# Foundations Test 2

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Foundations of Mathematics

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## 1

**Theorem 1.** Given $a \in \mathbb{Z}$, if $5 \mid 2a$ then $5 \mid a$

*Proof.* Let $a \in \mathbb{Z}$ such that $5 \mid 2a$. This can be written as $2a = 5k$ for some $k \in \mathbb{Z}$. Now we consider,

$$\begin{aligned} a &= 6a - 5a \\ &= 3(2a) - 5a \\ &= 3(5k) - 5a \text{ Using hypothesis} \\ &= 5(3k - a) \end{aligned}$$

Since $3k - a \in \mathbb{Z}$, this implies that $5 \mid a$.

$\square$

**Theorem 3.** For all $n \in \mathbb{N}$,$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

*Proof.* Let $n \in \mathbb{N}$. We proceed by induction.
**Base Case:** $n = 1$

$$\begin{aligned} 1 \cdot 2 &= \frac{1(1+1)(1+2)}{3} \\ 1 \cdot 2 &= \frac{1(1+1)(1+2)}{3} \\ 2 &= \frac{1(2)(3)}{3} \\ 2 &= \frac{6}{3} \\ 2 &= 2 \end{aligned}$$

**Inductive Hypothesis:** Assume that $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$ for some integer $k$.
**Inductive Step:** We must show that $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$. This can be simplified on the right hand side such that it can be restated as,

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} \\ &= \frac{(k^2 + 2k + k + 2)(k+3)}{3} \\ &= \frac{k^3 + 3k^2 + 2k^2 + 6k + k^2 + 3k + 2k + 6}{3} \\ &= \frac{k^3 + 6k^2 + 11k + 6}{3} \end{aligned}$$

We begin by adding $(k+1)(k+2)$ to both sides of the equation in the inductive hypothesis to get $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$. It can be then shown that,

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k^2 + k)(k+2) + 3(k^2 + 3k + 2)}{3} \\ &= \frac{k^3 + 2k^2 + k^2 + 2k + 3k^2 + 9k + 6}{3} \\ &= \frac{k^3 + 3k^2 + 2k + 3k^2 + 9k + 6}{3} \\ &= \frac{k^3 + 6k^2 + 11k + 6}{3} \end{aligned}$$

This is the same as the equation we got in the inductive step. Therefore, by process of induction, for all $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$. □

**Theorem 4.** Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv_n b$ then $ac \equiv_n bc$

*Proof.* Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. Let $a \equiv_n b$. Then by definition 7.79 there exists an integer $k$ such that $a - b = nk$. It can be shown that by multiplying each side of the equation by $c$ that $ac - bc = nkc$. Since $kc$ is an integer, $ac \equiv_n bc$ by definition 7.79. □

# 2

**Theorem 5.** For all integers $n \geq 0, 24 \mid (5^{2n} - 1)$.

*Proof.* Let $n \in \mathbb{Z}$ such that $n \geq 0$. We proceed by induction.
**Base Case:** $n = 0$

$$\begin{aligned} 5^{2(0)} - 1 &= 24k \\ 5^0 - 1 &= 24k \\ 1 - 1 &= 24k \\ 0 &= 24k \end{aligned}$$

which is true for all $k \in \mathbb{Z}$.
**Inductive Hypothesis:** Assume that $24 \mid (5^{2k} - 1)$ for some integer $k$. This can be restated as $5^{2k} - 1 = 24k$ for some integer $k$.
**Inductive Step:** We must show that $24 \mid (5^{2(k+1)} - 1)$. First, by multiplying each side of the inductive hypothesis by $5^2$ we get,

$$\begin{aligned} 5^2 \cdot (5^{2k} - 1) &= 24k \cdot 5^2 \\ 5^{2k+2} - 5^2 &= 24k \cdot 5^2 \end{aligned}$$

Secondly, we can then replace $5^2$ with 25 on the left hand side to yield,

$$5^{2k+2} - 25 = 24k \cdot 5^2$$

We then make the -25 into -1-24 which yields,

$$5^{2k+2} - 24 - 1 = 24k \cdot 5^2$$

Next, we add over the $-24$ from the left hand side to yield,

$$5^{2k+2} - 1 = 24k \cdot 5^2 + 24$$

Finally, factoring out the 24 from the right hand side of the equation yields,

$$5^{2k+2} - 1 = 24(k5^2 + 1)$$

Since $k5^2 + 1$ is an integer, we have shown that $24 \mid (5^{2(k+1)} - 1)$. Therefore, by process of induction, for all integers $n \geq 0, 24 \mid (5^{2n} - 1)$. □

**Theorem 6.** For all integers $x$, if $x^2 - 6x + 5$ is even, then $x$ is odd.

*Proof.* Let $x$ be an integer such that $x^2 - 6x + 5$ is even. We proceed by contraposition.
**Contrapositive:** If $x$ is even, then $x^2 - 6x + 5$ is odd.
Let $x$ be an even integer. Then by definition 2.1 there exists an integer $k$ such that $x = 2k$. We can then substitute $2k$ for $x$ in $x^2 - 6x + 5$ to yield,

$$\begin{aligned} (2k)^2 - 6(2k) + 5 &= 4k^2 - 12k + 5 \\ &= 2(2k^2 - 6k + 2) + 1 \end{aligned}$$

By definition 2.1, $2(2k^2 - 6k + 2)$ is an even integer. Therefore, $x^2 - 6x + 5$ is odd when $x$ is even. Therefore, by contraposition, for all integers $x$, if $x^2 - 6x + 5$ is even, then $x$ is odd. □

# 3

**Theorem 7.** Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $12a \not\equiv_n 12b$ then $a \nmid b$.

*Proof.* Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. For sake of contradiction, suppose $12a \not\equiv_n 12b$ and $n \mid 12$. By definition 2.1, there exists an integer $k$ such that $12 = nk$. By multiplying each side by $(a - b)$ we get,

$$12(a-b) = nk(a-b)$$
$$12a - 12b = n(ka - kb)$$

which is a contradiction because this would mean by definition 7.79 that $12a \equiv_n 12b$. Therefore, if $12a \not\equiv_n 12b$ then $a \nmid b$. $\square$

**Theorem 8.** For all $a, b \in \mathbb{Z}$, if $a$ is even and $b$ is odd, then 6 does not divide $a^2 + b^2$.

*Proof.* Let $a, b \in \mathbb{Z}$ such that by definition 2.1, there exists an integers $k, l$ such that $a = 2k$ and $b = 2l + 1$. For the sake of contradiction, suppose that $6 \mid (a^2 + b^2)$. It can be shown that,

$$\begin{aligned} a^2 + b^2 &= (2k)^2 + (2l+1)^2 \\ &= 4k^2 + 4l^2 + 4l + 1 \\ &= 2(2k^2 + 2l^2 + 2l) + 1 \end{aligned}$$

where $2k^2 + 2l^2 + 2l$ is an integer. Therefore, by definition 2.1, $a^2 + b^2$ is odd. This would imply that there exists an integer $m$ such that, $(a^2 + b^2) = 6m$, which can be rewritten as,

$$\frac{(a^2 + b^2)}{6} = m$$

Since $a^2 + b^2$ is odd, this implies $m$ is not an integer, which is a contradiction. Therefore, for all $a, b \in \mathbb{Z}$, if $a$ is even and $b$ is odd, then 6 does not divide $a^2 + b^2$.

$\square$

# 4

**Theorem 10.** If $A$,$B$ are sets, then $A \cap (B \setminus A) = \emptyset$

*Proof.* 6. Let $A, B$ be sets.
12. Suppose, for the sake of contradiction,
3. that $A \cap (B \setminus A) \neq \emptyset$.
5. Hence, there exists an $x \in A \cap (B \setminus A)$.

9. By the definition of intersection,
1. $x \in A$ and $x \in B \setminus A$.
7. By the definition of set difference,
11. $x \in B$ and $x \notin A$.
8. All together, this implies that $x \in A$
2. and $x \notin A$,
10. which is a contradiction.
4. Therefore, $A \cap (B \setminus A) = \emptyset$.

$\square$