A Modern Physics Review

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1.1 Representations of Spin Half

(a) Write $\hat{S}_z$ in the S_x basis, both in Dirac Notation and as a matrix. Hint: However you do this, you will need to know how $\hat{S}_z$ acts on the eignstates of S_x.

Find $\hat{S}_z$ in the S_x basis:

$$\hat{1}_x \hat{S}_z \hat{1}_x = (|+x\rangle\langle +x| + |-x\rangle\langle -x|)\, \hat{S}_z\, (|+x\rangle\langle +x| + |-x\rangle\langle -x|)$$

Need $\hat{S}_z$ (hint):

$$\begin{aligned} \hat{1}_z|\pm x\rangle &= |+z\rangle\langle +z|\pm x\rangle + |-z\rangle\langle -z|\pm x\rangle \\ &= \frac{1}{\sqrt{2}}\,(|+z\rangle| \pm |-z\rangle) \\ \implies \hat{S}_z|\pm x\rangle &= \frac{\hbar}{2\sqrt{2}}\,(\langle +z| \pm \langle -z|) \end{aligned}$$

So,

$$\begin{aligned} \hat{1}_x \hat{S}_z \hat{1}_x &= (|+x\rangle\langle +x| + |-x\rangle\langle -x|) \left(\frac{\hbar}{2}|-x\rangle\langle +x| + \frac{\hbar}{2}|+x\rangle\langle -x|\right) \\ &= \frac{\hbar}{2}\,(|+x\rangle\langle -x| + |-x\rangle\langle +x|) \\ \implies \mathbb{S}_z^{(S_x)} &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(b) Verify the equations (in the S_x basis), using your matrix representation for $\hat{S}_z$ in the S_x basis.

$$\hat{S}_z| \pm z\rangle = \pm\frac{\hbar}{2}| \pm z\rangle$$

$$\begin{aligned}
\mathbb{S}_z^{(S_x)}\vec{v}_{+z}^{(S_x)} &= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \frac{\hbar}{2}\vec{v}_{+z}^{(S_x)}
\end{aligned}$$

$$\hat{1}_x| - z\rangle = | + x\rangle\langle +x| - z\rangle + | - x\rangle\langle -x| - z\rangle$$

where

$$\langle +x| - z\rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle -x| - z\rangle = \frac{-1}{\sqrt{2}}$$

$$\begin{aligned}
\Longrightarrow \mathbb{S}_z^{(S_x)}\vec{v}_{-z}^{(S_x)} &= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= \frac{-\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \\
&= \frac{-\hbar}{2}\vec{v}_{-z}^{(S_x)}
\end{aligned}$$

(c) Write the projection operator $\hat{P}_{+z}$ in the S_x basis, both in Dirac Notation and as a matrix.

Find $\hat{P}^{(S_x)}_{+z}$ and $\hat{P}_{+z}$ in the S_x basis:

$$\begin{aligned}
\hat{P}^{(S_x)}_{+z} &= \hat{1}_x \hat{P}_{+z} \hat{1}_x \\
&= (|+x\rangle\langle +x| + |-x\rangle\langle -x|) \left(\frac{1}{\sqrt{2}} |+z\rangle\langle +x| + \frac{1}{\sqrt{2}} |+z\rangle\langle -x| \right) \\
&= \frac{1}{2} (|+x\rangle\langle +x| + |+x\rangle\langle -x| + |-x\rangle\langle +x| + |-x\rangle\langle -x|)
\end{aligned}$$

$$\mathbb{P}^{(S_x)}_{+z} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(d) Find eigenvalues and eigenvectors of your matrix representation of $\hat{P}_{+z}$ in the S_x basis, thus verifying that $\hat{P}_{+z}|+z\rangle = |+z\rangle$.

Diagnolize $\mathbb{P}^{(S_x)}_{+z}$ and verify $\mathbb{P}^{S_x}_{+z} \vec{v}^{S_x}_{+z} = \vec{v}^{S_x}_{+z}$.

$$\begin{aligned}
&\lambda_1 \lambda_2 = 0 \; ; \; \lambda_1 + \lambda_2 = 1 \\
&\implies \lambda_1 = 1 \text{ and } \lambda_2 = 0
\end{aligned}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

(e) Repeat parts (c) and (d) for $\hat{P}_{-z}$.