Axiomatic Potentialism

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Overview

Background

2 Warm Up: Height Potentialism

Height and Width Potentialism

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3 Height and Width Potentialism

Potentialism

Potentialism

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

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- Still deeper roots in mathematics in general.

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- **E**.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that S5 at a world V_{κ} is equivalent to Σ_3 correctness of κ .
- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and relate it to standard set theory.

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Logical Axioms

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Free FO logic

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$$\frac{\varphi \to \Box \psi}{\varphi \to \Box \forall x \psi}$$

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Standard modal model theory validates the rule

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Hence the need for free logic.

$$t: \mathcal{L}_0 \times V \to \mathcal{L}_{\in}, (\varphi, T) \mapsto \psi(T)$$

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Theorem

 $\mathsf{L} \vdash \varphi \text{ implies } \mathit{ZFC} \vdash \mathit{t}(\varphi)(\emptyset)$



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Linnebo Interpretation Theorem

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Proof: use mirroring.

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Mini-conclusion

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Equivalence

We have an exact proof-theoretic equivalence, $L \equiv ZFC$.

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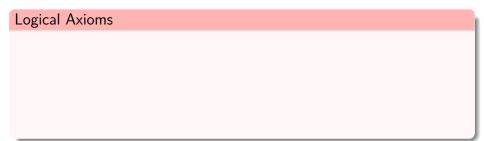
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■ M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond \uparrow}$.

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- M interprets ZFC⁻ under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves ¬Pow⁵.
- M proves $V = HC^{\diamond}$ and hence SOA^{\diamond} .
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any ℵ number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply w=z after all, so we have a contradiction.

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$$\square \forall z \square \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.



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- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for \Diamond_{\uparrow} will involve holding r fixed and climbing transitive sets in L[r];
- while our interpretation for \Diamond_{\leftarrow} will involve allowing new reals to be added but not extending the height of the transitive set parameter.

$$t: \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_{\in}, (\varphi, T, r) \mapsto \psi(T, r)$$

Let
$$M \models SOA + \Pi_1^1 PSP$$
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Theorem

 $M \vdash \varphi \text{ implies } T \vdash t(\varphi)(\emptyset, 0)$

References



John Smith (2012) Title of the publication

Journal Name 12(3), 45 - 678.

Thanks!