Exercises X

Exercise 1 (Galois Connection)**.** Remember that a Galois connection is defined by two monotonic functions $\alpha : C \to A$ and $\gamma : A \to C$ between partial orders $\leq$ on C and $\sqsubseteq$ on A, such that

$$\forall a, c. \quad \alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \qquad (*)$$

a) Show that the condition $(*)$ is equivalent to the conjunction of these two conditions:

$$\begin{array}{lrcl} \forall c. & c & \leq & \gamma(\alpha(c)) \\ \forall a. & \alpha(\gamma(a)) & \sqsubseteq & a \end{array}$$

b) Let α and γ satisfy the condition of a Galois connection. Show that the following three conditions are equivalent:

1. $\alpha(\gamma(a)) = a$ for all a
2. α is a surjective function
3. γ is an injective function

c) State the condition for $c = \gamma(\alpha(c))$ to hold for all c. When C is the set of sets of concrete states and A is a domain of static analysis, is it more reasonable to expect that $c = \gamma(\alpha(c))$ or $\alpha(\gamma(a)) = a$ to be satisfied, and why?

Exercise 2 (lub and glb)**.** Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound.
Prove that then every set $S \subseteq A$ has the least upper bound, or show a counterexample.
What about the lattice with three elements $\{0, 1_a, 1_b\}$ *the relations* $0 \leq 1_a$ *and* $0 \leq 1_b$*?* **Solution:** *Suppose we have some arbitrary set* $S \subseteq A$*, then we know that* $\sqcap S$ *exists.*
Let U *be the set of all its upper bounds, i.e.* $U = \{x | \forall y \in S. y \sqsubseteq x\}$*. Since every subset of* A *has a greatest lower bound, we know that* $a = \sqcap U$ *exists. Now we want to show that* a *is an upper bound on* S *and that it is the least one.*

We want to show that $\forall y \in S. y \sqsubseteq a$*.*
Let L *be the set of lower bounds on* U*, i.e.* $L = \{z | \forall x \in U. z \sqsubseteq x\}$*.*

Clearly, $S \subseteq L$*, since* U *are the upper bounds on* S*.*
Then a *is the greatest element in* L*, and thus* $\forall y \in S . y \sqsubseteq a$ *and so* a *is an upper bound on* S*.*

Now take any upper bound $x \in U$*. Since* $a = \sqcap U$*, we have* $a \sqsubseteq x$ *and so* a *is the least upper bound.* $\Diamond$

Exercise 3 (Lattices)**.** Consider algebraic structures with signature $(\vee, \wedge)$, each of arity 2, and satisfying the following axioms:

$$\begin{array}{c|c} x \vee y = y \vee x & x \wedge y = y \wedge x \\ (x \vee y) \vee z = x \vee (y \vee z) & (x \wedge y) \wedge z = x \wedge (y \wedge z) \\ x \vee x = x & x \wedge x = x \\ x \vee (x \wedge y) = x & x \wedge (x \vee y) = x \end{array}$$

a) Show that for any x and y, $x \wedge y = x$ if and only if $x \vee y = y$.

b) Define $x \leq y$ by $x \wedge y = x$. Show that $\leq$ is a partial order relation.

c) Show that $\vee$ and $\wedge$ are respectively the binary *greatest lower bound* and *least upper bound* for $\leq$.

Exercise 4 (post)**.** let S be any set, $r \subseteq S \times S$ a binary relation and $I \subseteq S$. Define $post : 2^S \to 2^S$ by $post(X) = I \cup r[X]$. Prove that *post* is monotonic. Does *post* admit a least fixed point?

Exercise 5. Partitioning

a) Show that for any set S, $(2^S, \subseteq)$ is a lattice.

b) Consider a set $S = P \times V$. For each set $g \in 2^{P \times V}$, define $\bar{g} : P \to 2^V$ by $\bar{g}(p) = \{v \mid (p, v) \in g\}$. Show that $\bar{g}$ is injective.

c) Consider the set of all functions $P \to 2^V$. Define a lattice on this set that is isomorphic to $(2^{P \times V}, \subseteq)$.

Remember: A lattice isomorphism between two lattices L_1 and L_2 is a bijective function $f : L_1 \to L_2$ such that $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee y) = f(x) \vee f(y)$ and $f(x) \leq f(y)$ holds if and only if $x \leq y$.