CS 4110

Programming Languages & Logics

Lecture 11
Weakest Preconditions

Review: Decorating Programs

$\{\textbf{true}\}$

$\mathsf{x} := \mathsf{m};$
$\mathsf{y} := 0;$
$\textbf{while}\ (\mathsf{n} < \mathsf{x})\ \textbf{do}\ ($
$\quad \mathsf{x} := \mathsf{x} - \mathsf{n};$
$\quad \mathsf{y} := \mathsf{y} + 1$
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$\{\mathsf{n} \times \mathsf{y} + \mathsf{x} = \mathsf{m}\}$

In other words, the program divides m by n, so y is the quotient and x is the remainder.

Generating Preconditions

To fill in a precondition:

$$\{\quad\}\ c\ \{Q\}$$

there are many possible preconditions—and some are more useful than others.

Weakest Preconditions

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More formally...

Definition (Weakest Liberal Precondition)

P is a weakest liberal precondition of c and Q written $wlp(c, Q)$ if:

$$\forall \sigma, I.\ \sigma \vDash_I P \iff (\mathcal{C}[\![c]\!]\ \sigma) \text{ undefined } \vee (\mathcal{C}[\![c]\!]\sigma) \vDash_I Q$$

Weakest Preconditions

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$$

where

$$
\begin{array}{rcl}
F_0(P) & = & \textbf{true} \\
F_{i+1}(P) & = & (\neg b \implies P) \wedge (b \implies \mathit{wlp}(c, F_i(P)))
\end{array}
$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

$\mathsf{p} := \mathsf{getPacket}();$
$\mathsf{processPacket}(\mathsf{p});$
$\textbf{assert } P_{\mathrm{safe}}$

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Applications of Weakest Preconditions

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$$\begin{array}{l} \mathsf{p} := \mathsf{getPacket}(); \\ \{P_{\text{filter}}(p)\}; \\ \mathsf{processPacket}(\mathsf{p}); \\ \{P_{\text{safe}}\} \end{array}$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

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Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
assert $P_{\mathrm{filter}}(p)$;
processPacket(p);

P_{filter} should be the *weakest* precondition to avoid ruling out legitimate inputs.

David Brumley, Hao Wang, Somesh Jha, and Dawn Song. “Creating Vulnerability Signatures Using Weakest Preconditions.” In *Computer Security Foundations* (CSF), 2007.

Properties of Weakest Preconditions

Lemma (Correctness of Weakest Preconditions)

$$\begin{aligned} &\forall c \in \mathbf{Com}, Q \in \mathbf{Assn}. \\ &\qquad \models \{wlp(c, Q)\}\, c\, \{Q\} \;\; \textit{and} \\ &\qquad \forall R \in \mathbf{Assn}.\ \models \{R\}\, c\, \{Q\}\ \textit{implies}\ (R \implies wlp(c, Q)) \end{aligned}$$

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Lemma (Provability of Weakest Preconditions)

$$\forall c \in \mathbf{Com}, Q \in \mathbf{Assn}.\ \vdash \{wlp(c, Q)\}\, c\, \{Q\}$$

Soundness and Completeness

Soundness: If we can prove it, then it's actually true.

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Definition (Soundness)

If $\vdash \{P\}\ c\ \{Q\}$ then $\models \{P\}\ c\ \{Q\}$.

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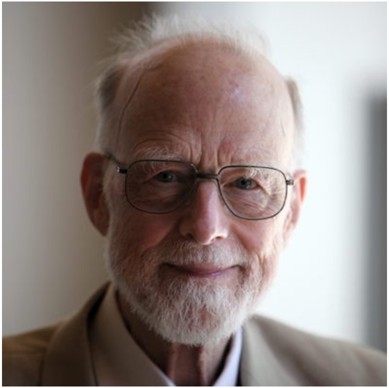
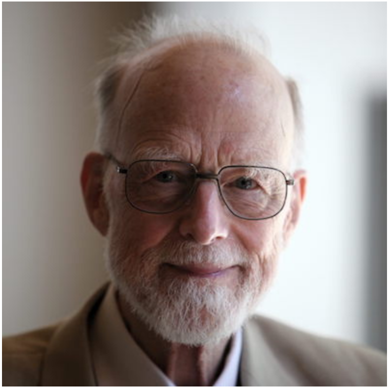
If $\models \{P\}\ c\ \{Q\}$ then $\vdash \{P\}\ c\ \{Q\}$.

vs.



Kurt Gödel

vs.



Sir Tony Hoare

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \mathbf{Assn}, c \in \mathbf{Com}.\ \vDash \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \mathbf{Assn}, c \in \mathbf{Com}.\ \vDash \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

Proof Sketch.

Let $\{P\}\, c\, \{Q\}$ be a valid partial correctness specification.

By the first Lemma we have $\vDash P \implies wlp(c, Q)$.

By the second Lemma we have $\vdash \{wlp(c, Q)\}\, c\, \{Q\}$.

We conclude $\vdash \{P\}\, c\, \{Q\}$ using the CONSEQUENCE rule. $\square$