Encoding the Bracket - An Overview of Lie Algebras

Alexander Jansing, Chaskin Saroff

Oswego State University
Department of Mathematics

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Conversely, if [,] is skew-symmetric, then $[x,x] + [x,x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces 2 = 0 and we can deduce nothing about [x,x].

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A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

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$\mathbb{R}^3$ as a non-Abelian Lie Algebra

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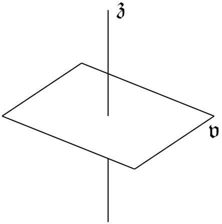

Figure: The Heisenberg Algebra, $\mathfrak{h}_3$

Inner Products

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This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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Examples of Cross Product over $\mathbb{R}^3$

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$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall that for any $z \in \mathfrak{z}$, the map $j_z : \mathfrak{v} \to \mathfrak{v}$ is defined by the identity

$$\langle y, j_z(x) \rangle_{\mathfrak{v}} = \langle z, [x, y] \rangle_{\mathfrak{z}}. \tag{7}$$

We wish to find a matrix J_z that represents the map j_z. Since j_z maps $\mathfrak{v}$ to itself, J_z should be a matrix in $\mathbb{R}^{n \times n}$. Rewriting (**??**) in matrix form, we get

$$\begin{aligned}
\langle y, j_z(x) \rangle_{\mathfrak{v}} &= \langle z, [x, y] \rangle_{\mathfrak{z}} \\
y^T E(J_z x) &= z^{\mathtt{t}}(x^T L y) \\
y^T (E J_z) x &= z^{\mathtt{t}}(-y^T L x) \\
&= -z^{\mathtt{t}}(y^T L x) \\
&= -y^T (z^{\mathtt{t}} L) x \\
&= y^T (z^{\mathtt{t}}(-L)) x \\
&= y^T (z^{\mathtt{t}} L^{\mathrm{T}}) x.
\end{aligned}$$

The equation $y^T(EJ_z)x = y^T(z^{\mathrm{t}} L^{\mathrm{T}})x$ must hold for arbitrary vectors x and y in $\mathfrak{v}$, implying that

$$EJ_z = z^{\mathrm{t}} L^{\mathrm{T}}.$$

Since $\det(E) \neq 0$, it is invertible and

$$J_z = E^{-1}(z^{\mathrm{t}} L^{\mathrm{T}}). \tag{8}$$

This gives us a method to compute the j-map for any z using only matrix computations. Furthermore, if $z = z_k$ is a basis vector of $\mathfrak{z}$, then $z_k^{\mathrm{t}} L^{\mathrm{T}}$ is just $(L^k)^{\mathrm{T}}$. Thus, for basis vectors z_k, the map j_{z_k} is represented by the matrix

$$J_{z_k} = E^{-1}(L^k)^{\mathrm{T}}.$$

Since the j-maps are linear, knowing how they act on a basis of $\mathfrak{z}$ is good enough. Indeed, we should be able to define a stack of j-maps by

$$J = E^{-1} L^{\mathrm{T}}. \tag{9}$$

The j-map j_z *should* then be $J_z = z^{\mathrm{t}} J$.

Using the programming language, sage, we created a program that would compute the j-maps of an arbitrary Lie algebra.

TikZ

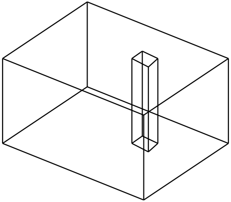

Figure: The L-Stack

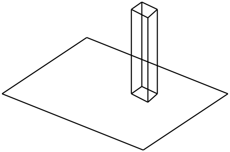

Figure: The L-Stack