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Restricting infections on graph classes

Supervised by Ana Karolinna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

Introduction

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- The diffusion of innovation, (mis-)information, and memes.

> **Information**
>
> Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.
>
> - The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
> - The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

Introduction

Several phenomena happen in a **contagion-like manner**.

- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.
- Even bedbugs seem to spread from hotel to hotel via travelers (Barabási and Pósfai 2016).

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- Compartments are states;
- Each agent/individual belongs to one state.

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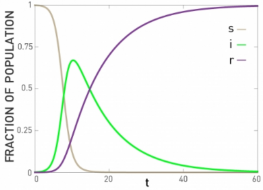
Example (SIR)



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- The homogeneous mixing hypothesis is **false**.
- The **structure of the contact network** is what facilitates the contagion.

Information

Studies have shown that the airport mobility network provides reliable information for the prediction and control of airborne epidemics like H1N1 (Soriano-Paños et al. 2022).

Modeling Contagion on Graphs: Threshold Model

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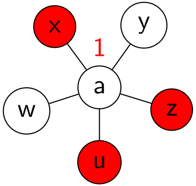

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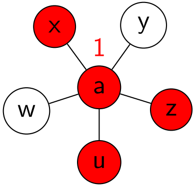

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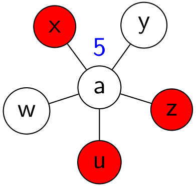
  - Bigger thresholds: laggards / greater resistance.



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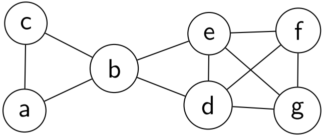

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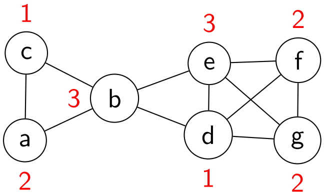

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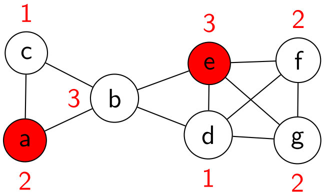

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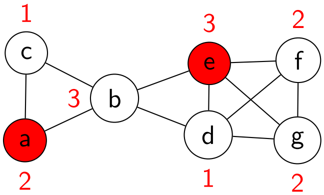

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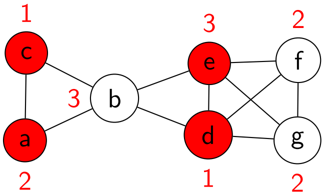

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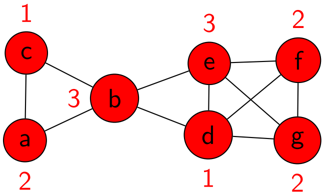
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

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  - Decision version is NP-complete even for bipartite graphs.
  - NP-complete even for k-regular graphs if we require S to infect all vertices of G – **Target Set Selection** (TSS) (Dreyer and Roberts 2009).

Problems arising from the Threshold Model

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

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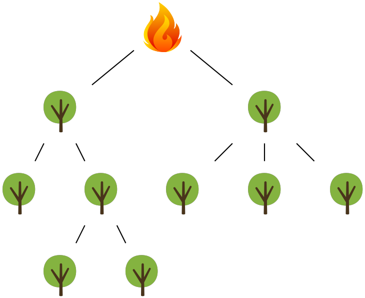

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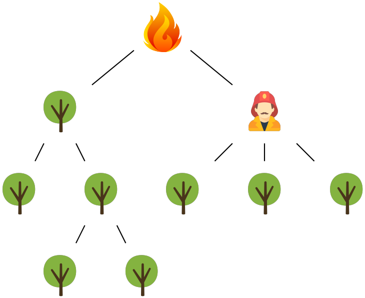

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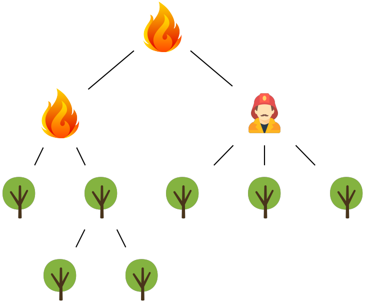

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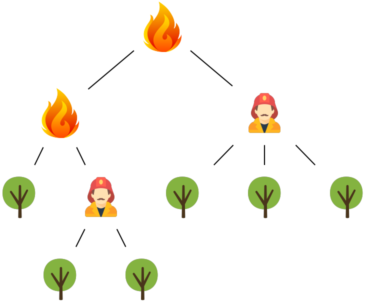

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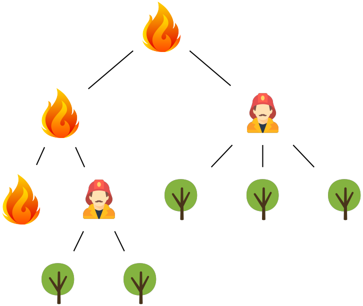

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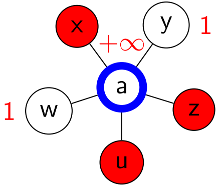
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter? **No, but sort of...**

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



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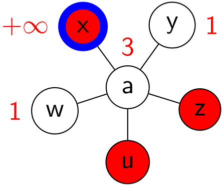

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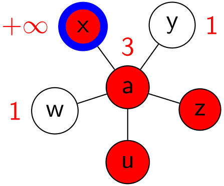

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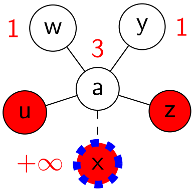

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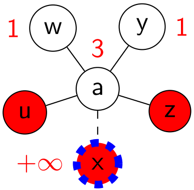

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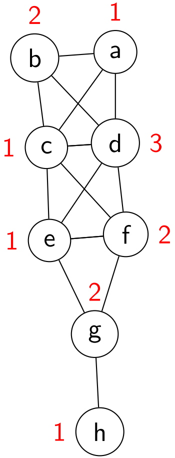

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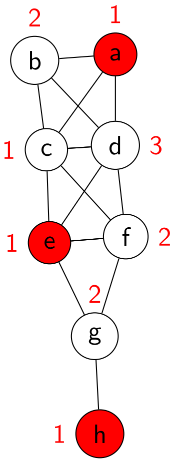

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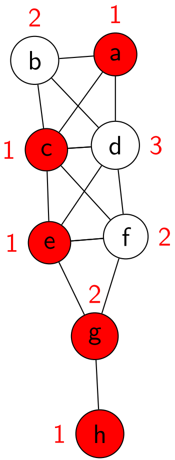

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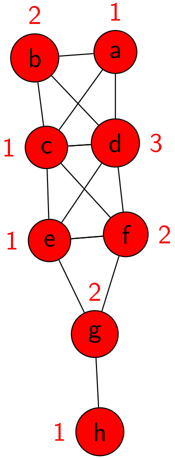

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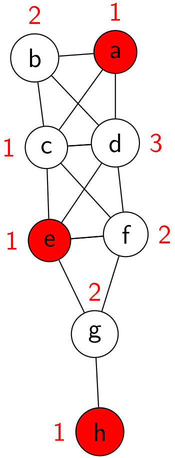

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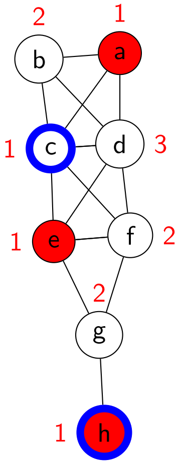
We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



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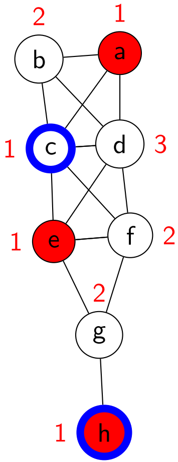
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.



Remark

Notice that we must have $k \geq |S|$.

But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

> *Input*: $x \in \Sigma^*$.
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Definition

A **parameter** is a function $\kappa : \Sigma^* \to \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) =$ "Number of variables of φ", where φ is a CNF formula.

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p-INDEPENDENT-SET
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

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But what is the motivation behind this?

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  - Robotics: the number of degrees of freedom in motion planning problems is usually ≤ 10.
- This means that **parameters of the problem matter for its tractability**.

Parameterized Complexity

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

- CLIQUE is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- VERTEX COVER is in FPT parameterized by k.

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  - Hitting Set and Dominating Set are W$[2]$-hard;
  - We can show other problems are W$[t]$-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Influence Immunization Bounding was introduced by (Cordasco, Gargano, and Rescigno 2023).

Parameter	**Hardness**	
k	W[1]-hard	$t(v) = 1 \quad \forall v$
l	W[1]-hard	$t(v) = 1 \quad \forall v$
$k + l$	FPT	
$\|S\| + l$	W[2]-hard	Bipartite graphs
$k + \|S\|$	FPT	
$\Delta(G) + l$	W[2]-hard	$t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard	
$nd(G)$	W[1]-hard	
$k + nd(G)$	FPT	
$l + nd(G)$	FPT	
$\min(\Delta(G), k) + tw(G) + l$	FPT	

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \to \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y, the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a k-**immunizing set**.

Definition

The k-**immunization number of** (G, t), denoted by

$$\mathsf{Im}(G, t, k)$$

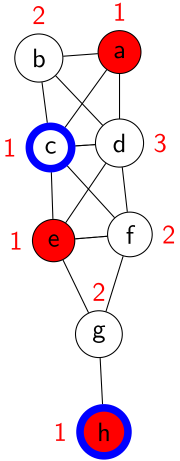
, is size of a **mimimum** k-**immunizing set** of (G, t).

Influence Immunization Bounding (cont.)

Consider the graph on the right.

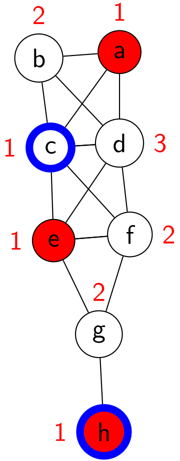
Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.



Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.

Definition

If $k = |S|$, then we say that $\mathrm{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t), and denote it by $\mathrm{In}(G, t)$.

Example

$\mathrm{In}(G, t) = \mathrm{Im}(G, t, k = 3) = 2$.

Influence Immunization Bounding (cont.)

Consider the graph on the right.

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Definition

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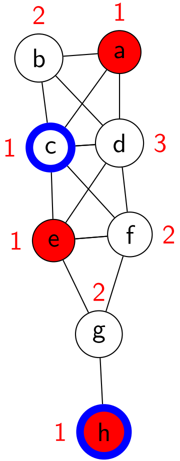
Example

$\text{In}(G, t) = \text{Im}(G, t, k = 3) = 2$.

Remark

For any suitable k, we have that

$$\text{Im}(G, t, k) \leq \text{In}(G, t)$$



Influence Immunization Bounding (cont.)

We can also define a **restricted version** of IIB, which we call $\textsc{r}\text{-}\mathrm{IIB}$.

Restricted Influence Immunization Bounding ($\textsc{r}\text{-}\mathrm{IIB}$)
Input: A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$, seed set $S \subseteq V(G)$, and $k, l \in \mathbb{N}$.
Question: Decide whether there exists $Y \subseteq V(G) \setminus S$ such that $|Y| \leq l$ and Y is a k-immunizing set of G.

- $\textsc{r}\text{-}\mathrm{IIB}$ is also W$[1]$-hard when parameterized by k or by l.

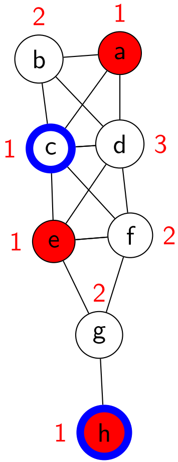
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- **$Y = \{c, h\}$ is <u>not</u> a restricted k-immunizing set of (G, t).**.



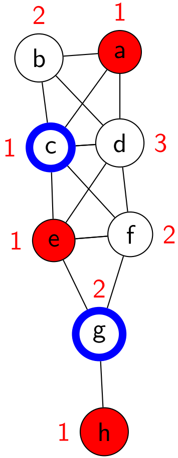
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G,t) if Y is a k-immunizing set of (G,t) and $Y \cap S = \emptyset$.

- $Y = \{c,h\}$ **is not a restricted k-immunizing set of** (G,t)..
- $Y = \{c,g\}$ **is a restricted k-immunizing set of** (G,t).



Influence Immunization Bounding (cont.)

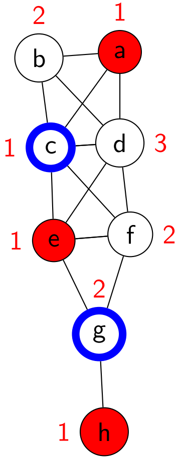
Definition

The **restricted k-immunization number of** (G, t), denoted by

$$\mathrm{Im}_r(G, t, k)$$

, is the size of a **mimimum restricted k-immunizing set** of (G, t).

- The **restricted inhibition number** $\mathrm{In}_r(G, t)$ is analogous.
- **But why study the restricted version?**



Influence Immunization Bounding (cont.)

The restricted version gives an upper bound for the original problem.

Remark

For any suitable k, we have that

$$\mathsf{Im}(G, t, k) \leq \mathsf{Im}_r(G, t, k)$$

- A restricted k-immunizing set for (G, t) is also a (unrestricted) k-immunizing set for (G, t).