Sequent Calculus and LISA

December 2, 2021

Formal Proofs

Writing proofs of mathematics Theorems:

Student Proof	Mathematician Proof
Not Formal	Formal

Formal Proofs

Writing proofs of mathematics Theorems:

Student Proof	Mathematician Proof	Proof Assistant Proof
Not Formal	~~Formal~~ Not Formal	Formal

Hilbert Systems

Hilbert Systems are an axiom-oriented formalisation of Logic, either propositional or first order. There is a single deduction rule, modus ponens:

$$\frac{\phi \quad \phi \to \psi}{\psi}$$

And everything else is expressed through axioms:

$$\phi \to \phi$$

$$\phi \to (\psi \to \psi)$$

$$(\phi \to (\psi \to \xi)) \to ((\phi \to \psi) \to (\phi \to \xi))$$

$$(\neg\phi \to \neg\psi) \to (\psi \to \phi)$$

$$\forall x\,(\phi) \to \phi[x := t]$$

$$\forall x\,(\phi \to \psi) \to (\forall x\,(\phi) \to \forall x\,(\psi))$$

$$\phi \to \forall x\,(\phi)$$

+ equality axioms...

Alternative System

Those systems are simple, but impractical. Alternative: No (logical) axioms, many deduction rules.
$\implies$ Natural Deduction, Sequent Calculus
Those systems are much more extendable and adapted to the way human do mathematics

Sequent Calculus

A sequent is formally a pair of sets of formulas, represented:

$$\Gamma \vdash \Pi$$

Deduction rules have 0, 1 or 2 premises and one conclusion. For example:

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Sigma \vdash \psi, \Pi}{\Gamma, \Sigma \vdash \phi \wedge \psi, \Delta, \Pi}\ \texttt{RightAnd}$$

Where ϕ, ψ represent arbitrary formulas and $\Gamma, \Sigma, \Delta, \Psi$ sets of formulas. When working with sets of formulas notation Γ, Σ means $\Gamma \cup \Sigma$ whereas, e.g., $\phi \wedge \psi, \Delta, \Pi$ means $\{\phi \wedge \psi\} \cup \Delta \cup \Pi$.

Sequent Calculus

Informally, sequents represent statements about what should be provable from what assumptions, so $\phi_1, \phi_2 \vdash \psi_1$ should mean that formula $\phi_1 \wedge \phi_2 \rightarrow \psi_1$ is valid.
In general, the semantics of a sequent $\phi_1 ..., \phi_n \vdash \psi_1, ..., \psi_m$ for $m, n \geq 0$ is that the following formula is valid:

$$(true \wedge \phi_1 \wedge ... \wedge \phi_n) \rightarrow (false \vee \psi_1 \vee ... \vee \psi_m)$$

i.e. the left hand sides are assumptions and the right hand side are (possible) conclusions.
The conjunct true and disjunct false clarify the meaning when there is nothing on one or both sides. For example, the empty sequent ($\vdash$) is equivalent to false.

Deduction Rules

$$\frac{}{\phi \vdash \phi}\ \texttt{Hypothesis}$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Sigma, \phi \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \texttt{Cut}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}\ \texttt{LeftAnd}$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Sigma \vdash \psi, \Pi}{\Gamma, \Sigma \vdash \phi \wedge \psi, \Delta, \Pi}\ \texttt{RightAnd}$$

$$\frac{\Gamma, \phi \vdash \Delta \qquad \Sigma, \psi \vdash \Pi}{\Gamma, \Sigma, \phi \vee \psi \vdash \Delta, \Pi}\ \texttt{LeftOr}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta}\ \texttt{RightOr}$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Sigma, \psi \vdash \Pi}{\Gamma, \Sigma, \phi \rightarrow \psi \vdash \Delta, \Pi}\ \texttt{LeftImplies}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Sigma \vdash \phi \rightarrow \psi, \Delta}\ \texttt{RightImplies}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}\ \texttt{LeftNot}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\ \texttt{RightNot}$$

$$\frac{\Gamma, \phi[t/x] \vdash \Delta}{\Gamma, \forall x.\phi \vdash \Delta}\ \texttt{LeftForall}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \forall x.\phi, \Delta}\ \texttt{RightForall*}$$

$$\frac{\Gamma, \phi \vdash \Delta}{\Gamma, \exists x.\phi \vdash \Delta}\ \texttt{LeftExists*}$$

$$\frac{\Gamma \vdash \phi[t/x], \Delta}{\Gamma \vdash \exists x.\phi, \Delta}\ \texttt{RightExists}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta}\ \texttt{LeftWeakening}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Pi, \Delta}\ \texttt{RightWeakening}$$

Example: Pierce's Law

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\phi \vdash \phi}\;\texttt{Hypothesis}
    }{\phi \vdash \phi, \psi}\;\texttt{RightWeakening}
  }{\vdash \phi, (\phi \to \psi)}\;\texttt{RightImplies}
  \qquad
  \dfrac{}{\phi \vdash \phi}\;\texttt{Hypothesis}
}{
  \dfrac{(\phi \to \psi) \to \phi \vdash \phi}{\vdash ((\phi \to \psi) \to \phi) \to \phi}\;\texttt{RightImplies}
}\;\texttt{LeftImplies}
$$

Example

$$\forall x.P(x) \to P(f(x)) \vdash \forall x.P(x) \to P(f(f(x)))$$

Example

$$\dfrac{\dfrac{P(x), (\forall x. P(x) \to P(f(x))) \vdash P(f(f(x)))}{\forall x. P(x) \to P(f(x)) \vdash P(x) \to P(f(f(x)))}\ \texttt{RightImplies}}{\forall x. P(x) \to P(f(x)) \vdash \forall x. P(x) \to P(f(f(x)))}\ \texttt{RightForall}$$

Example

$$
\begin{array}{cl}
P(x), P(x) \to P(f(x)), P(f(x)) \to P(f(f(x))) \vdash P(f(f(x))) & \\ \hline
P(x), (\forall x.P(x) \to P(f(x))), P(f(x)) \to P(f(f(x))) \vdash P(f(f(x))) & \texttt{LeftForall} \\ \hline
P(x), (\forall x.P(x) \to P(f(x))), (\forall x.P(x) \to P(f(x))) \vdash P(f(f(x))) & \texttt{LeftForall} \\ \hline
P(x), (\forall x.P(x) \to P(f(x))) \vdash P(f(f(x))) & \\ \hline
\forall x.P(x) \to P(f(x)) \vdash P(x) \to P(f(f(x))) & \texttt{RightImplies} \\ \hline
\forall x.P(x) \to P(f(x)) \vdash \forall x.P(x) \to P(f(f(x))) & \texttt{RightForall}
\end{array}
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \texttt{Hypothesis}}{P(x), P(f(x)) \to P(f(f(x))) \vdash P(x), P(f(f(x)))}\ \texttt{Weakening} \qquad \ldots \vdash \ldots}{P(x), P(x) \to P(f(x)), P(f(x)) \to P(f(f(x))) \vdash P(f(f(x)))}\ \texttt{LeftImplies}}{P(x), (\forall x.P(x) \to P(f(x))), P(f(x)) \to P(f(f(x))) \vdash P(f(f(x)))}\ \texttt{LeftForall}}{P(x), (\forall x.P(x) \to P(f(x))), (\forall x.P(x) \to P(f(x))) \vdash P(f(f(x)))}\ \texttt{LeftForall}}{P(x), (\forall x.P(x) \to P(f(x))) \vdash P(f(f(x)))}}{\forall x.P(x) \to P(f(x)) \vdash P(x) \to P(f(f(x)))}\ \texttt{RightImplies}}{\forall x.P(x) \to P(f(x)) \vdash \forall x.P(x) \to P(f(f(x)))}\ \texttt{RightForall}
$$

Cut Elimination

Gentzen's Cut Elimination theorem is the main property of Sequent Calculus.

- Every provable sequent is provable without the `Cut` rule.
- Doesn't seem so much, but it's a very important theorem of Logic and Proof Theory

Why is this usefull: For proof analysis. Except for cut, no rule can remove a bit of a formula, only add (subformula property).

Proofs

Sequent Calculus is particularly appreciated for proof analysis and proof search: Except for the cut rule, all rules are responsible for the introduction of a single symbol on the left or right. Moreover, no form

To prove a statement, start from bottom and decompose formulas in the "only possible way". If there is a non-trivial leaf left, the proof fails. Otherwise it succeeds.

Note: For quantifiers, we still have to make choice.

Cut Elimination Application: Consistency

Definition

A Theory is **inconsistent** if it proves every statement. Otherwise, the theory is said to be consistent.

From the empty sequent, you can prove everything, using the weakening rule. Cut elimination directly implies that the empty sequent is not provable:
You have to start from something (`hypothesis`) and you can't remove formulas. Hence no proof concludes with the empty sequent.

Interlude: Intuitionistic Logic

Intuitionistic Logic is a weakening of Classical Logic (either propositional or first order) where the Law of Excluded middle

$$\phi \vee \neg\phi$$

or equivalently Double Negation Elimination, don't hold.

- Non-constructive: In classical logic, if you prove $\phi \vee \psi$, you can't in general produce a proof of either ϕ or ψ.
- In Intuitionistic Logic, your guaranteed that if you prove $\phi \vee \psi$, either ϕ or ψ indeed have a proof.
- It is possible to develop many branches of mathematics without using those laws.

Cut Elimination Application: Intuitionistic Logic

There are no rule in sequent calculus assert directly double negation elimination or excluded middle. Instead, a theorem is intuitionistic if and only if it has a proof with at most one formula on the right of a sequent.

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\ \texttt{Hypothesis}}{\vdash \neg\phi, \phi}\ \texttt{RightNeg}}{\neg\neg\phi \vdash \phi}\ \texttt{LeftNeg}$$

Sequent Calculus In Practice

Main

```
0 0 .R. Axiom      |- ∀∀(∀((z$1∈x$3)⇔(z$1∈y$2))⇔(x$2=y$1))
1 .Hypo.        ∀(∀((z$1∈{x,y})⇔(z$1∈y$2))⇔({x,y}=y$1)) |- ∀(∀((z$1∈{x,y})⇔(z$1∈y$2))⇔({x,y}=y$1))
2 .Left ∀ 1     ∀∀(∀((z$1∈x$3)⇔(z$1∈y$2))⇔(x$2=y$1)) |- ∀(∀((z$1∈{x,y})⇔(z$1∈y$2))⇔({x,y}=y$1))
3 .Cut 0,2     |- ∀(∀((z$1∈{x,y})⇔(z$1∈y$2))⇔({x,y}=y$1))
4 .Hypo.        (∀((z$1∈{x,y})⇔(z$1∈{y,x}))⇔({x,y}={y,x})) |- (∀((z$1∈{x,y})⇔(z$1∈{y,x}))⇔({x,y}={y,x}))
5 .Left ∀ 4     ∀(∀((z$1∈{x,y})⇔(z$1∈y$2))⇔({x,y}=y$1)) |- (∀((z$1∈{x,y})⇔(z$1∈{y,x}))⇔({x,y}={y,x}))
6 .Cut 3,5     |- (∀((z$1∈{x,y})⇔(z$1∈{y,x}))⇔({x,y}={y,x}))
1 0 .0 ..R. Axiom      |- ∀∀∀((z$1∈{x$3,y$2})⇔((x$3=z$1)∨(y$2=z$1)))
1 ..Hypo.        ∀∀((z$1∈{x,y$2})⇔((x=z$1)∨(y$2=z$1))) |- ∀∀((z$1∈{x,y$2})⇔((x=z$1)∨(y$2=z$1)))
2 ..Left ∀ 1     ∀∀∀((z$1∈{x$3,y$2})⇔((x$3=z$1)∨(y$2=z$1))) |- ∀∀((z$1∈{x,y$2})⇔((x=z$1)∨(y$2=z$1)))
3 ..Cut 0,2     |- ∀∀((z$1∈{x,y$2})⇔((x=z$1)∨(y$2=z$1)))
4 ..Hypo.        ∀((z$1∈{x,y})⇔((x=z$1)∨(y=z$1))) |- ∀((z$1∈{x,y})⇔((x=z$1)∨(y=z$1)))
5 ..Left ∀ 4     ∀∀((z$1∈{x,y$2})⇔((x=z$1)∨(y$2=z$1))) |- ∀((z$1∈{x,y})⇔((x=z$1)∨(y=z$1)))
6 ..Cut 3,5     |- ∀((z$1∈{x,y})⇔((x=z$1)∨(y=z$1)))
7 ..Hypo.        ((z∈{x,y})⇔((x=z)∨(y=z))) |- ((z∈{x,y})⇔((x=z)∨(y=z)))
8 ..Left ∀ 7     ∀((z$1∈{x,y})⇔((x=z$1)∨(y=z$1))) |- ((z∈{x,y})⇔((x=z)∨(y=z)))
9 ..Cut 6,8     |- ((z∈{x,y})⇔((x=z)∨(y=z)))
1 .0 ..R. Axiom      |- ∀∀∀((z$1∈{x$3,y$2})⇔((x$3=z$1)∨(y$2=z$1)))
1 ..Hypo.        ∀∀((z$1∈{y,y$2})⇔((y=z$1)∨(y$2=z$1))) |- ∀∀((z$1∈{y,y$2})⇔((y=z$1)∨(y$2=z$1)))
2 ..Left ∀ 1     ∀∀∀((z$1∈{x$3,y$2})⇔((x$3=z$1)∨(y$2=z$1))) |- ∀∀((z$1∈{y,y$2})⇔((y=z$1)∨(y$2=z$1)))
3 ..Cut 0,2     |- ∀∀((z$1∈{y,y$2})⇔((y=z$1)∨(y$2=z$1)))
4 ..Hypo.        ∀((z$1∈{y,x})⇔((y=z$1)∨(x=z$1))) |- ∀((z$1∈{y,x})⇔((y=z$1)∨(x=z$1)))
5 ..Left ∀ 4     ∀∀((z$1∈{y,y$2})⇔((y=z$1)∨(y$2=z$1))) |- ∀((z$1∈{y,x})⇔((y=z$1)∨(x=z$1)))
6 ..Cut 3,5     |- ∀((z$1∈{y,x})⇔((y=z$1)∨(x=z$1)))
7 ..Hypo.        ((z∈{y,x})⇔((y=z)∨(x=z))) |- ((z∈{y,x})⇔((y=z)∨(x=z)))
8 ..Left ∀ 7     ∀((z$1∈{y,x})⇔((y=z$1)∨(x=z$1))) |- ((z∈{y,x})⇔((y=z)∨(x=z)))
9 ..Cut 6,8     |- ((z∈{y,x})⇔((y=z)∨(x=z)))
10 ..R. SubstIff 9 (((y=z)∨(x=z))⇔((x=z)∨(y=z))) |- ((z∈{y,x})⇔((x=z)∨(y=z)))
2 .R. SubstIff 0 ((z∈{y,x})⇔((x=z)∨(y=z))) |- ((z∈{x,y})⇔(z∈{y,x}))
3 .Cut 1,2    (((y=z)∨(x=z))⇔((x=z)∨(y=z))) |- ((z∈{x,y})⇔(z∈{y,x}))
4 .L. Axiom 3    |- ((z∈{x,y})⇔(z∈{y,x}))
5 .Right ∀ 4      |- ∀((z$1∈{x,y})⇔(z$1∈{y,x}))
2 Hypo.        ∀((z$1∈{x,y})⇔(z$1∈{y,x})) |- ∀((z$1∈{x,y})⇔(z$1∈{y,x}))
3 Hypo.        ({x,y}={y,x}) |- ({x,y}={y,x})
4 Left → 2,3    (∀((z$1∈{x,y})⇔(z$1∈{y,x}))⇒({x,y}={y,x})), ∀((z$1∈{x,y})⇔(z$1∈{y,x})) |- ({x,y}={y,x})
```