Solutions to Exercises 2

Exercise 1 (Traces)**.**

1. Since there are n variables, which can take two values, there are 2^n possible states.

2. As in class, $\bar{s} = (s_1, \dots s_n)$ and exponent are used to represent the variables in the different states (i.e. $\bar{s}^k = (s_1^k, \dots s_n^k)$).

   - We note
   $$\bar{v}^k = (s_1^k, \dots, s_n^k, a_1^k, \dots, a_m^k, x_1^k, \dots, x_\ell^k)$$
   $$F_{(=k)} = \exists \bar{v}^0, \dots, \bar{v}^{k-1}, \bar{s}^k.\ \mathit{Init}\left[\bar{s} := \bar{s}^0\right] \wedge \left(\bigwedge_{i=0}^{k-1} R_i\right) \wedge \left(\bar{s}^k = \bar{s}\right)$$
   Another solution:
   $$\begin{aligned}&\exists \bar{v}^0, \dots, \bar{v}^{k-1}.\\ &\quad \mathit{Init}\left[\bar{s} := \bar{s}^0\right] \wedge \left(\bigwedge_{i=0}^{k-2} R_i\right) \wedge R\left[(\bar{s}, \bar{a}, \bar{x}, \bar{s}') := (\bar{s}^{k-1}, \bar{a}^{k-1}, \bar{x}^{k-1}, \bar{s})\right]\end{aligned}$$
   - $F_{(\leq k)} = \bigvee_{i=0}^{k} F_{(=i)}$
   - The "intuitive" solution is $F_{(\leq \infty)}$, however this formula is infinite. Using question (1), it can be reduced to $F_{(\text{reachable})} = F_{(\leq 2^n - 1)}$
   - $F_{(=k,\text{Err})} = \exists \bar{v}^0, \dots, \bar{v}^{k-1}, \bar{s}^k.\ \left(\bar{s}^0 = \bar{s}\right) \wedge \left(\bigwedge_{i=0}^{k-1} R_i\right) \wedge E\left[\bar{s} := \bar{s}^k\right]$
   - $F_{(\leq k,\text{Err})} = \bigvee_{i=0}^{k} F_{(=i,\text{Err})}$
   - $F_{(\text{stuck})} = \neg \exists \bar{a}, \bar{x}, \bar{s}'.\ R$

3. - $\neg \exists \bar{s}.\ F_{(\text{reachable})} \wedge E$
   - $\neg \exists \bar{s}.\ F_{(\text{reachable})} \wedge F_{(\text{stuck})}$
   - $\forall \bar{s}.\ F_{(\leq k)}$
   - Note that we are only allowed to quantify on propositional variables. As such, a formula such as
   $$\forall k.\ \exists \bar{s}.\ F_{(=k)} \wedge \neg F_{(\text{stuck})}$$

is not allowed, as k ranges (implicitly) over $\mathbb{N}$.

Consider instead

$$\exists \bar{s}.\ F_{(=2^n)}$$

if such a state ($\bar{s}$) exists, the underlying trace contains $2^n + 1$ states (since it is 2^n steps long). By the pigeon hole principle, one of these states must appear twice. This means that there is a part of the trace which can be repeated infinitely ("pumped"), implying the existence of an infinite trace (see the proof of the pumping lemma for regular languages which relies on the same principle).

Exercise 2 (Tseytin Transformation)**.**

1. Any expression of the type $y_k \leftrightarrow x_i * x_j$ is easily convertible to a CNF.

$$\begin{aligned}
y_k \leftrightarrow x_i \vee x_j &\equiv (y_k \rightarrow x_i \vee x_j) \wedge (x_i \vee x_j \rightarrow y_k) \\
&\equiv (\neg y_k \vee x_i \vee x_j) \wedge (\neg x_i \vee y_k) \wedge (\neg x_j \vee y_k) \\
y_k \leftrightarrow x_i \wedge x_j &\equiv (y_k \rightarrow x_i \wedge x_j) \wedge (x_i \wedge x_j \rightarrow y_k) \\
&\equiv (\neg y_k \vee x_i) \wedge (\neg y_k \vee x_j) \wedge (\neg x_i \vee \neg x_j \vee y_k) \\
y_k \leftrightarrow x_i \rightarrow x_j &\equiv y_k \leftrightarrow \neg x_i \vee x_j \\
&\equiv (\neg y_k \vee \neg x_i \vee x_j) \wedge (x_i \vee y_k) \wedge (\neg x_j \vee y_k) \\
y_k \leftrightarrow \neg x_i &\equiv (y_k \rightarrow \neg x_i) \wedge (\neg x_i \rightarrow y_k) \\
&\equiv (\neg y_k \vee \neg x_i) \wedge (x_i \vee y_k)
\end{aligned}$$

Let's first define the mapping between sub formulas and fresh variables:

$$\begin{aligned}
&y_1 \leftrightarrow \neg b \\
&y_2 \leftrightarrow a \vee y_1 \\
&y_3 \leftrightarrow b \rightarrow c \\
&y_4 \leftrightarrow \neg y_3 \\
&y_5 \leftrightarrow y_2 \wedge y_4 \\
&y_6 \leftrightarrow y_5 \rightarrow a
\end{aligned}$$

The Tseytin transformation of the formula is therefore:

$$\begin{aligned}
&(y_1 \leftrightarrow \neg b) \wedge (y_2 \leftrightarrow a \vee y_1) \wedge (y_3 \leftrightarrow b \rightarrow c) \wedge (y_4 \leftrightarrow \neg y_3) \wedge \\
&(y_5 \leftrightarrow y_2 \wedge y_4) \wedge (y_6 \leftrightarrow y_5 \rightarrow a) \wedge y_6 \\
=&(\neg y_1 \vee \neg b) \wedge (b \vee y_1) \wedge (\neg y_2 \vee a \vee y_1) \wedge (\neg a \vee y_2) \wedge (\neg y_1 \vee y_2) \wedge \\
&(\neg y_3 \vee \neg b \vee c) \wedge (b \vee y_3) \wedge (\neg c \vee y_3) \wedge (\neg y_4 \vee \neg y_3) \wedge (y_3 \vee y_4) \wedge \\
&(\neg y_5 \vee y_2) \wedge (\neg y_5 \vee y_4) \wedge (\neg y_2 \vee \neg y_4 \vee y_5) \wedge (\neg y_6 \vee \neg y_5 \vee a) \wedge \\
&(y_5 \vee y_6) \wedge (\neg a \vee y_6) \wedge (\neg y_6 \vee \neg y_7) \wedge (y_6 \vee y_7) \wedge y_7
\end{aligned}$$

2. We first show that they might not be equivalent by providing a small counterexample. As written above, the Tseytin transform of the formula $\neg x$ is $(y \leftrightarrow \neg x) \wedge y$. The value of $\neg x$ when $x = 0$ is 1, whereas the value of its Tseytin transformation when $x = 0, y = 0$, is 0. Therefore the two formulas are not equivalent.

   Let's now prove that a formula is satisfiable if and only if its Tseytin transformation is satisfiable as well.

   Let φ be a formula and $V = \{x_i \mid x_i \in \mathrm{FV}(\varphi)\}$ its set of variables. One can notice (and prove by induction) that the Tseytin transformation of φ is equivalent to:

$$y_n \wedge \bigwedge_{i=1}^{n} (y_i \leftrightarrow \psi_i)$$

   where $\{\psi_i\}_{0 \leq i \leq n}$ are all the (non-atomic) subformulas of φ; in particular, $\psi_n = \varphi$. Therefore given an assignment α of V, a clause $(y_i \leftrightarrow \psi_i)$ is true if and only if one assigns the truth value of $\psi_i\,[V := \alpha]$ to y_i. Since these clauses can be true whatever is the value of α, we have that the Tseytin transformation of φ is satisfiable if and only if y_n is satisfiable. Since we have a clause $y_n \leftrightarrow \varphi$ (due to the fact that $\psi_n = \varphi$), y_n is satisfiable if and only if φ is satisfiable. Therefore φ and its Tseytin transformation are equisatisfiable.

3. F is a valid formula if and only if $\neg F$ is false for any variable assignment, i.e. it is unsatisfiable. A formula is unsatisfiable if and only if its Tseytin transformation is unsatisfiable as well (by the previous question). Therefore F is a valid formula if and only if the Tseytin transformation of $\neg F$ is unsatisfiable.

Exercise 3 (SAT solving)**.** By taking the CNF found in the above exercise and applying resolution, we obtain the following derivation trees:

$$\dfrac{\neg a \lor y_6 \qquad \dfrac{y_7 \quad \neg y_6 \lor \neg y_7}{\neg y_6}}{\neg a} \qquad \dfrac{b \lor y_3 \qquad \dfrac{\neg y_3 \lor \neg y_4 \qquad \dfrac{y_4 \lor \neg y_5 \qquad \dfrac{y_5 \lor y_6 \qquad \dfrac{y_7 \quad \neg y_6 \lor \neg y_7}{\neg y_6}}{y_5}}{y_4}}{\neg y_3}}{b}$$

$$\dfrac{\dfrac{\neg y_1 \lor \neg b \quad \neg y_2 \lor a \lor y_1}{\neg b \lor a \lor \neg y_2} \qquad \dfrac{\neg y_5 \lor y_2 \qquad \dfrac{y_5 \lor y_6 \qquad \dfrac{y_7 \quad \neg y_6 \lor \neg y_7}{\neg y_6}}{y_5}}{y_2}}{\neg b \lor a} \qquad \dfrac{b \qquad \dfrac{\neg b \lor a \quad \neg a}{\neg b}}{F}$$

Exercise 4 (First-order structures)**.**

1. Let $I = (D, \alpha)$ be an interpretation of $\mathcal{L}$, for which $\alpha(B) = b$ and $\alpha(T) = t$. For notational convenience we will denote $\alpha(L)$ by L_α

$$\begin{aligned}
[\![\mathsf{Min}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(b,b), L_\alpha(b,u) \text{ and } L_\alpha(b,t)\\
[\![\mathsf{Max}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(b,t), L_\alpha(u,t) \text{ and } L_\alpha(t,t)\\
[\![\mathsf{Tot}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(u,u) \text{ (by choosing } x = u \text{ and } y = u)\\
[\![\mathsf{Tra}]\!]_I = 1 \quad &\Longrightarrow (L_\alpha(t,b) \implies L_\alpha(t,u))
\end{aligned}$$

This greatly reduce the number of choices, restricting L_α to be either:

a)

L_α	b	u	t
b	1	1	1
u	1	1	1
t	1	1	1

b)

L_α	b	u	t
b	1	1	1
u	1	1	1
t	0	0	1

c)

L_α	b	u	t
b	1	1	1
u	0	1	1
t	0	1	1

d)

L_α	b	u	t
b	1	1	1
u	0	1	1
t	0	0	1

All the options satisfy reflexivity and transitivity as a consequence of the axioms. However, only *d)* satisfies the anti symmetry property. Therefore only d) is a partial order.

2.

a)

$E_{\alpha'}$	b	u	t
b	1	1	1
u	1	1	1
t	1	1	1

b)

$E_{\alpha'}$	b	u	t
b	1	1	0
u	1	1	0
t	0	0	1

c)

$E_{\alpha'}$	b	u	t
b	1	0	0
u	0	1	1
t	0	1	1

d)

$E_{\alpha'}$	b	u	t
b	1	0	0
u	0	1	0
t	0	0	1

In all the cases, $E_{\alpha'}$ is an equivalence relation.

3. $E_{\alpha'}$ is always an equivalence relation, whatever is the domain.

   Let D' be an arbitrary set and $I' = (D', \alpha')$.

   - **Reflexivity**: $[\![\mathsf{Both}]\!]_{I'} = 1 \implies \forall x \in D'. E_{\alpha'}(x,x) \leftrightarrow L_{\alpha'}(x,x)$ (by choosing $y = x$).

     We also know that, $[\![\mathsf{Tot}]\!]_{I'} = 1 \implies \forall x \in D'.\ L_{\alpha'}(x,x)$

     Therefore $\forall x \in D'.\ E_{\alpha'}(x,x)$

   - **Symmetry**:

     $[\![\mathsf{Both}]\!]_{I'} = 1 \implies \forall x \in D'. \forall y \in D'. E_{\alpha'}(x,y) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x))$

     We also know that for $x, y \in D'$ arbitrary $L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x) \equiv L_{\alpha'}(y,x) \wedge L_{\alpha'}(x,y)$. Therefore we have that for $\forall x \in D'. \forall y \in D'. E_{\alpha'}(x,y) \leftrightarrow E_{\alpha'}(y,x)$ is a tautology.

   - **Transitivity**: $[\![\mathsf{Both}]\!]_{I'} = 1$ implies that for x, y, z arbitrary in D'
     $$E_{\alpha'}(x,y) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x))$$
     and
     $$E_{\alpha'}(y,z) \leftrightarrow (L_{\alpha'}(y,z) \wedge L_{\alpha'}(z,y))$$
     Therefore
     $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,z) \wedge L_{\alpha'}(z,y) \wedge L_{\alpha'}(y,x)$$
     which implies
     $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \rightarrow (L_{\alpha'}(x,z) \wedge L_{\alpha'}(z,x))$$
     and finally
     $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \rightarrow E_{\alpha'}(x,z)$$

4. One of the solution consists in choosing $D_1 = {}^{D'}\!/_{E_{\alpha'}}$, the set of equivalence classes of $E_{\alpha'}$, $\alpha_1 = (B_1, T_1, L_1)$ where $B_1 = [b]_{E_{\alpha'}}$, $T_1 = [t]_{E_{\alpha'}}$ and

$$L_1 := \left\{ \left([x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \right) : L_{\alpha'}(x,y) \text{ where } x, y \in D' \right\}$$

Reflexivity and transitivity are immediate as they have been proven for $L_{\alpha'}$.

Antisymmetry comes from the fact that for arbitrary $[x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \in D_1$

$$\begin{aligned}
& L_1 \left([x]_{E_{\alpha'}} [y]_{E_{\alpha'}} \right) \wedge L_1 \left([x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \right) \\
\implies & L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x) \\
\implies & E_{\alpha'}(x,y) \\
\implies & [x]_{E_{\alpha'}} = [y]_{E_{\alpha'}}
\end{aligned}$$

Therefore L_1 is a partial order.