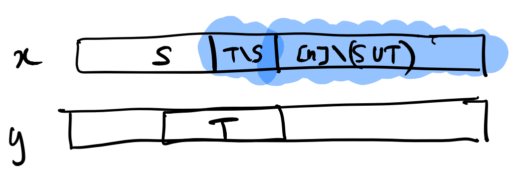
Let $a = \max\limits_{i \in S\setminus T} |x_i|$, $b = \min\limits_{j \in T\setminus S} |x_j|$

$$a - b \leq \frac{2\varepsilon}{\sqrt{k}} E$$

$$\begin{aligned}
\|x_{S\setminus T}\|_2^2 &\leq a^2 |S\setminus T| \\
&\leq \left(b + \frac{2\varepsilon}{\sqrt{k}} E\right)^2 |S\setminus T| \\
&\leq \left(\frac{\|x_{T\setminus S}\|_2}{\sqrt{|S\setminus T|}} + \frac{2\varepsilon}{\sqrt{k}} E\right)^2 |S\setminus T| \\
&\leq \left(\|x_{T\setminus S}\|_2 + 2\varepsilon E\right)^2 \qquad |S\setminus T| \leq k \\
&= \|x_{T\setminus S}\|_2^2 + 4\varepsilon E \underbrace{\|x_{T\setminus S}\|_2}_{\leq E} + 4\varepsilon^2 E^2 \\
&\leq \|x_{T\setminus S}\|_2^2 + 8\varepsilon E^2 \qquad (\varepsilon^2 \leq \varepsilon)
\end{aligned}$$

$$\begin{aligned}
\|x - z\|_2^2 &= \|(x-z)_T\|^2 + \|x_{S\setminus T}\|^2 + \|x_{[n]\setminus(S\cup T)}\|^2 \\
&\leq \varepsilon^2 E^2 + 8\varepsilon E^2 + \underbrace{\|x_{T\setminus S}\|_2^2 + \|x_{[n]\setminus(S\cup T)}\|^2}_{}
\end{aligned}$$

$$\|x_{T\setminus S}\|_2^2 + \|x_{[n]\setminus(S\cup T)}\|_2^2 = E^2$$



$$\|x - z\|_2^2 \leq \varepsilon^2 E^2 + 8\varepsilon E^2 + E^2 \leq (1 + 10\varepsilon) E^2$$

$$\|x - z\|_2 \leq (1 + 5\varepsilon) E$$

Next time: ℓ_0-sampling

Priority Sampling

Given stream of items with weights $w_1 \dots w_n$

Want to store representative sample S of the items with weights $\hat{w}_i$

Given query $I \subset [n]$, we would like to estimate

$$w_I = \sum_{i \in I} w_i \quad \text{by} \quad \sum_{i \in I \cap S} \hat{w}_i$$

1. For data $w_1 \dots w_n$ sample $u_i \in [0,1]$ uniformly at random independently
2. Priority $q_i = \frac{w_i}{u_i}$
3. Keep set S_K of largest K priorities seen so far as well as value τ of $(K+1)^{th}$ largest priority
4. $\hat{w}_i = \begin{cases} \max\{\tau, w_i\} & \text{if } i \in S_K \\ 0 & \text{otherwise} \end{cases}$

Given I, output $\hat{w}_I = \sum_{j \in I \cap S_K} \hat{w}_j$

Lemma: $\mathbb{E}[\hat{w}_i] = w_i$

Proof: Let $A(\tau')$ be event that $(K+1)$th highest priority is τ'

Given $A(\tau')$, for all $i \in S_K$, priority $q_i = \frac{w_i}{u_i} > \tau'$

weight $\hat{w}_i = \max(\tau', w_i)$

$i \notin S_K$, $q_i < \tau'$, $w_i' = 0$

$\mathbb{P}(i \in S \mid A(\tau'))$

① $w_i > \tau'$: $\mathbb{P}(i \in S_K \mid A(\tau')) = 1$, $\hat{w}_i = w_i$ $\quad \mathbb{E}[\hat{w}_i \mid A(\tau')] = w_i$

② $w_i \leq \tau'$: $\mathbb{P}(i \in S_K \mid A(\tau')) = \mathbb{P}(u_i \leq \frac{w_i}{\tau'}) = \frac{w_i}{\tau'}$

$\hat{w}_i = \tau'$

$\mathbb{E}[\hat{w}_i \mid A(\tau')] = w_i$

$$\mathbb{E}[\hat{w}_i] = w_i$$

$$\mathrm{Cov}[\hat{w}_i \ \hat{w}_j] = 0$$

Lemma $\mathbb{E}[\hat{w}_i \ \hat{w}_j] = w_i w_j$

<u>Lemma</u> $\mathbb{E}\left[\prod_{i \in I} \hat{w}_i\right] = \prod_{i=1}^{K} w_i \qquad |I| \leq K$

$= 0 \qquad |I| > K$

<u>Pf</u>: $|I| \leq K$

Let τ'' be the $(K - |I| + 1)$ th highest priority among the elements $j \neq i$

$A(\tau'')$: corresponding event

$$\mathbb{E}\left[\prod_{i \in I} \hat{w}_i \mid A(\tau'')\right] = \prod_{i \in I} w_i$$

Base case: $|I| = 1$

Case 1: $\exists h \in I \quad w_h > \tau''$ Clearly $h \in S_K$, $\hat{w}_h = w_h$

$$E\left[\prod_{i \in I} \hat{w}_i \mid A(\tau'')\right] = w_h \cdot \mathbb{E}\left[\prod_{i \in I \setminus \{h\}} \hat{w}_i \mid A(\tau'')\right]$$

Apply induction to $I \setminus \{h\}$

Case 2: For all $h \in I$, $w_h < \tau''$

Let q be min priority among items in I

If $q < \tau''$, then $\hat{w}_j = 0$ for some $j \in I$

$\Rightarrow$ product will be 0

Consider case when $q \geq \tau''$

$\prod_{i \in I} \left(\frac{w_i}{\tau''}\right) \leftarrow$ Pr all elements $i \in I$ have priority $> \tau''$

$\hat{w}_i = \tau''$

$$\mathbb{E}\left[\prod_{i \in I} \hat{w}_i\right] = \prod_{i \in I} (\tau'') \cdot \prod_{i \in I} \left(\frac{w_i}{\tau''}\right) = \prod_{i \in I} w_i$$