Foundations Midterm

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Foundations of Mathematics

March 13, 2023

Theorem 1. Assume $a, b \in \mathbb{Z}$. If $a^2\left(b^2 - 2b\right)$ is odd, then a and b is odd.

Proof. Let a and b be odd and $a^2\left(b^2 - 2b\right)$ be odd. By definition 2.1 if a and b are even then it can be shown,

$$\begin{aligned} a^2\left(b^2 - 2b\right) &= 2k^2[(2j)^2 - 2\left(2j\right)^2] \\ &= \left(4k^2\right)\left[4j^2 - 4j\right] \\ &= 2\left(2k^2\right)\left(2j^2 - 2j\right), \end{aligned}$$

which is an even number. This is a false statement. Therefore if $a^2\left(b^2 - 2b\right)$ is odd then a and b must also be odd. $\square$

Theorem 2. Given an integer a, if $7|4a$ then $7|a$.

Proof. Let a be an integer and $7|4a$. By definition 2.1, since $7 \nmid 4$ then 7 must divide a for $7|4a$ to be true. Therefore, if $7|4a$ then $7|a$. $\square$

Theorem 3. If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

Proof. Let $n \in \mathbb{Z}$ and $5n^2 + 3n + 7$ is odd. It can then be shown that,

$$5n^2 + 3n + 7 = 2k \text{ for some integer k} \tag{1}$$

$$\frac{5n^2}{2} + \frac{3n}{2} + \frac{7}{2} = k \tag{2}$$

$$\tag{3}$$

This is false statement because $\frac{7}{2}$ doesn't allow $\frac{5n^2}{2} + \frac{3n}{2} + \frac{7}{2}$ to be an integer for any n. Therefore, if $n \in \mathbb{Z}$ then $5n^2 + 3n + 7$ is odd. $\square$

Theorem 5. Assume $a \in \mathbb{Z}$. Then $a^2 \mid a$ if and only if $a \in \{-1, 0, 1\}$

Proof. Let $a \in \mathbb{Z}$. Using proof by cases we will show that $a^2 \mid a$ if and only if $a \in \{-1, 0, 1\}$.

Firstly, we will show if $a^2 \mid a$ then $a \in \{-1, 0, 1\}$. Consider the case a is non-zero and greater than 1. Then $a^2 \mid a$ implies $\frac{a}{a^2} \in \mathbb{Z}$ which is false. Similarly, when a is non-zero and less than -1, then $a^2 \mid a$ implies $\frac{a}{a^2} \in \mathbb{Z}$ which is false. Hence, a must be either -1, 0, or 1 for $a^2 \mid a$ to be true.

Secondly, we will show if $a \in \{-1, 0, 1\}$ then $a^2 \mid a$. We can prove this is true for each element directly by using the definition of divisibility for some integer k,

$$\begin{aligned}
&\text{Case 1: } a = -1 \\
&-1^2 = 1k \\
&\text{Case 2: } a = 0 \\
&\ \ 0^2 = 0k \\
&\text{Case 3: } a = 1 \\
&\ \ 1^2 = 1k
\end{aligned}$$

All three cases are true, therefore $a^2 \mid a$ if and only if $a \in \{-1, 0, 1\}$. $\square$

Theorem 6. If A, B, C, D are sets and $C \subseteq A$ and $D \subseteq B$, then $D \backslash A \subseteq B \backslash C$.

Proof. Let A, B, C, D be sets and $C \subseteq A$ and $D \subseteq B$. Let $x \in D \backslash A$. Then $x \in D$ and by definition 3.5, $x \in B$. Furthermore, because $C \subseteq A$ it follows that $x \in B \backslash C$. Therefore $D \backslash A \subseteq B \backslash C$.

Theorem 7. If we define the sets $A = \{12a + 4b : a, b \in \mathbb{Z}\}$ and $B = \{4c : c \in \mathbb{Z}\}$, then $A \subseteq B$.

Proof. Let $A = \{12a + 4b : a, b \in \mathbb{Z}\}$ and $B = \{4c : c \in \mathbb{Z}\}$. Let $x \in A$ such that $x = 12a + 4b$. It can be shown that,

$$\begin{aligned}
x &= 12a + 4b \\
&= 4(3a + b) \\
&= 4k \text{ for some integer k}
\end{aligned}$$

Hence, $4k \in B$. Therefore, if $A = \{12a + 4b : a, b \in \mathbb{Z}\}$ and $B = \{4c : c \in \mathbb{Z}\}$, then $A \subseteq B$. $\square$

Theorem 8. Theorem 8. Given integers a, b, and c, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

Proof. :
8. Let $a, b, c \in \mathbb{Z}$
3. such that $a^2 \mid b$ and $b^3 \mid c$.
11. By the definition of divisibility, this implies that there exists integers k and ℓ such that,
6. $b = a^2 k$ and $c = b^3 \ell$.
10. Now, substituting b^3, we see that,
2.

$$\begin{aligned} c &= b^3 \ell \\ &= a^6 k^3 \ell \\ &= a^6 \text{ m}, \end{aligned}$$

5. where $m = k^3 \ell$ is an integer.
1. Cubing both sides of $b = a^2 k$ we obtain,
4. $b^3 = a^6 k^3$.
7. Hence $c = a^6 m$,
9. and $a^6 \mid c$ by the definition of divisibility.

□
