CS 4110

Programming Languages & Logics

Lecture 15
Encodings

Review: λ-calculus

Syntax

$$\begin{array}{lcl} e & ::= & x \mid e_1\, e_2 \mid \lambda x.\, e \\ v & ::= & \lambda x.\, e \end{array}$$

Semantics

$$\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

Rewind: Currying

This is just a function that returns a function:

$$\mathsf{ADD} \triangleq \lambda x.\, \lambda y.\, x + y$$

$$\mathsf{ADD}\ 38 \to \lambda y.\, 38 + y$$

$$\mathsf{ADD}\ 38\ 4 = (\mathsf{ADD}\ 38)\ 4 \to 42$$

Informally, you can think of it as a *curried* function that takes two arguments, one after the other.

But that's just a way to get intuition. The λ-calculus only has one-argument functions.

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$e ::= x \mid \lambda x.\, e \mid e_1\, e_2$$
$$v ::= \lambda x.\, e$$

$$\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

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$$E ::= [\cdot] \mid E\, e \mid v\, E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$E_1 = [\cdot]\,(\lambda x.\, x)$$
$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\,\lambda x.\, x$$

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$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\ \lambda x.\, x$$

$$E_2 = (\lambda z.\, z\, z)\ [\cdot]$$
$$E_2[\lambda x.\ \lambda y.\, x] = (\lambda z.\, z\, z)\ (\lambda x.\ \lambda y.\, x)$$

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$$E_3 = ([\cdot]\ \lambda x.\, x\, x)\ ((\lambda y.\, y)\ (\lambda y.\, y))$$
$$E_3[\lambda f.\ \lambda g.\, f\, g] = ((\lambda f.\ \lambda g.\, f\, g)\ \lambda x.\, x\, x)\ ((\lambda y.\, y)\ (\lambda y.\, y))$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

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With this syntax:

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

The small-step rules are:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

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But the small-step rules are the same:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, e' \to e\{e'/x\}}\ \beta$$

Encodings

The pure λ-calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure λ-calculus. We can however encode objects, such as booleans, and integers.

Booleans

We need to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as follows:

$$\text{AND TRUE FALSE} = \text{FALSE}$$

$$\text{NOT FALSE} = \text{TRUE}$$

$$\text{IF TRUE } e_1\ e_2 = e_1$$

$$\text{IF FALSE } e_1\ e_2 = e_2$$

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$$\text{TRUE} \triangleq \lambda x.\, \lambda y.\, x$$
$$\text{FALSE} \triangleq \lambda x.\, \lambda y.\, y$$

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$$\text{NOT} \triangleq$$

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$$\text{NOT} \triangleq \lambda b.\ b\ \text{FALSE}\ \text{TRUE}$$

$$\text{AND} \triangleq \lambda b_1.\ \lambda b_2.\ b_1\ b_2\ \text{FALSE}$$

$$\text{OR} \triangleq \lambda b_1.\ \lambda b_2.\ b_1\ \text{TRUE}\ b_2$$

Church Numerals

Let's encode the natural numbers!

We'll write $\overline{n}$ for the encoding of the number n. The central function we'll need is a *successor* operation:

$$\text{SUCC } \overline{n} = \overline{n+1}$$

Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$
\begin{aligned}
\overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\
\overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\
\overline{2} &\triangleq \lambda f.\, \lambda x.\, f\, (f\, x)
\end{aligned}
$$

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We can write a successor function that "inserts" another application of f:

$$\mathsf{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\, (n\, f\, x)$$

Addition

Given the definition of SUCC, we can define addition. Intuitively, the natural number $n_1 + n_2$ is the result of applying the successor function n_1 times to n_2.

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$$\text{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1 \text{ SUCC } n_2$$