

Axiomatic Potentialism

Chris Scambler

All Souls College,
Oxford University

chris.scambler@all-souls.ox.ac.uk

November 30, 2021

Overview

- 1 Background
- 2 Warm Up: Height Potentialism
- 3 Height and Width Potentialism
- 4 Concluding Remarks

Table of Contents

1 Background

2 Warm Up: Height Potentialism

3 Height and Width Potentialism

4 Concluding Remarks

Potentialism

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.
- Ideas like this have deep roots in set theory, e.g. Zermelo and even Cantor

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.
- Ideas like this have deep roots in set theory, e.g. Zermelo and even Cantor
- Still deeper roots in mathematics in general.

- The recent literature has seen two branches of study here:

- The recent literature has seen two branches of study here:
 - ① Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.

- The recent literature has seen two branches of study here:
 - ① Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
 - ② Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.

Potentialism

- The recent literature has seen two branches of study here:
 - ① Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
 - ② Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.

Potentialism

- The recent literature has seen two branches of study here:
 - ① Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
 - ② Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ .

Potentialism

- The recent literature has seen two branches of study here:
 - ① Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
 - ② Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ .
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

Table of Contents

- 1 Background
- 2 Warm Up: Height Potentialism**
- 3 Height and Width Potentialism
- 4 Concluding Remarks

Motivation

Motivation

- Imagine one has the ability to take things and make a set containing them.

Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is able to do this arbitrarily many times.

Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is able to do this arbitrarily many times.
- Axiomatize this conception and work out its non-modal counterpart.

The Language \mathcal{L}_0

- object variables x, y, z

The Language \mathcal{L}_0

- object variables x, y, z
- plural variables X, Y, Z

The Language \mathcal{L}_0

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$

The Language \mathcal{L}_0

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- \square

The Language \mathcal{L}_0

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- \square
- \in

Axioms for the theory L

Logical Axioms

Axioms for the theory L

Logical Axioms

- 1 Free FO logic

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box \forall X \Diamond \exists x[Set(x, X)]$

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box \forall X \Diamond \exists x[Set(x, X)]$
- 3 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box \forall X \Diamond \exists x[Set(x, X)]$
- 3 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box \forall X \Diamond \exists x[\text{Set}(x, X)]$
- 3 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$
- 5 A modal translation of replacement

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg \text{Set}(y, X) \quad (2)$$

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg \text{Set}(y, X) \quad (2)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg \text{Set}(y, X) \quad (3)$$

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg \text{Set}(y, X) \quad (2)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg \text{Set}(y, X) \quad (3)$$

Hence the need for free logic.

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg \text{Set}(y, X) \quad (2)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg \text{Set}(y, X) \quad (3)$$

Hence the need for free logic.

$$UI = \forall x [\forall y \varphi y \rightarrow \varphi x];$$

Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg \text{Set}(y, X) \quad (1)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg \text{Set}(y, X) \quad (2)$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg \text{Set}(y, X) \quad (3)$$

Hence the need for free logic.

$$UI = \forall x [\forall y \varphi y \rightarrow \varphi x];$$

Instantiation requires assumption of existence.

Relative Consistency

- In fact ZFC interprets L.

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T [t(\varphi)(T)]$

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T [t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T [t(\varphi)(T)]$

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T [t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T [t(\varphi)(T)]$
- $t(\Box \varphi)(t) := \forall S \supseteq T [Tran(S) \rightarrow t(\varphi)(S)]$

Relative Consistency

- In fact ZFC interprets L.

$$t : \mathcal{L}_0 \times V \rightarrow \mathcal{L}_\infty, (\varphi, T) \mapsto \psi(T)$$

- assign plural variables odd numbered variables $t(X)$.
- membership claims = id, commutes with propositional connectives
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T [t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T [t(\varphi)(T)]$
- $t(\Box \varphi)(t) := \forall S \supseteq T [Tran(S) \rightarrow t(\varphi)(S)]$

Theorem

$L \vdash \varphi$ implies $ZFC \vdash Tran(X) \rightarrow t(\varphi)(X)$

Converse Translation

Mirroring theorem

Converse Translation

Mirroring theorem

For φ in \mathcal{L}_\in , let φ^\diamond be the result of prefixing all universal quantifiers by a \Box (and existential quantifiers by \Diamond .) Then we have

Converse Translation

Mirroring theorem

For φ in \mathcal{L}_\in , let φ^\diamond be the result of prefixing all universal quantifiers by a \Box (and existential quantifiers by \Diamond .) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\diamond \vdash_L \varphi^\diamond$$

Converse Translation

Mirroring theorem

For φ in \mathcal{L}_\in , let φ^\diamond be the result of prefixing all universal quantifiers by a \Box (and existential quantifiers by \Diamond .) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\diamond \vdash_L \varphi^\diamond$$

Note on replacement $^\diamond$.

Converse Translation

Mirroring theorem

For φ in \mathcal{L}_\in , let φ^\diamond be the result of prefixing all universal quantifiers by a \Box (and existential quantifiers by \Diamond .) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\diamond \vdash_L \varphi^\diamond$$

Note on replacement $^\diamond$.

Linnebo Interpretation Theorem

$L \vdash ZFC^\diamond$.

Converse Translation

Mirroring theorem

For φ in \mathcal{L}_\in , let φ^\diamond be the result of prefixing all universal quantifiers by a \Box (and existential quantifiers by \Diamond .) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\diamond \vdash_L \varphi^\diamond$$

Note on replacement $^\diamond$.

Linnebo Interpretation Theorem

$L \vdash ZFC^\diamond$.

Proof: use mirroring.

Axioms for the theory L

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF
- 3 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box \forall X \Diamond \exists x[\text{Set}(x, X)]$
- 3 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$
- 5 A modal translation of replacement

Mini-conclusion

Mini-conclusion

Equivalence

We have an exact proof-theoretic equivalence, $L \equiv ZFC$.

Mini-conclusion

Equivalence

We have an exact proof-theoretic equivalence, $L \equiv ZFC$.

Bi-interpretation

In fact, $ZFC \vdash Tran(X) \rightarrow t(\phi^\diamond)(X) \leftrightarrow \phi$, and conversely.

Table of Contents

- 1 Background
- 2 Warm Up: Height Potentialism
- 3 Height and Width Potentialism**
- 4 Concluding Remarks

Motivation

Motivation

- Imagine one has the ability to take things and make a set containing them.

Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;

Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;
- Or, equivalently, to take some things and add an enumerating function.

Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;
- Or, equivalently, to take some things and add an enumerating function.
- Axiomatize this conception and work out its non-modal counterpart.

The Language \mathcal{L}_1

- object variables x, y, z

The Language \mathcal{L}_1

- object variables x, y, z
- plural variables X, Y, Z

The Language \mathcal{L}_1

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$

The Language \mathcal{L}_1

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- $\Box_{\uparrow}, \Box_{\leftarrow}, \Box$

The Language \mathcal{L}_1

- object variables x, y, z
- plural variables X, Y, Z
- $\wedge, \neg, \forall, =$
- $\Box_{\uparrow}, \Box_{\leftarrow}, \Box$
- \in

Axioms for the theory M

Logical Axioms

Axioms for the theory M

Logical Axioms

- 1 Free FO logic

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box\uparrow\varphi$, same for $\Box\leftarrow$.
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box\uparrow\varphi$, same for $\Box\leftarrow$.
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box\forall X\Diamond_{\uparrow}\exists x[\text{Set}(x, X)]$

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box\forall X\Diamond_{\uparrow}\exists x[\text{Set}(x, X)]$
- 3 $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box\forall X\Diamond_{\uparrow}\exists x[\text{Set}(x, X)]$
- 3 $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond_{\uparrow}\exists X\Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box\forall X\Diamond_{\uparrow}\exists x[\text{Set}(x, X)]$
- 3 $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond_{\uparrow}\exists X\Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$
- 5 $\Box\forall\mathbb{P}, X[D(\mathbb{P}, X) \rightarrow \Diamond_{\leftarrow}\exists g[Fmeets(g, X)]]$

Axioms for the theory M

Logical Axioms

- 1 Free FO logic
- 2 S4.2 modal logic + CBF for each modal
- 3 $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for \Box_{\leftarrow} .
- 4 $\forall z[Xz \leftrightarrow Yz] \rightarrow X = Y$, rigidity, Choice, Comp

Set-theoretic axioms

- 1 Extensionality, \in -rigidity, foundation
- 2 $\Box\forall X\Diamond_{\uparrow}\exists x[\text{Set}(x, X)]$
- 3 $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
- 4 $\Diamond_{\uparrow}\exists X\Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$
- 5 $\Box\forall \mathbb{P}, X[D(\mathbb{P}, X) \rightarrow \Diamond_{\leftarrow}\exists g[Fmeets(g, X)]]$
- 6 A modal translation of replacement

Some basic facts

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.
- M interprets ZFC^- under the translation $\varphi \mapsto \varphi^{\diamond}$.

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.
- M interprets ZFC^- under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves $\neg \text{Pow}^{\diamond}$.

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^\diamond \uparrow$.
- M interprets ZFC^- under the translation $\varphi \mapsto \varphi^\diamond$.
- M proves $\neg Pow^\diamond$.
- M proves $V = HC^\diamond$ and hence SOA^\diamond .

Some basic facts

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.
- M interprets ZFC^- under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves $\neg \text{Pow}^{\diamond}$.
- M proves $V = \text{HC}^{\diamond}$ and hence SOA^{\diamond} .
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any \aleph number whose existence is provable in ZFC.

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond_{\leftarrow} \exists y [y \subseteq x \wedge y = z] \wedge \Box_{\uparrow} \neg \exists y [y = z])$$

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond_{\leftarrow} \exists y [y \subseteq x \wedge y = z] \wedge \Box_{\uparrow} \neg \exists y [y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond \leftarrow \exists y [y \subseteq x \wedge y = z] \wedge \Box \uparrow \neg \exists y [y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

By comprehension,

$$\Diamond \exists x \Psi(x, z) \wedge \exists X \forall y [Xy \leftrightarrow y \in z]$$

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond \leftarrow \exists y [y \subseteq x \wedge y = z] \wedge \Box \uparrow \neg \exists y [y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

By comprehension,

$$\Diamond \exists x \Psi(x, z) \wedge \exists X \forall y [Xy \leftrightarrow y \in z]$$

By height potentialism/rigidity,

$$\Diamond \exists x \Psi(x, z) \wedge \Diamond \uparrow \exists w \forall y [y \in w \leftrightarrow y \in z]$$

Inconsistency?

The axioms imply

$$\Diamond \exists x (\Diamond \leftarrow \exists y [y \subseteq x \wedge y = z] \wedge \Box \uparrow \neg \exists y [y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

By comprehension,

$$\Diamond \exists x \Psi(x, z) \wedge \exists X \forall y [Xy \leftrightarrow y \in z]$$

By height potentialism/rigidity,

$$\Diamond \exists x \Psi(x, z) \wedge \Diamond \uparrow \exists w \forall y [y \in w \leftrightarrow y \in z]$$

But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)
Natural solution: restrict comp to closed form:

$$\Box \forall z \Box \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)
Natural solution: restrict comp to closed form:

$$\Box \forall z \Box \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

Amounts to restricting ourselves to parameters that exist at the world of evaluation.

Relative Consistency

Intuitive idea

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $\text{ZFC}^- + \text{V} = \text{HC}$. Replacement is formulated as collection.)

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $ZFC^- + V = HC$. Replacement is formulated as collection.)

- We will use the fact that $T = SOA + \Pi_1^1\text{-PSP} \equiv ZFC$, and in fact T proves that $L[r]$ is a model of ZFC for every real r .

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $ZFC^- + V = HC$. Replacement is formulated as collection.)

- We will use the fact that $T = SOA + \Pi_1^1\text{-PSP} \equiv ZFC$, and in fact T proves that $L[r]$ is a model of ZFC for every real r .
- Our translation will be doubly parameterized, once by a real and once by a transitive set.

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $ZFC^- + V = HC$. Replacement is formulated as collection.)

- We will use the fact that $T = SOA + \Pi_1^1\text{-PSP} \equiv ZFC$, and in fact T proves that $L[r]$ is a model of ZFC for every real r .
- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for \Diamond_\uparrow will involve holding r fixed and climbing transitive sets in $L[r]$;

Relative Consistency

Intuitive idea

(From now on, I will ignore the difference between SOA and $ZFC^- + V = HC$. Replacement is formulated as collection.)

- We will use the fact that $T = SOA + \Pi_1^1\text{-PSP} \equiv ZFC$, and in fact T proves that $L[r]$ is a model of ZFC for every real r .
- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for \diamond_{\uparrow} will involve holding r fixed and climbing transitive sets in $L[r]$;
- while our interpretation for \diamond_{\leftarrow} will involve allowing new reals to be added but not extending the height of the transitive set parameter.

Relative consistency

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \rightarrow t(\varphi)(T, r)]$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x\varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X\varphi)(T, r) := \forall x \subseteq T[x \in L[r] \rightarrow t(\varphi)(T, r)]$
- $t(\Box_{\uparrow}\varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \rightarrow t(\varphi)(S, r)]$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x\varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X\varphi)(T, r) := \forall x \subseteq T[x \in L[r] \rightarrow t(\varphi)(T, r)]$
- $t(\Box_{\uparrow}\varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \rightarrow t(\varphi)(S, r)]$
- $t(\Box_{\leftarrow}\varphi)(T, r) := \forall s[r \in L[s] \rightarrow \forall S[S \leq T \wedge S \in L[s] \rightarrow t(\varphi)(S, s)]]$

Relative consistency

Let $M \models SOA + \Pi_1^1PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \rightarrow \mathcal{L}_\infty, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x\varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X\varphi)(T, r) := \forall x \subseteq T[x \in L[r] \rightarrow t(\varphi)(T, r)]$
- $t(\Box_{\uparrow}\varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \rightarrow t(\varphi)(S, r)]$
- $t(\Box_{\leftarrow}\varphi)(T, r) := \forall s[r \in L[s] \rightarrow \forall S[S \leq T \wedge S \in L[s] \rightarrow t(\varphi)(S, s)]]$

Theorem

$M \vdash \varphi$ implies $T \vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\varphi)(X, r)$

Converse Translation

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas.

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas. One can also show by forcing that “ $\diamond \mathbb{R}^{L[r]}$ is countable.”

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas. One can also show by forcing that “ $\diamond \mathbb{R}^{L[r]}$ is countable.” This yields the \diamond -translation of “ $\mathbb{R}^{L[r]}$ exists for every r , and is countable.”

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas. One can also show by forcing that “ $\diamond \mathbb{R}^{L[r]}$ is countable.” This yields the \diamond -translation of “ $\mathbb{R}^{L[r]}$ exists for every r , and is countable.” By a result of Solovay, if only countably many reals are constructible from r , then the $\Pi_1^1[r]$ PSP holds.

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas. One can also show by forcing that “ $\diamond \mathbb{R}^{L[r]}$ is countable.” This yields the \diamond -translation of “ $\mathbb{R}^{L[r]}$ exists for every r , and is countable.” By a result of Solovay, if only countably many reals are constructible from r , then the $\Pi_1^1[r]$ PSP holds. Hence by mirroring $\Pi_1^1[r] PSP^\diamond$.

Converse Translation

Theorem

$$M \vdash \Pi_1^1 PSP^\diamond$$

Proof (sketch)

Given any possible real r , one can show $\diamond \exists x [x = \mathbb{R}^{L[r]}]$ using the \diamond_\uparrow translation of ZFC and some absoluteness lemmas. One can also show by forcing that “ $\diamond \mathbb{R}^{L[r]}$ is countable.” This yields the \diamond -translation of “ $\mathbb{R}^{L[r]}$ exists for every r , and is countable.” By a result of Solovay, if only countably many reals are constructible from r , then the $\Pi_1^1[r]$ PSP holds. Hence by mirroring $\Pi_1^1[r] PSP^\diamond$. But r is arbitrary.

Table of Contents

- 1 Background
- 2 Warm Up: Height Potentialism
- 3 Height and Width Potentialism
- 4 Concluding Remarks**

Conclusions

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.
The latter is in fact equiconsistent with ZFC. $M \equiv L \equiv ZFC$.

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.
The latter is in fact equiconsistent with ZFC. $M \equiv L \equiv ZFC$.

Bi-interpretation

However, $SOA + \Pi_1^1PSP$
 $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\phi^\diamond)(X, r) \leftrightarrow \phi$; whether the converse holds is still open.

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.
The latter is in fact equiconsistent with ZFC. $M \equiv L \equiv ZFC$.

Bi-interpretation

However, $SOA + \Pi_1^1PSP$
 $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\phi^\diamond)(X, r) \leftrightarrow \phi$; whether the converse holds is still open.

Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity.

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.
The latter is in fact equiconsistent with ZFC. $M \equiv L \equiv ZFC$.

Bi-interpretation

However, $SOA + \Pi_1^1PSP$
 $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\phi^\diamond)(X, r) \leftrightarrow \phi$; whether the converse holds is still open.

Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity.

The 'first order picture' corresponding to the height + width potentialist view may be second order arithmetic + topological regularity.

Conclusions

Equivalence

We have an exact proof-theoretic equivalence, $M \equiv SOA + \Pi_1^1PSP$.
The latter is in fact equiconsistent with ZFC. $M \equiv L \equiv ZFC$.

Bi-interpretation

However, $SOA + \Pi_1^1PSP$
 $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \rightarrow t(\phi^\diamond)(X, r) \leftrightarrow \phi$; whether the converse holds is still open.

Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity.

The 'first order picture' corresponding to the height + width potentialist view may be second order arithmetic + topological regularity.

What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!