Recovering the Metric - An Overview of Lie Algebras

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Conversely, if [,] is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces 2 = 0 and we can deduce nothing about [x,x].

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A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \mathbf{0} \; \forall x \in \mathfrak{g}.$$

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$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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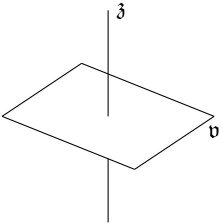

Figure : The Heisenberg Algebra, $\mathfrak{h}$

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\ ,\ \rangle$. (Confusing, I know.)

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This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?
$[e_i, e_j] = -[e_j, e_i]$

The Lie Bracket as a Matrix with Vector entries

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It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} \mathbf{0} & \boxed{L_{12}} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] && (7)\\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] && (8)\\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] && (9)\\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] && (10)\\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22} && (11)
\end{aligned}$$

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