Data Structures and Algorithms SS20 // Gian Hiltbrunner

Algorithms

Order of Complexity

$$1,\ \log\log n,\ \sqrt{\log n},\ \log\sqrt{n},\ \log n,\ \sqrt{n},\ n,\ n\log n,\ n^2,\ \binom{n}{3} \in n^3,\ n^c,\ 2^n,\ n!$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 \Rightarrow f \in \mathcal{O}(g), \mathcal{O}(f) \subsetneq \mathcal{O}(g); \lim_{n\to\infty}\frac{f(n)}{g(n)} = C > 0 (C \text{ constant})$$

$$\Rightarrow f \in \Theta(g); \frac{f(n)}{g(n)} \underset{n\to\infty}{\to} \infty g \in \mathcal{O}(f), \mathcal{O}(g) \subsetneq \mathcal{O}(f); \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Maximum Subarray Alogrithm

Algorithm 1: Inductive Maximum Subarray

Input : $(a_1, a_2, ..-, a_n)$
Output: $\max 0, \max_{i,j} \sum_{k=i}^{j} a_k$
for $i = 1, \ldots, n$ **do**
  $R \leftarrow R + a_i$
  if $R < 0$ **then**
    $R \leftarrow 0$
  end
  if $R > M$ **then**
    $M \leftarrow R$
  end
end
return *M*

Runtime : $\Theta(n)$

Searching

Linear Search

Best case: 1 comparison; Worst case: n comparisons
Expected: $\mathrm{E}(x) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2} \in \Theta(n)$

Binary Search

divide and conquer approach $\to \Theta(\log n)$ Works with two pointers l and r. If $l > r$ the search was without result.

Algorithm 2: Breadth-first search

Input : A graph G and a starting vertex root of G
Output: The parent links trace the shortest path back to root
let Q be a queue; label root as discovered; Q.enqueue(root)
while *Q is not empty* **do**
  v := Q.dequeue() **if** *v is the goal* **then**
    return v
  end
  for *all edges from v to w in G.adjacentEdges(v)* **do**
    if *w is not labeled as discovered* **then**
      label w as discovered
      w.parent := v; Q.enqueue(w)
    end
  end
end

Selecting

Blum's Algorithm

A good pivot can be selected using the median-of-medians-algorithm

1. Consider groups of 5 elements
2. Compute the median for each group (trivial)
3. Recursively compute medians

Pivot

Algorithm 3: Selection via Pivot

Input : Array A of length n with pivot p
Output: A partitioned around p with position of p
$l \leftarrow 1$
$r \leftarrow n$ **while** $l \leq r$ **do**
  while $A[l] < p$ **do**
    $l \leftarrow l + 1$
  end
  while $A[r] > p$ **do**
    $r \leftarrow r - 1$
  end
  swap(A[l],A[r]) **if** $A[l] = A[r]$ **then**
    $l \leftarrow l + 1$
  end
end
return $l - 1$

Algorithm 4: Quickselect

Input : Array A of length n; $1 \leq k \leq n$
$x \leftarrow$ RandomPivot(A)
$m \leftarrow$ Partition(A,x)
if $k < m$ **then**
  return *Quickselect(A[0..m-1],k)*
end
if *k > m* **then**
  return *Quickselect(A[m+1..n],k)* **else**
    return $A[k]$
  end
end

Sorting

- **Bubblesort**: Always swap if $A[i-1] > A[i]$. In each round, the max in the unsorted part will move to the right (like a bubble). $\Theta(n^2)$ stable
- **Selection sort**: swap the smallest element in the unsorted part with the most right element of the sorted part. $\Theta(n^2)$ unstable

```
arr[] = 64 25 12 22 11
// Place min at beginning
11 25 12 22 64
// Place min at beginning
11 12 25 22 64 ...
```

- **Insertion sort**: Determine the insertion position of element i. $\Theta(n^2)$ stable

```
1: Iterate over the array (curr).
2: Compare curr to predecessor (pre).
3: If curr < pre,
compare it to the elements before.
Larger elements are moved back 1 pos.
```

- **Merge sort**: At least two parts of the Array are already sorted. Iterative merging of the already sorted bits. - $\Theta(n \log n)$, $\Theta(n)$ storage, stable, needs intermediate storage for the merging step

Quicksort

Algorithm 5: Quicksort

Input : Array A of length n
Output: Array A sorted
if $n > 1$ **then**
  Choose Pivot $p \in A$ $k \leftarrow$ Partition(A,p)
  Quicksort(A[1,...,k-1])
  Quicksort(A[k+1,...,n])
end

Algorithm 6: Partition

Input : Array A, that contains the pivot p in A[l, . . . , r] at least once.
Output: Array A partitioned in [l, . . . , r] around p. Returns position of p.
while $l < r$ **do**
  while $A[l] < p$ **do**
    $l = l + 1$
  end
  while $A[r] > p$ **do**
    $r = r - 1$
  end
  swap$(A[l], A[r])$
  if $A[l] = A[r]$ **then**
    $l = l + 1$
  end
end
return $l - 1$

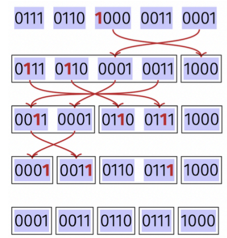
Runtime: in the mean $\mathcal{O}(n \cdot \log \cdot n)$, worst case $\Theta(n^2)$ if worst pivots are selected each time.

Radix Sort

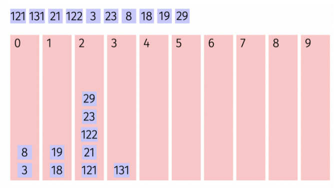
n-locks for n-keys $\in \mathcal{O}(n)$. We have m-adic binary numbers, so two categories to sort the numbers into. Used for numbers (and strings via UTF-8/ASCii)

Bucket Sort

Create a number of buckets. Sort e.g. after decimality into buckets and sort those buckets then. Can be implemented via linked list or a dynamic list(heap?).



Radix Sort



Bucket Sort

Hashing

Basics

Common: $h(k) = k \mod m$
Often: $m = 2^k - 1$
Linear Probing: $S(k) = (h(k), h(k)+1, ..., h(k)+m-1)$

$\bmod m$ *Issue*: Primary clustering, long contiguous areas of used entries.
Quadratic Probing: $S(k) = (h(k), h(k)+1, h(k)-1, h(k)+4, ...) \mod m$ *Issue*: Secondary clustering, traversal of the same probing sequence.
Double Hashing: $S(k) = (h(k), h(k)+h'(k), h(k)+2h'(k), ..., h(k)+(m-1)h'(k)) \mod m$

Trees

Trees are connected, directional and acyclic graphs.

Removing a child

- No children - Remove the node
- 1 child - Replace by the only child
- 2 children - Replace by the symmetric descendent

Ways of traversal

Preorder

v, then $T_{left}(v)$, next $T_{right}(v)$

Postorder

T_{left}, then T_{right}, next v

Inorder

T_{left}, then v, next $T_{left} \rightarrow$ ascending sequence.

Heaps

Keys are strictly larger/smaller depending on Max- or Minheap.

Insertion

Inserting a key into a heap can possibly violate the heap settings - Is reinstated by successive rising up.

Heap Sort

Every subtree is a heap - inductive sorting from below. $\rightarrow \mathcal{O}(n \cdot \log n)$

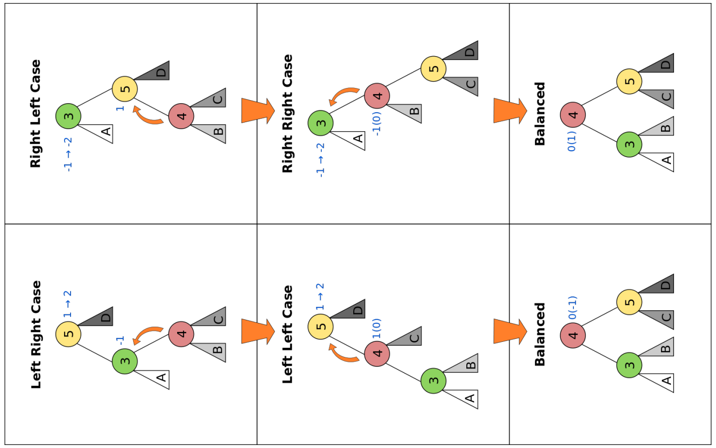
Quadtrees

Partitioning a subsection into 4 equal parts. If there are too many objects stored in one node, we split the node into four children. Objects that are falling on a border are stored in the parent node.

AVL trees

AVL trees guarantee a runtime of $\mathcal{O}(\log n)$
$bal(v) := h(T_r(v)) - h(T_l(v))$
AVL condition: $\forall v \in V : bal(v) \in \{-1, 0, 1\}$

Rebalancing AVL trees



Dynamic Programming

Samples

One-dimensional
Problem: Finding the longest possible combination of downwards ski slopes with lengths l_i. The slopes connect the stations with heights h_i.

1. **Table**: $n \times 1$
2. **Entry**: $[i]$: longest descent that ends in i.
3. **Calculation**: $D[i] = 0, \forall i = 1, ..., n$ and $D[i] = \max_{Slope(j,i)} \{D[j] + l(j,i)\}$
4. **Order**: `for i in (1, n); D[i]`
5. **Result**: $\max(D)$
6. **Reconstruction**: Recursively walk back from result and check $D[i] = D[j] + l(j,i)$ for all slopes (j,i)

Two-dimensional
Problem: Finding the smallest possible value of an expression (n values a_i and $n-1$ operators s_i) using optimal bracket placement.

1. **Table**: $n \times n$: Only upper right triangular matrix is used.
2. **Entry**: $[i,j]$: smallest possible value of sub-expression from value a_i to a_j.
3. **Calculation**: $A_{i,i} = a_i; 1 \leq i \leq n$ and $A_{i,j} = \min_{i \leq k \leq j} \{A_{i,k-1} \langle s_{k-1} \rangle A_{k,j}\}; 1 \leq i \leq j \leq n$
4. **Order**: `for s in (0, n-1); for i in (1, n-s); A[i,i+s]`
5. **Result**: `A[1,n]`
6. **Reconstruction**: Recursively walk back and check $A_{i,j} = A_{i,k-1} \langle s_{k-1} \rangle A_{k,j}$

Graphs

Basics

Connected: Graph where there is a connecting path (not edge) between each pair of nodes.
Complete: Graph where there is an edge between each pair of nodes.

Algorithm 7: Depth First Search

```
Input  : A graph G and a vertex v of G
Output: All vertices reachable from v labeled as discovered
label v as discovered
for E ∈ G.adjacentEdges(v) (lexicographic) do
    if vertex w is not labeled as discovered then
        DFS(G, w)
    end
end
```

Topological Sorting

A directed graph has a topological sorting if it is acyclic.
Idea We successively prune our graph by removing elements that have 0 entry edges (and then update the entry edges of the successors to find the next one.

Algorithm 8: Topological Sorting

```
A[v] contains number of entry edges of vertex v (calculate by setting
  A[w] = 0 and then loop through (v,w) ∈ E and set A[w]+ = 1
for v ∈ V where A[v] == 0 do
    Push(S, v)
end
i=0;
while S! = {} do
    v ← pop(S); ord[v] ← i
    i++;
    for (v, w) in E do
        A[w] ← A[w] − 1 ; //decrease incoming for all successors
        if A[w] == 0 then
            push(S,w)
        end
    end
end
if i = |V| then
    return SUCCESS
end
else
    return “Cycle detected”
end
```

Shortest Path

On either directed or non-directed, weighted graph, find the shortest distance between a point A and all the other points in the graph.

Dijkstra

Algorithm 9: Dijkstra

```
Input : G = (V, E, source)
create vertex set Q //as a queue / min heap;
for u ∈ V do
    dist[u] ← INFINITY;
    prev[u] ← UNDEFINED;
    Q.insert(u);
end
dist[source] = 0;
while Q not empty do
    u = Q.ExtractMin();
    for v in Neighbors of u still in Q do
        alt = dist[u] + length(u, v);
        if alt < dist[v] then
            dist[v] = alt;
            prev[v] = u;
            Q.DecreasePriority(v, alt);
        end
    end
end
```

Runtime of Dijkstra

- any data structure: $\mathcal{O}(|V| \cdot T_{em} + |E| \cdot T_{dp})$
- with an array or linked list $\mathcal{O}(|V|^2 + |E|) = \mathcal{O}(|V|^2)$
- dense graph in adjacency list $\mathcal{O}(|V|^2 log|V|)$ since $|E| = |V|^2$ and DecreaseKey $log(|V|)$
- sparse connected graph in adjacency list/ stored in binary tree $\mathcal{O}(|E|log|V|)$

A-Star

A^* is an extension of Dijkstra's algorithm using a additional heuristic to guide the direction of the search. Like Dijkstra, A* works by making a lowest-cost path tree from the start node to the target node but it uses a function $f(n)$ as an estimate of the total cost using this path.

$$f(n) = g(n) + h(n)$$

$f(n)$:total estimated cost of path through node n, $g(n)$: cost so far to reach node n, $h(n)$: estimated cost from n to goal.
A good heuristic $h(n)$ is the Manhattan distance.

Bellman-Ford

Instead of optimizing the order in which vertices are processed, Bellman-Ford simply relaxes all the edges $|V| - 1$ times and hence runs in $\mathcal{O}(|V||E|)$ time.
If we want to limit the number of edges that we pass trough for our solution we can limit the outer loop to only run k times.

Algorithm 10: Bellman-Ford

```
Input : G = (V, E, source)
for u ∈ V do
    dist[u] ← INFINITY;
    prev[u] ← UNDEFINED;
end
dist[source] = 0;
for u in |V| do
    for v in Neighbors of u do
        alt = dist[u] + length(u, v);
        if alt < dist[v] then
            dist[v] = alt;
            prev[v] = u;
        end
    end
end
for each edge (u, v) with weight w in |E| do
    if dist[u] + w < dist[v] then
        error "Graph contains a negative-weight cycle"
    end
end
```

Runtime of Bellman Ford

- $\mathcal{O}(|E| \cdot |V|)$

Floyd-Warshall

Goal is to find the shortest path between all pairwise edges in a Graph G.

Algorithm 11: Floyd-Warshall

```
Input : G = (V, E)
let G dist be a |V|×|V| array of minimum distances initialized to ∞
for each edge (u, v) do
    dist[u][v] ← w(u, v) // The weight of the edge (u, v)
end
for each vertex v do
    dist[v][v] ← 0
end
for k from 1 to |V| do
    for i from 1 to |V| do
        for j from 1 to |V| do
            if dist[i][j] > dist[i][k] + dist[k][j] then
                dist[i][j] ← dist[i][k] + dist[k][j];
            end
        end
    end
end
```

Runtime of Floyd-Warshall

- $\mathcal{O}(|V|^3)$

Johnson's Algorithm

Find the shortest paths between all pairs of vertices in an edge-weighted (negative), directed graph. Negative cycles are not allowed. It uses Bellman-Ford to remove all negative weights and then applies Dijkstra on the graph.
The runtime is given by $\mathbf{O}(|V|^2 \log |V| + |V||E|)$. Thus when the graph is sparse the algorithm is faster than Floyd-Warshall which solves the same problem in $\mathbf{O}(|V|^3)$.

1. New node q is added to the graph connected by zero-weight edges to each of the other nodes.
2. Bellman-Ford is used starting from the new vertex q to find the minimum weight from q to each vertex v. If a negative cycle is detected the algorithm terminates.
3. The original edges are reweighted using the values computed in the Bellman-Ford step. $w'(u, v) = w(u, v) + h(u) - h(v)$
4. q is removed and Dijkstra is used to find the shortest paths from each node s to every other vertex in the reweighted graph. The original distance is computed by adding $h(v) - h(u)$.

Choice of algorithm

- No weights or all equal weights → BFS ($\Theta(|V| + |E|)$)
- Only positive weights → Dijkstra with Fibonacci Heap ($\mathcal{O}(|V| \cdot \log(|V|) + |E|)$)
- Some negative weights → Bellman Ford ($\mathcal{O}(|E| \cdot |V|^2)$)
- All pairs of shortest paths.
  - V times Dijkstra. If negative edges, recreate graph with Johnson first $\mathcal{O}(|E| \cdot |V|log|V|)$
  - Floyd-Warshall. $\mathcal{O}(|V|^3)$
  - Johnsons in a sparse graph. $\mathbf{O}(|V|^2 \log |V| + |V||E|)$

Minimum Spanning Tree

Given is a undirected weighted connected graph $G(V, E)$. Searched is a minimum spanning tree:

- Tree: connected and acyclic
- Spanning tree: All vertices $v \in V$ are connected.
- minimal: $c(T) = \min \sum_{e \in E} c(e)$

Kruskal algorithm

Algorithm 12: Kruskal

```
Sort edges increasingly after their weight: c(e_1) ≤ c(e_2) ≤ ...c(e_m)
A ← ∅ for k = 1 to m do
    if A ∪ e_k then
        A ← A ∪ e_k
    end
end
```

Starts with the smallest edge! Edges that would create a cycle are subsequently discarded in the process → exam question.
Runtime: $\mathcal{O}(E \log E)$

Jarnik (Prims) Algorithm

Algorithm 13: Jarnik Algorithm

```
start with v ∈ V A ← ∅
S ← v_0 for i = 1 to |V| do
    choose cheapest (u, v) with u ∈ S and v ∉ S
    A ← A ∪ (u, v)
    S ← S ∪ v
end
```

Main difference to Kruskal is, that it starts at $v \in V$ and chooses the cheapest edge from there.
Runtime: $\mathcal{O}(E + V \log V)$ with fibonacci heaps.

UnionFind

Find(x): Find the node x, go to the root of this subtree and return it. Union: Add the smaller subtree as a child to the larger subtree.

Max Flow / Min Cut

Given a flow network, determine the maximal flow allowed. The cut of the Graph G(S,T) into a source graph S and a sink graph T with the smallest capacity (min cut) will have the same capicity as the maximal flow.

Ford-Fulkerson

Algorithm 14: Ford-Fulkerson

for $(u, v) \in E$ **do**
  $f(u, v) = 0$;
end
$//G_f$ describes network capacities minus the existing flows
while *Path p exists from s to t in residual network G_f* **do**
  $c_f(p) \leftarrow min(c_f(u, v) \in p)$;
  //increase the flow along this path
  for *edge $e(u, v) \in p$* **do**
    $f(e) \leftarrow f(e) + c_f(p)$;
    $c_f(e) \leftarrow c_f(e) - c_f(p)$;
  end
end

Edmonds-Karp

Edmonds-Karp implements the Ford-Fulkerson algorithm by using a BFS search on the residual network.

Runtime of Ford-Fulkerson with Integers If $f*$ is the maximum flow in the graph then, $\mathcal{O}(|E| \cdot f*)$, because the flow needs to increase by at least 1 in each iteration and each can be done in $\mathcal{O}(|E|)$ time.

Runtime of Edmonds-Karp $\mathcal{O}(|V||E|)$ iterations, each of which can be done in $\mathcal{O}(|E|)$ times, so $\mathcal{O}(|V||E|^2)$

Classes of Problems

Shortest-Path Problem

- Representation of simple graph with nodes representing the actual states (e.g. city).
- Representation of state space with nodes representing the current state of the system. (e.g. city and money left) $\rightarrow$ City is connected to neighbouring cities with the states that the system can take from here. ($A_{5\$} \underset{-2\$}{\rightarrow} B_{3\$}$)
- Cycle detection problem - e.g. figuring out if we can generate ∞ revenue. The Bellman–Ford algorithm will converge after iterating through the edges at most $|V| - 1$ times (as there cannot be more edges in a shortest path) if and only if there is no such negative cycle.

Bipartite Matching Problem

- Two classes of nodes that need to be matched in an optimal fashion. Use either Ford-Fulkerson or Edmonds-Karp depending on the runtime.

Minimum Spanning Tree Problem

- Finding the minimal tree to connect all nodes of a tree/subtree. Use Kruskal or Prim for a dense graph.

Failure Resilience Problem

- Edge failure resilience - Maximal flow -1 between A and B gives the number of edges that can be removed before the connection fails.
- Node failure resilience - Replace each node with an edge - Model failure state by this edge weight.

Parallel Programming

Amdahl assumes a fixed relative sequential portion (λ), Gustafson assumes a fixed absolute sequential part.
Amdahl: $S_A = \frac{1}{\lambda + \frac{1-\lambda}{p}}$ **Gustafson**: $S_G = p - \lambda(p - 1)$

Speedup calculation

$T_p \leq \frac{T_1}{p} + T_\infty$ | $S_p \geq \frac{T_1}{T_p}$
$T_\infty =$ longest single path | $S_\infty = \frac{T_1}{T_\infty}$

Performance Model

We have p processors and the corresponding execution time T_p. T_∞: The span of the execution network or longest path. Thus the time needed if we have an infinite number of processors.

$$\text{Parallelism} = T_1/T_\infty$$

Lower Bound Laws

$$T_p \geq T_1/p \quad \text{Work law}$$
$$T_p \geq T_\infty \quad \text{Span law}$$

Parallel Programming in C++

`std::mutex`

- Owned when `lock` was called until `unlock` is called.
- When owned all other threads block (halt) when `lock` is called.

`std::unique_lock`

```
std::unique_lock<std::mutex> lck (mtx);//Locked
lck.unlock();
```

- In locked state upon construction unless deferred using `std::defer_lock`.
- Will handle unlocking upon destruction like `std::lock_guard` but additionally provided locking and unlocking capabilities.

`std::condition_variable`

```
std::condition_variable cv;
std::unique_lock<std::mutex> lk(m);
cv.wait(lk, []{return x == 1;});
lk.unlock();
cv.notify_one();
cv.notify_all();
```

- `std::condition_variable` takes a `std::unique_lock<std::mutex>` which protects the shared variable.
- Releases the `std::mutex` and executes a wait operation on the current thread if the condition does not hold.
- Upon `notify_all` or `notify_one` wakeup it will reacquire the mutex atomically and check the condition.

Race Conditions

Data Race

Bad synchronisation of a shared resource, e.g. two writing processes at the same time.

Bad Interleaving

Unlucky order of execution of e.g. two threads even though the shared resource is otherwise well synchronised.

Complexities

Algorithm	Time Complexity Best	Time Complexity Average	Time Complexity Worst	Space Complexity Worst
Quicksort	$\Omega(n \cdot log(n))$	$\Theta(n \cdot log(n))$	$\mathcal{O}(n^2)$	$\mathcal{O}(log(n))$
Mergesort	$\Omega(n \cdot log(n))$	$\Theta(n \cdot log(n))$	$\mathcal{O}(n \cdot log(n))$	$\mathcal{O}(n)$
Heapsort	$\Omega(n \cdot log(n))$	$\Theta(n \cdot log(n))$	$\mathcal{O}(n \cdot log(n))$	$\mathcal{O}(1)$
Bubble Sort	$\Omega(n)$	$\Theta(n^2)$	$\mathcal{O}(n^2)$	$\mathcal{O}(1)$
Insertion Sort	$\Omega(n)$	$\Theta(n^2)$	$\mathcal{O}(n^2)$	$\mathcal{O}(1)$
Selection Sort	$\Omega(n^2)$	$\Theta(n^2)$	$\mathcal{O}(n^2)$	$\mathcal{O}(1)$
Shell Sort	$\Omega(n \cdot log(n))$	$\Theta(n \cdot log(n)^2)$	$\mathcal{O}(n \cdot log(n)^2)$	$\mathcal{O}(1)$
Bucket Sort	$\Omega(n + k)$	$\Theta(n + k)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n)$
Radix Sort	$\Omega(n \cdot k)$	$\Theta(n \cdot k)$	$\mathcal{O}(n \cdot k)$	$\mathcal{O}(n + k)$

Data Structure	**Time Complexity**				
Average	Average				
	Access	Search	Insertion	Deletion	
Array	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	
Stack	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	
Queue	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	
Linked-List	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	
Skip-List	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	
Hash-Table	N/A	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	
Binary Search Tree	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	
AVL Tree	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	$\Theta(log(n))$	
	Worst				Space Complexity
	Access	Search	Insertion	Deletion	Worst
Array	$\mathcal{O}(1)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
Stack	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$
Queue	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$
Linked-List	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$
Skip-List	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n \cdot log(n))$
Hash-Table	N/A	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
Binary Search Tree	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
AVL Tree	$\mathcal{O}(log(n))$	$\mathcal{O}(log(n))$	$\mathcal{O}(log(n))$	$\mathcal{O}(log(n))$	$\mathcal{O}(n)$