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Restricting infections on graph classes

Supervised by Ana Karolinna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

Introduction

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- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.

Information

Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.

- The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
- The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

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- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.
- Even bedbugs seem to spread from hotel to hotel via travelers (Barabási and Pósfai 2016).

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- Compartments are states;
- Each agent/individual belongs to one state.

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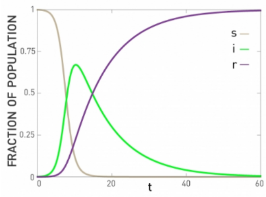
Example (SIR)



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The assumption that every individual from one compartment has the same chance of interacting with an individual from another compartment.

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- The homogeneous mixing hypothesis is **false**.
- The **structure of the contact network** is what facilitates the contagion.

Information

Studies have shown that the airport mobility network provides reliable information for the prediction and control of airborne epidemics like H1N1 (Soriano-Paños et al. 2022).

Modeling Contagion on Graphs: Threshold Model

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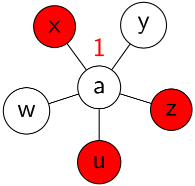

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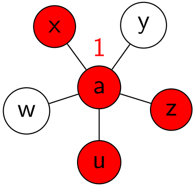

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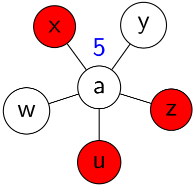
  - Bigger thresholds: laggards / greater resistance.



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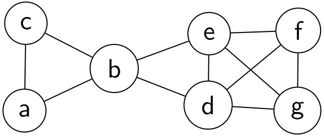

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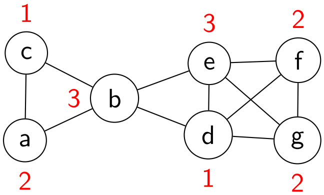

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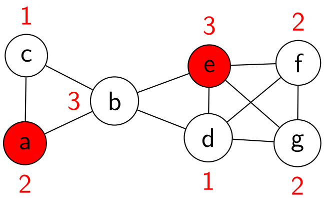

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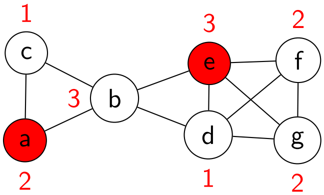

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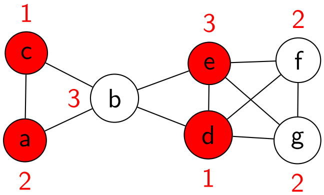

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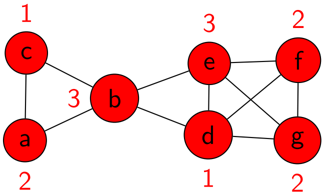
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

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  - Decision version is NP-complete even for bipartite graphs.
  - NP-complete even for k-regular graphs if we require S to infect all vertices of G – **Target Set Selection** (TSS) (Dreyer and Roberts 2009).

Problems arising from the Threshold Model

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

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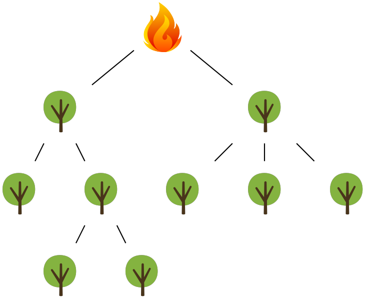

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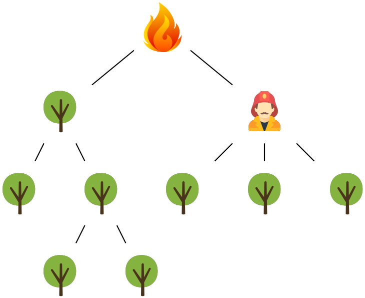

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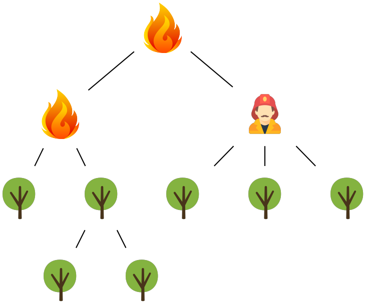

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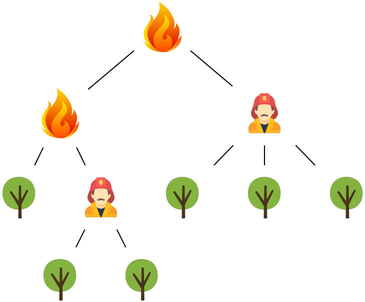

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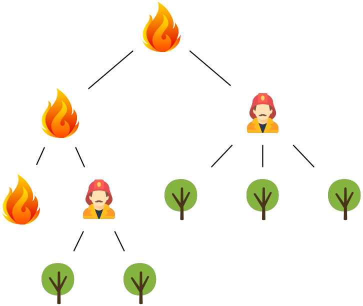

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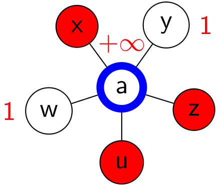
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter? **No, but sort of...**

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



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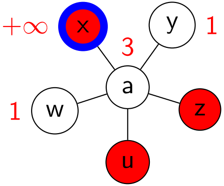

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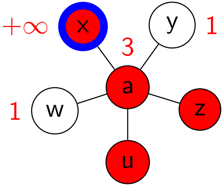

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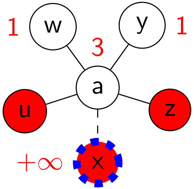

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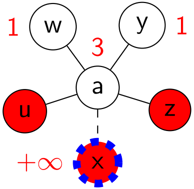

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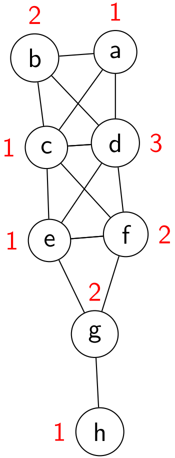

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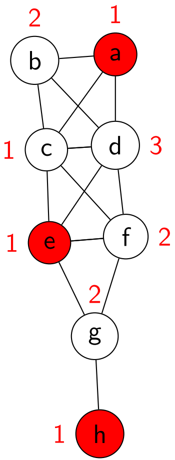

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Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

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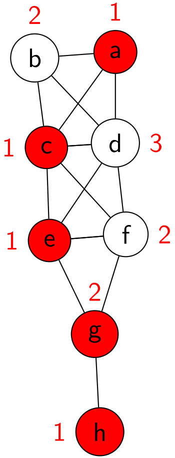

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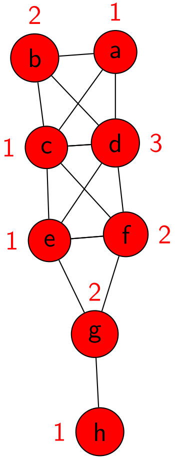

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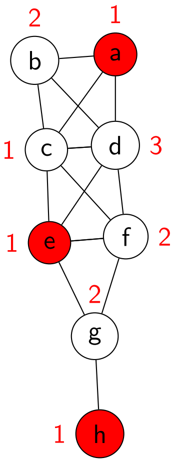

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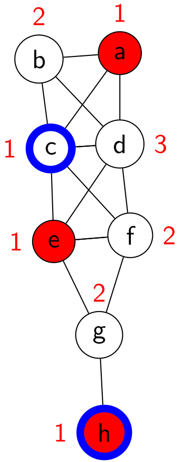
We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



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We want to find a immunizing set $Y \subseteq V(G)$ such that:

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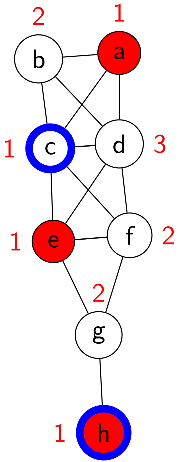
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.



Remark

Notice that we must have $k \geq |S|$.

But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

Input: $x \in \Sigma^*$.
Question: Decide whether $x \in Q$.

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Definition

A **parameter** is a function $\kappa : \Sigma^* \to \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) =$ "Number of variables of φ", where φ is a CNF formula.

Parameterized Complexity

Now we can define a **parameterized problem**.

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p-Independent-Set
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

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But what is the motivation behind this?

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  - VLSI design: the number of circuit layers is usually ≤ 10;
  - Computational biology: real instances of DNA chain reconstruction have special properties, e.g., treewidth ≤ 11;
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  - Robotics: the number of degrees of freedom in motion planning problems is usually ≤ 10.
- This means that **parameters of the problem matter for its tractability**.

Parameterized Complexity

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

- CLIQUE is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- Vertex Cover is in FPT parameterized by k.

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  - Hitting Set and Dominating Set are W[2]-hard;

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  - Hitting Set and Dominating Set are W[2]-hard;
  - We can show other problems are W[t]-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Influence Immunization Bounding was introduced by (Cordasco, Gargano, and Rescigno 2023).

Parameter	**Hardness**	
k	W[1]-hard	$t(v) = 1 \quad \forall v$
l	W[1]-hard	$t(v) = 1 \quad \forall v$
$k + l$	FPT	
$\lvert S\rvert + l$	W[2]-hard	Bipartite graphs
$k + \lvert S\rvert$	FPT	
$\Delta(G) + l$	W[2]-hard	$t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard	
$nd(G)$	W[1]-hard	
$k + nd(G)$	FPT	
$l + nd(G)$	FPT	
$\min(\Delta(G), k) + tw(G) + l$	FPT	

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \to \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y, the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a k-**immunizing set**.

Definition

The k-**immunization number of** (G, t), denoted by

$$\mathrm{Im}(G, t, k)$$

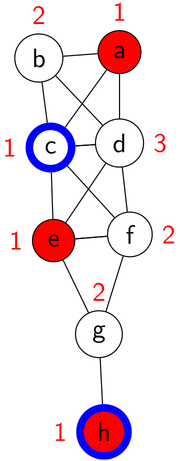
, is size of a **mimimum** k-**immunizing set** of (G, t).

Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\text{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

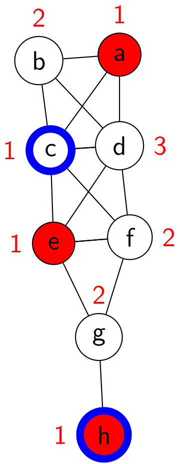
$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.

Definition

If $k = |S|$, then we say that $\mathrm{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t), and denote it by $\mathrm{In}(G, t)$.

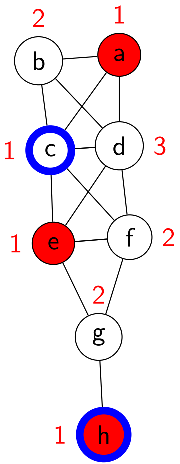
Example

$\mathrm{In}(G, t) = \mathrm{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.



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Example

$\text{In}(G, t) = \text{Im}(G, t, k = 3) = 2$.

Remark

For any suitable k, we have that

$$\text{Im}(G, t, k) \leq \text{In}(G, t)$$

Influence Immunization Bounding (cont.)

We can also define a **restricted version** of IIB, which we call $\textsc{r}\text{-}\mathrm{IIB}$.

> **Restricted Influence Immunization Bounding** ($\textsc{r}\text{-}\mathrm{IIB}$)
> *Input*: A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$, seed set $S \subseteq V(G)$, and $k, l \in \mathbb{N}$.
> *Question*: Decide whether there exists $Y \subseteq V(G) \setminus S$ such that $|Y| \leq l$ and Y is a k-immunizing set of G.

- $\textsc{r}\text{-}\mathrm{IIB}$ is also W[1]-hard when parameterized by k or by l.

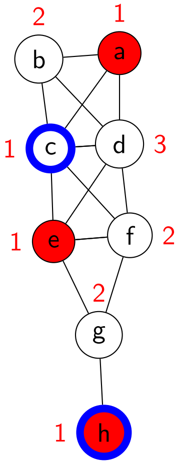
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- **$Y = \{c, h\}$ is not a restricted k-immunizing set of (G, t)..**



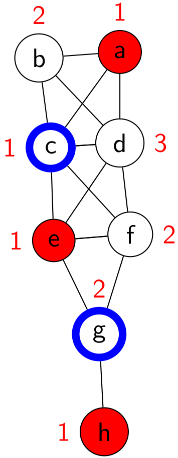
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- **$Y = \{c, h\}$ is not a restricted k-immunizing set of (G, t)..**
- **$Y = \{c, g\}$ is a restricted k-immunizing set of (G, t).**



Influence Immunization Bounding (cont.)

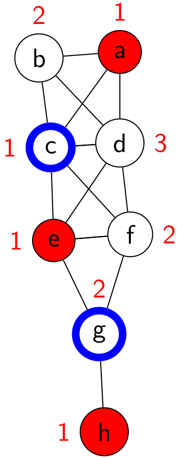
Definition

The **restricted k-immunization number of** (G, t), denoted by

$$\mathrm{Im}_r(G, t, k)$$

, is the size of a **mimimum restricted k-immunizing set** of (G, t).

- The **restricted inhibition number** $\mathrm{In}_r(G, t)$ is analogous.
- **But why study the restricted version?**



Influence Immunization Bounding (cont.)

The restricted version gives an upper bound for the original problem.

Remark

For any suitable k, we have that

$$\mathrm{Im}(G, t, k) \leq \mathrm{Im}_r(G, t, k)$$

- A restricted k-immunizing set for (G, t) is also a (unrestricted) k-immunizing set for (G, t).

Section 6: **Our Results**

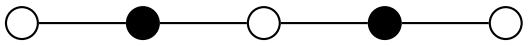
Part 1

Paths and Complete Graphs with $t(v) = c$

Paths with $t(v) = 2$

Definition

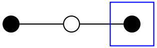
Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then $P = v_1 v_2 \ldots v_k$ is a S-**alternating path** of G if P is a path of G with *at least 3 vertices* and the vertices of P alternate with respect to their membership in S.



Paths with $t(v) = 2$

- **Main observation:** for each S-alternating path of length 4, **we need only one vertex to inhibit** the infection.

Figure: When the S-alternating paths has length 4, we can inhibit the infection by immunizing only one vertex.



(a) The infection stops by immunizing any vertex. The immunized vertex is shown in blue.

(b) When the S-alternating path has length 3 and starts with a uninfected vertex, the infection does not spread.

Paths with $t(v) = 2$

- We can split the path into S-alternating paths of length 4.

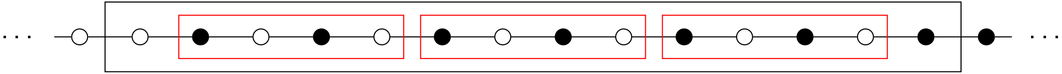


Figure: Splitting a bigger S-alternating path into pieces of length 4 starting with infected vertices.

Proposition

Let $G = P_n$ with $t(v) = 2$ for all $v \in V(G)$. Then, for any $k \in \mathbb{N}$,

$$Im(G, t, k) \leq In(G, t) \leq \lceil \frac{n}{4} \rceil$$

Exact value for paths with $t(v) = 2$

Definition

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then:

- $P = v_1 v_2 v_3$ is a $\bullet \circ \bullet$-path if P is a **maximal** S**-alternating path** of length exactly 3 such that $v_1, v_3 \in S$.
- $P = v_1 v_2 v_3 v_4$ is a $\bullet \circ \bullet\circ$-path if P is a S-alternating path of length 4 such that $v_1, v_3 \in S$.

Part 2

Polynomial Algorithm for the k-Immunization Number in Trees

Part 3

Polynomial Algorithm the Inhibition Number when $t(v) = 1$

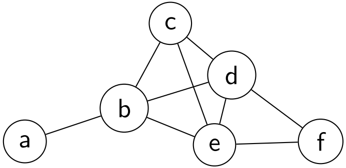
Part 4

Hardness for Chordal Graphs

Hardness for Chordal Graphs

Definition

A graph G is *chordal* if every cycle C of length at least 4 in G has a *chord*: an edge between no consecutive vertices of C.



Chordal graphs are a superclass of several important graph classes:

- Trees;
- Split Graphs;
- Interval Graphs.

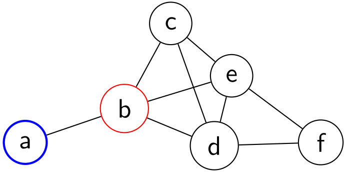
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

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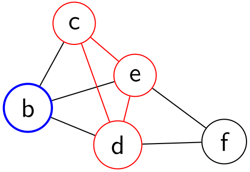

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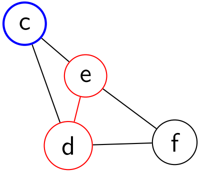

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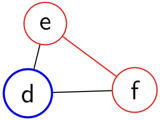

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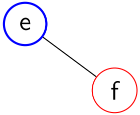

- Perfect elimination ordering: f, e, d, c, b, a.

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- Perfect elimination ordering: f, e, d, c, b, a.

Hardness for Chordal Graphs

Theorem (Dirac 1961)

A graph G is chordal if and only if G has a perfect elimination ordering.

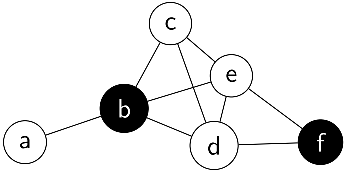
Theorem (Dirac 1961)

Let G be a graph. If G is chordal and not complete, then G has two non-adjacent simplicial vertices.

Hardness for Chordal Graphs

Definition

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ has at least one neighbor in D.



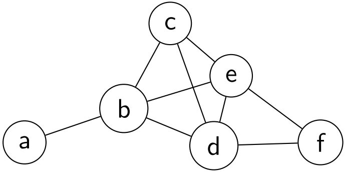
k-Dominating Set is:

- W[1]-hard for chordal graphs parameterized by k (Liu and Song 2009);
- NP-hard for split graphs (Bertossi 1984).

Hardness for Chordal Graphs

Proposition

Influence Immunization Bounding *is* W[1]*-hard parameterized by the maximum number of infected vertices* k *on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \emptyset$;
- $U = \emptyset$;

Let $\langle G, k \rangle$ be an instance of k-Dominating Set in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Define $t(s) = d_G(s)$;
5. Repeat steps (2-5) on $G - N_G[s]$.

Part 5

Hardness for Bipartite Graphs

Part 6

Hardness for Planar Bipartite Subcubic Graphs

Part 7

Pathwidth Based Bounds