A1-Final_CSaben

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Problem 1. Determine which of the following operations are associative. In each case, either write a proof or give a counter-example.

(a) the operation $*$ on $\mathbb{Z}$ defined by $a * b = a - b$;

Consider the following counter-example: $a = 1, b = 2, c = 3$. Then,

$$\begin{aligned}(a * b) * c &= (1 - 2) * 3 \\ &= -1 * 3 \\ &= -1 - 3 \\ &= -4\end{aligned}$$

and,

$$\begin{aligned}a * (b * c) &= 1 * (2 - 3) \\ &= 1 * -1 \\ &= 1 - (-1) \\ &= 1 + 1 \\ &= 2.\end{aligned}$$

Therefore, $(a * b) * c \neq a * (b * c)$ and the operation $*$ is not associative.

(b) the operation $*$ on $\mathbb{R}$ defined by $a * b = a + 2b + ab$;

Consider the following counter-example: $a = 1, b = 2, c = 3$. Then,

$$\begin{aligned}(a * b) * c &= (1 + 2(2) + 1(2)) * 3 \\ &= (1 + 4 + 2) * 3 \\ &= 7 * 3 \\ &= 7 + 2(3) + 7(3) \\ &= 7 + 6 + 21 \\ &= 34\end{aligned}$$

and,

$$\begin{aligned} a * (b * c) &= 1 * (2 + 2(3) + 1(2)) \\ &= 1 * (2 + 6 + 2) \\ &= 1 * 10 \\ &= 1 + 2(10) + 1(10) \\ &= 1 + 20 + 10 \\ &= 31. \end{aligned}$$

Therefore, $(a * b) * c \neq a * (b * c)$ and the operation $*$ is not associative.

(c) the operation $*$ on $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ defined by $a * b = \frac{a}{b}$;
Consider the following counter-example, $a = 1, b = 2, c = 3$. Then,

$$\begin{aligned} (a * b) * c &= (\frac{1}{2}) * 3 \\ &= \frac{\frac{1}{2}}{3} \\ &= \frac{1}{6} \end{aligned}$$

and,

$$\begin{aligned} a * (b * c) &= 1 * (\frac{2}{3}) \\ &= \frac{1}{\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

Therefore, $(a * b) * c \neq a * (b * c)$ and the operation $*$ is not associative.

(d) the operation $*$ on $\mathbb{Z}$ defined by $a * b = a + b - 2$.

Proof. Let $a, b, c \in \mathbb{Z}$. Then,

$$\begin{aligned} (a * b) * c &= (a + b - 2) * c \\ &= (a + b - 2) + c - 2 \\ &= a + b + c - 4 \end{aligned}$$

and,

$$\begin{aligned} a * (b * c) &= a * (b + c - 2) \\ &= a + (b + c - 2) - 2 \\ &= a + b + c - 4 \end{aligned}$$

Since $(a * b) * c = a * (b * c)$, the operation $*$ is associative. □

Problem 2. Consider the set $G = \{x \in \mathbb{Q} : x \neq 1\}$. Define an operation $*$ on G by

$$a * b = a + b - ab \quad (a, b \in G).$$

(a) Show that G is closed under $*$, that is, that for all $a, b \in G, a * b \in G$.

Proof. Let $a, b \in G$. It is obvious that $a+b-ab \in \mathbb{Q}$. It remains to show that $a+b-ab \neq 1$. Suppose for the sake of contradiction that $a+b-ab = 1$. Then,

$$\begin{aligned} a+b-ab &= 1 \\ a-1+b-ab &= 0 \\ (a-1)+(b-ab) &= 0 \\ (a-1)(1-b) &= 0, \end{aligned}$$

which implies that $a-1=0$ or $1-b=0$. If $a-1=0$, then $a=1$, which is a contradiction since $a \in G$. If $1-b=0$, then $b=1$, which is a contradiction since $b \in G$. Therefore, G is closed under $*$. □

(b) Prove that $\langle G, * \rangle$ is a group.

Proof. Let $a, b, c \in G$. From part (a) we know that G is closed under $*$. It remains to show that $*$ is associative and that G has an identity element and every element of G has an inverse. To show associativity, we must show that $(a*b)*c = a*(b*c)$. This is true since:

$$\begin{aligned} (a*b)*c &= (a+b-ab)*c \\ &= (a+b-ab)+c-(a+b-ab)c \\ &= a+b-ab+c-ac-bc+abc \\ &= a+b+c-ab-ac-bc+abc \\ &= a+(b+c-bc)-a(b+c-bc) \\ &= a*(b+c-bc) \\ &= a*(b*c). \end{aligned}$$

Consider $0 \in G$. Then, $a*0 = a+0-a0 = a$. Also, $0*a = 0+a-0a = a$. Therefore, 0 is the identity element of G. To show that every element of G has an inverse, we must show that for every $a \in G$, there exists $a' \in G$ such that $a*a' = 0$ and $a'*a = 0$. To determine a', we must solve the equation $a+a'-aa' = 0$ for a'. Thus,

$$\begin{aligned} a+a'-aa' &= 0 \\ a'-aa' &= -a \\ a'(1-a) &= -a \\ a' &= \frac{-a}{1-a} \\ a' &= \frac{a}{a-1}. \end{aligned}$$

Then firstly,

$$
\begin{aligned}
a * (\frac{a}{a-1}) &= a + \frac{a}{a-1} - a(\frac{a}{a-1}) \\
&= a + \frac{a}{a-1} - \frac{a^2}{a-1} \\
&= \frac{a^2 - a(a-1) + a(a-1) - a^2}{a-1} \\
&= \frac{0}{a-1} \\
&= 0.
\end{aligned}
$$

Also secondly,

$$
\begin{aligned}
a * (\frac{a}{a-1}) &= \frac{a}{a-1} + a - a(\frac{a}{a-1}) \\
&= \frac{a}{a-1} + a - \frac{a^2}{a-1} \\
&= \frac{a - a^2 + a(a-1) - a^2}{a-1} \\
&= \frac{0}{a-1} \\
&= 0.
\end{aligned}
$$

Therefore, $\frac{a}{a-1}$ is the inverse of a in G. Thus, $\langle G, * \rangle$ is a group.

□

Problem 3. Let G be any group. Prove that for all $a, b \in G$, if $ab = e$, then $ba = e$.

Proof. Let $a, b \in G$. Suppose that $ab = e$. Then,

$$
\begin{aligned}
ab &= e \\
a^{-1}ab &= a^{-1}e \\
eb &= a^{-1}e \\
bb^{-1} &= a^{-1}b^{-1} \\
e &= a^{-1}b^{-1} \\
bae &= baa^{-1}b^{-1} \\
ba &= beb^{-1} \\
ba &= bb^{-1} \\
ba &= e.
\end{aligned}
$$

Therefore, if $ab = e$, then $ba = e$. □