Axiomatic Potentialism

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Overview

1. Background
2. Warm Up: Height Potentialism
3. Height and Width Potentialism
4. Concluding Remarks

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- Still deeper roots in mathematics in general.

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- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ.
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Instantiation requires assumption of existence.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(\emptyset)$

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Proof: use mirroring.

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Bi-interpretation

In fact, ZFC $\vdash t(\phi^\diamond)(\emptyset) \leftrightarrow \phi$, and conversely.

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- M proves $\neg Pow^{\diamond}$.
- M proves $V = HC^{\diamond}$ and hence $SOA^{\diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

Resolution

The argument just sketched uses comprehension with arbitrary parameters:

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

M $\vdash \varphi$ implies T $\vdash t(\varphi)(\emptyset, 0)$

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What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!