Exercises 6

Exercise 1 (Galois Connection)**.** Remember that a Galois connection is defined by two monotonic functions $\alpha : C \to A$ and $\gamma : A \to C$ between partial orders $\leq$ on C and $\sqsubseteq$ on A, such that

$$\forall a, c. \quad \alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \qquad (*)$$

a) Show that the condition $(*)$ is equivalent to the conjunction of these two conditions:

$$\begin{array}{lrcl} \forall c. & c & \leq & \gamma(\alpha(c)) \\ \forall a. & \alpha(\gamma(a)) & \sqsubseteq & a \end{array}$$

b) Let α and γ satisfy the condition of a Galois connection. Show that the following three conditions are equivalent:

   1. $\alpha(\gamma(a)) = a$ for all a
   2. α is a surjective function
   3. γ is an injective function

c) State the condition for $c = \gamma(\alpha(c))$ to hold for all c. When C is the set of sets of concrete states and A is a domain of static analysis, is it more reasonable to expect that $c = \gamma(\alpha(c))$ or $\alpha(\gamma(a)) = a$ to be satisfied, and why?

Exercise 2 (lub and glb)**.** Let $(A, \sqsubseteq)$ be a partial order such that every set $S \subseteq A$ has the greatest lower bound.
Prove that then every set $S \subseteq A$ has the least upper bound, or show a counterexample.
What about the lattice with three elements $\{0, 1_a, 1_b\}$ *the relations* $0 \leq 1_a$ *and* $0 \leq 1_b$*?*

Exercise 3 (Lattices)**.** Consider algebraic structures with signature $(\vee, \wedge)$, each of arity 2, and satisfying the following axioms:

$$\begin{array}{c|c} x \vee y = y \vee x & x \wedge y = y \wedge x \\ (x \vee y) \vee z = x \vee (y \vee z) & (x \wedge y) \wedge z = x \wedge (y \wedge z) \\ x \vee x = x & x \wedge x = x \\ x \vee (x \wedge y) = x & x \wedge (x \vee y) = x \end{array}$$

a) Show that for any x and y, $x \wedge y = x$ if and only if $x \vee y = y$.

b) Define $x \leq y$ by $x \wedge y = x$. Show that $\leq$ is a partial order relation.

c) Show that $\wedge$ and $\vee$ are respectively the binary *greatest lower bound* and *least upper bound* for $\leq$.

Exercise 4 (post)**.** let S be any set, $r \subseteq S \times S$ a binary relation and $I \subseteq S$. Define $post : 2^S \to 2^S$ by $post(X) = I \cup r[X]$. Prove that $post$ is monotonic. Does $post$ admit a least fixed point?