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Restricting infections on graphs

Immunization problems

Supervised by Ana Karolinna Maia and Carlos Vinícius G. C. Lima

Introduction

Several phenomena happen in a **contagion-like manner**.

- Viruses or bacteria;
- Innovation, (mis-)information, and memes. (Ryan and Gross 1950; Coleman, Katz, and Menzel 1957)

Modeling Contagion on Graphs: Threshold Model

- Let's consider each vertex as an individual.
  - A vertex is either **active (infected)** or **inactive (susceptible)**.
- The individuals change their state based on their neighbors' state.

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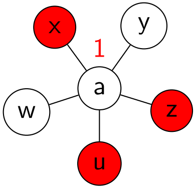

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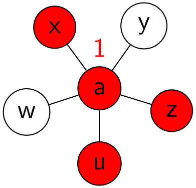

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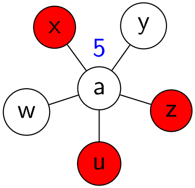

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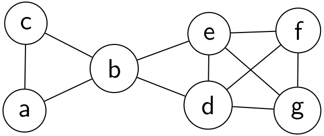

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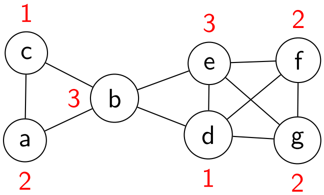

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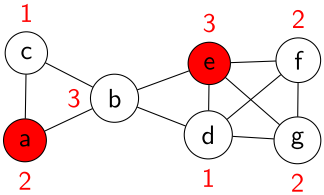

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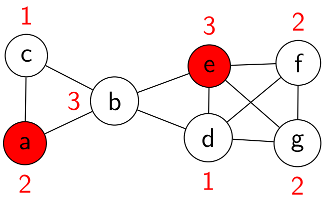

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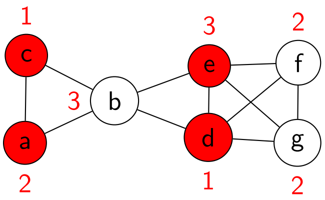

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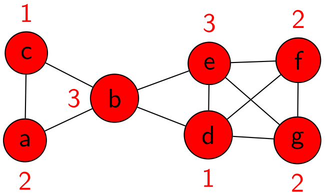
- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

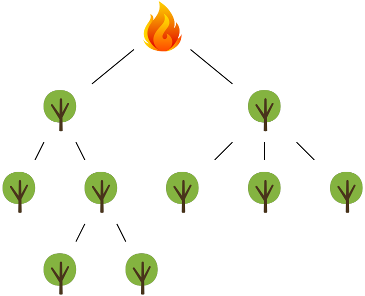
Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003). NP-complete.
- **Target Set Selection** (TSS) (Dreyer and Roberts 2009). NP-complete.
- Variations:
  - Directed version;
  - Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
  - TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
  - TSS with activation time equal to 1 (Araújo and Sampaio 2023);

Problems arising from the Threshold Model

Immunization problems:

- **Firefighter**;



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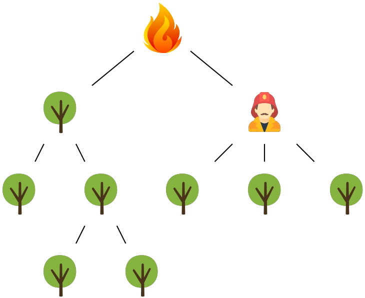

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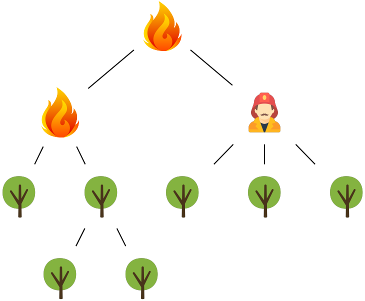

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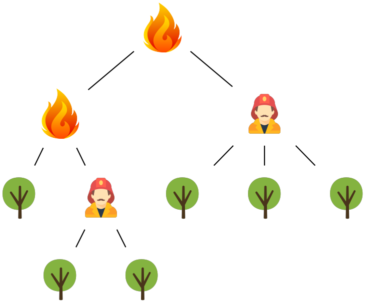

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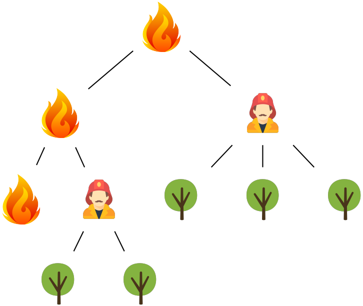
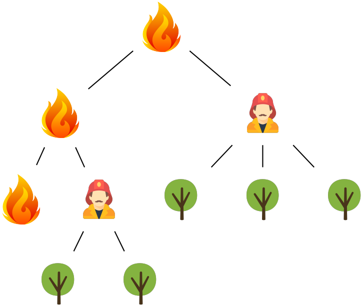


Problems arising from the Threshold Model

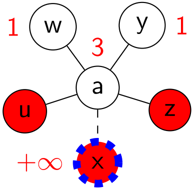
Immunization problems:

- **Firefighter**;
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter?



First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;
- **Option 2: make the vertex invisible – remove it.**



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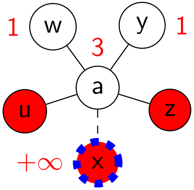

Remark (Option 2)

In this option, immunized vertices do not infect nor get infected.

Influence Immunization Bounding

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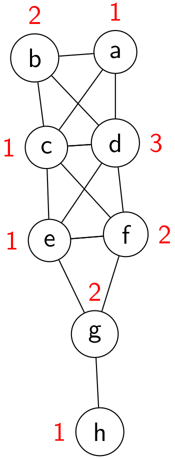

Influence Immunization Bounding

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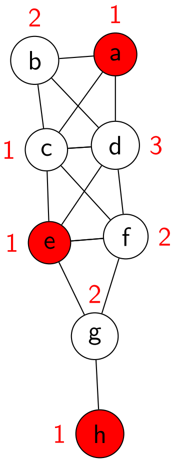

Remark

Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

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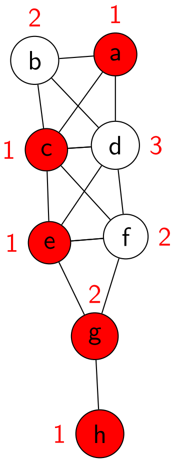

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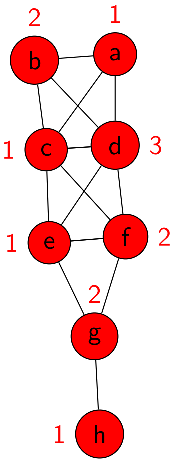

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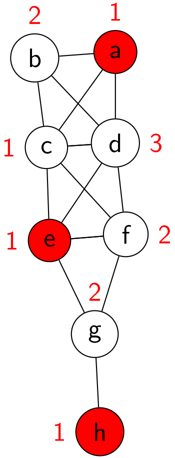
Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;
- Naturals k and l.



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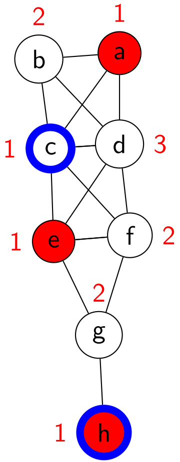
- Naturals k and l. **Let** $k = 3$ **and** $l = 2$**.**

We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



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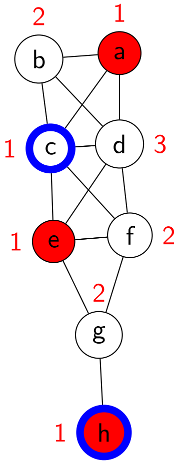
- $|Y| \leq l$; and
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.

Remark

Notice that we must have $k \geq |S|$.



But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

Definition

A *parameterized problem* is a pair (P, κ), such that P is a decision problem and κ is a parameter for P.

p-Independent-Set
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

Parameters of the problem matter for its tractability.

Parameterized Complexity

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

- Clique is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- Vertex Cover is in FPT parameterized by k.

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- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;
  - **Hypothesis: k-Clique is not in FPT.**
- The W[t]-hardness is the parameterized analogous of NP-hardness.
  - k-Clique is W[1]-hard;
  - Hitting Set and Dominating Set are W[2]-hard;
  - We can show other problems are W[t]-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Gennaro Cordasco, Luisa Gargano, and Adele A. Rescigno (Apr. 2023). "Immunization in the Threshold Model: A Parameterized Complexity Study". In: *Algorithmica* 85.11, pp. 3376–3405. ISSN: 1432-0541. DOI: 10.1007/s00453-023-01118-y

Parameter	**Hardness**	
k	W[1]-hard	$t(v) = 1 \quad \forall v$
l	W[1]-hard	$t(v) = 1 \quad \forall v$
$k + l$	FPT	
$\lvert S\rvert + l$	W[2]-hard	Bipartite graphs
$k + \lvert S\rvert$	FPT	
$\Delta(G) + l$	W[2]-hard	$t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard	
$nd(G)$	W[1]-hard	
$k + nd(G)$	FPT	
$l + nd(G)$	FPT	
$\min(\Delta(G), k) + tw(G) + l$	FPT	

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \to \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y, the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a k-**restricting set**.

Definition

The k-**restricting number of** (G, t), denoted by

$$\mathfrak{R}(G, t, k)$$

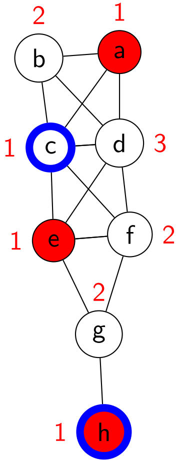
, is size of a **mimimum** k-**restricting set** of (G, t).

Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-restricting set of (G, t).
$\mathfrak{R}(G, t, k = 3) = 2$.



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Definition

If $k = |S|$, then we say that $\mathfrak{R}(G, t, k = |S|)$ is the **total restricting number of** (G, t), and denote it by $\mathfrak{R}_T(G, t)$.

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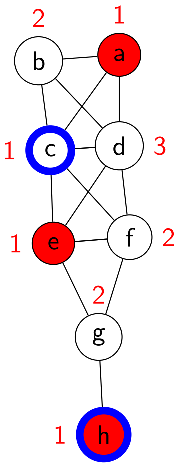
$\mathfrak{R}_T(G, t) = \mathfrak{R}(G, t, k = 3) = 2$.



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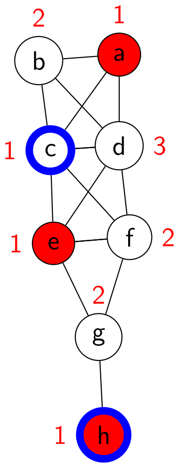
Remark

For any suitable k, we have that

$$\mathfrak{R}(G, t, k) \leq \mathfrak{R}_T(G, t)$$



Related Work

- Fedor V. Fomin, Petr A. Golovach, and Janne H. Korhonen (2013). "On the Parameterized Complexity of Cutting a Few Vertices from a Graph". In: *Mathematical Foundations of Computer Science 2013*. Ed. by Krishnendu Chatterjee and Jirí Sgall. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 421–432. ISBN: 978-3-642-40313-2
- Ara Hayrapetyan et al. (2005). "Unbalanced Graph Cuts". In: *Algorithms – ESA 2005*. Springer Berlin Heidelberg, pp. 191–202. ISBN: 9783540319511. DOI: 10.1007/11561071_19. URL: http://dx.doi.org/10.1007/11561071_19
- Hermish Mehta and Daniel Reichman (2022). *Local treewidth of random and noisy graphs with applications to stopping contagion in networks*. eprint: arXiv:2204.07827

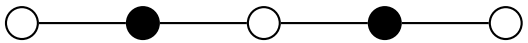
- Firefighter is very conceptually related (Anshelevich et al. 2010);
- Game theoretical approach (Aspnes, Chang, and Yampolskiy 2006; P.-A. Chen, David, and Kempe 2010; Moscibroda, Schmid, and Wattenhofer 2006; Meier et al. 2014);
- Spectral graph theory (Ahmad et al. 2020; C. Chen et al. 2016; Chakrabarti et al. 2008; Tariq et al. 2017).

Our Results

Paths with $t(v) = 2$

Definition

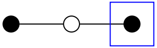
Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then $P = v_1 v_2 \dots v_k$ is a S-**alternating path** of G if P is a path of G with *at least 3 vertices* and the vertices of P alternate with respect to their membership in S.



Paths with $t(v) = 2$

- **Main observation:** for each S-alternating path of length 4, **we need only one vertex to inhibit** the infection.

Figure: When the S-alternating paths has length 4, we can inhibit the infection by immunizing only one vertex.



(a) The infection stops by immunizing any vertex. The immunized vertex is shown in blue.

(b) When the S-alternating path has length 3 and starts with a uninfected vertex, the infection does not spread.

Paths with $t(v) = 2$

- We can split the path into S-alternating paths of length 4.

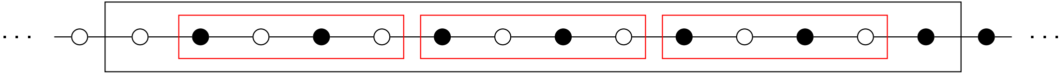


Figure: Splitting a bigger S-alternating path into pieces of length 4 starting with infected vertices.

Proposition

Let $G = P_n$ with $t(v) = 2$ for all $v \in V(G)$. Then, for any $k \in \mathbb{N}$,

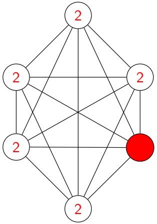
$$Im(G, t, k) \leq In(G, t) \leq \lceil \frac{n}{4} \rceil$$

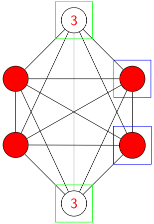
Complete graphs with $t(v) = c$

Proposition

Let $G \cong K_n$ be a complete graph on n vertices with thresholds $t(v) = c$ for all $v \in V(G)$, $S \subseteq V(G)$ a seed set of G, and $k \in \mathbb{N}$. Then:

$$Im(G, t, k) = \begin{cases} \min\{|S| - c + 1, n - k\}, & \textit{if } |S| \geq c \\ 0, & \textit{otherwise.} \end{cases}$$

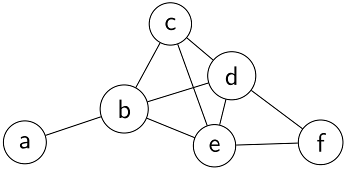




Hardness for Chordal Graphs

Definition

A graph G is *chordal* if every cycle C of length at least 4 in G has a *chord*: an edge between no consecutive vertices of C.



Chordal graphs are a superclass of several important graph classes:

- Trees;
- Split Graphs;
- Interval Graphs.

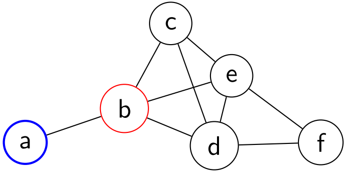
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

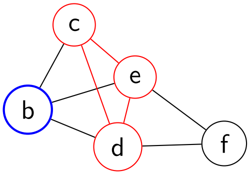
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



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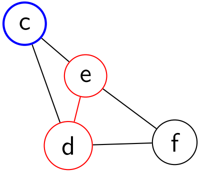

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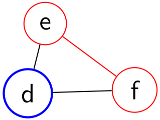

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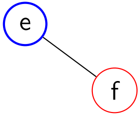

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- Perfect elimination ordering: f, e, d, c, b, a.

Hardness for Chordal Graphs

Theorem (Dirac 1961)

A graph G is chordal if and only if G has a perfect elimination ordering.

Theorem (Dirac 1961)

Let G be a graph. If G is chordal and not complete, then G has two non-adjacent simplicial vertices.

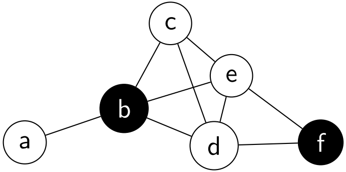
Theorem (Dirac 1961)

If a graph G is chordal, then any induced subgraph of G is chordal.

Hardness for Chordal Graphs

Definition

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ has at least one neighbor in D.



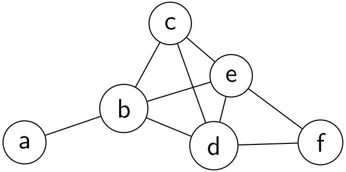
k-Dominating Set is:

- W[1]-hard for chordal graphs parameterized by k (Liu and Song 2009);
- NP-hard for split graphs (Bertossi 1984).

Hardness for Chordal Graphs

Proposition

Influence Immunization Bounding is W[1]-hard parameterized by the maximum number of immunized vertices I on chordal graphs. Moreover, it is NP-complete on split graphs.



- $S = \emptyset$;
- $U = \emptyset$;

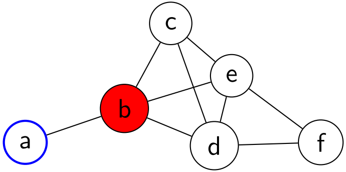
Let $\langle G, k \rangle$ be an instance of k-Dominating Set in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

Proposition

INFLUENCE IMMUNIZATION BOUNDING *is* W[1]*-hard parameterized by the maximum number of immunized vertices l on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \{b\}$;
- $U = \{a\}$;

Let $\langle G, k\rangle$ be an instance of k-DOMINATING SET in which G is chordal.

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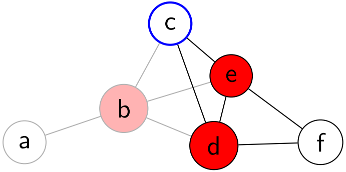

- $S = \{b, d, e\}$;
- $U = \{a, c\}$;

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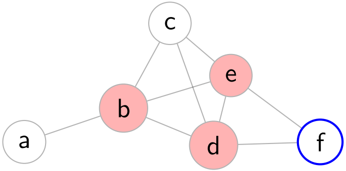

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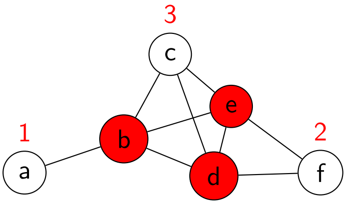

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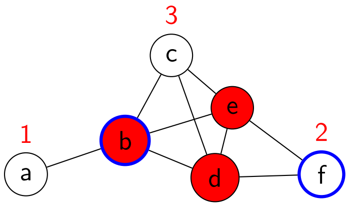
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

- **G has a dominating set of size $\leq k$ if and only if (G, t, S) has a $|S|$-imunizing set of size $l \leq k$.**



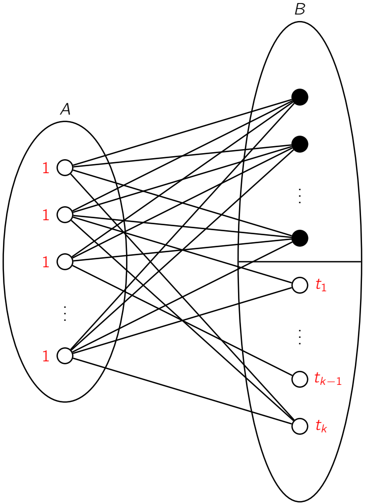
- IIB is W[1]-hard for chordal graphs parameterized by l, and NP-complete for split graphs.

Hardness for Bipartite Graphs

Cordasco, Gargano, and Rescigno 2023 showed that IIB is W[2]-hard parameterized by $|S| + l$ even on bipartite graphs.

Proposition

IIB *is* NP*-complete on bipartite graphs even if A is entirely susceptible and B is entirely infected.*



Hardness for Bipartite Graphs

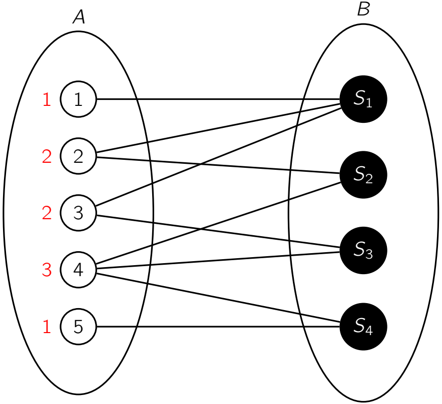
- Reduction from SET COVER.
  - Universe set $\mathcal{U} = \{a_1, a_2, \dots, a_n\}$;
  - $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$;
  - Parameter $c \in \mathbb{N}$.

1. Create a bipartite graph G such that $V(G) = \mathcal{U} \cup \mathcal{S}$ and $E(G) = \{a_i S_j \mid a_i \in S_j\}$;
2. Define $k = |\mathcal{S}|$, $l = c$, and set the seed set $S = \mathcal{S}$;
3. Set $t(a_i) = d_G(a_i)$, for all $i \in [n]$.

Hardness for Bipartite Graphs

Example

- $\mathcal{U} = \{1, 2, 3, 4, 5\}$;
- $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2, 4\}, S_3 = \{3, 4\}, S_4 = \{4, 5\}\}$; and
- $c = 2$.



Hardness for Bipartite Graphs

Example

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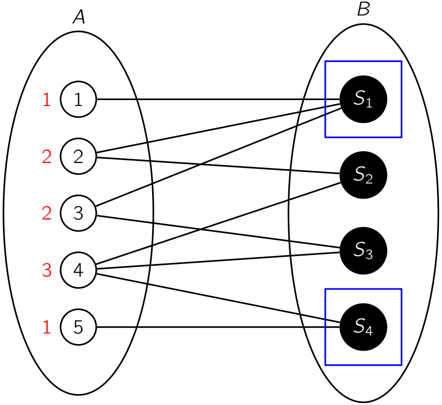
- $c = 2$.

Proposition

$(\mathcal{U}, \mathcal{S})$ has a set cover of size $\leq c$ if and only if (G, t) has a $|\mathcal{S}|$-immunizing set of size $\leq c$.

- The problem remains hard even on this restricted instance for bipartite graphs.



Hardness for Planar Bipartite Subcubic Graphs

Definition

A graph G is planar if it can be drawn on the plane without crossing edges.

Theorem (Wagner 1937)

A graph G is planar if and only if G does not have a K_5 minor nor a $K_{3,3}$ minor.

Hardness for Planar Bipartite Subcubic Graphs

- In Restricted 3-Sat, we are given a 3-Sat CNF formula in which every variable **occurs *exactly* 3 times**: twice as positive and once as negative.
  - r-3-Sat is NP-complete (Dahlhaus et al. 1994).

Example

$$\varphi = (u \vee v \vee \overline{w}) \wedge (\overline{u} \vee v \vee x) \wedge (u \vee \overline{x} \vee \overline{y}) \wedge (\overline{v} \vee w \vee x) \wedge (w \vee z \vee y) \wedge (y \vee \overline{z}) \wedge (z)$$

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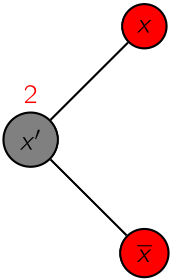
Proposition

r-3-Sat $\leq_P$ *(Planar Bipartite Subcubic)* IIB.

Hardness for Planar Bipartite Subcubic Graphs

- Given a R-3-SAT formula φ, let:
  - $V(\varphi)$ the set of variables of φ;
  - $\mathcal{C}(\varphi)$ the set of clauses of φ.
- Let's create a graph G_φ with thresholds t_φ. For each variable $x \in V(\varphi)$, we are going to add the following gadget to G_φ:

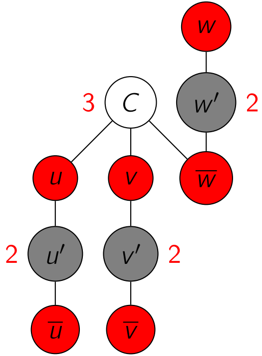


Hardness for Planar Bipartite Subcubic Graphs

- For each clause $C \in \mathcal{C}(\varphi)$, we are going to add a vertex in G_φ and connect it to its literals, setting $t_\varphi(C) = |C|$.

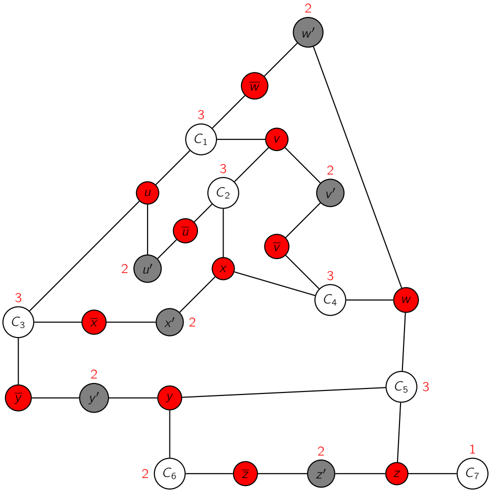
Example

$C = (u \vee v \vee \overline{w})$.



Hardness for Planar Bipartite Subcubic Graphs

- Example of G_φ for the R-3-SAT formula
  $\varphi = (u \vee v \vee \overline{w}) \wedge (\overline{u} \vee v \vee x) \wedge (u \vee \overline{x} \vee \overline{y}) \wedge (\overline{v} \vee w \vee x) \wedge (w \vee z \vee y) \wedge (y \vee \overline{z}) \wedge (z)$.



Hardness for Planar Bipartite Subcubic Graphs

- Suppose φ has a satisfying assignment $\mathcal{S} : V(\varphi) \to \{\textsc{True}, \textsc{False}\}$.

Proposition

$Y = (\{x \mid \mathcal{S}(x) = \textsc{True}\} \cup \{\overline{x} \mid \mathcal{S}(x) = \textsc{False}\})$ is a $(2|V(\varphi)|)$-immunizing set for (G_φ, t_φ) and $|Y| \leq |V(\varphi)|$.

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- Now, suppose Y is a $(2|V(\varphi)|)$-immunizing set for (G_φ, t_φ) and $|Y| \leq |V(\varphi)|$.

Proposition

For every $x \in V(\varphi)$, either $x \in Y$ or $\overline{x} \in Y$.

- Thus, we can build a satisfying assignment for φ from Y.

Hardness for Planar Bipartite Subcubic Graphs

Proposition

G_φ is planar bipartite subcubic.

- Clause vertices have degree at most 3, positive literals vertices have degree 3, negative literals vertices have degree 2, and auxiliary vertices have degree 2.

Hardness for Planar Bipartite Subcubic Graphs

Proposition

G_φ is planar bipartite subcubic.

- Clause vertices have degree at most 3, positive literals vertices have degree 3, negative literals vertices have degree 2, and auxiliary vertices have degree 2.
- Let A be the set of auxiliary vertices. Then we have the partition (X, Y) of $V(G_\varphi)$ such that $X = A \cup \mathcal{C}(\varphi)$ and $Y = \{x, \overline{x} \mid x \in V(\varphi)\}$.

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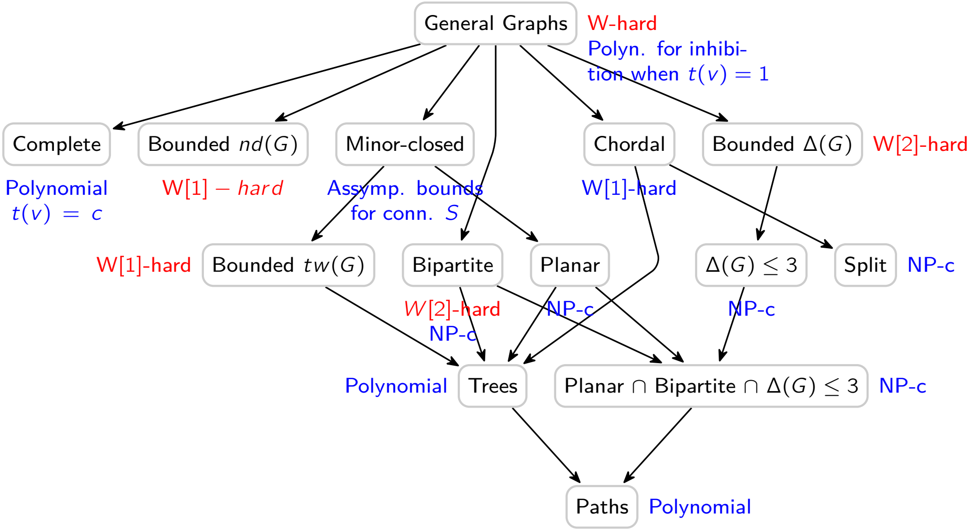
- Since G_φ is subcubic, it cannot have a K_5 minor.
- In order to have a $K_{3,3}$ minor, we should have 3 clauses connected with 3 equal positive literals. Since each positive literal appears only twice, G_φ is planar.

Section 8: **Conclusion and Future Work**


General Graphs
W-hard
Polyn. for inhibition when $t(v) = 1$
Complete
Polynomial $t(v) = c$
Bounded $nd(G)$
W[1] $- hard$
Minor-closed
Assymp. bounds for conn. S
Chordal
W[1]-hard
Bounded $\Delta(G)$
W[2]-hard
W[1]-hard
Bounded $tw(G)$
Bipartite
W[2]-hard
NP-c
Planar
NP-c
$\Delta(G) \leq 3$
NP-c
Split
NP-c
Polynomial
Trees
Planar $\cap$ Bipartite $\cap$ $\Delta(G) \leq 3$
NP-c
Paths
Polynomial

Obrigado.

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