Solutions to Exercises 1, Part I

Exercise 1. Any logical constant and connector can be simulated with Sheffer strokes. In fact:

$$\begin{aligned}
\neg\varphi \qquad\qquad\qquad & \equiv \varphi \uparrow \varphi \\
1 \equiv \varphi \uparrow \neg\varphi \qquad & \equiv \varphi \uparrow (\varphi \uparrow \varphi) \\
0 \equiv \neg 1 \qquad\qquad & \equiv (\varphi \uparrow (\varphi \uparrow \varphi)) \uparrow (\varphi \uparrow (\varphi \uparrow \varphi)) \\
\varphi_0 \wedge \varphi_1 \equiv \neg(\varphi_0 \uparrow \varphi_1) \;\; & \equiv (\varphi_0 \uparrow \varphi_1) \uparrow (\varphi_0 \uparrow \varphi_1) \\
\varphi_0 \vee \varphi_1 \equiv (\neg\varphi_0 \uparrow \neg\varphi_1) & \equiv ((\varphi_0 \uparrow \varphi_0) \uparrow (\varphi_1 \uparrow \varphi_1)) \\
\varphi_0 \to \varphi_1 \equiv \neg\varphi_0 \vee \varphi_1 \quad & \equiv (((\varphi_0 \uparrow \varphi_0) \uparrow (\varphi_0 \uparrow \varphi_0)) \uparrow (\varphi_1 \uparrow \varphi_1))
\end{aligned}$$

Exercise 2. First we rewrite the formulas using only the allowed symbols:

1. $((a \to b) \to a) \to a \equiv \neg(\neg(\neg a \vee b) \vee a) \vee a$
2. $\neg(a \wedge b) \to (\neg a \vee \neg b) \equiv \neg\neg(a \wedge b) \vee (\neg a \vee \neg b)$
3. $((\neg a \to b) \wedge (a \to b)) \to b \equiv \neg((\neg\neg a \vee b) \wedge (\neg a \vee b)) \vee b$

Then we derive $\neg F \vdash 0$ (we collapse sequences of Simp):

1.

$$\dfrac{\dfrac{\dfrac{\neg(\neg(\neg(\neg a \vee b) \vee a) \vee a) \quad \neg(\neg(\neg(\neg a \vee b) \vee a) \vee a)}{(\neg(\neg(\neg(\neg 0 \vee b) \vee 0) \vee 0)) \vee (\neg(\neg(\neg(\neg 1 \vee b) \vee 1) \vee 1))}\text{CA (on a)}}{0 \vee (\neg(\neg(\neg b \vee 1) \vee 1))}\textsc{Simp}}{\neg(\neg(\neg b \vee 1) \vee 1)}\textsc{Simp}$$

Note that, if we are being pedantic, we cannot simplify further as the simplification rules only cover cases where a literal is on the left.

$$\dfrac{\dfrac{\dfrac{\neg(\neg(\neg b \vee 1) \vee 1) \quad \neg(\neg(\neg b \vee 1) \vee 1)}{(\neg(\neg(\neg 0 \vee 1) \vee 1)) \vee (\neg(\neg(\neg 1 \vee 1) \vee 1))}\text{CA (on b)}}{0 \vee 0}\textsc{Simp}}{0}\textsc{Simp}$$

2.

$$\dfrac{\dfrac{\dfrac{\neg\,(\neg\neg(a \wedge b) \vee (\neg a \vee \neg b)) \quad \neg\,(\neg\neg(a \wedge b) \vee (\neg a \vee \neg b))}{(\neg\,(\neg\neg(0 \wedge b) \vee (\neg 0 \vee \neg b))) \vee (\neg\,(\neg\neg(1 \wedge b) \vee (\neg 1 \vee \neg b)))}\text{CA (on a)}}{0 \vee (\neg\,(\neg\neg b \vee \neg b))}\textsc{Simp}}{\neg\,(\neg\neg b \vee \neg b)}\textsc{Simp}$$

$$\dfrac{\dfrac{\neg\,(\neg\neg b \vee \neg b) \quad \neg\,(\neg\neg b \vee \neg b)}{(\neg\,(\neg\neg 0 \vee \neg 0)) \vee (\neg\,(\neg\neg 1 \vee \neg 1))}\text{CA (on b)}}{0}\textsc{Simp}$$

3.

$$\dfrac{\dfrac{\neg\,(\neg((\neg\neg a \vee b) \wedge (\neg a \vee b)) \vee b) \quad \neg\,(\neg((\neg\neg a \vee b) \wedge (\neg a \vee b)) \vee b)}{(\neg\,(\neg((\neg\neg 0 \vee b) \wedge (\neg 0 \vee b)) \vee b)) \vee (\neg\,(\neg((\neg\neg 1 \vee b) \wedge (\neg 1 \vee b)) \vee b))}\text{CA (on a)}}{\neg\,(\neg(b \wedge 1) \vee b) \vee \neg\,(\neg b \vee b)}\textsc{Simp}$$

$$\dfrac{\dfrac{\neg\,(\neg(b \wedge 1) \vee b) \vee \neg\,(\neg b \vee b) \quad \neg\,(\neg(b \wedge 1) \vee b) \vee \neg\,(\neg b \vee b)}{(\neg\,(\neg(0 \wedge 1) \vee 0) \vee \neg\,(\neg 0 \vee 0)) \vee (\neg\,(\neg(1 \wedge 1) \vee 1) \vee \neg\,(\neg 1 \vee 1))}\text{CA (on b)}}{0}\textsc{Simp}$$

Exercise 3.

1. Let's first express f1 and f2 as propositional formulas. **if** x **then** y **else** z means that when x is true then y is true and when x is not true z is true. In propositional logic this becomes $(x \to y) \wedge (\neg x \to z)$. By repeating the process and reducing them to CNF, we get the following formulas for f1 and f2:

$$\begin{aligned}
\text{f1(a, b, c)} &\equiv ((a \vee b) \to (((b \wedge \neg a) \to (b \wedge c)) \wedge (\neg(b \wedge \neg a) \to (\neg b \wedge c)))) \wedge (\neg(a \vee b) \to c) \\
&\equiv (\neg(a \vee b) \vee ((\neg(b \wedge \neg a) \vee (b \wedge c)) \wedge ((b \wedge \neg a) \vee (\neg b \wedge c)))) \wedge ((a \vee b) \vee c) \\
&\equiv (\neg a \vee \neg b) \wedge c
\end{aligned}$$

$$\begin{aligned}
\text{f2(a, b, c)} &\equiv (c \to (a \to \neg b) \wedge (\neg a \to 1)) \wedge (\neg c \to 0) \\
&\equiv (\neg c \vee (\neg a \vee \neg b) \wedge (a \vee 1)) \wedge (c \vee 0) \\
&\equiv (\neg a \vee \neg b) \wedge c
\end{aligned}$$

which proves that they do always produce the same output. One could also have computed the truth tables of each formula and check that they were equal.

2. Since $a \uparrow b \equiv$ **if** (**if** a **then** b **else false**) **then false else true**, and any formula can be written with Sheffer strokes, any formula can also be written only using **if then else**.

   By noting that $\varphi \equiv$ **if** a **then** $\varphi[a := 1]$ **else** $\varphi[a := 0]$ where $a \in \mathrm{FV}(\varphi)$ and by applying the identity recursively we can express any formula with **if then else** with only one variable in the condition. The proof goes by induction on the number of variables in the formula, and by noting that if φ has n variables and $a \in \mathrm{FV}(\varphi)$, then $\varphi[a := 0]$ and $\varphi[a := 1]$ contain only $n - 1$ variables.

Exercise 4. Remember that for any formulas P, Q, and R, the formula $P \rightarrow Q \rightarrow R$ is parsed as $P \rightarrow (Q \rightarrow R)$ and is equivalent to $(P \wedge Q) \rightarrow R$.

1. If $P \wedge Q$ is true, then so is Q.
2. We consider two cases. If $P \rightarrow Q$ holds, then we are done, otherwise we have $P \wedge \neg Q$, and we are done as well.
3. Counterexample: $P = \top$, $Q = \bot$, $R = \bot$.
4. Counterexample: $P = \top$, $Q = \top$, $R = \bot$.
5. If R and $\neg R$ are true, then we have a contradiction and Q is true as well.
6. A formula always implies itself: $(P \rightarrow Q) \rightarrow (P \rightarrow Q)$.
7. (Peirce's law) We consider two cases. If P is true, then the whole formula is true. Otherwise, $(P \rightarrow Q)$ is true, meaning the implication $((P \rightarrow Q) \rightarrow P)$ is false. This makes the whole formula true as well.
8. Counterexample: $P = \top$, $Q = \bot$.
9. Assume $(\neg Q \rightarrow \neg P)$ and P are true. Assume now (by contradiction) that Q is false. Then, we would have that P is false, which is a contradiction. Therefore Q is true and so is the whole formula.
10. Counterexample: $P = \top$, $Q = \top$, $R = \bot$.
11. Assume the three disjunctions are true. We want to show that P is true as well. Consider the first two disjunctions (the third one is not needed). In either of them, if the left-hand-side (P) is true, then we are done. Otherwise, it means their right-hand-sides Q and $\neg Q$ are both true, which is a contradiction.
12. Counterexample: $P = \top$, $Q = \bot$.
13. Assume $\neg(P \wedge Q)$ and P are both true. If Q is true as well, then, we have $P \wedge Q$ is true, which is a contradiction. Therefore Q is false, which is what we needed to prove.