CS 4110

Programming Languages & Logics

Lecture 21
Advanced Types

Review

We've developed a type system for the λ-calculus and mathematical tools for proving its type soundness.

We also know how to extend the λ-calculus with new language features.

Today, we'll extend our *type system* with features commonly found in real-world languages: products, sums, references, and exceptions.

Products (Pairs)

Syntax

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e$$
$$v ::= \cdots \mid (v_1, v_2)$$

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Semantics

$$E ::= \cdots \mid (E, e) \mid (v, E) \mid \#1\, E \mid \#2\, E$$

$$\frac{}{\#1\,(v_1, v_2) \to v_1} \qquad \frac{}{\#2\,(v_1, v_2) \to v_2}$$

Product Types

$$\tau_1 \times \tau_2$$

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

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$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\, e : \tau_1}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#2\, e : \tau_2}$$

Sums (Tagged Unions)

Syntax

$$e ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ e \mid \mathsf{inr}_{\tau_1+\tau_2}\ e \mid (\mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3)$$

$$v ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ v \mid \mathsf{inr}_{\tau_1+\tau_2}\ v$$

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$$\frac{}{\mathsf{case}\ \mathsf{inl}_{\tau_1+\tau_2}\ v\ \mathsf{of}\ e_2 \mid e_3 \rightarrow e_2\ v}$$

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Sum Types

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$$\frac{\Gamma \vdash \mathsf{e} : \tau_1 + \tau_2 \quad \Gamma \vdash \mathsf{e}_1 : \tau_1 \to \tau \quad \Gamma \vdash \mathsf{e}_2 : \tau_2 \to \tau}{\Gamma \vdash \mathsf{case}\ \mathsf{e}\ \mathsf{of}\ \mathsf{e}_1 \mid \mathsf{e}_2 : \tau}$$

Example

let $f = \lambda a : \textbf{int} + (\textbf{int} \to \textbf{int}).$ case a of $(\lambda y.\, y + 1) \mid (\lambda g.\, g\, 35)$ in

let $h = \lambda x : \textbf{int}.\, x + 7$ in

$f\, (\mathsf{inr}_{\textbf{int}+(\textbf{int}\to\textbf{int})}\, h)$

References

Syntax

$$e ::= \cdots \mid \mathsf{ref}\ e \mid !e \mid e_1 := e_2 \mid \ell$$
$$v ::= \cdots \mid \ell$$

References

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$$
\begin{aligned}
e &::= \cdots \mid \mathsf{ref}\, e \mid !e \mid e_1 := e_2 \mid \ell \\
v &::= \cdots \mid \ell
\end{aligned}
$$

Semantics

$$E ::= \cdots \mid \mathsf{ref}\, E \mid !E \mid E := e \mid v := E$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\, v\rangle \to \langle \sigma[\ell \mapsto v], \ell\rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\ !\ell\rangle \to \langle \sigma, v\rangle}$$

$$\frac{}{\langle \sigma, \ell := v\rangle \to \langle \sigma[\ell \mapsto v], v\rangle}$$

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Store Typings

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$$\frac{\Sigma(\ell) = \tau}{\Gamma, \Sigma \vdash \ell : \tau\ \textbf{ref}}$$

Reference Types Metatheory

Definition

Store σ is *well-typed* with respect to typing context Γ and store typing Σ, written $\Gamma, \Sigma \vdash \sigma$, if $dom(\sigma) = dom(\Sigma)$ and for all $\ell \in dom(\sigma)$ we have $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$.

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Theorem (Type soundness)

If $\cdot, \Sigma \vdash e : \tau$ and $\cdot, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \to^* \langle e', \sigma' \rangle$ then either e' is a value, or there exists e'' and σ'' such that $\langle e', \sigma' \rangle \to \langle e'', \sigma'' \rangle$.

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Lemma (Preservation)

If $\Gamma, \Sigma \vdash e : \tau$ and $\Gamma, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \rightarrow \langle e', \sigma' \rangle$ then there exists some $\Sigma' \supseteq \Sigma$ such that $\Gamma, \Sigma' \vdash e' : \tau$ and $\Gamma, \Sigma' \vdash \sigma'$.

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Using references, we (re)gain the ability define recursive functions!

$$\textbf{let } r = \text{ref } \lambda x\!:\!\textbf{int}.\, 0 \textbf{ in}$$

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$$
\begin{aligned}
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&\textbf{let } f = (\lambda x\!:\!\textbf{int}.\, \textbf{if } x = 0 \textbf{ then } 1 \textbf{ else } x \times (!r)\,(x-1)) \textbf{ in}\\
&\textbf{let } a = (r := f) \textbf{ in}
\end{aligned}
$$

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$$\begin{array}{l}
\textbf{let } r = \mathrm{ref}\ \lambda x{:}\,\textbf{int}.\ 0 \textbf{ in} \\
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f\ 5
\end{array}$$

Fixed Points

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Semantics

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$$\frac{}{\mathsf{fix}\, \lambda x\!:\!\tau.\, e \to e\{(\mathsf{fix}\, \lambda x\!:\!\tau.\, e)/x\}}$$

Fixed Points

Syntax

$$e ::= \cdots \mid \mathsf{fix}\; e$$

Semantics

$$E ::= \cdots \mid \mathsf{fix}\; E$$

$$\frac{}{\mathsf{fix}\; \lambda x\!:\!\tau.\, e \to e\{(\mathsf{fix}\; \lambda x\!:\!\tau.\, e)/x\}}$$

The typing rule for fix is on the homework...