Recovering the Metric - An Overview of Lie Agebra's

Chaskin Saroff

Oswego State University
Department of Mathematics

16 September 2014

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad \text{and} \quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad$ and $\quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in V$.

Definition

A *Lie Algebra* is a (finite-dimensional) vector space $\mathfrak{g}$ together with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies three properties:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. and
3. $[x + y, z] = [x, z] + [y, z] \quad$ and $\quad [\alpha x, y] = [x, \alpha y] = \alpha [x, y]$

Property 1 is known as *alternating*, property 2 is known as the *Jacobi Identity* and property 3 is known as *bilinearity*.

Property 1 and 3 imply another property called *skew symmetry*:

1. $[x, y] = -[y, x]$ for all $x, y \in V$.

The product $[\ ,\ ]$ is known as a *Lie bracket* on V.

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$
\begin{aligned}
0 &= [x+y, x+y] && (1)\\
&= [x+y, x] + [x+y, y] && (2)\\
&= [x, x] + [y, x] + [x+y, y] && (3)\\
&= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\
0 &= [x, y] + [y, x] && (5)\\
\implies [x, y] &= -[y, x] && (6)
\end{aligned}
$$

skew-symmetry

By bilinearity, every alternating product is also skew-symmetric, regardless of the characteristic of the underlying field. Indeed, if [,] is alternating then

$$\begin{aligned} 0 &= [x+y, x+y] && (1)\\ &= [x+y, x] + [x+y, y] && (2)\\ &= [x, x] + [y, x] + [x+y, y] && (3)\\ &= [x, x] + [y, x] + [x, y] + [y, y] && (4)\\ 0 &= [x, y] + [y, x] && (5)\\ \implies [x, y] &= -[y, x] && (6) \end{aligned}$$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

1. $\langle x, y \rangle = \langle y, x \rangle$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

1. $\langle x, y \rangle = \langle y, x \rangle$
2. If $\langle x, y \rangle = 0 \quad \forall y \in \mathfrak{g}$
   then $x = \vec{0}$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

1. $\langle x, y \rangle = \langle y, x \rangle$
2. If $\langle x, y \rangle = 0 \quad \forall y \in \mathfrak{g}$
   then $x = \vec{0}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
   and $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle x, \alpha y \rangle$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

1. $\langle x, y \rangle = \langle y, x \rangle$
2. If $\langle x, y \rangle = 0 \quad \forall y \in \mathfrak{g}$
   then $x = \vec{0}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
   and $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle x, \alpha y \rangle$
4. $\langle x, x \rangle \geq 0$
   and $\langle x, x \rangle = 0 \implies x = \vec{0}$

An *inner product* on the vector $\mathfrak{g}$ is a *symmetric, non-degenerate, bilinear and positive definite* function on $\mathfrak{g}$. That is, an inner product on $\mathfrak{g}$ is a function $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satifies the following properties: For $x, y, z \in \mathfrak{g}$

1. $\langle x, y \rangle = \langle y, x \rangle$
2. If $\langle x, y \rangle = 0 \quad \forall y \in \mathfrak{g}$
   then $x = \vec{0}$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
   and $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle x, \alpha y \rangle$
4. $\langle x, x \rangle \geq 0$
   and $\langle x, x \rangle = 0 \implies x = \vec{0}$

A First Example

Let V be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

A First Example

Let V be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

A First Example

Let V be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

A First Example

Let V be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

A First Example

Let V be any vector space, and define

$$[x, y] = 0 \text{ for all } x, y \in V.$$

Clearly [,] is alternating:

$$[x, x] = 0 \text{ for all } x \in V.$$

And the Jacobi Identity is trivial:

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] &= [0, z] + [0, x] + [0, y] \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$(x \times y) \times z + (y \times z) \times x + (z \times x) \times y$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned} &(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ &\quad = \quad (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z \end{aligned}$$

$\mathbb{R}^3$ as a non-Abelian Lie Algebra

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The *cross product* of x with y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The cross product is skew symmetric:

$$x \times y = -y \times x \quad \text{for all } x, y \in \mathbb{R}^3.$$

Using some identities from Calc III, we can show that the cross product satisfies the Jacobi Identity:

$$\begin{aligned} &(x \times y) \times z + (y \times z) \times x + (z \times x) \times y \\ &\quad = (x \cdot z)y - (y \cdot z)x + (y \cdot x)z - (z \cdot x)y + (z \cdot y)x - (x \cdot y)z \\ &\quad = 0. \end{aligned}$$

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

The basis vectors x, y, and z can be represented by the following matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

The basis vectors x, y, and z can be represented by the following matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{``}-\text{''} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{h}_3$: The Heisenberg Algebra

Consider a 3-dimensional vector space with basis vectors x, y, z. Define a Lie bracket on this space by

$$[x, y] = z,$$

with all other brackets equal to 0. This is called the *Heisenberg Algebra*, and is denoted by $\mathfrak{h}_3$.

The basis vectors x, y, and z can be represented by the following matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{``}-\text{''} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- $\mathfrak{h}_3$ is *not* a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ because it doesn't have the same bracket.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z.

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

$\mathfrak{h}_{2k+1}$: Higher Dimensional Heisenberg Algebras

Higher odd-dimensional analogues of the Heisenberg algebra can also be defined.

Consider $\mathbb{R}^{2k+1}$, $k \geq 1$, with basis vectors $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$, and z. Define a Lie bracket on these generators *via*

$$[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij}, \quad \text{and} \quad [z, \ ] = 0.$$

A general element of this space can be represented by a matrix of the form

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & z \\ 0 & 1 & 0 & \cdots & y_1 \\ 0 & 0 & 1 & \cdots & y_2 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y\rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$:

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \, , \, \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$; then $x \cdot y = x^T y$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$; then $x \cdot y = x^T y$.

Example The *Minkowski inner product* on $\mathbb{R}^n$:

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \ , \ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$; then $x \cdot y = x^T y$.

Example The *Minkowski inner product* on $\mathbb{R}^n$: Let $x = (t, x_1, x_2, x_3)$ and $y = (s, y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^4$.

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \ , \ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$; then $x \cdot y = x^T y$.

Example The *Minkowski inner product* on $\mathbb{R}^n$: Let $x = (t, x_1, x_2, x_3)$ and $y = (s, y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^4$. Then the Minkowski inner product of x and y is given by

$$\langle x, y \rangle = -(ts) + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Inner Products

An *inner product* on a vector space V is a non-degenerate, symmetric, bilinear function $\langle \ , \ \rangle : V \times V \to \mathbb{R}$.

1. If $\langle x, y \rangle = 0$ for all $y \in V$, then x must be 0;
2. $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, z \in V$, $a, b, \in \mathbb{R}$.

Example The *dot product* on $\mathbb{R}^n$: Let $x, y \in \mathbb{R}^n$; then $x \cdot y = x^T y$.

Example The *Minkowski inner product* on $\mathbb{R}^n$: Let $x = (t, x_1, x_2, x_3)$ and $y = (s, y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^4$. Then the Minkowski inner product of x and y is given by

$$\langle x, y \rangle = -(ts) + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

This is the inner product that gives rise to Einstein's Special Relativity!

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y]\,.$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y] , z \rangle = \langle y, j(x) z \rangle$$

for all $y, z \in \mathfrak{g}$.

Adjoints of Adjoints

If you put an inner product on a Lie algebra (rather than just a vector space), then you will have two different types of products on the space.

The *adjoint representation* of a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ on itself is the function defined by

$$\mathrm{ad}_x y = [x, y] .$$

Fixing $x \in \mathfrak{g}$, $\mathrm{ad}_x = [x,\ ]$ becomes a function from $\mathfrak{g}$ to itself. One can then define a map $j(x) : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$\langle [x, y] , z \rangle = \langle y, j(x) z \rangle$$

for all $y, z \in \mathfrak{g}$.

The map $j(x)$ is called the *adjoint* to ad_x with respect to the inner product $\langle\ ,\ \rangle$. (Confusing, I know.)

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z\rangle = \langle y, j(x)z\rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z\rangle = \langle y, \mathrm{ad}_x^\dagger z\rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y]\,, z\rangle = \langle y, j(x)z\rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z\rangle = \langle y, \mathrm{ad}_x^{\dagger} z\rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^{\dagger} : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

j-maps

The equation that we used to define $j(x)$,

$$\langle [x, y], z \rangle = \langle y, j(x)z \rangle$$

could be re-written to look like

$$\langle \mathrm{ad}_x y, z \rangle = \langle y, \mathrm{ad}_x^\dagger z \rangle.$$

Now we could define $j(x) = \mathrm{ad}_x^\dagger : \mathfrak{g} \to \mathfrak{g}$.

The j-maps are extremely useful because they simultaneously encode information about the algebraic structure and geometric structure of the Lie algebra $\mathfrak{g}$.

This makes the computation of many complicated geometric objects–such as curvatures–into "simple" calculations in linear algebra!

Lie Algebra Seminar

There is a "Group Study Seminar" on Lie Algebras that meets every week!

Day: Thursdays

Time: 5:30 – 6:30 pm

Room: Math Commons

We'll start with a "review" of vector spaces, then spend a lot of time building examples. Eventually we'll prove some theorems, and study a little geometry (time permitting).