A study of Lie Algebras

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Abstract: In this project, I examine what Lie Algebras are, what operators are defined on them and how to represent those operators as matrices. I also examine how to use linear algebra to apply these operators to vectors. Examples of these operators are described in algebraic form, and an algorithm is given for converting them into matrix form.

1 Preliminaries

Definition 1.1 A *Lie Algebra* is a vector space, $\mathfrak{g}$, with a non-associative multiplication called the *Lie Bracket* $[\ ,\ ]$ defined on it.

The Lie Bracket is a binary operator $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following three properties:
for $x, y, z \in \mathfrak{g}$ and $\alpha \in \mathbb{R}$:

1. $[x, x] = \mathbf{0}$,
2. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$,
3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = \mathbf{0}$.

It follows directly from (1) and (2) that $[\ ,\ ]$ is skew-symmetric; *i.e.* $[x, y] = -[y, x]$
By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field.
If $[\ ,\ ]$ is alternating, then

$$\begin{aligned}
\mathbf{0} &= [x + y, x + y] && (1)\\
&= [x + y, x] + [x + y, y] && (2)\\
&= [x, x] + [y, x] + [x + y, y] && (3)\\
&= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\
\mathbf{0} &= [x, y] + [y, x] && (5)\\
\implies [x, y] &= -[y, x]. && (6)
\end{aligned}$$

Conversely, if $[\ ,\ ]$ is skew-symmetric, then

$$\begin{aligned}
[x, x] &= -[x, x] && (7)\\
[x, x] + [x, x] &= \mathbf{0} && (8)\\
2([x, x]) &= \mathbf{0}. && (9)
\end{aligned}$$

Thus, as long as $2 \neq 0$, $[\ ,\ ]$ is alternating.

Definition 1.2 The *center* of a Lie Algebra, $\mathfrak{g}$ is the set

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0}\ \forall x \in \mathfrak{g}\}.$$

Thus z is in the center of $\mathfrak{g}$ if and only if

$$[z, x] = \mathbf{0}\ \forall x \in \mathfrak{g}.$$

The *non-center* is the set $\mathfrak{v}$, given by $\mathfrak{v} = \mathfrak{g} - \mathfrak{z}$.

Definition 1.3 A Lie Algebra $\mathfrak{g}$ is called *one-step nilpotent*, or *abelian*, if and only if

$$\left[x, y\right] = \mathbf{0} \ \forall x, y \in \mathfrak{g}.$$

This is equivalent to

$$\mathfrak{z} = \mathfrak{g}.$$

It turns out that the center is not only a subspace of $\mathfrak{g}$ and therefore a vector space, but it is also a one-step nilpotent Lie Algebra. This is because the Lie Bracket is defined by an identity, and the center inherits its properties.

Definition 1.4 A Lie Algebra $\mathfrak{g}$ is called *two-step nilpotent* if and only if

$$[[x, y], z] = \mathbf{0} \ \forall x, y, z \in \mathfrak{g} \text{ and } \mathfrak{z} \neq \mathfrak{g}.$$

This means that applying the bracket twice to any arbitrary vectors in $\mathfrak{g}$ always returns the zero vector. It also implies the existance of two vectors such that one application of the lie bracket returns a non-zero element, i.e. abelian Lie algebras are not two-step nilpotent.

Let's look at an example of a Lie Algebra that is neither abelian nor two-step nilpotent.

Example 1.5 It turns out that $\mathbb{R}^3$ with the cross product is a Lie algebra with the cross product as its Lie bracket. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors of $\mathbb{R}^3$.
The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is alternating:

$$x \times x = (x_2 x_3 - x_3 x_2, x_3 x_1 - x_1 x_3, x_1 x_2 - x_2 x_1) = (0, 0, 0) = \mathbf{0};$$

skew symmetric:

$$x \times y = -y \times x \ \text{ for all } x, y \in \mathbb{R}^3;$$

and satisfies the Jacobi identity:

$$\begin{aligned} & (x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ = \ & (x \cdot z) y - (y \cdot z) x + (y \cdot x) z - (z \cdot x) y + (z \cdot y) x - (x \cdot y) z \\ = \ & \mathbf{0}. \end{aligned}$$

Example 1.6 The *Heisenberg Algebra*, $\mathfrak{h}_3$, is a specific type of Lie algebra that is spanned by three vectors e_1, e_2, z.
Its Lie bracket is defined on this basis by:
$[e_1, e_2] = z$,
$[e_1, z] = \mathbf{0}$,
$[e_2, z] = \mathbf{0}$.

The vectors e_1 and e_2 are both in the non-center, $\mathfrak{v}$, because when the Lie bracket is applied on both elements together it returns z instead of the zero vector. The vector z is clearly in the center because multiplying it by any basis vector always returns zero. In order to show that a vector is in the center it is enough to show that its product on all other bases is $\mathbf{0}$, which is the case for z. For a similar reason, all linear combinations of basis vectors that are in the center are also in the center.

This means that all scalar multiples of z are in $\mathfrak{z}$ and all linear combinations of e_1 and e_2 are in $\mathfrak{v}$. Taking $\mathfrak{z}$ as a vector space by itself looks like a line, while $\mathfrak{v}$ by itself looks like a plane. Taking the Lie bracket of two items in the plane returns something in the line, while the Lie bracket of something in the line with anything returns the zero vector.

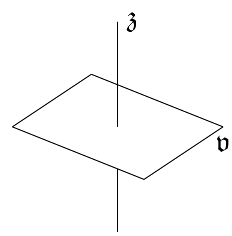


Figure 1: The Heisenberg Algebra, $\mathfrak{h}_3$

Now that we have explored some examples of Lie algebras, let's examine another operator on $\mathfrak{g}$.

Definition 1.7 An *inner product* on the vector $\mathfrak{g}$ is a *real-valued, symmetric, non-degenerate, bilinear, positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satisfies the following properties:
for $x, y, z \in \mathfrak{g}$ and $\alpha \in \mathbb{R}$.

1. $\langle x, y\rangle = \langle y, x\rangle$,
2. If $\langle x, y\rangle = 0 \quad \forall y \in \mathfrak{g}$ then $x = \mathbf{0}$,
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\alpha\langle x, y\rangle = \langle \alpha x, y\rangle = \langle x, \alpha y\rangle$,
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \implies x = \mathbf{0}$.

A Lie algebra with an inner product defined on it is called a *metric Lie algebra.*

Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

It's easy to verify that the dot product is an example of an inner product.

Definition 1.8 For any $x \in \mathfrak{g}$, the *adjoint representation* of $\mathfrak{g}$ on itself is the function

$$\mathrm{ad}_x(y) = [x, y]\ \forall y \in \mathfrak{g}$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. If $\mathfrak{g}$ is two-step nilpotent, then for all $x \in \mathfrak{g}$, ad_x is a linear function $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{z}$.

Definition 1.9 For any $z \in \mathfrak{z}$ of a two-step nilpotent Lie algebra, the *j-map* is the function $j_z : \mathfrak{v} \to \mathfrak{v}$ defined by the function

$$\langle [x, y], z\rangle = \langle y, j_z(x)\rangle$$

for all $x, y \in \mathfrak{g}$.

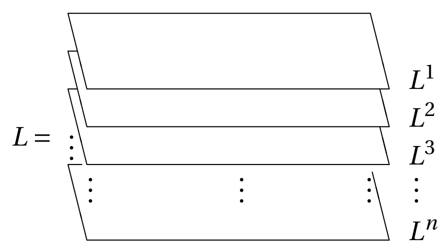
What this means is that for each vector in the center, there is a map from an element in the non-center to another element in the non-center. Once we explore how to represent these operators in matrix form, it will become clear how to untangle the j-map from the above equation.

2 Lie Operators in Matrix Form

In order to make concise, easy calculations with these algebraic operators, we will encode these operators as matrices. Consider the Lie Bracket, with bases $\{e_1, e_2, \ldots, e_n\}$. Formally, let $L_{ij} = [e_i, e_j]$. These L_{ij} can be arranged into a matrix, L.

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}.$$

So L_{ij} is the output of the Lie Bracket on the i^{th} basis vector with the j^{th} basis vector. Notice that L is a matrix with vector entries. Since L_{ij} is a vector, we need some way of indexing it. We will denote its k^{th} entry as L_{ij}^k. Now, since L is a matrix of vectors, it can actually be viewed differently (as a vector or "stack" of matrices indexed by k.)



This schematic picture represents L as a vector of matrices.

$$L = \begin{pmatrix} L^1 \\ L^2 \\ \vdots \\ L^n \end{pmatrix}.$$

By describing how the basis vectors interact under the Lie Bracket, L fully describes how *all* vectors interact under the Lie Bracket. To demonstrate this, let's examine the two-dimensional example.

Example 2.1 Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with $x, y \in \mathfrak{g}$, defined on the basis by
$x = w_1 e_1 + w_2 e_2$,
$y = v_1 e_1 + v_2 e_2$.

$$\begin{aligned}\text{Then } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\ &= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\ &= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}\end{aligned}$$

It turns out that $[x, y]$ can be written even more concisely using matrix multiplication.

Example 2.2 As before, let's examine the two dimensional example.
Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Recall from before that

$$\begin{aligned}[x, y] &= (w_1 v_1 L_{11} + w_2 v_1 L_{21} + w_1 v_2 L_{12} + w_2 v_2 L_{22}) \\ &= (w_1 L_{11} + w_2 L_{21}, w_1 L_{12} + w_2 L_{22}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (w_1\, w_2) \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = x^T L y.\end{aligned}$$

That is $[x, y] = x^T L y$.

This example extends into higher dimensions. Thus the Lie bracket of x, with y can be fully represented by a matrix multiplication.

Recall that the Lie bracket has some additional properties. Namely,
$[e_i, e_i] = \mathbf{0}$, and
$[e_i, e_j] = -[e_j, e_i]$.
So

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}.$$

Example 2.3 Let's look at the L that represents the cross product over $\mathbb{R}^3$. Recall that the cross product is defined on the basis by
$[e_1, e_2] = e_3$,
$[e_1, e_3] = -e_2$,
$[e_2, e_3] = e_1$.
We obtain,

$$L = \begin{pmatrix} \mathbf{0} & e_3 & -e_2 \\ -e_3 & \mathbf{0} & e_1 \\ e_2 & -e_1 & \mathbf{0} \end{pmatrix}.$$

The center of $\mathbb{R}^3$ with the cross product is only $\{\mathbf{0}\}$, since all rows of L have a non-zero element.

Example 2.4 We can represent the three dimensional Heisenberg algebra's Lie bracket as well. The bracket is given by $[e_1, e_2] = e_3$ for bases $e_1, e_2, e_3 \in \mathfrak{h}_3$, and all other brackets zero.

$$L = \begin{pmatrix} \mathbf{0} & e_3 & \mathbf{0} \\ -e_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The center is the span of $\{e_3\}$ because the third row is the only row with entries that are all $\mathbf{0}$.

We can write the inner product as a matrix as well.
Let $\langle e_i, e_j \rangle = E_{ij}$, and arrange these values into a matrix,

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}.$$

Recall that the Lie bracket returns a vector, while the inner product returns a scalar. This means that while L was a matrix of vectors, E is simply a standard matrix of numbers.
Now, the inner product has additional properties just as L did. In particular,
$\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$, $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0 \implies x = \mathbf{0}$.
Furthermore, $\mathbf{0}$ cannot be a basis vector because it is linearly dependent with all vectors.
So $E_{ii} > 0$ and therefore,

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{12} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1n} & E_{2n} & \cdots & E_{nn} \end{pmatrix}.$$

It turns out that $\langle x, y\rangle$ can be computed in the same way as $[x, y]$; that is $\langle x, y\rangle = x^T E y$. This multiplication returns a scalar.

Example 2.5 Let's examine the dot product on $\mathbb{R}^3$ as a matrix representation. It's easy to verify based on the definition that

$$\langle e_i, e_j\rangle = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore E is given by,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The dot product's E is the identity matrix, so $x \cdot y = x^T E y = x^T I y = x^T y$.
This is an unsurprising result since we defined dot product this way.
Remember that 2-step nilpotent metric Lie algebras have another operator defined on them, the j-maps:

$$\langle y, j_z(x)\rangle_{\mathfrak{v}} = \langle z, [x, y]\rangle_{\mathfrak{z}}.$$

Remember also that there is a specific j-map for each element of the center. We can now use E and L to construct J. Since we will only consider 2-step nilpotent Lie algebras, we can "shrink" E and L to only include interactions between elements of the non-center, $\mathfrak{v}$. This is because all interactions with the center are trivial. The length of $\mathfrak{v}$ is n, making L and E both $(n \times n)$ matrices.
Suppose $\mathfrak{z} = \text{span}\{z_1, z_2, \ldots, z_m\}$. Then for any z_k, the j-map $j_{z_k} : \mathfrak{v} \to \mathfrak{v}$ is given by

$$\begin{aligned} \langle y, j_{z_k}(x)\rangle_{\mathfrak{v}} &= \langle z_k, [x, y]\rangle_{\mathfrak{z}} \\ y^T E(J_{z_k} x) &= z_k^T (x^T L y) \\ y^T (E J_{z_k}) x &= y^T (L^k)^T x. \end{aligned}$$

Since the equation is equal for arbitrary x and y in $\mathfrak{v}$, the insides must be equal, i.e. $E J_{z_k} = (L^k)^T$.

Since $\det(E) \neq 0$ by the non-degeneracy of the inner product, we may solve for J_{z_k} to obtain

$$J_{z_k} = E^{-1} (L^k)^T \in \mathbb{R}^{n \times n}.$$

Now, if $z = \zeta_1 z_1 + \zeta_2 z_2 + \cdots + \zeta_m z_m$, where ζ_i are coefficients of the linear combination of z_i, then the map j_z is represented by the matrix

$$J_z = \zeta_1 J_{z_1} + \zeta_2 J_{z_2} + \cdots + \zeta_m J_{z_m}.$$

The j-maps of a 2-step nilpotent Lie algebra can be described by a vector of matrices of the same type as L. This is a consequence of the bilinearity of both E and L.

If we let $J^k = J_{z_k}$, then

$$J = \begin{pmatrix} J^1 \\ J^2 \\ \vdots \\ J^m \end{pmatrix}$$

is an m-dimensional vector of $(n \times n)$ matrices.
Then for $z = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T \in \mathfrak{z}$, the map j_z is represented by

$$J_z = z^T J.$$

Example 2.6 Consider the Heisenberg algebra $\mathfrak{h}_3$. Restricting L to $\mathfrak{v}$ and letting E be the dot product on $\mathfrak{v}$, we obtain

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The j-map $j_{e_3} : \mathfrak{v} \to \mathfrak{v}$ is then given by

$$J = E^{-1} L^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let's look at another example.

Example 2.7 Suppose we change the inner product to

$$E = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Then the j-map becomes

$$J = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} \\ \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

3 Conclusion

We can now represent an arbitrary Lie algebra's operators in matrix form and use simple matrix multiplication to apply these operators on arbitrary vectors. We can also easily compute the j-maps for a given L and E. Future work may include

1. Finding the structural properties of J,
2. Computing E given L and J,
3. Figuring out for which J's, E can be computed.

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