

CS 4110

# Programming Languages & Logics

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## Lecture 17 De Bruijn, Combinators



# de Bruijn Notation

Another way to avoid the tricky issues with substitution is to use a *nameless* representation of terms.

$$e ::= n \mid \lambda.e \mid e e$$

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$$e ::= n \mid \lambda.e \mid e e$$

Abstractions have lost their variables!

Variables are replaced with numerical indices!

# Examples

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Here are some terms written in standard and de Bruijn notation:

**Standard**

$\lambda x. x$

$\lambda z. z$

**de Bruijn**

$\lambda. 0$

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$\lambda x. \lambda y. x$

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$\lambda x. x$

$\lambda z. z$

$\lambda x. \lambda y. x$

$\lambda x. \lambda y. \lambda s. \lambda z. x s (y s z)$

## de Bruijn

$\lambda. 0$

$\lambda. 0$

$\lambda. \lambda. 1$

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$(\lambda x. x x) (\lambda x. x x)$

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# Free variables

To represent a  $\lambda$ -expression that contains free variables in de Bruijn notation, we need a way to map the free variables to integers.

We will work with respect to a map  $\Gamma$  from variables to integers called a *context*.

## Examples:

Suppose that  $\Gamma$  maps  $x$  to 0 and  $y$  to 1.

- Representation of  $xy$  is 0 1
- Representation of  $\lambda z. x y z$  is  $\lambda. 1\ 2\ 0$

# Shifting

To define substitution, we will need an operation that shifts by  $i$  the variables above a cutoff  $c$ :

$$\begin{aligned}\uparrow_c^i(n) &= \begin{cases} n & \text{if } n < c \\ n + i & \text{otherwise} \end{cases} \\ \uparrow_c^i(\lambda.e) &= \lambda.(\uparrow_{c+1}^i e) \\ \uparrow_c^i(e_1 e_2) &= (\uparrow_c^i e_1) (\uparrow_c^i e_2)\end{aligned}$$

The cutoff  $c$  keeps track of the variables that were bound in the original expression and so should not be shifted.

The cutoff is 0 initially.

# Substitution

Now we can define substitution:

$$\begin{aligned}n\{e/m\} &= \begin{cases} e & \text{if } n = m \\ n & \text{otherwise} \end{cases} \\(\lambda.e_1)\{e/m\} &= \lambda.e_1\{(\uparrow_0^1 e)/m + 1\} \\(e_1 e_2)\{e/m\} &= (e_1\{e/m\}) (e_2\{e/m\})\end{aligned}$$

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The  $\beta$  rule for terms in de Bruijn notation is just:

$$\overline{(\lambda.e_1) e_2 \rightarrow \uparrow_0^{-1} (e_1\{\uparrow_0^1 e_2/0\})} \beta$$

# Example

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Consider the term  $(\lambda u. \lambda v. u x) y$  with respect to a context where  $\Gamma(x) = 0$  and  $\Gamma(y) = 1$ .

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which, in standard notation (with respect to  $\Gamma$ ), is the same as  $\lambda v. y x$ .

# Combinators

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Another way to avoid the issues having to do with free and bound variable names in the  $\lambda$ -calculus is to work with closed expressions or *combinators*.

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Another way to avoid the issues having to do with free and bound variable names in the  $\lambda$ -calculus is to work with closed expressions or *combinators*.

With just three combinators, we can encode the entire  $\lambda$ -calculus.

$$K = \lambda x. \lambda y. x$$

$$S = \lambda x. \lambda y. \lambda z. x z (y z)$$

$$I = \lambda x. x$$

# Combinators

We can even define independent evaluation rules that don't depend on the  $\lambda$ -calculus at all.

Behold the “SKI-calculus”:

$$K\ e_1\ e_2 \rightarrow e_1$$

$$S\ e_1\ e_2\ e_3 \rightarrow e_1\ e_3\ (e_2\ e_3)$$

$$I\ e \rightarrow e$$

You would never want to program in this language—it doesn't even have variables!—but it's exactly as powerful as the  $\lambda$ -calculus.

# Bracket Abstraction

The function  $[x]$  that takes a combinator term  $M$  and builds another term that behaves like  $\lambda x.M$ :

$$\begin{aligned}[x] x &= I \\ [x] N &= K N && \text{where } x \notin fv(N) \\ [x] N_1 N_2 &= S ([x] N_1) ([x] N_2)\end{aligned}$$

The idea is that  $([x] M) N \rightarrow M\{N/x\}$  for every term  $N$ .

# Bracket Abstraction

We then define a function  $(e)^*$  that maps a  $\lambda$ -calculus expression to a combinator term:

$$\begin{aligned}(x)^* &= x \\ (e_1 e_2)^* &= (e_1)^* (e_2)^* \\ (\lambda x. e)^* &= [\lambda] (e)^*\end{aligned}$$

# Example

As an example, the expression  $\lambda x. \lambda y. x$  is translated as follows:

$$\begin{aligned} & (\lambda x. \lambda y. x)^* \\ = & [x] (\lambda y. x)^* \\ = & [x] ([y] x) \\ = & [x] (K x) \\ = & (S ([x] K) ([x] x)) \\ = & S (K K) I \end{aligned}$$

No variables in the final combinator term!

# Example

We can check that this behaves the same as our original  $\lambda$ -expression by seeing how it evaluates when applied to arbitrary expressions  $e_1$  and  $e_2$ .

$$\begin{aligned} & (\lambda x. \lambda y. x) e_1 e_2 \\ \rightarrow & (\lambda y. e_1) e_2 \\ \rightarrow & e_1 \end{aligned}$$

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$$\begin{aligned} & (\lambda x. \lambda y. x) e_1 e_2 \\ \rightarrow & (\lambda y. e_1) e_2 \\ \rightarrow & e_1 \end{aligned}$$

and

$$\begin{aligned} & (S (K K) I) e_1 e_2 \\ \rightarrow & (K K e_1) (I e_1) e_2 \\ \rightarrow & K e_1 e_2 \\ \rightarrow & e_1 \end{aligned}$$

# SKI Without I

Looking back at our definitions...

$$K e_1 e_2 \rightarrow e_1$$

$$S e_1 e_2 e_3 \rightarrow e_1 e_3 (e_2 e_3)$$

$$I e \rightarrow e$$

...I isn't strictly necessary. It behaves the same as S K K.



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...*I* isn't strictly necessary. It behaves the same as *S K K*.

Our example becomes:

$$S (K K) (S K K)$$