Solutions 2

Exercise 1. 1. Since there are n variables, which can take two values, there are 2^n possible states.

2. As in class, $\bar{s} = (s_1, \dots s_n)$ and exponent are used to represent the variables in the different states (i.e. $\bar{s}^k = (s_1^k, \dots s_n^k)$).

   - We note
$$\bar{v}^k = (s_1^k, \dots, s_n^k, a_1^k, \dots, a_m^k, x_1^k, \dots, x_\ell^k)$$
$$F_{(=k)} = \exists \bar{v}^0, \dots, \bar{v}^{k-1}, \bar{s}^k.\ \mathit{Init}\left[\bar{s} := \bar{s}^0\right] \wedge \left(\bigwedge_{i=0}^{k-1} R_i\right) \wedge \left(\bar{s}^k = \bar{s}\right)$$
     Another solution:
$$\exists \bar{v}^0, \dots, \bar{v}^{k-1}.$$
$$\mathit{Init}\left[\bar{s} := \bar{s}^0\right] \wedge \left(\bigwedge_{i=0}^{k-2} R_i\right) \wedge R\left[(\bar{s}, \bar{a}, \bar{x}, \bar{s}') := (\bar{s}^{k-1}, \bar{a}^{k-1}, \bar{x}^{k-1}, \bar{s})\right]$$
   - $F_{(\leq k)} = \bigvee_{i=0}^{k} F_{(=i)}$
   - The "intuitive" solution is $F_{(\leq \infty)}$, however this formula is infinite. Using question (1), it can be reduced to $F_{(\text{reachable})} = F_{(\leq 2^n - 1)}$
   - $F_{(=k,\text{Err})} = \exists \bar{v}^0, \dots, \bar{v}^{k-1}, \bar{s}^k.\ \left(\bar{s}^0 = \bar{s}\right) \wedge \left(\bigwedge_{i=0}^{k-1} R_i\right) \wedge E\left[\bar{s} := \bar{s}^k\right]$
   - $F_{(\leq k,\text{Err})} = \bigvee_{i=0}^{k} F_{(=i,\text{Err})}$
   - $F_{(\text{stuck})} = \neg \exists \bar{a}, \bar{x}, \bar{s}'.\ R$

3. - $\neg \exists \bar{s}.\ F_{(\text{reachable})} \wedge E$
   - $\neg \exists \bar{s}.\ F_{(\text{reachable})} \wedge F_{(\text{stuck})}$
   - $\forall \bar{s}.\ F_{(\leq k)}$
   - Note that we are only allowed to quantify on propositional variables. As such, a formula such as
$$\forall k.\ \exists \bar{s}.\ F_{(=k)} \wedge \neg F_{(\text{stuck})}$$
     is not allowed, as k ranges (implicitly) over $\mathbb{N}$.

Consider instead

$$\exists \bar{s}.\ F_{(=2^n)}$$

if such a state ($\bar{s}$) exists, the underlying trace contains $2^n + 1$ states (since it is 2^n steps long). By the pigeon hole principle, one of these states must appear twice. This means that there is a part of the trace which can be repeated infinitely ("pumped"), implying the existence of an infinite trace (see the proof of the pumping lemma for regular languages[1] which relies on the same principle).

Exercise 2. 1. Any expression of the type $y_k \leftrightarrow x_i * x_j$ is easily convertible to a CNF.

$$\begin{aligned}
y_k \leftrightarrow x_i \vee x_j &\equiv (y_k \rightarrow x_i \vee x_j) \wedge (x_i \vee x_j \rightarrow y_k) \\
&\equiv (\neg y_k \vee x_i \vee x_j) \wedge (\neg x_i \vee y_k) \wedge (\neg x_j \vee y_k) \\
y_k \leftrightarrow x_i \wedge x_j &\equiv (y_k \rightarrow x_i \wedge x_j) \wedge (x_i \wedge x_j \rightarrow y_k) \\
&\equiv (\neg y_k \vee x_i) \wedge (\neg y_k \vee x_j) \wedge (\neg x_i \vee \neg x_j \vee y_k) \\
y_k \leftrightarrow x_i \rightarrow x_j &\equiv y_k \leftrightarrow \neg x_i \vee x_j \\
&\equiv (\neg y_k \vee \neg x_i \vee x_j) \wedge (x_i \vee y_k) \wedge (\neg x_j \vee y_k) \\
y_k \leftrightarrow \neg x_i &\equiv (y_k \rightarrow \neg x_i) \wedge (\neg x_i \rightarrow y_k) \\
&\equiv (\neg y_k \vee \neg x_i) \wedge (x_i \vee y_k)
\end{aligned}$$

Let's first define the mapping between sub formulas and fresh variables:

$$\begin{aligned}
&y_1 \leftrightarrow \neg b \\
&y_2 \leftrightarrow a \vee y_1 \\
&y_3 \leftrightarrow b \rightarrow c \\
&y_4 \leftrightarrow \neg y_3 \\
&y_5 \leftrightarrow y_2 \wedge y_4 \\
&y_6 \leftrightarrow y_5 \rightarrow a
\end{aligned}$$

The Tseytin transformation of the formula is therefore:

$$\begin{aligned}
&(y_1 \leftrightarrow \neg b) \wedge (y_2 \leftrightarrow a \vee y_1) \wedge (y_3 \leftrightarrow b \rightarrow c) \wedge (y_4 \leftrightarrow \neg y_3) \wedge \\
&(y_5 \leftrightarrow y_2 \wedge y_4) \wedge (y_6 \leftrightarrow y_5 \rightarrow a) \wedge y_6 \\
=&(\neg y_1 \vee \neg b) \wedge (b \vee y_1) \wedge (\neg y_2 \vee a \vee y_1) \wedge (\neg a \vee y_2) \wedge (\neg y_1 \vee y_2) \wedge \\
&(\neg y_3 \vee \neg b \vee c) \wedge (b \vee y_3) \wedge (\neg c \vee y_3) \wedge (\neg y_4 \vee \neg y_3) \wedge (y_3 \vee y_4) \wedge \\
&(\neg y_5 \vee y_2) \wedge (\neg y_5 \vee y_4) \wedge (\neg y_2 \vee \neg y_4 \vee y_5) \wedge (\neg y_6 \vee \neg y_5 \vee a) \wedge \\
&(y_5 \vee y_6) \wedge (\neg a \vee y_6) \wedge (\neg y_6 \vee \neg y_7) \wedge (y_6 \vee y_7) \wedge y_7
\end{aligned}$$

[1] John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman. *Introduction to automata theory, languages, and computation.* Addison-Wesley, 2006, pp. 128–129.

2. We first show that they might not be equivalent by providing a small counterexample. As written above, the Tseytin transform of the formula $\neg x$ is $(y \leftrightarrow \neg x) \wedge y$. The value of $\neg x$ when $x = 0$ is 1, whereas the value of its Tseytin transformation when $x = 0, y = 0$, is 0. Therefore the two formulas are not equivalent.

   Let's now prove that a formula is satisfiable if and only if its Tseytin transformation is satisfiable as well.

   Let φ be a formula and $V = \{x_i \mid x_i \in \mathrm{FV}(\varphi)\}$ its set of variables. One can notice (and prove by induction) that the Tseytin transformation of φ is equivalent to:

   $$y_n \wedge \bigwedge_{i=1}^{n} (y_i \leftrightarrow \psi_i)$$

   where $\{\psi_i\}_{0 \leq i \leq n}$ are all the (non-atomic) subformulas of φ; in particular, $\psi_n = \varphi$. Therefore given an assignment α of V, a clause $(y_i \leftrightarrow \psi_i)$ is true if and only if one assigns the truth value of $\psi_i\,[V := \alpha]$ to y_i. Since these clauses can be true whatever is the value of α, we have that the Tseytin transformation of φ is satisfiable if and only if y_n is satisfiable. Since we have a clause $y_n \leftrightarrow \varphi$ (due to the fact that $\psi_n = \varphi$), y_n is satisfiable if and only if φ is satisfiable. Therefore φ and its Tseytin transformation are equisatisfiable.

3. F is a valid formula if and only if $\neg F$ is false for any variable assignment, i.e. it is unsatisfiable. A formula is unsatisfiable if and only if its Tseytin transformation is unsatisfiable as well (by the previous question). Therefore F is a valid formula if and only if the Tseytin transformation of $\neg F$ is unsatisfiable.

Exercise 3. By taking the CNF found in the above exercise and applying resolution, we obtain the following derivation trees:

$$\dfrac{\neg a \lor y_6 \qquad \dfrac{y_7 \qquad \neg y_6 \lor \neg y_7}{\neg y_6}}{\neg a}
\qquad
\dfrac{b \lor y_3 \qquad \dfrac{\neg y_3 \lor \neg y_4 \qquad \dfrac{y_4 \lor \neg y_5 \qquad \dfrac{y_5 \lor y_6 \qquad \dfrac{y_7 \qquad \neg y_6 \lor \neg y_7}{\neg y_6}}{y_5}}{y_4}}{\neg y_3}}{b}$$

$$\dfrac{\dfrac{\neg y_1 \lor \neg b \qquad \neg y_2 \lor a \lor y_1}{\neg b \lor a \lor \neg y_2} \qquad \dfrac{\neg y_5 \lor y_2 \qquad \dfrac{y_5 \lor y_6 \qquad \dfrac{y_7 \qquad \neg y_6 \lor \neg y_7}{\neg y_6}}{y_5}}{y_2}}{\neg b \lor a}
\qquad
\dfrac{b \qquad \dfrac{\neg b \lor a \qquad \neg a}{\neg b}}{F}$$

Exercise 4. 1. Let $I = (D, \alpha)$ be an interpretation of $\mathcal{L}$, for which $\alpha(B) = b$ and $\alpha(T) = t$. For notational convenience we will denote $\alpha(L)$ by L_α

$$\begin{aligned}
[\![\mathsf{Min}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(b,b), L_\alpha(b,u) \text{ and } L_\alpha(b,t)\\
[\![\mathsf{Max}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(b,t), L_\alpha(u,t) \text{ and } L_\alpha(t,t)\\
[\![\mathsf{Tot}]\!]_I = 1 \quad &\Longrightarrow L_\alpha(u,u) \text{ (by choosing } x = u \text{ and } y = u)\\
[\![\mathsf{Tra}]\!]_I = 1 \quad &\Longrightarrow (L_\alpha(t,b) \implies L_\alpha(t,u))
\end{aligned}$$

This greatly reduce the number of choices, restricting L_α to be either:

a)

L_α	b	u	t
b	1	1	1
u	1	1	1
t	1	1	1

b)

L_α	b	u	t
b	1	1	1
u	1	1	1
t	0	0	1

c)

L_α	b	u	t
b	1	1	1
u	0	1	1
t	0	1	1

d)

L_α	b	u	t
b	1	1	1
u	0	1	1
t	0	0	1

All the options satisfy reflexivity and transitivity as a consequence of the axioms. However, only *d)* satisfies the anti symmetry property. Therefore only d) is a partial order.

2.

a)

$E_{\alpha'}$	b	u	t
b	1	1	1
u	1	1	1
t	1	1	1

b)

$E_{\alpha'}$	b	u	t
b	1	1	0
u	1	1	0
t	0	0	1

c)

$E_{\alpha'}$	b	u	t
b	1	0	0
u	0	1	1
t	0	1	1

d)

$E_{\alpha'}$	b	u	t
b	1	0	0
u	0	1	0
t	0	0	1

In all the cases, $E_{\alpha'}$ is an equivalence relation.

3. $E_{\alpha'}$ is always an equivalence relation, whatever is the domain.

Let D' be an arbitrary set and $I' = (D', \alpha')$.

- **Reflexivity**: $[\![\mathsf{Both}]\!]_{I'} = 1 \implies \forall x \in D'. E_{\alpha'}(x,x) \leftrightarrow L_{\alpha'}(x,x)$ (by choosing $y = x$).

  We also know that, $[\![\mathsf{Tot}]\!]_{I'} = 1 \implies \forall x \in D'.\ L_{\alpha'}(x,x)$

  Therefore $\forall x \in D'.\ E_{\alpha'}(x,x)$

- **Symmetry**:

  $$[\![\mathsf{Both}]\!]_{I'} = 1 \implies \forall x \in D'. \forall y \in D'.\ E_{\alpha'}(x,y) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x))$$

  We also know that for $x, y \in D'$ arbitrary $L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x) \equiv L_{\alpha'}(y,x) \wedge L_{\alpha'}(x,y)$. Therefore we have that for $\forall x \in D'.\ \forall y \in D'.\ E_{\alpha'}(x,y) \leftrightarrow E_{\alpha'}(y,x)$ is a tautology.

- **Transitivity**: $[\![\mathsf{Both}]\!]_{I'} = 1$ implies that for x, y, z arbitrary in D'

  $$E_{\alpha'}(x,y) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,x))$$

  and

  $$E_{\alpha'}(y,z) \leftrightarrow (L_{\alpha'}(y,z) \wedge L_{\alpha'}(z,y))$$

  Therefore

  $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \leftrightarrow (L_{\alpha'}(x,y) \wedge L_{\alpha'}(y,z) \wedge L_{\alpha'}(z,y) \wedge L_{\alpha'}(y,x)$$

  which implies

  $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \rightarrow (L_{\alpha'}(x,z) \wedge L_{\alpha'}(z,x))$$

  and finally

  $$E_{\alpha'}(x,y) \wedge E_{\alpha'}(y,z) \rightarrow E_{\alpha'}(x,z)$$

4. One of the solution consists in choosing $D_1 = D' / E_{\alpha'}$, the set of equivalence classes of $E_{\alpha'}$, $\alpha_1 = (B_1, T_1, L_1)$ where $B_1 = [b]_{E_{\alpha'}}$, $T_1 = [t]_{E_{\alpha'}}$ and

$$L_1 := \left\{ \left([x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \right) : L_{\alpha'}(x, y) \text{ where } x, y \in D' \right\}$$

Reflexivity and transitivity are immediate as they have been proven for $L_{\alpha'}$.

Antisymmetry comes from the fact that for arbitrary $[x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \in D_1$

$$\begin{aligned}
& L_1 \left([x]_{E_{\alpha'}} [y]_{E_{\alpha'}} \right) \wedge L_1 \left([x]_{E_{\alpha'}}, [y]_{E_{\alpha'}} \right) \\
\implies & L_{\alpha'}(x, y) \wedge L_{\alpha'}(y, x) \\
\implies & E_{\alpha'}(x, y) \\
\implies & [x]_{E_{\alpha'}} = [y]_{E_{\alpha'}}
\end{aligned}$$

Therefore L_1 is a partial order.