CS 4110

Programming Languages & Logics

Lecture 2
Introduction to Semantics

Semantics

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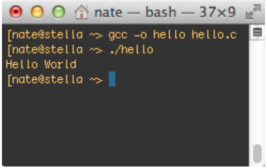
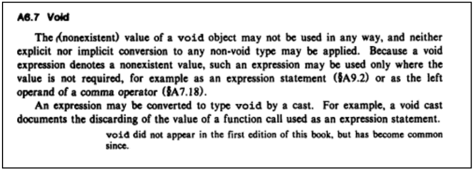
Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...




A8.7 Void

The (nonexistent) value of a void object may not be used in any way, and neither explicit nor implicit conversion to any non-void type may be applied. Because a void expression denotes a nonexistent value, such an expression may be used only where the value is not required, for example as an expression statement (§A9.2) or as the left operand of a comma operator (§A7.18).

An expression may be converted to type void by a cast. For example, a void cast documents the discarding of the value of a function call used as an expression statement.

void did not appear in the first edition of this book, but has become common since.

...but none of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
- Denotational: $[\![e]\!]$
  - Model program as mathematical objects
  - Useful for theoretical foundations
- Axiomatic $\vdash \{\phi\}\ e\ \{\psi\}$
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

Arithmetic Expressions

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BNF Grammar:

$$\begin{array}{rl} e ::= & x \\ | & n \\ | & e_1 + e_2 \\ | & e_1 * e_2 \\ | & x := e_1 \,;\; e_2 \end{array}$$

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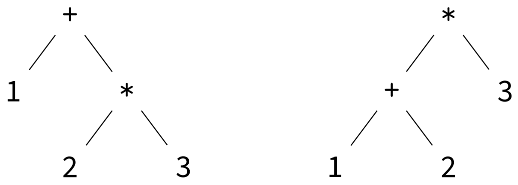

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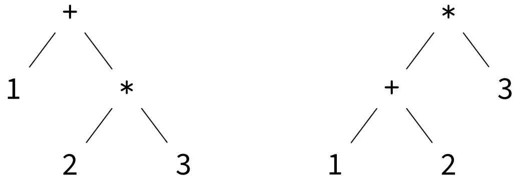

In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

Representing Expressions

BNF Grammar:

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\begin{aligned}
e ::= {} & x \\
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OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var ”foo”, Int 1))

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Java:

```
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var(”foo”), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to ...?

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- $x + 1$ evaluates to error?

The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

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Some Important Relations

- empty: $\emptyset$
- total: $A \times B$
- identity on A: $\{(a, a) \mid a \in A\}$.
- composition $R; S$: $\{(a, c) \mid \exists b. \, (a, b) \in R \wedge (b, c) \in S\}$

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The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. Formally:

$$\mathrm{image}(f) \triangleq \{f(a) \mid a \in A\}$$

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Given two functions $f : A \to B$ and $g : B \to C$, the composition of f and g is defined by: $(g \circ f)(x) = g(f(x))$ Note order!

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

Operational Semantics

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More formally:

$$\begin{array}{rcl} \textbf{Store} & \triangleq & \textbf{Var} \rightharpoonup \textbf{Int} \\ \textbf{Config} & \triangleq & \textbf{Store} \times \textbf{Exp} \end{array}$$

(A store is a *partial* function from variables to integers.)

Operational Semantics

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Question: How should we define this relation?

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which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in$ "$\rightarrow$".

Question: How should we define this relation? Remember that there are an infinite number of configurations and possible steps!

Inference Rules

Answer: Define it inductively, using inference rules:

$$\frac{\text{premise}_1 \qquad \text{premise}_2 \qquad \cdots}{\text{conclusion}}\ \textsc{Name}$$

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An inference rule defines an implication: if all the premises hold, then the conclusion also holds.

Formally, "$\rightarrow$" is the smallest relation that is closed under all the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \to \langle \sigma, n \rangle} \textsc{Var}$$

Addition

$$\frac{p = m \mathbin{\color{blue}{+}} n}{\langle \sigma, n + m \rangle \to \langle \sigma, p \rangle}\ \textsc{Add}$$

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Multiplication

$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle}\ \textsc{Mul}$$

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$$\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e_2' \rangle}{\langle \sigma, n * e_2 \rangle \rightarrow \langle \sigma', n * e_2' \rangle}\ \textsc{RMul}$$

Assignment

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n\,;\; e_2 \rangle \to \langle \sigma', e_2 \rangle}\ \textsc{Assgn}$$

Notation: $\sigma[x \mapsto n]$ is a *new* (partial) function that mostly behaves like σ, except that it maps x to n.

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$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1\,;\ e_2 \rangle \to \langle \sigma', x := e_1'\,;\ e_2 \rangle}\ \textsc{Assgn1}$$

Operational Semantics

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \textsc{Var}$$

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$$\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \textsc{Add}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e_1' * e_2 \rangle} \textsc{LMul}$$

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$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n \,;\; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle} \textsc{Assgn}$$

Multi-Step Evaluation

We can define the multi-step evaluation relation, written $\rightarrow^*$, as the reflexive and transitive closure of the small-step evaluation relation.

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle}\ \textsc{Refl}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \qquad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle}\ \textsc{Trans}$$