Introduction to Game Theory

3b. Extensive-Form Games

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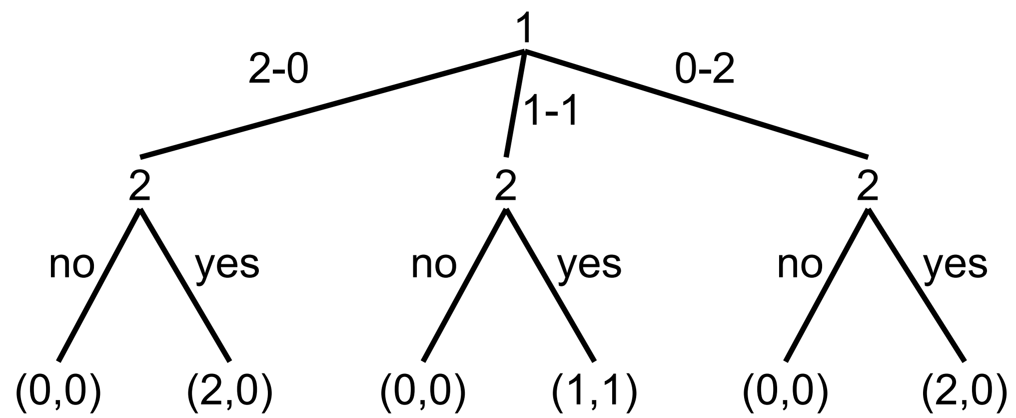
The Sharing Game

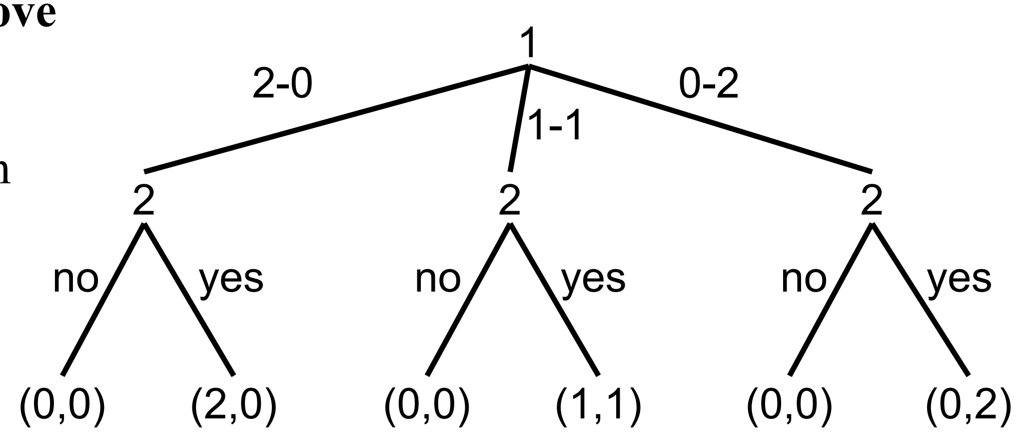
- Suppose agents 1 and 2 are two children
- Someone offers them two cookies, but only if they can agree how to share them
- Agent 1 chooses one of the following options:
  - Agent 1 gets 2 cookies, agent 2 gets 0 cookies
  - They each get 1 cookie
  - Agent 1 gets 0 cookies, agent 2 gets 2 cookies
- Agent 2 chooses to accept or reject the split:
  - Accept =>
    they each get their cookies(s)
  - Otherwise, neither gets any



Extensive Form

- The sharing game is a game in **extensive form**
  - A game representation that makes the temporal structure explicit
  - Doesn't assume agents act simultaneously
- Extensive form can be converted to normal form, so previous results carry over
  - But there are additional results that depend on the temporal structure

- In a perfect-information game, the extensive form is a **game tree**:
  - Nonterminal node = place where an agent chooses an action
  - Edge = an available **action** or **move**
  - Terminal node = a final outcome
    - At each terminal node h, each agent i has a utility $u_i(h)$

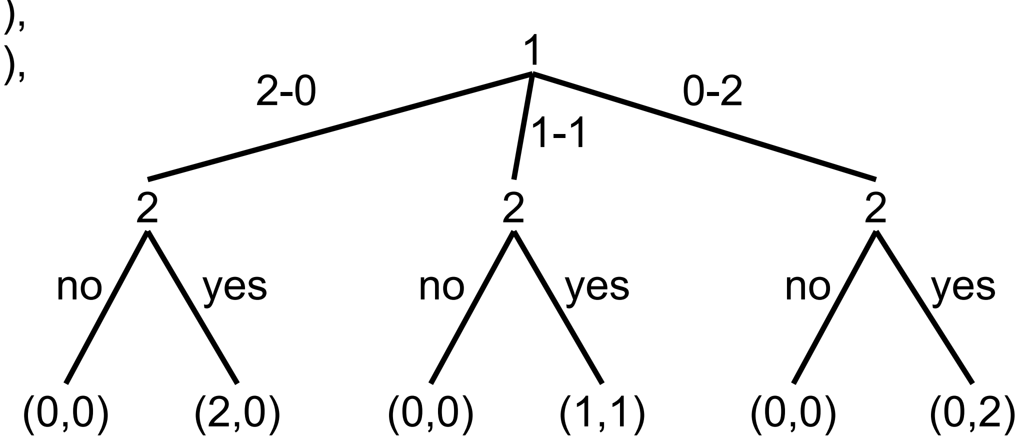


Pure Strategies

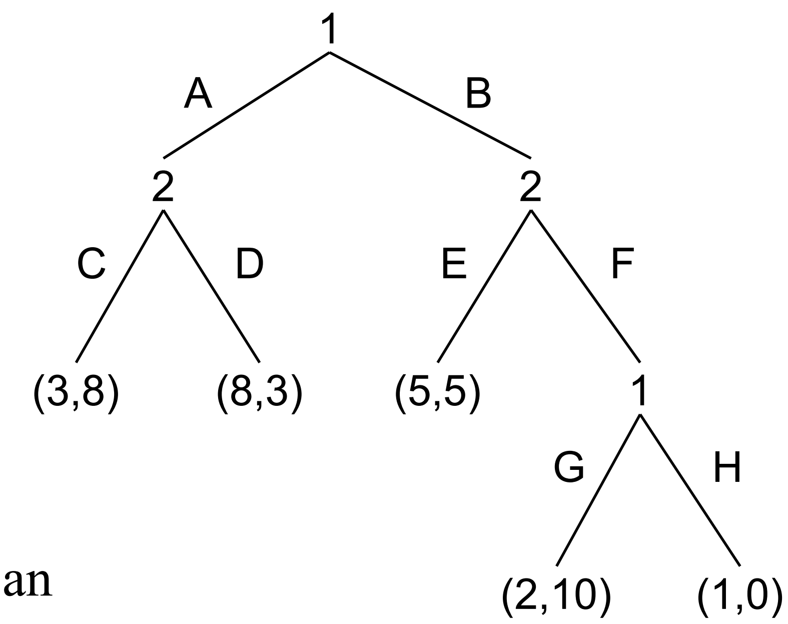
- Pure strategy for agent *i* in a perfect-information game:
  - specifies which action to take at every node where it's *i*'s choice

Sharing game:

- Agent 1 has 3 pure strategies:
  - S_1 = {2-0, 1-1, 0-2}
- Agent 2 has 8 pure strategies:
  - S_2 = {(yes, yes, yes), (yes, yes, no),
    (yes, no, yes), (yes, no, no),
    (no, yes, yes), (no, yes, no),
    (no, no, yes), (no, no, no)}



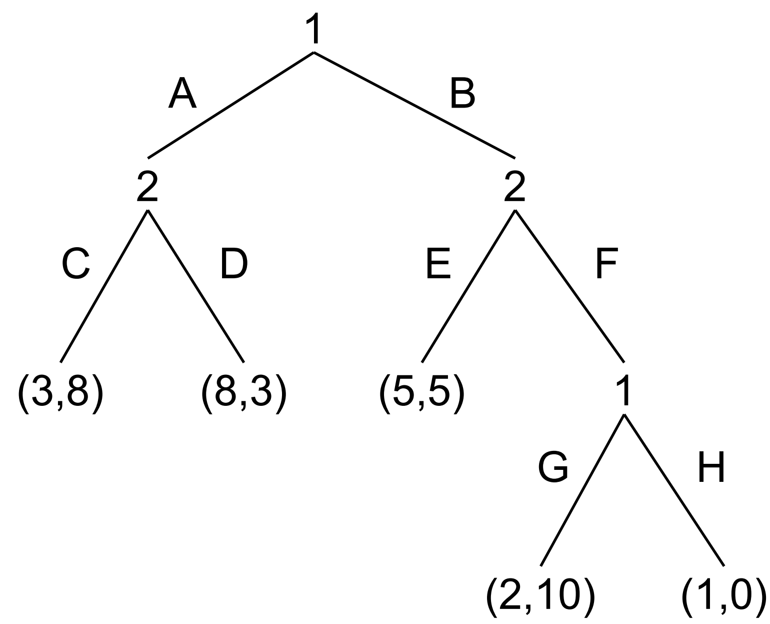
Extensive form vs. normal form



- Every game tree corresponds to an equivalent normal-form game
- The first step is to get all of the agents' pure strategies
- An agent's complete strategy must specify an action at every node where it's the agent's move
- Example: the game tree shown here
  - Agent 1 has four pure strategies:
    - $s_1 = \{(A, G), (A, H), (B, G), (B, H)\}$
      - Must include (A, G) and (A, H), even though action A makes the G-versus-H choice moot
  - Agent 2 also has four pure strategies:
    - $s_2 = \{(C, E), (C, F), (D, E), (D, F)\}$

Extensive form vs. normal form

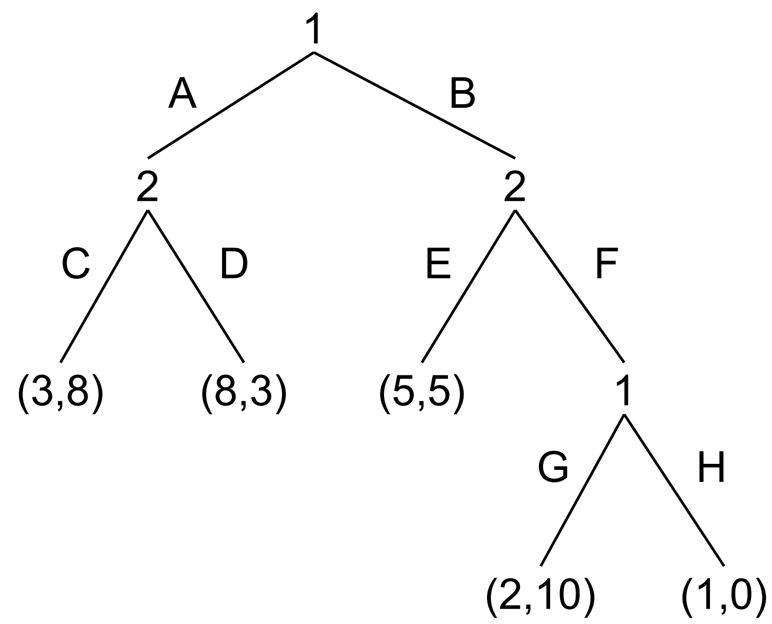
- Once we have all of the pure strategies, we can rewrite the game in normal form
- Converting to normal form introduces redundancy
  - 16 outcomes in the payoff matrix, versus 5 outcomes in the game tree
  - Payoff (3,8) occurs
    - once in the game tree
    - four times in the payoff matrix
- This can cause an exponential blowup



	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	3,8	8,3	8,3
(A,H)	3,8	3,8	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	5,5	1,0	5,5	1,0

Nash Equilibrium

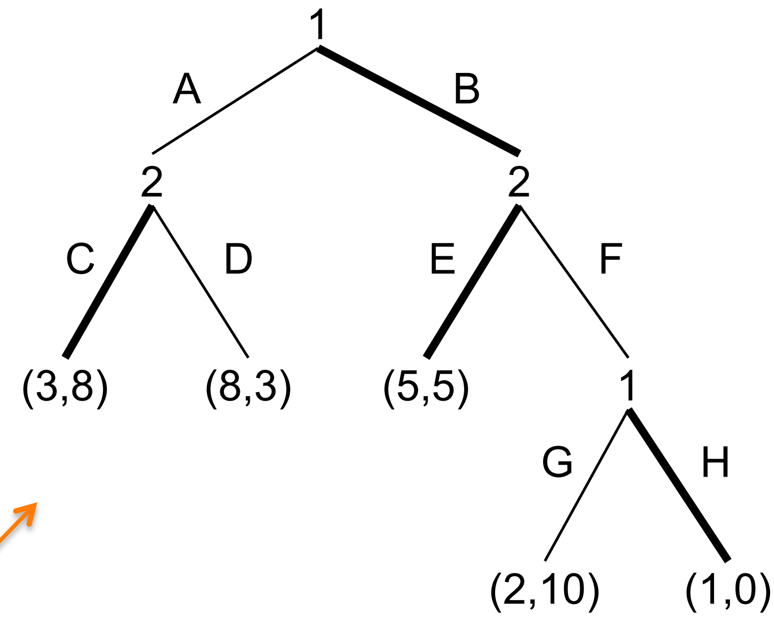
- **Theorem.** Every perfect-information game in extensive form has a pure-strategy Nash equilibrium
  - This theorem has been attributed to Zermelo (1913), but there's some controversy about that
- Intuition:
  - Agents take turns, and everyone sees what's happened so far before making a move
  - So never need to introduce randomness into action selection to find an equilibrium
- In our example, there are three pure-strategy Nash equilibria



	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3,8	$\boxed{3,8}$	8,3	8,3
(A,H)	3,8	$\boxed{3,8}$	8,3	8,3
(B,G)	5,5	2,10	5,5	2,10
(B,H)	$\boxed{5,5}$	1,0	5,5	1,0

Nash Equilibrium

- The concept of a Nash equilibrium can be too weak for use in extensive-form games
- Recall that our example has three pure-strategy Nash equilibria:
  - {(A,G), (C,F)}
  - {(A,H), (C,F)}
  - {(B,H), (C,E)}
- Here is {(B,H), (C,E)} with the game in extensive form



	(C,E)	(C,F)	(D,E)	(D,F)
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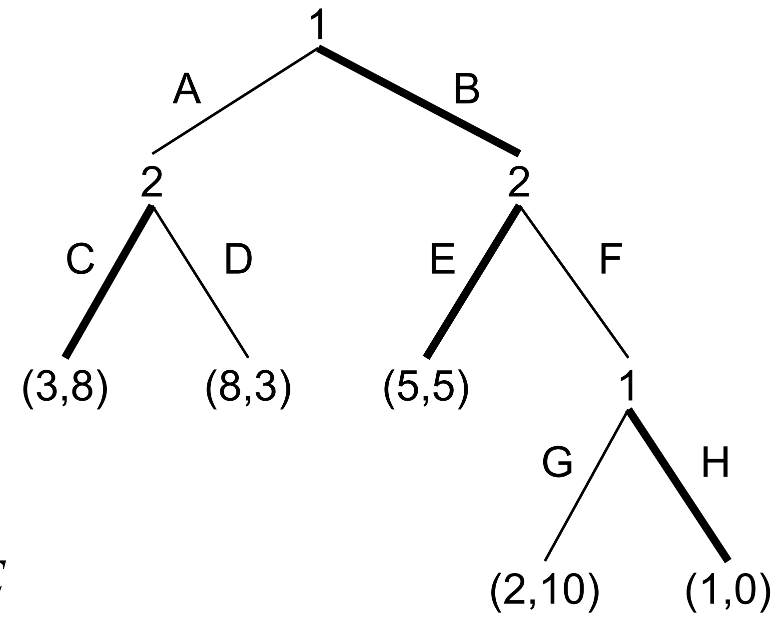
Nash Equilibrium



- If agent 1 used (*B,G*) instead of (*B,H*)
  - Then agent 2's best response would be (*C,F*), not (*C,E*)
- When agent 1 plays *B*
  - The only reason for agent 2 to choose *E* is if agent 1 has already committed to *H* rather than *G*
- This behavior by agent 1 is a *threat*:
  - By committing to choose *H*, which is harmful to agent 2, agent 1 can make agent 2 avoid that part of the tree
  - Thus agent 1 gets a payoff of 5 instead of 2
- But is agent 1's threat credible?
  - If agent 2 plays *F*, would agent 1 *really* play *H* rather than *G*?
  - It would reduce agent 1's own utility

Subgame-Perfect Equilibrium

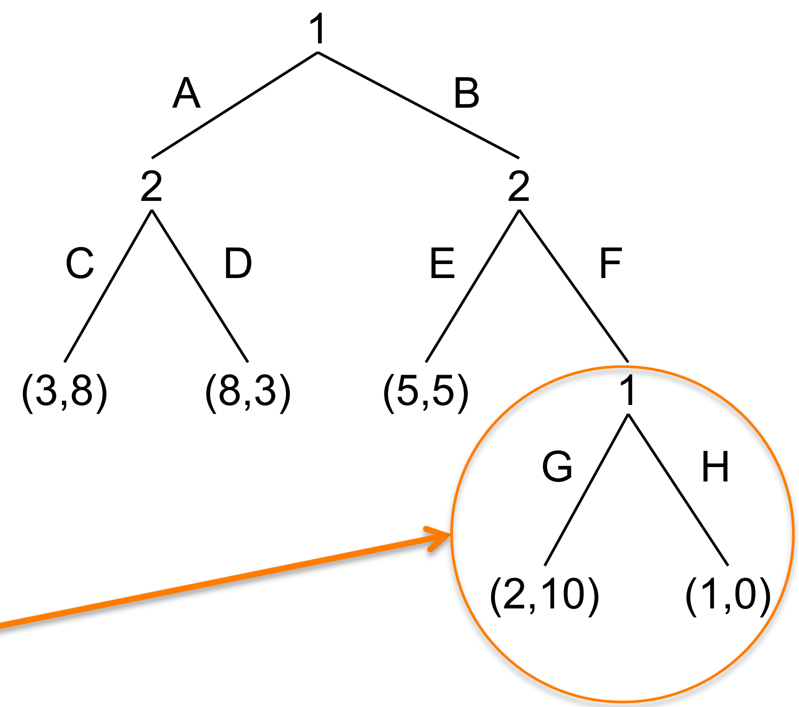
- Given a perfect-information extensive-form game G, the **subgame** of G rooted at node h is the restriction of G to the descendants of h

- Now we can define a refinement of the Nash equilibrium that eliminates noncredible threats

- A **subgame-perfect equilibrium** (SPE) is a strategy profile S such that for every subgame G' of G, the restriction of S to G' is a Nash equilibrium of G'
  - Since G itself is is a subgame of G, every SPE is also a Nash equilibrium

- Every perfect-information extensive-form game has at least 1 SPE
  - Can prove this by induction on the height of the game tree

Example



- Recall that we have three Nash equilibria:

  $\{(A, G), (C, F)\}$

  $\{(A, H), (C, F)\}$

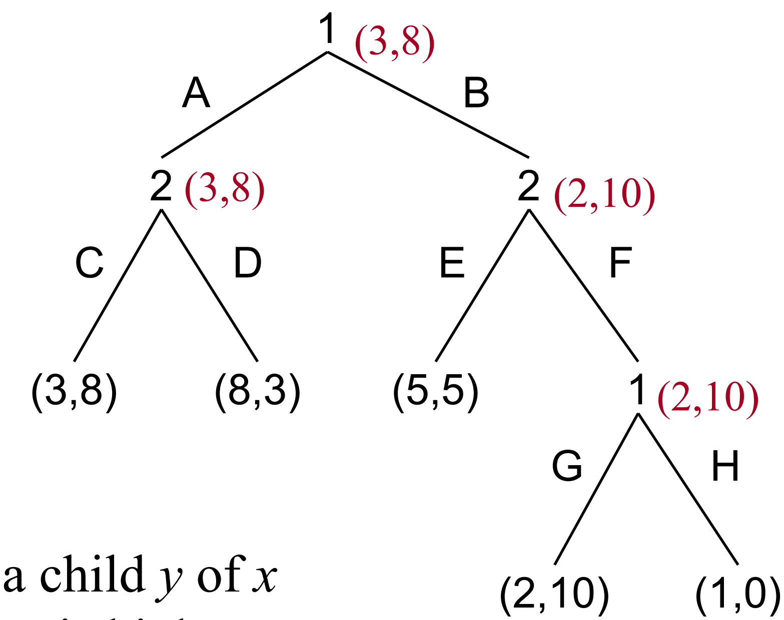
  $\{(B, H), (C, E)\}$

- Consider this subgame:
  - For agent 1,

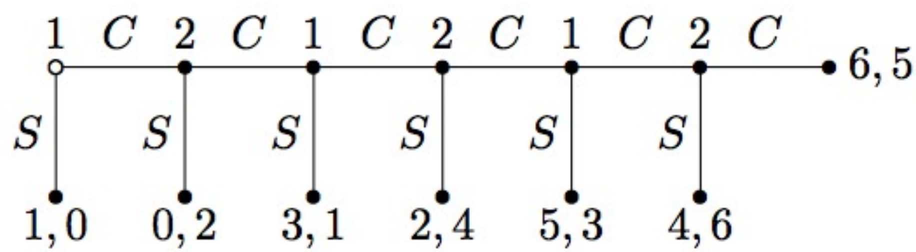
    G strictly dominates H
  - Thus H can't be part of a Nash equilibrium
  - This excludes $\{(A, H), (C, F)\}$ and $\{(B, H), (C, E)\}$
  - Just one subgame-perfect equilibrium
    - $\{(A, G), (C, F)\}$

Backward Induction

- To find subgame-perfect equilibria, we can use **backward induction**
- Identify the equilibria in the bottom-most nodes
  - Assume they'll be played if the game ever reaches these nodes
- For each node x, recursively compute a vector $v_x = (v_{x1}, \ldots, v_{xn})$ that gives every agent's equilibrium utility
  - At each node x,
    - If i is the agent to move, then i's equilibrium action is to move to a child y of x for which i's equilibrium utility v_{yi} is highest
    - Thus $v_x = v_y$



Let's Play a Game

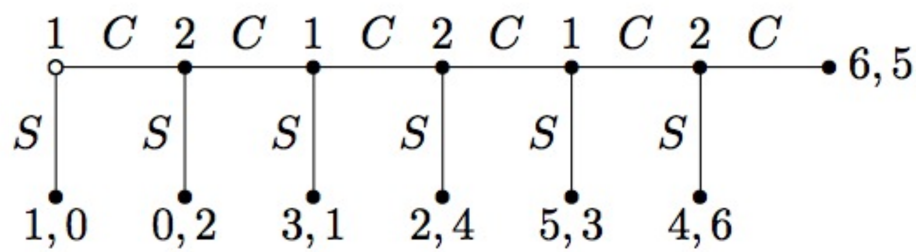


- I need two volunteers to play the game shown here:
  - One to be Agent 1
  - One to be Agent 2

- Whenever it's your turn to move, you have two possible moves:
  - *C* (continue) and *S* (stop)
- Agent 1 makes the first move
- At each terminal node, the payoffs are as shown

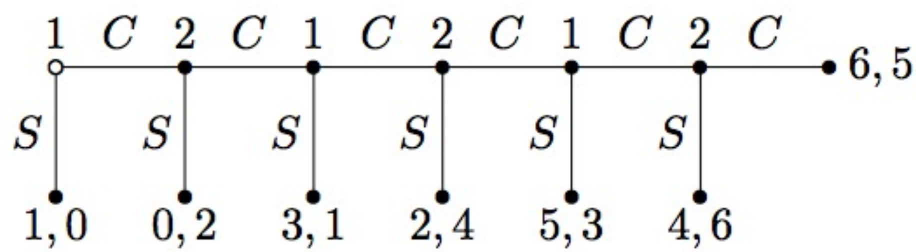
A Problem with Backward Induction

The Centipede Game



- Can extend this game to any length
- The payoffs are constructed in such a way that for each agent, the only SPE is always to choose *S*
- This equilibrium isn't intuitively appealing
  - Seems unlikely that an agent would choose *S* near the start of the game
  - If the agents continue the game for several moves, they'll both get higher payoffs
  - In lab experiments, subjects continue to choose *C* until close to the end of the game

A Problem with Backward Induction



- Suppose agent 1 chooses *C*
- If you're agent 2, what do you do?
  - SPE analysis says you should choose *S*
  - But SPE analysis also says you should never have gotten here at all
  - How to amend your beliefs and course of action based on this event?
- Fundamental problem in game theory
  - Differing accounts of it, depending on
    - the probabilistic assumptions made
    - what is common knowledge (whether there is common knowledge of rationality)
    - how to revise our beliefs in the face of an event with probability 0

Backward Induction in Zero-Sum Games

- Backward induction works much better in zero-sum games
  - No zero-sum version of the Centipede Game, because we can't have increasing payoffs for both players
- Only need one number: agent 1's payoff (= negative of agent 2's payoff)
- Propagate agent 1's payoff up to the root
  - At each node where it's agent 1's move,
    the value is the maximum of the labels of its children
  - At each node where it's agent 2's move,
    the value is the minimum of the labels of its children
  - The root's label is the **value** of the game (from the Minimax Theorem)

- In practice, it may not be possible to generate the entire game tree
  - E.g., extensive-form representation of chess has about 10^{150} nodes
- Need a heuristic search algorithm

Summary

- Extensive-form games
  - relation to normal-form games
  - Nash equilibria
  - subgame-perfect equilibria
  - backward induction
    - The Centipede Game
  - backward induction in zero-sum games