

Recovering the Metric - An Overview of Lie Algebras

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Conversely, if $[,]$ is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that $2[x, x] = 0$. Now we see that this implies $[x, x] = 0$ so long as our field is not of characteristic 2, for in those spaces $2 = 0$ and we can deduce nothing about $[x, x]$.

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A vector, z , of \mathfrak{g} is said to be in the center of \mathfrak{g} if

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\mathbb{R}^3 as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

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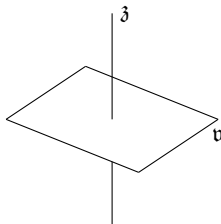


Figure : The Heisenberg Algebra, \mathfrak{h}

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This makes the computation of many complicated geometric objects—such as curvatures—into “simple” calculations in linear algebra!

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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This means that the Lie Bracket is fully described by the matrix, L

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j -map as a matrix

Recall that for any $z \in \mathfrak{z}$, the map $j_z : \mathfrak{v} \rightarrow \mathfrak{v}$ is defined by the identity

$$\langle y, j_z(x) \rangle_{\mathfrak{v}} = \langle z, [x, y] \rangle_{\mathfrak{z}}. \quad (7)$$

We wish to find a matrix J_z that represents the map j_z . Since j_z maps \mathfrak{v} to itself, J_z should be a matrix in $\mathbb{R}^{n \times n}$. Rewriting (7) in matrix form, we get

$$\begin{aligned} \langle y, j_z(x) \rangle_{\mathfrak{v}} &= \langle z, [x, y] \rangle_{\mathfrak{z}} \\ y^T E(J_z x) &= z^t (x^T L y) \\ y^T (E J_z) x &= z^t (-y^T L x) \\ &= -z^t (y^T L x) \\ &= -y^T (z^t L) x \\ &= y^T (z^t (-L)) x \\ &= y^T (z^t L^T) x. \end{aligned}$$

The equation $y^T(EJ_z)x = y^T(z^t L^T)x$ must hold for arbitrary vectors x and y in \mathfrak{v} , implying that

$$EJ_z = z^t L^T.$$

Since $\det(E) \neq 0$, it is invertible and

$$J_z = E^{-1}(z^t L^T). \quad (8)$$

This gives us a method to compute the j -map for any z using only matrix computations. Furthermore, if $z = z_k$ is a basis vector of \mathfrak{z} , then $z_k^t L^T$ is just $(L^k)^T$. Thus, for basis vectors z_k , the map j_{z_k} is represented by the matrix

$$J_{z_k} = E^{-1}(L^k)^T.$$

Since the j -maps are linear, knowing how they act on a basis of \mathfrak{z} is good enough. Indeed, we should be able to define a stack of j -maps by

$$J = E^{-1}L^T. \quad (9)$$

The j -map j_z *should* then be $J_z = z^\dagger J$.

What we did

Using the programming language, sage, we created a program that would compute the j -maps of an arbitrary Lie algebra.

TikZ

Acknowledgements

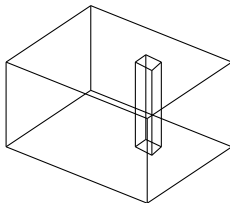


Figure : The L-Stack

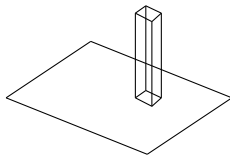


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