Axiomatic Potentialism

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3. Height and Width Potentialism

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- Still deeper roots in mathematics in general.

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- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Instantiation requires assumption of existence.

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Proof: use mirroring.

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This is essentially what Button calls ‘near-synonymy’.

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- M proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

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Amounts to 'restricting' ourselves to parameters that exist at the world of evaluation.

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- The Γ-PSP implies no formula in Γ defines a well-order of the reals.
- Γ-definable sets are thus 'regular' or 'well behaved'.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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M $\vdash \varphi$ implies T $\vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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The exact details of $\Diamond_{\leftarrow}$ are incidental anyway; if we formulate the theory with just $\Diamond$ and $\Diamond_{\uparrow}$, we will get near synonymy.

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- In the modal theory, this corresponds to the $\Diamond_{\uparrow}$-possibility of large cardinals.

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- The potentialist perspective enshrined in M, and the regularity perspective in T, provide two tightly equivalent ways of fleshing out the alternative viewpoint.

Thanks!