Matrix Computations in Lie Algebras

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$$\begin{aligned} 0 &= [x+y, x+y] && (1)\\ &= [x+y, x] + [x+y, y] && (2)\\ &= [x,x] + [y,x] + [x+y, y] && (3)\\ &= [x,x] + [y,x] + [x,y] + [y,y] && (4)\\ 0 &= [x,y] + [y,x] && (5)\\ \implies [x,y] &= -[y,x] && (6) \end{aligned}$$

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Conversely, if [,] is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that 2[x,x] $=$ 0. Now we see that this implies [x,x] $=$ 0 so long as our field is not of characteristic 2, for in those spaces 2 $=$ 0 and we can deduce nothing about [x,x].

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Thus z, is in the center of $\mathfrak{g}$ if and only if

$$[z, x] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

The *non-center*, $\mathfrak{v}$, is given by $\mathfrak{v} = \mathfrak{g} - \mathfrak{z}$

A First Example: One-Step Nilpotent Lie Algebras

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2. Applying the bracket once to any arbitrary set of two vectors does not always return $\mathbf{0}$.

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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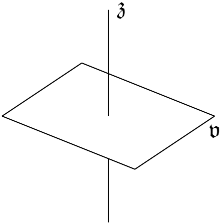

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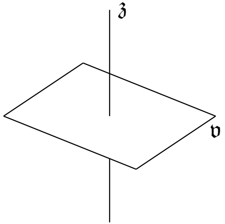

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Can anyone tell me if the Heisenberg Algebra is one-step nilpotent, two-step nilpotent, neither, or both?

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This makes the computation of many seemingly complicated objects "simple" calculations in linear algebra!

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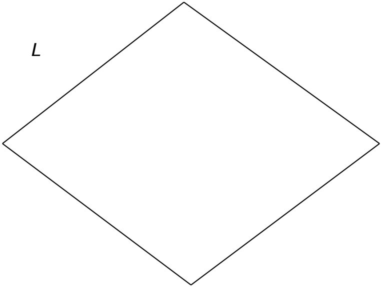

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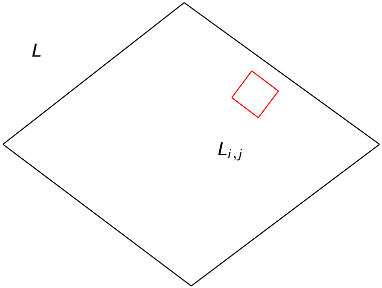

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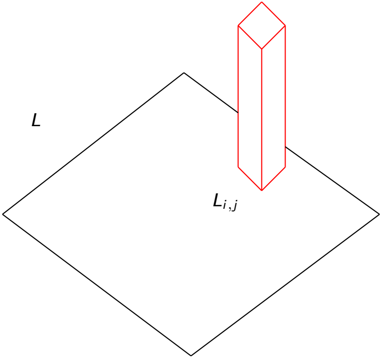
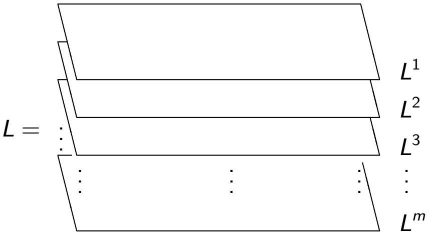


Figure : The Lie Bracket as a Matrix, L

The Lie Bracket as a vector of matrices

It can also be represented by a “stack” of matrices



A vector of matrices.

$$L_{ij} = \begin{pmatrix} L_{ij}^1 \\ L_{ij}^2 \\ \vdots \\ L_{ij}^m \end{pmatrix}.$$

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These L_{ij} can then be arranged in an $n \times n$ matrix

$$L = (L_{ij}) = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}.$$

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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$$\text{So}\,[x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2]$$

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$$\begin{aligned} \text{So}\,[x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\ &= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \end{aligned}$$

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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2]
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&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
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&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
\end{aligned}$$

This means that the Lie Bracket is fully described by the matrix, L

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That is $[x, y] = x^T L y$

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What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

Inner Product as a Matrix

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Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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Recall also that the inner product has additional properties.

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$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{12} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1n} & E_{2n} & \cdots & E_{nn} \end{pmatrix}$$

Recall also that the inner product has additional properties.
Postive Definite: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \implies x = \mathbf{0}$
And $\mathbf{0}$ canot be a basis vector because it is linearly dependent with all vectors.
So $e_{ii} > 0$

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{12} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1n} & E_{2n} & \cdots & E_{nn} \end{pmatrix}$$

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Dot Product on $\mathbb{R}^3$

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$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix representation of the j-maps

Let $\mathfrak{n}$ be a 2-step nilpotent Lie Algebra with $\mathfrak{v}$ as the non-center and $\mathfrak{z}$ as the center.
Recall that the Lie bracket on a 2-step nilpotent Lie algebra is a bilinear map $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$.
So for every $z \in \mathfrak{z}$, one can define a linear transformation $j_z : \mathfrak{v} \to \mathfrak{v}$ by the identity

$$\langle y, j_z(x) \rangle_{\mathfrak{v}} = \langle z, [x, y] \rangle_{\mathfrak{z}}.$$

We can then use the same methods as before to construct matrix representations of $\langle\ ,\ \rangle_{\mathfrak{v}}$ and $[\ ,\ ]$ to find a matrix representation for j_z.

Matrix representation of the j-maps

By linearity, we will know how to construct any j-map, if we know the j-maps corresponding to basis vectors of $\mathfrak{z}$.
Suppose $\mathfrak{z} = \text{span}\{z_1, z_2, \ldots, z_m\}$.
Then for any z_k, the j-map $j_{z_k} : \mathfrak{v} \to \mathfrak{v}$ is given by

$$\begin{aligned}
\langle y, j_{z_k}(x) \rangle_{\mathfrak{v}} &= \langle z_k, [x, y] \rangle_{\mathfrak{z}} \\
y^T E(J_{z_k} x) &= z_k^T (x^T L y) \\
y^T (E J_{z_k}) x &= y^T (L^k)^T x
\end{aligned}$$

taking advantage of some clever stack-matrix manipulations.

Matrix representation of the j-maps

Since $y^T(EJ_{z_k})x = y^T(L^k)^T x$ for arbitrary x and y in $\mathfrak{v}$, we deduce that $EJ_{z_k} = (L^k)^T$.
Since $\det(E) \neq 0$, we may solve for J_{z_k} to obtain

$$J_{z_k} = E^{-1}(L^k)^T \in \mathbb{R}^{n\times n}.$$

If $z = \zeta_1 z_1 + \zeta_2 z_2 + \cdots + \zeta_m z_m$, where ζ_i are coefficients of the linear combination of z_i, then the map j_z is represented by the matrix

$$J_z = \zeta_1 J_{z_1} + \zeta_2 J_{z_2} + \cdots + \zeta_m J_{z_m}.$$

The matrix J as a stack

The j-maps of a 2-step nilpotent Lie algebra can be described by a stack of matrices of the same type as L.
Indeed, if we let $J^k = J_{z_k}$,
then

$$J = \begin{pmatrix} J^1 \\ J^2 \\ \vdots \\ J^m \end{pmatrix}$$

is an $(m \times 1)$ stack of $(n \times n)$ matrices.
Then for $z = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T \in \mathfrak{z}$, the map j_z is represented by

$$J_z = z^T J.$$

The equation $y^T(EJ_z)x = y^T(z^{\mathtt{t}}L^{\mathrm{T}})x$ must hold for arbitrary vectors x and y in $\mathfrak{v}$, implying that

$$EJ_z = z^{\mathtt{t}}L^{\mathrm{T}}.$$

Since $\det(E) \neq 0$, it is invertible and

$$J_z = E^{-1}(z^{\mathtt{t}}L^{\mathrm{T}}). \tag{7}$$

This gives us a method to compute the j-map for any z using only matrix computations. Furthermore, if $z = z_k$ is a basis vector of $\mathfrak{z}$, then $z_k^{\mathtt{t}}L^{\mathrm{T}}$ is just $(L^k)^{\mathrm{T}}$. Thus, for basis vectors z_k, the map j_{z_k} is represented by the matrix

$$J_{z_k} = E^{-1}(L^k)^{\mathrm{T}}.$$

Since the j-maps are linear, knowing how they act on a basis of $\mathfrak{z}$ is good enough. Indeed, we should be able to define a stack of j-maps by

$$J = E^{-1} L^{\mathrm{T}}. \tag{8}$$

The j-map j_z *should* then be $J_z = z^{\mathrm{t}} J$.

What we did

Using the programming language, sage, we created a program that would compute the *j*-maps of an arbitrary Lie algebra.

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