Writing Assignment 7

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Foundations of Mathematics

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Theorem WA 7.1. If $n \in \mathbb{N}$, then $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$. This will be a total of one proof.

Proof. Let $n \in \mathbb{N}$. We proceed by induction,

$$\frac{1}{1^2} \leq 2 - \frac{1}{1}$$
$$1 = 1$$

Next, we proceed to our inductive step. Assume that for $k \in \mathbb{N}$, we have

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

By adding $\frac{1}{(n+1)^2}$ to both sides of the inequality, we get

$$\begin{aligned}
1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&\leq \frac{2(k+1)^2k - (k+1)^2 + k}{k(k+1)^2} \\
&\leq \frac{2k^3 + 4k^2 + 2k - k^2 - k}{k(k+1)^2} \text{ (drop -1 in numerator)} \\
&\leq \frac{2k^3 + 4k^2 + 2k - k(k+1)}{k(k+1)^2} \\
&\leq \frac{2(k^2+1)^2k}{k(k+1)^2} - \frac{k(k+1)}{k(k+1)^2} \\
&\leq 2 - \frac{1}{k+1}
\end{aligned}$$

It can be seen that in the $k+1$ step that,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Thus, by induction, we have proven that for $n \in \mathbb{N}$, we have

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

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