Foundations Test 2

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Foundations of Mathematics

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Theorem 1. Given $a \in \mathbb{Z}$, if $5 \mid 2a$ then $5 \mid a$

Proof. Let $a \in \mathbb{Z}$ and $5 \mid 2a$. □

Theorem 3. For all $n \in \mathbb{N}$, $1{\cdot}2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Proof. Let $n \in \mathbb{N}$. We proceed by induction.
Base Case: $n = 1$

$$1 \cdot 2 = \frac{1(1+1)(1+2)}{3} \tag{1}$$
$$1 \cdot 2 = \frac{1(1+1)(1+2)}{3} \tag{2}$$
$$2 = \frac{1(2)(3)}{3} \tag{3}$$
$$2 = \frac{6}{3} \tag{4}$$
$$2 = 2 \tag{5}$$

Inductive Hypothesis: Assume that $1{\cdot}2+2{\cdot}3+\cdots+k(k+1) = \frac{k(k+1)(k+2)}{3}$ for some integer k.
Inductive Step: We must show that $1{\cdot}2+2{\cdot}3+\cdots+k(k+1)+(k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$.
This can be simplified on the right hand side such that it can be restated as,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} \tag{6}$$
$$= \frac{(k^2+2k+k+2)(k+3)}{3} \tag{7}$$
$$= \frac{k^3+3k^2+2k^2+6k+k^2+3k+2k+6}{3} \tag{8}$$
$$= \frac{k^3+6k^2+11k+6}{3} \tag{9}$$

We begin by adding $(k+1)(k+2)$ to both sides of the equation in the inductive hypothesis to get $1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2)=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$. It can be then shown that,

$$\begin{aligned}
1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) &= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2) && (10)\\
&= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} && (11)\\
&= \frac{(k^2+k)(k+2)+3(k^2+3k+2)}{3} && (12)\\
&= \frac{k^3+2k^2+k^2+2k+3k^2+9k+6}{3} && (13)\\
&= \frac{k^3+3k^2+2k+3k^2+9k+6}{3} && (14)\\
&= \frac{k^3+6k^2+11k+6}{3} && (15)\\
& && (16)
\end{aligned}$$

This is the same as the equation we got in the inductive step. Therefore, by process of induction, for all $n\in\mathbb{N}$, $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$. □

Theorem 4. Let $a,b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$. If $a\equiv_n b$ then $ac\equiv_n bc$

Proof. Let $a,b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$. Let $a\equiv_n b$. Then by definition 7.79 there exists an integer k such that $a-b=nk$. It can be shown that by multiplying each side of the equation by c that $ac-bc=nkc$. Since kc is an integer, $ac\equiv_n bc$ by definition 7.79. □

Theorem 5. For all integers $n\geq 0, 24\mid(5^{2n}-1)$.

Proof. Let $n\in\mathbb{Z}$ such that $n\geq 0$. We proceed by induction.
Base Case: $n=0$

$$\begin{aligned}
5^{2(0)}-1 &= 24k && (17)\\
5^0-1 &= 24k && (18)\\
1-1 &= 24k && (19)\\
0 &= 24k && (20)
\end{aligned}$$

Inductive Hypothesis: Assume that $24\mid(5^{2k}-1)$ for some integer k. This can be restated as $5^{2k}-1=24k$ for some integer k.
Inductive Step: We must show that $24\mid(5^{2(k+1)}-1)$. We begin by multiplying each side of the inductive hypothesis by 5^{2k} to get,

$$\begin{aligned}
5^4\cdot(5^{2k}-1) &= 24k\cdot 5^4 && (21)\\
5^{2(k+1)}-5^4 &= 24k\cdot 5^4 && (22)
\end{aligned}$$

We then add $24k$ to both sides of the equation to get,

$$5^{2(k+1)} - 5^4 + 24k = 24k \cdot 5^4 + 24k \tag{23}$$
$$5^{2(k+1)} - 5^4 + 24k = 24k(5^4 + 1) \tag{24}$$

□

Theorem 6.

Proof. □

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Proof. □

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Theorem 10. If A,B are sets, then $A \cap (B \setminus A) = \emptyset$

Proof. 6. Let A, B be sets.
12. Suppose, for the sake of contradiction,
3. that $A \cap (B \setminus A) \neq \emptyset$.
5. Hence, there exists an $x \in A \cap (B \setminus A)$.
9. By the definition of intersection,
1. $x \in A$ and $x \in B \setminus A$.
7. By the definition of set difference,
11. $x \in B$ and $x \notin A$.
8. All together, this implies that $x \in A$
2. and $x \notin A$,
10. which is a contradiction.
4. Therefore, $A \cap (B \setminus A) = \emptyset$.

□