

Recovering the Metric - An Overview of Lie Algebras

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Conversely, if $[,]$ is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that $2[x, x] = 0$. Now we see that this implies $[x, x] = 0$ so long as our field is not of characteristic 2, for in those spaces $2 = 0$ and we can deduce nothing about $[x, x]$.

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A vector, z , of \mathfrak{g} is said to be in the center of \mathfrak{g} if

$$[x, z] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

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\mathbb{R}^3 as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

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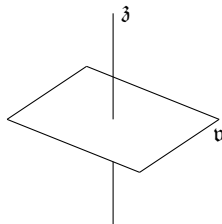


Figure : The Heisenberg Algebra, \mathfrak{h}

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product \langle , \rangle . (Confusing, I know.)

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This makes the computation of many complicated geometric objects—such as curvatures—into “simple” calculations in linear algebra!

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases e_1, e_2, \dots, e_n Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

$$[e_i, e_j] = -[e_j, e_i]$$

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Recall from before the adjoint map, $\text{ad}_{e_1} : \mathfrak{g} \rightarrow \mathfrak{g}$ to itself. Can anyone see a matrix representation of ad_{e_1} ?

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It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?

Notice that we can retrieve information about the bases.

$$[e_1, e_2] = L_{12}$$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

We can write $[x, y]$ as a linear combination their component vectors's brackets.

Let $x, y \in \mathfrak{g}$ with

$$x = w_1 e_1 + w_2 e_2$$

$$y = v_1 e_1 + v_2 e_2$$

$$\text{So } [x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \quad (7)$$

$$= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \quad (8)$$

$$= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \quad (9)$$

$$= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \quad (10)$$

$$= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22} \quad (11)$$

TikZ

Acknowledgements

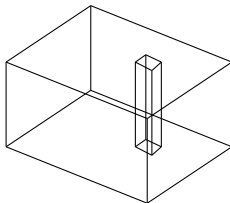


Figure : The L-Stack

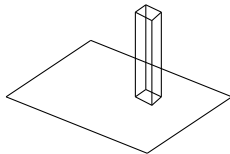


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