Axiomatic Potentialism

Chris Scambler

All Souls College, Oxford University

chris.scambler@all-souls.ox.ac.uk

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Overview

- Background
- 2 Warm Up: Height Potentialism
- Height and Width Potentialism
- Concluding Remarks

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- 3 Height and Width Potentialism
- 4 Concluding Remarks

Potentialism

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is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

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- Still deeper roots in mathematics in general.

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- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- The model-theoretic side has seen thorough examination in the recent literature
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and work out its non-modal counterpart.

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Instantiation requires assumption of existence.



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Note: convention on variables T, S.



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Proof: use mirroring.

Axioms for the theory L

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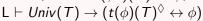
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This is essentially what Button calls 'near-synonymy'.

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- M proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any ℵ number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply w=z after all, so we have a contradiction.

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Amounts to 'restricting' ourselves to parameters that exist at the world of evaluation.

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- Our interpretation for \Diamond_{\uparrow} will involve holding r fixed and climbing transitive sets in L[r];
- while our interpretation for \Diamond_{\leftarrow} will involve allowing new reals to be added but not extending the height of the transitive set parameter.

$$t: \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_{\in}, (\varphi, T, r) \mapsto \psi(T, r)$$

Let $M \models SOA + \Pi_1^1 PSP$.

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Let $M \models SOA + \Pi_1^1 PSP$.

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 $M \vdash \varphi \text{ implies } T \vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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A peculiar sort of necker cube effect. Of philosophical interest?

Thanks!