Exploring Reachable States

Transition system: $M = (S, I, r, A)$, where $I \subseteq S$, $r \subseteq S \times A \times S$

$\bar{r} = \{(s, s') \mid \exists a \in A. (s, a, s') \in r\}$

Similarly to $post(X)$ previously, define:

$$G : 2^S \to 2^S$$
$$G(X) = I \cup \bar{r}[X]$$

What properties does function G have?

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G is monotonic!
Indeed. Say $S \subseteq S'$. Then $\bar{r}[S] \subseteq \bar{r}[S']$, so $I \cup \bar{r}[S] \subseteq I \cup \bar{r}[S']$.
That is, $G(S) \subseteq G(S')$.

Define $G^0(X) = X$, $G^{n+1}(X) = G(G^n(X))$.

What $G^n(\emptyset)$ is

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$\ldots$

$$G^n(\emptyset) = \bigcup_{k=0}^{n-1} \bar{r}^k[I]$$

All states reachable in less than n steps!

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All states reachable in less than n steps!
Thus,

$$Reach(M) = \bigcup_{k \geq 0} \overline{r}^k[I] = \bigcup_{n \geq 0} G^n(\emptyset)$$

Sequence of States

Consider the infinite sequence $G^i(\emptyset)$ for all i:

$$\emptyset, G(\emptyset), G(G(\emptyset)), \ldots, G^n(\emptyset), \ldots,$$

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Suppose that S is **finite**. What must happen? $G(G^n(\emptyset)) = G^n(\emptyset)$

- we cannot get strictly bigger for infinitely many steps
- once $G^{n+1}(\emptyset) = G^n(\emptyset)$ we get $G^{n+k}(\emptyset) = G^n(\emptyset)$, for all $k \geq 0$ (sequence becomes constant)

A Reachability Procedure

def findXstar(S,I,r,A) =
  def G(X) = $I \cup \bar{r}[X]$
  var X = ∅; **var** GX = G(X)
  while GX != X **do**
    X = GX; GX = G(X)
  end while
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What do we know about any set X^* where for some n, $X^* = G^n(\emptyset)$ and $G(X^*) = X^*$?

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What do we know about any set X^* where for some n, $X^* = G^n(\emptyset)$ and $G(X^*) = X^*$?

$$Reach(M) = \bigcup_{i \geq 0} G^i(\emptyset) = \underbrace{G^0(\emptyset) \cup \ldots G^{n-1}(\emptyset)}_{\subseteq X*} \cup \underbrace{G^n(\emptyset)}_{=X*} \cup \underbrace{G^{n+1}(\emptyset) \cup \ldots}_{=X*} = X^*$$

Stops in step that exceeds the length of the longest trace of non-repeating states.
(Need not terminate for infinite systems.)

Fixed Point of a Function

Given a function on some set, $f : S \longrightarrow S$, a value $x \in S$ is a fixed point iff $f(x) = x$.

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$$I \cup \bar{r}[X] \subseteq X$$

Equivalent to:

1. $I \subseteq X$
2. $\bar{r}[X] \subseteq X$, that is, if $s \in X$ and $(s, s') \in \bar{r}$, then $s' \in X$

Inductive invariants are precisely sets X for which $G(X) \subseteq X$ (called postfix points)

How to implement X and $G(X)$?

def findXstar(S,I,r,A) $=$
  def G(X) $= I \cup \bar{r}[X]$
  var X $= \emptyset$; **var** GX $=$ G(X)
  while GX != X **do**
    X $=$ GX; GX $=$ G(X)
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We can use:

- **explicit-state model checking**: sets of states, e.g. hash tables

  val GX $=$
    for (s $\leftarrow$ X,
        s' $\leftarrow \bar{r}[\{s\}]$)
    yield s'

- **symbolic model checking**: formulas and their normal forms, such as BDDs (binary decision diagrams)

Symbolic Algorithm

Instead of a set $X_1 \subseteq S$, we use a formula X that is true precisely for states in X_1

var X = False; **var** GX = G(X)
while SAT($GX \wedge \neg X$) **do**
  X = GX; GX = G(X)
end while
X

- Empty set is formula False.
- Checking $GX \neq X$ can be replaced by checking $\neg(GX \subseteq X)$
  - the other inclusion always holds
  - If GX and X are formulas with free variables, $\neg(GX \subseteq X)$ becomes satisfiability of $GX \wedge \neg X$
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- Why not check GX != X? Without further assumptions, there are infinitely many equivalent formulas, loop would not need to terminate even in a finite-state system
- It remains to implement $G(X)$ on formulas.

Symbolic G

$$
\begin{array}{rcl}
GX_1 & = & I \quad \cup \quad \bar{r}[X_1] \\
s \in GX_1 & \iff & s \in I \quad \vee \quad s \in r[X_1] \\
s \in GX_1 & \iff & s \in I \quad \vee \quad \exists s_0.\ s_0 \in X_1 \wedge \exists a.(s_0, a, s) \in r
\end{array}
$$

We use formulas X, GX over Boolean variables $\bar{s}$. Relation R has variables $\bar{s}, \bar{a}, \bar{s}'$, formula *Init* stands for the set I. Let R' denote $\exists \bar{a}.R[\bar{s} := \bar{s}_0, \bar{s}' := \bar{s}]$
We define:

$$G(X) = Init \ \vee \ \exists \bar{s}_0.(X[\bar{s} := \bar{s}_0] \wedge R')$$

If we want to keep formulas to be quantifier-free, we need to eliminate $\exists \bar{s}_0$ at every step (exponential blowup, substitute all truth values):

$eliminate((x_1; \bar{x}), F) = eliminate(\bar{x}, F[x_1 := 0] \vee F[x_1 := 1])$
$eliminate((), F) = F$

Symbolic Algorithm with Non-Normalized Formulas

def G(X) = $\mathit{Init} \lor \mathit{eliminate}(\bar{s}_0,\ X[\bar{s} := \bar{s}_0] \land R')$
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To keep formulas smaller, we can try to simplify and normalize formulas at every step.

Disjunctive normal form → similar to explicit-state model checking.

A popular alternative normal form: BDDs (binary decision diagrams)