A3-Final_CSaben

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1. (6 points) Consider the permutations $f, g, h \in S_6$ given in problem A1 on page 75 .

(a) Write f, g, and h in cycle notation. (For example, (154)(26) is an element of S_6 written in cycle notation. Note that the permutation (154)(26) fixes 3 , but the 1-cycle (3), which is the identity in S_6, can be skipped.)

Problem 1a. Answers to 1a are as follows:

$$\begin{aligned} f &= (162)(45) \\ g &= (123)(46) \\ h &= (1362) \end{aligned}$$

(b) Calculate $f^3, g^{-1} \circ h \circ f^{-1}$, and g^{-2}. Use cycle notation in all calculations. (For example, you should write $(154)(26) \circ (16)(24)(35) = (12)(3465)$.)

Problem 1b. Answers to 1b are as follows:

$$\begin{aligned} f^3 &= (45) \\ g^{-1} \circ h \circ f^{-1} &= (134562) \\ g^{-2} &= (123) \end{aligned}$$

2. (7 points) Let $n \geq 2$. Denote by B_n the set of all odd permutations in S_n. Define $f : A_n \to B_n$ by

$$f(\alpha) = (12) \circ \alpha \quad (\alpha \in A_n)\,.$$

(a) Prove that the codomain of f is indeed B_n; that is, if $\alpha \in A_n$, then $f(\alpha) \in B_n$. Hint: If $\alpha \in A_n$, then $\alpha = (a_1b_1) \circ \cdots \circ (a_kb_k)$, where k is even.

Proof. 2a Recall A_n is the alternating group consisting of all even permutations in S_n. Let $\alpha \in A_n$ such that α is an even permutation. Then, $\alpha = (a_1b_1) \circ \cdots \circ (a_kb_k)$, where k is even. Since $f(\alpha) = (12) \circ \alpha$ is a composition of $k+1$ transpositions, it is clear that $f(\alpha)$ is an odd permutation and therefore $f(\alpha) \in B_n$. Thus, the codomain of f is indeed B_n. □

(b) Prove that f is bijective.

Proof. 2b We will show that f is both injective and surjective to ensure injectivity.

(injection) Suppose $a_1, a_2 \in A_n$. Then,

$$\begin{aligned}(12) \circ a_1 &= (12) \circ a_2 \\ (21)(12) \circ a_1 &= (21)(12) \circ a_2 \\ a_1 &= a_2\end{aligned}$$

(surjective) Consider $\alpha = (21) \circ y$. Then, $n \neq 1$ since 1 is odd and A_n is an even permutation. $n \nleq 1$ since the cardinality of A_n is positive and nonempty. Thus $\alpha \in A$ and $f(21) \circ y) = (12) \circ (21) \circ y = y$. Hence, f is surjective.

Therefore f is bijective.

□

(c) Use (b) and the fact that $|S_n| = n\,!$ to show that A_n has $n!/2$ elements.

Hints: (i) $S_n = A_n \cup B_n$ and $A_n \cap B_n = \emptyset$; (ii) if X and Y are any finite sets, then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

Proof. 2c Recall that $|s_n| = n!|$, $S_n = A_n \cup B_n$, and $A_n \cap B_n = \emptyset$. Thus, A_n and B_n are disjoint sets. Note,

$$\begin{aligned}|A_n \cup B_n| &= |A_n| + |B_n| - |A_n \cap B_n| \\ |A_n \cup B_n| &= |A_n| + |B_n| - 0 \\ |A_n \cup B_n| &= |A_n| + |B_n|\end{aligned}$$

which we can restate as,

$$n! = |A_n| + |B_n|.$$

Note that since $f : A_n \to B_n$ is bijective that implies that $|A_n| = |B_n|$. Let $k = |A_n| = |B_n|$. Then,

$$\begin{aligned}k + k &= n! \\ 2k &= n! \\ k &= \frac{n!}{2}\end{aligned}$$

Thus, $|A_n| = \frac{n!}{2}$. □

3. (7 points) Let $Z_7^* = \{1, 2, 3, 4, 5, 6\}$. Then $\langle \mathbb{Z}_7^*, \cdot \rangle$, where $\cdot$ is the multiplication modulo 7 (see Exercise D2 on page 99), is a group (take this as given). For example, $3 \cdot 4 = 5$ and $6 \cdot 6 = 1$.

Prove that $\langle \mathbb{Z}_6, + \rangle$, the group of integers modulo 6 , is isomorphic to $\langle \mathbb{Z}_7^*, \cdot \rangle$.

Hint: Use the table method. Take the solution (page 363) to Exercise C4 on page 99 as a model. In the table for $\mathbb{Z}_6$, order the elements of $\mathbb{Z}_6 : 0, 1, 2, 3, 4, 5$. In the table for $\mathbb{Z}_7^*$, order the elements of $\mathbb{Z}_7^*$ appropriately.

You must present both tables and an isomorphism f (in two-row notation), as on page 363 .

To confirm that your f is an isomorphism, use the tables to show that $f(1+4) = f(1) \cdot f(4)$, $f(2+5) = f(2) \cdot f(5), f(3+3) = f(3) \cdot f(3), f(3+4) = f(3) \cdot f(4)$, and $f(5+5) = f(5) \cdot f(5)$.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Table 1: Table for $\langle \mathbb{Z}_6, + \rangle$

$\cdot$	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2	6	4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

Table 2: Table for $\langle \mathbb{Z}_7^*, \cdot \rangle$

Let $f : \mathbb{Z}_6 \to \mathbb{Z}_7^*$ be defined by,

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix}.$$

By inspection, f transforms the table of the table $\langle \mathbb{Z}_6, + \rangle$ into the table $\langle \mathbb{Z}_7^*, \cdot \rangle$. Thus f is an isomorphism from $\langle \mathbb{Z}_6, + \rangle$ to $\langle \mathbb{Z}_7^*, \cdot \rangle$.

Confirming with examples;

$$f(1+4) = f(5) = 5 \text{ and } f(1) \cdot f(4) = 3 \cdot 5 = 5$$
$$f(2+5) = f(1) = 3 \text{ and } f(2) \cdot f(5) = 2 \cdot 5 = 3$$
$$f(3+3) = f(0) = 1 \text{ and } f(3) \cdot f(3) = 6 \cdot 6 = 1$$
$$f(3+4) = f(1) = 3 \text{ and } f(3) \cdot f(4) = 6 \cdot 4 = 3$$
$$f(5+5) = f(4) = 4 \text{ and } f(5) \cdot f(5) = 5 \cdot 5 = 4$$