CS 584 – Machine Learning

Topic: Maximum Likelihood and Bayesian Estimation

Mustafa Bilgic

🔗 http://www.cs.iit.edu/~mbilgic

🐦 https://twitter.com/bilgicm

PARAMETER ESTIMATION

- Imagine we have a thumbtack
- Flip it, and it comes as heads or tails

heads   tails

- P(Heads) = θ, P(Tails) = 1- θ
- Assume we flip it $(a + b)$ times and it comes head a times. What is θ if
  - $a = 4, b = 6$
  - $a = 42, b = 58$
  - $a = 407\ b = 593$
- Can you prove your answers?
- Can you associate a confidence score with your estimates?

WE WILL SEE TWO APPROACHES

1. Maximum likelihood estimation
2. Bayesian estimation

3

# Maximum Likelihood Estimation

MAXIMUM LIKELIHOOD ESTIMATION (MLE) – FOR THE THUMBTACK EXAMPLE

- Assume we have a set of thumbtack tosses

  - $\mathcal{D} = \{d_1, d_2, \ldots, d_m\}$ where a are Heads and b are Tails

- Hypothesis space: $[0, 1]$

- Find a "good" scoring $\theta \in [0, 1]$

- Let $f(\theta : \mathcal{D})$ be the score of θ given $\mathcal{D}$, where a high score is a "good" score

- Learning: $\operatorname*{argmax}_{\theta \in [0,1]} f(\theta : \mathcal{D})$

- What is $f(\theta : \mathcal{D})$?

- The space [0, 1] is infinite. How do we search this space efficiently?

LIKELIHOOD

- What is the probability, or *likelihood*, of seeing the sequence H, T, T, H, H?

  - $\theta * (1 - \theta) * (1 - \theta) * \theta * \theta = \theta^3 (1 - \theta)^2$
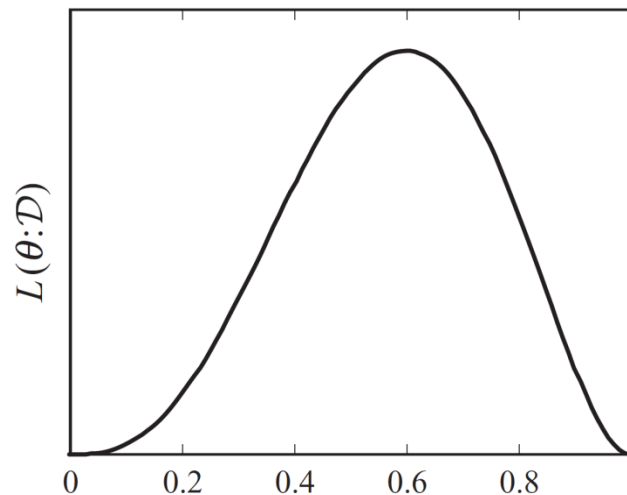
When is $L(\theta : \mathcal{D})$ maximum?

LIKELIHOOD/LOG-LIKELIHOOD

- Number of heads = a, number of tails = b

- Likelihood: $L(\theta:\mathcal{D}) = \theta^a(1-\theta)^b$

- Log-likelihood: $l(\theta:\mathcal{D}) = a\log\theta + b\log(1-\theta)$

- Note that $L(\theta:\mathcal{D})$ achieves its maximum for θ that maximizes $l(\theta:\mathcal{D})$

- Find θ that maximizes the log-likelihood

- Take derivate of $l(\theta:\mathcal{D})$ w.r.t. θ and set it to zero

7

Bayesian Estimation

8

Bayesian Estimation

- MLE gives the same estimate of $\theta = 0.4$, if we have 4 H and 6 T, as well as 4M H and 6M T

- In Bayesian estimation, rather than a single θ,
  - We will have a prior belief about θ: $\boldsymbol{p(\theta)}$
  - The posterior distribution over θ: $\boldsymbol{p(\theta \mid \mathcal{D})}$
  - The probability distribution for the next toss: $\boldsymbol{P(d_{m+1} \mid \mathcal{D})}$

POSTERIOR: $p(\theta \mid \mathcal{D})$

$$p(\theta \mid \mathcal{D}) = \frac{p(\theta)P(\mathcal{D} \mid \theta)}{P(\mathcal{D})}$$

$P(\mathcal{D})$ does not depend on θ. Hence, it can be treated as a constant from the perspective of θ.

$$p(\theta \mid \mathcal{D}) \propto p(\theta)P(\mathcal{D} \mid \theta)$$

Next, assume each data point is independent given θ: $d_i \perp d_j \mid \theta$

$$P(\mathcal{D}|\theta) = P(d_1|\theta)P(d_2|\theta)P(d_3|\theta) \cdots P(d_m|\theta) = \prod_{i=1}^{m} P(d_i \mid \theta)$$

Hence, the posterior becomes

$$p(\theta \mid \mathcal{D}) \propto p(\theta) \prod_{i=1}^{m} P(d_i \mid \theta)$$

10

PREDICTION: $P(d_{m+1} \mid \mathcal{D})$

$$P(d_{m+1}|D) = \int_0^1 P(d_{m+1}|\theta, \mathcal{D})p(\theta|\mathcal{D})d\theta$$

Assuming $d_i \perp d_j \mid \theta$:

$$P(d_{m+1}|D) = \int_0^1 P(d_{m+1}|\theta)p(\theta|\mathcal{D})d\theta$$

Using the posterior equation from the previous slide:

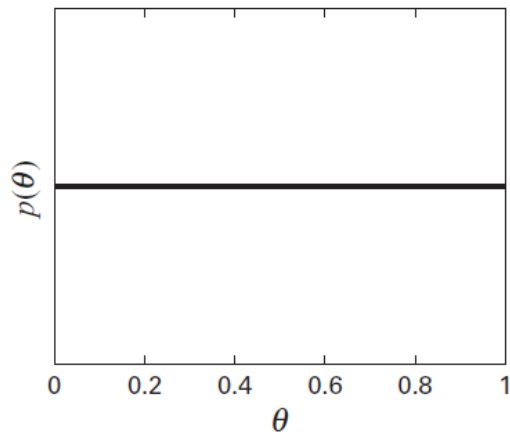$$P(d_{m+1}|D) \propto \int_0^1 P(d_{m+1}|\theta)p(\theta)\prod_{i=1}^m P(d_i \mid \theta)\,d\theta$$

UNIFORM PRIOR

- Assume a Heads and b Tails

- Assume a uniform prior over θ. That is, $p(\theta) = 1$

- What is $P(d_{m+1} = Heads \mid \mathcal{D})$?
  - (a+1)/(a+b+2)

- What is $p(\theta \mid \mathcal{D})$?
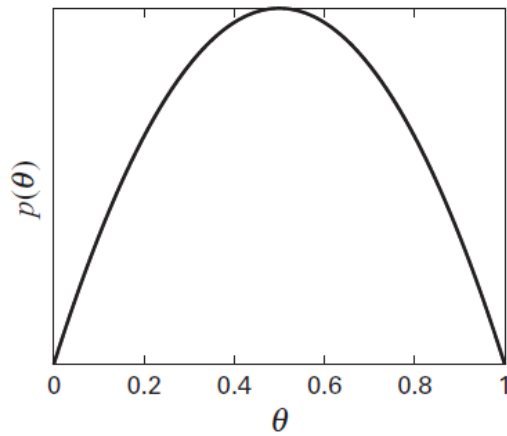 - Beta(a+1, b+1)

BETA DISTRIBUTION

- $\theta \sim \text{Beta}(\alpha,\beta)$ if $p(\theta) = \gamma\theta^{\alpha-1}(1-\theta)^{\beta-1}$ where γ is a normalizing constant

- Mean: $\alpha/(\alpha+\beta)$

- Mode: $(\alpha-1)/(\alpha+\beta-2)$

- Note that the mode is closer to the mean when α and β are large

- Read more at

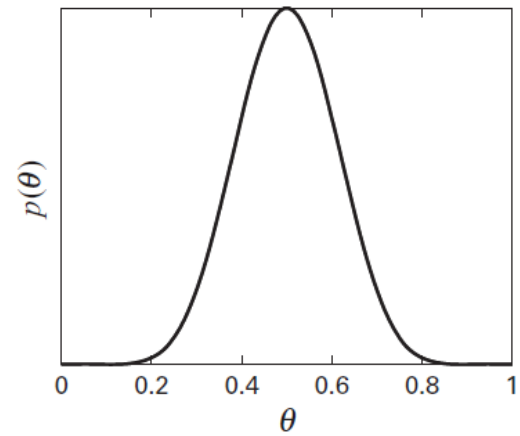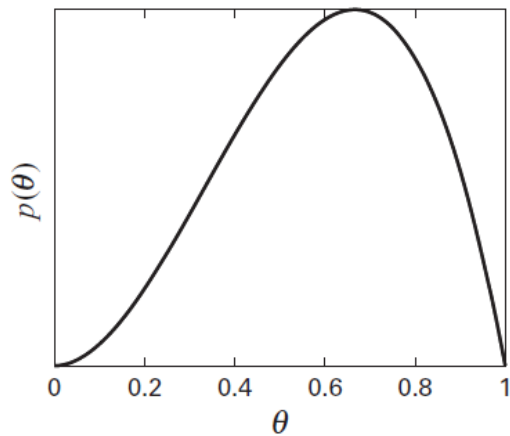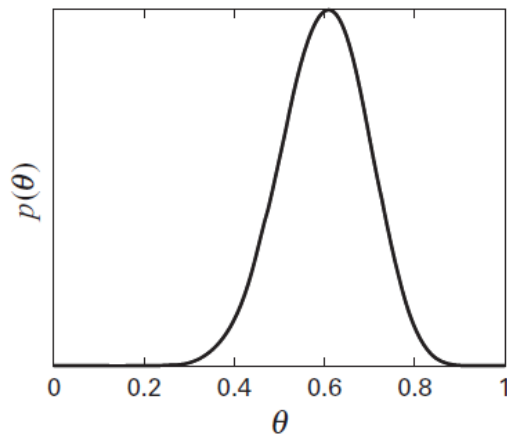 - https://en.wikipedia.org/wiki/Beta_distribution
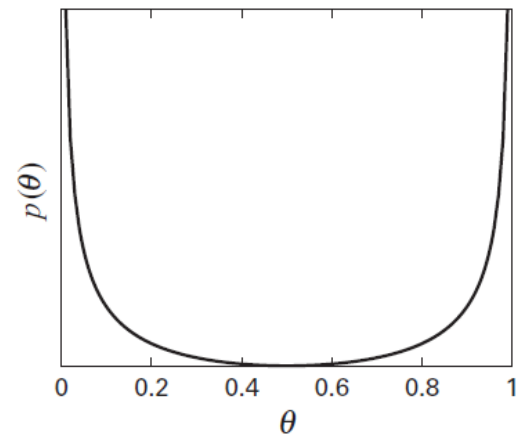
13

BETA DISTRIBUTION

Beta(1,1)   Beta(2,2)   Beta(10,10)

Beta(3,2)   Beta(15,10)   Beta(0.5,0.5)

CS 584 – Machine Learning – Illinois Institute of Technology

BETA DISTRIBUTION

- What is $P(d_{M+1}=True \mid d_1,\ldots,d_M)$ if the prior is Beta(α,β)?

  - $P(X[M+1]=\text{True} \mid D) = (a + \alpha) / (a + b + \alpha + \beta)$

- What is the posterior, $P(\theta \mid D)$, if the prior is Beta(α,β)?

  - $P(\theta \mid D) = \text{Beta}(a + \alpha, b + \beta)$

- α and β work like pseudo-counts for the positive and negative cases respectively

- What values to choose for α and β?

  - It depends on our belief and the strength of our belief

15

Dirichlet priors

- Generalizes the Beta distribution for multinomials

$$\theta \sim Dirichlet(\alpha_1,\ldots,\alpha_K) \text{ if } P(\theta) \propto \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}$$

- What is $P(d_{M+1}=v_i \mid D)$ if the prior is Dirichlet?

  - $P(d_{M+1}=v_i \mid D) = (n_i + \alpha_i) / (\mid D \mid + \alpha)$ where n_i is the number of times the ith case appears in D and $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_K$

- What is the posterior, $P(\theta \mid D)$, if the prior is Dirichlet?

  - $P(\theta \mid D) = Dirichlet(n_1 + \alpha_1, n_2 + \alpha_2, \ldots, n_K + \alpha_K)$

16