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Restricting infections on graph classes

Supervised by Ana Karolinnna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

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Information

Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.

- » The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
- » The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

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- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.
- Even bedbugs seem to spread from hotel to hotel via travelers (Barabási and Pósfai 2016).

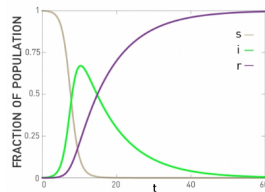
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Example (SIR)



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- › The homogeneous mixing hypothesis is **false**.
- › The **structure of the contact network** is what facilitates the contagion.



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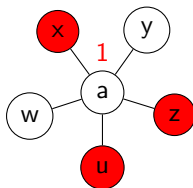
Studies have shown that the airport mobility network provides reliable information for the prediction and control of airborne epidemics like H1N1 (Soriano-Paños et al. 2022).

Modeling Contagion on Graphs: Threshold Model

- Let's consider each vertex as an individual.
 - A vertex is either **active (infected)** or **inactive (susceptible)**.

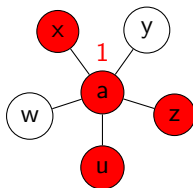
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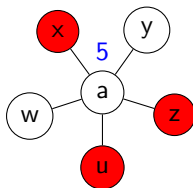
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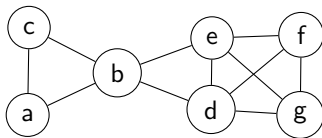
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 - Bigger thresholds: laggards / greater resistance.



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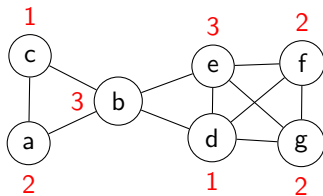
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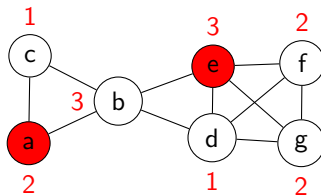
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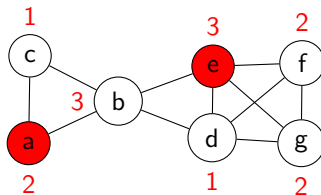
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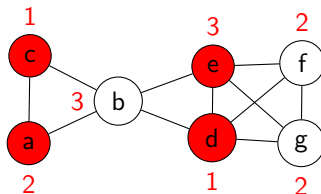
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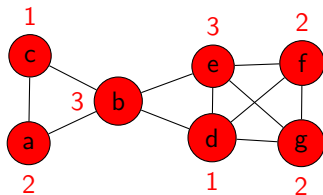
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- A **graph** $G = (V, E)$;
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- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t -**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

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- **Influence Maximization (IM)** (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.
 - » Decision version is NP-complete even for bipartite graphs.
 - » NP-complete even for k -regular graphs if we require S to infect all vertices of G – **Target Set Selection (TSS)** (Dreyer and Roberts 2009).

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

Problems arising from the Threshold Model

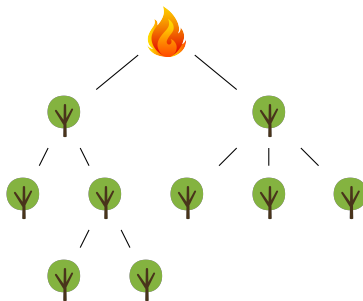
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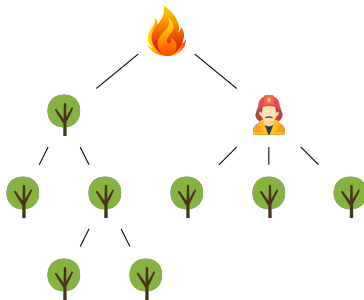


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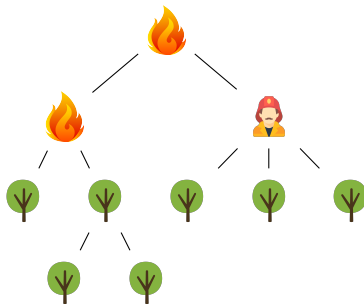


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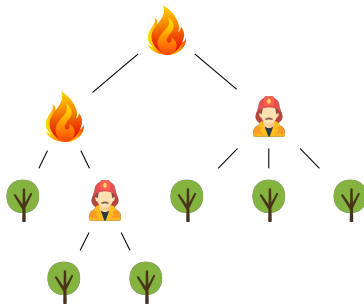


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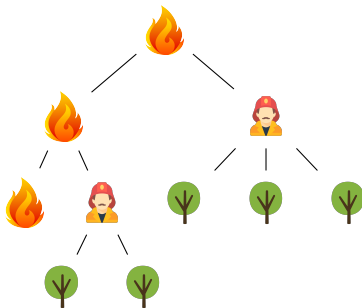


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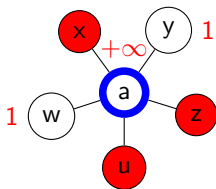


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- **Influence Immunization Bounding (IIB);**
 - A generalization of Firefighter? **No, but sort of...**

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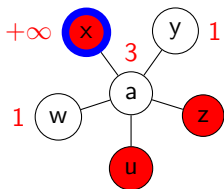
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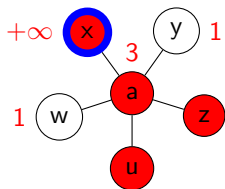
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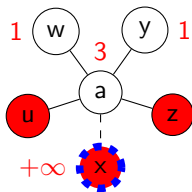


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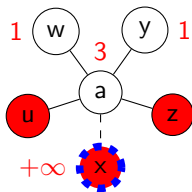


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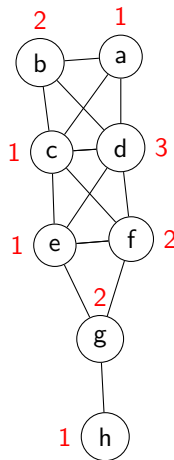
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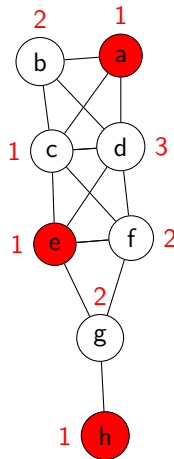


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Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

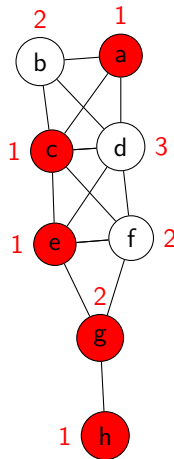


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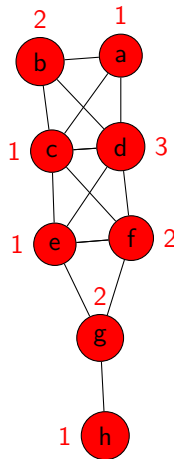


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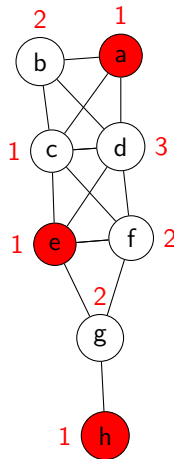
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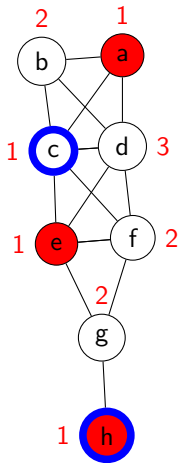
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We want to find a immunizing set $Y \subseteq V(G)$ such that:

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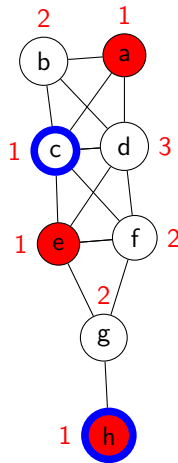
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- $|Y| \leq l$; and
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.

Remark

Notice that we must have $k \geq |S|$.



But before we proceed...

A little of **Parameterized Complexity**.

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

Input: $x \in \Sigma^*$.

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Definition

A **parameter** is a function $\kappa : \Sigma^* \rightarrow \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) = \text{"Number of variables of } \varphi\text{"}$, where φ is a CNF formula.

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p-INDEPENDENT-SET

Input: A graph G and $k \in \mathbb{N}$.

Question: Decide whether G has an independent set of cardinality k .

Parameter: k .

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 - › Robotics: the number of degrees of freedom in motion planning problems is **usually ≤ 10** .
- › This means that **parameters of the problem matter for its tractability**.

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ) .

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

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- **CLIQUE** is in XP parameterized by k : enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot \text{poly}(|x|))$$

- VERTEX COVER is in FPT parameterized by k .

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 - ›› We can show other problems are $W[t]$ -hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Influence Immunization Bounding was introduced by (Cordasco, Gargano, and Rescigno 2023).

Parameter	Hardness
k	W[1]-hard $t(v) = 1 \quad \forall v$
l	W[1]-hard $t(v) = 1 \quad \forall v$
$k + l$	FPT
$ S + l$	W[2]-hard Bipartite graphs
$k + S $	FPT
$\Delta(G) + l$	W[2]-hard $t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard
$nd(G)$	W[1]-hard
$k + nd(G)$	FPT
$l + nd(G)$	FPT
$\min(\Delta(G), k) + tw(G) + l$	FPT

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \rightarrow \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y , the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a **k -immunizing set**.

Definition

The **k -immunization number of (G, t)** , denoted by

$$\text{Im}(G, t, k)$$

, is size of a **minimum k -immunizing set** of (G, t) .

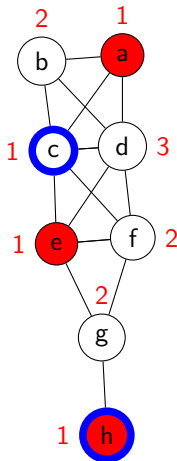
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Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t) .

$\text{Im}(G, t, k = 3) = 2$.



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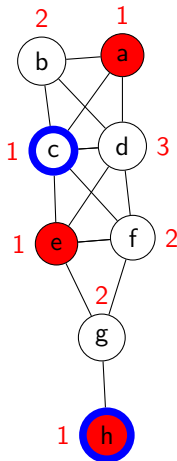
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If $k = |S|$, then we say that $\text{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t) , and denote it by $\text{In}(G, t)$.

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$\text{In}(G, t) = \text{Im}(G, t, k = 3) = 2$.



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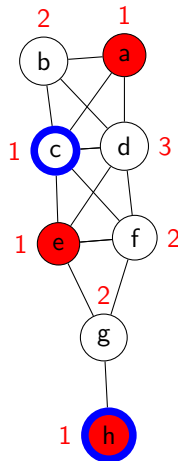
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$\text{In}(G, t) = \text{Im}(G, t, k = 3) = 2$.

Remark

For any suitable k , we have that

$$\text{Im}(G, t, k) \leq \text{In}(G, t)$$



We can also define a **restricted version** of IIB, which we call R-IIB.

Restricted Influence Immunization Bounding (R-IIB)

Input: A graph $G = (V, E)$ with thresholds $t : V(G) \rightarrow \mathbb{N}$, seed set $S \subseteq V(G)$, and $k, l \in \mathbb{N}$.

Question: Decide whether there exists $Y \subseteq V(G) \setminus S$ such that $|Y| \leq l$ and Y is a k -immunizing set of G .

- R-IIB is also $W[1]$ -hard when parameterized by k or by l .

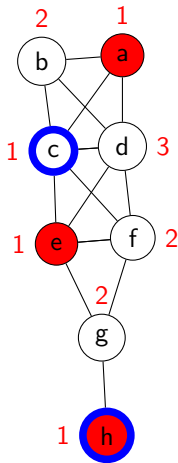
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k -immunizing set** of (G, t) if Y is a k -immunizing set of (G, t) and $Y \cap S = \emptyset$.

- $Y = \{c, h\}$ is not a restricted k -immunizing set of (G, t) .



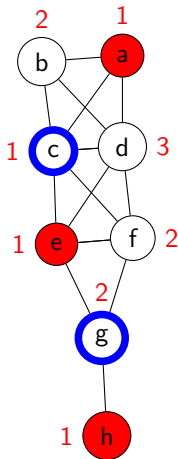
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k -immunizing set** of (G, t) if Y is a k -immunizing set of (G, t) and $Y \cap S = \emptyset$.

- $Y = \{c, h\}$ is not a restricted k -immunizing set of (G, t) .
- $Y = \{c, g\}$ is a restricted k -immunizing set of (G, t) .



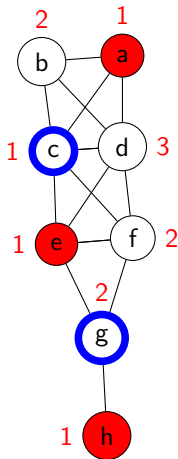
Definition

The **restricted k -immunization number** of (G, t) , denoted by

$$\text{Im}_r(G, t, k)$$

, is the size of a **minimum restricted k -immunizing set** of (G, t) .

- The **restricted inhibition number** $\text{In}_r(G, t)$ is analogous.
- But why study the restricted version?



The restricted version gives an upper bound for the original problem.

Remark

For any suitable k , we have that

$$\text{Im}(G, t, k) \leq \text{Im}_r(G, t, k)$$

- A restricted k -immunizing set for (G, t) is also a (unrestricted) k -immunizing set for (G, t) .

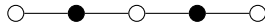
Section 6: **Our Results**

Part 1

Paths and Complete Graphs with $t(v) = c$

Definition

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G . Then $P = v_1 v_2 \dots v_k$ is a **S -alternating path** of G if P is a path of G with *at least 3 vertices* and the vertices of P alternate with respect to their membership in S .



- **Main observation:** for each S -alternating path of length 4, we need only one vertex to inhibit the infection.

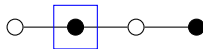
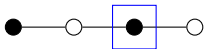
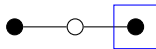


Figure: When the S -alternating path has length 4, we can inhibit the infection by immunizing only one vertex.



- (a) The infection stops by immunizing any vertex. The immunized vertex is shown in blue.



- (b) When the S -alternating path has length 3 and starts with a uninfected vertex, the infection does not spread.

- We can split the path into S -alternating paths of length 4.

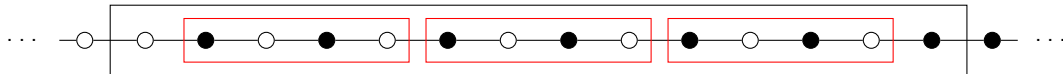


Figure: Splitting a bigger S -alternating path into pieces of length 4 starting with infected vertices.

Proposition

Let $G = P_n$ with $t(v) = 2$ for all $v \in V(G)$. Then, for any $k \in \mathbb{N}$,

$$Im(G, t, k) \leq In(G, t) \leq \lceil \frac{n}{4} \rceil$$

Definition

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G . Then:

- $P = v_1 v_2 v_3$ is a $\bullet \circ \bullet$ -path if P is a **maximal S -alternating path** of length exactly 3 such that $v_1, v_3 \in S$.
- $P = v_1 v_2 v_3 v_4$ is a $\bullet \circ \bullet \circ$ -path if P is a S -alternating path of length 4 such that $v_1, v_3 \in S$.

Part 2

Polynomial Algorithm for the k - Immunization Number in Trees

Part 3

Polynomial Algorithm the Inhibition Number when $t(v) = 1$

Part 4

Hardness for Chordal Graphs

Part 5

Hardness for Bipartite Graphs

Part 6

Hardness for Planar Bipartite Subcubic Graphs

Part 7

Pathwidth Based Bounds