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So for every $z \in \mathfrak{z}$, one can define a linear transformation $j_z : \mathfrak{v} \to \mathfrak{v}$ by the identity

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We can then use the same methods as before to construct matrix representations of $\langle\ ,\ \rangle_{\mathfrak{v}}$ and $[\ ,\ ]$ to find a matrix representation for j_z.

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taking advantage of some clever stack-matrix manipulations.

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$$J_z = z^T J.$$

Some Examples

to be done...

We wrote some code in Sage to help us compute the j-maps for a couple of interesting Lie algebras.