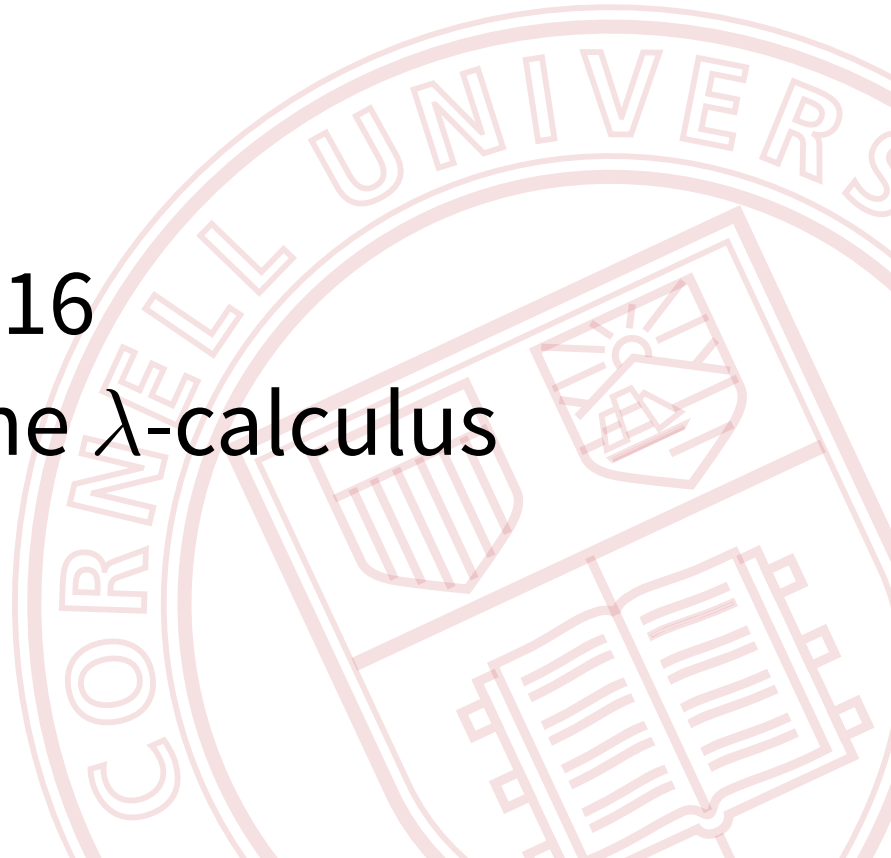
CS 4110

Programming Languages & Logics



Lecture 16
Programming in the λ-calculus

Review: Church Booleans

We can encode TRUE, FALSE, and IF, as:

$$\text{TRUE} \triangleq \lambda x.\, \lambda y.\, x$$
$$\text{FALSE} \triangleq \lambda x.\, \lambda y.\, y$$
$$\text{IF} \triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\ t\ f$$

This way, IF behaves how it ought to:

$$\text{IF TRUE } v_t\ v_f \rightarrow^* v_t$$
$$\text{IF FALSE } v_t\ v_f \rightarrow^* v_f$$

Review: Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$\begin{aligned} \overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\ \overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\ \overline{2} &\triangleq \lambda f.\, \lambda x.\, f\, (f\, x) \end{aligned}$$

We can define other functions on integers:

$$\mathrm{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\, (n\, f\, x)$$

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Recursive Functions

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We'd like to write it like this...

$$\text{FACT} \triangleq \lambda n.\ \text{IF}\ (\text{ISZERO}\ n)\ 1\ (\text{TIMES}\ n\ (\text{FACT}\ (\text{PRED}\ n)))$$

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$$\text{FACT} \triangleq \lambda n.\, \text{IF}\ (\text{ISZERO}\ n)\ 1\ (\text{TIMES}\ n\ (\text{FACT}\ (\text{PRED}\ n)))$$

In slightly more readable notation this is...

$$\text{FACT} \triangleq \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times \text{FACT}\ (n - 1)$$

...but this is an equation, not a definition!

Recursion removal trick

We can perform a “trick” to define a function FACT that satisfies the recursive equation on the previous slide.

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Define a new function FACT′ that takes a function f as an argument. Then, for “recursive” calls, it uses $f\ f$:

$$\text{FACT}' \triangleq \lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((f\ f)\ (n - 1))$$

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Then define FACT as FACT′ applied to itself:

$$\text{FACT} \triangleq \text{FACT}'\ \text{FACT}'$$

Example

Let’s try evaluating FACT on 3...

FACT 3

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$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') \ 3$

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$$\begin{aligned}\text{FACT } 3 &= (\text{FACT}' \ \text{FACT}')\ 3 \\ &= ((\lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\, f)\ (n-1)))\ \text{FACT}')\ 3\end{aligned}$$

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&\to (\lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\text{FACT}' \ \text{FACT}')\ (n-1)))\ 3
\end{aligned}
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&\to (\lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((\mathrm{FACT}'\ \mathrm{FACT}')\ (n-1)))\ 3 \\
&\to \textbf{if}\ 3 = 0\ \textbf{then}\ 1\ \textbf{else}\ 3 \times ((\mathrm{FACT}'\ \mathrm{FACT}')\ (3-1))
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&\to 3 \times ((\mathsf{FACT}'\ \mathsf{FACT}')\ (3-1)) \\
&= 3 \times (\mathsf{FACT}\ (3-1))
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&= 3 \times (\mathsf{FACT}\ (3-1)) \\
&\to \ldots \\
&\to 3 \times 2 \times 1 \times 1
\end{aligned}
$$

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&\to \ldots \\
&\to 3 \times 2 \times 1 \times 1 \\
&\to^* 6
\end{aligned}
$$

Fixed point combinators

Our “trick” requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f'\ f'$.

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Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1))$$

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Recall that if g if a fixed point of G, then $G\,g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$g\,5 = (G\,g)\,5$$

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Fixed point combinators

How can we generate the fixed point of *G*?

In denotational semantics, finding fixed points took a lot of math. In the λ-calculus, we just need a suitable combinator...

Y Combinator

The (infamous) Y combinator is defined as

$$\mathsf{Y} \triangleq \lambda f.\, (\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))$$

We say that Y is a *fixed point combinator* because Y f is a fixed point of f (for any lambda term f).

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What happens when we evaluate Y G under CBV?

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

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Example

Let’s see Z in action, on our function *G*.

FACT

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$$
\begin{array}{rl}
 & \text{FACT} \\
= & \text{Z } G
\end{array}
$$

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$$
\begin{aligned}
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= & (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\, (n-1))) \\
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\to & \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1 \\
 & \qquad \textbf{else}\ n \times ((\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y)\,(n-1))
\end{array}
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=_\beta & \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\lambda y.\,(\mathsf{Z}\,G)\,y)\,(n-1)
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= & \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\mathsf{FACT}\, (n-1))
\end{array}
$$

Other fixed point combinators

There are many (indeed infinitely many) fixed-point combinators. Here's a cute one:

$$\mathsf{Y}_k \triangleq (\mathsf{L}\,\mathsf{L})$$

where

$$\mathsf{L} \triangleq \lambda abcdefghijklmnopqstuvwxyzr.\ (r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r))$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

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We know that $\Theta\, f$ is a fixed point of f, so we have

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$$\Theta\, f = f\,(\Theta\, f).$$

We can write the following recursive equation:

$$\Theta = \lambda f.\, f\,(\Theta\, f)$$

Now use the recursion removal trick:

$$\begin{aligned} \Theta' &\triangleq \lambda t.\, \lambda f.\, f\,(t\, t\, f) \\ \Theta &\triangleq \Theta'\, \Theta' \end{aligned}$$

θ Example

$$\mathsf{FACT} = \Theta\, G$$

θ Example

$$\begin{aligned}\mathsf{FACT} &= \Theta\, G \\ &= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G\end{aligned}$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\,G
\end{aligned}
$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\,G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G)
\end{aligned}
$$

θ Example

$$
\begin{aligned}
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&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
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&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G) \\
&= G\,(\Theta\, G)
\end{aligned}
$$

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$$
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&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G) \\
&= G\,(\Theta\, G) \\
&= (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1)))\,(\Theta\, G) \\
&\to \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((\Theta\, G)\,(n-1)) \\
&= \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\mathsf{FACT}\,(n-1))
\end{aligned}
$$

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$e ::= x \mid \lambda x.\, e \mid e_1\, e_2$$
$$v ::= \lambda x.\, e$$

$$\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E\ e \mid v\ E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$E_1 = [\cdot]\,(\lambda x.\, x)$$
$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\,\lambda x.\, x$$

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$$E_1 = [\cdot]\,(\lambda x.\, x)$$
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$$E_2 = (\lambda z.\, z\, z)\,[\cdot]$$
$$E_2[\lambda x.\, \lambda y.\, x] = (\lambda z.\, z\, z)\,(\lambda x.\, \lambda y.\, x)$$

Examples

$$E_1 = [\cdot]\,(\lambda x.\, x)$$
$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\,\lambda x.\, x$$

$$E_2 = (\lambda z.\, z\, z)\,[\cdot]$$
$$E_2[\lambda x.\, \lambda y.\, x] = (\lambda z.\, z\, z)\,(\lambda x.\, \lambda y.\, x)$$

$$E_3 = ([\cdot]\,\lambda x.\, x\, x)\,((\lambda y.\, y)\,(\lambda y.\, y))$$
$$E_3[\lambda f.\, \lambda g.\, f\, g] = ((\lambda f.\, \lambda g.\, f\, g)\,\lambda x.\, x\, x)\,((\lambda y.\, y)\,(\lambda y.\, y))$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

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With this syntax:

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

The small-step rules are:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\, e$$

But the small-step rules are the same:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, e' \to e\{e'/x\}}\ \beta$$

Multi-Argument λ-calculus

Let's define a version of the λ-calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \ldots, x_n.\, e \mid e_0\, e_1\ \ldots\ e_n$$

Multi-Argument λ-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \ \ldots \ v_{i-1} \, E \, e_{i+1} \ \ldots \ e_n$$

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x_1, \ldots, x_n . \, e_0) \, v_1 \ \ldots \ v_n \to e_0\{v_1/x_1\}\{v_2/x_2\} \ldots \{v_n/x_n\}} \ \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 \, e_1 \ \ldots \ e_n$ from left to right.

Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

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We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

In definitional translation, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

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Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

$$\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x_1, \ldots, x_n.\, e]\!] &= \lambda x_1.\ \ldots\ \lambda x_n.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_0\ e_1\ e_2\ \ldots\ e_n]\!] &= (\ldots((\mathcal{T}[\![e_0]\!]\ \mathcal{T}[\![e_1]\!])\ \mathcal{T}[\![e_2]\!])\ldots\ \mathcal{T}[\![e_n]\!])
\end{aligned}$$

This translation *curries* the multi-argument λ-calculus.