Encoding the Bracket - An Overview of Lie Algebras

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$$\begin{aligned} 0 &= [x+y, x+y] && (1)\\ &= [x+y, x] + [x+y, y] && (2)\\ &= [x, x] + [y, x] + [x+y, y] && (3)\\ &= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\ 0 &= [x, y] + [y, x] && (5)\\ \implies [x, y] &= -[y, x] && (6) \end{aligned}$$

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Conversely, if [,] is skew-symmetric, then $[x,x] + [x,x] = 0$ implies that 2[x,x] $=$ 0. Now we see that this implies [x,x] $=$ 0 so long as our field is not of characteristic 2, for in those spaces 2 $=$ 0 and we can deduce nothing about [x,x].

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Thus z, is in the center of $\mathfrak{g}$ if and only if

$$[z, x] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

The *non-center*, $\mathfrak{v}$, is given by $\mathfrak{v} = \mathfrak{g} - \mathfrak{z}$

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1. Applying the bracket twice to any arbitrary set of three vectors in the Algebra, always returns $\mathbf{0}$.
2. Applying the bracket once to any arbitrary set of two vectors does not always return $\mathbf{0}$.

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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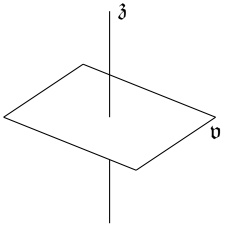

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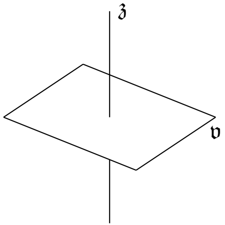

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Can anyone tell me if the Heisenberg Algebra is one-step nilpotent, two-step nilpotent, neither, or both?

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This makes the computation of many seemingly complicated objects "simple" calculations in linear algebra!

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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$$\begin{aligned} \text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \end{aligned}$$

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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2]
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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2]
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&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
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This means that the Lie Bracket is fully described by the matrix, L

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What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

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Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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Postive Definite: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \implies x = \mathbf{0}$
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$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

j-map as a matrix

Recall that for any $z \in \mathfrak{z}$, the map $j_z : \mathfrak{v} \to \mathfrak{v}$ is defined by the identity

$$\langle y, j_z(x) \rangle_{\mathfrak{v}} = \langle z, [x, y] \rangle_{\mathfrak{z}}. \tag{7}$$

We wish to find a matrix J_z that represents the map j_z. Since j_z maps $\mathfrak{v}$ to itself, J_z should be a matrix in $\mathbb{R}^{n \times n}$. Rewriting (7) in matrix form, we get

$$\begin{aligned}
\langle y, j_z(x) \rangle_{\mathfrak{v}} &= \langle z, [x, y] \rangle_{\mathfrak{z}} \\
y^T E(J_z x) &= z^{\mathtt{t}}(x^T L y) \\
y^T (E J_z) x &= z^{\mathtt{t}}(-y^T L x) \\
&= -z^{\mathtt{t}}(y^T L x) \\
&= -y^T (z^{\mathtt{t}} L) x \\
&= y^T (z^{\mathtt{t}}(-L)) x \\
&= y^T (z^{\mathtt{t}} L^{\mathrm{T}}) x.
\end{aligned}$$

The equation $y^T(EJ_z)x = y^T(z^{\mathtt{t}}L^{\mathrm{T}})x$ must hold for arbitrary vectors x and y in $\mathfrak{v}$, implying that

$$EJ_z = z^{\mathtt{t}}L^{\mathrm{T}}.$$

Since $\det(E) \neq 0$, it is invertible and

$$J_z = E^{-1}(z^{\mathtt{t}}L^{\mathrm{T}}). \tag{8}$$

This gives us a method to compute the j-map for any z using only matrix computations. Furthermore, if $z = z_k$ is a basis vector of $\mathfrak{z}$, then $z_k^{\mathtt{t}}L^{\mathrm{T}}$ is just $(L^k)^{\mathrm{T}}$. Thus, for basis vectors z_k, the map j_{z_k} is represented by the matrix

$$J_{z_k} = E^{-1}(L^k)^{\mathrm{T}}.$$

Since the j-maps are linear, knowing how they act on a basis of $\mathfrak{z}$ is good enough. Indeed, we should be able to define a stack of j-maps by

$$J = E^{-1}L^{\mathrm{T}}. \tag{9}$$

The j-map j_z *should* then be $J_z = z^{\mathrm{t}}J$.

What we did

Using the programming language, sage, we created a program that would compute the j-maps of an arbitrary Lie algebra.

TikZ

Acknowledgements

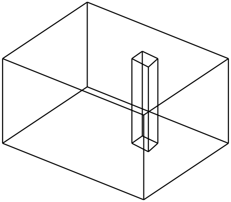

Figure: The L-Stack

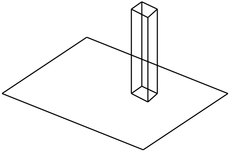

Figure: The L-Stack