Boundary condition treatment with the compressible method of characteristics

1 Basics

The quantities conserved along their respective characteristic lines are

$$\alpha = a + \frac{\gamma - 1}{2} v, \tag{1}$$

$$\beta = a - \frac{\gamma - 1}{2} v, \tag{2}$$

where a is the isentropic wave propagation speed, γ is the ratio of specific heats and v is the fluid velocity. Furthermore, the slopes of the β = constant and the α = constant lines are $a - v$ and $a + v$, respectively.

2 Interpolation methods

2.1 Origin of the characteristic line

To update the state variables at the boundaries, one must locate the characteristic line which crosses the boundary exactly at $t + \Delta t$. The problem is illustrated for the left side boundary by Figure 1.

2.1.1 Left side

For the left side we have

$$\frac{x_{\mathrm{R}} - x_0}{t_{j+1} - t_j} = \frac{x_{\mathrm{R}} - x_0}{\Delta t} = -(v_{\mathrm{R}} - a_{\mathrm{R}}), \tag{3}$$

where the v_R and a_R has to be interpolated by

$$v_{\mathrm{R}} = v_0 + \frac{v_1 - v_0}{\Delta x} x_{\mathrm{R}}, \tag{4}$$

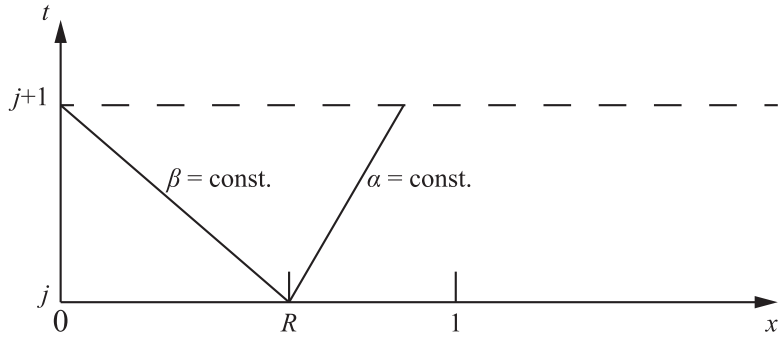

Figure 1: The position to be located.

$$a_{\mathrm{R}} = a_0 + \frac{a_1 - a_0}{\Delta x} x_{\mathrm{R}}. \tag{5}$$

This results in

$$x_{\mathrm{R}} - x_0 = \frac{(a_0 - v_0)\Delta t}{\left(1 + \frac{v_1 - v_0}{\Delta x}\Delta t - \frac{a_1 - a_0}{\Delta x}\Delta t\right)} \tag{6}$$

2.1.2 Right side

Similarly, for the right side we have

$$\frac{x_N - x_{\mathrm{L}}}{t_{j+1} - t_j} = \frac{x_N - x_{\mathrm{L}}}{\Delta t} = v_{\mathrm{L}} + a_{\mathrm{L}}, \tag{7}$$

where N is the number of nodes and v_L and a_L are interpolated by

$$v_{\mathrm{L}} = v_N + \frac{v_{N-1} - v_N}{\Delta x}\left(x_N - x_{\mathrm{L}}\right), \tag{8}$$

$$a_{\mathrm{L}} = a_N + \frac{a_{N-1} - a_N}{\Delta x}\left(x_N - x_{\mathrm{L}}\right). \tag{9}$$

This results in

$$x_N - x_{\mathrm{L}} == \frac{(a_N + v_N)\Delta t}{\left(1 - \frac{v_{N-1} - v_N}{\Delta x}\Delta t - \frac{a_{N-1} - a_N}{\Delta x}\Delta t\right)}. \tag{10}$$

2.1.3 Generalised

Note that Equations (6) and (10) have similar forms. It is feasible to generalise them to

$$\xi = \frac{(a_{\mathrm{bou}} - \Theta v_{\mathrm{bou}})\Delta t}{1 + \Theta\frac{v_{\mathrm{adj}} - v_{\mathrm{bou}}}{\Delta x}\Delta t - \frac{a_{\mathrm{adj}} - a_{\mathrm{bou}}}{\Delta x}\Delta t}, \tag{11}$$

$$\Theta = \begin{cases} +1, & \text{left side} \\ -1, & \text{right side.} \end{cases} \tag{12}$$

where ξ is the distance to the boundary and the "bou" and "adj" indices refer to the corresponding quantity on the boundary and the adjacent points, respectively. The state variables can be interpolated using

$$v_\xi = v_{\mathrm{bou}} + \frac{v_{\mathrm{adj}} - v_{\mathrm{bou}}}{\Delta x}\xi, \tag{13}$$

$$a_\xi = a_{\mathrm{bou}} + \frac{a_{\mathrm{adj}} - a_{\mathrm{bou}}}{\Delta x}\xi, \tag{14}$$

$$p_\xi = p_{\mathrm{bou}} + \frac{p_{\mathrm{adj}} - p_{\mathrm{bou}}}{\Delta x}\xi. \tag{15}$$

2.2 Location of the fluid packet reaching the boundary

For some boundary conditions (outflow and valve), it is required to know where the fluid packet — that is on the boundary at t_{j+1} — was at t_j. The following subsections propose two methods for this.

2.2.1 Zero local and convective accelerations

Let us assume that the velocity distribution near the boundary is linear, i.e.

$$v_\zeta = v_{\text{bou}} + \frac{v_{\text{adj}} - v_{\text{bou}}}{\Delta x}\zeta. \tag{16}$$

From now on assuming that both the convective and the local accelerations are zero, we get

$$v_\zeta \Delta t = \zeta. \tag{17}$$

Finally, combining these equations gives

$$\zeta = \frac{v_{\text{bou}}\Delta t}{1 - \frac{v_{\text{adj}} - v_{\text{bou}}}{\Delta x}\Delta t}. \tag{18}$$

2.2.2 Zero local and linear convective acceleration

Similarly, a linear velocity distribution is assumed, i.e.

$$v(x) = v_{\text{bou}} + \frac{v_{\text{adj}} - v_{\text{bou}}}{\Delta x}x = v_{\text{bou}} + bx. \tag{19}$$

However, this time let us take into account the convective acceleration. This leads to the differential equation

$$\dot{x} = v_{\text{bou}} + bx \tag{20}$$

with the initial condition

$$x(0) = 0. \tag{21}$$

The analytical solution is

$$x(t) = \left(\mathrm{e}^{bt} - 1\right)\frac{v_{\text{bou}}}{b}. \tag{22}$$

With this notation it follows that $x(\Delta t) = \zeta$, i.e. the distance covered by the particle during the time step is exactly the distance from its original position to the boundary, that is

$$\zeta = \left(\mathrm{e}^{b\Delta t} - 1\right)\frac{v_{\text{bou}}}{b}. \tag{23}$$

3 Closed pipe boundary condition

The main principles behind the closed pipe boundary condition are

- the velocity at the pipe end is always zero,
- α or β are conserved along the characteristic line,
- the change of state on the boundary is isentropic in time,
- the ideal gas law can be applied.

For the state variables the method is

$$v_{\text{bou}}^{j+1} = 0, \tag{24}$$

$$a_{\text{bou}}^{j+1} = a_\xi - \Theta\frac{\gamma - 1}{2}v_\xi, \tag{25}$$

$$T_{\text{bou}}^{j+1} = \frac{\left(a_{\text{bou}}^{j+1}\right)^2}{\gamma \mathrm{R}}, \tag{26}$$

$$\rho_{\text{bou}}^{j+1} = \left(\frac{T_{\text{bou}}^{j+1}}{T_{\text{bou}}^{j}}\right)^{\frac{1}{\gamma-1}} \rho_{\text{bou}}^{j}, \tag{27}$$

$$p_{\text{bou}}^{j+1} = \rho_{\text{bou}}^{j+1} \mathrm{R} T_{\text{bou}}^{j+1}, \tag{28}$$

$$e_{\text{bou}}^{j+1} = c_v T_{\text{bou}}^{j+1}, \tag{29}$$

where c_v is the isochoric specific heat capacity and the superscript indices refer to the time step. Similarly to before, Θ depends on the side, i.e.

$$\Theta = \begin{cases} +1, & \text{left side} \\ -1, & \text{right side.} \end{cases} \tag{30}$$

4 Reservoir boundary condition

4.1 Inflow from the reservoir

The main principles behind the inflow boundary condition are

- the inflow is isentropic,
- α or β are conserved along the characteristic line,
- the ideal gas law can be applied.

For simplicity, variables without indices refer to those on the boundary at the new time step. First, the energy equation is

$$c_p T_{\text{t,res}} = c_p T + \frac{v^2}{2}. \tag{31}$$

Using that $a = \sqrt{\gamma R T}$ and $c_p = \frac{\gamma}{\gamma-1} R$ it gives

$$a_{\text{t,res}}^2 = a^2 + \frac{\gamma - 1}{2} v^2. \tag{32}$$

The conserved quantity (depending on Θ, i.e. the side) is

$$a_\xi - \Theta \frac{\gamma - 1}{2} v_\xi = K = a - \Theta \frac{\gamma - 1}{2} v \tag{33}$$

Equations (32) and (33) give a second order equation, which when solved for the velocity gives

$$v = \frac{-B + \Theta\sqrt{B^2 - 4AC}}{2A}, \tag{34}$$

with

$$A = \left(\frac{\gamma - 1}{2}\right)^2 + \frac{\gamma - 1}{2}, \quad B = \Theta(\gamma - 1)K, \quad C = K^2 - a_{\text{t,res}}^2. \tag{35}$$

For inflow ($v > 0$), we need $K < a_{t,res}$. With this, the new isentropic wave propagation speed can be obtained from Equation (33), i.e.

$$a = K + \Theta\frac{\gamma - 1}{2}v. \tag{36}$$

The temperature simply follows with

$$T = \frac{a^2}{\gamma \mathrm{R}}. \tag{37}$$

As the inflow is isentropic,

$$p = \left(\frac{T}{T_{\mathrm{t,res}}}\right)^{\frac{\gamma}{\gamma-1}} p_{\mathrm{t,res}}. \tag{38}$$

Finally,

$$\rho = \frac{p}{\mathrm{R}T}, \tag{39}$$

$$e = c_v T + \frac{v^2}{2}. \tag{40}$$

4.2 Outflow to the reservoir

The main principles behind the outflow boundary condition are

- the static pressure at the boundary equals the total pressure in the reservoir
- α or β are conserved along the characteristic line,
- the ideal gas law can be applied,
- the change of state of the fluid packet arriving at the boundary is isentropic.

Again, the variables without indices refer to those on the boundary at the new time step. As stated above, the pressure is

$$p = p_{\mathrm{t,res}}, \tag{41}$$

and with this

$$a = \left(\frac{p}{p_\zeta}\right)^{\frac{\gamma-1}{2\gamma}} a_\zeta. \tag{42}$$

The velocity is updated using the conservation of K (see above) along the characteristic line, i.e.

$$v = \frac{2(a - K)}{\Theta(\gamma - 1)}, \tag{43}$$

and the temperature is

$$T = \frac{a^2}{\gamma \mathrm{R}}. \tag{44}$$

The rest are

$$\rho = \frac{p}{\mathrm{R}T}, \tag{45}$$

$$e = c_v T + \frac{v^2}{2}. \tag{46}$$

4.3 Determining whether there is inflow or outflow

4.3.1 Left side

On the left side we have

$$\beta = a - \frac{\gamma - 1}{2} v = \beta_{\mathrm{R}} = a_{\mathrm{R}} - \frac{\gamma - 1}{2} v_{\mathrm{R}}, \tag{47}$$

$$\alpha = a + \frac{\gamma - 1}{2} v \tag{48}$$

If there is inflow, i.e. $v > 0$, then from the energy equation it follows that $a < a_{\mathrm{t,res}}$. By adding and subtracting α and β we get

$$\alpha + \beta = 2a \leq 2a_{\mathrm{t,res}}, \tag{49}$$

$$\alpha - \beta = (\gamma - 1)v \geq 0. \tag{50}$$

Adding these two gives

$$a_{\mathrm{t,res}} \geq \beta = \beta_{\mathrm{R}}. \tag{51}$$

4.3.2 Right side

Similarly we have

$$\alpha = a + \frac{\gamma - 1}{2} v = \alpha_{\mathrm{L}} = a_{\mathrm{L}} + \frac{\gamma - 1}{2} v_{\mathrm{L}}, \tag{52}$$

$$\beta = a - \frac{\gamma - 1}{2} v. \tag{53}$$

As the velocity is positive if the fluid flows from the left side to the right side, in the case of inflow $v < 0$. However, from the energy equation it still holds that $a < a_{\mathrm{t,res}}$. Again, let us take

$$\alpha + \beta = 2a \leq 2a_{\mathrm{t,res}}, \tag{54}$$

$$\alpha - \beta = (\gamma - 1)v \leq 0. \tag{55}$$

Adding these two gives

$$a_{\mathrm{t,res}} \geq \alpha = \alpha_{\mathrm{L}}. \tag{56}$$

4.3.3 Generalised

It can be seen that Equations (51) and (56) are quite similar: both state that the isentropic wave propagation speed in the reservoir is greater than or equal to the respective conserved quantity. Generally, it can be said that the condition for inflow is

$$a_{\mathrm{t,res}} \geq K = a_{\xi} - \Theta \frac{\gamma - 1}{2} v_{\xi}. \tag{57}$$

5 Valve boundary condition

mi történik, ha visszafelé áramlás van? The main principles behind the valve boundary condition are

- the mass flow rate equation for the valve must be satisfied,
- α or β are conserved along the characteristic line,
- the ideal gas law can be applied,
- the change of state of the fluid packet arriving at the boundary is isentropic.

As this system of equations does not have a closed form solution, an iterative method was employed, which looks as follows. The equation for the mass flow rate is

$$\dot{m}_{\text{new}} = C_{\text{D}} A_{\text{ref}}(x) \sqrt{K \rho_{\text{old}} p_{\text{old}}}, \tag{58}$$

with

$$A_{\text{ref}}(x) = D_{\text{pipe}} \pi x, \tag{59}$$

and $x/D > 0.25$-nél is ezzel számolok

$$K = \begin{cases} \gamma \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} & \text{if} \quad p_v/p_b > 1.8929 \\ \frac{2\gamma}{\gamma-1}\left(\left(\frac{p_{\text{b}}}{p_{\text{old}}}\right)^{\frac{2}{\gamma}} - \left(\frac{p_{\text{b}}}{p_{\text{old}}}\right)^{\frac{\gamma+1}{\gamma}}\right) & \text{otherwise,} \end{cases} \tag{60}$$

where p_{b} is the backpressure, i.e. the pressure downstream of the valve. This equals to the mass flow rate at the end of the pipe, from which the new velocity is

$$v_{\text{new}} = -\Theta \frac{\dot{m}}{\rho_{\text{old}} A_{\text{pipe}}}. \tag{61}$$

By utilizing that K is conserved on the characteristic line we get

$$a_{\text{new}} = K + \Theta \frac{\gamma - 1}{2} v_{\text{new}} = a_{\xi} + \Theta \frac{\gamma - 1}{2} \left(v_{\text{new}} - v_{\xi}\right), \tag{62}$$

from which the new temperature is

$$T_{\text{new}} = \frac{a_{\text{new}}^2}{\gamma \text{R}}, \tag{63}$$

and the new pressure is

$$p_{\text{new}} = \left(\frac{a_{\text{new}}}{a_{\zeta}}\right)^{\frac{2\gamma}{\gamma-1}} p_{\zeta}. \tag{64}$$

Finally, the new density and specific energy are

$$\rho_{\text{new}} = \frac{p_{\text{new}}}{\text{R} T_{\text{new}}}, \tag{65}$$

$$e_{\text{new}} = c_v T_{\text{new}} + \frac{v_{\text{new}}^2}{2}. \tag{66}$$

The iteration process is initialized with the state variables from the previous time step and is repeated until convergence is achieved.

a kódban a nyomás egy lépésen belüli relatív megváltozását nézem