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Restricting infections on graph classes

Supervised by Ana Karolinna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

Introduction

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- The diffusion of innovation, (mis-)information, and memes.

Information

Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.

- The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
- The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

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- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.
- Even bedbugs seem to spread from hotel to hotel via travelers (Barabási and Pósfai 2016).

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- Compartments are states;
- Each agent/individual belongs to one state.

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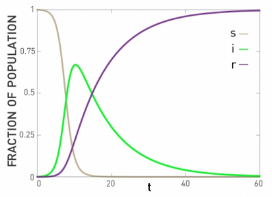
Example (SIR)



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The assumption that every individual from one compartment has the same chance of interacting with an individual from another compartment.

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- The homogeneous mixing hypothesis is **false**.
- The **structure of the contact network** is what facilitates the contagion.

Information

Studies have shown that the airport mobility network provides reliable information for the prediction and control of airborne epidemics like H1N1 (Soriano-Paños et al. 2022).

Modeling Contagion on Graphs: Threshold Model

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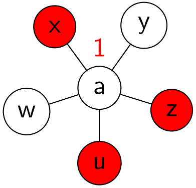

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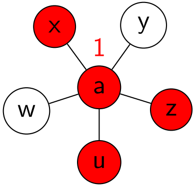

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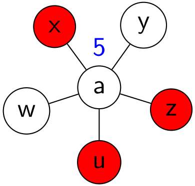
  - Bigger thresholds: laggards / greater resistance.



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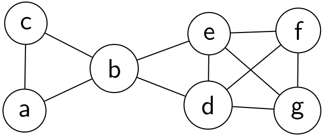

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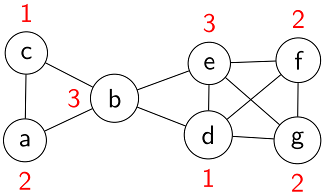

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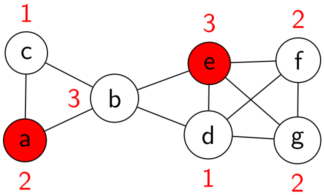

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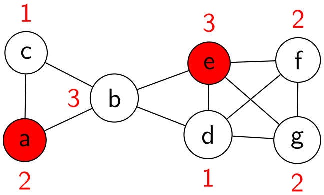

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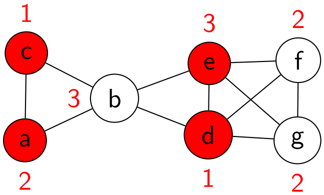

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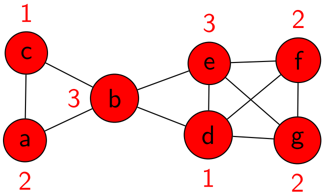
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

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  - Decision version is NP-complete even for bipartite graphs.
  - NP-complete even for k-regular graphs if we require S to infect all vertices of G – **Target Set Selection** (TSS) (Dreyer and Roberts 2009).

Problems arising from the Threshold Model

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

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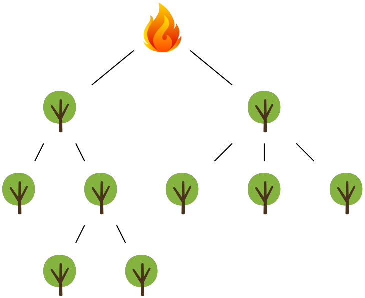

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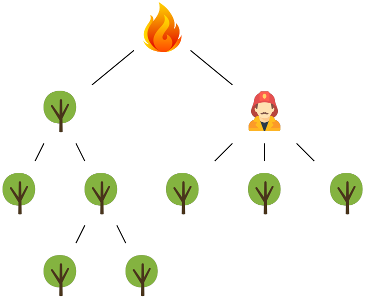

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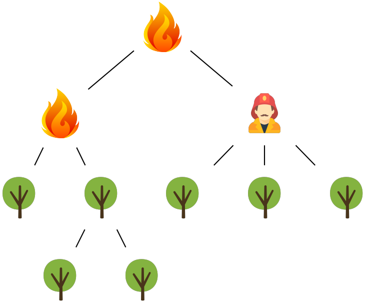

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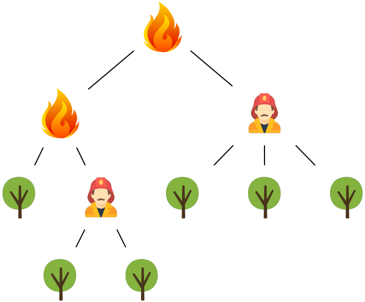

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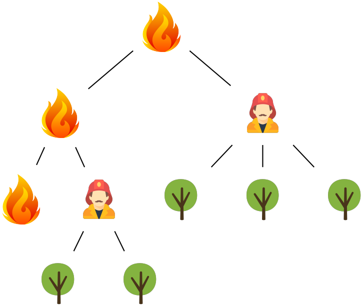

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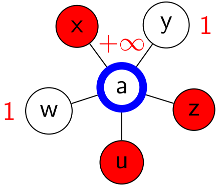
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter? **No, but sort of...**

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



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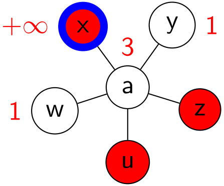

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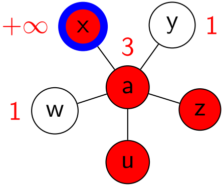

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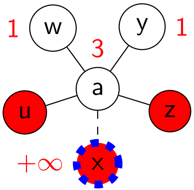

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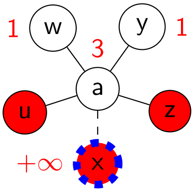

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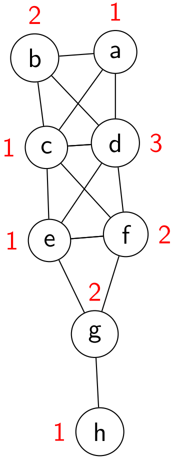

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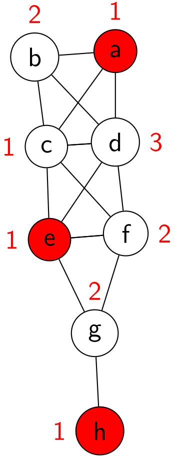

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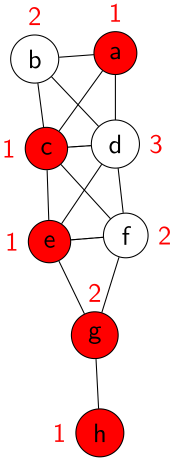

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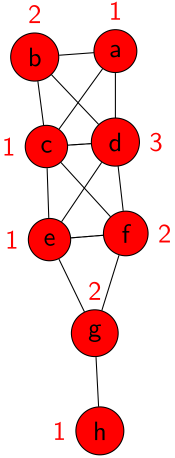

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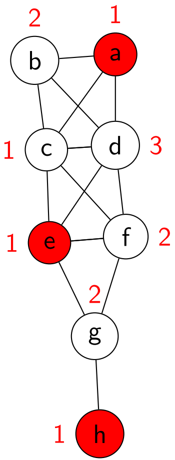

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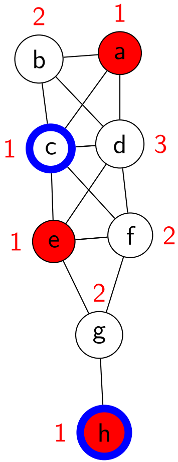
We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



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We want to find a immunizing set $Y \subseteq V(G)$ such that:

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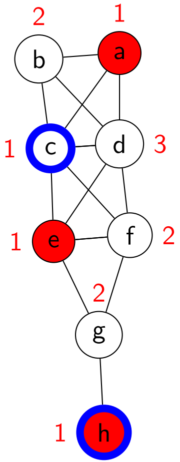
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.



Remark

Notice that we must have $k \geq |S|$.

But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

Input: $x \in \Sigma^*$. *Question*: Decide whether $x \in Q$.

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Definition

A **parameter** is a function $\kappa : \Sigma^* \to \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) =$ "Number of variables of φ", where φ is a CNF formula.

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p-INDEPENDENT-SET
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

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- This means that **parameters of the problem matter for its tractability**.

Parameterized Complexity

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

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- Clique is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- Vertex Cover is in FPT parameterized by k.

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  - We can show other problems are W$[t]$-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Influence Immunization Bounding was introduced by (Cordasco, Gargano, and Rescigno 2023).

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- W$[2]$-hard parameterized by $\Delta(G) + l$ even if $t(v) \leq 2 \quad \forall v \in V(G)$.

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- W$[1]$-hard parameterized by the treewidth $tw(G)$;
- W$[2]$-hard parameterized by $|S| + l$ even on bipartite graphs;
- W$[2]$-hard parameterized by $\Delta(G) + l$ even if $t(v) \leq 2 \quad \forall v \in V(G)$.
- FPT parameterized by $k + l$;
- FPT parameterized by $k + |S|$;
- FPT parameterized by $min(\Delta(G), k) + tw(G) + l$;
- FPT parameterized by $k + nd(G)$;
- FPT parameterized by $l + nd(G)$.