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Restricting infections on graph classes

Supervised by Ana Karolinna Maia de Oliveira and Carlos Vinícius Gomes Costa Lima

Introduction

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- Spreading of pathogens, like viruses or bacteria;
- The diffusion of innovation, (mis-)information, and memes.

Information

Studies found that most people adopt innovative ideas based on the experience of their neighbors or community.

- The adoption of hybrid seed corn among farmers (Ryan and Gross 1950);
- The adoption of tetracycline by physicians (Coleman, Katz, and Menzel 1957).

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- The homogeneous mixing hypothesis is **false**.
- The **structure of the contact network** is what facilitates the contagion.

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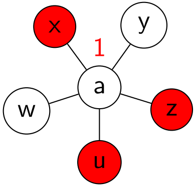
  - Smaller thresholds: early adopters / low immunity;



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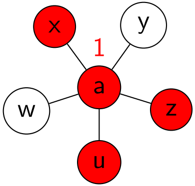

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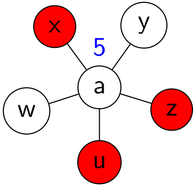
  - Bigger thresholds: laggards / greater resistance.



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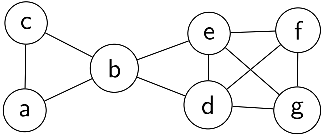

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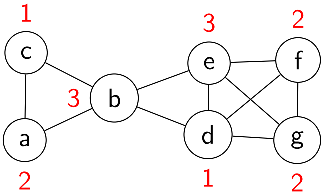

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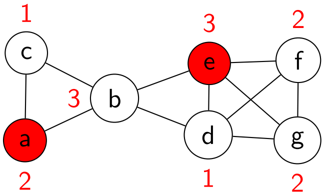

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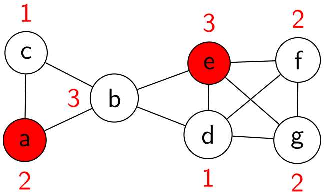

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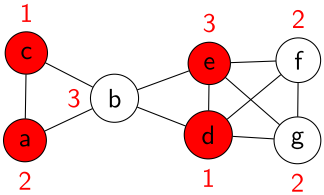

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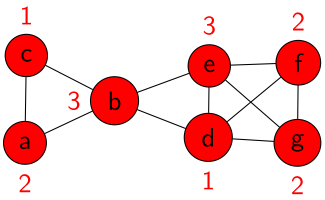
- The diffusion happens in **discrete time steps**.
- Once a vertex is infected, it **stays infected** – we call it a t-**irreversible process**.



Problems arising from the Threshold Model

Several computational problems arise from the Threshold Model.

- **Influence Maximization** (IM) (Kempe, Kleinberg, and Tardos 2003): find a seed set of bounded cardinality that maximizes the number of infected vertices.

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  - Decision version is NP-complete even for bipartite graphs.
  - NP-complete even for k-regular graphs if we require S to infect all vertices of G – **Target Set Selection** (TSS) (Dreyer and Roberts 2009).

Problems arising from the Threshold Model

Variations of these problems:

- Directed version;
- Majority version: $t(v) = \lceil \frac{d(v)}{2} \rceil \quad \forall v \in V(G)$;
- TSS with maximum activation time (Keiler et al. 2023; Flocchini et al. 2003; Marcilon and Sampaio 2018);
- TSS with activation time equal to 1 (Araújo and Sampaio 2023);

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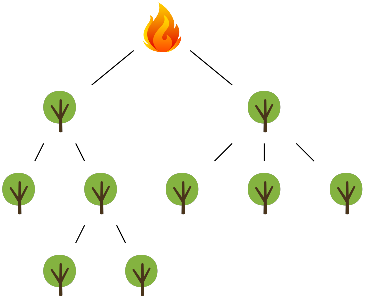

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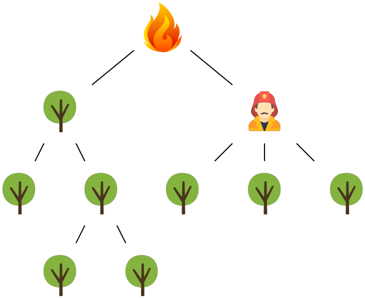

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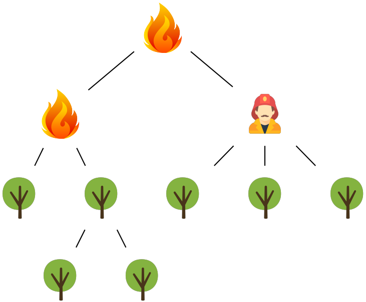

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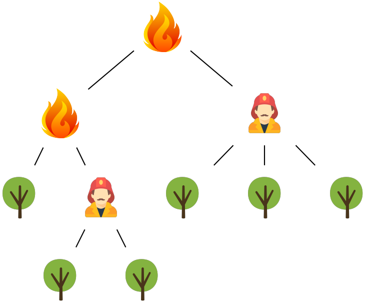

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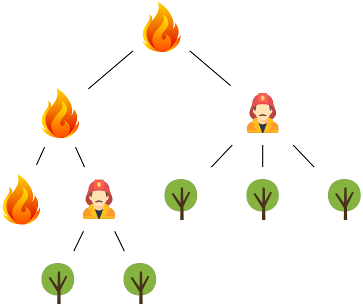

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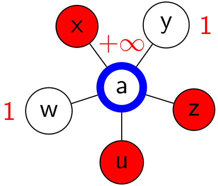
- **Influence Immunization Bounding** (IIB);
  - A generalization of Firefighter? **No, but sort of...**

Influence Immunization Bounding

First, we need to define what it means to **immunize a vertex**.

- Option 1: raise the vertex threshold above its degree;



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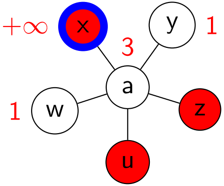

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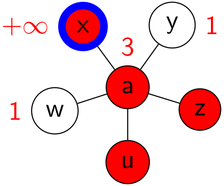

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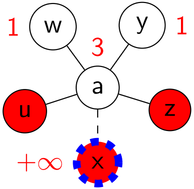

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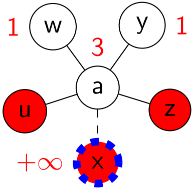

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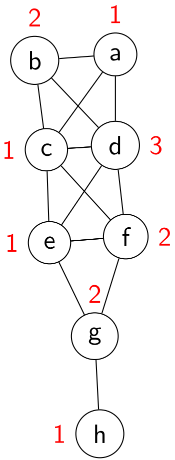

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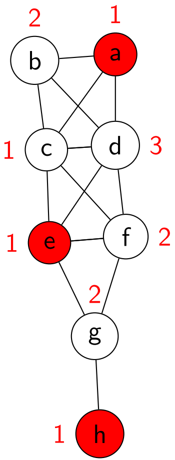

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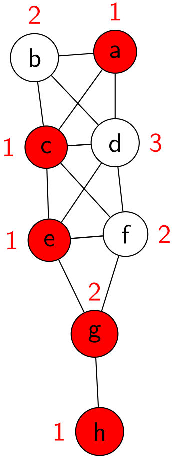

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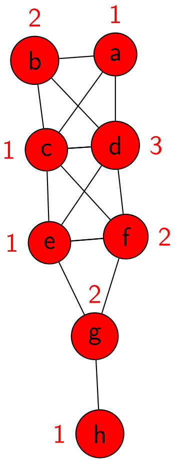

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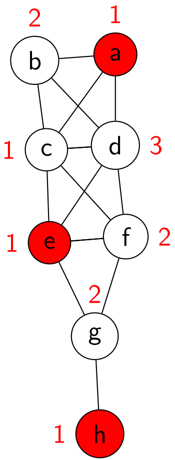

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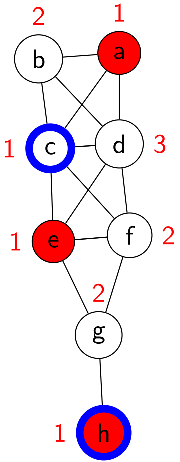
We want to find a immunizing set $Y \subseteq V(G)$ such that:

- $|Y| \leq l$; and



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We want to find a immunizing set $Y \subseteq V(G)$ such that:

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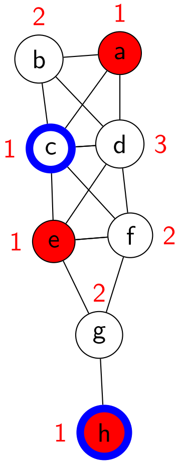
- By immunizing Y at time $\tau = 0$, the **infection gets restricted to at most k vertices**.



Remark

Notice that we must have $k \geq |S|$.

But before we proceed...
A little of **Parameterized Complexity**.

Parameterized Complexity

We can define a (classical) decision problem as follows:

- Let Σ be a finite alphabet, and $Q \subseteq \Sigma^*$.

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Definition

A **parameter** is a function $\kappa : \Sigma^* \to \mathbb{N}$ that takes the input of a problem to the naturals.

Example

A parameter for SAT can be $\kappa(\varphi) =$ "Number of variables of φ", where φ is a CNF formula.

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Now we can define a **parameterized problem**.

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p-Independent-Set
Input: A graph G and $k \in \mathbb{N}$.
Question: Decide whether G has an independent set of cardinality k.
Parameter: k.

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But what is the motivation behind this?

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  - Computational biology: real instances of DNA chain reconstruction have special properties, e.g., treewidth ≤ 11;
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  - Robotics: the number of degrees of freedom in motion planning problems is usually ≤ 10.
- This means that **parameters of the problem matter for its tractability**.

Parameterized Complexity

Let $x \in \Sigma^*$ be an instance of a parameterized problem (P, κ).

Definition (XP)

(P, κ) is **slicewise polynomial** if it admits an algorithm which running time is

$$O(|x|^{\kappa(x)})$$

- CLIQUE is in XP parameterized by k: enumerate all subsets of k vertices and check if they form a clique.

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Definition (FPT)

(P, κ) is **fixed-parameter tractable** if it admits an algorithm which running time is

$$O(f(\kappa(x)) \cdot poly(|x|))$$

- Vertex Cover is in FPT parameterized by k.

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  - Hitting Set and Dominating Set are W[2]-hard;
  - We can show other problems are W[t]-hard by using FPT-reductions.

Influence Immunization Bounding (cont.)

Gennaro Cordasco, Luisa Gargano, and Adele A. Rescigno (Apr. 2023). "Immunization in the Threshold Model: A Parameterized Complexity Study". In: *Algorithmica* 85.11, pp. 3376–3405. ISSN: 1432-0541. DOI: 10.1007/s00453-023-01118-y

Parameter	**Hardness**	
k	W[1]-hard	$t(v) = 1 \quad \forall v$
l	W[1]-hard	$t(v) = 1 \quad \forall v$
$k + l$	FPT	
$\lvert S\rvert + l$	W[2]-hard	Bipartite graphs
$k + \lvert S\rvert$	FPT	
$\Delta(G) + l$	W[2]-hard	$t(v) \leq 2 \quad \forall v$
$tw(G)$	W[1]-hard	
$nd(G)$	W[1]-hard	
$k + nd(G)$	FPT	
$l + nd(G)$	FPT	
$\min(\Delta(G), k) + tw(G) + l$	FPT	

Influence Immunization Bounding (cont.)

Let $G = (V, E)$ be a graph with thresholds $t : V(G) \to \mathbb{N}$ and seed set $S \subseteq V(G)$.

Definition

Let $Y \subseteq V(G)$. If, by immunizing the vertices of Y, the infection in (G, t) is restricted to **at most** k vertices, we say that Y is a k-**immunizing set**.

Definition

The k-**immunization number of** (G, t), denoted by

$$\mathrm{Im}(G, t, k)$$

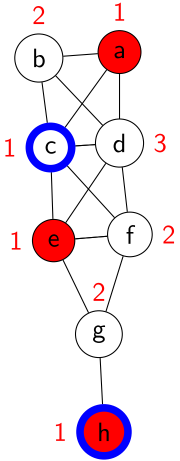
, is size of a **mimimum** k-**immunizing set** of (G, t).

Influence Immunization Bounding (cont.)

Consider the graph on the right.

Example

$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.



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Example

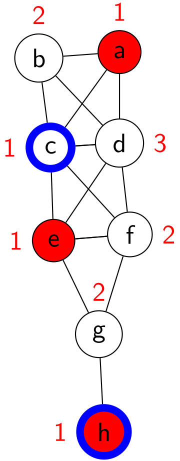
$Y = \{c, h\}$ is a 3-immunizing set of (G, t).
$\mathrm{Im}(G, t, k = 3) = 2$.

Definition

If $k = |S|$, then we say that $\mathrm{Im}(G, t, k = |S|)$ is the **inhibition number of** (G, t), and denote it by $\mathrm{In}(G, t)$.

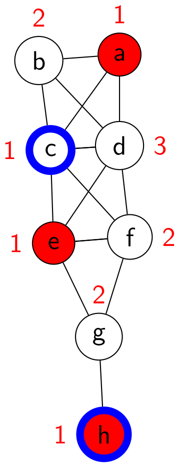
Example

$\mathrm{In}(G, t) = \mathrm{Im}(G, t, k = 3) = 2$.



Influence Immunization Bounding (cont.)

Consider the graph on the right.



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Example

$\text{In}(G, t) = \text{Im}(G, t, k = 3) = 2$.

Remark

For any suitable k, we have that

$$\text{Im}(G, t, k) \leq \text{In}(G, t)$$

Influence Immunization Bounding (cont.)

We can also define a **restricted version** of IIB, which we call R-IIB.

> **Restricted Influence Immunization Bounding** (R-IIB)
> *Input*: A graph $G = (V, E)$ with thresholds $t : V(G) \to \mathbb{N}$, seed set $S \subseteq V(G)$, and $k, l \in \mathbb{N}$.
> *Question*: Decide whether there exists $Y \subseteq V(G) \setminus S$ such that $|Y| \leq l$ and Y is a k-immunizing set of G.

- R-IIB is also W[1]-hard when parameterized by k or by l.

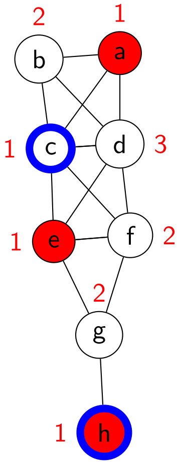
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted k-immunizing set** of (G, t) if Y is a k-immunizing set of (G, t) and $Y \cap S = \emptyset$.

- **$Y = \{c, h\}$ is not a restricted k-immunizing set of (G, t)..**



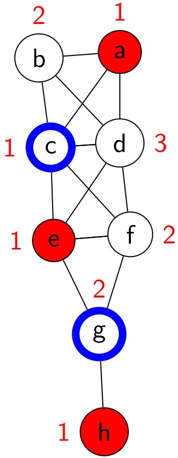
Influence Immunization Bounding (cont.)

Consider the graph on the right.

Definition

$Y \subseteq V(G)$ is a **restricted** k**-immunizing set** of (G,t) if Y is a k-immunizing set of (G,t) and $Y \cap S = \emptyset$.

- $Y = \{c, h\}$ **is** **not** **a restricted** k**-immunizing set of** (G,t)..
- $Y = \{c, g\}$ **is a restricted** k**-immunizing set of** (G,t).



Influence Immunization Bounding (cont.)

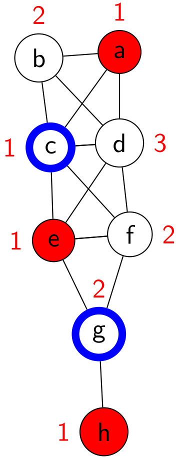
Definition

The **restricted k-immunization number of** (G, t), denoted by

$$\mathrm{Im}_r(G, t, k)$$

, is the size of a **mimimum restricted k-immunizing set** of (G, t).

- The **restricted inhibition number** $\mathrm{In}_r(G, t)$ is analogous.
- **But why study the restricted version?**



Influence Immunization Bounding (cont.)

The restricted version gives an upper bound for the original problem.

Remark

For any suitable k, we have that

$$\mathrm{Im}(G, t, k) \leq \mathrm{Im}_r(G, t, k)$$

- A restricted k-immunizing set for (G, t) is also a (unrestricted) k-immunizing set for (G, t).

Section 6: **Our Results**

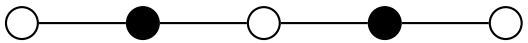
Part 1

Paths and Complete Graphs with $t(v) = c$

Paths with $t(v) = 2$

Definition

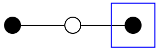
Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then $P = v_1 v_2 \dots v_k$ is a S-**alternating path** of G if P is a path of G with *at least 3 vertices* and the vertices of P alternate with respect to their membership in S.



Paths with $t(v) = 2$

- **Main observation:** for each S-alternating path of length 4, **we need only one vertex to inhibit** the infection.

Figure: When the S-alternating paths has length 4, we can inhibit the infection by immunizing only one vertex.



(a) The infection stops by immunizing any vertex. The immunized vertex is shown in blue.

(b) When the S-alternating path has length 3 and starts with a uninfected vertex, the infection does not spread.

Paths with $t(v) = 2$

- We can split the path into S-alternating paths of length 4.

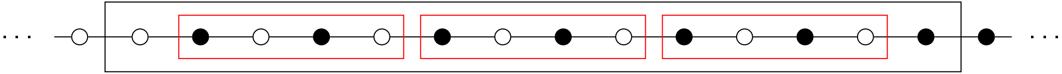


Figure: Splitting a bigger S-alternating path into pieces of length 4 starting with infected vertices.

Proposition

Let $G = P_n$ with $t(v) = 2$ for all $v \in V(G)$. Then, for any $k \in \mathbb{N}$,

$$Im(G, t, k) \leq In(G, t) \leq \lceil \frac{n}{4} \rceil$$

Exact value for paths with $t(v) = 2$

Definition

Let $G = (V, E)$ be a graph and $S \subseteq V(G)$ a seed set of G. Then:

- $P = v_1 v_2 v_3$ is a $\bullet \circ \bullet$-path if P is a **maximal S-alternating path** of length exactly 3 such that $v_1, v_3 \in S$.
- $P = v_1 v_2 v_3 v_4$ is a $\bullet \circ \bullet\circ$-path if P is a S-alternating path of length 4 such that $v_1, v_3 \in S$.

Part 2

Polynomial Algorithm for the k-Immunization Number in Trees

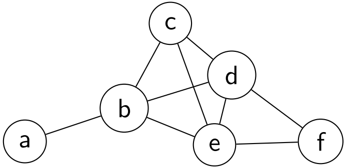
Part 3

Hardness for Chordal Graphs

Hardness for Chordal Graphs

Definition

A graph G is *chordal* if every cycle C of length at least 4 in G has a *chord*: an edge between no consecutive vertices of C.



Chordal graphs are a superclass of several important graph classes:

- Trees;
- Split Graphs;
- Interval Graphs.

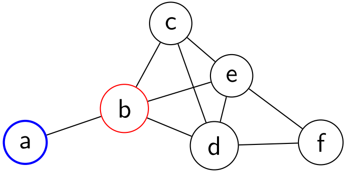
Hardness for Chordal Graphs

Definition

Given a graph G, a vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G(v)$ is a clique.

Definition

Given a graph G, an ordering $v_1, \ldots, v_n$ of $V(G)$ is a *perfect elimination ordering* if for all $i \in [n]$, v_i is a simplicial vertex of $G[\{v_1, \ldots, v_i\}]$.



- Perfect elimination ordering: f, e, d, c, b, a.

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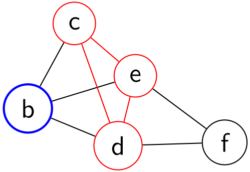

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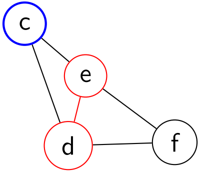

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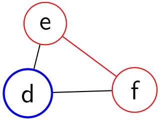

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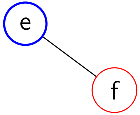

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- Perfect elimination ordering: f, e, d, c, b, a.

Hardness for Chordal Graphs

Theorem (Dirac 1961)

A graph G is chordal if and only if G has a perfect elimination ordering.

Theorem (Dirac 1961)

Let G be a graph. If G is chordal and not complete, then G has two non-adjacent simplicial vertices.

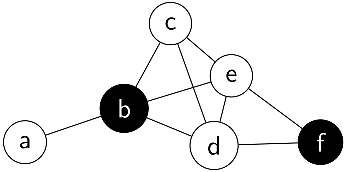
Theorem (Dirac 1961)

If a graph G is chordal, then any induced subgraph of G is chordal.

Hardness for Chordal Graphs

Definition

Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ has at least one neighbor in D.



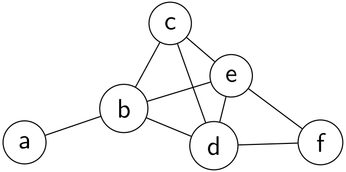
k-Dominating Set is:

- W[1]-hard for chordal graphs parameterized by k (Liu and Song 2009);
- NP-hard for split graphs (Bertossi 1984).

Hardness for Chordal Graphs

Proposition

Influence Immunization Bounding is W[1]*-hard parameterized by the maximum number of immunized vertices l on chordal graphs. Moreover, it is* NP*-complete on split graphs.*



- $S = \emptyset$;
- $U = \emptyset$;

Let $\langle G, k \rangle$ be an instance of k-Dominating Set in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

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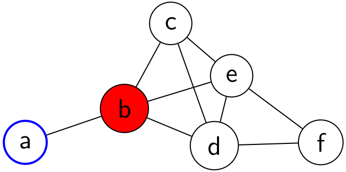

- $S = \{b\}$;
- $U = \{a\}$;

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Hardness for Chordal Graphs

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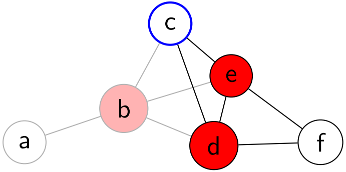

- $S = \{b, d, e\}$;
- $U = \{a, c\}$;

Let $\langle G, k \rangle$ be an instance of k-DOMINATING SET in which G is chordal.

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2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
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Hardness for Chordal Graphs

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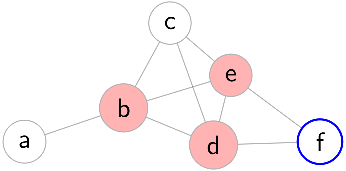

- $S = \{b, d, e\}$;
- $U = \{a, c, f\}$;

Let $\langle G, k \rangle$ be an instance of k-Dominating Set in which G is chordal.

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Hardness for Chordal Graphs

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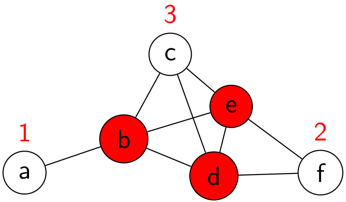


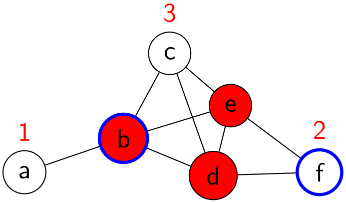
- $S = \{b, d, e\}$;
- $U = \{a, c, f\}$;

Let $\langle G, k\rangle$ be an instance of k-Dominating Set in which G is chordal.

1. Initialize $S = \emptyset$ and $U = \emptyset$;
2. Find a simplicial vertex s of G;
3. Update $S = S \cup N_G(s)$ and $U = U \cup \{s\}$;
4. Repeat steps (2-5) on $G[V(G) \setminus N_G[s]]$.
5. Define $t(v) = d_G(v)$ for all v;

Hardness for Chordal Graphs

- **G has a dominating set of size $\leq k$ if and only if (G, t, S) has a $|S|$-imunizing set of size $\leq l = k$.**



- We conclude that IIB is W[1]-hard for chordal graphs parameterized by l, and also NP-complete for split graphs.

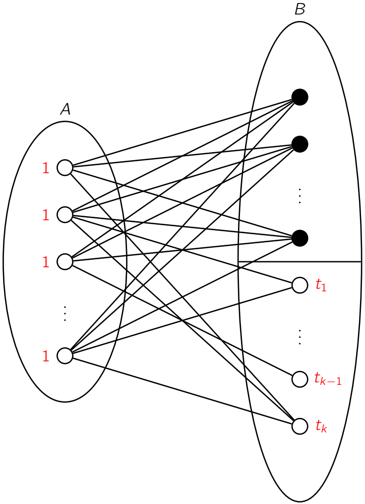
Part 4

Hardness for Bipartite Graphs

Hardness for Bipartite Graphs

Cordasco, Gargano, and Rescigno 2023 showed that IIB is $\mathrm{W}[2]$-hard parameterized by $|S| + l$ even on bipartite graphs.

- Their graph looked like this. **The part B has mixed susceptible and infected vertices.**



Proposition

IIB *is* NP*-complete on bipartite graphs even if A is entirely susceptible and B is entirely infected.*

Hardness for Bipartite Graphs

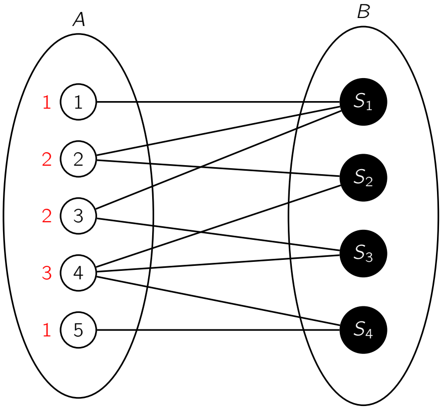
- On the SET COVER problem, we are given a universe set $\mathcal{U} = \{a_1, a_2, \ldots, a_n\}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of $\mathcal{U}$, and a parameter $c \in \mathbb{N}$, and the goal is to find $C \subset \mathcal{S}$ such that $|C| \leq c$ and the union of all sets in C is equal to $\mathcal{U}$.

1. Create a bipartite graph G such that $V(G) = \mathcal{U} \cup \mathcal{S}$ and $E(G) = \{a_i S_j \mid a_i \in S_j\}$;
2. Define $k = |\mathcal{S}|$, $l = c$, and set the seed set $S = \mathcal{S}$;
3. Set $t(a_i) = d_G(a_i)$, for all $i \in [n]$.

Hardness for Bipartite Graphs

Example

- $\mathcal{U} = \{1, 2, 3, 4, 5\}$;
- $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2, 4\}, S_3 = \{3, 4\}, S_4 = \{4, 5\}\}$; and
- $c = 2$.



Hardness for Bipartite Graphs

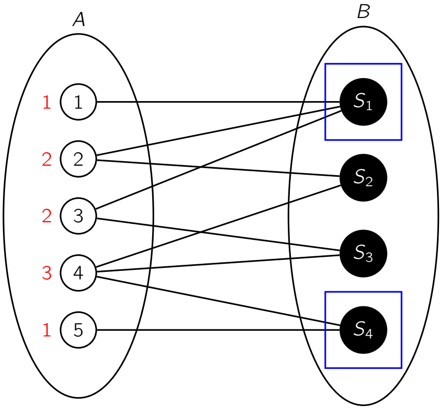
Example

- $\mathcal{U} = \{1, 2, 3, 4, 5\}$;
- $\mathcal{S} = \{S_1 = \{1, 2, 3\}, S_2 = \{2, 4\}, S_3 = \{3, 4\}, S_4 = \{4, 5\}\}$; and
- $c = 2$.

Proposition

$(\mathcal{U}, \mathcal{S})$ has a set cover of size $\leq c$ if and only if (G, t) has a $|\mathcal{S}|$-immunizing set of size $\leq c$.

- The problem remains hard even on this restricted instance for bipartite graphs.



Part 5

Hardness for Planar Bipartite Subcubic Graphs

Hardness for Planar Bipartite Subcubic Graphs

Definition

A graph G is planar if it can be drawn on the plane without crossing edges.

Theorem (Wagner 1937)

A graph G is planar if and only if G does not have a K_5 minor nor a $K_{3,3}$ minor.

Hardness for Planar Bipartite Subcubic Graphs

- In RESTRICTED 3-SAT, we are given a 3-Sat CNF formula in which every variable **occurs *exactly* 3 times**: twice as positive and once as negative.
  - R-3-SAT is NP-complete (Dahlhaus et al. 1994).

Example

$\varphi = (u \vee v \vee \overline{w}) \wedge (\overline{u} \vee v \vee x) \wedge (u \vee \overline{x} \vee \overline{y}) \wedge (\overline{v} \vee w \vee x) \wedge (w \vee z \vee y) \wedge (y \vee \overline{z}) \wedge (z)$

Hardness for Planar Bipartite Subcubic Graphs

- In Restricted 3-Sat, we are given a 3-Sat CNF formula in which every variable **occurs *exactly* 3 times**: twice as positive and once as negative.
  - r-3-Sat is NP-complete (Dahlhaus et al. 1994).

Example

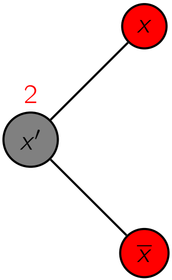
$\varphi = (u \lor v \lor \overline{w}) \land (\overline{u} \lor v \lor x) \land (u \lor \overline{x} \lor \overline{y}) \land (\overline{v} \lor w \lor x) \land (w \lor z \lor y) \land (y \lor \overline{z}) \land (z)$

Proposition

r-3-Sat $\leq_P$ *(Planar Bipartite Subcubic)* IIB.

Hardness for Planar Bipartite Subcubic Graphs

- Given a $\textsc{r-3-Sat}$ formula φ, let:
  - $V(\varphi)$ the set of variables of φ;
  - $\mathcal{C}(\varphi)$ the set of clauses of φ.
- Let's create a graph G_φ with thresholds t_φ. For each variable $x \in V(\varphi)$, we are going to add the following gadget to G_φ:

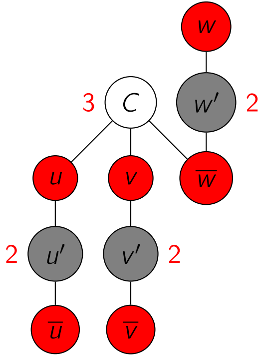


Hardness for Planar Bipartite Subcubic Graphs

- For each clause $C \in \mathcal{C}(\varphi)$, we are going to add a vertex in G_φ and connect it to its literals, setting $t_\varphi(C) = |C|$.

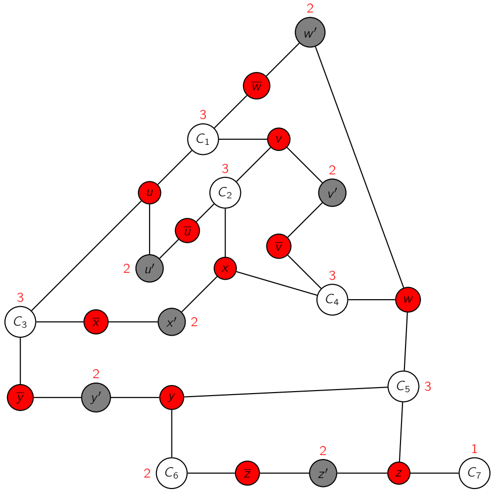
Example

$C = (u \vee v \vee \overline{w})$.



Hardness for Planar Bipartite Subcubic Graphs

- Example of G_φ for the R-3-SAT formula $\varphi = (u \vee v \vee \overline{w}) \wedge (\overline{u} \vee v \vee x) \wedge (u \vee \overline{x} \vee \overline{y}) \wedge (\overline{v} \vee w \vee x) \wedge (w \vee z \vee y) \wedge (y \vee \overline{z}) \wedge (z)$.



Hardness for Planar Bipartite Subcubic Graphs

- Suppose φ has a satisfying assignment $\mathcal{S} : V(\varphi) \to \{\textsc{True}, \textsc{False}\}$.

Proposition

$Y = (\{x \mid \mathcal{S}(x) = \textsc{True}\} \cup \{\overline{x} \mid \mathcal{S}(x) = \textsc{False}\})$ is a $(2|V(\varphi)|)$-immunizing set for (G_φ, t_φ) and $|Y| \leq |V(\varphi)|$.

Hardness for Planar Bipartite Subcubic Graphs

- Suppose φ has a satisfying assignment $\mathcal{S} : V(\varphi) \to \{\textsc{True}, \textsc{False}\}$.

Proposition

$Y = (\{x \mid \mathcal{S}(x) = \textsc{True}\} \cup \{\overline{x} \mid \mathcal{S}(x) = \textsc{False}\})$ is a $(2|V(\varphi)|)$-immunizing set for (G_φ, t_φ) and $|Y| \leq |V(\varphi)|$.

- Now, suppose Y is a $(2|V(\varphi)|)$-immunizing set for (G_φ, t_φ) and $|Y| \leq |V(\varphi)|$.

Proposition

For every $x \in V(\varphi)$, either $x \in Y$ or $\overline{x} \in Y$.

- Thus, we can build a satisfying assignment for φ from Y.

Hardness for Planar Bipartite Subcubic Graphs

Proposition

G_φ is planar bipartite subcubic.

- Clause vertices have degree at most 3, positive literals vertices have degree 3, negative literals vertices have degree 2, and auxiliary vertices have degree 2.

Hardness for Planar Bipartite Subcubic Graphs

Proposition

G_φ is planar bipartite subcubic.

- Clause vertices have degree at most 3, positive literals vertices have degree 3, negative literals vertices have degree 2, and auxiliary vertices have degree 2.
- Let A be the set of auxiliary vertices. Then we have the partition (X, Y) of $V(G_\varphi)$ such that $X = A \cup \mathcal{C}(\varphi)$ and $Y = \{x, \overline{x} \mid x \in V(\varphi)\}$.

Hardness for Planar Bipartite Subcubic Graphs

Proposition

G_φ is planar bipartite subcubic.

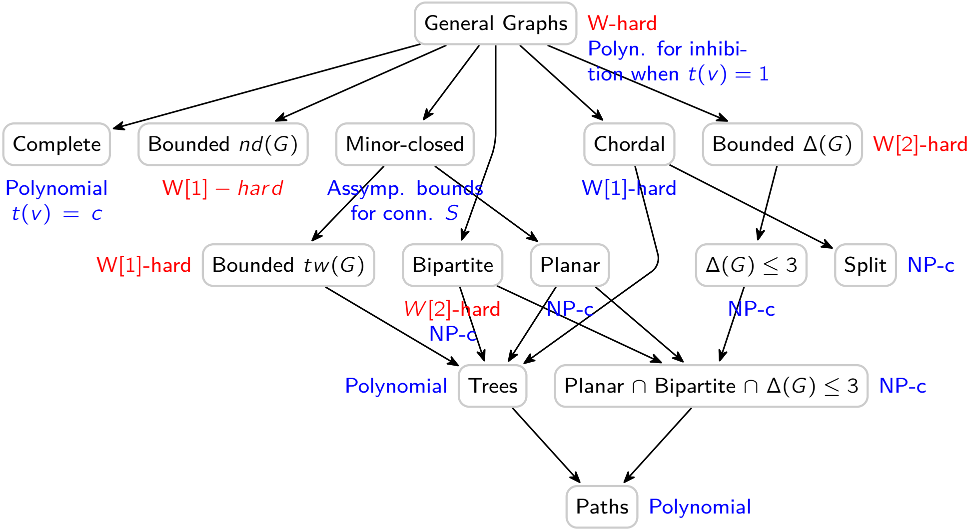
- Clause vertices have degree at most 3, positive literals vertices have degree 3, negative literals vertices have degree 2, and auxiliary vertices have degree 2.
- Let A be the set of auxiliary vertices. Then we have the partition (X, Y) of $V(G_\varphi)$ such that $X = A \cup \mathcal{C}(\varphi)$ and $Y = \{x, \overline{x} \mid x \in V(\varphi)\}$.
- Since G_φ is subcubic, it cannot have a K_5 minor.
- In order to have a $K_{3,3}$ minor, we should have 3 clauses connected with 3 equal positive literals. Since each positive literal appears only twice, G_φ is planar.

Part 6

Pathwidth Based Bounds

Section 7: **Conclusion and Future Work**

Conclusion and Future Work



Obrigado.

References I

- Araújo, Rafael and Rudini Sampaio (May 2023). “Domination and convexity problems in the target set selection model”. In: *Discrete Applied Mathematics* 330, pp. 14–23. ISSN: 0166-218X. DOI: 10.1016/j.dam.2022.12.021. URL: http://dx.doi.org/10.1016/j.dam.2022.12.021.
- Barabási, Albert-László and Márton Pósfai (2016). *Network science*. Cambridge: Cambridge University Press. ISBN: 9781107076266 1107076269. URL: http://barabasi.com/networksciencebook/.
- Bertossi, Alan A (July 1984). “Dominating sets for split and bipartite graphs”. en. In: *Inf. Process. Lett.* 19.1, pp. 37–40.
- Coleman, James, Elihu Katz, and Herbert Menzel (Dec. 1957). “The diffusion of an innovation among physicians”. In: *Sociometry* 20.4, p. 253.
- Cordasco, Gennaro, Luisa Gargano, and Adele A. Rescigno (Apr. 2023). “Immunization in the Threshold Model: A Parameterized Complexity Study”. In: *Algorithmica* 85.11, pp. 3376–3405. ISSN: 1432-0541. DOI: 10.1007/s00453-023-01118-y.
- Dahlhaus, E. et al. (Aug. 1994). “The Complexity of Multiterminal Cuts”. In: *SIAM Journal on Computing* 23.4, pp. 864–894. ISSN: 1095-7111. DOI: 10.1137/s0097539792225297. URL: http://dx.doi.org/10.1137/s0097539792225297.

References II

- Dirac, G A (Apr. 1961). "On rigid circuit graphs". en. In: *Abh. Math. Semin. Univ. Hambg.* 25.1-2, pp. 71–76.
- Dreyer, Paul A. and Fred S. Roberts (2009). "Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion". In: *Discrete Applied Mathematics* 157.7, pp. 1615–1627. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2008.09.012. URL: https://www.sciencedirect.com/science/article/pii/S0166218X08004393.
- Flocchini, Paola et al. (Apr. 2003). "On time versus size for monotone dynamic monopolies in regular topologies". In: *Journal of Discrete Algorithms* 1.2, pp. 129–150. ISSN: 1570-8667. DOI: 10.1016/s1570-8667(03)00022-4. URL: http://dx.doi.org/10.1016/S1570-8667(03)00022-4.
- Keiler, Lucas et al. (Oct. 2023). "Target set selection with maximum activation time". In: *Discrete Applied Mathematics* 338, pp. 199–217. ISSN: 0166-218X. DOI: 10.1016/j.dam.2023.06.004. URL: http://dx.doi.org/10.1016/j.dam.2023.06.004.

References III

Kempe, David, Jon Kleinberg, and Éva Tardos (2003). "Maximizing the spread of influence through a social network". In: *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '03. Washington, D.C.: Association for Computing Machinery, pp. 137–146. ISBN: 1581137370. DOI: 10.1145/956750.956769. URL: https://doi.org/10.1145/956750.956769.

Liu, Chunmei and Yinglei Song (July 2009). "Parameterized dominating set problem in chordal graphs: complexity and lower bound". en. In: *J. Comb. Optim.* 18.1, pp. 87–97.

Marcilon, Thiago and Rudini Sampaio (Jan. 2018). "The maximum time of 2-neighbor bootstrap percolation: Complexity results". In: *Theoretical Computer Science* 708, pp. 1–17. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2017.10.014. URL: http://dx.doi.org/10.1016/j.tcs.2017.10.014.

Ryan, Bryce and Neal Gross (1950). *Acceptance and diffusion of hybrid corn seed in two Iowa communities*.

Soriano-Paños, David et al. (Mar. 2022). "Modeling Communicable Diseases, Human Mobility, and Epidemics: A Review". In: *Annalen der Physik* 534.6. ISSN: 1521-3889. DOI: 10.1002/andp.202100482. URL: http://dx.doi.org/10.1002/andp.202100482.

References IV

Wagner, K (Dec. 1937). "Über eine Eigenschaft der ebenen Komplexe". de. In: *Math. Ann.* 114.1, pp. 570–590.