Theorem 7.88

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Theorem 7.88. For $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, $[a]_n = [b]_n$ if and only if a and b have the same remainder when divided by n.

Proof. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. Through the usage of the bi-conditional theorem it can be shown that $[a]_n = [b]_n$ if and only if a and b have the same remainder when divided by n.

$\Longrightarrow$

Let $[a]_n = [b]_n$ and $m \in [a]_n$, which then implies that $m \in [b]_n$. Let $-k \in \mathbb{Z}$, we can re-write m as follows,

$$\begin{aligned} m - a &= (-k)n \\ a &= m + kn \\ a &= kn + m \end{aligned}$$

By Theorem 6.7 (The Division Algorithm), a has the remainder m when divided by n. Let $-l \in \mathbb{Z}$, it can similarly be shown that,

$$\begin{aligned} m - b &= (-l)n \\ b &= m + kn \\ b &= ln + m \end{aligned}$$

which by Theorem 6.7 , b has the remainder m when divided by n. Therefore, if $[a]_n = [b]_n$, then a and b have the same remainder when divided by n.

$\Longleftarrow$

Let a and b have the same remainder when divided by n. Additionally, let $-d, -g \in \mathbb{Z}$ and $r \in \mathbb{Z}$ such that $a = n(-d) + r$ and $b = n(-g) + r$. Then,

$$\begin{aligned} a &= n(-d) + r \\ nd &= r - a \\ r - a &= nd \\ r - a &\in n\mathbb{Z} \\ r &\equiv_n a \\ r &\in [a]_n \end{aligned}$$

It can be similarly shown that $r \in [b]_n$. Therefore, $[a]_n = [b]_n$. Therefore, if a and b have the same remainder when divided by n, then $[a]_n = [b]_n$.

Thus, for $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, $[a]_n = [b]_n$ if and only if a and b have the same remainder when divided by n. $\square$