Recovering the Metric - An Overview of Lie Algebras

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&= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\
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Conversely, if [,] is skew-symmetric, then $[x,x] + [x,x] = 0$ implies that 2[x,x] = 0. Now we see that this implies [x,x] = 0 so long as our field is not of characteristic 2, for in those spaces 2 = 0 and we can deduce nothing about [x,x].

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A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \mathbf{0} \ \forall x \in \mathfrak{g}.$$

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$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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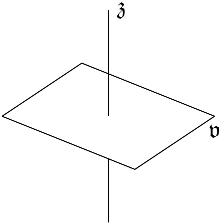

Figure : The Heisenberg Algebra, $\mathfrak{h}$

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\ ,\ \rangle$. (Confusing, I know.)

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This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?
$[e_i, e_j] = -[e_j, e_i]$

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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$$\text{So}\,[x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2]$$

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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
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This means that the Lie Bracket is fully described by the matrix, L

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What is the center of $\mathfrak{h}_3$?
The center is the span of $\{e_3\}$.

Inner Product as a Matrix

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Recall that $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$

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$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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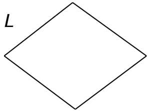

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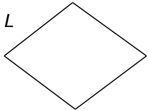
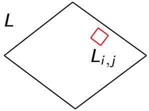


Now, let's pick out a single $L_{i,j}$ from that matrix.



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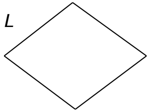

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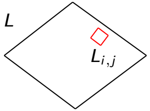
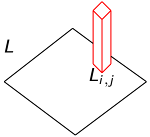


This is what is really lying behind that element!



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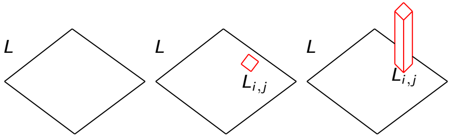

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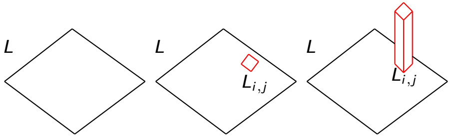

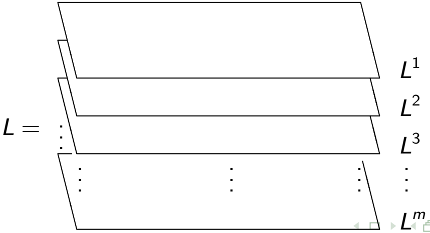

Acknowledgements