CS 4110

Programming Languages & Logics

Lecture 17
Definitional Translation & Continuations

Products and Let

Syntax

$$
\begin{aligned}
e ::= \; & x \\
\mid \; & \lambda x.\, e \\
\mid \; & e_1\; e_2 \\
\mid \; & (e_1, e_2) \\
\mid \; & \#1\; e \\
\mid \; & \#2\; e \\
\mid \; & \mathsf{let}\; x = e_1 \;\mathsf{in}\; e_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= \; & \lambda x.\, e \\
\mid \; & (v_1, v_2)
\end{aligned}
$$

Products and Let

Evaluation Contexts

$$
\begin{aligned}
E ::= & \; [\cdot] \\
& \mid E\, e \\
& \mid v\, E \\
& \mid (E, e) \\
& \mid (v, E) \\
& \mid \#1\, E \\
& \mid \#2\, E \\
& \mid \text{let}\, x = E \text{ in } e_2
\end{aligned}
$$

Products and Let

Semantics

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\; v \to e\{v/x\}}\;\beta$$

$$\frac{}{\#1\;(v_1, v_2) \to v_1} \qquad \frac{}{\#2\;(v_1, v_2) \to v_2}$$

$$\frac{}{\mathsf{let}\; x = v \;\mathsf{in}\; e \to e\{v/x\}}$$

Products and Let

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\ e_2]\!] &= \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![(e_1, e_2)]\!] &= (\lambda x.\, \lambda y.\, \lambda f.\, f\, x\, y)\ \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![\#1\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, x) \\
\mathcal{T}[\![\#2\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, y) \\
\mathcal{T}[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] &= (\lambda x.\, \mathcal{T}[\![e_2]\!])\ \mathcal{T}[\![e_1]\!]
\end{aligned}
$$

Laziness

Consider the call-by-name λ-calculus...

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & e_1\ e_2 \\
\mid {} & \lambda x.\ e
\end{aligned}
$$

$$v ::= \lambda x.\ e$$

Semantics

$$\frac{e_1 \rightarrow e_1'}{e_1\ e_2 \rightarrow e_1'\ e_2} \qquad \frac{}{(\lambda x.\ e_1)\ e_2 \rightarrow e_1\{e_2/x\}}\ \beta$$

Laziness

Translation

$$\begin{aligned}
\mathcal{T}[\![x]\!] &= x\,(\lambda y.\, y) \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\; e_2]\!] &= \mathcal{T}[\![e_1]\!]\,(\lambda z.\, \mathcal{T}[\![e_2]\!]) \qquad z \text{ is not a free variable of } e_2
\end{aligned}$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$\begin{aligned} e ::= {} & x \\ \mid {} & \lambda x.\, e \\ \mid {} & e_0\; e_1 \\ \mid {} & \mathsf{ref}\; e \end{aligned}$$

$$v ::= \lambda x.\, e$$

References

Syntax

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\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & \,!e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= \;& x \\
\mid\;& \lambda x.\, e \\
\mid\;& e_0\; e_1 \\
\mid\;& \mathsf{ref}\; e \\
\mid\;& \,!e \\
\mid\;& e_1 := e_2
\end{aligned}
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$$
v ::= \lambda x.\, e
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Syntax

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\mid {} & \lambda x.\, e \\
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\mid {} & e_1 := e_2 \\
\mid {} & \ell
\end{aligned}
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References

Evaluation Contexts

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\begin{aligned}
E ::= {} & [\cdot] \\
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& \mid E := e
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E ::= {} & [\cdot] \\
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\mid {} & \text{ref}\ E \\
\mid {} & \ !E \\
\mid {} & E := e \\
\mid {} & v := E
\end{aligned}
$$

References

Semantics

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, E[e] \rangle \to \langle \sigma', E[e'] \rangle} \qquad \frac{}{\langle \sigma, (\lambda x.\, e)\; v \rangle \to \langle \sigma, e\{v/x\} \rangle}\;\beta$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\; v \rangle \to \langle \sigma[\ell \mapsto v], \ell \rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\; !\ell \rangle \to \langle \sigma, v \rangle}$$

$$\frac{}{\langle \sigma, \ell := v \rangle \to \langle \sigma[\ell \mapsto v], v \rangle}$$

References

Translation

...left as an exercise to the reader. ;-)

Adequacy

How do we know if a translation is correct?

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Every target evaluation should represent a source evaluation...

Definition (Soundness)

$\forall e \in \mathbf{Exp}_{\text{src}}.$ if $\mathcal{T}[\![e]\!] \to^*_{\text{trg}} v'$ then $\exists v.\ e \to^*_{\text{src}} v$
and v' equivalent to v

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...and every source evaluation should have a target evaluation:

Definition (Completeness)

$\forall e \in \mathbf{Exp}_{\text{src}}.$ if $e \to^*_{\text{src}} v$ then $\exists v'.\ \mathcal{T}[\![e]\!] \to^*_{\text{trg}} v'$ and v' equivalent to v

Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

$$\mathcal{T}[\![\lambda x.\, e]\!] = \lambda x.\, \mathcal{T}[\![e]\!]$$
$$\mathcal{T}[\![e_1\ e_2]\!] = \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!]$$

What can go wrong with this approach?

Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers…
- …or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions

Example

Consider the following expression:

$$(\lambda x.\, x)\ ((1+2)+3)+4$$

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k_3 &= \lambda c.\, k_2\, (c + 2)
\end{aligned}$$

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k_3 &= \lambda c.\, k_2\, (c + 2)
\end{aligned}$$

The original expression is equivalent to $k_3\ 1$, or:

$$(\lambda c.\, (\lambda b.\, (\lambda a.\, (\lambda v.\, (\lambda x.\, x)\, v)\, (a + 4))\, (b + 3))\, (c + 2))\, 1$$

Example (Continued)

Recall that let $x = e$ in e' is syntactic sugar for $(\lambda x.\, e')\; e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$\begin{aligned}
&\text{let } c = 1 \text{ in} \\
&\text{let } b = c + 2 \text{ in} \\
&\text{let } a = b + 3 \text{ in} \\
&\text{let } v = a + 4 \text{ in} \\
&(\lambda x.\, x)\; v
\end{aligned}$$

CPS Transformation

We write $\mathcal{CPS}[\![e]\!]\, k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh".

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$

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CPS Transformation

$$\mathcal{CPS}[\![n]\!]\, k = k\, n$$
$$\mathcal{CPS}[\![e_1 + e_2]\!]\, k = \mathcal{CPS}[\![e_1]\!]\, (\lambda n.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda m.\, k\, (n + m)))$$

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\mathcal{CPS}[\![(e_1, e_2)]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w)))
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\mathcal{CPS}[\![\#1\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v))
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\mathcal{CPS}[\![x]\!]\,k &= k\,x \\
\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k')
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