Axiomatic Potentialism

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November 22, 2021

Overview

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- Still deeper roots in mathematics in general.

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- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(\emptyset)$

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Proof: use mirroring.

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Mini-conclusion

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Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

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- M interprets ZFC^{-} under the translation $\varphi \mapsto \varphi^{\Diamond}$.
- M proves $\neg Pow^{\Diamond}$.
- M proves $V = HC^{\Diamond}$ and hence $SOA^{\Diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

By height potentialism/rigidty,

$$\Diamond\exists x\Psi(x, z) \wedge \Diamond_{\uparrow}\exists w\forall y[y \in w \leftrightarrow y \in z]$$

But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

Resolution

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

M $\vdash \varphi$ implies T $\vdash t(\varphi)(\emptyset, 0)$

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Table of Contents

Conclusions

Equi Consistency bi-interpretation height-width fungibility: determinacy vs large cardinals

Thanks!