Last Lecture

- In two-person zero-sum game, maxmin iff equilibrium
  - Interchangeability
  - Unique equilibrium payoff
- Iff condition for a strategy profile being a mixed equilibrium
  - $\forall i, \sigma_{-i}$, Indifference between actions assigned positive prob.
  - Prefer positive supports over zero supports
- A procedure to find all mixed Nash: enumerate support!
- Correlated equilibrium: players randomize according to correlated event(s)
  - mixed Nash: players randomize according to independent events (i.e., mixed strategy), a special case of CE
  - equivalently: a mediator samples according to a distribution over action profiles, tells each player what action to play (nothing about others) $\Rightarrow$ each player wants to follow her advise
  - calculate this distribution is easy

Introduction

- The normal form game representation does not incorporate "order of plays"
- The extensive form is an alternative representation that makes the temporal structure explicit.
- Two variants:
    - perfect information extensive-form games
    - imperfect-information extensive-form games

Definition

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- Choice nodes and labels for these nodes:
  - Choice nodes: H is a set of non-terminal choice nodes

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  - Player function: $\rho : H \rightarrow N$ assigns to each non-terminal node h a player $i \in N$ who chooses an action at h

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- Terminal nodes: Z is a set of terminal nodes, disjoint from H

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  - Player function: $\rho : H \to N$
- Terminal nodes: Z
- Successor function: $\sigma : H \times A \to H \cup Z$ maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
  - The choice nodes form a tree, so we can identify a node with its history.

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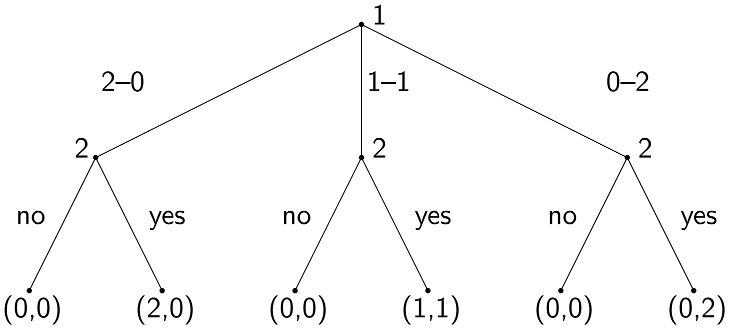
- Successor function: $\sigma : H \times A \rightarrow H \cup Z$
- Utility function: $u = (u_1, \ldots, u_n)$; $u_i : Z \rightarrow \mathbb{R}$ is a utility function for player i on the terminal nodes Z

Example: the sharing game



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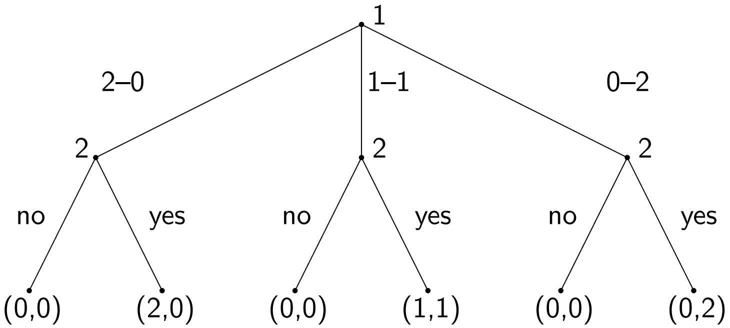

Pure Strategies

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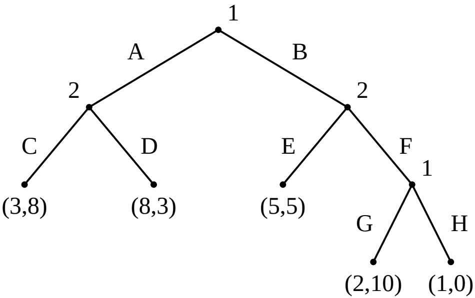
  - player 1: 3; player 2: 8
- Overall, a pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player.

Definition (pure strategies)

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consist of the cross product

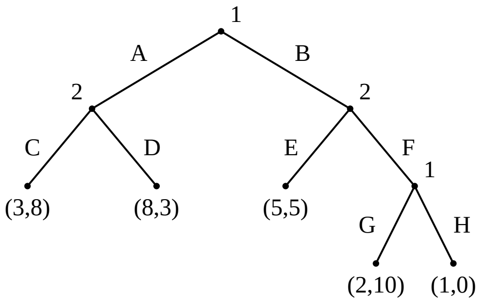
$$\underset{h \in H, \rho(h)=i}{\times} \chi(h)$$

Pure Strategies Example



What are the pure strategies for player 2?

Pure Strategies Example



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- $S_2 = \{(C,E);(C,F);(D,E);(D,F)\}$

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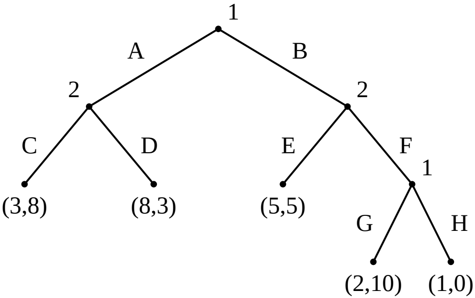

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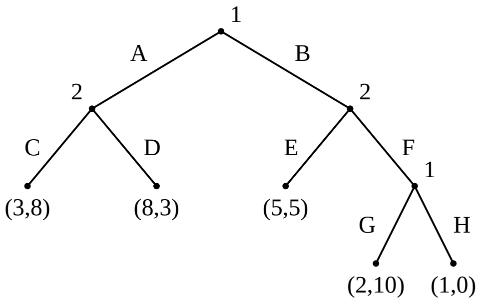

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- $S_2 = \{(C,E);(C,F);(D,E);(D,F)\}$

What are the pure strategies for player 1?

- $S_1 = \{(B,G);(B,H),(A,G),(A,H)\}$
- This is true even though, conditional on taking A, the choice between G and H will never have to be made

Nash Equilibria

Given our new definition of pure strategy, we are able to reuse our old definitions of:

- mixed strategies
- best response
- Nash equilibrium

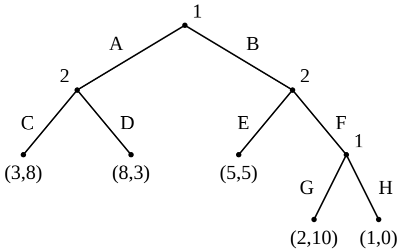
Theorem

Every perfect information game in extensive form has a PSNE

Prove later for a stronger result.

Induced Normal Form

- In fact, the connection to the normal form is even tighter
  - we can “convert” an extensive-form game into normal form



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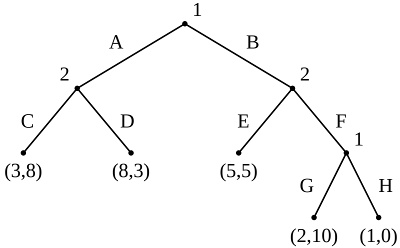

	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
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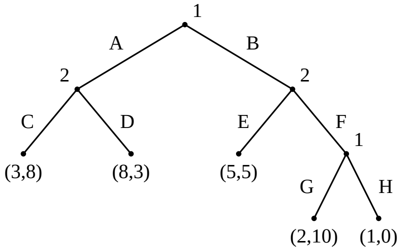

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- this illustrates the lack of compactness of the normal form
  - games aren't always this small
  - even here we write down 16 payoff pairs instead of 5

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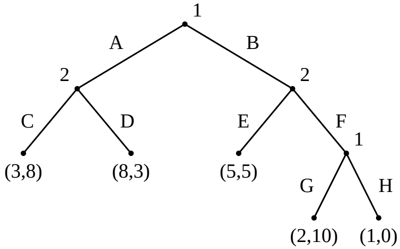

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- while we can write any extensive-form game as a NF, we can't do the reverse.
  - e.g., matching pennies cannot be written as a perfect-information extensive form game
  - imperfect information is needed!

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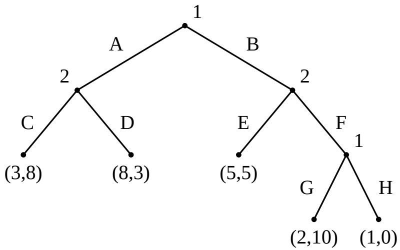

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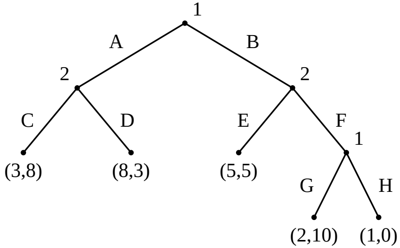

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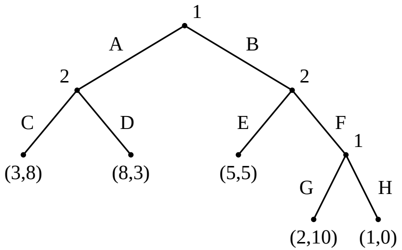

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