A4-Draft_CSaben

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1. (6 points) Let $\langle G_1, \cdot \rangle$ and $\langle G_2, * \rangle$ be groups with identities e_1 and e_2, respectively. Suppose that $f : G_1 \rightarrow G_2$ is an isomorphism. Prove that $f(e_1) = e_2$. Write - when you multiply elements of G_1, and $*$ when you multiply elements of G_2.

Proof. 1 Since $f : G_1 \rightarrow G_2$ is an isomorphism, that is, it preserves the operation the for any elements $a, b \in G_1$, we have $f(a \cdot b) = f(a) * fb$. Note that by definition, $f : G_1 \rightarrow G_2$ is also bijective, and so surjective. Let $b \in G_2$ is the image of some $a \in G_1$ since f is surjective. Then, $b = f(a)$ for some $a \in G_1$. Let $e_1 \in G_1$ and $e_2 \in G_2$ be the identities of each respectve group. Then,

$$\begin{aligned} f(e_1) * f(a) &= f(e_1 \cdot a) \\ &= f(a) \\ &= b \end{aligned}$$

and

$$\begin{aligned} f(e_1) * b &= b \\ &= e_2 * b. \end{aligned}$$

Thus, $f(e_1) * b = e_2 * b$ because $b = f(a)$, and $f(e_1) * f(a) = f(e_1 \cdot a)$ by the isomorphism property of f. Thus, $f(e_1) = e_2$. □

2. (6 points) Let G be a group and let $a \in G$ be an element of order 6 . Construct a Cayley table (written in LTEXX) for the cyclic subgroup $\langle a \rangle$ of G generated by a. Each element in your table must be written as a^r, where $r \in \{0, 1, \ldots, 5\}$. If you wish, you can write a^0 as e, and a^1 as a.

$\cdot$	e	a	a^2	a^3	a^4	a^5
e	e	a	a^2	a^3	a^4	a^5
a	a	a^2	a^3	a^4	a^5	e
a^2	a^2	a^3	a^4	a^5	e	a
a^3	a^3	a^4	a^5	e	a	a^2
a^4	a^4	a^5	e	a	a^2	a^3
a^5	a^5	e	a	a^2	a^3	a^4

3. (8 points) Let D_4 be the group of symmetries of the square.

(a) List all distinct cyclic subgroups of D_4. Write each cyclic subgroup as $\langle a \rangle = \{\ldots\}$, using the symbols from Class Notes for Chapter 7: $R_0, R_{90}, R_{180}, R_{270}, \rho_A, \rho_B, \rho_H$, and ρ_V. For example, $\langle \rho_A \rangle = \{R_{90}, R_{180}, \rho_V\}$ (that is incorrect). Do not repeat.

(b) Find a subgroup H of D_4 such that $H \neq D_4$ and H is not cyclic. Explain why your H is not cyclic.

Hint: See Assignment 2, but be careful if your table for D_4 was not correct.

(a)

- $\langle R_0 \rangle = \{R_0\}$
- $\langle R_{90} \rangle = \{R_0, R_{90}, R_{180}, R_{270}\}$
- $\langle R_{180} \rangle = \{R_0, R_{180}\}$
- $\langle R_{270} \rangle = \{R_0, R_{270}\}$
- $\langle \rho_A \rangle = \{R_0, \rho_A\}$
- $\langle \rho_B \rangle = \{R_0, \rho_B\}$
- $\langle \rho_H \rangle = \{R_0, \rho_H\}$
- $\langle \rho_V \rangle = \{R_0, \rho_V\}$

(b)

The subgroup H of D_4 such that $H \neq D_4$ and H is not cyclic is, $H = \{R_0, \rho_H, \rho_V, \rho_A\, \rho_B\}$. Recall, from (a),

- $\langle R_0 \rangle = \{R_0\}$
- $\langle \rho_A \rangle = \{R_0, \rho_A\}$
- $\langle \rho_B \rangle = \{R_0, \rho_B\}$
- $\langle \rho_H \rangle = \{R_0, \rho_H\}$
- $\langle \rho_V \rangle = \{R_0, \rho_V\}$.

Thus, no element in H generates H therefore this subgroup of D_4 is not cyclic.