Foundations Test 2

Clark Saben
Foundations of Mathematics

April 19, 2023

1

Theorem 1. If $n \in \mathbb{Z}$ and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

Proof. Let $n \in \mathbb{Z}$ and $n \geq 0$. We will show that $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ via induction. For our base case, $n = 0$,

$$\begin{aligned}\sum_{i=0}^{0} 0 \cdot 0! &= (0+1)! - 1 \\ &= 0\end{aligned}$$

which is true, so we proceed with our inductive step. Let us assume that for some $k \in \mathbb{Z}$ that $k \geq 0$ and $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$. It can be shown that by adding $(k+1)(k+1)!)$ to both sides of the equation we get,

$$\sum_{i=0}^{k} i \cdot i! + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

Next, by re-arranging with algebra (namely, the distributive property), we can see,

$$\begin{aligned}\sum_{i=0}^{k} i \cdot i! + (k+1)(k+1)! &= (1 + (k+1))(k+1)! - 1 \\ &= (k+2)(k+1)! - 1\end{aligned}$$

Now, notice that when we have $k = k+1$, we have,

$$\begin{aligned}\sum_{i=0}^{k+1} i \cdot i! &= ((k+1)+1)! - 1 \\ &= (k+2)(k+1)! - 1\end{aligned}$$

which is matches our result from the inductive step, so we have shown that $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ for all $n \in \mathbb{Z}$ and $n \geq 0$ by process of induction. $\square$

Theorem 2. The inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. We will show that the inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for all n via induction. For our base case, $n = 1$,

$$\begin{aligned} 2^1 &\leq 2^{1+1} - 2^{1-1} - 1 \\ 2 &\leq 4 - 1 - 1 \\ 2 &\leq 2 \end{aligned}$$

which is true, so we proceed with our inductive step. Let us assume that for some $k \in \mathbb{N}$ the inequality $2^k \leq 2^{k+1} - 2^{k-1} - 1$ holds for all k. Since 2^k is going to be less than $2^{k+1} - 2^{k-1} - 1$ it will also be less than that plus an additional $2^{k-1} + 1$. Hence we can correctly re-write our inequality from,

$$2^k \leq 2^{k+1} - 2^{k-1} - 1$$

to,

$$\begin{aligned} 2^k &\leq 2^{k+1} - 2^{k-1} - 1 + (2^{k-1} + 1) \\ 2^k &\leq 2^{k+1} \\ &\leq (2)2^k \end{aligned}$$

Clearly, for any natural number, k, $2^k \leq (2)2^k$. Therefore, the inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for all $n \in \mathbb{N}$.

□

Theorem 3. Define the relation R on $\mathbb{Z}$ such that xRy if and only if $3x - 5y$ is even. Then R is an equivalence relation.

Proof. Let x and $y \in \mathbb{Z}$ the define relation R on $\mathbb{Z}$ such that xRy if and only if $3x - 5y$ is even. We will show R is an equivalence relation.

To show R is reflexive, let $x \in \mathbb{Z}$. Then xRx if and only if $3x - 5x$ is even. By Definition 2.1 it can be seen that,

$$\begin{aligned} 3x - 5x &= 2(-k) \text{ (for some } -k \in \mathbb{Z}) \\ -2x &= -2k \end{aligned}$$

which is true for all $x \in \mathbb{Z}$, so R is reflexive.

To show R is symmetric, let x and $y \in \mathbb{Z}$. Then xRy if and only if $3x - 5y$ is even. By Definition 2.1 it can be seen that,

$$3x - 5y = 2(-k) \text{ (for some } -k \in \mathbb{Z})$$

We can add 2x-2y to both sides to get,

$$\begin{aligned}5x - 3y &= 2(-k) + 2x - 2y \\ &= -2(k - x + y)\end{aligned}$$

We can then multiply each side by (-1) to get,

$$\begin{aligned}(-1)5x - 3y &= (-1)(-2(k - x + y)) \\ -5x + 3y &= 2(k - x + y) \\ 3y - 5x &= 2(k - x + y)\end{aligned}$$

where $(k - x + y)$ is some integer. Therefore, since $3y - 5x$ is even (Definition 2.1), yRx, so R is symmetric.

To show R is transitive, let x,y,$z \in \mathbb{Z}$ and xRy and yRz. Then $3x - 5y$ is even and $3y - 5z$ is even. By Definition 2.1 it can be seen that,

$$\begin{aligned}3x - 5y &= 2(k) \text{ (for some } k \in \mathbb{Z}) \\ 3y - 5z &= 2(l) \text{ (for some } l \in \mathbb{Z})\end{aligned}$$

We can add the two equations to get,

$$\begin{aligned}3x - 5y + 3y - 5z &= 2(k) + 2(l) \\ 3x - 5z - 2y &= 2(k + l) \\ &= 2(k + l + y)\end{aligned}$$

We see that $3x - 5z$ is even, so xRz and R is transitive.

In conclusion, R is an equivalence relation since R is reflexive, symmetric, and transitive. □

2

Theorem 4. Let a,$b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv_n b$ then $a^2 \equiv_n b^2$. (recall the definition of $\equiv_n$ from Definition 7.79)

Proof. Let a,$b \in \mathbb{Z}$ and $n \in \mathbb{N}$. We will show that if $a \equiv_n b$ then $a^2 \equiv_n b^2$. By definition 7.79, $a \equiv_n b$ if and only if $a - b$ is divisible by n. It can thus be shown that,

$$a - b = nk \text{ (for some } k \in \mathbb{Z})$$

we can then multiply both sides by $a+b$ to get,

$$\begin{aligned} a^2 + ab - ba + b^2 &= (a+b)nk \\ a^2 - b^2 &= (a+b)nk \\ a^2 - b^2 &= np \ \boxed{\text{such that } p = (a+b)k} \end{aligned}$$

Therefore, $a^2 - b^2$ is divisible by n. Thus, $a^2 \equiv_n b^2$ if $a - b$ is divisible by n. □

Theorem 5. For an $n \geq 4$, one can obtain n dollars using only \$2 and \$5 bills.

Proof. Let $n \geq 4$. We will show that one can obtain n dollars using only \$2 and \$5 bills. In other words, there exist non-negative integers a and b such that $n = 2a + 5b$ for all $n \geq 4$. We will prove this via strong induction.

For our base case we will prove the cases for $n = 2$, 4, and 5 directly.
For $n = 2$, we can use one \$2 bill: $2 = 2 \cdot 1 + 5 \cdot 0$ $(a = 1, b = 0)$
For $n = 4$, we can use two \$2 bills: $4 = 2 \cdot 2 + 5 \cdot 0$ $(a = 2, b = 0)$
For $n = 5$, we can use one \$5 bill: $5 = 2 \cdot 0 + 5 \cdot 1$ $(a = 0, b = 1)$

For our inductive hypothesis, we will assume that for $5 \leq k \leq m$ can be obtained using only \$2 and \$5 bills such that $k, m \in \mathbb{Z}$.

We will now prove that $k+1$ can be obtained using only \$2 and \$5 bills. Since $k-1$ can be obtained using only \$2 and \$5 bills, we can see that,

$$k - 1 = 2a + 5b \text{ (for some } a, b \in \mathbb{Z})$$

which can be rewritten as,

$$(k+1) - 2 = 2a + 5b$$

Hence, by adding adding a \$2 bill to the \$2 and \$5 bills used to obtain $k-1$, we can obtain $k+1$ using only \$2 and \$5 bills.

Thus, we have shown via strong induction that for any $n \geq 4$, one can obtain n dollars using only \$2 and \$5 bills.

□

Theorem 6. Define $\Psi = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$ and define a relation $\sim$ on Ψ via $(a, b) \sim (c, d)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation.

Proof. Let $\Psi = \{(a,b) : a,b \in \mathbb{Z}, b \neq 0\}$ and define a relation $\sim$ on Ψ via $(a,b) \sim (c,d)$ if and only if $ad = bc$. To show that $\sim$ is an equivalence relation, we must show that $\sim$ is reflexive, symmetric, and transitive.

To show that $\sim$ is reflexive, we must show that for all $(a,b) \in \Psi$, $(a,b) \sim (a,b)$. It can be shown that,

$$\begin{aligned}(a,b) \sim (a,b) &\Leftrightarrow ab = ba \\ &\Leftrightarrow ab = ab \text{ (commutative property)}\end{aligned}$$

Therefore, $\sim$ is reflexive.

To show that $\sim$ is symmetric, we must show that for all $(a,b) \in \Psi$ and $(c,d) \in \Psi$, if $(a,b) \sim (c,d)$ then $(c,d) \sim (a,b)$. It can be shown that,

$$(a,b) \sim (c,d) \Leftrightarrow ad = bc$$

and that,

$$\begin{aligned}(c,d) \sim (a,b) &\Leftrightarrow cb = da \\ &\Leftrightarrow ad = bc \text{ (commutative property)}\end{aligned}$$

Therefore, $\sim$ is symmetric.

To show that, $\sim$ is transitive, we must show that for all $(a,b) \in \Psi$, $(c,d) \in \Psi$, and $(e,f) \in \Psi$, if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ then $(a,b) \sim (e,f)$. It can be shown that,

$$\begin{aligned}ad &= bc \\ fad &= fbc\end{aligned}$$

Since $(c,d) \sim (e,f)$, $cf = de$. We can re-write f as $\frac{de}{c}$. We can finally show that,

$$\begin{aligned}adf &= bc(\frac{de}{c}) \\ adf &= bde \\ af &= be,\end{aligned}$$

which would follow from $(a,b) \sim (e,f)$. Therefore, $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation.

□

3

Problem 1. Consider the following relation:
$R = \{(a, a), (a, b), (a, d), (b, d), (c, c), (d, b), (d, c)\}$

Problem 1. .1 What elements must be add to R to make it reflexive?

To be reflexive, (b, b) and (d, d) must be added to R.

Problem 1. .1 What elements must be add to R to make it symmetric?

To be symmetric, (b, a), (d, a) and (c, d) must be added to R.