Axiomatic Potentialism

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3. Height and Width Potentialism

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- Still deeper roots in mathematics in general.

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- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ.
- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and relate it to standard set theory.

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Hence the need for free logic.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(\emptyset)$

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Proof: use mirroring.

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Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

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- M interprets ZFC^- under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves $\neg Pow^{\diamond}$.
- M proves $V = HC^{\diamond}$ and hence $SOA^{\diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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The axioms imply

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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

Resolution

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And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

M $\vdash \varphi$ implies T $\vdash t(\varphi)(\emptyset, 0)$

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References

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Thanks!