Recovering the Metric - An Overview of Lie Algebras

Alexander Jansing, Chaskin Saroff

Oswego State University
Department of Mathematics

30 April 2015

OSWEGO
STATE UNIVERSITY OF NEW YORK

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad$ and $\quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in V$.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in V$.

The product $[\ ,\ ]$ is known as a *Lie bracket* on V.

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$\begin{aligned}
0 &= [x+y, x+y] && (1)\\
&= [x+y, x] + [x+y, y] && (2)\\
&= [x, x] + [y, x] + [x+y, y] && (3)\\
&= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\
0 &= [x, y] + [y, x] && (5)\\
\implies [x, y] &= -[y, x] && (6)
\end{aligned}$$

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$\begin{aligned} 0 &= [x+y, x+y] && (1)\\ &= [x+y, x] + [x+y, y] && (2)\\ &= [x,x] + [y,x] + [x+y, y] && (3)\\ &= [x,x] + [y,x] + [x,y] + [y,y] && (4)\\ 0 &= [x,y] + [y,x] && (5)\\ \implies [x,y] &= -[y,x] && (6) \end{aligned}$$

Conversely, if [,] is skew-symmetric, then $[x,x] + [x,x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces 2 = 0 and we can deduce nothing about [x,x].

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \ \forall x \in \mathfrak{g}\}.$$

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \; \forall x \in \mathfrak{g}\}.$$

A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \mathbf{0} \; \forall x \in \mathfrak{g}.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

This Algebra is called *Abelian* or *One-Step Nilpotent*

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

This Algebra is called *Abelian* or *One-Step Nilpotent*

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \text{ for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$(x \times y) \times z + (y \times z) \times x + (z \times x) \times y$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned}&(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ &\quad = \quad (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z\end{aligned}$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned}
&(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\
&\quad = \quad (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z \\
&\quad = \quad 0.
\end{aligned}$$

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

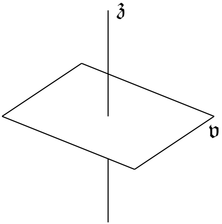

Figure : The Heisenberg Algebra, $\mathfrak{h}$

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \;] = 0.$$

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \; , \; \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$:

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y] , z \rangle = \langle y, j(x) z \rangle$$

for all $y, z \in \mathfrak{g}$.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y] , z \rangle = \langle y, j(x) z \rangle$$

for all $y, z \in \mathfrak{g}$.

The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\ ,\ \rangle$. (Confusing, I know.)

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x) z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z\rangle = \langle y, j(x)z\rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z\rangle = \langle y, \mathrm{ad}_x^\dagger z\rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z\rangle = \langle y, j(x)z\rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z\rangle = \langle y, \mathrm{ad}_x^\dagger z\rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form.

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

What is special about $[e_i, e_i]$?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

What is special about $[e_i, e_i]$?
$[e_i, e_i] = \mathbf{0}$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?
$[e_i, e_j] = -[e_j, e_i]$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

Recall from before the adjoint map, $\mathrm{ad}_{e_1} : \mathfrak{g} \to \mathfrak{g}$ to itself. Can anyone see a matrix representation of ad_{e1}?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \boxed{\mathbf{0} \quad L_{12} \quad \cdots \quad L_{1n}} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

Recall from before the adjoint map, $\mathrm{ad}_{e_1} : \mathfrak{g} \to \mathfrak{g}$ to itself. Can anyone see a matrix representation of ad_{e1}?

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ \boxed{}- L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets. Let's look at an example of a 2 dimensional space

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\text{So } [x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2]$$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned} \text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \end{aligned}$$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned} \text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \end{aligned}$$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So}\,[x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2]
\end{aligned}$$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So}\,[x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
\end{aligned}$$

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
\end{aligned}$$

This means that the Lie Bracket is fully described by the matrix, L

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example.

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

$$(x^T L)y = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

$$(x^T L)y = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 v_1 L_{11} + w_2 v_1 L_{12} + w_1 v_2 L_{12} + w_2 v_2 L_{22} \end{pmatrix}$$

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example. Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

$$(x^T L)y = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 v_1 L_{11} + w_2 v_1 L_{12} + w_1 v_2 L_{12} + w_2 v_2 L_{22} \end{pmatrix}$$

That is $[x, y] = x^T L y$

Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?

Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?
Recall that a Lie Algebra is Abelian if

Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?
Recall that a Lie Algebra is Abelian if
$[x, y] = \mathbf{0}\ \forall x, y \in \mathbb{R}^3$

Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?
Recall that a Lie Algebra is Abelian if
$[x, y] = \mathbf{0}\ \forall x, y \in \mathbb{R}^3$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?
Recall that a Lie Algebra is Abelian if
$[x, y] = \mathbf{0} \ \forall x, y \in \mathbb{R}^3$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

What is the center of Abelian $\mathbb{R}^3$

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?

$[e_1, e_2] = e_3$

$[e_1, e_3] = e_2$

$[e_2, e_3] = e_1$

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?

$[e_1, e_2] = e_3$
$[e_1, e_3] = e_2$
$[e_2, e_3] = e_1$

$$\begin{pmatrix} \mathbf{0} & e_3 & -e_2 \\ -e_3 & \mathbf{0} & e_1 \\ e_2 & -e_1 & \mathbf{0} \end{pmatrix}$$

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?

$[e_1, e_2] = e_3$
$[e_1, e_3] = e_2$
$[e_2, e_3] = e_1$

$$\begin{pmatrix} \mathbf{0} & e_3 & -e_2 \\ -e_3 & \mathbf{0} & e_1 \\ e_2 & -e_1 & \mathbf{0} \end{pmatrix}$$

What is the center of $\mathbb{R}^3$ with the cross product?

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$\mathfrak{h}_3$ is two-step nilpotent

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$\mathfrak{h}_3$ is two-step nilpotent
$[[x, y], z] \ \forall x, y, z \in \mathfrak{h}_3$

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?

$\mathfrak{h}_3$ is two-step nilpotent

$[[x, y], z] \;\; \forall x, y, z \in \mathfrak{h}_3$

$$\begin{pmatrix} \mathbf{0} & e_3 & \mathbf{0} \\ -e_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$\mathfrak{h}_3$ is two-step nilpotent
$[[x, y], z] \ \forall x, y, z \in \mathfrak{h}_3$

$$\begin{pmatrix} \mathbf{0} & e_3 & \mathbf{0} \\ -e_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

What is the center of $\mathfrak{h}_3$?

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$\mathfrak{h}_3$ is two-step nilpotent
$[[x, y], z] \;\; \forall x, y, z \in \mathfrak{h}_3$

$$\begin{pmatrix} \mathbf{0} & e_3 & \mathbf{0} \\ -e_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.

$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}$$

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}$$

Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}$$

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.

$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{12} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1n} & E_{1n-1} & \cdots & E_{nn} \end{pmatrix}$$

Dot Product on $\mathbb{R}^3$

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.

Dot Product on $\mathbb{R}^3$

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = 0$

Dot Product on $\mathbb{R}^3$

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = 0$
$\langle e_i, e_i \rangle = 1$

Dot Product on $\mathbb{R}^3$

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = 0$
$\langle e_i, e_i \rangle = 1$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TikZ

Acknowledgements

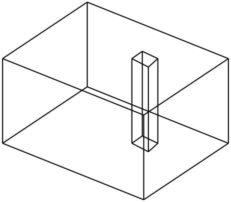

Figure : The L-Stack

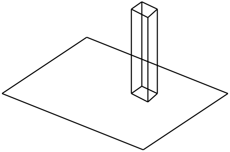

Figure : The L-Stack