CS 5435

Introduction to Public-Key Cryptography

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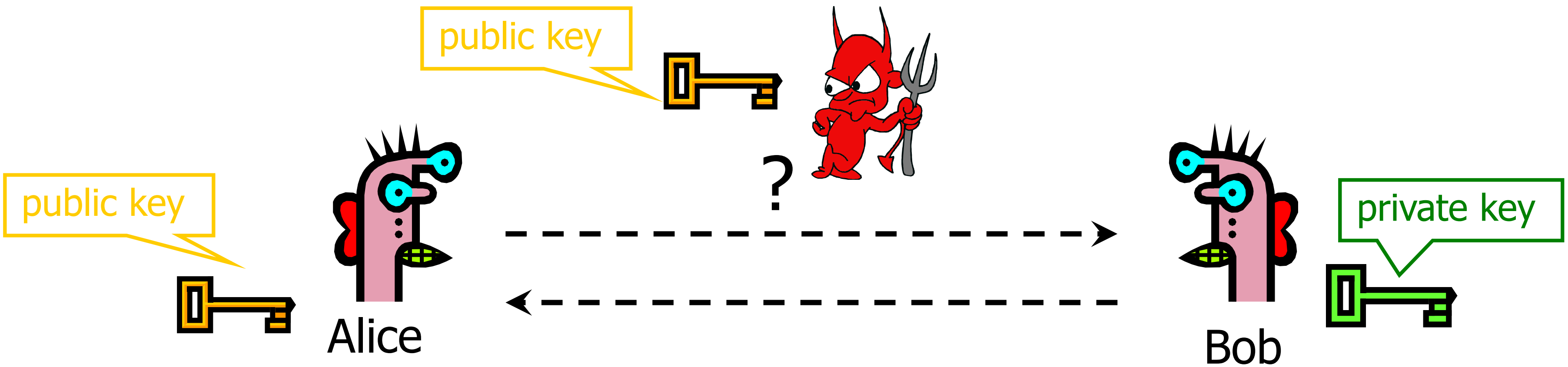
Public-Key Cryptography



Given: Everybody knows Bob's public key

- How is this achieved in practice?

Only Bob knows the corresponding private key

Goals: 1. Alice wants to send a message that only Bob can read

2. Bob wants to send a message that only Bob could have written

Applications of Public-Key Crypto

Encryption for confidentiality

- Anyone can encrypt a message
  - With symmetric crypto, must know the secret key to encrypt
- Only someone who knows the private key can decrypt
- Secret keys are only stored in one place

Digital signatures for authentication

- Only someone who knows the private key can sign

Session key establishment

- Exchange messages to create a secret session key
- Then switch to symmetric cryptography (why?)

Public-Key Encryption

Key generation: computationally easy to generate a pair (public key PK, private key SK)

Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$

Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M

- Infeasible to learn anything about M from C without SK
- Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the $[1,n]$ interval that are relatively prime to n

- Two numbers are relatively prime if their greatest common divisor (gcd) is 1

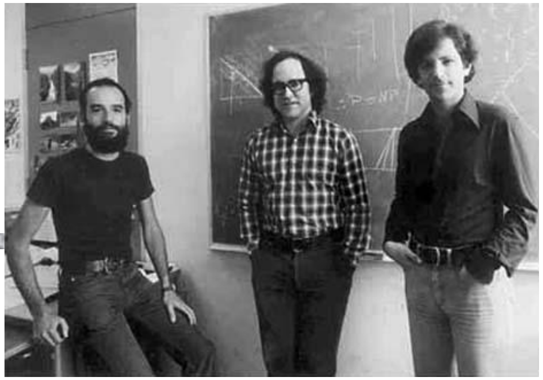
Euler's theorem:

if $a \in Z_n^*$, then $a^{\varphi(n)} \equiv 1 \bmod n$

Special case: <u>Fermat's Little Theorem</u>

if p is prime and $\gcd(a,p)=1$, then $a^{p-1} \equiv 1 \bmod p$

RSA Cryptosystem



[Rivest, Shamir, Adleman 1977]

Key generation:

- Generate large primes p, q
  - At least 2048 bits each… need primality testing!
- Compute $n=pq$
  - Note that $\varphi(n)=(p-1)(q-1)$
- Choose small e, relatively prime to $\varphi(n)$
  - Typically, $e=3$ (may be vulnerable) or $e=2^{16}+1=65537$ (why?)
- Compute unique d such that $ed \equiv 1 \bmod \varphi(n)$
- Public key = (e,n); private key = d

Encryption of m: $c = m^e \bmod n$

Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

Why RSA Decryption Works

$e \cdot d \equiv 1 \bmod \varphi(n)$

Thus $e \cdot d = 1+k \cdot \varphi(n) = 1+k(p-1)(q-1)$ for some k

If $\gcd(m,p)=1$, then by Fermat's Little Theorem, $m^{p-1} \equiv 1 \bmod p$

Raise both sides to the power $k(q-1)$ and multiply by m, obtaining $m^{1+k(p-1)(q-1)} \equiv m \bmod p$

Thus $m^{ed} \equiv m \bmod p$

By the same argument, $m^{ed} \equiv m \bmod q$

Since p and q are distinct primes and $p \cdot q=n$,

$m^{ed} \equiv m \bmod n$

Why Is RSA Secure?

RSA problem: given c, $n=pq$, and e such that $\gcd(e,(p-1)(q-1))=1$, find m such that $m^e=c \bmod n$

- In other words, recover m from ciphertext c and public key (n,e) by taking e^{th} root of c modulo n
- There is no known efficient algorithm for doing this

Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$

If factoring is easy, then RSA problem is easy, but may be possible to break RSA without factoring n

Factoring Records

RSA-x is an RSA challenge modulus of size x bits

Algorithm	Year	Algorithm	Time
RSA-400	1993	Quadratic sieve	830 MIPS years
RSA-478	1994	Quadratic sieve	5000 MIPS years
RSA-515	1999	Number-field sieve	8000 MIPS years
RSA-768	2009	Number-field sieve	~2.5 years

Nowadays, minimal recommended size is 2048-bit modulus

Exponentiation in $O(\log N)$, and so size impacts performance

“Textbook” RSA Is Bad Encryption

Deterministic

- Attacker can guess plaintext, compute ciphertext, and compare for equality
- If messages are from a small set (for example, yes/no), can build a table of corresponding ciphertexts

Can tamper with encrypted messages

- Take an encrypted auction bid c and submit $c(101/100)^e \bmod n$ instead

Does not provide semantic security (security against chosen-plaintext attacks)

Integrity in RSA Encryption

"Textbook" RSA does not provide integrity

- Given encryptions of m_1 and m_2, attacker can create encryption of $m_1 \cdot m_2$
  - $(m_1^e) \cdot (m_2^e) \bmod n \equiv (m_1 \cdot m_2)^e \bmod n$
- Attacker can convert m into m^k without decrypting
  - $(m^e)^k \bmod n \equiv (m^k)^e \bmod n$

In practice, **OAEP** is used: instead of encrypting M, encrypt $M \oplus G(r) \,;\, r \oplus H(M \oplus G(r))$

- r is random and fresh, G and H are hash functions
- Resulting encryption is "plaintext-aware": infeasible to compute a valid encryption without knowing plaintext
  - … if hash functions are "good" and the RSA problem is hard

Key Exchange

pk

C = Enc(pk,K,R)

Have (pk,sk) pair

Choose fresh
symmetric key K

K <- Dec(sk,C)

Server picks long-lived (pk,sk) pair; pk sent to client

Client encrypts a key K using pk and some fresh randomness R

Ciphertext C sent to server; server decrypts using sk

Forward Secrecy?

Choose fresh
symmetric key K

pk

C = Enc(pk,K,R)

Record
encrypted
transcript

Have (pk,sk) pair

K <- Dec(sk,C)

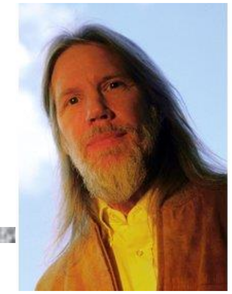
Sometime later… break in and steal sk

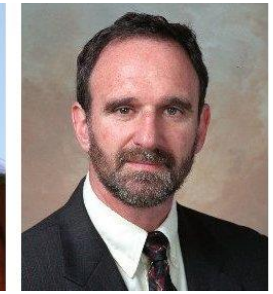
Can adversary recover K? Yes!

Have (pk,sk) pair

We want key exchange protocol that provides
forward secrecy: later compromises don't reveal
previous sessions

Diffie-Hellman Protocol

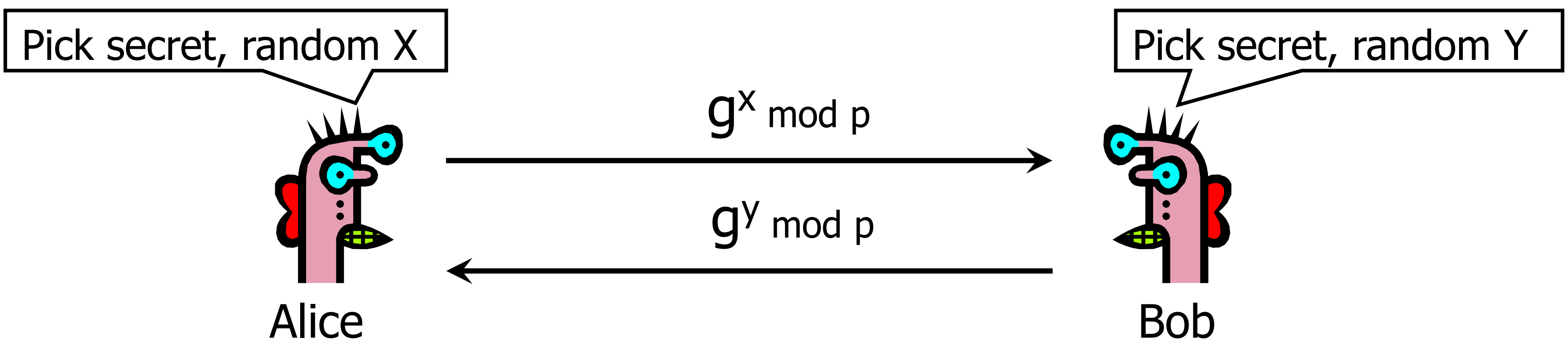




Alice and Bob never met and share no secrets

Public info: p and g

- p is a large prime number, g is a generator of Z_p^*
  - $Z_p^* = \{1, 2 \ldots p-1\}$; $\forall a \in Z_p^*\ \exists i$ such that $a = g^i \bmod p$



Compute $k=(g^y)^x=g^{xy} \bmod p$

Compute $k=(g^x)^y=g^{xy} \bmod p$

Why Is Diffie-Hellman Secure?

Discrete Logarithm (DL) problem:
given $g^x \bmod p$, it's hard to extract x

- There is no known efficient algorithm for doing this
- This is not enough for Diffie-Hellman to be secure!

Computational Diffie-Hellman (CDH) problem:
given g^x and g^y, it's hard to compute $g^{xy} \bmod p$

- … unless you know x or y, in which case it's easy

Decisional Diffie-Hellman (DDH) problem:

given g^x and g^y, it's hard to tell the difference between $g^{xy} \bmod p$ and $g^r \bmod p$ where r is random

Properties of Diffie-Hellman

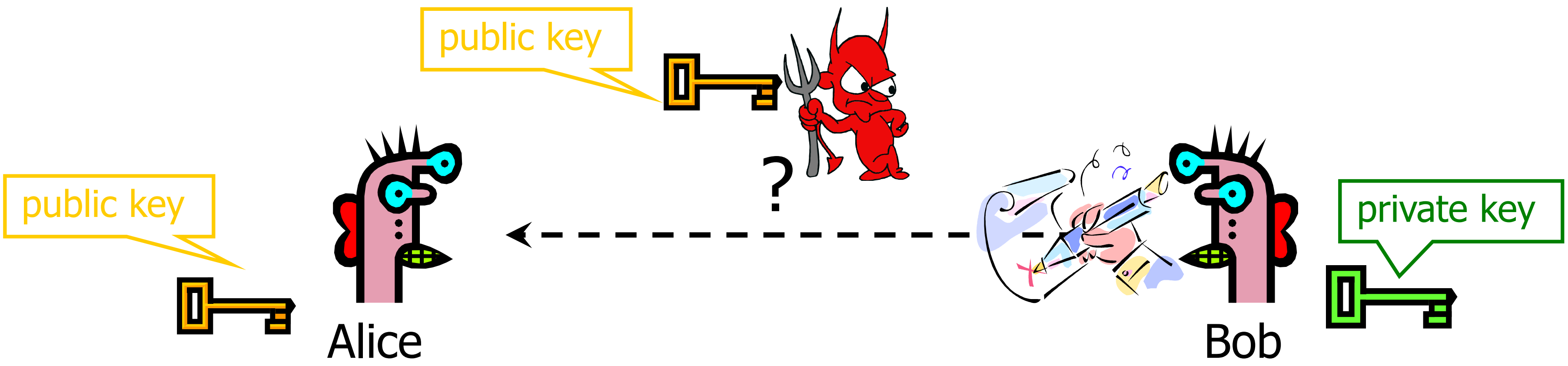
Assuming DDH problem is hard, Diffie-Hellman protocol is a secure key establishment protocol against <u>passive</u> attackers

- Eavesdropper can’t tell the difference between the established key and a random value
- Can use the new key for symmetric cryptography

Basic Diffie-Hellman protocol does not provide authentication

- IPsec combines Diffie-Hellman with signatures, anti-DoS cookies, etc.

Digital Signatures: Basic Idea



<u>Given</u>: Everybody knows Bob's public key
Only Bob knows the corresponding private key

<u>Goal</u>: Bob sends a "digitally signed" message
1. To compute a signature, must know the private key
2. To verify a signature, only need the public key

RSA Signatures

Public key is (n,e), private key is d

To sign message m: $s = hash(m)^d \bmod n$

- Signing and decryption are the same mathematical operation in RSA

To verify signature s on message m:

$s^e \bmod n = (hash(m)^d)^e \bmod n = hash(m)$

- Verification and encryption are the same mathematical operation in RSA

Message must be hashed and padded (why?)

Digital Signature Algorithm (DSA)

U.S. government standard (1991-94)

- Modification of the ElGamal signature scheme (1985)

Key generation:

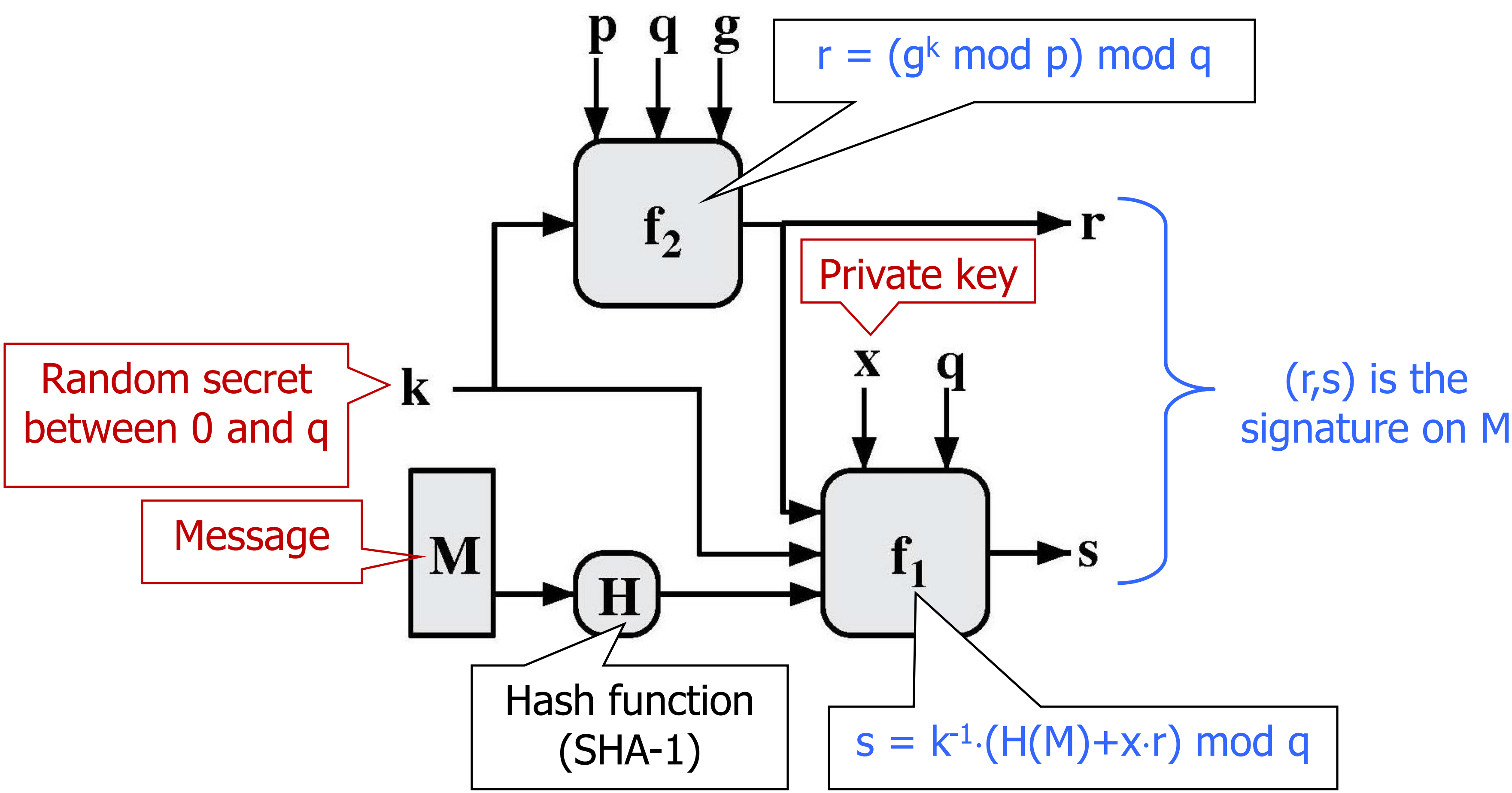
- Generate large primes p, q such that q divides p-1
  - $2^{159} < q < 2^{160}$, $2^{511+64t} < p < 2^{512+64t}$ where $0 \leq t \leq 8$
- Select $h \in Z_p{}^*$ and compute $g = h^{(p-1)/q} \bmod p$
- Select random x such $1 \leq x \leq q-1$, compute $y = g^x \bmod p$

Public key: $(p, q, g, g^x \bmod p)$, private key: x

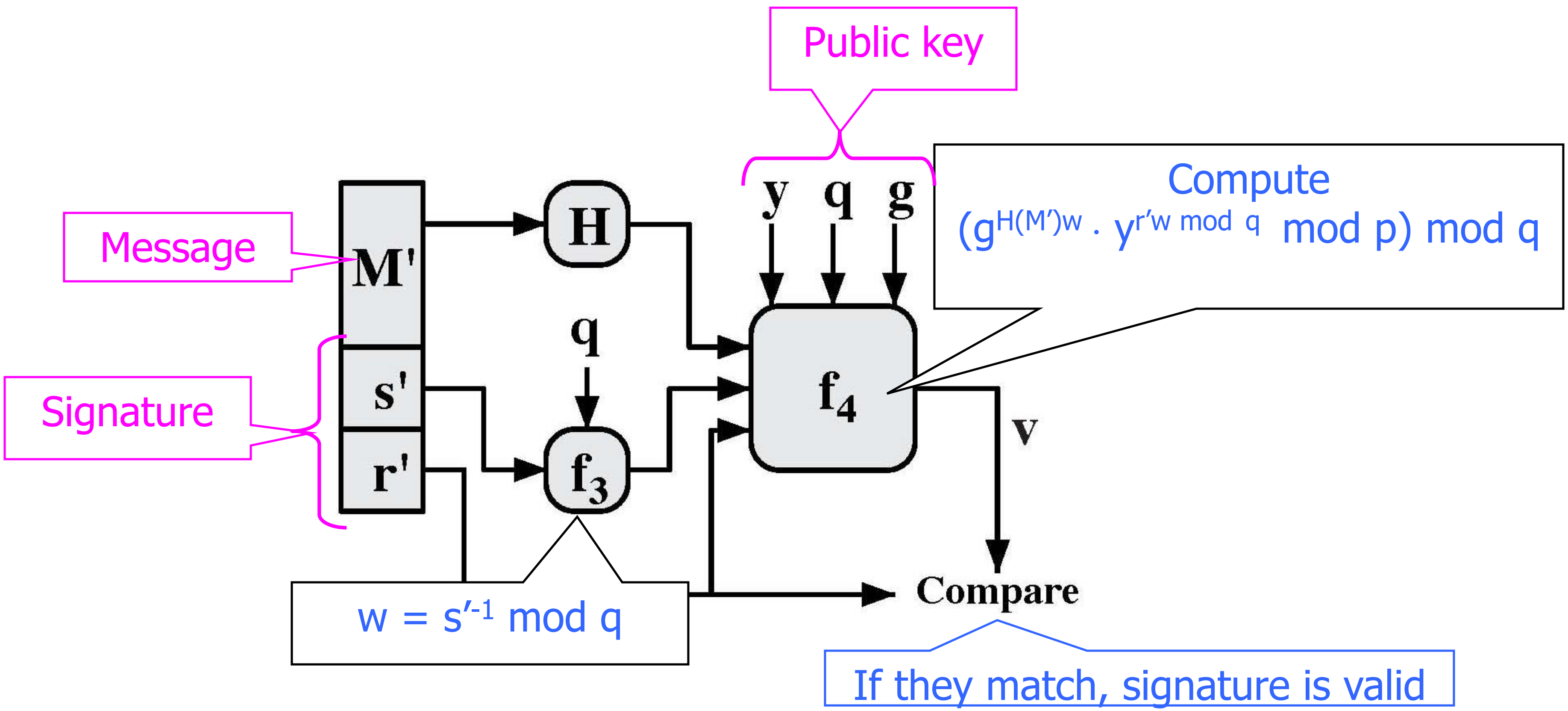
Security of DSA requires hardness of discrete log

- If one can take discrete logarithms, then can extract x (private key) from $g^x \bmod p$ (public key)

DSA: Signing a Message



DSA: Verifying a Signature



Why DSA Verification Works

If (r,s) is a valid signature, then

$r \equiv (g^k \bmod p) \bmod q$; $s \equiv k^{-1} \cdot (H(M)+x \cdot r) \bmod q$

Thus $H(M) \equiv -x \cdot r + k \cdot s \bmod q$

Multiply both sides by $w = s^{-1} \bmod q$

$H(M) \cdot w + x \cdot r \cdot w \equiv k \bmod q$

Exponentiate g to both sides

$(g^{H(M) \cdot w + x \cdot r \cdot w} \equiv g^k) \bmod p \bmod q$

In a valid signature, $g^k \bmod p \bmod q = r$, $g^x \bmod p = y$

Verify $g^{H(M) \cdot w} \cdot y^{r \cdot w} \equiv r \bmod p \bmod q$

Security of DSA

Can’t create a valid signature without private key

Can’t change or tamper with signed message

If the same message is signed twice, signatures are different

- Each signature is based in part on random secret k

Secret k must be different for each signature!

- If k is leaked or if two messages re-use the same k, attacker can recover secret key x and forge any signature from then on

PS3 Epic Fail

Sony uses ECDSA (DSA on elliptic curves) to sign authorized software for Playstation 3

- … with the same random value in every signature

Trivial to extract master signing key and sign any homebrew software – perfect "jailbreak" for PS3

Announced by George "Geohot" Hotz and Fail0verflow team in Dec 2010

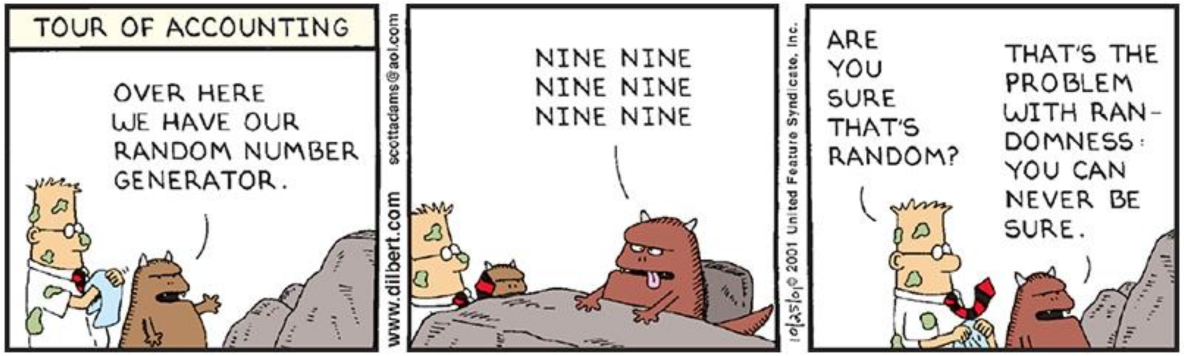
Q: Why didn't Sony just revoke the key?

0XFFFFFFFF EVERY TIME IS 0XDEADBEEF —

How a months-old AMD microcode bug destroyed my weekend [UPDATED]

AMD shipped Ryzen 3000 with a serious microcode bug in its random number generator.

JIM SALTER - 10/29/2019, 7:00 AM



Disadvantages of Public-Key Crypto

Calculations are 2-3 orders of magnitude slower

- Modular exponentiation is an expensive computation
- Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
  - SSL, IPsec, most other systems based on public crypto

Keys are longer

- 2048 bits (RSA) rather than 128 bits (AES)

Relies on unproven number-theoretic assumptions

- Factoring, RSA problem, discrete logarithm problem, decisional Diffie-Hellman problem…