

Axiomatic Potentialism

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Overview

- 1 Background
- 2 Warm Up: Height Potentialism
- 3 Height and Width Potentialism
- 4 Concluding Remarks

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- Still deeper roots in mathematics in general.

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- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Instantiation requires assumption of existence.

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Theorem

$L \vdash \varphi$ implies $ZFC \vdash \forall X [Tran(X) \rightarrow t(\varphi)(X)]$

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Proof: use mirroring.

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This is something like what Button calls ‘near-synonymy’.

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- M proves $V = \text{HC}^{\diamond}$ and hence SOA^{\diamond} .
- M proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any \aleph number whose existence is provable in ZFC.

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$$\Diamond \exists x \Psi(x, z) \wedge \Diamond \uparrow \exists w \forall y [y \in w \leftrightarrow y \in z]$$

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond \leftarrow \exists y [y \subseteq x \wedge y = z] \wedge \Box \uparrow \neg \exists y [y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

By comprehension,

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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

The argument just sketched uses comprehension with arbitrary parameters:

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- Our interpretation for \diamond_{\uparrow} will involve holding r fixed and climbing transitive sets in $L[r]$;
- while our interpretation for \diamond_{\leftarrow} will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

$M \vdash \varphi$ implies $T \vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!