Lecture 1

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1 2/21/23 lecture

1. $V = 0$ everywhere (trivial)
2. more meaningful solutions agree with the boundary conditions(b.c.) and vary in some region as $V(x, y, z)$.

Dirichlet b.c.:

1. $V(x = 5m) = 0 = m(5) + b => b = -5m$
2. $V(x = 1) = 4 = m(1) + (-5m) = -4m => m = -1$

Hence,

$V(x) = (5 - x)$ Volts
$E_x = -1$ V/m

- Solution is generally a linear function
- consider $V(x + a) = mx + b + ma$
- and $V(x - a) = mx + b - ma$

$V(x)$ is the average of these expressions (explicitly shown below).

$V(x) = \frac{1}{2}(V(x + a) + V(x - a))$

Laplace's equations is like an averageing instruction (1D, 2D, 3D).

$\nabla^2 V = 0$
$\implies$ No maxium or minimum inside of the volume of the space. And the only extrema exists at the boundary

i.e. $\left(\frac{dV}{dn} = \frac{-\sigma}{\epsilon_0}\right)$

There are two types of b.c.'s:

1. Dirichlet b.c. : V is fixed by some external means at the surface dV (e.g. grounded)
2. Neumann b.c. : The value of $\vec{\nabla}V.\hat{n}$ at the surface AV is fixed (the normal derivate is fixed, i.e. $\frac{dV}{dn} =$ is fixed

There are actually two uniqueness theorems but we will only be using the first one in this class.

1. Theorem 1: With either b.c. (1) or (2) chosen, at surface dV, there is a unique solution $V(x, y, z)$ to Laplace's equation in a region of space.

Proof:
Suppose we have a function $f = V_1 - V_2$, where V_1 and V_2 satisfy $\nabla^2 V_i = 0$.

$$\begin{aligned}\int_V \vec{\nabla} \cdot \left(f\vec{\nabla}f\right) d^3x &= \int_V \left(\vec{\nabla}f \cdot \vec{\nabla}f) + f\left(\nabla^2 f\right)\right) d^3x \\ &= \int_V \left(\vec{\nabla}f \cdot \vec{\nabla}f\right) d^3x \text{ Div. Thm} \\ &= \int_{dV} f\vec{\nabla}f \cdot d\vec{a}\end{aligned}$$

$\nabla^2 f$ becomes zero, hence the simplification

A) Suppose we pick a situation where Dirichlet b.c. conditions are used. Then, V_1 and V_2 are both solutions to Laplace's equation. Hence, they have the same value at the boundary $(dV\,(V_1 = V_2 = V_{fixed}))$

$f = 0$ in volume V too, or else on extrema would occur inside region.

Suppose $f > 0$ inside V, in order to go to zero at dV or f would have some max insde V $(\nabla^2 f \neq 0)$: contradiction.

$\implies f = 0$ or $V_1 = V_2$ is unique.

B) Suppose we pick a situation where Neumann b.c. conditions are used. Then, $\vec{\nabla}f = 0$ at dV (since $\vec{\nabla}V_1.\hat{n}_{=}\vec{\nabla}V_2.\hat{n}$)

Within Volume V $\vec{\nabla} f = 0$ too since,
$\oint_{aV} f \vec{\nabla} f \cdot d\vec{a} = 0 \implies \int_V |\vec{\nabla} f|^2 d^3x$
since $\vec{\nabla} f = 0$ everywhere

Here, we have a gauge freedom. This can be seen because $V_1 = V_2 + V_0$ for some V_0 (constant).

- Physics only cares about how the V changes ($\vec{E} = -\vec{\nabla} V$))

e.g. of B) Parallel Plates

Say, $|\vec{B}| = |\frac{\mathbf{\nabla} V}{d}| = 1000$ V/m

$\vec{E}$ will always be in direction of lower potential.
Separation of variables

Goal: Find $V(x, y, z)$ such that $\nabla^2 V = 0$

In 3D we have 6 total b.c.'s.

Working under assumption of $V(x, y, z) = f(x^1)g(x^2)h(x^3)$ s.t. superscripts represent different x's

The solutions depend on coordinate system:

1. Cartesian: x,y,z or $x^1, x^2, x^3 \implies$ sines/cosines, exponentials
2. Cylindrical: s,ρ,z $\implies$ Bessel functions: J^n, K^n
3. Spherical: r, θ, ϕ $\implies$ Legendre Polynomials (i.e. new trancendental functions)

1. Cartesian

- $c < 0$, $c < -k^2$ $\implies$ f" $\overline{f = -k^2}$ s.t. k is constant

e.g. 2D)