

# Foundations Test 2

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Foundations of Mathematics

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## 1

**Theorem 1.** If  $n \in \mathbb{Z}$  and  $n \geq 0$ , then  $\sum_{i=0}^n i \cdot i! = (n+1)! - 1$ .

**Theorem 2 (complete).** The inequality  $2^n \leq 2^{n+1} - 2^{n-1} - 1$  holds for all  $n \in \mathbb{N}$ .

*Proof.* Let  $n \in \mathbb{N}$ . We will show that the inequality  $2^n \leq 2^{n+1} - 2^{n-1} - 1$  holds for all  $n$  via induction. For our base case,  $n = 1$ ,

$$\begin{aligned} 2^1 &\leq 2^{1+1} - 2^{1-1} - 1 \\ 2 &\leq 4 - 1 - 1 \\ 2 &\leq 2 \end{aligned}$$

which is true, so we proceed with our inductive step. Let us assume that for some  $k \in \mathbb{N}$  the inequality  $2^k \leq 2^{k+1} - 2^{k-1} - 1$  holds for all  $k$ . Since  $2^k$  is going to be less than  $2^{k+1} - 2^{k-1} - 1$  it will also be less than that plus an additional  $2^{k-1} + 1$ . Hence we can correctly re-write our inequality from,

$$2^k \leq 2^{k+1} - 2^{k-1} - 1$$

to,

$$\begin{aligned} 2^k &\leq 2^{k+1} - 2^{k-1} - 1 + (2^{k-1} + 1) \\ 2^k &\leq 2^{k+1} \\ &\leq (2)2^k \end{aligned}$$

Clearly, for any natural number,  $k$ ,  $2^k \leq (2)2^k$ . Therefore, the inequality  $2^n \leq 2^{n+1} - 2^{n-1} - 1$  holds for all  $n \in \mathbb{N}$ . □

**Theorem 3 (complete).** Define the relation  $R$  on  $\mathbb{Z}$  such that  $xRy$  if and only if  $3x - 5y$  is even. Then  $R$  is an equivalence relation.

*Proof.* Let  $x$  and  $y \in \mathbb{Z}$  the define relation  $R$  on  $\mathbb{Z}$  such that  $xRy$  if and only if  $3x - 5y$  is even. We will show  $R$  is an equivalence relation.

To show  $R$  is reflexive, let  $x \in \mathbb{Z}$ . Then  $xRx$  if and only if  $3x - 5x$  is even. By definition 2.4 it can be seen that,

$$\begin{aligned} 3x - 5x &= 2(-k) \text{ (for some } -k \in \mathbb{Z}) \\ -2x &= -2k \end{aligned}$$

which is true for all  $x \in \mathbb{Z}$ , so  $R$  is reflexive.

To show  $R$  is symmetric, let  $x$  and  $y \in \mathbb{Z}$ . Then  $xRy$  if and only if  $3x - 5y$  is even. By definition 2.4 it can be seen that,

$$3x - 5y = 2(-k) \text{ (for some } -k \in \mathbb{Z})$$

We can add  $2x-2y$  to both sides to get,

$$\begin{aligned} 5x - 3y &= 2(-k) + 2x - 2y \\ &= -2(k - x + y) \end{aligned}$$

We can then multiply each side by  $(-1)$  to get,

$$\begin{aligned} (-1)5x - 3y &= (-1)(-2(k - x + y)) \\ -5x + 3y &= 2(k - x + y) \\ 3y - 5x &= 2(k - x + y) \end{aligned}$$

where  $(k - x + y)$  is some integer. Therefore, since  $3y - 5x$  is even (Definition 2.4),  $yRx$ , so  $R$  is symmetric.

To show  $R$  is transitive, let  $x, y, z \in \mathbb{Z}$  and  $xRy$  and  $yRz$ . Then  $3x - 5y$  is even and  $3y - 5z$  is even. By definition 2.4 it can be seen that,

$$\begin{aligned} 3x - 5y &= 2(k) \text{ (for some } k \in \mathbb{Z}) \\ 3y - 5z &= 2(l) \text{ (for some } l \in \mathbb{Z}) \end{aligned}$$

We can add the two equations to get,

$$\begin{aligned} 3x - 5y + 3y - 5z &= 2(k) + 2(l) \\ 3x - 5z - 2y &= 2(k + l) \\ &= 2(k + l + y) \end{aligned}$$

We see that  $3x - 5z$  is even, so  $xRz$  and  $R$  is transitive.

In conclusion,  $R$  is an equivalence relation since  $R$  is reflexive, symmetric, and transitive.  $\square$

## 2

**Theorem 4 (complete but seems wrong).** Let  $a, b \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . If  $a \equiv_n b$  then  $a^2 \equiv_n b^2$ . (recall the definition of  $\equiv_n$  from Definition 7.79)

*Proof.* Let  $a, b \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . We will show that if  $a \equiv_n b$  then  $a^2 \equiv_n b^2$ . By definition 7.79,  $a \equiv_n b$  if and only if  $a - b$  is divisible by  $n$ . It can thus be shown that,

$$a - b = nk \text{ (for some } k \in \mathbb{Z})$$

we can then multiply both sides by  $a + b$  to get,

$$\begin{aligned} a^2 + ab - ba + b^2 &= (a + b)nk \\ a^2 - b^2 &= (a + b)nk \\ a^2 - b^2 &= np \quad \boxed{\text{such that } p = (a + b)k} \end{aligned}$$

Therefore,  $a^2 - b^2$  is divisible by  $n$ . Thus,  $a^2 \equiv_n b^2$  if  $a - b$  is divisible by  $n$ .  $\square$

**Theorem 5.** For an  $n \geq 4$ , one can obtain  $n$  dollars using only \$2 and \$5 bills.

**Theorem 6 (complete).** Define  $\Psi = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$  and define a relation  $\sim$  on  $\Psi$  via  $(a, b) \sim (c, d)$  if and only if  $ad = bc$ . Then  $\sim$  is an equivalence relation.

*Proof.* Let  $\Psi = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$  and define a relation  $\sim$  on  $\Psi$  via  $(a, b) \sim (c, d)$  if and only if  $ad = bc$ . To show that  $\sim$  is an equivalence relation, we must show that  $\sim$  is reflexive, symmetric, and transitive.

To show that  $\sim$  is reflexive, we must show that for all  $(a, b) \in \Psi$ ,  $(a, b) \sim (a, b)$ . It can be shown that,

$$\begin{aligned} (a, b) \sim (a, b) &\Leftrightarrow ab = ba \\ &\Leftrightarrow ab = ab \text{ (commutative property)} \end{aligned}$$

Therefore,  $\sim$  is reflexive.

To show that  $\sim$  is symmetric, we must show that for all  $(a, b) \in \Psi$  and  $(c, d) \in \Psi$ , if  $(a, b) \sim (c, d)$  then  $(c, d) \sim (a, b)$ . It can be shown that,

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

and that,

$$\begin{aligned} (c, d) \sim (a, b) &\Leftrightarrow cb = da \\ &\Leftrightarrow ad = bc \text{ (commutative property)} \end{aligned}$$

Therefore,  $\sim$  is symmetric.

To show that,  $\sim$  is transitive, we must show that for all  $(a, b) \in \Psi$ ,  $(c, d) \in \Psi$ , and  $(e, f) \in \Psi$ , if  $(a, b) \sim (c, d)$  and  $(c, d) \sim (e, f)$  then  $(a, b) \sim (e, f)$ . It can be shown that,

$$\begin{aligned} ad &= bc \\ fad &= fbc \end{aligned}$$

Since  $(c, d) \sim (e, f)$ ,  $cf = de$ . We can re-write  $f$  as  $\frac{de}{c}$ . We can finally show that,

$$\begin{aligned} adf &= bc\left(\frac{de}{c}\right) \\ adf &= bde \\ af &= be, \end{aligned}$$

which would follow from  $(a, b) \sim (e, f)$ . Therefore,  $\sim$  is transitive.

Therefore,  $\sim$  is an equivalence relation.

□

### 3 (complete)

**Problem 1.** Consider the following relation:

$$R = \{(a, a), (a, b), (a, d), (b, d), (c, c), (d, b), (d, c)\}$$

**Problem 1.** .1 What elements must be add to  $R$  to make it reflexive?

To be reflexive,  $(b, b)$  and  $(d, d)$  must be added to  $R$ .

**Problem 1.** .1 What elements must be add to  $R$  to make it symmetric?

To be symmetric,  $(b, a)$ ,  $(d, a)$  and  $(c, d)$  must be added to  $R$ .