



## 1 Polymorphism in OCaml

In languages like OCaml, programmers don't have to annotate their programs with  $\forall X. \tau$  or  $e [\tau]$ . Both are automatically inferred by the compiler, although the programmer can specify types explicitly if desired.

For example, we can write

```
let double f x = f (f x)
```

and OCaml will figure out that the type is

```
('a → 'a) → 'a → 'a
```

which is roughly equivalent to the System F type

$$\forall A. (A \rightarrow A) \rightarrow A \rightarrow A$$

We can also write

```
double (fun x → x+1) 7
```

and OCaml will infer that the polymorphic function `double` is instantiated at the type `int`.

The polymorphism in ML is not, however, exactly like the polymorphism in System F. ML restricts what types a type variable may be instantiated with. Specifically, type variables can not be instantiated with polymorphic types. Also, polymorphic types are not allowed to appear on the left-hand side of arrows—i.e., a polymorphic type cannot be the type of a function argument. This form of polymorphism is known as *let-polymorphism* (due to the special role played by `let` in ML), or *prenex polymorphism*. These restrictions ensure that *type inference* is decidable.

An example of a term that is typable in System F but not typable in ML is the self-application expression  $\lambda x. x x$ . Try typing

```
fun x → x x
```

in the top-level loop of OCaml, and see what happens...

## 2 Type Inference

In the simply-typed lambda calculus, we explicitly annotate the type of function arguments:  $\lambda x:\tau. e$ . These annotations are used in the typing rule for functions.

$$\frac{\Gamma, x:\tau \vdash e:\tau'}{\Gamma \vdash \lambda x:\tau. e:\tau \rightarrow \tau'}$$

Suppose that we didn't want to provide type annotations for function arguments. We would need to guess a  $\tau$  to put into the type context.

Can we still type check our program without these type annotations? For the simply typed-lambda calculus (and many of the extensions we have considered so far), the answer is yes: we can *infer* (or *reconstruct*) the types of a program.

Let's consider an example to see how this type inference could work.

$$\lambda a. \lambda b. \lambda c. \text{if } a(b+1) \text{ then } b \text{ else } c$$

Since the variable  $b$  is used in an addition, the type of  $b$  must be **int**. The variable  $a$  must be some kind of function, since it is applied to the expression  $b+1$ . Since  $a$  has a function type, the type of the expression  $b+1$  (i.e., **int**) must be  $a$ 's argument type. Moreover, the result of the function application ( $a(b+1)$ ) is used as the test of a conditional, so it had better be the case that the result type of  $a$  is also **bool**. So the type of  $a$  should be **int**  $\rightarrow$  **bool**. Both branches of a conditional should return values of the same type, so the type of  $c$  must be the same as the type of  $b$ , namely **int**.

We can write the expression with the reconstructed types:

$$\lambda a:\mathbf{int} \rightarrow \mathbf{bool}. \lambda b:\mathbf{int}. \lambda c:\mathbf{int}. \text{if } a(b+1) \text{ then } b \text{ else } c$$

## 2.1 Constraint-based typing

We now present an algorithm that, given a typing context  $\Gamma$  and an expression  $e$ , produces a set of *constraints*—equations between types (including type variables)—that must be satisfied in order for  $e$  to be well-typed in  $\Gamma$ . We introduce *type variables*, which are just placeholders for types. We let metavariables  $X$  and  $Y$  range over type variables. The language we will consider is the lambda calculus with integer constants and addition. We assume that all function definitions contain a type annotation for the argument, but this type may simply be a type variable  $X$ .

$$\begin{aligned} e &::= x \mid \lambda x:\tau. e \mid e_1 e_2 \mid n \mid e_1 + e_2 \\ \tau &::= \mathbf{int} \mid X \mid \tau_1 \rightarrow \tau_2 \end{aligned}$$

To formally define type inference, we introduce a new typing relation:

$$\Gamma \vdash e:\tau \mid C$$

Intuitively, if  $\Gamma \vdash e:\tau \mid C$ , then expression  $e$  has type  $\tau$  provided that every constraint in the set  $C$  is satisfied.

We define the judgment  $\Gamma \vdash e:\tau \mid C$  with inference rules and axioms. When read from bottom to top, these inference rules provide a procedure that, given  $\Gamma$  and  $e$ , calculates  $\tau$  and  $C$  such that  $\Gamma \vdash e:\tau \mid C$ .

$$\begin{aligned} \text{CT-VAR} & \frac{x:\tau \in \Gamma}{\Gamma \vdash x:\tau \mid \emptyset} & \text{CT-INT} & \frac{}{\Gamma \vdash n:\mathbf{int} \mid \emptyset} \\ \text{CT-ADD} & \frac{\Gamma \vdash e_1:\tau_1 \mid C_1 \quad \Gamma \vdash e_2:\tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2:\mathbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \mathbf{int}, \tau_2 = \mathbf{int}\}} \end{aligned}$$

$$\text{CT-Abs} \frac{\Gamma, x:\tau_1 \vdash e:\tau_2 \mid C}{\Gamma \vdash \lambda x:\tau_1. e:\tau_1 \rightarrow \tau_2 \mid C}$$

$$\text{CT-App} \frac{\Gamma \vdash e_1:\tau_1 \mid C_1 \quad \Gamma \vdash e_2:\tau_2 \mid C_2 \quad C' = C_1 \cup C_2 \cup \{\tau_1 = \tau_2 \rightarrow X\}}{\Gamma \vdash e_1 e_2:X \mid C'} \quad X \text{ fresh}$$

Note that we must be careful with the choice of type variables—in particular, the type variable in the rule CT-APP must be chosen appropriately.

## 2.2 Unification

So what does it mean for a set of constraints to be satisfied? To answer this question, we define *type substitutions* (or just *substitutions*, when it's clear from context). A type substitution is a finite map from type variables to types. For example, we write  $[X \mapsto \mathbf{int}, Y \mapsto \mathbf{int} \rightarrow \mathbf{int}]$  for the substitution that maps type variable  $X$  to  $\mathbf{int}$ , and type variable  $Y$  to  $\mathbf{int} \rightarrow \mathbf{int}$ . Note that the same variable may occur in both the domain and range of a substitution. In that case, the intention is that the substitutions are performed simultaneously. For example the substitution  $[X \mapsto \mathbf{int}, Y \mapsto (\mathbf{int} \rightarrow X)]$  maps  $Y$  to  $\mathbf{int} \rightarrow X$ .

More formally, we define substitution of type variables as follows.

$$\sigma(X) = \begin{cases} \tau & \text{if } X \mapsto \tau \in \sigma \\ X & \text{if } X \text{ not in the domain of } \sigma \end{cases}$$

$$\sigma(\mathbf{int}) = \mathbf{int}$$

$$\sigma(\tau \rightarrow \tau') = \sigma(\tau) \rightarrow \sigma(\tau')$$

Note that we don't need to worry about avoiding variable capture, since there are no constructs in the language that bind type variables. If we had polymorphic types  $\forall X. \tau$  from the polymorphic lambda calculus, we would need to be concerned with this.

Given two substitutions  $\sigma$  and  $\sigma'$ , we write  $\sigma \circ \sigma'$  for their composition:  $(\sigma \circ \sigma')(\tau) = \sigma(\sigma'(\tau))$ .

### 2.2.1 Unification

Constraints are of the form  $\tau = \tau'$ . We say that a substitution  $\sigma$  *unifies* constraint  $\tau = \tau'$  if  $\sigma(\tau) = \sigma(\tau')$ . We say that substitution  $\sigma$  *satisfies* (or *unifies*) set of constraints  $C$  if  $\sigma$  unifies every constraint in  $C$ .

For example, the substitution  $\sigma = [X \mapsto \mathbf{int}, Y \mapsto (\mathbf{int} \rightarrow \mathbf{int})]$  unifies the constraint

$$X \rightarrow (X \rightarrow \mathbf{int}) = \mathbf{int} \rightarrow Y$$

since

$$\sigma(X \rightarrow (X \rightarrow \mathbf{int})) = \mathbf{int} \rightarrow (\mathbf{int} \rightarrow \mathbf{int}) = \sigma(\mathbf{int} \rightarrow Y)$$

So to solve a set of constraints  $C$ , we need to find a substitution that unifies  $C$ . More specifically, suppose that  $\Gamma \vdash e:\tau \mid C$ ; a solution for  $(\Gamma, e, \tau, C)$  is a pair  $\sigma, \tau'$  such that  $\sigma$  satisfies  $C$  and  $\sigma(\tau) = \tau'$ . If there are no substitutions that satisfy  $C$ , then we know that  $e$  is not typeable.

### 2.2.2 Unification algorithm

To calculate solutions to constraint sets, we use the idea, due to Hindley and Milner, of using *unification* to check that the set of solutions is non-empty, and to find a “best” solution (from which all other solutions can be easily generated). The unification algorithm is defined as follows:

$$\begin{aligned}
 \text{unify}(\emptyset) &= [] && \text{(the empty substitution)} \\
 \text{unify}(\{\tau = \tau'\} \cup C') &= \text{if } \tau = \tau' \text{ then} \\
 &\quad \text{unify}(C') \\
 &\quad \text{else if } \tau = X \text{ and } X \text{ not a free variable of } \tau' \text{ then} \\
 &\quad \quad \text{unify}(C' \{\tau' / X\}) \circ [X \mapsto \tau'] \\
 &\quad \text{else if } \tau' = X \text{ and } X \text{ not a free variable of } \tau \text{ then} \\
 &\quad \quad \text{unify}(C' \{\tau / X\}) \circ [X \mapsto \tau] \\
 &\quad \text{else if } \tau = \tau_o \rightarrow \tau_1 \text{ and } \tau' = \tau'_o \rightarrow \tau'_1 \text{ then} \\
 &\quad \quad \text{unify}(C' \cup \{\tau_o = \tau'_o, \tau_1 = \tau'_1\}) \\
 &\quad \text{else} \\
 &\quad \quad \text{fail}
 \end{aligned}$$

The check that  $X$  is not a free variable of the other type ensures that the algorithm doesn’t produce a cyclic substitution (e.g.,  $X \mapsto (X \rightarrow X)$ ), which doesn’t make sense with the finite types we currently have.

The unification algorithm always terminates. (How would you go about proving this?) Moreover, it produces a solution if and only if a solution exists. The solution found is the most general solution, in the sense that if  $\sigma = \text{unify}(C)$  and  $\sigma'$  is a solution to  $C$ , then there is some  $\sigma''$  such that  $\sigma' = (\sigma'' \circ \sigma)$ .