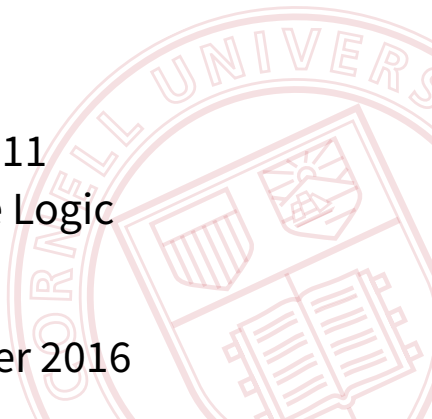# CS 4110

Programming Languages & Logics

Lecture 11
More Hoare Logic

19 September 2016

Announcements

- New TA with new office hours (welcome back, Andrew!)
  - Monday usually; Friday this week
- Homework 2 returned
  - Out of 36, $\bar{x} = 28.5$, $\sigma = 6.3$, median 29.5

A Recipe for Induction Over Derivations

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   d. Does the goal have premises from the same relation? If not, this is a base case. Reason directly.
   e. If so, this is an inductive case. Apply P to those subderivations you marked with vertical dots. Write down the resulting conclusion. Use that fact to prove $P(\mathcal{D})$ for this derivation.

Overview

Last time
- Hoare Logic

Today
- "Decorated" programs
- Weakest Preconditions

Review: Hoare Logic

$$\frac{}{\vdash \{P\} \textbf{ skip } \{P\}} \text{ S\textsc{kip}} \qquad \frac{}{\vdash \{P[a/x]\} \, x := a \, \{P\}} \text{ A\textsc{ssign}}$$

$$\frac{\vdash \{P\} \, c_1 \, \{R\} \qquad \vdash \{R\} \, c_2 \, \{Q\}}{\vdash \{P\} \, c_1; c_2 \, \{Q\}} \text{ S\textsc{eq}}$$

$$\frac{\vdash \{P \wedge b\} \, c_1 \, \{Q\} \qquad \vdash \{P \wedge \neg b\} \, c_2 \, \{Q\}}{\vdash \{P\} \textbf{ if } b \textbf{ then } c_1 \textbf{ else } c_2 \, \{Q\}} \text{ I\textsc{f}}$$

$$\frac{\vdash \{P \wedge b\} \, c \, \{P\}}{\vdash \{P\} \textbf{ while } b \textbf{ do } c \, \{P \wedge \neg b\}} \text{ W\textsc{hile}}$$

$$\frac{\models P \Rightarrow P' \qquad \vdash \{P'\} \, c \, \{Q'\} \qquad \models Q' \Rightarrow Q}{\vdash \{P\} \, c \, \{Q\}} \text{ C\textsc{onsequence}}$$

Decorated Programs

Observation: Once we've identified loop invariants and uses of consequence, the structure of a Hoare logic is determined!

Notation: Can write proofs by "decorating" programs with:
- A precondition ($\{P\}$)
- A postcondition ($\{Q\}$)
- Invariants ($\{I\}$**while** b **do** c)
- Uses of consequence $\{R\} \Rightarrow \{S\}$
- Assertions between sequences $c_1; \{T\}c_2$

A decorated program describes a valid Hoare logic proof if the rest of the proof tree's structure is implied. (Caveats: Invariants are constrained, etc.)

Example: Decorated Factorial

$$\{x = n \land n > 0\}$$

```
y := 1;
while x > 0 do {
      y := y * x;
      x := x - 1
}
```

$$\{y = n!\}$$

Example: Decorated Factorial

$$\{x = n \land n > 0\} \Rightarrow$$
$$\{1 = 1 \land x = n \land n > 0\}$$
$$y := 1;$$
$$\{y = 1 \land x = n \land n > 0\} \Rightarrow$$
$$\{y * x! = n! \land x \geq 0\}$$
while $x > 0$ **do** {
$$\quad \{y * x! = n! \land x > 0 \land x \geq 0\} \Rightarrow$$
$$\quad \{y * x * (x - 1)! = n! \land (x - 1) \geq 0\}$$
$$\quad y := y * x;$$
$$\quad \{y * (x - 1)! = n! \land (x - 1) \geq 0\}$$
$$\quad x := x - 1$$
$$\quad \{y * x! = n! \land x \geq 0\}$$
}
$$\{y * x! = n! \land (x \geq 0) \land \neg(x > 0)\} \Rightarrow$$
$$\{y = n!\}$$

Informal Rules for Decoration

Check whether a decorated program represents a valid proof using local consistency checks.

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Check whether a decorated program represents a valid proof using local consistency checks.

For **skip**, the precondition and postcondition should be the same:

$$\{P\}$$
$$\textbf{skip}$$
$$\{P\}$$

Informal Rules for Decoration

For sequences, $\{P\}\ c_1\ \{R\}$ and $\{R\}\ c_2\ \{Q\}$ must be (recursively) locally consistent:

$$\{P\}$$
$$c_1;$$
$$\{R\}$$
$$c_2$$
$$\{Q\}$$

Informal Rules for Decoration

Assignment should use the substitution from the rule:

$$\{P[a/x]\}$$
$$x := a$$
$$\{P\}$$

Informal Rules for Decoration

An **if** is locally consistent when both branches are locally consistent after adding the branch condition to each:

$$\{P\}$$
if b **then**
$$\{P \wedge b\}$$
$$c_1$$
$$\{Q\}$$
else
$$\{P \wedge \neg b\}$$
$$c_2$$
$$\{Q\}$$
$$\{Q\}$$

Informal Rules for Decoration

Decorate a **while** with the loop invariant:

$$\{P\}$$
$$\textbf{while } b \textbf{ do}$$
$$\{P \wedge b\}$$
$$c$$
$$\{P\}$$
$$\{P \wedge \neg b\}$$

Informal Rules for Decoration

To capture the CONSEQUENCE rule, you can always write a (valid) implication:

$$\{P\} \Rightarrow$$
$$\{Q\}$$

Example

```
{                         }
while (0 < y) do (
   x := x + 1;
   y := y − 1
)
{            }
```

Example

$\{x = m \ \wedge \ y = n \wedge 0 \leq n\}$

while $(0 < y)$ **do** (
  $x := x + 1;$
  $y := y - 1$
)

$\{x = m + n\}$

Example

$$\{x = m \land y = n \land 0 \le n\} \Rightarrow$$
$$\{I\}$$
while $(0 < y)$ **do** (
$$\quad \{I \land 0 < y\} \Rightarrow$$
$$\quad \{I[y - 1/y][x + 1/x]\}$$
$$\quad x := x + 1;$$
$$\quad \{I[y - 1/y]\}$$
$$\quad y := y - 1$$
$$\quad \{I\}$$
$$)$$
$$\{I \land 0 \not< y\} \Rightarrow$$
$$\{x = m + n\}$$

Where I is $(x = m + n - y) \land 0 \le y$.

Example

```
{     }
while (x ≠ 0) do (
  x := x − 1
)
{     }
```

Example

$\{\textbf{true}\}$
while $(x \neq 0)$ **do** (
    $x := x - 1$
)
$\{x = 0\}$

Example

$\{ \qquad \}$
y := 1
while $(0 < x)$ **do** (
  x := x − 1;
  y := y ∗ 2
)
$\{ \qquad \}$

Example

$\{x = n \wedge 0 \leq n\}$

y := 1
while $(0 < x)$ **do** (
   x := x − 1;
   y := y ∗ 2
)
$\{y = 2^n\}$