Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Mohammad Abdulaziz, Ph.D.

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Functional Data Structures

Exercise Sheet 9

Exercise 9.1 Indicate Unchanged by Option

Write an insert function for red-black trees that either inserts the element and returns a new tree, or returns None if the element was already in the tree

fun *ins'* :: "$'a$::*linorder* $\Rightarrow$ $'a$ *rbt* $\Rightarrow$ $'a$ *rbt option*"
lemma "*invc t* $\Longrightarrow$ *case ins' x t of None* $\Rightarrow$ *ins x t* $=$ *t* | *Some t'* $\Rightarrow$ *ins x t* $=$ *t'*"

Exercise 9.2 Joining 2-3-Trees

Write a join function for 2-3-trees: The function shall take two 2-3-trees l and r and an element x, and return a new 2-3-tree with the inorder-traversal $l\ x\ r$.

Write two functions, one for the height of l being greater, the other for the height of r being greater.

height r greater

fun *joinL* :: "$'a$ *tree23* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree23* $\Rightarrow$ $'a$ *upI*"
lemma *complete_joinL*: "$[\![$ *complete l*; *complete r*; *height l* $\leq$ *height r* $]\!]$ $\Longrightarrow$
complete (*treeI* (*joinL l x r*)) $\wedge$ *height*(*joinL l x r*) $=$ *height r*"
lemma *inorder_joinL*: "$[\![$ *complete l*; *complete r*; *height l* $\leq$ *height r* $]\!]$ $\Longrightarrow$ *inorder* (*treeI* (*joinL l x r*)) $=$ *inorder l* @*x* # *inorder r*"

height l greater

fun *joinR* :: "$'a$ *tree23* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree23* $\Rightarrow$ $'a$ *upI*"
lemma *complete_joinR*: "$[\![$ *complete l*; *complete r*; *height l* $\geq$ *height r* $]\!]$ $\Longrightarrow$
complete (*treeI* (*joinR l x r*)) $\wedge$ *height*(*joinR l x r*) $=$ *height l*"

Note the generalization: We augmented the lemma with a statement about the height of the result.

lemma *inorder_joinR*: "$[\![$ *complete l*; *complete r*; *height l* $\geq$ *height r* $]\!]$ $\Longrightarrow$ *inorder* (*treeI* (*joinR l x r*)) $=$ *inorder l* @*x* # *inorder r*"

Combine both functions

fun *join* :: "$'a$ *tree23* $\Rightarrow$ $'a$ $\Rightarrow$ $'a$ *tree23* $\Rightarrow$ $'a$ *tree23*"

lemma "$\llbracket$ *complete l*; *complete r* $\rrbracket \Longrightarrow$ *complete* (*join l x r*)"
lemma "$\llbracket$ *complete l*; *complete r* $\rrbracket \Longrightarrow$ *inorder* (*join l x r*) = *inorder l* @*x* # *inorder r*"

Homework 9.1 2-3 Tree to Red-Black Tree

Submission until Friday, 26. 6. 2020, 10:00am.

In this task you are to define a function *mk_rbt* which constructs a red-black tree that contains the members of a given 2-3 tree.

fun *mk_rbt* :: "*'a tree23* ⇒ *'a rbt*" **where**

Show that the inorder traversal of the tree constructed by *list_to_rbt* is the same as the given list

lemma *mk_rbt_inorder_btree*: "*Tree2.inorder* (*mk_rbt t*) = *Tree23.inorder t*"

Show that the color of the root node is always black

lemma *mk_rbt_color_btree*: "*color* (*mk_rbt t*) = *Black*"

Show that the returned tree satisfies the height invariant.

lemma *mk_rbt_invh_btree*: "*Tree23.complete t* $\Longrightarrow$ *invh* (*mk_rbt t*)"

Show that the returned tree satisfies the color invariant.

lemma *mk_rbt_invc_btree*: "*invc* (*mk_rbt t*)"

Homework 9.2 Red-Black Tree Property

Submission until Friday, 26. 6. 2020, 10:00am.

In a red-black tree, all paths from a root to any leaf traverse the same number of black nodes. In this exercise you are required to prove that. Consider the following function:

fun *bhs* :: "*'a rbt* ⇒ *nat set*" **where**
"*bhs Tree.Leaf* = {*0*}" |
"*bhs* (*Tree.Node l* (_,*c*) *r*) = (*let H* = *bhs l* ∪ *bhs r in if c*=*Black then Suc* ‘ *H else H*)"

Note that f ‘ s denotes the image of a function f on a set s. With that in mind, the above function encodes the set of numbers of black nodes traversed in all paths from the root to any of the leaves. Prove the following lemma, which formalises the fact that all paths starting at the root and ending at a leaf have the same number of black nodes.

corollary *invh_iff_bhs*: "*invh t* $\longleftrightarrow$ *bhs t* = {*bheight t*}"