



## 1 Overview

In this lecture we will extend the simply-typed  $\lambda$ -calculus with several features we saw earlier in the course, including products, sums, and references, as well as one new one.

### 1.1 Products

We have previously seen how to encode *products* into untyped  $\lambda$ -calculus.

$$\begin{aligned} e &::= \dots \mid (e_1, e_2) \mid \#1\ e \mid \#2\ e \\ v &::= \dots \mid (v_1, v_2) \end{aligned}$$

We defined congruence rules that determine the order of evaluation, using the following evaluation contexts.

$$E ::= \dots \mid (E, e) \mid (v, E) \mid \#1\ E \mid \#2\ E$$

We also defined two computation rules that determine how the pairing constructor and destructors interact.

$$\frac{}{\#1\ (v_1, v_2) \rightarrow v_1} \qquad \frac{}{\#2\ (v_1, v_2) \rightarrow v_2}$$

In simply-typed  $\lambda$ -calculus, the type of a product expression (or a *product type*) is a pair of types, written  $\tau_1 \times \tau_2$ . The typing rules for the product constructors and destructors are as follows:

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\ e : \tau_1} \qquad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#2\ e : \tau_2}$$

Note the similarities between these rules and the proof rules for conjunction in natural deduction. We will examine this relationship closely later in the course.

### 1.2 Sums

The next example, *sums*, are dual to products. Intuitively, a product holds two values, one of type  $\tau_1$ , and one of type  $\tau_2$ , while a sum holds a single value that is either of type  $\tau_1$  or of type  $\tau_2$ . The type of a sum is written  $\tau_1 + \tau_2$ . There are two constructors for sums, corresponding to whether we are constructing a sum with a value of  $\tau_1$  or a value of  $\tau_2$ .

$$\begin{aligned} e &::= \dots \mid \text{inl}_{\tau_1 + \tau_2}\ e \mid \text{inr}_{\tau_1 + \tau_2}\ e \mid \text{case } e_1 \text{ of } e_2 \mid e_3 \\ v &::= \dots \mid \text{inl}_{\tau_1 + \tau_2}\ v \mid \text{inr}_{\tau_1 + \tau_2}\ v \end{aligned}$$

There are congruence rules that determine the order of evaluation, as defined by the following evaluation contexts.

$$E ::= \dots \mid \text{inl}_{\tau_1+\tau_2} E \mid \text{inr}_{\tau_1+\tau_2} E \mid \text{case } E \text{ of } e_2 \mid e_3$$

There are also two computation rules that show how the constructors and destructors interact.

$$\overline{\text{case inl}_{\tau_1+\tau_2} v \text{ of } e_2 \mid e_3 \rightarrow e_2 v}$$

$$\overline{\text{case inr}_{\tau_1+\tau_2} v \text{ of } e_2 \mid e_3 \rightarrow e_3 v}$$

The type of a sum expression (or a *sum type*) is written  $\tau_1 + \tau_2$ . The typing rules for the sum constructors and destructor are the following.

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{inl}_{\tau_1+\tau_2} e : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{inr}_{\tau_1+\tau_2} e : \tau_1 + \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma \vdash e_1 : \tau_1 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2 \rightarrow \tau}{\Gamma \vdash \text{case } e \text{ of } e_1 \mid e_2 : \tau}$$

Let's see an example of a program that uses sum types.

```
let f = λa : int + (int → int). case a of (λy. y + 1) | (λg. g 35) in
let h = λx : int. x + 7 in
f (inrint+(int→int) h)
```

The function  $f$  takes argument  $a$ , which is a sum—that is, the actual argument for  $a$  will either be a value of type **int** or a value of type **int**  $\rightarrow$  **int**. We destruct the sum value with a case statement, which must be prepared to take either of the two kinds of values that the sum may contain. In this instance, we end up applying  $f$  to a value of type **int**  $\rightarrow$  **int** (i.e., a value injected into the right type of the sum), so the entire program ends up evaluating to 42.

### 1.3 References

Next we consider mutable references. Recall the syntax and semantics for references.

$$\begin{aligned} e &::= \dots \mid \text{ref } e \mid !e \mid e_1 := e_2 \mid \ell \\ v &::= \dots \mid \ell \\ E &::= \dots \mid \text{ref } E \mid !E \mid E := e \mid v := E \end{aligned}$$

$$\text{ALLOC} \frac{}{\langle \sigma, \text{ref } v \rangle \rightarrow \langle \sigma[\ell \mapsto v], \ell \rangle} \ell \notin \text{dom}(\sigma) \quad \text{DEREF} \frac{}{\langle \sigma, !\ell \rangle \rightarrow \langle \sigma, v \rangle} \sigma(\ell) = v$$

$$\text{ASSIGN} \frac{}{\langle \sigma, \ell := v \rangle \rightarrow \langle \sigma[\ell \mapsto v], v \rangle}$$

To extend the type system, we add a new type,  $\tau$  **ref**, to stand for the type of a location that contains a value of type  $\tau$ . For example the expression **ref** 7 has type **int ref**, since it evaluates to a location that contains a value of type **int**. Dereferencing a location of type  $\tau$  **ref** results in a value of type  $\tau$ , so  $!e$  has type  $\tau$  if  $e$  has type  $\tau$  **ref**. And for assignment  $e_1 := e_2$ , if  $e_1$  has type  $\tau$  **ref**, then  $e_2$  must have type  $\tau$ .

$$\tau ::= \dots \mid \tau \text{ ref}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \text{ref } e : \tau \text{ ref}} \quad \frac{\Gamma \vdash e : \tau \text{ ref}}{\Gamma \vdash !e : \tau} \quad \frac{\Gamma \vdash e_1 : \tau \text{ ref} \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 := e_2 : \tau}$$

Note that there is no typing rule for location values. What should the type of a location value  $\ell$  be? Clearly, it should be of type  $\tau \text{ ref}$ , where  $\tau$  is the type of the value contained in location  $\ell$ . But how do we know what value is contained in location  $\ell$ ? We could directly examine the store, but this would not be inefficient. In addition, examining the store directly may not give us a conclusive answer! Consider, for example, a store  $\sigma$  and location  $\ell$  where  $\sigma(\ell) = \ell$ ; what is the type of  $\ell$ ?

Instead, we introduce *store typings* to track the types of values stored in locations. Store typings are partial functions from locations to types. We use metavariable  $\Sigma$  to range over store typings. Our typing relation now becomes a relation over 4 entities: typing contexts, store typings, expressions, and types. We write  $\Gamma, \Sigma \vdash e : \tau$  when expression  $e$  has type  $\tau$  under typing context  $\Gamma$  and store typing  $\Sigma$ .

Our new typing rules for references are as follows. (Typing rules for other constructs are modified to take a store typing in the obvious way.)

$$\frac{\Gamma, \Sigma \vdash e : \tau}{\Gamma, \Sigma \vdash \text{ref } e : \tau \text{ ref}} \quad \frac{\Gamma, \Sigma \vdash e : \tau \text{ ref}}{\Gamma, \Sigma \vdash !e : \tau} \quad \frac{\Gamma, \Sigma \vdash e_1 : \tau \text{ ref} \quad \Gamma, \Sigma \vdash e_2 : \tau}{\Gamma, \Sigma \vdash e_1 := e_2 : \tau} \quad \frac{\Sigma(\ell) = \tau}{\Gamma, \Sigma \vdash \ell : \tau \text{ ref}}$$

So, how do we state type soundness? Our type soundness theorem for simply-typed lambda calculus said that if  $\Gamma \vdash e : \tau$  and  $e \rightarrow^* e'$  then  $e'$  is not stuck. But our operational semantics for references now has a store, and our typing judgment now has a store typing in addition to a typing context. We need to adapt the definition of type soundness appropriately. To do so, we define what it means for a store to be well-typed with respect to a typing context.

**Definition.** Store  $\sigma$  is *well-typed* with respect to typing context  $\Gamma$  and store typing  $\Sigma$ , written  $\Gamma, \Sigma \vdash \sigma$ , if  $\text{dom}(\sigma) = \text{dom}(\Sigma)$  and for all  $\ell \in \text{dom}(\sigma)$  we have  $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$ .

We can now state type soundness for our language with references.

**Theorem** (Type soundness). *If  $\cdot, \Sigma \vdash e : \tau$  and  $\cdot, \Sigma \vdash \sigma$  and  $\langle e, \sigma \rangle \rightarrow^* \langle e', \sigma' \rangle$  then either  $e'$  is a value, or there exists  $e''$  and  $\sigma''$  such that  $\langle e', \sigma' \rangle \rightarrow \langle e'', \sigma'' \rangle$ .*

We can prove type soundness for our language using the same strategy as for the simply-typed lambda calculus: using the preservation and progress lemmas. The progress lemma can be easily adapted for the semantics and type system for references. Adapting preservation is a little more involved, since we need to describe how the store typing changes as the store evolves. The rule **ALLOC** extends the store  $\sigma$  with a fresh location  $\ell$ , producing store  $\sigma'$ . Since  $\text{dom}(\Sigma) = \text{dom}(\sigma) \neq \text{dom}(\sigma')$ , it means that we will not have  $\sigma'$  well-typed with respect to typing store  $\Sigma$ .

Since the store can increase in size during the evaluation of the program, we also need to allow the store typing to grow as well.

**Lemma** (Preservation). *If  $\Gamma, \Sigma \vdash e : \tau$  and  $\Gamma, \Sigma \vdash \sigma$  and  $\langle e, \sigma \rangle \rightarrow \langle e', \sigma' \rangle$  then there exists some  $\Sigma' \supseteq \Sigma$  such that  $\Gamma, \Sigma' \vdash e' : \tau$  and  $\Gamma, \Sigma' \vdash \sigma'$ .*

We write  $\Sigma' \supseteq \Sigma$  to mean that for all  $\ell \in \text{dom}(\Sigma)$  we have  $\Sigma(\ell) = \Sigma'(\ell)$ . This makes sense if we think of partial functions as sets of pairs:  $\Sigma \equiv \{(\ell, v) \mid \ell \in \text{dom}(\Sigma) \wedge \Sigma(\ell) = v\}$ . Note that the preservation lemma states simply that there is some store type  $\Sigma' \supseteq \Sigma$ , but does not specify what

exactly that store typing is. Intuitively,  $\Sigma'$  will either be  $\Sigma$ , or  $\Sigma$  extended with a newly allocated location.

Interestingly, references are enough to recover Turing completeness. For example, to implement a recursive function  $f$  we can initialize a reference cell containing a dummy value for  $f$  and then “backpatch” it with the actual definition. For example, here is an implementation of the familiar factorial function, written using **let** expressions, conditionals, and natural numbers for clarity.

$$\begin{aligned} &\mathbf{let } r = \mathbf{ref } \lambda x. 0 \mathbf{ in} \\ &r := \lambda x : \mathbf{int}. \mathbf{if } x = 0 \mathbf{ then } 1 \mathbf{ else } x \times !r (x - 1) \end{aligned}$$

This trick is known as “Landin’s knot” after its inventor.

## 1.4 Fixed Points

Another way to obtain fixed points in the simply-typed lambda calculus is to simply add a new primitive **fix** to the language. The evaluation rules for the new primitive mimic the behavior of the fixed-point combinators we saw previously.

We extend the syntax with the new primitive operator. Intuitively,  $\mathbf{fix } e$  is the fixed-point of the function  $e$ . Note that  $\mathbf{fix } v$  is *not* a value.

$$e ::= \dots \mid \mathbf{fix } e$$

We extend the operational semantics for the new operator. There is a new evaluation context, and a new axiom.

$$E ::= \dots \mid \mathbf{fix } E \qquad \frac{}{\mathbf{fix } \lambda x : \tau. e \rightarrow e\{(\mathbf{fix } \lambda x : \tau. e)/x\}}$$

Note that we can define the  $\mathbf{letrec } x : \tau = e_1 \mathbf{ in } e_2$  construct in terms of the **fix** operator.

$$\mathbf{letrec } x : \tau = e_1 \mathbf{ in } e_2 \triangleq \mathbf{let } x = \mathbf{fix } \lambda x : \tau. e_1 \mathbf{ in } e_2$$

The typing rule for **fix** is left as an exercise.

Returning to our trusty factorial example, the following program implements the factorial function using the **fix** operator.

$$\mathbf{FACT} \triangleq \mathbf{fix } \lambda f : \mathbf{int} \rightarrow \mathbf{int}. \lambda n : \mathbf{int}. \mathbf{if } n = 0 \mathbf{ then } 0 \mathbf{ else } n \times (f (n - 1))$$

Note that we can write non-terminating computations for any type: the expression  $\mathbf{fix } \lambda x : \tau. x$  has type  $\tau$ , and does not terminate.

Although the **fix** operator is normally used to define recursive functions, it can be used to find fixed points of any type. For example, consider the following expression.

$$\begin{aligned} &\mathbf{fix } \lambda x : (\mathbf{int} \rightarrow \mathbf{bool}) \times (\mathbf{int} \rightarrow \mathbf{bool}). (\lambda n : \mathbf{int}. \mathbf{if } n = 0 \mathbf{ then } \mathbf{true} \mathbf{ else } (\#2 \ x) (n - 1), \\ &\lambda n : \mathbf{int}. \mathbf{if } n = 0 \mathbf{ then } \mathbf{false} \mathbf{ else } (\#1 \ x) (n - 1)) \end{aligned}$$

This expression defines a pair of mutually recursive functions; the first function returns **true** if and only if its argument is even; the second function returns **true** if and only if its argument is odd.

## 1.5 Exceptions

Many programming languages provide support for throwing and catching exceptions. We can model an extremely simple form of exceptions by extending the simply-typed  $\lambda$ -calculus with a single exception representing an error. We first extend the syntax of the language:

$$e ::= \dots \mid \mathbf{error} \mid \mathbf{try} \ e_1 \ \mathbf{with} \ e_2$$

We do not add **try** expressions to our evaluation contexts—doing so would allow exceptions to “jump over” handlers. Instead, we add a special propagation rule for **try**:

$$\frac{e_1 \rightarrow e'_1}{\mathbf{try} \ e_1 \ \mathbf{with} \ e_2 \rightarrow \mathbf{try} \ e'_1 \ \mathbf{with} \ e_2}$$

and rules for propagating and catching exceptions:

$$\frac{}{E[\mathbf{error}] \rightarrow \mathbf{error}} \qquad \frac{}{\mathbf{try} \ \mathbf{error} \ \mathbf{with} \ e \rightarrow e} \qquad \frac{}{\mathbf{try} \ v \ \mathbf{with} \ e \rightarrow v}$$

The typing rule for exceptions allows them to take *any* type, while the typing rule for try-with expressions requires both sub-expressions to have the same type:

$$\frac{}{\Gamma \vdash \mathbf{error} : \tau} \qquad \frac{\Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \mathbf{try} \ e_1 \ \mathbf{with} \ e_2 : \tau}$$

The first typing rule is extremely flexible, allowing errors to be thrown anywhere in a program. However, it is not hard to see that it causes the progress lemma to become false: the expression **error** is not a value but is stuck. Fortunately, we can prove the following weaker version, which is still strong enough to prove a useful form of type soundness.

**Lemma (Progress).** *If  $\vdash e : \tau$  then  $e$  is a value or  $e$  is **error** or there exists  $e'$  such that  $e \rightarrow e'$ .*

The preservation theorem remains unchanged.

The actual soundness theorem is as follows:

**Theorem 1 (Soundness).** *If  $\vdash e : \tau$  and  $e \rightarrow^* e'$  and  $e' \not\rightarrow$  then either  $e'$  is a value or  $e'$  is **error**.*