CS 4110

Programming Languages & Logics

Lecture 4
Inductive Proof and Large-Step Semantics

Review

So far we've:

- Defined a simple language of arithmetic expressions
- Formalized its semantics as a "small-step" relation:
  $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$

Review

So far we've:

- Defined a simple language of arithmetic expressions
- Formalized its semantics as a "small-step" relation: $\langle \sigma, e \rangle \to \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \to^* \langle \sigma', e' \rangle$

Today we'll:

- Prove some properties of the small-step relation by induction
- Develop an alternate semantics based on a "large-step" relation
- Prove the equivalence of the two semantics

Large-Step Semantics

Idea: Define a new relation that captures the *complete* evaluation of an expression.

Formally: Define a relation $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$. Our new $\Downarrow$ binary relation has this type:

$$\Downarrow \subseteq (\textbf{Store} \times \textbf{Exp}) \times (\textbf{Store} \times \textbf{Int})$$

Intuition: Completely evaluating the expression e in store σ produces the number n while updating the store to σ'.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \textsc{Var}$$

Integers

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle} \textsc{Int}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle}\ \textsc{Add}$$

Multiplication

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \textsc{Mul}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 \,;\; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \textsc{Assgn}$$

Large-Step Semantics

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Int} \qquad\qquad \frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Var}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle}\ \textsc{Add}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle}\ \textsc{Mul}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 \,;\, e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}\ \textsc{Assgn}$$

Example

Assume that $\sigma(bar) = 7$. Let $\sigma' = \sigma[foo \mapsto 3]$.

$$
\dfrac{
  \dfrac{}{\langle \sigma', foo \rangle \Downarrow \langle \sigma', 3 \rangle}\ \textsc{Var}
  \qquad
  \dfrac{}{\langle \sigma', bar \rangle \Downarrow \langle \sigma', 7 \rangle}\ \textsc{Var}
}{
  \langle \sigma', foo * bar \rangle \Downarrow \langle \sigma', 21 \rangle
}\ \textsc{Mul}
$$

$$
\dfrac{
  \dfrac{}{\langle \sigma, 3 \rangle \Downarrow \langle \sigma, 3 \rangle}\ \textsc{Int}
  \qquad
  \dfrac{
    \dfrac{}{\langle \sigma', foo \rangle \Downarrow \langle \sigma', 3 \rangle}\ \textsc{Var}
    \quad
    \dfrac{}{\langle \sigma', bar \rangle \Downarrow \langle \sigma', 7 \rangle}\ \textsc{Var}
  }{
    \langle \sigma', foo * bar \rangle \Downarrow \langle \sigma', 21 \rangle
  }\ \textsc{Mul}
}{
  \langle \sigma, foo := 3 \,;\, foo * bar \rangle \Downarrow \langle \sigma', 21 \rangle
}\ \textsc{Assgn}
$$

Equivalence

Theorem (Equivalence of small-step and large-step)

$$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \textit{ if and only if } \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

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To streamline the proof, we'll use the following multi-step relation:

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle}\ \textsc{Refl}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \qquad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle}\ \textsc{Trans}$$

Equivalence

Theorem (Equivalence of small-step and large-step)

$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$ *if and only if* $\langle \sigma, e \rangle \to^* \langle \sigma', n \rangle$

Lemma

1. *If* $\langle \sigma, e \rangle \to^* \langle \sigma', n \rangle$*, then:*
   - $\langle \sigma, e + e_2 \rangle \to^* \langle \sigma', n + e_2 \rangle$
   - $\langle \sigma, n_1 + e \rangle \to^* \langle \sigma', n_1 + n \rangle$
   - $\langle \sigma, e * e_2 \rangle \to^* \langle \sigma', n * e_2 \rangle$
   - $\langle \sigma, n_1 * e \rangle \to^* \langle \sigma', n_1 * n \rangle$
   - $\langle \sigma, x := e \,;\; e_2 \rangle \to^* \langle \sigma', x := n \,;\; e_2 \rangle$

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   - $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
   - $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
   - $\langle \sigma, x := e \,;\ e_2 \rangle \rightarrow^* \langle \sigma', x := n \,;\ e_2 \rangle$
2. *If* $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ *and* $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$, *then* $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$

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   - $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
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3. *If* $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$ *and* $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$*, then* $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$