Axiomatic Potentialism

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Overview

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Potentialism

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.
- Ideas like this have deep roots in set theory, e.g. Zermelo and even Cantor
- Still deeper roots in mathematics in general.

Potentialism

- The recent literature has seen two branches of study here:
  1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
  2. Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ.
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is able to do this arbitrarily many times.
- Axiomatize this conception and work out its non-modal counterpart.

Language

The Language $\mathcal{L}_0$

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- $\Box$
- $\in$

Axioms for the theory L

Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X\Diamond\exists x[Set(x, X)]$
3. $\Diamond\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond\exists X\Box\forall x[Xx \leftrightarrow x \subseteq y]$
5. A modal translation of replacement

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \to \Box\psi}{\varphi \to \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \to \forall y\Box\neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \to \Box\neg Set(y, X) \tag{2}$$

$$(Xx \leftrightarrow x \notin x) \to \Box\forall y\neg Set(y, X) \tag{3}$$

Hence the need for free logic.
UI $= \forall x[\forall y\varphi y \to \varphi x]$;
Instantiation requires assumption of existence.

Relative Consistency

- In fact ZFC interprets L.

$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$

- assign plural variables odd numbered variables $t(X)$.
- membership claims $=$ id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T[t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T[t(\varphi)(T)]$
- $t(\Box \varphi)(t) := \forall S \supseteq T[Tran(S) \to t(\varphi)(S)]$

Theorem

L $\vdash \varphi$ implies $ZFC \vdash Tran(X) \to t(\varphi)(X)$

Converse Translation

Mirroring theorem

For φ in $\mathcal{L}_{\in}$, let $\varphi^{\Diamond}$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^{\Diamond} \vdash_{\mathsf{L}} \varphi^{\Diamond}$$

Note on replacement$^{\Diamond}$.

Linnebo Interpretation Theorem

$\mathsf{L} \vdash ZFC^{\Diamond}$.

Proof: use mirroring.

Axioms for the theory L

Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X\Diamond\exists x[Set(x, X)]$
3. $\Diamond\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond\exists X\Box\forall x[Xx \leftrightarrow x \subseteq y]$
5. A modal translation of replacement

Mini-conclusion

Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

Bi-interpretation

In fact, ZFC $\vdash Tran(X) \rightarrow t(\phi^{\diamond})(X) \leftrightarrow \phi$, and conversely.

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Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;
- Or, equivalently, to take some things and add an enumerating function.
- Axiomatize this conception and work out its non-modal counterpart.

Language

The Language $\mathcal{L}_1$

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- $\Box_{\uparrow}$, $\Box_{\leftarrow}$, $\Box$
- $\in$

Axioms for the theory M

Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for X, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond_{\uparrow}\exists x[Set(x, X)]$
3. $\Diamond_{\uparrow}\exists X \Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond_{\uparrow}\exists X \Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$
5. $\Box\forall \mathbb{P}, X[D(\mathbb{P}, X) \to \Diamond_{\leftarrow}\exists g[Fmeets(g, X)]]$
6. A modal translation of replacement

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\Diamond\uparrow}$.
- M interprets ZFC^{-} under the translation $\varphi \mapsto \varphi^{\Diamond}$.
- M proves $\neg Pow^{\Diamond}$.
- M proves $V = HC^{\Diamond}$ and hence $SOA^{\Diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

Inconsistency?

The axioms imply

$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.
By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

By height potentialism/rigidty,

$$\Diamond\exists x\Psi(x, z) \wedge \Diamond_{\uparrow}\exists w\forall y[y \in w \leftrightarrow y \in z]$$

But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

Resolution

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)
Natural solution: restrict comp to closed form:

$$\Box \forall z \Box \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

Amounts to restricting ourselves to parameters that exist at the world of evaluation.

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $\mathsf{ZFC}^{-} + \mathsf{V} = \mathsf{HC}$. Replacement is formulated as collection.)

- We will use the fact that $\mathsf{T} = \mathsf{SOA} + \Pi^1_1\text{-}\mathsf{PSP} \equiv \mathsf{ZFC}$, and in fact T proves that $L[r]$ is a model of ZFC for every real r.
- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for $\Diamond_{\uparrow}$ will involve holding r fixed and climbing transitive sets in $L[r]$;
- while our interpretation for $\Diamond_{\leftarrow}$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership $=$ id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$
- $t(\Box_\uparrow \varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \to t(\varphi)(S, r)]$
- $t(\Box_\leftarrow \varphi)(T, r) := \forall s[r \in L[s] \to \forall S[S \leq T \wedge S \in L[s] \to t(\varphi)(S, s)]]$

Theorem

M $\vdash \varphi$ implies T $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \to t(\varphi)(X, r)$

Converse Translation

Theorem

$\mathsf{M} \vdash \Pi^1_1 PSP^\Diamond$

Proof (sketch)

Given any possible real r, one can show $\Diamond\exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_\uparrow$ translation of ZFC and some absoluteness lemmas. One can also show by forcing that "$\Diamond\mathbb{R}^{L[r]}$ is countable." This yields the $\Diamond$-translation of "$\mathbb{R}^{L[r]}$ exists for every r, and is coutnable." By a result of Solovay, if only countably many reals are constructible from r, then the $\Pi^1_1[r]$ PSP holds. Hence by mirroring $\Pi^1_1[r]PSP^\Diamond$. But r is arbitrary.

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Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $\mathrm{M} \equiv SOA + \Pi^1_1 PSP$. The latter is in fact equiconsistent with ZFC. $\mathrm{M} \equiv \mathrm{L} \equiv ZFC$.

Bi-interpretation

However, SOA $+ \Pi^1_1 PSP \vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\phi^\diamond)(X, r) \leftrightarrow \phi$; whether the converse holds is still open.

Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity.
The 'first order picture' corresponding to the height + width potentialist view may be second order arithmetic + topological regularity.
What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!