Foundations Test 2

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Foundations of Mathematics

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Theorem 1. If $n \in \mathbb{Z}$ and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

Theorem 2. The inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for all $n \in \mathbb{N}$.

Theorem 3. Define the relation R on $\mathbb{Z}$ such that xRy if and only if $3x - 5y$ is even. Then R is an equivalence relation.

2

Theorem 4. Let a,$b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv_n b$ then $a^2 \equiv_n b^2$. (recall the definition of $\equiv_n$ from Definition 7.79)

Theorem 5. For an $n \geq 4$, one can obtain n dollars using only \$2 and \$5 bills.

Theorem 6 Complete but gross. Define $\Psi = \{(a,b) : a,b \in \mathbb{Z}, b \neq 0\}$ and define a relation $\sim$ on Ψ via $(a,b) \sim (c,d)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation.

Let $\Psi = \{(a,b) : a,b \in \mathbb{Z}, b \neq 0\}$ and define a relation $\sim$ on Ψ via $(a,b) \sim (c,d)$ if and only if $ad = bc$. To show that $\sim$ is an equivalence relation, we must show that $\sim$ is reflexive, symmetric, and transitive.

To show that $\sim$ is reflexive, we must show that for all $(a,b) \in \Psi$, $(a,b) \sim (a,b)$.
It can be shown that,

$$\begin{aligned} (a,b) \sim (a,b) &\Leftrightarrow ab = ba \\ &\Leftrightarrow ab = ab \text{ (commutative property)} \end{aligned}$$

Therefore, $\sim$ is reflexive.

To show that $\sim$ is symmetric, we must show that for all $(a, b) \in \Psi$ and $(c, d) \in \Psi$, if $(a, b) \sim (c, d)$ then $(c, d) \sim (a, b)$.
It can be shown that,

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

and that,

$$\begin{aligned}(c, d) \sim (a, b) &\Leftrightarrow cb = da \\ &\Leftrightarrow ad = bc \text{ (commutative property)}\end{aligned}$$

Therefore, $\sim$ is symmetric.

To show that, $\sim$ is transitive, we must show that for all $(a, b) \in \Psi$, $(c, d) \in \Psi$, and $(e, f) \in \Psi$, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $(a, b) \sim (e, f)$.
It can be shown that,

$$\begin{aligned}ad &= bc \\ fad &= fbc \\ f &= \frac{de}{c} \\ adf &= bc(\frac{de}{c}) \\ adf &= bde \\ af &= be\end{aligned}$$

Therefore, $\sim$ is transitive.

3 (complete)

Problem 1. Consider the following relation:
$R = \{(a, a), (a, b), (a, d), (b, d), (c, c), (d, b), (d, c)\}$

Problem 1. .1 What elements must be add to R to make it reflexive?

To be reflexive, (b, b) and (d, d) must be added to R.

Problem 1. .1 What elements must be add to R to make it symmetric?

To be symmetric, (b, a), (d, a) and (c, d) must be added to R.