

Asian and European Option Specifics– Pricing, Estimation, and Comparison Using R

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Abstract

In this article, I present methods to efficiently estimate the price and the probability of exercise for vanilla and exotic options. In addition, I am using these results to compare the empirical delta between European and average rate Asian options.

Results confirm that average rate exotic options generally reduce the variability of the option's price, probability of exercise, and the option's delta when compared to vanilla options. Hence, offering average rate options might be beneficial when delta hedging the option portfolio is of concern.

This research is 100% reproducible. All R source code is publicly available on my personal [GitHub page](#).¹

Keywords: Exotic Option Pricing, Monte Carlo Simulation, Delta Hedging.

¹URL: <https://github.com/csatzky>

1 Simulating Vanilla Options

Obtaining an option's price using Monte-Carlo simulation is straightforward. However, simulating the delta of an option is more complex. In the following, I am first reviewing the underlying price process of the Black-Scholes model. Secondly, I present a simple, lightweight R function to simulate this price process. Thirdly, after carefully dissecting the Black-Scholes equation, I am presenting a simulation approach on how to estimate an option's delta.

1.1 Geometric Brownian Motion

Under the Black-Scholes differential equation, the stock price S follows a geometric Brownian motion (GBM). The stock price at time t can be expressed as:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B_t\right) \quad (1)$$

Where:

- S_t Stock price at time t
- μ Annual rate of return of the risky asset S
- σ Standard deviation of μ
- B_t Wiener process, $B_t \sim N(0, t), \forall t$

1.2 Simulating A Geometric Brownian Motion

Let's denote Δt the fixed step size $\Delta t = t_i - t_{i-1} \forall i$, such that:

$$S_{t_i} = S_{t_{i-1}} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z\right) \quad (2)$$

Where $Z \sim N(0, 1)$.

To simulate one GBM path efficiently, I am using vectorization and two lines of R code. The in-built R function `cumprod()` cumulatively multiplies the incremental factors in the GBM evolution.

```
gbm_path <- function(s0, mu, sigma, T, dt = 1/(T-1)) {  
  
  # draw from the standard-normal distribution  
  epsilon <- rnorm(T-1)  
  
  # compute GBM path from t=0 to t=T  
  cumprod(c(s0, exp((mu-sigma*sigma/2)*dt + sigma*epsilon*sqrt(dt))))  
}
```

Below, I am using the `gbm_path()` function to simulate 10 GBM paths sharing the following characteristics:

S_0	= 100	Stock price in $t = 0$ is equal to \$100
T	= 252	Simulate 252 trading days, i.e. 1 calendar year from S_0 to S_T
μ	= 0.07	Stock S has an expected, continuously compounded, annual return of 7%
σ	= 0.3	Stock S has constant, annual volatility of 30%
Δt	= $1/251$	If $t = 0$ to $t = T$ represents one year, then $\Delta t = 1/(T - 1)$

Figure 1: 10 Simulated GBM Paths with Equal Characteristics



1.2.1 Code Cross-Check

When simulating a GBM path with $S_0 = 100$, $\mu = 0.07$, and $T - t_0$ resembling one calendar year, the expected stock price at expiry, $E[S_T]$ is equal to:

$$E[S_T] = S_0 \exp(\mu) = 100 \exp(0.07) = 107.2508$$

To test the correctness of the `gbm_path()` function, below I am simulating $n = 100,000$ GBM paths to estimate S_T .

```
# for reproducibility
set.seed(1)

# simulate 100k GBMs with low sigma
gbms <- replicate(100000,
  gbm_path(s0 = 100, mu = 0.07, sigma = 0.0005, T = 252))

# compute average stock price at expiry and 95% CIs
ST <- mean(tail(gbms,1))
sd <- sd(tail(gbms,1))
ci_upper <- ST + (sd/sqrt(100000))*pnorm(0.975)
ci_lower <- ST - (sd/sqrt(100000))*pnorm(0.975)
```

Table 1: Simulation Results for Stock Price at Expiry

\widehat{S}_T	95% CI
107.2508	[107.25099, 107.25071]
$n = 0.1\text{m}, S_0 = \$100, \mu = 0.07, \sigma = 0.0005.$	

The simulation confirms that the `gbm_path()` function yields an unbiased estimate of S_T .

1.3 Option Pricing using Monte-Carlo Simulation

Under the Black-Scholes model, the investor is invariant to risk. Hence, risky assets are discounted at the risk-free rate, r . Likewise, risky assets are expected to grow at the risk-free rate, r . Hence, when simulating GBM paths, μ has no relevance and is set to $\mu = r$ in the `gbm_path()` function.

Obtaining the price for a European call option is straightforward. First, let's simulate several GBM paths. Secondly, compute the pay-off at expiry. Thirdly, discount all pay-offs to time $t = 0$. The average of all present values of the simulated pay-offs is equal in expectation to the option price obtained via the Black-Scholes formula:

$$V_t^{CE} = S_t F_N(d_1) - X \exp^{-r(T-t)} F_N(d_2) \quad (3)$$

Where:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$F_N()$: Cumulative distribution function of the standard normal distribution

σ : Annualized volatility of the share price S_t

$T - t$: Time to expiry in years

S_t : Share price at time t

X : Strike price of the option

r : Annualized, continuously compounded risk-free rate

Note that at the option's inception, $t = 0$, and for expiry in one year, $T = 1$, the terms simplify to $T - t = 1$ and $S_t = S_0$.

1.3.1 Dissecting the Black-Scholes Equation's Terms

In the following, I am developing a method to obtain estimates for a vanilla option's delta, i.e. the first derivative of the option price with respect to the share price. Delta is of crucial importance as it is used to hedge the option price against changes in the underlying asset.

To develop an approach to simulate an option's delta, it is useful to first interpret the

terms in Black-Scholes equation (3). In line with Nielsen (1992)², the terms resemble the following:

$$\underbrace{V_t^C}_{\text{European Call Price at time } t} = \underbrace{S_t}_{\text{Present Value of } \mathbb{E}[S_T]} \underbrace{F_N(d_1)}_{\frac{\partial V_t^C}{\partial S_t}} - \underbrace{X \exp^{-r(T-t)}}_{\text{Present Value of Strike Price}} \underbrace{F_N(d_2)}_{\mathbb{P}(S_T > X)} \quad (4)$$

Note:

1. As established in the previous section, due to risk-neutrality, $S_t = \mathbb{E}[S_T] \exp^{-r(T-t)}$, and likewise, $\mathbb{E}[S_T] = S_t \exp^{r(T-t)}$.
2. It is a common misconception to interpret the option's delta as the probability of the option ending up in the money at expiry. When it comes to a European call option, $\text{Delta}^C = F_N(d_1)$ *overestimates* the true probability, $\mathbb{P}(S_T > X) = F_N(d_1 - \sigma\sqrt{T-t})$. This is due to the missing volatility term, $\sigma\sqrt{T-t}$. Given delta, this probability can be calculated using the standard normal quantile function, $Q_N(p)$:

$$\mathbb{P}(S_T > X) = F_N\left(Q_N(\text{Delta}^C) - \sigma\sqrt{T-t}\right)$$

In R, this can be computed as `pnorm(qnorm(d) - sigma*sqrt(T-t))`, where $d = \text{Delta}^C$, $\text{sigma} = \sigma$ (e.g. implied volatility), and $T-t$ is the time to maturity in years.

In the following section, I am using (4) to simulate a European call option's delta.

²Lars Tyge Nielsen, "Understanding $N(d_1)$ and $N(d_2)$: risk adjusted probabilities in the black-scholes model", INSEAD, 1992.

1.3.2 Simulating a Vanilla Option's Delta

It is straightforward for most options to determine the terms in equation (4) using Monte-Carlo simulation. One possible exception, however, is the option's delta, which is given by $F_N(d_1)$ for a European call. More specifically,

1. S_t , X , r , and $T - t$ are all given.
2. $\mathbb{P}(S_T > X)$ can be estimated by computing the proportion of several GBM paths exceeding the strike price, X at expiry, T .
3. V_t^C , the option price at time t , can be estimated by discounting and averaging the simulated pay-offs at expiry T .

Hence, using (4) and Monte-Carlo simulation, the option's delta for a European call option can be estimated by:

$$\text{Delta}^C = \frac{\partial V_t^C}{\partial S_t} = \frac{1}{S_t} \left[V_t^C + X \exp^{-r(T-t)} \mathbb{P}(S_T > X) \right] \quad (5)$$

Note that S_t , X , r , T , and t are all exogenous and the two variables obtainable by simulation are $\mathbb{P}(S_T > X)$ and V_t^C . Hence, equation (5) reduces to a scaled sum of two variables of the form $aX + bY$. Thus, its variance is equal to:

$$\text{Var}[\hat{\text{Delta}}^C] = \frac{1}{S_t^2} \left[\text{Var}[\hat{V}_t^C] + b^2 \text{Var}[\hat{\mathbb{P}}(S_T > X)] + 2b \text{Cov}[\hat{V}_t^C, \hat{\mathbb{P}}(S_T > X)] \right] \quad (6)$$

Where:

$$b = X \exp^{-r(T-t)}$$

It follows the standard error for $\hat{\text{Delta}}^C$:

$$\text{SE}[\hat{\text{Delta}}^C] = \frac{\sqrt{\text{Var}[\hat{\text{Delta}}^C]}}{\sqrt{n}} \quad (7)$$

1.3.3 Application: Simulating the Price and Delta of a European Call

In the following, I am comparing the price, the delta, and the probability of exercise of a European call option obtained by the Black-Scholes equation to the estimates obtained by the simulation method as described in [the previous section](#).

Consider a European call option with the following characteristics:

S_0	= \$100	Stock price at $t = 0$ is \$100
σ	= 0.2	The underlying asset has 20% volatility
X	= \$100	The option's strike price is \$100
r	= 0.02	Risk-free rate is 2%
$T - t$	= 1	1 year to expiry

The table below compares the results obtained by Monte-Carlo simulation and the Black-Scholes equation.

Table 2: Black-Scholes Solution vs Monte-Carlo Estimates

Variable	Black-Scholes Solution	Monte-Carlo Simulation	
		Estimate	95% CI
V_0^C	8.916	8.913	[8.9034, 8.9222]
$\mathbb{P}(S_T > X)$	0.500	0.500	[0.4997, 0.5003]
Delta ^C	0.579	0.579	[0.5788, 0.5796]

$n = 1.5\text{m}$, $S_0 = \$100$, $r = 0.02$, $\sigma = 0.2$, $X = \$100$.

As seen in the table above, all variables' confidence intervals cover the Black-Scholes solutions. More specifically, the simulated results are accurate to at least two digits. The 95% confidence interval for the option price, V_0^C , has a total width of just \$0.02.

Also note the evidence for the claim that, $\mathbb{P}(S_T > X) = F_N(d_2)$, where $\mathbb{P}(S_T > X)$ is simulated by computing the proportion of GBMs ending up in the money $S_T > X = \$100$.

2 Asian Option Pricing

The key difference between Asian and vanilla options is how the pay-off depends on the underlying's price. More specifically, the pay-off of an Asian option is dependent on the *average* stock price, \bar{S}_T at finite, different time intervals. Assuming only one time interval, the pay-offs for Asian call and put options at expiry are as follows:

Average Rate Asian Call Option Pay-Off

$$C_T = \max\left(\frac{1}{T} \sum_{t=0}^T S_t - X, 0\right)$$

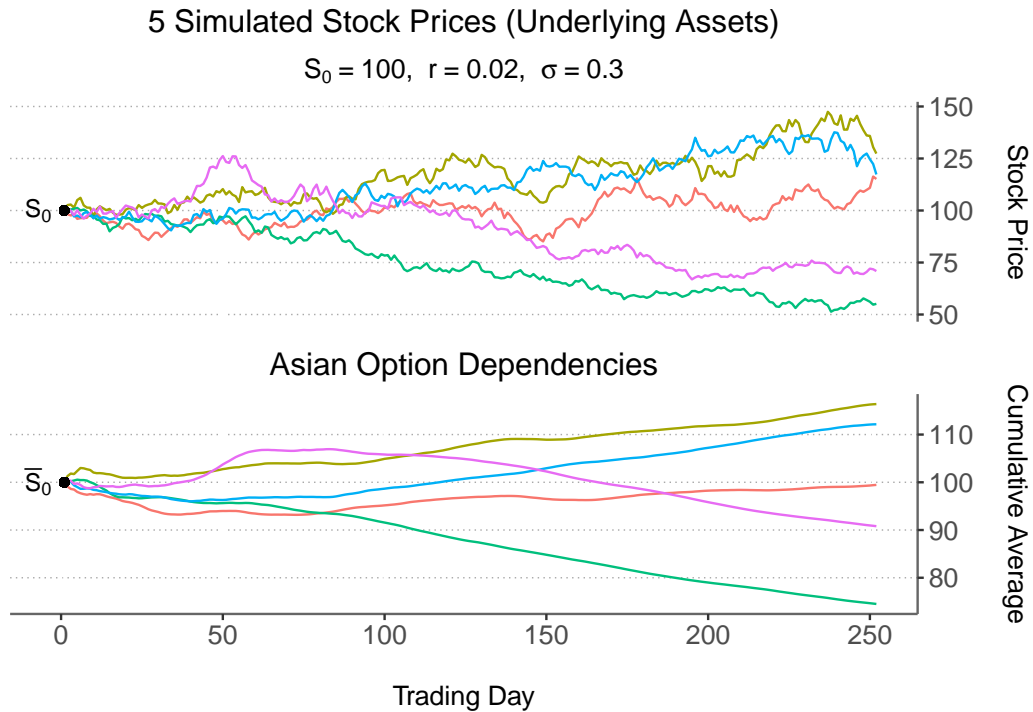
Average Rate Asian Put Option Pay-Off

$$P_T = \max\left(X - \frac{1}{T} \sum_{t=0}^T S_t, 0\right)$$

Hence, the pay-off of an Asian option is dependent on the share price over the *lifetime* of the option, whereas for vanilla options, the pay-off is only dependent on the (one) share price at expiry.

To illustrate this, let's simulate 5 GBMs, resembling 5 different assets over time. Below, the first plot resembles the actual price development of the underlying assets. The second plot, however, depicts the *cumulative arithmetic averages* of the underlying share prices. This is the price function the Asian option is dependent on.

Figure 2: Price Evolution of Underlying Assets Versus Asian Options' Dependencies



As seen in figure 2, the Asian option's stock price dependencies are far less volatile as compared to vanilla options, which are dependent on the actual share price. Hence there should also be far less variation in the Asian option's delta when compared to vanilla options, making delta-hedging generally more accurate.

2.1 Simulating an Asian Call Option's Price and Probability of Exercise

Assuming an *average rate*, one-period Asian call option, there are several similarities to a European call option. First, it spawns across one specific time period. Secondly, it can only be exercised at expiry. Thirdly, the pay-off is the difference between a function of the underlying asset's price and the fixed strike price X , or zero. The *only* difference is that the European call option depends on S_t and $\mathbb{E}[S_T]$, whereas the Asian call option depends on $\bar{S}_t = \frac{1}{t} \sum_{i=0}^t S_i$ and $\mathbb{E}[\bar{S}_T] = \mathbb{E}[\frac{1}{T} \sum_{i=0}^T S_i]$. Hence, at any point in time, the Asian option's price is at least partially dependent on the *past* prices of the underlying asset. Thus, it is sensible to distinguish between pricing an Asian option at inception, and pricing an Asian option after its inception, where $t > 0$.

Simulating the Price and Probability of Exercise at Inception of the Option

Simulating an Asian option's price, and the probability of exercise, $\mathbb{P}(S_T > X)$, is similar to the approach of a vanilla option, with one additional step (i.e. the average rate transformation of the simulated GBM paths). The proposed method is as follows:

1. Simulate several GBM paths. Result: Many estimates for S_T .
2. Compute the *arithmetic mean* of the simulated GBM paths. Result: Many estimates for \bar{S}_T .
3. From (2), compute the Asian call option's pay-off, i.e. $V_T^C = \max(\bar{S}_T - X, 0)$. Result: Many estimated pay-offs at expiry, T .
4. From (2), compute the proportion of $\bar{S}_T > X$ via $\text{mean}(\bar{S}_T > X)$. This yields the estimate for $\mathbb{P}(\bar{S}_T > X)$.
5. Discount the pay-offs in (3) from T to $t = 0$. Result: Many present values of the estimated future pay-offs.
6. The average of the present value of the future pay-offs estimates the Asian option price at time $t = 0$.

Simulating an Asian Option's Price and Probability of Exercise After Inception

At any point in time t , where $T > t > 0$, the expected value of the Asian option's price

dependency, \bar{S}_T , is a weighted sum of the following two elements:

$$\begin{aligned}\mathbb{E}[\bar{S}_T] &= w_1 \bar{S}_t + w_2 \mathbb{E}[\bar{S}_T] \\ \mathbb{E}[\bar{S}_T] &= w_1 \frac{1}{t+1} \sum_{i=0}^t S_i + w_2 \mathbb{E}\left[\frac{1}{T-t} \sum_{i=t+1}^T S_i\right]\end{aligned}\tag{8}$$

Where:

$$w_1 = t/T$$

$$w_2 = (T-t)/T$$

S_i = Price of risky asset at time $t = i$.

Algorithm:

1. Compute the arithmetic average price of the underlying asset from S_0 to S_t , where $t = 0$ is the date of the inception of the option, i.e. $\bar{S}_{0,t} = \frac{1}{t} \sum_{i=0}^t S_i$.
2. Simulate many GBM paths from S_t to S_T . Result: Many potential price paths from t to T .
3. For each price evolution in (2), compute the arithmetic average from t to T . Result: Many estimates for $\bar{S}_{t,T}$.
4. Multiply each $\bar{S}_{t,T}$ by $\frac{T-t}{T}$ and add $\frac{t}{T} \bar{S}_{0,t}$. Result: Many estimates for \bar{S}_T .
5. Compute the Asian call option's pay-off, i.e. $V_T^C = \max(S_T - X, 0)$. Result: Many estimates for pay-offs at time T .
6. For (5), compute the proportion of $S_T > X$ via `mean(ST_hats > X)`. This yields the estimate for $\mathbb{P}(S_T > X)$.
7. For (5), discount the estimated pay-offs from time T to t . Result: Many present values of the future estimated pay-offs.
8. The arithmetic average of (7) estimates the option price at time t .
9. Applying the Central Limit Theorem, it is straightforward to estimate a confidence interval for (8) by using the standard error $SE = \frac{s}{\sqrt{n}}$, where n is the number of simulated pay-offs in (7) and s is the sample standard deviation of (7).

For computational efficiency, instead of simulating GBM paths for each point in time, $[t = 0, t = 1, \dots, t = T]$, I am only once simulating 1,000,000 GBMs. To obtain estimates for $[t = 1, \dots, t = T]$, I am consecutively subtracting one day at a time from the end of these paths, and scaling them by S_t/S_0 .

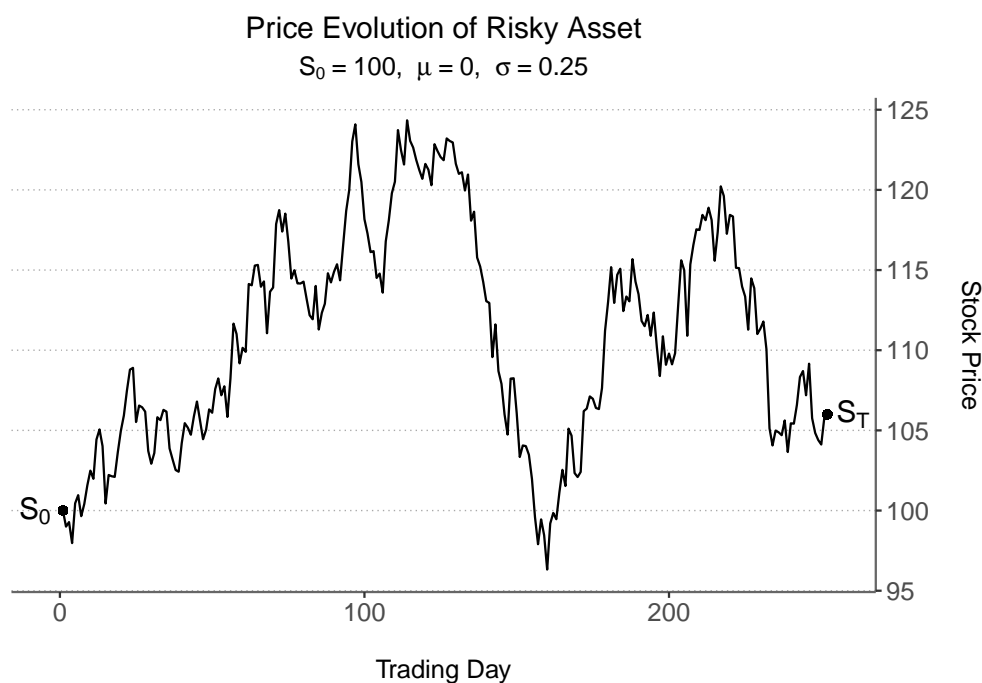
3 Application: Asian Option's Price, Probability of Exercise, and Empirical Delta for a Risky Asset

Consider one risky asset having the following characteristics:

$S_0 = \$100$ Stock price at $t = 0$ is \$100;
 $\sigma = 0.25$ The underlying asset has 25% volatility.

Assume that the risky asset has the following price evolution over 1 calendar year (252 trading days). Further assume that at any point in time t , the future price of the risky asset, S_{t+i} where $i > 0$, is unknown.

Figure 3: Price Evolution of Hypothetical Risky Asset

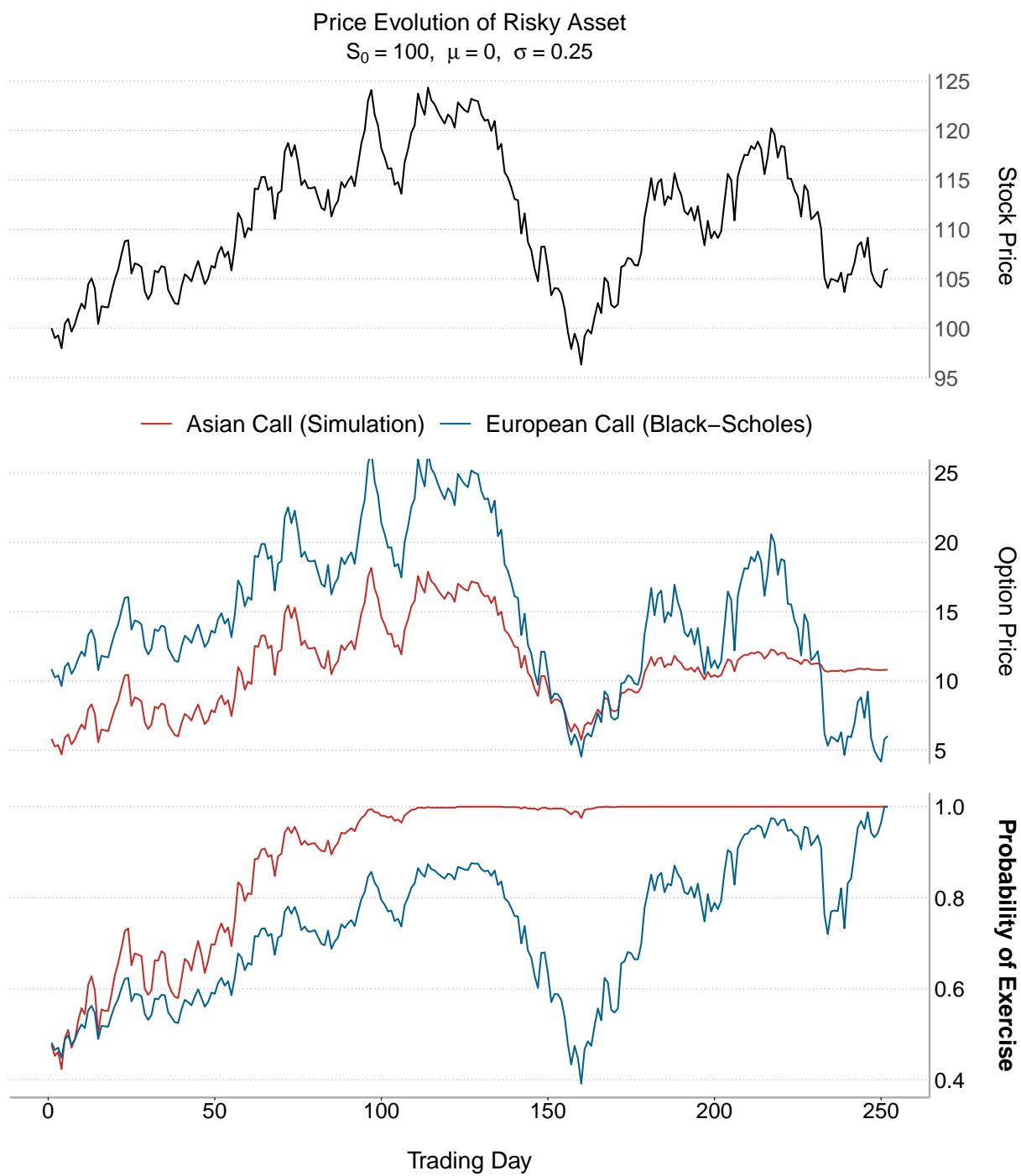


Furthermore, consider one European call option and one Asian call option, both having the same underlying as described above, sharing the following characteristics:

$X = \$100$ The option's strike price is \$100
 $r = 0.02$ Risk-free rate is 2%
 $T - t = 1$ 1 year to expiry

For the risky asset in figure 3, I am using the methods as described in the previous sections to simulate daily prices and probabilities of exercise for both a European and an average rate Asian call option.

Figure 4: European Versus Asian Call Price and Probability of Exercise



3.1 Empirical Delta

Mathematically, an option's delta is the first derivative of the option price with respect to the price of the underlying asset. The discrete-time equivalent is the change in the option price divided by the change in the underlying:

$$\begin{aligned}\text{Delta}_t &= \frac{\partial V_t}{\partial S_t} \\ &\approx \frac{\Delta V_t}{\Delta S_t} = \frac{V_t - V_{t-1}}{S_t - S_{t-1}}\end{aligned}\tag{9}$$

As opposed to the continuous-time derivative, the discrete-time delta in equation (9) is not bound by $[0, 1]$. Empirically, this delta represents the quantity fraction necessary to perfectly hedge a portfolio consisting of an option and its underlying asset. Thus, the empirical delta is found by setting the *change* in this portfolio, ΔPF , to zero:

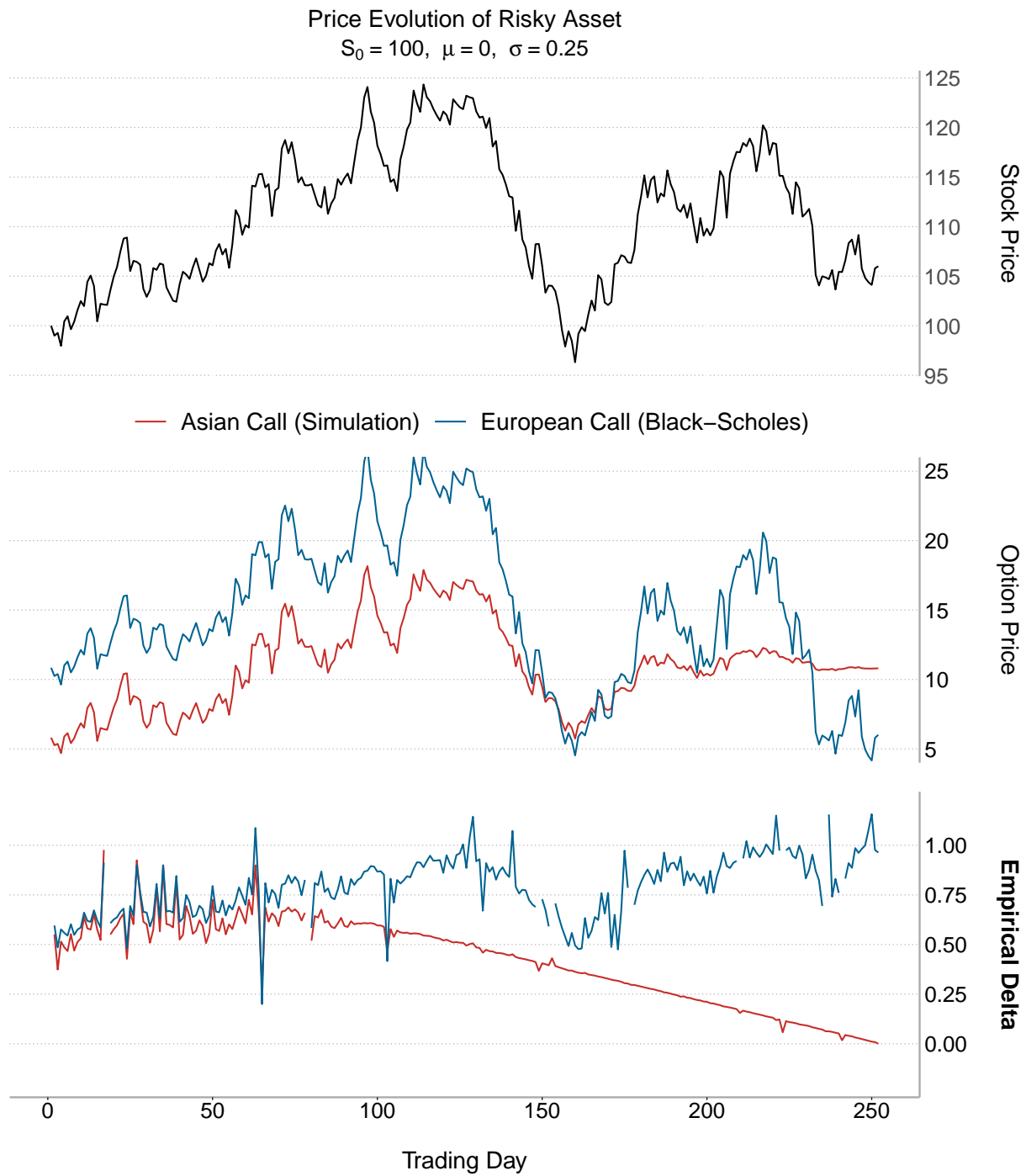
$$\begin{aligned}\Delta PF_t &= \Delta V_t + \text{Delta}_t \Delta S_t \\ 0 &= \Delta V_t + \text{Delta}_t \Delta S_t \\ \text{Delta}_t &= -\frac{\Delta V_t}{\Delta S_t}\end{aligned}$$

Thus, the negative value of the discrete-time delta in equation (9) is the empirical quantity of the underlying asset needed to perfectly hedge the option portfolio.

Below, I am comparing the empirical deltas between European options and Asian options for the same underlying asset as in figures 3 and 4.

Noteworthy, the average rate option's delta goes to zero as the option approaches expiry. As seen in equation (2.1), this is because the option's weight on past prices of the risky asset increases to one as time goes by. This mechanism also explains the behavior of the probability of exercise in figure 4. At a certain point in time, the Asian option's pay-off "sticks" to a specific value, and it becomes even more sticky as time goes by.

Figure 5: European Versus Asian Call Price and Empirical Delta



4 Conclusion

In this paper, I propose methods to estimate vanilla and exotic options' prices and probability of exercise using Monte Carlo simulation in R. The computational efficiency of these methods relies on;

1. Usage of vectorization, and
2. Recycling of the simulated Geometric Brownian Motion paths.

[Section 3](#) illustrates the differences between a European and an average rate Asian option for the same underlying asset. Due to the function applied on the price evolution of the underlying asset, the average rate Asian option generally exposes less variability in its price, probability of exercise, and delta. More specifically, the average rate Asian option has decreasing variability in all investigated dimensions as time goes by. When it comes to the Asian option's delta, it inevitably approaches zero over time, as the underlying's future prices have continuously less impact on the option's pay-off.

In general, the cumulative average transformation of the risky asset's price in the Asian option's pay-off greatly reduces uncertainty. This reduced risk is revealed in lower variability of the option price, delta, and probability of exercise. Naturally, this price transformation also the potential return, explaining the lower option price at inception.

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