Recovering the Metric - An Overview of Lie Agebra's

Chaskin Saroff

Oswego State University
Department of Mathematics

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Conversely, if [,] is skew-symmetric, then $[x,x] + [x,x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces 2 = 0 and we can deduce nothing about [x,x].

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A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \vec{0} \, \forall x \in \mathfrak{g}.$$

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$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

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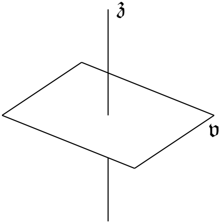

Figure : The Heisenberg Algebra, $\mathfrak{h}$

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\ ,\ \rangle$. (Confusing, I know.)

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This makes the computation of many complicated geometric objects–such as curvatures–into “simple” calculations in linear algebra!

Acknowledgements