

# Axiomatic Potentialism

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# Overview

- 1 Background
- 2 Warm Up: Height Potentialism
- 3 Height and Width Potentialism
- 4 Concluding Remarks

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# Potentialism

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- Still deeper roots in mathematics in general.

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- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.



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Instantiation requires assumption of existence.

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## Theorem

$L \vdash \varphi$  implies  $ZFC \vdash \forall X [Tran(X) \rightarrow t(\varphi)(X)]$

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Proof: use mirroring.

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This is something like what Button calls near-synonymy  
(near-near-synonymy, I guess.)

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- $M$  proves  $\neg Pow^{\diamond}$ .
- $M$  proves  $V = HC^{\diamond}$  and hence  $SOA^{\diamond}$ .

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- $M$  proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any  $\aleph$  number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply  $w = z$  after all, so we have a contradiction.

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.



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- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for  $\diamond_{\uparrow}$  will involve holding  $r$  fixed and climbing transitive sets in  $L[r]$ ;
- while our interpretation for  $\diamond_{\leftarrow}$  will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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## Theorem

$M \vdash \varphi$  implies  $T \vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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What happens with stronger regularity properties? How does PD/AD affect things?

# Thanks!