Axiomatic Potentialism

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Overview

Background

2 Warm Up: Height Potentialism

Height and Width Potentialism

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3 Height and Width Potentialism

Potentialism

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

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- Still deeper roots in mathematics in general.

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- **E**.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that S5 at a world V_{κ} is equivalent to Σ_3 correctness of κ .
- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and relate it to standard set theory.

The Language \mathcal{L}_0

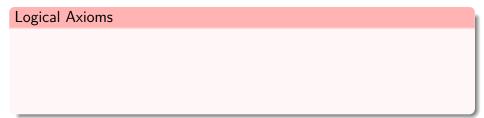
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Logical Axioms

Free FO logic

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- 2 S4.2 modal logic + CBF

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- § Ext for X, $\Diamond Xx \to \Box Xx$, $\Diamond \exists x[Xx \land x = y] \to \exists x[Xx \land x = y]$, Choice, Comp

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$$\frac{\varphi \to \Box \psi}{\varphi \to \Box \forall x \psi}$$

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Standard modal model theory validates the rule

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Hence the need for free logic.

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Proof: use mirroring.

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$$t: \mathcal{L}_0 \times V \to \mathcal{L}_{\in}, (\varphi, T) \mapsto \psi(T)$$

In fact ZFC interprets L as well.

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- $\qquad \mathsf{t}(\Box\varphi)(\alpha) := \forall \mathsf{S} \supseteq \mathsf{T}[\mathsf{\mathit{TranS}} \to \mathsf{t}(\varphi)(\mathsf{S})]$

Theorem

 $\mathsf{L} \vdash \varphi \text{ implies } \mathsf{ZFC} \vdash \mathsf{t}(\varphi)(\emptyset)$



Mini-conclusion

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Equivalence

We have an exact proof-theoretic equivalence, $L \equiv ZFC$.

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Inconsistency?

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A Model

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Connection to Standard Set Theory

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References



John Smith (2012) Title of the publication

Journal Name 12(3), 45 - 678.

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Thanks!