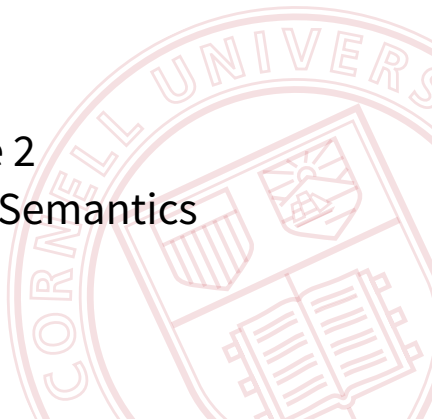# CS 4110

Programming Languages & Logics
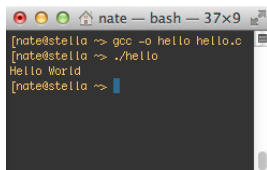
Lecture 2
Introduction to Semantics

Semantics

Question: What is the meaning of a program?

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Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...

...but none of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - ▶ Model program by execution on abstract machine
  - ▶ Useful for implementing compilers and interpreters

- Denotational: $[\![e]\!]$
  - ▶ Model program as mathematical objects
  - ▶ Useful for theoretical foundations

- Axiomatic $\vdash \{\phi\}\, e\, \{\psi\}$
  - ▶ Model program by the logical formulas it obeys
  - ▶ Useful for proving program correctness

Arithmetic Expressions

Syntax

A language of integer arithmetic expressions with assignment.

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$$n, m \in \textbf{Int}$$
$$e \in \textbf{Exp}$$

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\end{aligned}
$$

BNF Grammar:

$$
\begin{aligned}
e ::= \ &x \\
| \ &n \\
| \ &e_1 + e_2 \\
| \ &e_1 * e_2 \\
| \ &x := e_1 \ ; \ e_2
\end{aligned}
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Ambiguity

What expression does the string "$1 + 2 * 3$" describe?

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```
        +                      *
       / \                    / \
      1   *                  +   3
         / \                / \
        2   3              1   2
```
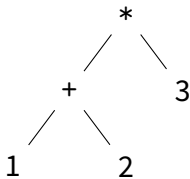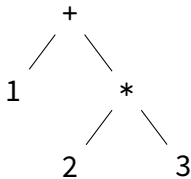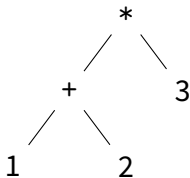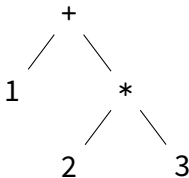
In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

Representing Expressions

BNF Grammar:

$$e ::= x$$
$$| n$$
$$| e_1 + e_2$$
$$| e_1 * e_2$$
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Representing Expressions

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$$e ::= x$$
$$| \; n$$
$$| \; e_1 + e_2$$
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$$| \; x := e_1 \; ; \; e_2$$

OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))

Representing Expressions

BNF Grammar:

$$e ::= x$$
$$\mid n$$
$$\mid e_1 + e_2$$
$$\mid e_1 * e_2$$
$$\mid x := e_1 ; \ e_2$$

Java:

```
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to …?

Quiz

- $7 + (4 * 2)$ evaluates to 15

Quiz

- $7 + (4 * 2)$ evaluates to 15
- $i := 6 + 1$; $2 * 3 * i$ evaluates to ...?

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- $x + 1$ evaluates to ...?

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- $x + 1$ evaluates to error?

Quiz

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- $i := 6 + 1 \; ; \; 2 * 3 * i$ evaluates to 42
- $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

The *product* of two sets A and B, written $A \times B$, contains all ordered pairs (a, b) with $a \in A$ and $b \in B$.

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Some Important Relations

- empty: $\emptyset$
- total: $A \times B$
- identity on A: $\{(a, a) \mid a \in A\}$.
- composition $R; S$: $\{(a, c) \mid \exists b. (a, b) \in R \wedge (b, c) \in S\}$

Functions

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When f is a function, we usually write $f : A \rightarrow B$ instead of $f \subseteq A \times B$.

The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. Formally:

$$\text{image}(f) \triangleq \{f(a) \,|\, a \in A\}$$

Some Important Functions

Given two functions $f : A \to B$ and $g : B \to C$, the composition of f and g is defined by: $(g \circ f)(x) = g(f(x))$ Note order!

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

Operational Semantics

Overview

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For our language, a configuration $\langle \sigma, e \rangle$ is a pair of:

- a store σ that records the values of variables,
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More formally:

$$
\begin{aligned}
\textbf{Store} \quad &\triangleq \quad \textbf{Var} \rightharpoonup \textbf{Int} \\
\textbf{Config} \quad &\triangleq \quad \textbf{Store} \times \textbf{Exp}
\end{aligned}
$$

(A store is a *partial* function from variables to integers.)

Operational Semantics

The small-step operational semantics itself is a relation on configurations—i.e., a subset of **Config** $\times$ **Config**.

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Question: How should we define this relation?

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which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in$ "$\to$".

Question: How should we define this relation? Remember that there are an infinite number of configurations and possible steps!

Inference Rules

Answer: Define it inductively, using inference rules:

$$\frac{\text{premise}_1 \quad \text{premise}_2 \quad \cdots}{\text{conclusion}} \text{ Name}$$

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An inference rule defines an implication: if all the premises hold, then the conclusion also holds.

Formally, "$\rightarrow$" is the smallest relation that is closed under all the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \text{ VAR}$$

Addition

$$\frac{p = m + n}{\langle \sigma, n + m \rangle \to \langle \sigma, p \rangle} \text{ Add}$$

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Multiplication

$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle} \; \text{MUL}$$

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$$\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n * e_2 \rangle \to \langle \sigma', n * e_2' \rangle} \ \text{RMul}$$

Assignment

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n \,;\, e_2 \rangle \to \langle \sigma', e_2 \rangle} \text{ Assgn}$$

Notation: $\sigma[x \mapsto n]$ is a *new* function that mostly behaves like σ, except that it maps *x* to *n*.

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Operational Semantics

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \to \langle \sigma, n \rangle} \text{ VAR} \qquad\qquad \frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle} \text{ LADD}$$

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