Writing Assignment 6

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Foundations of Mathematics

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Theorem WA 6.1. If $n \in \mathbb{N}$, then $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Proof. Let $n \in \mathbb{N}$. We will proceed by induction.
Firstly, when $n = 1$,

$$\begin{aligned}(2(1) - 1) &= (1)^2 \\ &= 1^2 \\ &= 1\end{aligned}$$

Next, to begin our inductive step, let us assume that for some k, such that $k \in \mathbb{N}$, $1 + 3 + 5 + ... + (2k - 1) = k^2$. It can be shown that when $n = k + 1$,

$$\begin{aligned}1 + 3 + 5 + ... + (2k - 1) + (2k + 2 - 1) &= (k + 1)^2 \\ &= k^2 + 2k + 1\end{aligned}$$

Finally, by adding $2k + 2 - 1$ to both sides of $1 + 3 + 5 + ... + (2k - 1) = k^2$, we get

$$\begin{aligned}1 + 3 + 5 + ... + (2k - 1) + (2k + 2 - 1) &= k^2 + (2k + 2 - 1) \\ &= k^2 + (2k + 2 - 1) \\ &= k^2 + 2k + 1\end{aligned}$$

Therefore, by induction if $n \in \mathbb{N}$, then $1 + 3 + 5 + ... + (2n - 1) = n^2$. $\square$

Theorem WA 6.2. For every $n \in \mathbb{N}$, $2^n + 1 \leq 3^n$.

Proof. Let $n \in \mathbb{N}$. We will proceed by induction.
Firstly, when $n = 1$,

$$\begin{aligned} 2^1 + 1 &= 3^1 \\ 3 &= 3^1 \\ 3 &= 3 \end{aligned}$$

Next, to begin our inductive step, let us assume that for some k, such that $k \in \mathbb{N}$, $2^k + 1 \leq 3^k$.
We can then show that when $n = k + 1$,

$$\begin{aligned} 2^{k+1} + 1 &= 3^{k+1} \\ &= (3)3^k \\ &= (3)(2^k + 1) \text{ (given our assumption above)} \\ &= (2 + 1)(2^k + 1) \\ &= 2^{k+1} + 2 + 2k + 1 \\ &= 2^{k+1} + 2k + 3 \end{aligned}$$

Clearly, $2^{k+1} + 1 \leq 2^{k+1} + 2k + 3$, so we can conclude by induction that for every $n \in \mathbb{N}$, $2^n + 1 \leq 2^{n+1}$.

$\square$