CS 4110

Programming Languages & Logics

Lecture 19
Continuations

Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

$$\begin{aligned}
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\ e_2]\!] &= \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!]
\end{aligned}$$

What can go wrong with this approach?

Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers...
- ...or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions

Example

Consider the following expression:

$$(\lambda x.\, x)\, ((3 * (1 + 2)) - 4)$$

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The original expression is equivalent to $k_3\ 1$, or:

$$(\lambda c.\, (\lambda b.\, (\lambda a.\, (\lambda v.\, (\lambda x.\, x)\, v)\, (a - 4))\, (3 * b))\, (c + 2))\, 1$$

Example (Continued)

Recall that let $x = e$ in e' is syntactic sugar for $(\lambda x.\, e')\; e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$
\begin{aligned}
&\text{let } c = 1 \text{ in} \\
&\text{let } b = c + 2 \text{ in} \\
&\text{let } a = 3 * b \text{ in} \\
&\text{let } v = a - 4 \text{ in} \\
&(\lambda x.\, x)\; v
\end{aligned}
$$

CPS Transformation

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\, k = k\, n$$

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CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$
$$\mathcal{CPS}[\![e_1 + e_2]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m)))$$

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\mathcal{CPS}[\![(e_1, e_2)]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w)))
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\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k')
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