

A Modern Physics Review

Clark Saben

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1.1 Representations of Spin Half

(a) Write \hat{S}_z in the S_x basis, both in Dirac Notation and as a matrix. Hint: However you do this, you will need to know how \hat{S}_z acts on the eigenstates of S_x .

Find \hat{S}_z in the S_x basis:

$$\hat{1}_x \hat{S}_z \hat{1}_x = (|+x\rangle\langle+x| + |-x\rangle\langle-x|) \hat{S}_z (|+x\rangle\langle+x| + |-x\rangle\langle-x|)$$

Need \hat{S}_z (hint):

$$\begin{aligned}\hat{1}_z |\pm x\rangle &= | + z \rangle \langle + z | \pm x \rangle + | - z \rangle \langle - z | \pm x \rangle \\ &= \frac{1}{\sqrt{2}} (| + z \rangle | \pm | - z \rangle) \\ \implies \hat{S}_z |\pm x\rangle &= \frac{\hbar}{2\sqrt{2}} (\langle + z | \pm \langle - z |)\end{aligned}$$

So,

$$\begin{aligned}\hat{1}_x \hat{S}_z \hat{1}_x &= (|+x\rangle\langle+x| + |-x\rangle\langle-x|) \left(\frac{\hbar}{2} |-x\rangle\langle+x| + \frac{\hbar}{2} |+x\rangle\langle-x| \right) \\ &= \frac{\hbar}{2} (|+x\rangle\langle-x| + |-x\rangle\langle+x|) \\ \implies \mathbb{S}_z^{(S_x)} &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

(b) Verify the equations (in the S_x basis),
using your matrix representation for \hat{S}_z in the S_x basis.

$$\hat{S}_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$$

$$\begin{aligned} \mathbb{S}_z^{(S_x)} \vec{v}_{+z}^{(S_x)} &= \frac{\hbar}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\hbar}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{\hbar}{2} \vec{v}_{+z}^{(S_x)} \end{aligned}$$

$$\hat{1}_x | -z \rangle = | +x \rangle \langle +x | -z \rangle + | -x \rangle \langle -x | -z \rangle$$

where

$$\langle +x | -z \rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle -x | -z \rangle = \frac{-1}{\sqrt{2}}$$

$$\begin{aligned} \Rightarrow \mathbb{S}_z^{(S_x)} \vec{v}_{-z}^{(S_x)} &= \frac{\hbar}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{\hbar}{2\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \frac{-\hbar}{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{-\hbar}{2} \vec{v}_{-z}^{(S_x)} \end{aligned}$$

(c) Write the projection operator \hat{P}_{+z} in the S_x basis, both in Dirac Notation and as a matrix.

Find $\hat{P}_{+z}^{(S_x)}$ and \hat{P}_{+z} in the S_x basis:

$$\begin{aligned}\hat{P}_{+z}^{(S_x)} &= \hat{1}_x \hat{P}_{+z} \hat{1}_x \\ &= (|+x\rangle\langle+x| + |-x\rangle\langle-x|) \left(\frac{1}{\sqrt{2}}|+z\rangle\langle+x| + \frac{1}{\sqrt{2}}|+z\rangle\langle-x| \right) \\ &= \frac{1}{2} (|+x\rangle\langle+x| + |+x\rangle\langle-x| + |-x\rangle\langle+x| + |-x\rangle\langle-x|)\end{aligned}$$

$$\mathbb{P}_{+z}^{(S_x)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(d) Find eigenvalues and eigenvectors of your matrix representation of \hat{P}_{+z} in the S_x basis, thus verifying that $\hat{P}_{+z}|+z\rangle = |+z\rangle$.

Diagonalize $\mathbb{P}_{+z}^{(S_x)}$ and verify $\mathbb{P}_{+z}^{S_x} \vec{v}_{+z}^{S_x} = \vec{v}_{+z}^{S_x}$.

$$\begin{aligned}\lambda_1 \lambda_2 &= 0 ; \lambda_1 + \lambda_2 = 1 \\ \implies \lambda_1 &= 1 \text{ and } \lambda_2 = 0\end{aligned}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

(e) Repeat parts (c) and (d) for \hat{P}_{-z} .