Axiomatic Potentialism

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Overview

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- Still deeper roots in mathematics in general.

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- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Instantiation requires assumption of existence.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash Tran(X) \to t(\varphi)(X)$

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Proof: use mirroring.

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This is something like what Button calls near-synonymy (near-near-synonmy, I guess.)

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- M proves $V = HC^{\Diamond}$ and hence $SOA^{\Diamond}$.
- M proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.
By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

By height potentialism/rigidty,

$$\Diamond\exists x\Psi(x, z) \wedge \Diamond_{\uparrow}\exists w\forall y[y \in w \leftrightarrow y \in z]$$

But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

Resolution

The argument just sketched uses comprehension with arbitrary parameters:

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

M $\vdash \varphi$ implies T $\vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!