A5-Draft_CSaben

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1. (7 points) Consider the alternating group A_4 and its subgroup $H = \langle (134) \rangle$.

Problem 1a. (a) Calculate the right cosets of H in A_4. Do not repeat. Each right coset must be written as $H\alpha = \{\ldots\}$, for example,

$$H(1) = \{(1), (134), (143)\}$$

Hint: Do not use any elements that are not in A_4. For example, $H(123)$ is a coset of H in A_4, but $H(12)$ is not.

$$\begin{aligned} H(1) &= \{(1), (134), (143)\} \\ H(123) &= \{(123), (1234), (124)\} \\ H(132) &= \{(132), (14)(23), (243)\} \\ H(421) &= \{(421), (423), (42)(13)\} \end{aligned}$$

Problem 1b. (b) Show that H is not a normal subgroup of A_4.

Hint: Again, do not use any elements that are not in A_4.

For H to be a normal subgroup of A_4, we must have $gH = Hg$ for all $g \in A_4$. However, we can see that $H(123) \neq (123)H$ since $(123)H = \{(123), (234), (14)(23)\}$ and $H(123) = \{(123), (1234), (124)\}$.

Problem 1c. (c) What is the index $(A_4 : H)$ of H in A_4 ? Justify your answer.

The index of H in A_4 is the number of right cosets of H in A_4. Since we have 4 cosets, the index is 4. In other words, index $= (A_4 : H) = 4$.

2. (7 points) Let $G = \{x \in \mathbb{R} : x \neq -1\}$. Define an operation $*$ on G by $a*b = a+b+ab$. For example, $2*3 = 2+3+2\cdot 3 = 11$. Then $\langle G, *\rangle$ is a group. (Take this as given.) Recall that $\mathbb{R}^*$ is the group of nonzero real numbers with ordinary multiplication.

Problem 2a. (a) Prove that $f : \mathbb{R}^* \to G$ defined by $f(x) = x - 1$ is a homomorphism.

Proof. Let $a, b \in \mathbb{R}^*$. Then we have

$$f(ab) = ab - 1$$

and,

$$\begin{aligned} f(a)f(b) &= (a-1)*(b-1) \\ &= (a-1)+(b-1)+(a-1)(b-1) \\ &= a+b-2+ab-a-b+1 \\ &= ab-1 \end{aligned}$$

Therefore, $f(ab) = f(a)f(b)$ and f is a homomorphism. □

Problem 2b. (b) Find the kernel of f. Justify your answer.

The kernel of f is the set of all elements in $\mathbb{R}^*$ that map to the identity in G. Since $f(x) = x - 1$, we have $f(x) = 0$ when $x = 1$. Thus, the kernel of f is $\{1\}$.

3. (6 points) Let H be any subgroup of G and let K be a normal subgroup of G. Define a subset S of G by

$$S = \{hk : h \in H \text{ and } k \in K\}$$

Problem 3. Prove that S is a subgroup of G.

Hint: Recall that K is a normal subgroup of G if and only if $Ka = aK$ for every $a \in G$.

Proof. Note that S is nonempty since H and K are nonempty. Note K is a normal subgroup of G if and only if $Ka = aK$ for every $a \in G$. Let $a, b \in S$. Then $a = hk$ and $b = h'k'$ for some $h, h' \in H$ and $k, k' \in K$. Then we have,

$$\begin{aligned} ab^{-1} &= (hk)(h'k')^{-1} \\ &= (hk)(k'^{-1}h'^{-1}) \\ &= h(kk'^{-1})h'^{-1} \end{aligned}$$

Since H is a subgroup of G, $h(kk'^{-1})h'^{-1} \in H$. Since K is a normal subgroup of G, $kk'^{-1} \in K$. So, $h(kk'^{-1})h'^{-1} \in H$ and $kk'^{-1} \in K$. Therefore, S is a subgroup of G. $\square$