

Recovering the Metric - An Overview of Lie Algebras

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Conversely, if $[,]$ is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that $2[x, x] = 0$. Now we see that this implies $[x, x] = 0$ so long as our field is not of characteristic 2, for in those spaces $2 = 0$ and we can deduce nothing about $[x, x]$.

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A vector, z , of \mathfrak{g} is said to be in the center of \mathfrak{g} if

$$[x, z] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

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\mathbb{R}^3 as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

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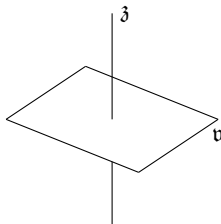


Figure : The Heisenberg Algebra, \mathfrak{h}

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product \langle , \rangle . (Confusing, I know.)

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This makes the computation of many complicated geometric objects—such as curvatures—into “simple” calculations in linear algebra!

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

$$[e_i, e_j] = -[e_j, e_i]$$

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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This means that the Lie Bracket is fully described by the matrix, L

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The center is the span of $\{e_3\}$.

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Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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Acknowledgements