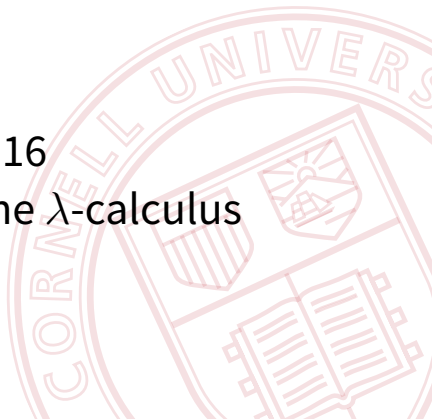# CS 4110

Programming Languages & Logics

Lecture 16
Programming in the λ-calculus

Review: Church Booleans

We can encode TRUE, FALSE, and IF, as:

$$\text{TRUE} \triangleq \lambda x.\, \lambda y.\, x$$
$$\text{FALSE} \triangleq \lambda x.\, \lambda y.\, y$$
$$\text{IF} \triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\, t\, f$$

This way, IF behaves how it ought to:

$$\text{IF TRUE } v_t\, v_f \rightarrow^* v_t$$
$$\text{IF FALSE } v_t\, v_f \rightarrow^* v_f$$

Review: Church Numerals

Church numerals encode a number *n* as a function that takes *f* and *x*, and applies *f* to *x* *n* times.

$$\overline{0} \triangleq \lambda f. \lambda x. x$$
$$\overline{1} \triangleq \lambda f. \lambda x. f x$$
$$\overline{2} \triangleq \lambda f. \lambda x. f (f x)$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n. \lambda f. \lambda x. f (n f x)$$

Review: Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$\overline{0} \triangleq \lambda f.\, \lambda x.\, x$$
$$\overline{1} \triangleq \lambda f.\, \lambda x.\, f\, x$$
$$\overline{2} \triangleq \lambda f.\, \lambda x.\, f\,(f\, x)$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\,(n\, f\, x)$$
$$\text{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1\ \text{SUCC}\ n_2$$

Review: Church Numerals

Church numerals encode a number *n* as a function that takes *f* and *x*, and applies *f* to *x* *n* times.

$$\overline{0} \triangleq \lambda f. \, \lambda x. \, x$$
$$\overline{1} \triangleq \lambda f. \, \lambda x. \, f \, x$$
$$\overline{2} \triangleq \lambda f. \, \lambda x. \, f \, (f \, x)$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n. \, \lambda f. \, \lambda x. \, f \, (n \, f \, x)$$
$$\text{PLUS} \triangleq \lambda n_1. \, \lambda n_2. \, n_1 \, \text{SUCC} \, n_2$$
$$\text{TIMES} \triangleq \lambda n_1. \, \lambda n_2. \, n_1 \, (\text{PLUS} \, n_2) \, \overline{0}$$

Review: Church Numerals

Church numerals encode a number *n* as a function that takes *f* and *x*, and applies *f* to *x* *n* times.

$$\overline{0} \triangleq \lambda f.\, \lambda x.\, x$$
$$\overline{1} \triangleq \lambda f.\, \lambda x.\, f\, x$$
$$\overline{2} \triangleq \lambda f.\, \lambda x.\, f\, (f\, x)$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\, (n\, f\, x)$$
$$\text{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1\, \text{SUCC}\, n_2$$
$$\text{TIMES} \triangleq \lambda n_1.\, \lambda n_2.\, n_1\, (\text{PLUS}\, n_2)\, \overline{0}$$
$$\text{ISZERO} \triangleq \lambda n.\, n\, (\lambda z.\, \text{FALSE})\, \text{TRUE}$$

Recursive Functions

How would we write recursive functions like factorial?

Recursive Functions

How would we write recursive functions like factorial?

We'd like to write it like this…

$$\text{FACT} \triangleq \lambda n.\ \text{IF } (\text{ISZERO } n)\ 1\ (\text{TIMES } n\ (\text{FACT } (\text{PRED } n)))$$

Recursive Functions

How would we write recursive functions like factorial?

We'd like to write it like this...

$$\text{FACT} \triangleq \lambda n.\, \text{IF} \,(\text{ISZERO} \, n)\, 1\, (\text{TIMES} \, n \,(\text{FACT} \,(\text{PRED} \, n)))$$

In slightly more readable notation this is...

$$\text{FACT} \triangleq \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times \text{FACT} \,(n - 1)$$

...but this is an equation, not a definition!

Recursion removal trick

We can perform a "trick" to define a function FACT that satisfies the recursive equation on the previous slide.

Recursion removal trick

We can perform a "trick" to define a function FACT that satisfies the recursive equation on the previous slide.

Define a new function FACT$'$ that takes a function f as an argument. Then, for "recursive" calls, it uses $f\ f$:

$$\text{FACT}' \triangleq \lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\ f)\ (n - 1))$$

Recursion removal trick

We can perform a "trick" to define a function FACT that satisfies the recursive equation on the previous slide.

Define a new function FACT$'$ that takes a function f as an argument. Then, for "recursive" calls, it uses $f\ f$:

$$\text{FACT}' \triangleq \lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\ f)\ (n - 1))$$

Then define FACT as FACT$'$ applied to itself:

$$\text{FACT} \triangleq \text{FACT}'\ \text{FACT}'$$

Example

Let's try evaluating FACT on 3...

FACT 3

Example

Let's try evaluating FACT on 3...

FACT 3 = (FACT′ FACT′) 3

Example

Let's try evaluating FACT on 3...

$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') \, 3$$
$$= ((\lambda f. \, \lambda n. \, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f f) \, (n - 1))) \, \text{FACT}') \, 3$$

Example

Let's try evaluating FACT on 3...

$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') \, 3$$
$$= ((\lambda f. \, \lambda n. \textbf{ if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f f) \, (n - 1))) \text{ FACT}') \, 3$$
$$\rightarrow (\lambda n. \textbf{ if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\text{FACT}' \text{ FACT}') \, (n - 1))) \, 3$$

Example

Let's try evaluating FACT on 3...

$$FACT\ 3 = (FACT'\ FACT')\ 3$$
$$= ((\lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\,f)\,(n-1)))\ FACT')\ 3$$
$$\rightarrow (\lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((FACT'\ FACT')\,(n-1)))\ 3$$
$$\rightarrow \textbf{if } 3 = 0 \textbf{ then } 1 \textbf{ else } 3 \times ((FACT'\ FACT')\,(3-1))$$

Example

Let's try evaluating FACT on 3...

$$FACT\ 3 = (FACT'\ FACT')\ 3$$
$$= ((\lambda f.\ \lambda n.\ \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((f\,f)\,(n-1)))\ FACT')\ 3$$
$$\rightarrow (\lambda n.\ \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((FACT'\ FACT')\,(n-1)))\ 3$$
$$\rightarrow \textbf{if}\ 3 = 0\ \textbf{then}\ 1\ \textbf{else}\ 3 \times ((FACT'\ FACT')\,(3-1))$$
$$\rightarrow 3 \times ((FACT'\ FACT')\,(3-1))$$

Example

Let's try evaluating FACT on 3...

$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}')\, 3$$
$$= ((\lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f f)\, (n - 1)))\, \text{FACT}')\, 3$$
$$\rightarrow (\lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\text{FACT}' \text{ FACT}')\, (n - 1)))\, 3$$
$$\rightarrow \textbf{if } 3 = 0 \textbf{ then } 1 \textbf{ else } 3 \times ((\text{FACT}' \text{ FACT}')\, (3 - 1))$$
$$\rightarrow 3 \times ((\text{FACT}' \text{ FACT}')\, (3 - 1))$$
$$= 3 \times (\text{FACT }(3 - 1))$$

Example

Let's try evaluating FACT on 3...

$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') \, 3$$
$$= ((\lambda f. \, \lambda n. \, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f f) \, (n - 1))) \text{ FACT}') \, 3$$
$$\rightarrow (\lambda n. \, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\text{FACT}' \text{ FACT}') \, (n - 1))) \, 3$$
$$\rightarrow \textbf{if } 3 = 0 \textbf{ then } 1 \textbf{ else } 3 \times ((\text{FACT}' \text{ FACT}') \, (3 - 1))$$
$$\rightarrow 3 \times ((\text{FACT}' \text{ FACT}') \, (3 - 1))$$
$$= 3 \times (\text{FACT} \, (3 - 1))$$
$$\rightarrow \ldots$$
$$\rightarrow 3 \times 2 \times 1 \times 1$$

Example

Let's try evaluating FACT on 3...

$$\text{FACT } 3 = (\text{FACT}' \text{ FACT}') \, 3$$
$$= ((\lambda f. \, \lambda n. \, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f f) \, (n - 1))) \, \text{FACT}') \, 3$$
$$\rightarrow (\lambda n. \, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\text{FACT}' \text{ FACT}') \, (n - 1))) \, 3$$
$$\rightarrow \textbf{if } 3 = 0 \textbf{ then } 1 \textbf{ else } 3 \times ((\text{FACT}' \text{ FACT}') \, (3 - 1))$$
$$\rightarrow 3 \times ((\text{FACT}' \text{ FACT}') \, (3 - 1))$$
$$= 3 \times (\text{FACT} \, (3 - 1))$$
$$\rightarrow \ldots$$
$$\rightarrow 3 \times 2 \times 1 \times 1$$
$$\rightarrow^* 6$$

Fixed point combinators

Our "trick" requires following human-readable instructions.
Write a different function f' that takes itself as an argument and
uses self-application for recursive calls, and then define f as $f'\ f'$.

Fixed point combinators

Our "trick" requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as f' f'.

There is another way: fixed points!

Fixed point combinators

Our "trick" requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f'\ f'$.

There is another way: fixed points!

Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\ \lambda n.\ \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n - 1))$$

Fixed point combinators

Our "trick" requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f'\ f'$.

There is another way: fixed points!

Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f(n-1))$$

Recall that if g if a fixed point of G, then $G\ g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$g\ 5 = (G\ g)\ 5$$

Fixed point combinators

Our "trick" requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f'\ f'$.

There is another way: fixed points!

Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\ (n-1))$$

Recall that if g if a fixed point of G, then $G\ g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$g\ 5 = (G\ g)\ 5$$
$$\rightarrow^* 5 \times (g\ 4)$$

Fixed point combinators

Our "trick" requires following human-readable instructions.
Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f'\ f'$.

There is another way: fixed points!

Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f(n-1))$$

Recall that if g if a fixed point of G, then $G\ g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$\begin{aligned}
g\ 5 &= (G\ g)\ 5 \\
&\rightarrow^* 5 \times (g\ 4) \\
&= 5 \times ((G\ g)\ 4)
\end{aligned}$$

Fixed point combinators

How can we generate the fixed point of *G*?

In denotational semantics, finding fixed points took a lot of math. In the λ-calculus, we just need a suitable combinator...

Y Combinator

The (infamous) Y combinator is defined as

$$Y \triangleq \lambda f. \, (\lambda x. \, f \, (x \, x)) \, (\lambda x. \, f \, (x \, x))$$

We say that Y is a *fixed point combinator* because Y f is a fixed point of f (for any lambda term f).

Y Combinator

The (infamous) Y combinator is defined as

$$Y \triangleq \lambda f. \, (\lambda x. \, f \, (x \, x)) \, (\lambda x. \, f \, (x \, x))$$

We say that Y is a *fixed point combinator* because Y f is a fixed point of f (for any lambda term f).

What happens when we evaluate Y G under CBV?

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

$$Z \triangleq \lambda f. \, (\lambda x. \, f \, (\lambda y. \, x \, x \, y)) \, (\lambda x. \, f \, (\lambda y. \, x \, x \, y))$$

Example

Let's see Z in action, on our function *G*.

FACT

Example

Let's see Z in action, on our function *G*.

FACT
$=$ Z *G*

Example

Let's see Z in action, on our function *G*.

$$\text{FACT}$$
$$= \text{Z } G$$
$$= (\lambda f. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y))) G$$

Example

Let's see Z in action, on our function *G*.

$$
\begin{aligned}
& \text{FACT} \\
=\ & \text{Z } G \\
=\ & (\lambda f.\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y)))\, G \\
\rightarrow\ & (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))
\end{aligned}
$$

Example

Let's see Z in action, on our function *G*.

$$
\begin{aligned}
&\text{FACT} \\
={}& \text{Z } G \\
={}& (\lambda f. (\lambda x. f (\lambda y. x\, x\, y)) (\lambda x. f (\lambda y. x\, x\, y))) \, G \\
\rightarrow{}& (\lambda x. G (\lambda y. x\, x\, y)) (\lambda x. G (\lambda y. x\, x\, y)) \\
\rightarrow{}& G (\lambda y. (\lambda x. G (\lambda y. x\, x\, y)) (\lambda x. G (\lambda y. x\, x\, y)) \, y)
\end{aligned}
$$

Example

Let's see Z in action, on our function G.

$$
\begin{aligned}
&\text{FACT} \\
=\ & \text{Z } G \\
=\ & (\lambda f.\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y)))\, G \\
\to\ & (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y)) \\
\to\ & G\, (\lambda y.\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, y) \\
=\ & (\lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\, (n - 1))) \\
& \quad\ (\lambda y.\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, y)
\end{aligned}
$$

Example

Let's see Z in action, on our function G.

$$
\begin{aligned}
&\quad \text{FACT} \\
&= \quad Z\, G \\
&= \quad (\lambda f.\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y)))\, G \\
&\rightarrow \quad (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y)) \\
&\rightarrow \quad G\, (\lambda y.\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, y) \\
&= \quad (\lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\, (n-1))) \\
&\qquad\qquad (\lambda y.\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, y) \\
&\rightarrow \quad \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \\
&\qquad\quad \textbf{else } n \times ((\lambda y.\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, y)\, (n-1))
\end{aligned}
$$

Example

Let's see Z in action, on our function G.

$$
\begin{aligned}
& \text{FACT} \\
=\ & \text{Z}\,G \\
=\ & (\lambda f.\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y)))\,G \\
\rightarrow\ & (\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y)) \\
\rightarrow\ & G\,(\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
=\ & (\lambda f.\,\lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\,(n-1))) \\
& \qquad (\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
\rightarrow\ & \lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \\
& \qquad \textbf{else } n \times ((\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y)\,(n-1)) \\
=_\beta\ & \lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (\lambda y.\,(\text{Z}\,G)\,y)\,(n-1)
\end{aligned}
$$

Example

Let's see Z in action, on our function G.

$$
\begin{aligned}
&\quad \text{FACT} \\
&= \; Z\,G \\
&= \; (\lambda f.\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y)))\,G \\
&\rightarrow \; (\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y)) \\
&\rightarrow \; G\,(\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
&= \; (\lambda f.\,\lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\,(n-1))) \\
&\qquad\quad (\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
&\rightarrow \; \lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \\
&\qquad\quad \textbf{else } n \times ((\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y)\,(n-1)) \\
&=_\beta \; \lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (\lambda y.\,(Z\,G)\,y)\,(n-1) \\
&=_\beta \; \lambda n.\,\textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (Z\,G\,(n-1))
\end{aligned}
$$

Example

Let's see Z in action, on our function G.

$$
\begin{aligned}
&\quad \text{FACT} \\
&= \ Z\,G \\
&= \ (\lambda f.\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y)))\,G \\
&\rightarrow \ (\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y)) \\
&\rightarrow \ G\,(\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
&= \ (\lambda f.\,\lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1))) \\
&\qquad\quad (\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
&\rightarrow \ \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1 \\
&\qquad\quad \textbf{else}\ n \times ((\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y)\,(n-1)) \\
&=_\beta \ \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\lambda y.\,(Z\,G)\,y)\,(n-1) \\
&=_\beta \ \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (Z\,G\,(n-1)) \\
&= \ \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}\,(n-1))
\end{aligned}
$$

Other fixed point combinators

There are many (indeed infinitely many) fixed-point combinators. Here's a cute one:

$$Y_k \triangleq (\text{L L L L L L L L L L L L L L L L L L L L L L L L L L})$$

where

$$\text{L} \triangleq \lambda abcdefghijklmnopqstuvwxyzr.$$
$$(r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r))$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

We know that Θ f is a fixed point of f, so we have

$$\Theta\, f = f\, (\Theta\, f).$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

We know that Θf is a fixed point of f, so we have

$$\Theta f = f(\Theta f).$$

We can write the following recursive equation:

$$\Theta = \lambda f.\, f(\Theta f)$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

We know that Θf is a fixed point of f, so we have

$$\Theta f = f(\Theta f).$$

We can write the following recursive equation:

$$\Theta = \lambda f.\, f(\Theta f)$$

Now use the recursion removal trick:

$$\Theta' \triangleq \lambda t.\, \lambda f.\, f(t\, t\, f)$$
$$\Theta \triangleq \Theta'\, \Theta'$$

$$\mathsf{FACT} = \Theta\, G$$

$\mathsf{FACT} = \Theta\, G$
$\quad = ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G$

θ Example

$$\text{FACT} = \Theta\, G$$
$$= ((\lambda t.\ \lambda f.\ f\,(t\,t\,f))\,(\lambda t.\ \lambda f.\ f\,(t\,t\,f)))\, G$$
$$\rightarrow (\lambda f.\ f\,((\lambda t.\ \lambda f.\ f\,(t\,t\,f))\,(\lambda t.\ \lambda f.\ f\,(t\,t\,f))\, f))\, G$$

θ Example

$$\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, (\lambda t.\, \lambda f.\, f\, (t\, t\, f)))\, G \\
&\rightarrow (\lambda f.\, f\, ((\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, (\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, f))\, G \\
&\rightarrow G\, ((\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, (\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, G)
\end{aligned}$$

θ Example

$$\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\rightarrow (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\, G \\
&\rightarrow G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G) \\
&= G\,(\Theta\, G)
\end{aligned}$$

$$\begin{aligned}
\text{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\rightarrow (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, f))\, G \\
&\rightarrow G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, G) \\
&= G\,(\Theta\, G) \\
&= (\lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (f\,(n-1)))\,(\Theta\, G) \\
&\rightarrow \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((\Theta\, G)\,(n-1)) \\
&= \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times (\text{FACT}\,(n-1))
\end{aligned}$$

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$e ::= x \mid \lambda x.\, e \mid e_1\, e_2$$
$$v ::= \lambda x.\, e$$

$$\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2} \qquad \frac{e \rightarrow e'}{v\, e \rightarrow v\, e'}$$

$$\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}} \; \beta$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E\,e \mid v\,E$$

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E\,e \mid v\,E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$E_1 = [\cdot]\,(\lambda x.\,x)$$
$$E_1[\lambda y.\,y\,y] = (\lambda y.\,y\,y)\,\lambda x.\,x$$

Examples

$$E_1 = [\cdot]\,(\lambda x.\,x)$$
$$E_1[\lambda y.\,y\,y] = (\lambda y.\,y\,y)\,\lambda x.\,x$$

$$E_2 = (\lambda z.\,z\,z)\,[\cdot]$$
$$E_2[\lambda x.\,\lambda y.\,x] = (\lambda z.\,z\,z)\,(\lambda x.\,\lambda y.\,x)$$

Examples

$$E_1 = [\cdot]\,(\lambda x.\,x)$$
$$E_1[\lambda y.\,y\,y] = (\lambda y.\,y\,y)\,\lambda x.\,x$$

$$E_2 = (\lambda z.\,z\,z)\,[\cdot]$$
$$E_2[\lambda x.\,\lambda y.\,x] = (\lambda z.\,z\,z)\,(\lambda x.\,\lambda y.\,x)$$

$$E_3 = ([\cdot]\,\lambda x.\,x\,x)\,((\lambda y.\,y)\,(\lambda y.\,y))$$
$$E_3[\lambda f.\,\lambda g.\,f\,g] = ((\lambda f.\,\lambda g.\,f\,g)\,\lambda x.\,x\,x)\,((\lambda y.\,y)\,(\lambda y.\,y))$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

With this syntax:

$$E ::= [\cdot] \mid E\,e \mid v\,E$$

The small-step rules are:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\,e)\,v \to e\{v/x\}} \; \beta$$

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\ e$$

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\, e$$

But the small-step rules are the same:

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x.\, e)\, e' \rightarrow e\{e'/x\}} \; \beta$$