

CS 4110

Programming Languages & Logics

Lecture 23 Type Inference



Review: Polymorphic λ -Calculus

Syntax

$$\begin{aligned} e &::= n \mid x \mid \lambda x:\tau. e \mid e_1 e_2 \mid \Lambda X. e \mid e [\tau] \\ v &::= n \mid \lambda x:\tau. e \mid \Lambda X. e \end{aligned}$$

Dynamic Semantics

$$E ::= [\cdot] \mid E e \mid v E \mid E [\tau]$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']} \quad \frac{}{(\lambda x:\tau. e) v \rightarrow e\{v/x\}} \quad \frac{}{(\Lambda X. e) [\tau] \rightarrow e\{\tau/X\}}$$

Review: Polymorphic λ -Calculus

$$\frac{}{\Delta, \Gamma \vdash n : \mathbf{int}} \qquad \frac{\Gamma(x) = \tau}{\Delta, \Gamma \vdash x : \tau}$$

$$\frac{\Delta, \Gamma, x : \tau \vdash e : \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \qquad \frac{\Delta, \Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Delta, \Gamma \vdash e_2 : \tau}{\Delta, \Gamma \vdash e_1 e_2 : \tau'}$$

$$\frac{\Delta \cup \{X\}, \Gamma \vdash e : \tau}{\Delta, \Gamma \vdash \Lambda X. e : \forall X. \tau} \qquad \frac{\Delta, \Gamma \vdash e : \forall X. \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash e [\tau] : \tau' \{\tau / X\}}$$

Review: Polymorphic λ -Calculus

Polymorphism let us write a doubling function that works for *any* type of function:

$$\text{double} \triangleq \Lambda X. \lambda f:X \rightarrow X. \lambda x:X. f(fx).$$

The type of this expression is:

$$\forall X. (X \rightarrow X) \rightarrow X \rightarrow X$$

You can use the polymorphic function by providing a type:

$$\text{double } [\mathbf{int}] (\lambda n:\mathbf{int}. n + 1) 7$$

Type Inference

In languages like OCaml, programmers don't have to annotate their programs with $\forall X. \tau$ or $e [\tau]$.

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For example, we can write:

```
let double f x = f (f x)
```

and OCaml will figure out that the type is:

$$('a \rightarrow 'a) \rightarrow 'a \rightarrow 'a$$

which is equivalent to the same System F type:

$$\forall A. (A \rightarrow A) \rightarrow A \rightarrow A$$

Type Inference

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We can also write

```
double (fun x → x+1) 7
```

and OCaml will infer that the polymorphic function `double` is instantiated at the type `int`.

ML Polymorphism

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- Not prenex: $(\forall \alpha. \alpha \rightarrow \alpha) \rightarrow \mathbf{int}$

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Examples

- Prenex: $\forall \alpha. \alpha \rightarrow \alpha$
- Not prenex: $(\forall \alpha. \alpha \rightarrow \alpha) \rightarrow \mathbf{int}$

These restrictions have the following practical ramifications:

- Can't instantiate type variables with polymorphic types
- Can't put a polymorphic type on the left of an arrow

Example

These restrictions mean that certain terms that are typeable in System F are not typeable in ML!

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```
OCaml version 4.01.0
```

```
# fun x -> x x;;
```

```
Error: This expression has type 'a -> 'b  
      but an expression was expected of type 'a  
      The type variable 'a occurs inside 'a -> 'b
```

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Type inference for the STLC means guessing a τ in every abstraction in an *untyped* version:

$$\lambda x. e$$

to produce a *typed* program:

$$\lambda x:\tau. e$$

that we can use in the typing rule for functions.

Example

Here's an untyped program:

$\lambda a. \lambda b. \lambda c. \text{if } a (b + 1) \text{ then } b \text{ else } c$

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- the argument type of a must be the same as $b + 1$
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- the type of c must be the same as b

Putting all these pieces together:

$\lambda a : \mathbf{int} \rightarrow \mathbf{bool}. \lambda b : \mathbf{int}. \lambda c : \mathbf{int}. \text{if } a(b + 1) \text{ then } b \text{ else } c$

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$$\Gamma \vdash e : \tau \mid C$$

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We introduce a new judgment:

$$\Gamma \vdash e : \tau \mid C$$

Given a typing context Γ and an expression e , it generates a set of *constraints*—equations between types.

If these constraints are solvable, then e can be well-typed in Γ .

A solution to a set of constraints is a *type substitution* σ that, for each equation, makes both sides syntactically equal.

STLC for Type Inference

Let's define the type inference judgment for this STLC language:

$$\begin{aligned} e &::= x \mid \lambda x:\tau. e \mid e_1 e_2 \mid n \mid e_1 + e_2 \\ \tau &::= \mathbf{int} \mid X \mid \tau_1 \rightarrow \tau_2 \end{aligned}$$

You can use a type variable X wherever you want to have a type inferred.

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$$\frac{x:\tau \in \Gamma}{\Gamma \vdash x:\tau \mid \emptyset} \text{CT-VAR}$$

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$$\frac{\Gamma \vdash e_1:\tau_1 \mid C_1 \quad \Gamma \vdash e_2:\tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2:\mathbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \mathbf{int}, \tau_2 = \mathbf{int}\}} \text{CT-ADD}$$

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$$\frac{\Gamma, x:\tau_1 \vdash e:\tau_2 \mid C}{\Gamma \vdash \lambda x:\tau_1. e:\tau_1 \rightarrow \tau_2 \mid C} \text{CT-ABS}$$

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$$\frac{\begin{array}{l} \Gamma \vdash e_1:\tau_1 \mid C_1 \quad \Gamma \vdash e_2:\tau_2 \mid C_2 \\ X \text{ fresh} \quad C' = C_1 \cup C_2 \cup \{\tau_1 = \tau_2 \rightarrow X\} \end{array}}{\Gamma \vdash e_1 e_2:X \mid C'} \text{CT-APP}$$

Solving Constraints

A *type substitution* is a finite map from type variables to types.

Example: The substitution

$[X \mapsto \mathbf{int}, Y \mapsto \mathbf{int} \rightarrow \mathbf{int}]$

maps type variable X to \mathbf{int} and Y to $\mathbf{int} \rightarrow \mathbf{int}$.

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We don't need to worry about avoiding variable capture: all type variables are “free.”

Given two substitutions σ_1 and σ_2 , we write $\sigma_1 \circ \sigma_2$ for their composition: $(\sigma_1 \circ \sigma_2)(\tau) = \sigma_1(\sigma_2(\tau))$.

Unification

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We say that a substitution σ *unifies* constraint $\tau = \tau'$ if $\sigma(\tau) = \sigma(\tau')$.

We say that substitution σ *satisfies* (or *unifies*) set of constraints C if σ unifies every constraint in C .

Unification

If:

- $\Gamma \vdash e : \tau \mid C$, and
- σ satisfies C ,

then e has type τ' under Γ ,
where $\sigma(\tau) = \tau'$.

If there are no substitutions that satisfy C , then e is not typeable.

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If there are no substitutions that satisfy C , then e is not typeable.

So let's find a substitution σ that unifies a set of constraints C !

Unification Algorithm

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else if $\tau = X$ and X not a free variable of τ' then

$unify(C' \{\tau' / X\}) \circ [X \mapsto \tau']$

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else if $\tau = \tau_0 \rightarrow \tau_1$ and $\tau' = \tau'_0 \rightarrow \tau'_1$ then

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else if $\tau = \tau_0 \rightarrow \tau_1$ and $\tau' = \tau'_0 \rightarrow \tau'_1$ then

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else

fail

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The solution, if it exists, is the most general solution: if $\sigma = \text{unify}(C)$ and σ' is a solution to C , then there is some σ'' such that $\sigma' = (\sigma'' \circ \sigma)$.