

CS 4110

Programming Languages & Logics

Lecture 9 Axiomatic Semantics



Kinds of Semantics

Operational Semantics

- Describes *how* programs compute
- Relatively easy to define
- Close connection to implementations

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- Relatively easy to define
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Denotational Semantics

- Describes *what* programs compute
- Solid mathematical foundation
- Simplifies equational reasoning

Axiomatic Semantics

- Describes the *properties* programs satisfy
- Useful for reasoning about correctness

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- Proof rules for establishing the validity of properties with respect to programs

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- The value of z is prime

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Assertion Languages:

- First-order logic: $\forall, \exists, \wedge, \vee, x = y, R(x), \dots$
- Temporal or modal logic: $\Box, \Diamond, X, U, F, \dots$
- Special-purpose logics: Alloy, Sugar, Z3, etc.

Applications

- Proving correctness
- Documentation
- Test generation
- Symbolic execution
- Translation validation
- Bug finding
- Malware detection

Pre-Conditions and Post-conditions

Assertions are often used (informally) in code

```
/* Precondition:  $0 \leq i < A.length$  */  
/* Postcondition: returns  $A[i]$  */  
public int get(int i) {  
    return A[i];  
}
```

These assertions are useful as documentation or run-time checks, but there is no guarantee they are correct.

Idea: Let's make this rigorous by defining the semantics of the language in terms of pre-conditions and post-conditions!

Partial Correctness

Here's the IMP syntax:

$a \in \mathbf{Aexp}$	$a ::= x \mid n \mid a_1 + a_2 \mid a_1 \times a_2$
$b \in \mathbf{Bexp}$	$b ::= \mathbf{true} \mid \mathbf{false} \mid a_1 < a_2$
$c \in \mathbf{Com}$	$c ::= \mathbf{skip} \mid x := a \mid c_1; c_2$ $\mid \mathbf{if } b \mathbf{ then } c_1 \mathbf{ else } c_2 \mid \mathbf{while } b \mathbf{ do } c$

A **partial correctness statement** is a triple:

$$\{P\} c \{Q\}$$

Meaning: If P holds, and then c executes (and terminates), then Q holds afterward.

Partial Correctness

$$\{x = 21\} y := x \times 2 \{y = 42\}$$

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$$\{x = 21\} y := x \times 2 \{y = 42\}$$

$$\{x = n\} y := x \times 2 \{y = 2n\}$$

Question

Given the following partial correctness specification,

$$\{P\} \textbf{while } x < 0 \textbf{ do } x := x + 1 \{x \geq 0\}$$

which P makes it valid?

- A. **true**
- B. **false**
- C. $x \geq 0$
- D. All of the above.
- E. None of the above.

Question

Given the following partial correctness specification,

$$\{P\} \textbf{while } x < 0 \textbf{ do } x := x + 1 \{ \textbf{false} \}$$

which P makes it valid?

- A. **true**
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- C. $x \geq 0$
- D. All of the above.
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Total Correctness

Note that partial correctness specifications don't ensure that the program will terminate—this is why they are called “partial.”

Sometimes we need to know that the program will terminate.

A **total correctness statement** is a triple written with square brackets:

$$[P]c[Q]$$

Meaning: if P holds, then c will terminate and Q holds after c .

We'll focus mostly on partial correctness.

Example: Partial Correctness

```
{foo = 0 ∧ bar = i}  
baz := 0;  
while foo ≠ bar  
do  
    baz := baz - 2;  
    foo := foo + 1  
{baz = -2 × i}
```

Intuition: if we start with a store σ that maps `foo` to 0 and `bar` to an integer i , and if the execution of the command terminates, then the final store σ' will map `baz` to $-2i$.

Example: Total Correctness

```
[foo = 0  $\wedge$  bar =  $i \wedge i \geq 0$ ]  
baz := 0;  
while foo  $\neq$  bar  
do  
    baz := baz - 2;  
    foo := foo + 1  
[baz =  $-2 \times i$ ]
```

Intuition: if we start with a store σ that maps foo to 0 and bar to a non-negative integer i , then the execution of the command will terminate in a final store σ' will map baz to $-2i$.

Another Example

```
{foo = 0  $\wedge$  bar = i}  
baz := 0;  
while foo  $\neq$  bar  
do  
    baz := baz + foo;  
    foo := foo + 1  
{baz = i}
```

Is this partial correctness statement valid?

Assertions

We define a new language syntax to write assertions:

$$i \in \mathbf{LVar}$$

$$a \in \mathbf{Aexp} ::= x \mid i \mid n \mid a_1 + a_2 \mid a_1 \times a_2$$

$$\begin{aligned} P, Q \in \mathbf{Assn} ::= & \mathbf{true} \mid \mathbf{false} \\ & \mid a_1 < a_2 \\ & \mid P_1 \wedge P_2 \mid P_1 \vee P_2 \mid P_1 \Rightarrow P_2 \\ & \mid \neg P \mid \forall i. P \mid \exists i. P \end{aligned}$$

Assertions can introduce **logical variables**, which are different from program variables.

Note that every boolean expression b is also an assertion.

Satisfaction

Next we'll define what it means for a store σ to satisfy an assertion.

To do this, we need an **interpretation** for the logical variables, which is like the store for program variables:

$$/ : \mathbf{LVar} \rightarrow \mathbf{Int}$$

Satisfaction

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$$I : \mathbf{LVar} \rightarrow \mathbf{Int}$$

And a denotation function for assertion arithmetic expressions, $\mathcal{A}_i \llbracket a \rrbracket$, that's almost the same as for ordinary arithmetic:

$$\mathcal{A}_i \llbracket n \rrbracket(\sigma, I) = n$$

$$\mathcal{A}_i \llbracket x \rrbracket(\sigma, I) = \sigma(x)$$

$$\mathcal{A}_i \llbracket i \rrbracket(\sigma, I) = I(i)$$

$$\mathcal{A}_i \llbracket a_1 + a_2 \rrbracket(\sigma, I) = \mathcal{A}_i \llbracket a_1 \rrbracket(\sigma, I) + \mathcal{A}_i \llbracket a_2 \rrbracket(\sigma, I)$$

Satisfaction

Next we define the satisfaction relation for assertions, \models_I :

Definition (Assertion satisfaction)

$\sigma \models_I \mathbf{true}$	(always)
$\sigma \models_I a_1 < a_2$	if $\mathcal{A}_i[[a_1]](\sigma, l) < \mathcal{A}_i[[a_2]](\sigma, l)$
$\sigma \models_I P_1 \wedge P_2$	if $\sigma \models_I P_1$ and $\sigma \models_I P_2$
$\sigma \models_I P_1 \vee P_2$	if $\sigma \models_I P_1$ or $\sigma \models_I P_2$
$\sigma \models_I P_1 \Rightarrow P_2$	if $\sigma \not\models_I P_1$ or $\sigma \models_I P_2$
$\sigma \models_I \neg P$	if $\sigma \not\models_I P$
$\sigma \models_I \forall i. P$	if $\forall k \in \text{Int}. \sigma \models_{l[i \mapsto k]} P$
$\sigma \models_I \exists i. P$	if $\exists k \in \text{Int}. \sigma \models_{l[i \mapsto k]} P$

Satisfaction

Next we define what it means for a command c to satisfy a partial correctness statement.

Definition (Partial correctness statement satisfiability)

A partial correctness statement $\{P\} c \{Q\}$ is satisfied in store σ and interpretation I , written $\sigma \models_I \{P\} c \{Q\}$, if:

$$\forall \sigma'. \text{ if } \sigma \models_I P \text{ and } \mathcal{C}[\![c]\!]\sigma = \sigma' \text{ then } \sigma' \models_I Q$$

Validity

Definition (Assertion validity)

An assertion P is valid (written $\models P$) if it is valid in any store, under any interpretation: $\forall \sigma, I. \sigma \models_I P$

Definition (Partial correctness statement validity)

A partial correctness triple is valid (written $\models \{P\} c \{Q\}$), if it is valid in any store and interpretation: $\forall \sigma, I. \sigma \models_I \{P\} c \{Q\}$.

Now we know what we mean when we say “assertion P holds” or “partial correctness statement $\{P\} c \{Q\}$ is valid.”

Proving Specifications

How do we show that $\{P\} c \{Q\}$ holds?

We know that $\{P\} c \{Q\}$ is valid if it holds for all stores and interpretations: $\forall \sigma, I. \sigma \models_I \{P\} c \{Q\}$.

Showing that $\sigma \models_I \{P\} c \{Q\}$ requires reasoning about the denotation of c (because of the definition of satisfaction).

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We can do this manually, but there is a better way!

We can use a set of inference rules and axioms, called *Hoare rules*, to directly derive valid partial correctness statements without having to reason about stores, interpretations, and the execution of c .