Recovering the Metric - An Overview of Lie Agebra's

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OSWEGO
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Conversely, if [,] is skew-symmetric, then $[x, x] + [x, x] = 0$ implies that $2[x,x] = 0$. Now we see that this implies $[x,x] = 0$ so long as our field is not of characteristic 2, for in those spaces $2 = 0$ and we can deduce nothing about $[x,x]$.

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A vector, z, of $\mathfrak{g}$ is said to be in the center of $\mathfrak{g}$ if

$$[x, z] = \mathbf{0} \; \forall x \in \mathfrak{g}.$$

One-Step Nilpotence

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$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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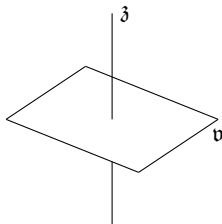with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

Figure : The Heisenberg Algebra, $\mathfrak{h}$

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$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

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A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

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The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\,,\,\rangle$. (Confusing, I know.)

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This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

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Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?

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$$L = \begin{pmatrix} \mathbf{0} & L_{12} & \cdots & L_{1n} \\ L_{21} & \mathbf{0} & \cdots & L_{2n} \\ \vdots & & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & \mathbf{0} \end{pmatrix}$$

Given $[e_i, e_j]$, what do we know about $[e_j, e_i]$?
$[e_i, e_j] = -[e_j, e_i]$

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So it will be enough find a way to write all Lie Brackets as a linear combination of basis brackets.

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$$\text{So } [x, y] = [w_1 e_1 + w_2 e_2, v_1 e_1 + v_2 e_2]$$
$$= [w_1 e_1, v_1 e_1 + v_2 e_2] + [w_2 e_2, v_1 e_1 + v_2 e_2]$$

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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2]
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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2]
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&= [w_1 e_1, v_1 e_1] + [w_1 e_1, v_2 e_2] + [w_2 e_2, v_1 e_1] + [w_2 e_2, v_2 e_2] \\
&= w_1 v_1 [e_1, e_1] + w_1 v_2 [e_1, e_2] + w_2 v_1 [e_2, e_1] + w_2 v_2 [e_2, e_2] \\
&= w_1 v_1 L_{11} + w_1 v_2 L_{12} + w_2 v_1 L_{21} + w_2 v_2 L_{22}
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This means that the Lie Bracket is fully described by the matrix, L

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That is $[x, y] = x^T L y$

Examples of Cross Product over $\mathbb{R}^3$

Some text

Acknowledgements