Basics of Game Theory

Jacob W. Crandall

February 11, 2008

1 What is Game Theory?

Game theory is a mathematical method for predicting human behavior in strategic, multi-player situations. Game theory assumes that each player[1] forms rational beliefs about what other players (associates) in the game will do and then chooses a response to those strategies that maximizes its own payoffs. Under the assumption of *rationality*, game theory assumes that play will be at (or will converge to) an equilibrium.

In these notes, we will discuss the basic solution concepts of game theory. In particular, we will talk about maximin (or minimax), the best response, Nash equilibria, strategic dominance, and pareto optimality. But first, some notation.

2 Notation

Let A_i be the set of actions available to agent i, and let $A = (A_1, \ldots, A_n)$, where n is the number of players in the game, be the set of *joint actions* that can be played in the game. Thus, $\mathbf{a} \in A$ is a vector $\mathbf{a} = (a_1, \ldots, a_n)$, where $a_i \in A_i$. Let $\mathbf{a}_{-1}$ denote the joint action taken by all agents except agent i.

A *strategy* (or *policy*) for agent i is a probability distribution π_i over its action set A_i[2]. When all probability is placed on a single action, the strategy is said to be a *pure strategy.* Otherwise, the strategy is said to be a *mixed strategy.* A *joint strategy* played by n agents is denoted by $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$. Also, let $\boldsymbol{\pi}_{-i}$ denote the joint strategy of all agents except agent i.

Central to game theory is the matrix game, which is defined by a set of matrices $R = \{r_1, \ldots, r_n\}$. Let $r_i(\mathbf{a})$ be payoff to player i when the joint action $\mathbf{a}$ is played[3]. Also, let $r_i(\boldsymbol{\pi})$ be the payoff to player i when the joint strategy $\boldsymbol{\pi}$ is played, and let $r_i(\pi_i, \boldsymbol{\pi}_{-i})$ be the expected payoff to player i when it plays π_i and when the the other agents play $\boldsymbol{\pi}_{-i}$.

3 Solution Concepts

It would be really nice if we could identify a way to play every game. That is, it would be really nice if you could give me any payoff matrix, and I could use a certain set of mathematical rules to decide how to play the game. That is exactly what game theorists have tried to do. From this work, a number of solution concepts have been developed. Such solution concepts are supposed to indicate/predict how people will behave when they play a generic game. Throughout the class, we will refer to these concepts a lot, so we should be sure to understand them.

[1] As always, we will use the terms *player* and *agent* interchangeably.

[2] Here, we assume a normal-form (or matrix) game. Recall that any extensive-form game can be converted to a normal-form game.

[3] Notice that this notation assumes that payoffs are deterministic (namely, a single payoff for each joint action). We could easily remove this assumption, but we won't for simplicity.

3.1 The Maximin Solution

One of the most basic properties of every game is the *maximin solution* (or maximin strategy). The maximin solution is the strategy that *maximizes a player's minimum expected payoff*. Note that there is an alternate set of terminology we can use (and is often used in the literature). Rather than speak of maximizing our minimum expected payoff, we can talk of minimizing our maximum expected costs. This is known as the *minimax solution*. Thus, the terms minimax and maximin can be used interchangeably (and we will often do so – often for no good reason other than habit).

Let's take a look at one of the most studied matrix games of all time: the prisoner's dilemma.

	cooperate	defect
cooperate	3, 3	0, 5
defect	5, 0	1, 1

In the prisoner's dilemma, both players are faced with the choice of *cooperating* or *defecting*. If both players cooperate, they both receive a relatively high payoff (which is 3 in this case). However, if one of the players cooperates and the other defects, the defector receives a very high payoff (called the *temptation payoff*, which is 5 in this case), and the cooperator receives a very low payoff (called the *succor's payoff*, which is 0 in this case). If both players defect, then both receive a low payoff (which is 1 in this case).

So what should you do in this game? Well, there are a lot of ways to look at it, but if you want to play conservatively, you might want to invoke the maximin solution concept, which follows from the following reasoning. If you play *cooperate*, the worst you can do is get a payoff of 0 (thus, we say that the *security* of cooperating is 0). Likewise, if you play *defect*, the worst you can do is get a payoff of 1 (*security* of defecting is 1). Alternately, we can form a mixed strategy over the two actions. However, it just so happens in this game that no mixed strategy has higher security than defecting, so the maximin strategy in this game is to defect. This means that the *maximin value* (which is the minimum payoff I can receive when I play the maximin strategy) is 1.

Let's look at another example, which we'll call the fighters and bombers game. This game, which is described by a fellow named Casti, comes from air combat during World War II. In that day of slower bombers, fighter aircraft typically adopted the strategy of swooping down on the bombers from above from the direction of the sun (so they would be difficult to spot). To combat this strategy, the gunners on the bombers put on their sunglasses and looked into the sun for fighter aircraft. This gave fighter aircraft the chance to attack the bombers by simply flying directly at the bombers from below. If the fighter was not spotted, (s)he succeeded. However, if (s)he was spotted, then it was certain death.

We can capture this game with a payoff matrix, where the fighter aircraft is the row player and the bomber is the column player (this is a zero-sum game, so we only show the payoffs to the fighter aircraft).

	Look Up	Look Down
Sun Attack	0.95	1
Bottom Attack	1	0

In this case, the security of the Sun Attack is 0.95, while the security of the Bottom Attack is zero. This means that in the pure-strategy space of the game, the maximin solution concept says that a fighter aircraft should attack from the sun. However, the fighter aircraft can minimize its expected costs even further if it uses a mixed strategy. Namely, if a fighter aircraft uses the Sun Attack with probability $p = \frac{20}{21}$ and the Bottom Attack with probability $1 - p = \frac{1}{21}$, it guarantees itself an expected payoff of approximately 0.9524 (which is the maximin value for the game). Let's go over where that value comes from.

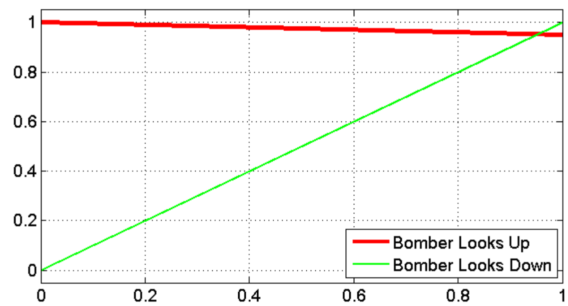
If the bomber plays Look Up, then the fighter's expected payoff given p (the probability that the fighter plays Attack Down) is given by

$$f(p|\text{LookUp}) = 0.95p + 1(1 - p). \tag{1}$$

Likewise, if the the bomber plays Look Down, the fighter's expected payoff is

$$f(p|\text{LookDown}) = 1p + 0(1 - p) = p. \tag{2}$$

Using these two functions, we can find out what value p should be for the player to maximize its minimum expected payoff. First, consider the following figure, which shows both payoff functions. The figure shows



that the fighter can maximize its minimum cost if it finds the point at which the lines meet. We can solve for this point by setting the two equations equal to each other and solving for p.

$$f(p|\text{LookUp}) = f(p|\text{LookDown}) \tag{3}$$

$$0.95p + 1(1-p) = p \tag{4}$$

$$p = \frac{1}{1.05} = \frac{20}{21} \tag{5}$$

Let's give a formal definition of the maximin solution and value. Formally, the maximin solution is

$$\mu_i^m = \arg\max_{\pi_i} \min_{a_{-i} \in A_{-i}} r_i(\pi_i, a_{-i}), \tag{6}$$

and the maximin value is

$$\pi_i^m = \max_{\pi_i} \min_{a_{-i} \in A_{-i}} r_i(\pi_i, a_{-i}). \tag{7}$$

Thus, given knowledge of its payoff matrix R_i, player i can calculate her/his/its maximin solution and value using linear programming.

However, even though the maximin value is the highest payoff you can guarantee yourself without the cooperation of your associates, you might be able to do much better on average than the maximin strategy if you can either outsmart your associates or get them to cooperator or compromise with you (in a game that is not fully competitive). So we need other solution concepts as well.

3.2 Best Response

Another basic solution concept in multi-agent choice is to play the strategy that gives you the highest payoff given your associates's strategies. That is exactly what the notion of the *best response* suggests. Suppose that you are player i, and your associates[4] are playing the strategy $\boldsymbol{\pi}_{-i}$. Then the your best response is the strategy π_i^*, such that $r_i(\pi_i^*, \boldsymbol{\pi}_{-i}) \geq r_i(\pi_i, \boldsymbol{\pi}_{-i})$ for all π_i.

As a side comment, we will see throughout this class that the best response idea has had a huge impact on multi-agent learning algorithms (and rightly so). If you know what your associates are going to do, why not get the highest payoff you can get (i.e., why not play a best response)? Taking this one step further, you might reason that if you think you know what your associates are going to do, why not play a best response to that belief? While this obviously is not an unreasonable idea, it has two problems. The first problem is that your belief may be wrong, which might expose you to terrible risks. Second (and perhaps more importantly), we will see that this "best-response" approach can be quite unproductive in a repeated game when associates (i.e., humans or anything else) are learning/adapting.

[4] We use the terms *associates* and *opponents* to denote the other players in the game.

3.3 Nash Equilibrium

John Nash's identification of the Nash equilibrium (which exists, he showed, in every game) concept has had perhaps the single biggest impact on game theory. Simply put, in a Nash equilibrium, no player has an incentive to unilaterally deviate from its current strategy. Put another way, if each player plays a best response to the strategies of all other players, we have have a Nash equilibrium. But let us illustrate the concept with examples.

We return to the prisoner's dilemma game:

	cooperate	defect
cooperate	3, 3	0, 5
defect	5, 0	1, 1

Let's identify a Nash equilibrium of this game by going through each cell of the game. Suppose that both players strategies are to cooperate. Well, this is not a Nash equilibrium since the row player could benefit by changing his/her/its action to defect (so could the column player for that matter). Now supposed that the row player's strategy is defect, and the column player's strategy is cooperate. Then, the column player could benefit by unilaterally changing its strategy, hence this is not a Nash equilibrium either. However, if both players' strategy is to defect, than neither player can benefit by choosing to cooperate. Thus, the joint action $(defect, defect)$ is a Nash equilibrium.

Let's look at another game (a coordination game):

	a	b
A	2, 2	0, 0
B	0, 0	1, 1

By inspection (such as we did for the prisoner's dilemma), we can conclude that both the joint actions (A, a) and (B, b) are Nash equilibria since in both cases, neither player can benefit by unilaterally changing its strategy. Note, however, that this illustrates that not all Nash equilibria are created equally. Some give better payoffs than others (and some players might have different preference orderings over Nash equilibrium).

While all the Nash equilibria we have identified so far for these two games are pure strategy Nash equilibrium, they need not be so. In fact, there is also a third Nash equilibrium in the above coordination game in which both players play mixed strategies. Can you find it?

Nash showed that every game has at least one Nash equilibrium. However, as far as we are aware, there is no method for finding a Nash equilibrium in polynomial time for a generic game. Recently, Papadimitriou showed that computing a Nash equilibrium in the general case is *hard*. :)

Here are a couple more observations about the Nash equilibrium as a solution concept.

- In constant-sum games, the maximin solution is a Nash equilibrium of the game. In fact, it is the unique Nash equilibrium of constant-sum games as long is there is not more than one maximin solution (which occurs only when two strategies have the same security level).

- Since a game can have multiple Nash equilibrium, this concept does not tell us how to play a game (or how we would guess others would play the game). This poses another question: Given multiple Nash equilibrium, which one should (or will) be played?

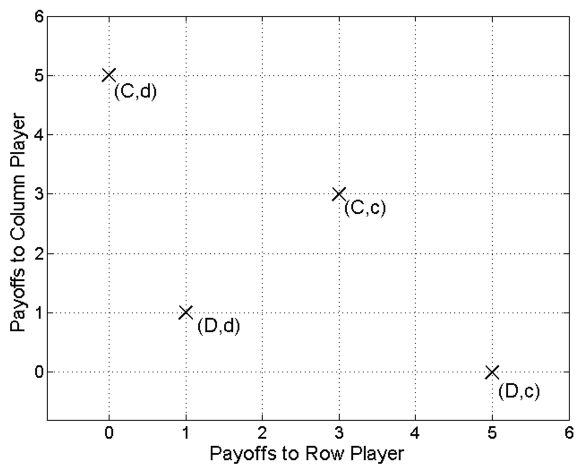
3.4 Strategic Dominance

Strategic dominance is another powerful solution concept that can be used in many games, but not all. Loosely, an action is *strategically dominated* if it never produces higher payoffs and (at least) sometimes produces lower payoffs than some other action. An action is *strategically dominant* if it strategically dominates all other actions. For example, in the prisoner's dilemma, the action *defect* strategically dominates *cooperate* in the one-shot game.

The concept of strategic dominance (or just dominance, as we will sometimes call it) can be used in some games (called *iterative dominance solvable games*) to compute a Nash equilibrium. We'll talk more about these kinds of games in class.

3.5 Pareto Optimality

Technically, pareto optimality is not a solution concept per se, but it can be an important attribute in determining what solution the players should play (or learn to play). Loosely, a pareto optimal (also sometimes called pareto efficient) solution is a solution for which there exists no other solution that gives every player in the game a higher payoff. For example, consider the joint payoffs of the prisoner's dilemma shown in the figure below.



In this game, each outcome is pareto optimal except the Nash equilibrium solution (D, d).

But let's give a more formal definition

Definition 1. *(Pareto Dominated) A solution* $\boldsymbol{\pi}$ *is strictly pareto dominated if there exists a joint action* $\mathbf{a} \in A$ *for which* $r_i(\mathbf{a}) > r_i(\boldsymbol{\pi})$ *for all* i *and weakly pareto dominated if there exists a joint action* $\mathbf{a} \in A$, $\mathbf{a} \neq \boldsymbol{\pi}$ *for which* $r_i(\mathbf{a}) \geq r_i(\boldsymbol{\pi})$ *for all* i.

Definition 2. *(Pareto Efficient) A solution* $\boldsymbol{\pi}$ *is weakly pareto efficient (PE) if it is not strictly pareto dominated and strictly PE if it is not weakly pareto dominated.*