A2-Final_CSaben
Abstract Algebra 1

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Problem 1. Let G and H be groups and consider their direct product $G \times H$. Suppose that e is the identity in G. Prove that the subset $S = \{(e, h) : h \in H\}$ of $G \times H$ is a subgroup of $G \times H$.

Proof.

1. $S \neq \emptyset$ since for some $h_1 \in H$, $(e, h_1) \in S$.
2. Let $(e, h_1), (e, h_2) \in S$ for some $h_1, h_2 \in H$. Then $(e, h_1) \cdot (e, h_2) = (e, h_1 h_2) \in S$ since $h_1 h_2 \in H$.
3. Let $(e, h) \in S$. Then $(e, h)^{-1} = (e, h^{-1}) \in S$ since $h^{-1} \in H$.

Therefore, S is a subgroup of $G \times H$.

□

Problem 2. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined in Exercise B4 on page 63 . Prove that f is an element of the symmetric group $S_{\mathbb{Z}}$.

Hints: To prove that f is injective, suppose that $f(n_1) = f(n_2)$, where $n_1, n_2 \in \mathbb{Z}$. Then, to show that $n_1 = n_2$, you will have to consider cases about possible parities (even or odd) of n_1 and n_2. You will also have to consider cases to prove that f is surjective.

Proof. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(n) = n + 1$ if n is even and $f(n) = n - 1$ if n is odd. We will show that f is injective and surjective.

1. Let $n_1, n_2 \in \mathbb{Z}$ such that $f(n_1) = f(n_2)$. Then we have two possible cases:

   (a) If n_1 and n_2 are both even, then $n_1 + 1 = n_2 + 1$ which implies $n_1 = n_2$.

   (b) If n_1 and n_2 are both odd, then $n_1 - 1 = n_2 - 1$ which implies $n_1 = n_2$.

   Also, we have two impossible cases:

   (a) If n_1 is odd and n_2 is even, then $n_1 - 1 = n_2 + 1$ which is impossible since we assume $f(n_1) = f(n_2)$.

   (b) If n_1 is even and n_2 is odd, then $n_1 + 1 = n_2 - 1$ which is impossible since we assume $f(n_1) = f(n_2)$.

Thus, $n_1 = n_2$ when n_1 and n_2 are both even or both odd, in other words, $f(n_1) = f(n_2) \implies n_1 = n_2$. Therefore, f is injective.

2. Let $n \in \mathbb{Z}$. Then we have two cases:

   (a) If n is even, then $f(n-1) = n - 1 + 1 = n$ for some $m = n - 1 \in \mathbb{Z}$.

   (b) If n is odd, then $f(n+1) = n + 1 - 1 = n$ for some $m = n + 1 \in \mathbb{Z}$.

   Therefore, f is surjective since for all $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $f(m) = n$.

Thus, f is a bijection and therefore an element of $S_{\mathbb{Z}}$.

$\square$

Problem 3. Let D_4 be the group of symmetries of the square. Construct a Cayley table for D_4. In the table, use the symbols from Class Notes for Chapter 7. In the left column and top row, order the symbols as follows: $R_0, R_{90}, R_{180}, R_{270}, \rho_A, \rho_B, \rho_H, \rho_V$. The table must be created in ^{2} TEX. In this problem, I only need to see the table.

$\circ$	R_0	R_{90}	R_{180}	R_{270}	ρ_A	ρ_B	ρ_H	ρ_V
R_0	R_0	R_{90}	R_{180}	R_{270}	ρ_A	ρ_B	ρ_H	ρ_V
R_{90}	R_{90}	R_{180}	R_{270}	R_0	ρ_V	ρ_H	ρ_A	ρ_B
R_{180}	R_{180}	R_{270}	R_0	R_{90}	ρ_B	ρ_A	ρ_V	ρ_H
R_{270}	R_{270}	R_0	R_{90}	R_{180}	ρ_H	ρ_V	ρ_B	ρ_A
ρ_A	ρ_A	ρ_H	ρ_B	ρ_V	R_0	R_{180}	R_{270}	R_{90}
ρ_B	ρ_B	ρ_V	ρ_A	ρ_H	R_{180}	R_0	R_{270}	R_{90}
ρ_H	ρ_H	ρ_B	ρ_V	ρ_A	R_{270}	R_{90}	R_0	R_{180}
ρ_V	ρ_V	ρ_A	ρ_H	ρ_B	R_{90}	R_{270}	R_{180}	R_0