Theorem 4.18

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Foundations of Mathematics

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Theorem 4.18. For all integers $n \geq 3, 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-2)\left(n^2+2n+3\right)}{3}$

Proof. Let $n \geq 3$. We will prove that $2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-2)\left(n^2+2n+3\right)}{3}$ by utilizing induction.

Base case: $n = 3$

$$\begin{aligned}(3-1)\,3 &= \frac{(3-2)\,(3^2+2(3)+3))}{3}\\ 6 &= 6\end{aligned}$$

Next, for some $n = k$ such that $k \in \mathbb{N}$ and $k \geq 3$, we assume that $2\cdot3+3\cdot4+\cdots+(k-1)\cdot k = \frac{(k-2)\left(k^2+2k+3\right)}{3}$
It can be shown that for $n = k+1$ that if we add zero we get the following,

$$\begin{aligned}2 \cdot 3 + 3 \cdot 4 + \cdots + (k-1) \cdot k + ((k)(k+1) - (k)(k+1)) &= (k-2)\,\frac{(k^2+2k+3)}{3}\\ 2 \cdot 3 + 3 \cdot 4 + \cdots + (k-1) \cdot k + (k)(k+1) &= (k)(k+1)\left((k-2)\,\frac{(k^2+2k+3)}{3}\right)\\ &= k^2 + k + \frac{k^3 - k - 6}{3}\\ &= \frac{3k^2 + 2k - 6}{3}\end{aligned}$$

We can show by substituting $k+1$ for k in our original equation, that for $k \geq 3, 2 \cdot 3 + 3 \cdot 4 + \cdots + (k-1) \cdot k + (k)(k+1) = \frac{(k-2)\left(k^2+2k+3\right)}{3}$, that the right hand side can be simplified to match our preceeding equation. This can be shown as follows,

$$\frac{(k-2)\,(k^2+2k+3)}{3} = \frac{3k^2+2k-6}{3}$$

Thus, we can conclude that by induction that $2\cdot3+3\cdot4+\cdots+(n-1)\cdot n = \frac{(n-2)\left(n^2+2n+3\right)}{3}$. □