Samuel Santos
csamuelssm@alu.ufc.br

UFC
Universidade Federal do Ceará

December 2024

Restricting infections on graph classes

Supervised by Ana Karolinna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

Introduction

Several phenomena happen in a **contagion-like manner**.

- Spreading of pathogens, like viruses or bacteria;

Introduction

Several phenomena happen in a **contagion-like manner**.

- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.

> **Information**
>
> Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.
>
> - The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
> - The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

Introduction

Several phenomena happen in a **contagion-like manner**.

- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.
- Even bedbugs seem to spread from hotel to hotel via travelers (Barabási and Pósfai 2016).

Introduction

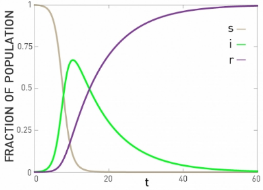
The early models built to study such phenomena are the so-called *compartmental models*.

- Compartments are states;
- Each agent/individual belongs to one state.

The early models built to study such phenomena are the so-called *compartmental models*.

- Compartments are states;
- Each agent/individual belongs to one state.

Example (SIR)



The early models built to study such phenomena are the so-called *compartmental models*.

- Compartments are states;
- Each agent/individual belongs to one state.

Definition (Homogeneous Mixing Hypothesis)

The assumption that every individual from one compartment has the same chance of interacting with an individual from another compartment.

Introduction

The early models built to study such phenomena are the so-called *compartmental models*.

- Compartments are states;
- Each agent/individual belongs to one state.

Definition (Homogeneous Mixing Hypothesis)

The assumption that every individual from one compartment has the same chance of interacting with an individual from another compartment.

- The homogeneous mixing hypothesis is **false**.

Introduction

The early models built to study such phenomena are the so-called *compartmental models*.

- Compartments are states;
- Each agent/individual belongs to one state.

Definition (Homogeneous Mixing Hypothesis)

The assumption that every individual from one compartment has the same chance of interacting with an individual from another compartment.

- The homogeneous mixing hypothesis is **false**.
- The **structure of the contact network** is what facilitates the contagion.

Information

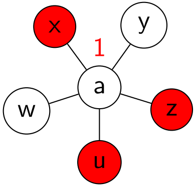
Studies have shown that the airport mobility network provides reliable information for the prediction and control of airborne epidemics like H1N1 (Soriano-Paños et al. 2022).

Modeling Contagion on Graphs: Threshold Model

- Let's consider each vertex as an individual.
  - A vertex is either **active (infected)** or **inactive (susceptible)**.

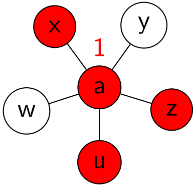
Modeling Contagion on Graphs: Threshold Model

- Let's consider each vertex as an individual.
  - A vertex is either **active (infected)** or **inactive (susceptible)**.
- The individuals change their state based on their neighbors' state.
  - A threshold value represents **how susceptible** an individual is to the contagious agent.
  - Smaller thresholds: early adopters / low immunity;



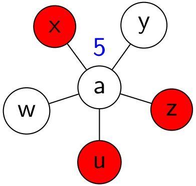
Modeling Contagion on Graphs: Threshold Model

- Let's consider each vertex as an individual.
  - A vertex is either **active (infected)** or **inactive (susceptible)**.
- The individuals change their state based on their neighbors' state.
  - A threshold value represents **how susceptible** an individual is to the contagious agent.
  - Smaller thresholds: early adopters / low immunity;



Modeling Contagion on Graphs: Threshold Model

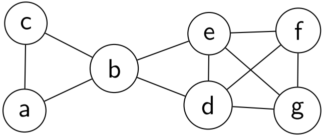
- Let's consider each vertex as an individual.
  - A vertex is either **active (infected)** or **inactive (susceptible)**.
- The individuals change their state based on their neighbors' state.
  - A threshold value represents **how susceptible** an individual is to the contagious agent.
  - Smaller thresholds: early adopters / low immunity;
  - Bigger thresholds: laggards / greater resistance.



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

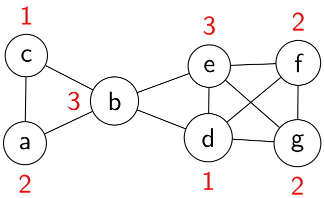
- A **graph** $G = (V, E)$;



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

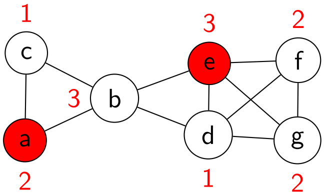
- A **graph** $G = (V, E)$;
- A **threshold function** $t : V(G) \to \mathbb{N}$;



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

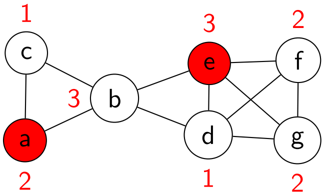
- A **graph** $G = (V, E)$;
- A **threshold function** $t : V(G) \to \mathbb{N}$;
- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

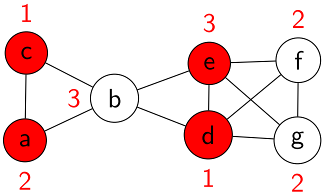
- A **graph** $G = (V, E)$;
- A **threshold function** $t : V(G) \to \mathbb{N}$;
- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.
- The diffusion happens in **discrete time steps**.



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

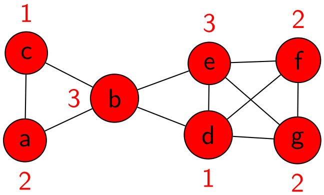
- A **graph** $G = (V, E)$;
- A **threshold function** $t : V(G) \rightarrow \mathbb{N}$;
- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.
- The diffusion happens in **discrete time steps**.



Modeling Contagion on Graphs: Threshold Model

In the threshold model, we are given:

- A **graph** $G = (V, E)$;
- A **threshold function** $t : V(G) \to \mathbb{N}$;
- A set of **initial infected vertices** – the *seed set* $S \subseteq V(G)$.
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.
  - Decision version is NP-complete even for bipartite graphs.

Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.
  - Decision version is NP-complete even for bipartite graphs.
  - NP-complete even for k-regular graphs if we require S to infect all vertices of G – **Target Set Selection** (TSS) (Dreyer and Roberts 2009).

Problems arising from the Threshold Model

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

Problems arising from the Threshold Model

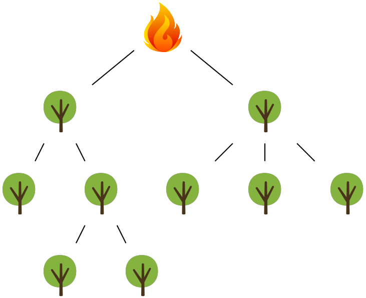
Immunization problems:

- **Firefighter**;

Problems arising from the Threshold Model

Immunization problems:

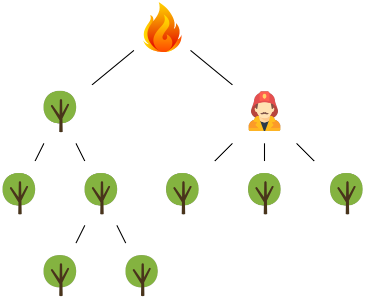
- **Firefighter**;
  - A fire breaks at some vertex and will spread;



Problems arising from the Threshold Model

Immunization problems:

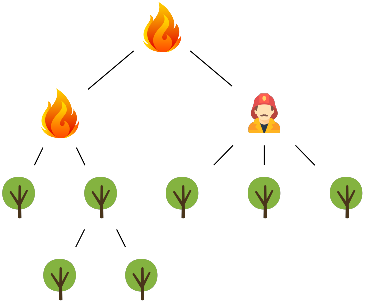
- **Firefighter**;
  - A fire breaks at some vertex and will spread;
  - The goal is to place firefighters to protect the vertices from the fire.



Problems arising from the Threshold Model

Immunization problems:

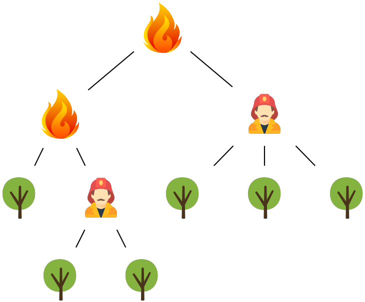
- **Firefighter**;
  - A fire breaks at some vertex and will spread;
  - The goal is to place firefighters to protect the vertices from the fire.



Problems arising from the Threshold Model

Immunization problems:

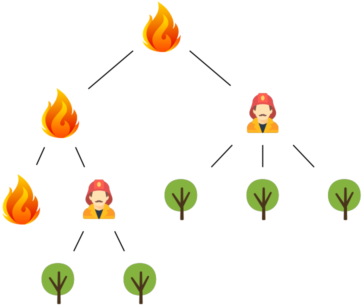
- **Firefighter**;
  - A fire breaks at some vertex and will spread;
  - The goal is to place firefighters to protect the vertices from the fire.



Problems arising from the Threshold Model

Immunization problems:

- **Firefighter**;
  - A fire breaks at some vertex and will spread;
  - The goal is to place firefighters to protect the vertices from the fire.
  - NP-complete **even for trees!** (And a bunch of other classes)



Problems arising from the Threshold Model

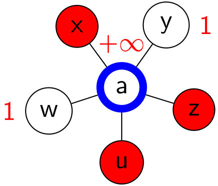
Immunization problems:

- **Firefighter**;
  - A fire breaks at some vertex and will spread;
  - The goal is to place firefighters to protect the vertices from the fire.
  - NP-complete **even for trees!** (And a bunch of other classes)
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter? **No, but sort of...**

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

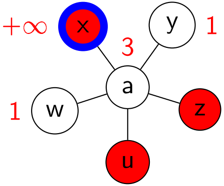
- Option 1: raise the vertex threshold above its degree;



Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



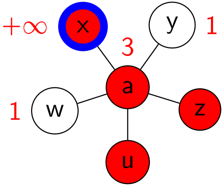
Remark (Option 1)

Infected vertices can also be immunized. In this option, the immunized infected vertices still **can infect others**.

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



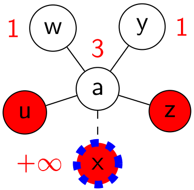
Remark (Option 1)

Infected vertices can also be immunized. In this option, the immunized infected vertices still **can infect others**.

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;
- Option 2: make the vertex invisible – remove it.



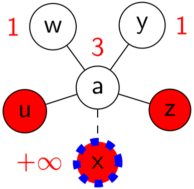
Remark (Option 2)

In this option, immunized vertices do not infect nor get infected.

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;
- **Option 2: make the vertex invisible – remove it.**



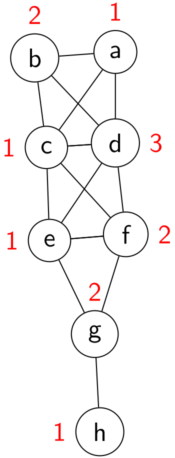
Remark (Option 2)

In this option, immunized vertices do not infect nor get infected.

Influence Immunization Bounding

Given:

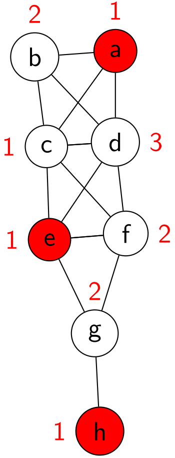
- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;



Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;



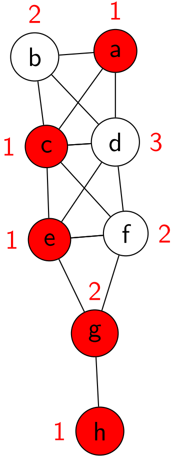
Remark

Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;



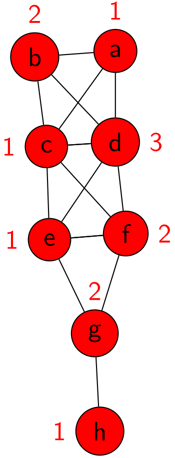
Remark

Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;



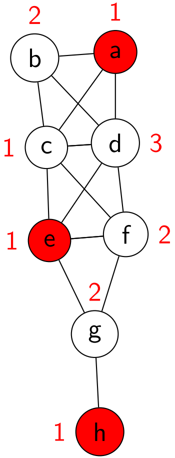
Remark

Notice that $S = \{a, e, h\}$ is a *target set* for this graph.

Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;
- Naturals k and l.



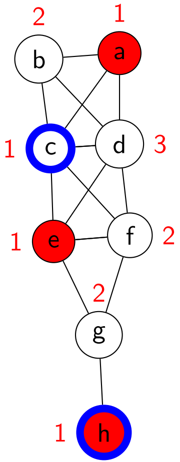
Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;
- Naturals k and l. **Let** $k = 3$ **and** $l = 2$**.**

We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



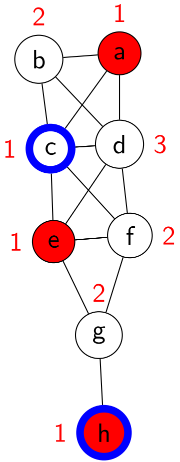
Influence Immunization Bounding

Given:

- A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$;
- A seed set $S \subseteq V(G)$;
- Naturals k and l. **Let $k = 3$ and $l = 2$.**

We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.



Remark

Notice that we must have $k \geq |S|$.

But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

Input: $x \in \Sigma^*$. *Question*: Decide whether $x \in Q$.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

Input: $x \in \Sigma^*$.
Question: Decide whether $x \in Q$.

Definition

A **parameter** is a function $\kappa : \Sigma^* \to \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) =$ "Number of variables of φ", where φ is a CNF formula.

Now we can define a **parameterized problem**.

Definition

A *parameterized problem* is a pair (P, κ), such that P is a decision problem and κ is a parameter for P.

Parameterized Complexity

Now we can define a **parameterized problem**.

Definition

A *parameterized problem* is a pair (P, κ), such that P is a decision problem and κ is a parameter for P.

p-INDEPENDENT-SET
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

Parameterized Complexity

But what is the motivation behind this?

- NP-hard problems cannot have **all instances** solved in polynomial time, unless P = NP.

Parameterized Complexity

But what is the motivation behind this?

- NP-hard problems cannot have **all instances** solved in polynomial time, unless $\mathsf{P} = \mathsf{NP}$.
- But in practice, **only a subset** of them is relevant.
  - VLSI design: the number of circuit layers is usually ≤ 10;
  - Computational biology: real instances of DNA chain reconstruction have special properties, e.g., treewidth ≤ 11;
  - Robotics: the number of degrees of freedom in motion planning problems is usually ≤ 10.

But what is the motivation behind this?

- NP-hard problems cannot have **all instances** solved in polynomial time, unless P $=$ NP.
- But in practice, **only a subset** of them is relevant.
  - VLSI design: the number of circuit layers is usually ≤ 10;
  - Computational biology: real instances of DNA chain reconstruction have special properties, e.g., treewidth ≤ 11;
  - Robotics: the number of degrees of freedom in motion planning problems is usually ≤ 10.
- This means that **parameters of the problem matter for its tractability**.

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

- Clique is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- Vertex Cover is in FPT parameterized by k.

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;
- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;
- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;
  - **Hypothesis: k-Clique is not in FPT.**

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;
- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;
  - **Hypothesis: k-Clique is not in FPT.**
- The W[t]-hardness is the parameterized analogous of NP-hardness.
  - k-Clique is W[1]-hard;

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;
- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;
  - **Hypothesis: k-Clique is not in FPT.**
- The W[t]-hardness is the parameterized analogous of NP-hardness.
  - k-Clique is W[1]-hard;
  - Hitting Set and Dominating Set are W[2]-hard;

- The class FPT is the parameterized analogous to the class P;
- The class para-NP is the parameterized analogous to the class NP;
- k-Clique is the parameterized analogous to 3-Sat;
  - No one has managed to find a FPT algorithm;
  - **Hypothesis:** k**-Clique is not in** FPT**.**
- The W[t]-hardness is the parameterized analogous of NP-hardness.
  - k-Clique is W[1]-hard;
  - Hitting Set and Dominating Set are W[2]-hard;
  - We can show other problems are W[t]-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Gennaro Cordasco, Luisa Gargano, and Adele A. Rescigno (Apr. 2023). "Immunization in the Threshold Model: A Parameterized Complexity Study". In: *Algorithmica* 85.11, pp. 3376–3405. ISSN: 1432-0541. DOI: 10.1007/s00453-023-01118-y

Parameter	**Hardness**	
k	W[1]-hard	$t(v) = 1 \quad \forall v$
l	W[1]-hard	$t(v) = 1 \quad \forall v$
$k + l$	FPT	
$\|S\| + l$	W[2]-hard	Bipartite graphs
$k + \|S\|$	FPT	
$\Delta(G) + l$	W[2]-hard	$t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard	
$nd(G)$	W[1]-hard	
$k + nd(G)$	FPT	
$l + nd(G)$	FPT	
$\min(\Delta(G), k) + tw(G) + l$	FPT	

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \to \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y, the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a k-**immunizing set**.

Definition

The k-**immunization number of** (G, t), denoted by

$$\mathrm{Im}(G, t, k)$$

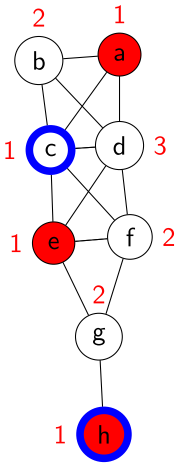
, is size of a **mimimum** k-**immunizing set** of (G, t).

Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\text{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

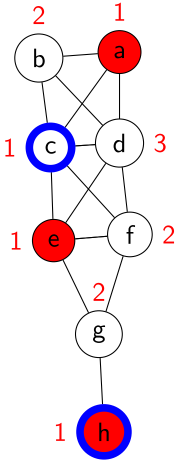
$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.

Definition

If $k = |S|$, then we say that $\mathrm{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t), and denote it by $\mathrm{In}(G, t)$.

Example

$\mathrm{In}(G, t) = \mathrm{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.

Definition

If $k = |S|$, then we say that $\mathrm{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t), and denote it by $\mathrm{In}(G, t)$.

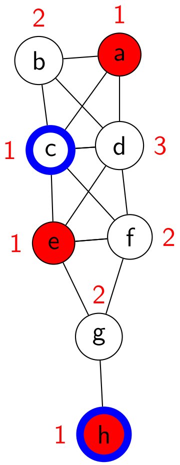
Example

$\mathrm{In}(G, t) = \mathrm{Im}(G, t, k = 3) = 2$.

Remark

For any suitable k, we have that

$$\mathrm{Im}(G, t, k) \leq \mathrm{In}(G, t)$$



Influence Immunization Bounding (cont.)

We can also define a **restricted version** of IIB, which we call R-IIB.

Restricted Influence Immunization Bounding (R-IIB)
Input: A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$, seed set $S \subseteq V(G)$, and $k, l \in \mathbb{N}$.
Question: Decide whether there exists $Y \subseteq V(G) \setminus S$ such that $|Y| \leq l$ and Y is a k-immunizing set of G.

- R-IIB is also W[1]-hard when parameterized by k or by l.

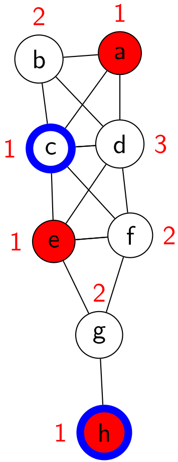
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- $Y = \{c, h\}$ **is not a restricted k-immunizing set of** (G, t)..



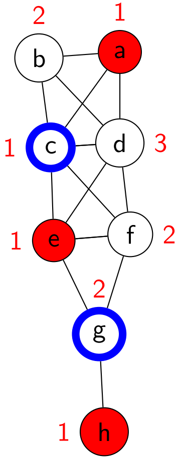
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- $Y = \{c, h\}$ **is not a restricted k-immunizing set of** (G, t)**..**
- $Y = \{c, g\}$ **is a restricted k-immunizing set of** (G, t)**.**



Influence Immunization Bounding (cont.)

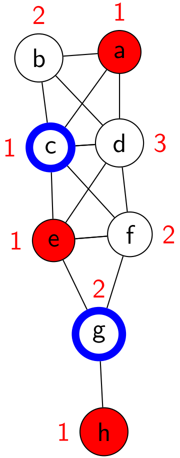
Definition

The **restricted k-immunization number of** (G, t), denoted by

$$\mathrm{Im}_r(G, t, k)$$

, is the size of a **mimimum restricted k-immunizing set** of (G, t).

- The **restricted inhibition number** $\mathrm{In}_r(G, t)$ is analogous.
- **But why study the restricted version?**



Influence Immunization Bounding (cont.)

The restricted version gives an upper bound for the original problem.

Remark

For any suitable k, we have that

$$\mathrm{Im}(G, t, k) \leq \mathrm{Im}_r(G, t, k)$$

- A restricted k-immunizing set for (G, t) is also a (unrestricted) k-immunizing set for (G, t).

Section 6: **Our Results**

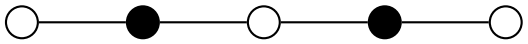
Part 1

Paths and Complete Graphs with $t(v) = c$

Paths with $t(v) = 2$

Definition

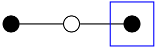
Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then $P = v_1 v_2 \ldots v_k$ is a S-**alternating path** of G if P is a path of G with *at least 3 vertices* and the vertices of P alternate with respect to their membership in S.



Paths with $t(v) = 2$

- **Main observation:** for each S-alternating path of length 4, **we need only one vertex to inhibit** the infection.

Figure: When the S-alternating paths has length 4, we can inhibit the infection by immunizing only one vertex.



(a) The infection stops by immunizing any vertex. The immunized vertex is shown in blue.

(b) When the S-alternating path has length 3 and starts with a uninfected vertex, the infection does not spread.

Paths with $t(v) = 2$

- We can split the path into S-alternating paths of length 4.

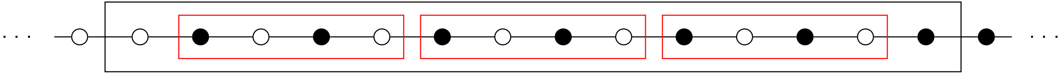


Figure: Splitting a bigger S-alternating path into pieces of length 4 starting with infected vertices.

Proposition

Let $G = P_n$ with $t(v) = 2$ for all $v \in V(G)$. Then, for any $k \in \mathbb{N}$,

$$Im(G, t, k) \leq In(G, t) \leq \lceil \frac{n}{4} \rceil$$

Exact value for paths with $t(v) = 2$

Definition

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then:

- $P = v_1 v_2 v_3$ is a $\bullet \circ \bullet$-path if P is a **maximal S-alternating path** of length exactly 3 such that $v_1, v_3 \in S$.
- $P = v_1 v_2 v_3 v_4$ is a $\bullet \circ \bullet \circ$-path if P is a S-alternating path of length 4 such that $v_1, v_3 \in S$.

Part 2

Polynomial Algorithm for the k-Immunization Number in Trees

Part 3

Polynomial Algorithm the Inhibition Number when $t(v) = 1$

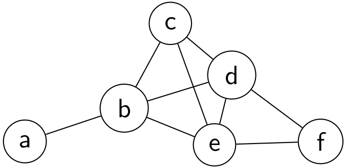
Part 4

Hardness for Chordal Graphs

Hardness for Chordal Graphs

Definition

A graph G is *chordal* if every cycle C of length at least 4 in G has a *chord*: an edge between no consecutive vertices of C.



Chordal graphs are a superclass of several important graph classes:

- Trees;
- Split Graphs;
- Interval Graphs.

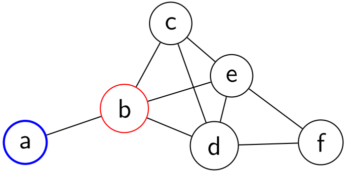
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

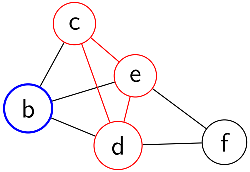
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

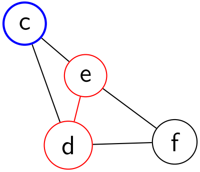
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

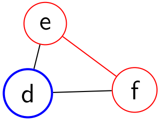
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

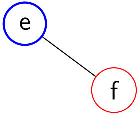
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \dots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \dots, v_i\}]$.

- Perfect elimination ordering: f, e, d, c, b, a.

Hardness for Chordal Graphs

Theorem (Dirac 1961)

A graph G is chordal if and only if G has a perfect elimination ordering.

Theorem (Dirac 1961)

Let G be a graph. If G is chordal and not complete, then G has two non-adjacent simplicial vertices.

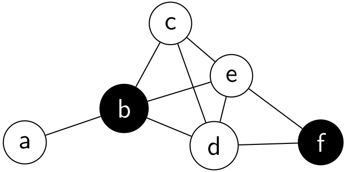
Theorem (Dirac 1961)

If a graph G is chordal, then any induced subgraph of G is chordal.

Hardness for Chordal Graphs

Definition

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ has at least one neighbor in D.



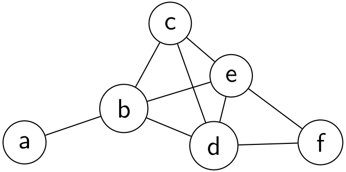
k-Dominating Set is:

- W[1]-hard for chordal graphs parameterized by k (Liu and Song 2009);
- NP-hard for split graphs (Bertossi 1984).

Hardness for Chordal Graphs

Proposition

INFLUENCE IMMUNIZATION BOUNDING *is* W[1]*-hard parameterized by the maximum number of infected vertices* k *on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \emptyset$;
- $U = \emptyset$;

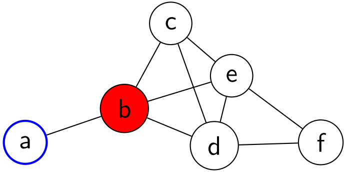
Let $\langle G, k\rangle$ be an instance of k-DOMINATING SET in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

Proposition

Influence Immunization Bounding is W[1]*-hard parameterized by the maximum number of infected vertices k on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \{b\}$;
- $U = \{a\}$;

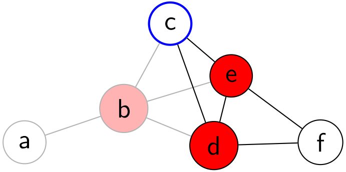
Let $\langle G, k \rangle$ be an instance of k-Dominating Set in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

Proposition

INFLUENCE IMMUNIZATION BOUNDING *is* W[1]*-hard parameterized by the maximum number of infected vertices* k *on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \{b, d, e\}$;
- $U = \{a, c\}$;

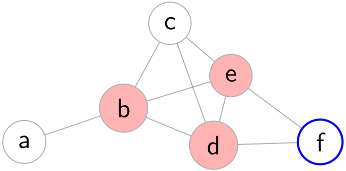
Let $\langle G, k\rangle$ be an instance of k-DOMINATING SET in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

Proposition

INFLUENCE IMMUNIZATION BOUNDING *is* W[1]*-hard parameterized by the maximum number of infected vertices k on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \{b, d, e\}$;
- $U = \{a, c, f\}$;

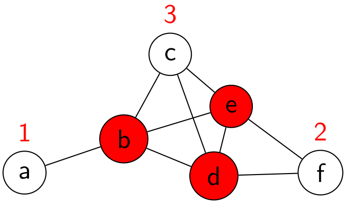
Let $\langle G, k\rangle$ be an instance of k-DOMINATING SET in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

Proposition

INFLUENCE IMMUNIZATION BOUNDING *is* W[1]*-hard parameterized by the maximum number of infected vertices* k *on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



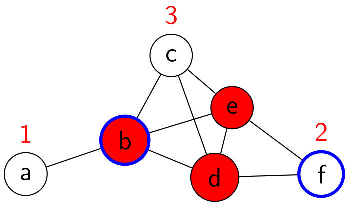
- $S = \{b, d, e\}$;
- $U = \{a, c, f\}$;

Let $\langle G, k\rangle$ be an instance of k-DOMINATING SET in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

- **G has a dominating set of size $\leq k$ if and only if (G, t, S) has a $|S|$-imunizing set of size $\leq k$.**



- We conclude that IIB is W[1]-hard for chordal graphs parameterized by k, and also NP-complete for split graphs.

Part 5

Hardness for Bipartite Graphs

Part 6

Hardness for Planar Bipartite Subcubic Graphs

Hardness for Planar Bipartite Subcubic Graphs

Definition

A graph G is planar if it can be drawn on the plane without crossing edges.

Hardness for Planar Bipartite Subcubic Graphs

- In RESTRICTED 3-SAT, we are given a 3-Sat CNF formula in which every variable **occurs *exactly* 3 times**: twice as positive and once as negative.
  - R-3-SAT is NP-complete (Dahlhaus et al. 1994).

Example

$\varphi = (u \vee v \vee \overline{w}) \wedge (\overline{u} \vee v \vee x) \wedge (u \vee \overline{x} \vee \overline{y}) \wedge (\overline{v} \vee w \vee x) \wedge (w \vee z \vee y) \wedge (y \vee \overline{z}) \wedge (z)$

- We show that R-3-SAT $\leq_P$ (Planar Bipartite Subcubic) IIB

Part 7

Pathwidth Based Bounds

References I

- Araújo, Rafael and Rudini Sampaio (May 2023). "Domination and convexity problems in the target set selection model". In: *Discrete Applied Mathematics* 330, pp. 14–23. ISSN: 0166-218X. DOI: 10.1016/j.dam.2022.12.021. URL: http://dx.doi.org/10.1016/j.dam.2022.12.021.
- Barabási, Albert-László and Márton Pósfai (2016). *Network science*. Cambridge: Cambridge University Press. ISBN: 9781107076266 1107076269. URL: http://barabasi.com/networksciencebook/.
- Bertossi, Alan A (July 1984). "Dominating sets for split and bipartite graphs". en. In: *Inf. Process. Lett.* 19.1, pp. 37–40.
- Coleman, James, Elihu Katz, and Herbert Menzel (Dec. 1957). "The diffusion of an innovation among physicians". In: *Sociometry* 20.4, p. 253.
- Cordasco, Gennaro, Luisa Gargano, and Adele A. Rescigno (Apr. 2023). "Immunization in the Threshold Model: A Parameterized Complexity Study". In: *Algorithmica* 85.11, pp. 3376–3405. ISSN: 1432-0541. DOI: 10.1007/s00453-023-01118-y.
- Dahlhaus, E. et al. (Aug. 1994). "The Complexity of Multiterminal Cuts". In: *SIAM Journal on Computing* 23.4, pp. 864–894. ISSN: 1095-7111. DOI: 10.1137/s0097539792225297. URL: http://dx.doi.org/10.1137/s0097539792225297.

References II

Dirac, G A (Apr. 1961). "On rigid circuit graphs". en. In: *Abh. Math. Semin. Univ. Hambg.* 25.1-2, pp. 71–76.

Dreyer, Paul A. and Fred S. Roberts (2009). "Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion". In: *Discrete Applied Mathematics* 157.7, pp. 1615–1627. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2008.09.012. URL: https://www.sciencedirect.com/science/article/pii/S0166218X08004393.

Flocchini, Paola et al. (Apr. 2003). "On time versus size for monotone dynamic monopolies in regular topologies". In: *Journal of Discrete Algorithms* 1.2, pp. 129–150. ISSN: 1570-8667. DOI: 10.1016/s1570-8667(03)00022-4. URL: http://dx.doi.org/10.1016/S1570-8667(03)00022-4.

Keiler, Lucas et al. (Oct. 2023). "Target set selection with maximum activation time". In: *Discrete Applied Mathematics* 338, pp. 199–217. ISSN: 0166-218X. DOI: 10.1016/j.dam.2023.06.004. URL: http://dx.doi.org/10.1016/j.dam.2023.06.004.

References III

Kempe, David, Jon Kleinberg, and Éva Tardos (2003). "Maximizing the spread of influence through a social network". In: *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '03. Washington, D.C.: Association for Computing Machinery, pp. 137–146. ISBN: 1581137370. DOI: 10.1145/956750.956769. URL: https://doi.org/10.1145/956750.956769.

Liu, Chunmei and Yinglei Song (July 2009). "Parameterized dominating set problem in chordal graphs: complexity and lower bound". en. In: *J. Comb. Optim.* 18.1, pp. 87–97.

Marcilon, Thiago and Rudini Sampaio (Jan. 2018). "The maximum time of 2-neighbor bootstrap percolation: Complexity results". In: *Theoretical Computer Science* 708, pp. 1–17. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2017.10.014. URL: http://dx.doi.org/10.1016/j.tcs.2017.10.014.

Ryan, Bryce and Neal Gross (1950). *Acceptance and diffusion of hybrid corn seed in two Iowa communities.*

Soriano-Paños, David et al. (Mar. 2022). "Modeling Communicable Diseases, Human Mobility, and Epidemics: A Review". In: *Annalen der Physik* 534.6. ISSN: 1521-3889. DOI: 10.1002/andp.202100482. URL: http://dx.doi.org/10.1002/andp.202100482.