Axiomatic Potentialism

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Overview

- Background
- 2 Warm Up: Height Potentialism
- Height and Width Potentialism
- Concluding Remarks

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- 3 Height and Width Potentialism
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Potentialism

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is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

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- Still deeper roots in mathematics in general.

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- **E**.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that S5 at a world V_{κ} is equivalent to Σ_3 correctness of κ .
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and work out its non-modal counterpart.

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Set-theoretic axioms

■ Extensionality, ∈-rigidity, foundation

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Instantiation requires assumption of existence.



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Theorem

 $L \vdash \varphi \text{ implies } ZFC \vdash Tran(X) \rightarrow t(\varphi)(X)$

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Linnebo Interpretation Theorem

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Proof: use mirroring.



Axioms for the theory L

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We have an exact proof-theoretic equivalence, $L \equiv ZFC$.

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Bi-interpretation

In fact, ZFC $\vdash Tran(X) \rightarrow t(\phi^{\diamond})(X) \leftrightarrow \phi$, and conversely.

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- M proves ¬Pow⁵.
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- M proves it is possible for the continuum to exist and have a cardinality at least as great as any N number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply w = z after all, so we have a contradiction.

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$$\square \forall z \square \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

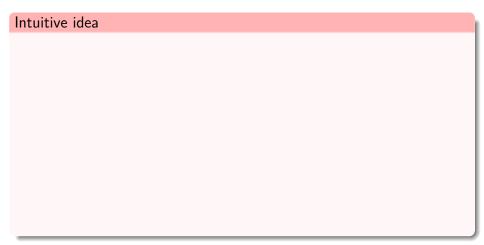
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Amounts to restricting ourselves to parameters that exist at the world of evaluation.



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(From now on, I will ignore the difference between SOA and $ZFC^-+V=HC$. Replacement is formulated as collection.)

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- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for \Diamond_{\uparrow} will involve holding r fixed and climbing transitive sets in L[r];
- while our interpretation for \Diamond_{\leftarrow} will involve allowing new reals to be added but not extending the height of the transitive set parameter.

Let $M \models SOA + \Pi_1^1 PSP$.

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Theorem

 $\mathsf{M} \vdash \varphi \text{ implies } \mathsf{T} \vdash \mathbb{R}(r) \land \mathit{Tran}(X) \land X \in \mathit{L}[r] \rightarrow \mathit{t}(\varphi)(X,r)$

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Theorem

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Proof (sketch)

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What happens with stronger regularity properties? How does \mbox{PD}/\mbox{AD} affect things?

Thanks!