

Exercises 6

Exercise 1 (Galois Connection). Remember that a Galois connection is defined by two monotonic functions $\alpha : C \rightarrow A$ and $\gamma : A \rightarrow C$ between partial orders \leq on C and \sqsubseteq on A , such that

$$\forall a, c. \quad \alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \quad (*)$$

- a) Show that the condition $(*)$ is equivalent to the conjunction of these two conditions:

$$\forall c. \quad c \leq \gamma(\alpha(c)) \quad (1)$$

$$\forall a. \quad \alpha(\gamma(a)) \sqsubseteq a \quad (2)$$

- b) Let α and γ satisfy the condition of a Galois connection. Show that the following three conditions are equivalent:

1. $\alpha(\gamma(a)) = a$ for all a
2. α is a surjective function
3. γ is an injective function

- c) State the condition for $c = \gamma(\alpha(c))$ to hold for all c . When C is the set of sets of concrete states and A is a domain of static analysis, is it more reasonable to expect that $c = \gamma(\alpha(c))$ or $\alpha(\gamma(a)) = a$ to be satisfied, and why?

Exercise 2 (lub and glb). Let (A, \sqsubseteq) be a partial order such that every set $S \subseteq A$ has the greatest lower bound.

Prove that then every set $S \subseteq A$ has the least upper bound, or show a counterexample.

What about the lattice with three elements $\{0, 1_a, 1_b\}$ the relations $0 \leq 1_a$ and $0 \leq 1_b$?

Exercise 3 (Lattices). Consider algebraic structures with signature (\vee, \wedge) , each of arity 2, and satisfying the following axioms:

$x \vee y = y \vee x$	$(\vee\text{-Com})$	$x \wedge y = y \wedge x$	$(\wedge\text{-Com})$
$(x \vee y) \vee z = x \vee (y \vee z)$	$(\vee\text{-Assoc})$	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	$(\wedge\text{-Assoc})$
$x \vee x = x$	$(\vee\text{-Idem})$	$x \wedge x = x$	$(\wedge\text{-Idem})$
$x \vee (x \wedge y) = x$	$(\vee\text{-Abs})$	$x \wedge (x \vee y) = x$	$(\wedge\text{-Abs})$

- a) Show that for any x and y , $x \wedge y = x$ if and only if $x \vee y = y$.
- b) Define $x \leq y$ by $x \wedge y = x$. Show that \leq is a partial order relation.
- c) Show that \wedge and \vee are respectively the binary *greatest lower bound* and *least upper bound* for \leq .

Exercise 4 (post). let S be any set, $r \subseteq S \times S$ a binary relation and $I \subseteq S$. Define $post : 2^S \rightarrow 2^S$ by $post(X) = I \cup r[X]$. Prove that $post$ is monotonic. Does $post$ admit a least fixed point?

Exercise 5. Partitioning

- a) Show that for any set S , $(2^S, \subseteq)$ is a lattice.
- b) Consider a set $S = P \times V$. For each set $g \in 2^{P \times V}$, define $\bar{g} : P \rightarrow 2^V$ by $\bar{g}(p) = \{v \mid (p, v) \in g\}$. Show that the bar function $\bar{\cdot}$ defines a bijection between $2^{P \times V}$ and $P \rightarrow 2^V$.
- c) Consider the set of all functions $P \rightarrow 2^V$. Define a lattice on this set that is isomorphic to $(2^{P \times V}, \subseteq)$.