CS 4110

Programming Languages & Logics

Lecture 11
Weakest Preconditions

21 September 2016

Announcements

- Homework #3 due tonight at 11:59pm
- Homework #4 out now

Review: Decorating Programs

```
{true}
x := m;
y := 0;
while (n < x) do (
    x := x − n;
    y := y + 1
)
{                     }
```

Review: Decorating Programs

```
{true}

x := m;
y := 0;
while (n < x) do (
    x := x − n;
    y := y + 1
)

{n × y + x = m}
```

In other words, the program divides m by n, so y is the quotient and x is the remainder.

Review: Decorating Programs

```
{true}

x := m;
y := 0;
{$n \times y + x = m$}
while ($n < x$) do (
    {$n \times y + x = m \land n < x$}
    x := x − n;
    y := y + 1
    {$n \times y + x = m$}
)
{$n \times y + x = m \land x \leq n$}

{$n \times y + x = m$}
```

Review: Decorating Programs

```
{true}
$\{n \times 0 + m = m\}$
x := m;
$\{n \times 0 + x = m\}$
y := 0;
$\{n \times y + x = m\}$
while (n < x) do (
    $\{n \times y + x = m \;\land\; n < x\}$
    $\{n \times (y + 1) + (x - n) = m\}$
    x := x - n;
    $\{n \times (y + 1) + x = m\}$
    y := y + 1
    $\{n \times y + x = m\}$
)
$\{n \times y + x = m \;\land\; x \leq n\}$
$\{n \times y + x = m\}$
```

Review: Decorating Programs

```
$\{\textbf{true}\} \Rightarrow \{n \times 0 + m = m\}$
$x := m;$
$\{n \times 0 + x = m\}$
$y := 0;$
$\{n \times y + x = m\}$
$\textbf{while}\ (n < x)\ \textbf{do}\ ($
    $\{n \times y + x = m \ \wedge\ n < x\} \Rightarrow \{n \times (y + 1) + (x - n) = m\}$
    $x := x - n;$
    $\{n \times (y + 1) + x = m\}$
    $y := y + 1$
    $\{n \times y + x = m\}$
$)$
$\{n \times y + x = m \ \wedge\ x \leq n\} \Rightarrow \{n \times y + x = m\}$
```

Generating Preconditions

To fill in a precondition:

$$\{\quad\}\ c\ \{Q\}$$

there are many possible preconditions—and some are more useful than others.

Weakest Preconditions

Intuition: The weakest liberal precondition for c and Q is the weakest assertion P such that $\{P\}\ c\ \{Q\}$ is valid.

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More formally...

Definition (Weakest Liberal Precondition)

P is a weakest liberal precondition of c and Q written $wlp(c, Q)$ if:

$$\forall \sigma, I.\ \sigma \vDash_I P \iff (\mathcal{C}[\![c]\!]\ \sigma) \text{ undefined } \vee (\mathcal{C}[\![c]\!]\sigma) \vDash_I Q$$

Weakest Preconditions

$$wlp(\mathbf{skip}, P) \quad = \quad P$$

Weakest Preconditions

$$\begin{array}{rcl} wlp(\textbf{skip}, P) & = & P \\ wlp(\mathsf{x} := a, P) & = & P[a/\mathsf{x}] \end{array}$$

Weakest Preconditions

$$\begin{array}{rcl} wlp(\textbf{skip}, P) & = & P \\ wlp(\mathsf{x} := a, P) & = & P[a/\mathsf{x}] \\ wlp((c_1; c_2), P) & = & wlp(c_1, wlp(c_2, P)) \end{array}$$

Weakest Preconditions

$$\begin{array}{rcl}
wlp(\textbf{skip}, P) & = & P \\
wlp(\mathsf{x} := a, P) & = & P[a/\mathsf{x}] \\
wlp((c_1; c_2), P) & = & wlp(c_1, wlp(c_2, P)) \\
wlp(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, P) & = & (b \implies wlp(c_1, P)) \wedge \\
 & & (\neg b \implies wlp(c_2, P))
\end{array}$$

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wlp(\textbf{skip}, P) & = & P \\
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wlp(\textbf{while}\ b\ \textbf{do}\ c, P) & = & \bigwedge_i F_i(P)
\end{array}$$

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 & & (\neg b \implies wlp(c_2, P)) \\
wlp(\textbf{while } b \textbf{ do } c, P) & = & \bigwedge_i F_i(P)
\end{array}$$

where

$$\begin{array}{rcl}
F_0(P) & = & \textbf{true} \\
F_{i+1}(P) & = & (\neg b \implies P) \wedge (b \implies wlp(c, F_i(P)))
\end{array}$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

```
p := getPacket();
processPacket(p);
assert $P_{\text{safe}}$
```

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p := getPacket();
processPacket(p);
{$P_{\text{safe}}$}
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Applications of Weakest Preconditions

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```
p := getPacket();
{$P_{\text{filter}}(p)$};
processPacket(p);
{$P_{\text{safe}}$}
```

Applications of Weakest Preconditions

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Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

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p := getPacket();
assert $P_{\text{filter}}(p)$;
processPacket(p);
```

P_{filter} should be the *weakest* precondition to avoid ruling out legitimate inputs.

David Brumley, Hao Wang, Somesh Jha, and Dawn Song. “Creating Vulnerability Signatures Using Weakest Preconditions.” In *Computer Security Foundations* (CSF), 2007.

Properties of Weakest Preconditions

Lemma (Correctness of Weakest Preconditions)

$$\begin{aligned}
&\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\\
&\qquad \vDash \{wlp(c, Q)\}\, c\, \{Q\} \;\; \textit{and}\\
&\qquad \forall R \in \textbf{Assn}.\; \vDash \{R\}\, c\, \{Q\}\; \textit{implies}\; (R \implies wlp(c, Q))
\end{aligned}$$

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\end{aligned}$$

Lemma (Provability of Weakest Preconditions)

$$\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\; \vdash \{wlp(c, Q)\}\, c\, \{Q\}$$

Soundness and Completeness

Soundness: If we can prove it, then it’s actually true.

Completeness: If it’s true, then a proof exists.

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Definition (Soundness)

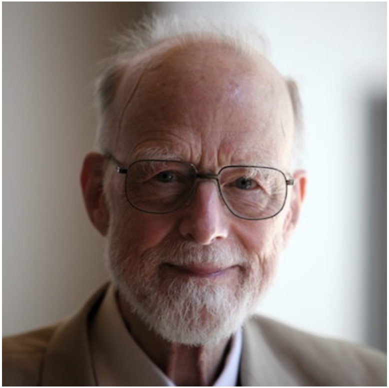
If $\vdash \{P\}\ c\ \{Q\}$ then $\models \{P\}\ c\ \{Q\}$.

Completeness: If it's true, then a proof exists.

Definition (Completeness)

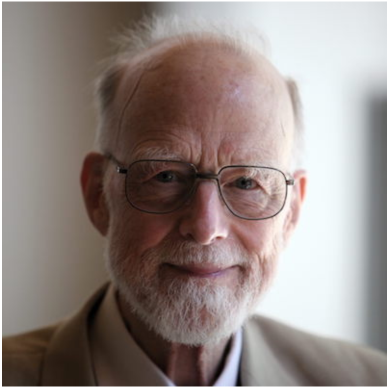
If $\models \{P\}\ c\ \{Q\}$ then $\vdash \{P\}\ c\ \{Q\}$.

vs.



Kurt Gödel

vs.



Sir Tony Hoare

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \models \{P\}\, c\, \{Q\}\ \textit{implies}\ \vdash \{P\}\, c\, \{Q\}.$

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \models \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

Proof Sketch.

Let $\{P\}\, c\, \{Q\}$ be a valid partial correctness specification.

By the first Lemma we have $\models P \implies wlp(c, Q)$.

By the second Lemma we have $\vdash \{wlp(c, Q)\}\, c\, \{Q\}$.

We conclude $\vdash \{P\}\, c\, \{Q\}$ using the CONSEQUENCE rule. □