Matrix Representations in Lie Algebras

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Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = \mathbf{0}$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = \mathbf{0}$ for all $x, y, z \in \mathfrak{g}$. and
3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$ $\forall x, y, z \in \mathfrak{g}$ and $\alpha \in \mathbb{R}$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 describes *bilinearity*.

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Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 describes *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

The product $[\ ,\ ]$ is known as a *Lie bracket* on $\mathfrak{g}$.

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \ \forall x \in \mathfrak{g}\}.$$

Thus z, is in the center of $\mathfrak{g}$ if and only if

$$[z, x] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

The *non-center*, $\mathfrak{v}$, is given by $\mathfrak{v} = \mathfrak{g} - \mathfrak{z}$

A First Example: One-Step Nilpotent Lie Algebras

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = \mathbf{0} \text{ for all } x, y \in \mathfrak{g}.$$

Clearly [,] is alternating:

$$[x, x] = \mathbf{0} \text{ for all } x \in \mathfrak{g}.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [\mathbf{0}, z] + [\mathbf{0}, x] + [\mathbf{0}, y] \\ &= \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}. \end{aligned}$$

This Algebra is called *Abelian* or *One-Step Nilpotent*.
Everything in the Abelian case is is the center!

Two-Step Nilpotent

A Lie Algebra, $\mathfrak{g}$ is called *two-step nilpotent* if and only if

$$[[x, y], z] = \mathbf{0} \; \forall x, y, z \in \mathfrak{g} \text{ and } \mathfrak{z} \neq \mathfrak{g}$$

1. Applying the bracket twice to any arbitrary set of three vectors in the Algebra, always returns $\mathbf{0}$.
2. Applying the bracket once to any arbitrary set of two vectors does not always return $\mathbf{0}$.

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned} &(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ &\quad = \quad (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z \\ &\quad = \quad 0. \end{aligned}$$

Is the $\mathbb{R}^3$ non-Abelian case one-step nilpotent, two-step nilpotent, or neither?

$\mathfrak{h}_3$: The Heisenberg Algebra

We can construct another example from $\mathbb{R}^3$ by defining a different Lie Bracket. Consider a 3-dimensional vector space with basis vectors e_1, e_2, e_3. Define a Lie bracket on this space by

$$[e_1, e_2] = e_3,$$

with all other brackets equal to 0.
This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

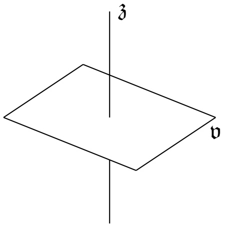

Figure : The Heisenberg Algebra, $\mathfrak{h}_3$

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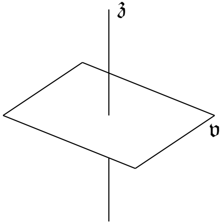

Figure : The Heisenberg Algebra, $\mathfrak{h}_3$

Can anyone tell me if the Heisenberg Algebra is one-step nilpotent, two-step nilpotent, neither, or both?

Inner Products

An *inner product* on a vector space $\mathfrak{g}$ is a non-degenerate, symmetric, bilinear, positive-definite function $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in \mathfrak{g}$, then x must be $\mathbf{0}$;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in \mathfrak{g}$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in \mathfrak{g}$, $a, b, \in \mathbb{R}$.
4. $\langle x, x\rangle \geq 0\ \forall x \in \mathfrak{g}$ and $\langle x, x\rangle = 0 \implies x = \mathbf{0}$

A Lie Algebra with an inner product is called a *Metric Lie Algebra*.

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Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

j-maps

For two-step nilpotent algebras, we can define a single map that encode the data of both $\left[,\right]$ and $\langle,\rangle$.
In order to combine the interaction between the center and the non-center of a two-step nilpotent lie algebra, we can define a map, $j_z : \mathfrak{v} \to \mathfrak{v}$.
j_z is defined by the equation

$$\langle [x, y] \, , z\rangle = \langle y, j_z x\rangle$$

The *j*-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

This makes the computation of many seemingly complicated objects “simple” calculations in linear algebra!

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$
Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

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We will denote the k^{th} entry of L_{ij} as L_{ij}^k.

The Lie Bracket as a Matrix with Vector entries

Recall that the Lie Bracket is a matrix of vectors.

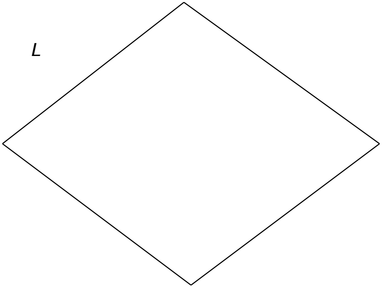

Figure : The Lie Bracket as a Matrix, L

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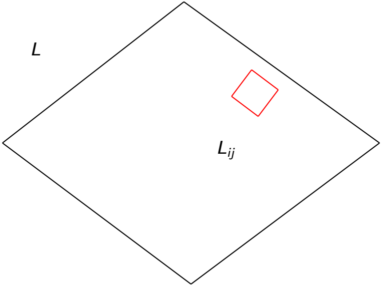

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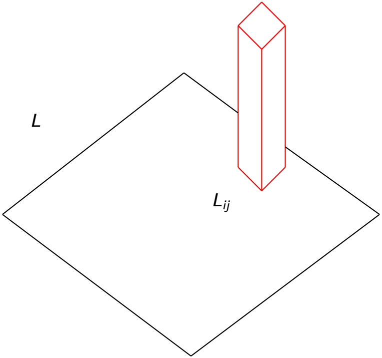

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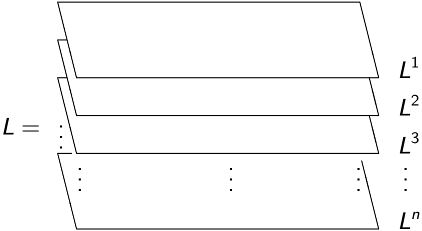
The Lie Bracket as a vector of matrices

It can also be represented by a “stack” of matrices



A vector of matrices.

$$L = \begin{pmatrix} L^1 \\ L^2 \\ \vdots \\ L^n \end{pmatrix}.$$

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L can then be arranged in an $n \times n$ matrix

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}.$$

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} L_{11} & \boxed{L_{12}} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination of their component vectors' brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So}\,[x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
\end{aligned}$$

This means that the Lie Bracket is fully described by the matrix, L

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example.
Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

$$(x^T L)y = \begin{pmatrix} w_1 L_{11} + w_2 L_{12}, w_1 L_{12} + w_2 L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 v_1 L_{11} + w_2 v_1 L_{12} + w_1 v_2 L_{12} + w_2 v_2 L_{22} \end{pmatrix}$$

That is $[x, y] = x^T L y$

The Lie Bracket as a Matrix with Vector entries

Let's take another look at L
Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$
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What is special about $[e_i, e_i]$?

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What is special about $[e_i, e_i]$?
$[e_i, e_i] = \mathbf{0}$

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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$[e_i, e_j] = -[e_j, e_i]$

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Consider the map $[e_1, \,] : \mathfrak{g} \to \mathfrak{g}$.
Can anyone see the matrix representation of $[e_1, \,]$?

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Example Abelian $\mathbb{R}^3$

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Recall that a Lie Algebra is Abelian if
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What is the center of Abelian $\mathbb{R}^3$?

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?

$[e_1, e_2] = e_3$
$[e_1, e_3] = -e_2$
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$$\begin{pmatrix} \mathbf{0} & e_3 & -e_2 \\ -e_3 & \mathbf{0} & e_1 \\ e_2 & -e_1 & \mathbf{0} \end{pmatrix}$$

What is the center of $\mathbb{R}^3$ with the cross product?

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$[e_1, e_2] = e_3$ for bases $e_1, e_2, e_3 \in \mathfrak{h}_3$
And all other brackets zero

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What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}$$

Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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And $\mathbf{0}$ cannot be a basis vector because it is linearly dependent with all vectors.
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Dot Product on $\mathbb{R}^3$

The Dot Product can be represented in this way.
$\langle e_i, e_j \rangle = 0$
$\langle e_i, e_i \rangle = 1$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix representation of the j-maps

Let $\mathfrak{n}$ be a 2-step nilpotent Lie Algebra with $\mathfrak{v}$ as the non-center and $\mathfrak{z}$ as the center.
Recall that the Lie bracket on a 2-step nilpotent Lie algebra is a bilinear map $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$.
So for every $z \in \mathfrak{z}$, one can define a linear transformation $j_z : \mathfrak{v} \to \mathfrak{v}$ by the identity

$$\langle y, j_z(x)\rangle_{\mathfrak{v}} = \langle z, [x, y]\rangle_{\mathfrak{z}}.$$

We now use the E and L that we just constructed to complete matrices for J.

Matrix representation of the j-maps

By linearity, we will know how to construct any j-map, if we know the j-maps corresponding to basis vectors of $\mathfrak{z}$.
Suppose $\mathfrak{z} = \operatorname{span}\{z_1, z_2, \ldots, z_m\}$.
Then for any z_k, the j-map $j_{z_k} : \mathfrak{v} \to \mathfrak{v}$ is given by

$$\begin{aligned} \langle y, j_{z_k}(x) \rangle_{\mathfrak{v}} &= \langle z_k, [x, y] \rangle_{\mathfrak{z}} \\ y^T E(J_{z_k} x) &= z_k^T (x^T L y) \\ y^T (E J_{z_k}) x &= y^T (L^k)^T x \end{aligned}$$

taking advantage of some clever stack-matrix manipulations.

Matrix representation of the j-maps

Since $y^T(EJ_{z_k})x = y^T(L^k)^T x$ for arbitrary x and y in $\mathfrak{v}$, we deduce that $EJ_{z_k} = (L^k)^T$.
Since $\det(E) \neq 0$, we may solve for J_{z_k} to obtain

$$J_{z_k} = E^{-1}(L^k)^T \in \mathbb{R}^{n \times n}.$$

If $z = \zeta_1 z_1 + \zeta_2 z_2 + \cdots + \zeta_m z_m$, where ζ_i are coefficients of the linear combination of z_i, then the map j_z is represented by the matrix

$$J_z = \zeta_1 J_{z_1} + \zeta_2 J_{z_2} + \cdots + \zeta_m J_{z_m}.$$

The matrix J as a stack

The j-maps of a 2-step nilpotent Lie algebra can be described by a stack of matrices of the same type as L.
Indeed, if we let $J^k = J_{z_k}$,
then

$$J = \begin{pmatrix} J^1 \\ J^2 \\ \vdots \\ J^m \end{pmatrix}$$

is an $(m \times 1)$ stack of $(n \times n)$ matrices.
Then for $z = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T \in \mathfrak{z}$, the map j_z is represented by

$$J_z = z^T J.$$

Examples of j-maps

We used Sage to help us compute the j-maps for some examples of 2-step nilpotent Lie algebras.

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First, consider the Heisenberg algebra $\mathfrak{h}_3$. Restricting L to $\mathfrak{v}$ and letting E be the dot product on $\mathfrak{v}$, we obtain

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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The j-map $j_{e_3} : \mathfrak{v} \to \mathfrak{v}$ is then given by

$$J = E^{-1} L^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This is easy to verify by hand.

Examples of j-maps

Suppose we change the inner product to

$$E = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Then the j-map becomes

$$J = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} \\ \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

This is also not too hard to verify by hand.

Examples of j-maps

As a second example, consider the 6-dimensional algebra spanned by the vectors $\{e_1, e_2, e_3, e_4, z_1, z_2\}$, with non-trivial brackets

$$\begin{aligned} [e_1, e_3] &= z_1 \\ [e_2, e_4] &= z_1 \\ [e_1, e_4] &= z_2 \\ [e_2, e_3] &= z_2 \end{aligned}$$

The Lie bracket is represented by the matrices

$$L^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad L^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Examples of j-maps

If E is the dot product, then the j-maps for this algebra are

$$J^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Examples of j-maps

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Again, easy to verify by hand. They're just $(L^1)^T$ and $(L^2)^T$.

Examples of j-maps

However, if we let E be something a little more exotic, like

$$E = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{3} & 0 \\ -\frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{6} & 0 & 1 \end{pmatrix},$$

which has $\det(E) = \frac{607}{1296} > 0$,

Examples of j-maps

then the j-maps become

$$J^1 = \frac{1}{607}\begin{pmatrix} -582 & -90 & -936 & -540 \\ -756 & -192 & -540 & -1152 \\ 1179 & 126 & 582 & 756 \\ 126 & 639 & 90 & 192 \end{pmatrix}, \quad \text{and}$$

$$J^2 = \frac{1}{607}\begin{pmatrix} -90 & -582 & -540 & -936 \\ -192 & -756 & -1152 & -540 \\ 126 & 1179 & 756 & 582 \\ 639 & 126 & 192 & 90 \end{pmatrix}.$$

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You wouldn't want to calculate these by hand!

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