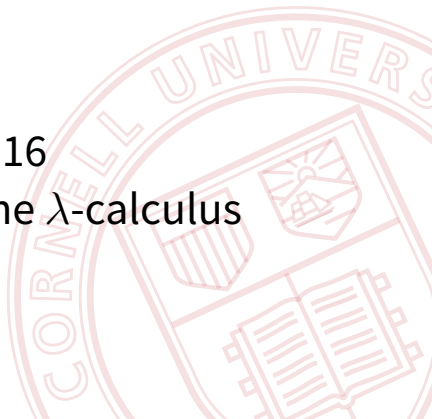


CS 4110

Programming Languages & Logics

Lecture 16 Programming in the λ -calculus



Review: Church Booleans

We can encode TRUE, FALSE, and IF, as:

$$\text{TRUE} \triangleq \lambda x. \lambda y. x$$

$$\text{FALSE} \triangleq \lambda x. \lambda y. y$$

$$\text{IF} \triangleq \lambda b. \lambda t. \lambda f. b \ t \ f$$

This way, IF behaves how it ought to:

$$\text{IF TRUE } v_t \ v_f \rightarrow^* v_t$$

$$\text{IF FALSE } v_t \ v_f \rightarrow^* v_f$$

Review: Church Numerals

Church numerals encode a number n as a function that takes f and x , and applies f to x n times.

$$\overline{0} \triangleq \lambda f. \lambda x. x$$

$$\overline{1} \triangleq \lambda f. \lambda x. f x$$

$$\overline{2} \triangleq \lambda f. \lambda x. f(f x)$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n. \lambda f. \lambda x. f(n f x)$$

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$$\text{TIMES} \triangleq \lambda n_1. \lambda n_2. n_1 (\text{PLUS } n_2) \bar{0}$$

$$\text{ISZERO} \triangleq \lambda n. n (\lambda z. \text{FALSE}) \text{TRUE}$$

Recursive Functions

How would we write recursive functions like factorial?

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We'd like to write it like this...

$$\text{FACT} \triangleq \lambda n. \text{IF } (\text{ISZERO } n) \text{ 1 } (\text{TIMES } n \text{ (FACT (PRED } n)))$$

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In slightly more readable notation this is...

$$\text{FACT} \triangleq \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n \times \text{FACT } (n - 1)$$

...but this is an equation, not a definition!

Recursion removal trick

We can perform a “trick” to define a function FACT that satisfies the recursive equation on the previous slide.

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Define a new function FACT':

$$\text{FACT}' \triangleq \lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n \times (ff(n - 1))$$

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Define a new function FACT':

$$\text{FACT}' \triangleq \lambda f. \lambda n. \text{if } n = 0 \text{ then } 1 \text{ else } n \times (ff(n - 1))$$

Then define FACT as FACT' applied to itself:

$$\text{FACT} \triangleq \text{FACT}' \text{ FACT}'$$

Example

Let's try evaluating FACT on 3...

FACT 3

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$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') 3$$

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So we have a technique for writing recursive functions: write a function f' that takes itself as an argument and define f as $f' f$.

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Consider factorial again. It is a fixed point of the following:

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Fixed point combinators

How can we generate the fixed point of G ?

In denotational semantics, finding fixed points took a lot of math. In the λ -calculus, we just need a suitable combinator...

Y Combinator

The (infamous) Y combinator is defined as

$$Y \triangleq \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

We say that Y is a *fixed point combinator* because $Y f$ is a fixed point of f (for any lambda term f).

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What happens when we evaluate Y G under CBV?

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

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Example

Let's see Z in action, on our function G .

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Example

Let's see Z in action, on our function G.

```
FACT
= Z G
= (λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))) G
→ (λx. G (λy. x x y)) (λx. G (λy. x x y))
→ G (λy. (λx. G (λy. x x y)) (λx. G (λy. x x y)) y)
= (λf. λn. if n = 0 then 1 else n × (f (n - 1)))
    (λy. (λx. G (λy. x x y)) (λx. G (λy. x x y)) y)
```


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= (λf. λn. if n = 0 then 1 else n × (f (n - 1)))
    (λy. (λx. G (λy. x x y)) (λx. G (λy. x x y)) y)
→ λn. if n = 0 then 1
    else n × ((λy. (λx. G (λy. x x y)) (λx. G (λy. x x y)) y) (n - 1))
```

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Other fixed point combinators

There are many (indeed infinitely many) fixed-point combinators. Here's a cute one:

$$Y_k \triangleq (L L)$$

where

$$L \triangleq \lambda abcdefghijklmnopqrstuvwxyzr. \\ (r(thisisafixedpointcombinator))$$

Turing's Fixed Point Combinator

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We know that Θf is a fixed point of f , so we have

$$\Theta f = f(\Theta f).$$

We can write the following recursive equation:

$$\Theta = \lambda f. f(\Theta f).$$

Now use the recursion removal trick:

$$\begin{aligned}\Theta' &\triangleq \lambda t. \lambda f. f(t t f) \\ \Theta &\triangleq \Theta' \Theta'\end{aligned}$$

θ Example

$$\text{FACT} = \Theta G$$

θ Example

$$\begin{aligned}\text{FACT} &= \Theta G \\ &= ((\lambda t. \lambda f. f(t t f)) (\lambda t. \lambda f. f(t t f))) G\end{aligned}$$

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θ Example

$$\begin{aligned}\text{FACT} &= \Theta G \\ &= ((\lambda t. \lambda f. f(t t f)) (\lambda t. \lambda f. f(t t f))) G \\ &\rightarrow (\lambda f. f((\lambda t. \lambda f. f(t t f)) (\lambda t. \lambda f. f(t t f)) f)) G \\ &\rightarrow G((\lambda t. \lambda f. f(t t f)) (\lambda t. \lambda f. f(t t f)) G)\end{aligned}$$

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Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ -calculus.

Can we define a *real* programming language by translating everything in it into the λ -calculus?

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Can we define a *real* programming language by translating everything in it into the λ -calculus?

In **definitional translation**, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.

Review: Call-by-Value

Here are the syntax and CBV semantics of λ -calculus:

$$\begin{aligned} e &::= x \mid \lambda x. e \mid e_1 e_2 \\ v &::= \lambda x. e \end{aligned}$$

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \qquad \frac{e \rightarrow e'}{v e \rightarrow v e'}$$

$$\frac{}{(\lambda x. e) v \rightarrow e\{v/x\}} \beta$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the “interesting” reductions.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

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An evaluation context E is an expression with a “hole” in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

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We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e .

Examples

$$E_1 = [\cdot] (\lambda x. x)$$

$$E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x$$

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$$E_1 = [\cdot] (\lambda x. x)$$

$$E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x$$

$$E_2 = (\lambda z. z z) [\cdot]$$

$$E_2[\lambda x. \lambda y. x] = (\lambda z. z z) (\lambda x. \lambda y. x)$$

Examples

$$E_1 = [\cdot] (\lambda x. x)$$

$$E_1[\lambda y. y y] = (\lambda y. y y) \lambda x. x$$

$$E_2 = (\lambda z. z z) [\cdot]$$

$$E_2[\lambda x. \lambda y. x] = (\lambda z. z z) (\lambda x. \lambda y. x)$$

$$E_3 = ([\cdot] \lambda x. x x) ((\lambda y. y) (\lambda y. y))$$

$$E_3[\lambda f. \lambda g. f g] = ((\lambda f. \lambda g. f g) \lambda x. x x) ((\lambda y. y) (\lambda y. y))$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ -calculus with just two rules: one for evaluation contexts, and one for β -reduction.

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With this syntax:

$$E ::= [\cdot] \mid E e \mid v E$$

The small-step rules are:

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x. e) v \rightarrow e\{v/x\}} \beta$$

CBN With Evaluation Contexts

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$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x. e) e' \rightarrow e\{e'/x\}} \beta$$

Multi-Argument λ -calculus

Let's define a version of the λ -calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \dots, x_n. e \mid e_0 e_1 \dots e_n$$

Multi-Argument λ -calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \dots v_{i-1} E e_{i+1} \dots e_n$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x_1, \dots, x_n. e_0) v_1 \dots v_n \rightarrow e_0 \{v_1/x_1\} \{v_2/x_2\} \dots \{v_n/x_n\}} \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 e_1 \dots e_n$ from left to right.

Definitional Translation

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The multi-argument λ -calculus isn't any more expressive than the pure λ -calculus.

We can define a translation $\mathcal{T}[\cdot]$ that takes an expression in the multi-argument λ -calculus and returns an equivalent expression in the pure λ -calculus.

$$\mathcal{T}[x] = x$$

$$\mathcal{T}[\lambda x_1, \dots, x_n. e] = \lambda x_1. \dots \lambda x_n. \mathcal{T}[e]$$

$$\mathcal{T}[e_0 e_1 e_2 \dots e_n] = (\dots ((\mathcal{T}[e_0] \mathcal{T}[e_1]) \mathcal{T}[e_2]) \dots \mathcal{T}[e_n])$$

This translation *curries* the multi-argument λ -calculus.