CS 4110

Programming Languages & Logics

Lecture 7
Denotational Semantics Proofs

Determinism in Small-Step Semantics

Determinism: every configuration has at most one successor

$$\begin{aligned}&\forall e \in \textbf{Exp}.\ \forall \sigma, \sigma', \sigma'' \in \textbf{Store}.\ \forall e', e'' \in \textbf{Exp}.\\ &\quad \text{if } \langle \sigma, e\rangle \to \langle \sigma', e'\rangle \text{ and } \langle \sigma, e\rangle \to \langle \sigma'', e''\rangle\\ &\quad \text{then } e' = e'' \text{ and } \sigma' = \sigma''.\end{aligned}$$

A different property, which you can call confluence:

$$\begin{aligned}&\text{If } \langle \sigma, e\rangle \to^* \langle \sigma', e'\rangle \text{ and } \langle \sigma, e\rangle \to^* \langle \sigma'', e''\rangle\\ &\quad \text{and neither } \langle \sigma', e'\rangle \text{ nor } \langle \sigma'', e''\rangle \text{ can take a step}\\ &\quad \text{then } e' = e'' \text{ and } \sigma' = \sigma''.\end{aligned}$$

Kleene Fixed-Point Theorem

Definition (Scott Continuity)

A function F is *Scott-continuous* if for every chain $X_1 \subseteq X_2 \subseteq \ldots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

Kleene Fixed-Point Theorem

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Theorem (Kleene Fixed Point)

Let F be a Scott-continuous function. The least fixed point of F is $\bigcup_i F^i(\emptyset)$.

Denotational Semantics for IMP Commands

$$\mathcal{C}[\![\textbf{skip}]\!] = \{(\sigma, \sigma)\}$$

$$\mathcal{C}[\![x := a]\!] = \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\}$$

$$\mathcal{C}[\![c_1; c_2]\!] = \{(\sigma, \sigma') \mid \exists \sigma''. ((\sigma, \sigma'') \in \mathcal{C}[\![c_1]\!] \wedge (\sigma'', \sigma') \in \mathcal{C}[\![c_2]\!])\}$$

$$\mathcal{C}[\![\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2]\!] = \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_1]\!]\} \cup \{(\sigma, \sigma') \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_2]\!]\}$$

$$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \mathit{fix}(F)$$

where $F(f) = \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \cup \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge \exists \sigma''. ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'', \sigma') \in f\}$

Exercises

skip; *c* and *c*; **skip** are equivalent.

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$\mathcal{C}[\![\textbf{while true do skip}]\!]$