

1 Recap

Last lecture we saw how to unify types.

$$\begin{aligned}\text{Unify}(\emptyset) &\triangleq I \\ \text{Unify}(\alpha = \alpha, E) &\triangleq \text{Unify}(E) \\ \text{Unify}(\alpha = \tau, E) &\triangleq \{\tau/\alpha\} \cdot \text{Unify}(E\{\tau/\alpha\}), \quad \alpha \notin \text{FV}(\tau) \\ \text{Unify}(\sigma_1 \rightarrow \tau_1 = \sigma_2 \rightarrow \tau_2, E) &\triangleq \text{Unify}(\sigma_1 = \sigma_2, \tau_1 = \tau_2, E)\end{aligned}$$

where I is the identity substitution $\alpha \mapsto \alpha$. Substitutions are applied from left to right, so the composition ST means: do S first, then do T .

2 Polymorphic λ -Calculus

Suppose we have base types `int` and `bool`. The problem with the simple type inference mechanism that we have presented is that we do not have quite as much *polymorphism*¹ as we would like. For example, consider a program that binds a variable to the identity function, then applies it to an `int` and also to a `bool`.

$$\text{let } f = \lambda x. x \text{ in if } (f \text{ true}) \text{ then } (f \text{ 3}) \text{ else } (f \text{ 4}) \quad (1)$$

The type checker encounters the `bool` first and says that the function is of type `bool` \rightarrow `bool`, then gives an error when it sees the `int` parameter, whereas we really want it to be interpreted as type `bool` \rightarrow `bool` when applied to a `bool` parameter and `int` \rightarrow `int` when applied to an `int` parameter.

We can handle this by introducing a new type constructor that quantifies over types.

$$\tau ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \tau \mid \forall \alpha. \tau \quad (2)$$

The type $\forall \alpha. \tau$ can be viewed as a *polymorphic type* or *type schema*, a pattern with type variables that can be instantiated to obtain actual types. For example, the polymorphic type of the identity function will be the type schema

$$\forall \alpha. \alpha \rightarrow \alpha$$

and the type of the K combinator $\lambda xy. x$ will be

$$\forall \alpha. \forall \beta. \alpha \rightarrow \beta \rightarrow \alpha.$$

There will be rules that allow us to delay the instantiation of the type variables until the function is applied. Thus we can interpret the identity function as `int` \rightarrow `int` or `bool` \rightarrow `bool` depending on context.

The resulting language is called the *polymorphic λ -calculus*. In this new language, the terms and evaluation rules are the same, but the types are defined by (2). All the terms that were previously well-typed will still be well-typed, but there will be more well-typed terms than before; for example, (1).

¹Greek for “many forms”

3 Typing Rules

In addition to the old typing rules

$$\begin{array}{c} \Gamma \vdash n : \text{int} \quad (\text{and similarly for other constants}) \\[10pt] \frac{\Gamma \vdash e : \sigma \rightarrow \tau \quad \Gamma \vdash d : \sigma}{\Gamma \vdash (e \ d) : \tau} \end{array} \qquad \begin{array}{c} \Gamma, x : \tau \vdash x : \tau \\[10pt] \frac{\Gamma, x : \sigma \vdash e : \tau}{\Gamma \vdash \lambda x. e : \sigma \rightarrow \tau} \end{array}$$

we add the following two new rules for polymorphic types:

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash e : \forall \alpha. \tau} \quad (\alpha \notin FV(\Gamma)) \qquad \frac{\Gamma \vdash e : \forall \alpha. \tau}{\Gamma \vdash e : \tau \{\sigma/\alpha\}}$$

These are called the *generalization rule* and the *instantiation rule*, respectively.

The notation $\tau \{\sigma/\alpha\}$ refers to the safe substitution of the type σ for the type variable α in τ . Here the binding operator $\forall \alpha$ binds the type variable α in the same way that λx binds the variable x in λ -terms, and the notions of scope, free and bound variables are the same. In particular, one can α -convert type variables as necessary to avoid the capture of free type variables when performing substitutions.

The generalization rule includes the side condition $\alpha \notin FV(\Gamma)$. The idea here is that the type judgment $\Gamma \vdash e : \tau$ must hold without any assumptions involving α ; if so, then we can conclude that α could have been any type σ , and the type judgment $\Gamma \vdash e : \tau \{\sigma/\alpha\}$ would also hold.

4 Examples

Here is a derivation of the polymorphic type of K in this system.

$$\frac{\frac{\frac{x : \alpha, y : \beta \vdash x : \alpha}{x : \alpha \vdash \lambda y. x : \beta \rightarrow \alpha}}{\vdash \lambda x. \lambda y. x : \alpha \rightarrow \beta \rightarrow \alpha}}{\vdash \lambda x. \lambda y. x : \forall \beta. \alpha \rightarrow \beta \rightarrow \alpha}}{\vdash \lambda x. \lambda y. x : \forall \alpha. \forall \beta. \alpha \rightarrow \beta \rightarrow \alpha}$$

Starting from $x : \alpha, y : \beta \vdash x : \alpha$, two applications of the abstraction rule yield $\vdash \lambda x. \lambda y. x : \alpha \rightarrow \beta \rightarrow \alpha$, then two applications of the generalization rule yield $\vdash \lambda x. \lambda y. x : \forall \alpha. \forall \beta. \alpha \rightarrow \beta \rightarrow \alpha$.

Some terms are typable in this system that were not typable before. For example, the term $\lambda x. xx$ is typable:

$$\frac{\frac{\frac{x : \forall \alpha. \alpha \vdash x : \forall \alpha. \alpha}{x : \forall \alpha. \alpha \vdash x : \alpha \rightarrow \beta} \quad \frac{x : \forall \alpha. \alpha \vdash x : \forall \alpha. \alpha}{x : \forall \alpha. \alpha \vdash x : \alpha}}{\frac{x : \forall \alpha. \alpha \vdash xx : \beta}}{\vdash \lambda x. xx : (\forall \alpha. \alpha) \rightarrow \beta}}{\vdash \lambda x. xx : \forall \beta. (\forall \alpha. \alpha) \rightarrow \beta}$$

Unfortunately, this type is not too meaningful, because *nothing* has type $\forall \alpha. \alpha$. This type is said to be *uninhabited*, and we give it a name: **Void**. However, by a similar argument, we can show that $\lambda x. xx$ also has type $\forall \beta. (\forall \alpha. \alpha \rightarrow \alpha) \rightarrow (\beta \rightarrow \beta)$, which *is* meaningful.

$$\begin{array}{c}
\frac{x : \forall \alpha. \alpha \rightarrow \alpha \vdash x : \forall \alpha. \alpha \rightarrow \alpha}{x : \forall \alpha. \alpha \rightarrow \alpha \vdash x : (\beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta)} \quad \frac{x : \forall \alpha. \alpha \rightarrow \alpha \vdash x : \forall \alpha. \alpha \rightarrow \alpha}{x : \forall \alpha. \alpha \rightarrow \alpha \vdash x : \beta \rightarrow \beta} \\
\hline
\frac{x : \forall \alpha. \alpha \rightarrow \alpha \vdash xx : \beta \rightarrow \beta}{\vdash \lambda x. xx : (\forall \alpha. \alpha \rightarrow \alpha) \rightarrow (\beta \rightarrow \beta)} \\
\hline
\vdash \lambda x. xx : \forall \beta. (\forall \alpha. \alpha \rightarrow \alpha) \rightarrow (\beta \rightarrow \beta)
\end{array}$$

Although $\lambda x. xx$ is typable, the paradoxical combinator $\Omega = (\lambda x. xx)(\lambda x. xx)$ is not, and neither is the Y combinator. This is because the language is still strongly normalizing. This means that the polymorphic λ -calculus is not Turing complete, that is, it cannot simulate arbitrary Turing machines.

Worse, types inference is undecidable, so the programmer must sometimes provide types.

5 Let-Polymorphism

We can regain decidability of type inference by placing some restrictions on the use of the type quantifier $\forall \alpha$. Specifically, we will only allow it at the top level; that is, we will only allow polymorphic type expressions of the form $\forall \alpha_1 \dots \forall \alpha_n. \tau$, where τ is quantifier-free:

$$\begin{array}{ll}
\text{quantifier-free terms} & \tau ::= \text{int} \mid \text{bool} \mid \alpha \mid \tau_1 \rightarrow \tau_2 \\
\text{polymorphic terms} & \pi ::= \tau \mid \forall \alpha. \pi
\end{array}$$

We will also modify our rules so that it can only be introduced in the context of a **let** statement. Thus we will modify our definition of terms to include a **let** statement:

$$e ::= \dots \mid \text{let } x = e_1 \text{ in } e_2$$

and replace the generalization rule with the **let** rule

$$\frac{\Gamma \vdash d : \sigma \quad \Gamma, x : \forall \alpha_1 \dots \forall \alpha_n. \sigma \vdash e : \tau}{\Gamma \vdash \text{let } x = d \text{ in } e : \tau} \quad (\{\alpha_1, \dots, \alpha_n\} = FV(\sigma) - FV(\Gamma))$$

So type schemas are only used to type **let** expressions. For this reason, this approach is called *let-polymorphism*.

The type systems of OCaml and Haskell are based on let-polymorphism. We previously considered the expression $\text{let } x = d \text{ in } e$ to be syntactic sugar for $(\lambda x. e) d$, but in OCaml, the former may be typable in some cases when the latter is not:

```

# let f = fun x -> x in if (f true) then (f 3) else (f 4);;
- : int = 3
# (fun f -> if (f true) then (f 3) else (f 4)) (fun x -> x);;
Error: This expression has type int but an expression was expected of type
      bool

```

In theory, let-polymorphism can cause the type checker to run in exponential time, but in practice this is not a problem.

6 System F

In the Church-style simply-typed λ -calculus, we annotated binding occurrences of variables with their types. The corresponding version of the polymorphic λ -calculus is called *System F*. Here we explicitly abstract terms

with respect to types and explicitly instantiate by applying an abstracted term to a type. We augment the syntax with new terms and types:

$$e ::= \dots \mid \Lambda\alpha.e \mid e\tau \qquad \tau ::= b \mid \tau_1 \rightarrow \tau_2 \mid \alpha \mid \forall\alpha.\tau$$

where b are the base types (e.g., `int` and `bool`). The new terms are *type abstraction* and *type application*, respectively. Operationally, we have

$$(\Lambda\alpha.e)\tau \rightarrow e\{\tau/\alpha\}. \quad (3)$$

This just gives the rule for instantiating a type schema. Since these reductions only affect the types, they can be performed at compile time.

The typing rules for these constructs need a notion of well-formed type. We introduce a new environment Δ that maps type variables to their *kinds*. For now, there is only one kind, namely `type`, so Δ is a partial function with finite domain mapping type variables to $\{\text{type}\}$. Since the range is only a singleton, all Δ does for now is to specify a set of types, namely $\text{dom } \Delta$ (it will get more complicated later). As before, we use the notation $\Delta, \alpha : \text{type}$ for the partial function $\Delta[\text{type}/\alpha]$. For now, we just abbreviate this by Δ, α .

The type system has two classes of judgments:

$$\Delta \vdash \tau : \text{type} \qquad \Delta; \Gamma \vdash e : \tau$$

For now, we just abbreviate the former by $\Delta \vdash \tau$. These judgments just determine when τ is well-formed under the assumptions Δ . The typing rules for this class of judgments are:

$$\Delta, \alpha \vdash \alpha \qquad \Delta \vdash b \qquad \frac{\Delta \vdash \sigma \quad \Delta \vdash \tau}{\Delta \vdash \sigma \rightarrow \tau} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \forall\alpha.\tau}$$

Right now, all these rules do is use Δ to keep track of free type variables. One can show that $\Delta \vdash \tau$ iff $FV(\tau) \subseteq \text{dom } \Delta$.

The typing rules for the second class of judgments are:

$$\frac{\Delta \vdash \tau}{\Delta; \Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Delta; \Gamma \vdash e_0 : \sigma \rightarrow \tau \quad \Delta; \Gamma \vdash e_1 : \sigma}{\Delta; \Gamma \vdash (e_0 e_1) : \tau} \qquad \frac{\Delta; \Gamma, x : \sigma \vdash e : \tau \quad \Delta \vdash \sigma}{\Delta; \Gamma \vdash (\lambda x : \sigma. e) : \sigma \rightarrow \tau}$$

$$\frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \quad \Delta \vdash \sigma}{\Delta; \Gamma \vdash (e\sigma) : \tau\{\sigma/\alpha\}} \qquad \frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash (\Lambda\alpha.e) : \forall\alpha.\tau} \quad (\alpha \notin FV(\Gamma))$$

One can show that if $\Delta; \Gamma \vdash e : \tau$ is derivable, then τ and all types occurring in annotations in e are well-formed. In particular, $\vdash e : \tau$ only if e is a closed term and τ is a closed type, and all type annotations in e are closed types.

For example, the polymorphic identity function is $\Lambda\alpha.\lambda x : \alpha. x$, which has polymorphic type $\forall\alpha.\alpha \rightarrow \alpha$ according to the following proof:

$$\frac{\frac{\frac{\alpha \vdash \alpha}{\alpha; x : \alpha \vdash x : \alpha} \quad \alpha \vdash \alpha}{\alpha; \vdash (\lambda x : \alpha. x) : \alpha \rightarrow \alpha}}{\vdash (\Lambda\alpha.\lambda x : \alpha. x) : \forall\alpha.\alpha \rightarrow \alpha}$$

To apply this function to a value of a particular type, one must explicitly instantiate the type using (3):

$$((\Lambda\alpha.\lambda x : \alpha. x) \text{ int}) 3 \rightarrow (\lambda x : \text{int}. x) 3 \rightarrow 3.$$