Recovering the Metric - An Overview of Lie Agebra's

Chaskin Saroff

Oswego State University
Department of Mathematics

16 September 2014

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\, , \,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\, , \,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry* :

1. $[x, y] = - [y, x]$ for all $x, y \in V$.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

3. $[x + y, z] = [x, z] + [y, z]$ and $[\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in V$.

The product $[\ ,\ ]$ is known as a *Lie bracket* on V.

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$0 = [x + y, x + y] \tag{1}$$
$$= [x + y, x] + [x + y, y] \tag{2}$$
$$= [x, x] + [y, x] + [x + y, y] \tag{3}$$
$$= [x, x] + [y, x] + [x, y] + [y, y] \tag{4}$$
$$0 = [x, y] + [y, x] \tag{5}$$
$$\implies [x, y] = -[y, x] \tag{6}$$

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$0 = [x + y, x + y] \tag{1}$$
$$= [x + y, x] + [x + y, y] \tag{2}$$
$$= [x, x] + [y, x] + [x + y, y] \tag{3}$$
$$= [x, x] + [y, x] + [x, y] + [y, y] \tag{4}$$
$$0 = [x, y] + [y, x] \tag{5}$$
$$\implies [x, y] = -[y, x] \tag{6}$$

Conversely, if [,] is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces $2 = 0$ and we can deduce nothing about [x,x].

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \; \forall x \in \mathfrak{g}\}.$$

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \; \forall x \in \mathfrak{g}\}.$$

A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \mathbf{0} \; \forall x \in \mathfrak{g}.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \ \text{ for all } x, y \in V.$$

Clearly $[\ , \ ]$ is alternating:

$$[x, x] = 0 \ \text{ for all } x \in V.$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

Clearly $[\ ,\ ]$ is alternating:

$$[x, x] = 0 \quad \text{for all } x \in V.$$

And the Jacobi Identity is trivial:

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \quad \text{for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned}
[[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

Clearly $[\ ,\ ]$ is alternating:

$$[x, x] = 0 \quad \text{for all } x \in V.$$

And the Jacobi Identity is trivial:

$$
\begin{aligned}
[[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\
&= 0 + 0 + 0 = 0.
\end{aligned}
$$

This Algebra is called *Abelian* or *One-Step Nilpotent*

One-Step Nilpotence

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = 0 \quad \text{for all } x, y \in V.$$

Clearly $[\ ,\ ]$ is alternating:

$$[x, x] = 0 \quad \text{for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

This Algebra is called *Abelian* or *One-Step Nilpotent*

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$(x \times y) \times z + (y \times z) \times x + (z \times x) \times y$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$(x \times y) \times z + (y \times z) \times x + (z \times x) \times y$$
$$= (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$(x \times y) \times z + (y \times z) \times x + (z \times x) \times y$$
$$= (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z$$
$$= 0.$$

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

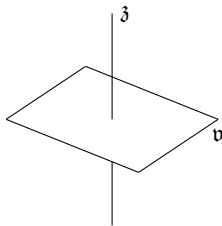with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

Figure : The Heisenberg Algebra, $\mathfrak{h}$

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \ , \ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$:

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\,,\,])$ on itself is the function defined by

$$\text{ad}_x y = [x, y].$$

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y].$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y].$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

for all $y, z \in \mathfrak{g}$.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\, , \,])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y].$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x, \,]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

for all $y, z \in \mathfrak{g}$.

The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle \, , \, \rangle$. (Confusing, I know.)

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathsf{ad}_x y, z \rangle = \langle y, \mathsf{ad}_x^\dagger z \rangle.$$

Now we could define $j(x) = \mathsf{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form.

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

What is special about $[e_i, e_i]$?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

What is special about $[e_i, e_i]$?
$[e_i, e_i] = \mathbf{0}$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?
$[e_i, e_j] = -[e_j, e_i]$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

Recall from before the adjoint map, $\mathrm{ad}_{e_1} : \mathfrak{g} \to \mathfrak{g}$ to itself. Can anyone see a matrix representation of ad_{e1}?

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$ Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

Recall from before the adjoint map, $\mathrm{ad}_{e_1} : \mathfrak{g} \to \mathfrak{g}$ to itself. Can anyone see a matrix representation of ad_{e1}?

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding to as the Lie Bracket "map"?

Notice that we can retrieve information about the bases.

$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} \mathbf{0} & \boxed{L_{12}} & \cdots & L_{1n} \\ -L_{12} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ -L_{1n} & -L_{2n} & \cdots & \mathbf{0} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

We can write $[x, y]$ as a linear combination their component vectors's brackets.
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\text{So } [x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \tag{7}$$
$$= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \tag{8}$$
$$= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \tag{9}$$
$$= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \tag{10}$$
$$= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22} \tag{11}$$

Acknowledgements