Matrix Representations in Lie Algebras

Alexander Jansing, Chaskin Saroff

Oswego State University
Department of Mathematics

5 May 2015

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = \mathbf{0}$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = \mathbf{0}$ for all $x, y, z \in \mathfrak{g}$. and
3. $[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y] \ \forall x, y, z \in \mathfrak{g}$ and $\alpha \in \mathbb{R}$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 describes *bilinearity*.

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Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 describes *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.

The product $[\ ,\ ]$ is known as a *Lie bracket* on $\mathfrak{g}$.

The Center

The *center* of a Lie Algebra, $\mathfrak{g}$ is

$$\mathfrak{z} = \{z \in \mathfrak{g} \mid [z, x] = \mathbf{0} \; \forall x \in \mathfrak{g}\}.$$

Thus z, is in the center of $\mathfrak{g}$ if and only if

$$[z, x] = \mathbf{0} \; \forall x \in \mathfrak{g}.$$

The *non-center*, $\mathfrak{v}$, is given by $\mathfrak{v} = \mathfrak{g} - \mathfrak{z}$

A First Example: One-Step Nilpotent Lie Algebras

Let $\mathfrak{g}$ be any vector space, and define

$$[x, y] = \mathbf{0} \text{ for all } x, y \in \mathfrak{g}.$$

Clearly [,] is alternating:

$$[x, x] = \mathbf{0} \text{ for all } x \in \mathfrak{g}.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [\mathbf{0}, z] + [\mathbf{0}, x] + [\mathbf{0}, y] \\ &= \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}. \end{aligned}$$

This Algebra is called *Abelian* or *One-Step Nilpotent*.
Everything in the Abelian case is is the center!

Two-Step Nilpotent

A Lie Algebra, $\mathfrak{g}$ is called *two-step nilpotent* if and only if

$$[[x, y], z] = \mathbf{0} \ \forall x, y, z \in \mathfrak{g} \text{ and } \mathfrak{z} \neq \mathfrak{g}$$

1. Applying the bracket twice to any arbitrary set of three vectors in the Algebra, always returns $\mathbf{0}$.
2. Applying the bracket once to any arbitrary set of two vectors does not always return $\mathbf{0}$.

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned} &(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ &\quad = (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z \\ &\quad = 0. \end{aligned}$$

Is the $\mathbb{R}^3$ non-Abelian case one-step nilpotent, two-step nilpotent, or neither?

$\mathfrak{h}_3$: The Heisenberg Algebra

We can construct another example from $\mathbb{R}^3$ by defining a different Lie Bracket. Consider a 3-dimensional vector space with basis vectors e_1, e_2, e_3. Define a Lie bracket on this space by

$$[e_1, e_2] = e_3,$$

with all other brackets equal to 0.
This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

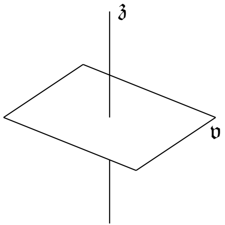

Figure : The Heisenberg Algebra, $\mathfrak{h}_3$

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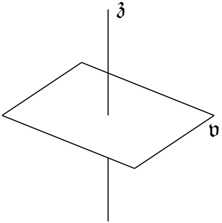

Figure : The Heisenberg Algebra, $\mathfrak{h}_3$

Can anyone tell me if the Heisenberg Algebra is one-step nilpotent, two-step nilpotent, neither, or both?

Inner Products

An *inner product* on a vector space $\mathfrak{g}$ is a non-degenerate, symmetric, bilinear, positive-definite function $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in \mathfrak{g}$, then x must be $\mathbf{0}$;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in \mathfrak{g}$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in \mathfrak{g}$, $a, b, \in \mathbb{R}$.
4. $\langle x, x \rangle \geq 0\ \forall x \in \mathfrak{g}$ and $\langle x, x \rangle = 0 \implies x = \mathbf{0}$

A Lie Algebra with an inner product is called a *Metric Lie Algebra*.

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Example The *dot product* on $\mathbb{R}^3$: Let $x, y \in \mathbb{R}^3$; then $x \cdot y = x^T y$. The dot product is given by

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

j-maps

For two-step nilpotent algebras, we can define a single map that encode the data of both $\left[\ ,\ \right]$ and $\langle\ ,\ \rangle$.
In order to combine the interaction between the center and the non-center of a two-step nilpotent lie algebra, we can define a map, $j_z : \mathfrak{v} \to \mathfrak{v}$.
j_z is defined by the equation

$$\langle [x, y]\, , z\rangle = \langle y, j_z x\rangle$$

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

This makes the computation of many seemingly complicated objects "simple" calculations in linear algebra!

The Lie Bracket as a Matrix with Vector entries

In order to do math with Lie Algebras on a computer, we would like to represent the Lie Bracket in a matrix form. Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$
Formally, $L_{ij} = [e_i, e_j]$

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

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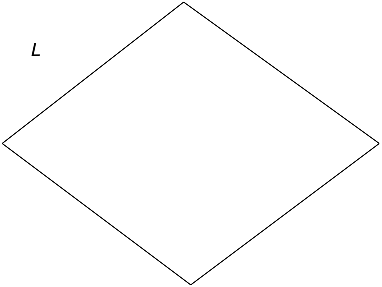
We will denote the k^{th} entry of L_{ij} as L_{ij}^k.

The Lie Bracket as a Matrix with Vector entries

Recall that the Lie Bracket is a matrix of vectors.



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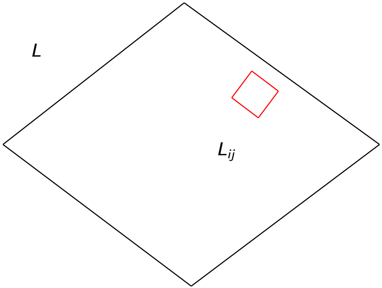

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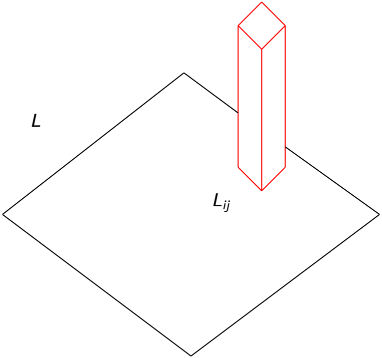

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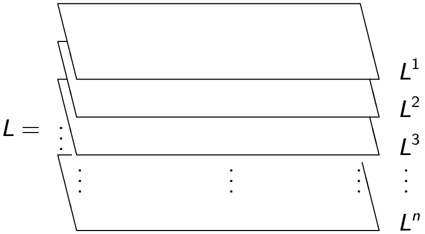
The Lie Bracket as a vector of matrices

It can also be represented by a “stack” of matrices



A vector of matrices.

$$L = \begin{pmatrix} L^1 \\ L^2 \\ \vdots \\ L^n \end{pmatrix}.$$

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L can then be arranged in an $n \times n$ matrix

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}.$$

It's nice to see that we can *encode* the bracket into a matrix representation, but how can we use that encoding as the Lie Bracket "map"?
Notice that we can retrieve information about the bases.
$[e_1, e_2] = L_{12}$

$$L = \begin{pmatrix} L_{11} & \boxed{L_{12}} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix}$$

So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

Lie Bracket as Linear Combination of Bases

We can write $[x, y]$ as a linear combination of their component vectors' brackets.
Let's look at an example of a 2 dimensional space
Let $x, y \in \mathfrak{g}$ with
$x = w_1 e_1 + w_2 e_2$
$y = v_1 e_1 + v_2 e_2$

$$\begin{aligned}
\text{So } [x, y] &= [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2] \\
&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
\end{aligned}$$

This means that the Lie Bracket is fully described by the matrix, L

Generalizing the Lie Bracket

As before, we'll investigate the 2 dimensional example.
Let

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

with

$$x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad y = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x^T L = \begin{pmatrix} w_1 w_2 \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} w_1 L_{11} + w_2 L_{21}, w_1 L_{12} + w_2 L_{22} \end{pmatrix}$$

$$(x^T L)y = \begin{pmatrix} w_1 L_{11} + w_2 L_{21}, w_1 L_{12} + w_2 L_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_1 v_1 L_{11} + w_2 v_1 L_{21} + w_1 v_2 L_{12} + w_2 v_2 L_{22} \end{pmatrix}$$

That is $[x, y] = x^T L y$

The Lie Bracket as a Matrix with Vector entries

Let's take another look at L
Consider the Lie Bracket, with bases $e_1, e_2, \ldots, e_n$
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What is special about $[e_i, e_i]$?

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What is special about $[e_i, e_i]$?
$[e_i, e_i] = \mathbf{0}$

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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$[e_i, e_j] = -[e_j, e_i]$

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Consider the map $[e_1, \,] : \mathfrak{g} \to \mathfrak{g}$.
Can anyone see the matrix representation of $[e_1, \,]$?

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Example Abelian $\mathbb{R}^3$

What does Abelian $\mathbb{R}^3$ look like in this form?
Recall that a Lie Algebra is Abelian if
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What is the center of Abelian $\mathbb{R}^3$?

Examples of Cross Product over $\mathbb{R}^3$

What does the the cross product on $\mathbb{R}^3$ look like in this form?
$[e_1, e_2] = e_3$
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$$\begin{pmatrix} \mathbf{0} & e_3 & -e_2 \\ -e_3 & \mathbf{0} & e_1 \\ e_2 & -e_1 & \mathbf{0} \end{pmatrix}$$

What is the center of $\mathbb{R}^3$ with the cross product?

$\mathfrak{h}_3$'s Lie Bracket

How could the three dimensional Heisenberg Algebra be represented?
$[e_1, e_2] = e_3$ for bases $e_1, e_2, e_3 \in \mathfrak{h}_3$
And all other brackets zero

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What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

Inner Product as a Matrix

The Inner Product can be represented by a matrix very similarly to the Lie Bracket.
$\langle e_i, e_j \rangle = E_{ij}$

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix}$$

Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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Dot Product on $\mathbb{R}^3$

The Dot Product can be represented in this way.
$\langle e_i, e_j \rangle = 0$
$\langle e_i, e_i \rangle = 1$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix representation of the j-maps

Let $\mathfrak{n}$ be a 2-step nilpotent Lie Algebra with $\mathfrak{v}$ as the non-center and $\mathfrak{z}$ as the center.
Recall that the Lie bracket on a 2-step nilpotent Lie algebra is a bilinear map $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$.
So for every $z \in \mathfrak{z}$, one can define a linear transformation $j_z : \mathfrak{v} \to \mathfrak{v}$ by the identity

$$\langle y, j_z(x) \rangle_{\mathfrak{v}} = \langle z, [x, y] \rangle_{\mathfrak{z}}.$$

We now use the E and L that we just constructed to complete matrices for J.

Matrix representation of the j-maps

By linearity, we will know how to construct any j-map, if we know the j-maps corresponding to basis vectors of $\mathfrak{z}$.
Suppose $\mathfrak{z} = \text{span}\{z_1, z_2, \ldots, z_m\}$.
Then for any z_k, the j-map $j_{z_k} : \mathfrak{v} \to \mathfrak{v}$ is given by

$$\begin{array}{rcl} \langle y, j_{z_k}(x)\rangle_{\mathfrak{v}} & = & \langle z_k, [x, y]\rangle_{\mathfrak{z}} \\ y^T E(J_{z_k} x) & = & z_k^T (x^T L y) \\ y^T (E J_{z_k}) x & = & y^T (L^k)^T x \end{array}$$

taking advantage of some clever stack-matrix manipulations.

Matrix representation of the j-maps

Since $y^T(EJ_{z_k})x = y^T(L^k)^T x$ for arbitrary x and y in $\mathfrak{v}$, we deduce that $EJ_{z_k} = (L^k)^T$.
Since $\det(E) \neq 0$, we may solve for J_{z_k} to obtain

$$J_{z_k} = E^{-1}(L^k)^T \in \mathbb{R}^{n\times n}.$$

If $z = \zeta_1 z_1 + \zeta_2 z_2 + \cdots + \zeta_m z_m$, where ζ_i are coefficients of the linear combination of z_i, then the map j_z is represented by the matrix

$$J_z = \zeta_1 J_{z_1} + \zeta_2 J_{z_2} + \cdots + \zeta_m J_{z_m}.$$

The matrix J as a stack

The j-maps of a 2-step nilpotent Lie algebra can be described by a stack of matrices of the same type as L.
Indeed, if we let $J^k = J_{z_k}$,
then

$$J = \begin{pmatrix} J^1 \\ J^2 \\ \vdots \\ J^m \end{pmatrix}$$

is an $(m \times 1)$ stack of $(n \times n)$ matrices.
Then for $z = (\zeta_1, \zeta_2, \ldots, \zeta_m)^T \in \mathfrak{z}$, the map j_z is represented by

$$J_z = z^T J.$$

Examples of j-maps

We used Sage to help us compute the j-maps for some examples of 2-step nilpotent Lie algebras.

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First, consider the Heisenberg algebra $\mathfrak{h}_3$. Restricting L to $\mathfrak{v}$ and letting E be the dot product on $\mathfrak{v}$, we obtain

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The j-map $j_{e_3} : \mathfrak{v} \to \mathfrak{v}$ is then given by

$$J = E^{-1} L^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This is easy to verify by hand.

Examples of j-maps

Suppose we change the inner product to

$$E = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Then the j-map becomes

$$J = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} \\ \frac{4}{3} & \frac{2}{3} \end{pmatrix}.$$

This is also not too hard to verify by hand.

Examples of j-maps

As a second example, consider the 6-dimensional algebra spanned by the vectors $\{e_1, e_2, e_3, e_4, z_1, z_2\}$, with non-trivial brackets

$$\begin{aligned}
[e_1, e_3] &= z_1 \\
[e_2, e_4] &= z_1 \\
[e_1, e_4] &= z_2 \\
[e_2, e_3] &= z_2
\end{aligned}$$

The Lie bracket is represented by the matrices

$$L^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad L^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Examples of j-maps

If E is the dot product, then the j-maps for this algebra are

$$J^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and } J^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

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Again, easy to verify by hand. They're just $(L^1)^T$ and $(L^2)^T$.

Examples of j-maps

However, if we let E be something a little more exotic, like

$$E = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{3} & 0 \\ -\frac{1}{4} & 1 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{6} & 0 & 1 \end{pmatrix},$$

which has $\det(E) = \frac{607}{1296} > 0$,

Examples of *j*-maps

then the *j*-maps become

$$J^1 = \frac{1}{607}\begin{pmatrix} -582 & -90 & -936 & -540 \\ -756 & -192 & -540 & -1152 \\ 1179 & 126 & 582 & 756 \\ 126 & 639 & 90 & 192 \end{pmatrix}, \quad \text{and}$$

$$J^2 = \frac{1}{607}\begin{pmatrix} -90 & -582 & -540 & -936 \\ -192 & -756 & -1152 & -540 \\ 126 & 1179 & 756 & 582 \\ 639 & 126 & 192 & 90 \end{pmatrix}.$$

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You wouldn't want to calculate these by hand!

Acknowledgements

Justin Ryan for introducing us to Lie Algebras and providing us with Tex Templates, papers and illustrations.

Jonathan Mckibbin for his illustration of the Heisenberg Algebra.