Exercises 5

Exercise 1 (Quantifier Elimination in PA)**.** Apply quantifier elimination as seen in the Lectures to the following formulas:

- $\exists x, y.\, 2x + 3y < 7 \land x < y$
- $\exists x, y.\, 2x + 3y < 7 \land y < x$
- $\exists x, y.\, 3x + 3y < 8 \land 8 < 3x + 2y$
- $\exists x, y.\, x = 2y \land \exists z. x = 3z$

Exercise 2 (Satisfiability algorithm for Presburger arithmetic)**.** Consider the formula $F(x)$ given by

$$F(x) = \bigwedge_{i=1} a_i < x \land \bigwedge_{j=1} x < b_j \land \bigwedge_{i=1} K_i | (x + t_i).$$

Recall that the terms a_i, b_j, t_i may in general contain other variables than x.

1. Assume all a_i, b_j, t_i are integer constants. Give an algorithm that, given any formula of the form above, returns:
   - a value for x, if such value exists, and
   - "UNSAT" if no such value exists
2. Give a recursive algorithm that, given a formula in the above form returns
   - one map from variables to integers for which formula evaluates to true, if such a map exists, and
   - "UNSAT" if no such map exists.

Exercise 3 (Quantifier elimination for rationals)**.** In this exercise we will devise a quantifier elimination method for rational numbers. We consider formulas over the signature $(\mathbb{Q}, <, \leq, =, +, -)$, i.e. with constant symbols among $\mathbb{Q}$, interpreted over the standard structure of rational numbers.

1. Show that for any formula F, there exists a formula F_1 such that

$$F \iff Q_1 x_1, \ldots, Q_n x_n, Q_{n+1} y. F_1$$

Where Q_i are either $\exists$ or $\neg\exists$, i.e. existential quantifiers that can be separated by negations and where F_1 is built only from $(\wedge, \vee, \mathbb{Q}, <, =, +, -, k \cdot \_)$. In particular it is quantifier-free and contains no negation!

2. Do we need to add the divisibility relation as in the PA case? Why?
3. Show that there exist a formula F_2 such that $F_1 \iff F_2$ and every atom of F_2 is of the form:
$$t < y$$
or
$$y < t$$
or
$$t = y$$
for some term t
4. Show that there exists a formula F_3 that is quantifier-free such that
$$(\exists y. F_2) \iff F_3$$

Exercise 4 (PA without divisibility)**.** Show that Presburger Arithmetic without the divisibility relationship does not admit quantifier elimination with the following steps:

1. Find a quantified formula of one free variable $F(y)$ such that $F(y)$ is true for infinitely many positive integers and false for infinitely many positive integers. I.e., $S_F = \{n \in \mathbb{N} | F(n)\}$ is infinite and $\mathbb{N} \setminus S_F$ is infinite.
2. Show that for any quantifier-free formula of one free variable $G(y)$, either S_G is finite or $\mathbb{N} \setminus S_G$ is finite.
3. Conclude.

Exercise 5 (Structure of sets)**.** Consider the structure $(\mathcal{P}(\mathbb{N}), \subseteq, = \cap, \cup, \_^c)$ whose base set is the set of all sets of natural numbers and where $\_^c$ denotes complement. Is it possible to eliminate quantifiers from arbitrary first order formulas on this structure? For example, $\exists B. A \subseteq B \wedge B \subseteq C$ is equivalent to $A \subseteq C$. Show a quantifier elimination procedure, or give an example of a quantified first-order logic formula that has no equivalent formula without quantifiers, and prove it.

Exercise 6 (A rational arithmetic formula)**.** Consider the following formula $G(x, z)$ where the variables range over rational numbers $\mathbb{Q}$:

$$\forall y.((x < y \wedge y < z) \longrightarrow \forall u.(x \neq u + u + u))$$

Find a quantifier-free formula equivalent to $G(x, z)$.