# Axiomatic Potentialism

Chris Scambler

All Souls College,
Oxford University

*chris.scambler@all-souls.ox.ac.uk*

November 23, 2021

# Overview

1. Background
2. Warm Up: Height Potentialism
3. Height and Width Potentialism
4. Concluding Remarks

# Table of Contents

# Potentialism

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.
- The idea has deep roots in set theory, e.g. Zermelo and even Cantor

# Potentialism

**Potentialism**

is the idea that a mathematical object (e.g. a set) is the sort of thing that may *merely possibly* exist.

- E.g. a geometric object as a figure one can construct
- A set as a certain sort of data structure one could assemble
- Or perhaps a structure that is instantiated given enough objects.
- The idea has deep roots in set theory, e.g. Zermelo and even Cantor
- Still deeper roots in mathematics in general.

# Potentialism

- The recent literature has seen two branches of study here:

# Potentialism

- The recent literature has seen two branches of study here:
  1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.

# Potentialism

- The recent literature has seen two branches of study here:
  1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
  2. Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.

# Potentialism

- The recent literature has seen two branches of study here:
    1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
    2. Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.

# Potentialism

- The recent literature has seen two branches of study here:
  1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
  2. Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of $V$ that $S5$ at a world $V_\kappa$ is equivalent to $\Sigma_3$ correctness of $\kappa$.

# Potentialism

- The recent literature has seen two branches of study here:
  1. Model-theoretic: study Kripke models whose worlds are structures with the accessibility relation (some refinement of) the substructure relation.
  2. Axiomatic: Develop axiom systems designed to characterize this or that form of potentialism directly, without appeal to models.
- In each case interesting questions arise concerning the relation between assertions in the modal framework and in first order set theory.
- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of $V$ that $S5$ at a world $V_\kappa$ is equivalent to $\Sigma_3$ correctness of $\kappa$.
- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

# Table of Contents

# Motivation

# Motivation

- Imagine one has the ability to take things and make a set containing them.

# Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is able to do this arbitrarily many times.

# Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is able to do this arbitrarily many times.
- Axiomatize this conception and relate it to standard set theory.

# Language

**The Language $\mathcal{L}_0$**

- object variables $x, y, z$

# Language

**The Language $\mathcal{L}_0$**

- object variables $x, y, z$
- plural variables $X, Y, Z$

# Language

## The Language $\mathcal{L}_0$

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$

# Language

## The Language $\mathcal{L}_0$

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$
- $\Box$

# Language

**The Language $\mathcal{L}_0$**

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$
- $\Box$
- $\in$

# Axioms for the theory L

## Logical Axioms

# Axioms for the theory L

## Logical Axioms

1. Free FO logic

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond\exists x[Set(x, X)]$

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box \forall X \Diamond \exists x[Set(x, X)]$

NB: $\Box \exists X \neg \exists y[Set(x, X)]$

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond\exists x[Set(x, X)]$
3. $\Diamond\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$

NB: $\Box\exists X\neg\exists y[Set(x, X)]$

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box \forall X \Diamond \exists x[Set(x, X)]$
3. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$

NB: $\Box \exists X \neg \exists y[Set(x, X)]$

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box \forall X \Diamond \exists x[Set(x, X)]$
3. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$
5. A modal translation of replacement

NB: $\Box \exists X \neg \exists y[Set(x, X)]$

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y\Box\neg Set(y, X) \tag{1}$$

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg Set(y, X) \tag{2}$$

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg Set(y, X) \tag{2}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg Set(y, X) \tag{3}$$

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \to \Box\psi}{\varphi \to \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \to \forall y\Box\neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \to \Box\neg Set(y, X) \tag{2}$$

$$(Xx \leftrightarrow x \notin x) \to \Box\forall y\neg Set(y, X) \tag{3}$$

Hence the need for free logic.

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \rightarrow \Box\psi}{\varphi \rightarrow \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \forall y \Box \neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \neg Set(y, X) \tag{2}$$

$$(Xx \leftrightarrow x \notin x) \rightarrow \Box \forall y \neg Set(y, X) \tag{3}$$

Hence the need for free logic.
UI $= \forall x[\forall y \varphi y \rightarrow \varphi x]$;

# Inconsistency?

Standard modal model theory validates the rule

$$\frac{\varphi \to \Box\psi}{\varphi \to \Box\forall x\psi}$$

$$(Xx \leftrightarrow x \notin x) \to \forall y \Box \neg Set(y, X) \tag{1}$$

$$(Xx \leftrightarrow x \notin x) \to \Box \neg Set(y, X) \tag{2}$$

$$(Xx \leftrightarrow x \notin x) \to \Box \forall y \neg Set(y, X) \tag{3}$$

Hence the need for free logic.
UI $= \forall x[\forall y \varphi y \to \varphi x]$;
Instantiation requires assumption of existence.

# Relative Consistency

- In fact ZFC interprets L.

## Relative Consistency

- In fact ZFC interprets L.

$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$

# Relative Consistency

- In fact ZFC interprets L.

$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$

- assign plural variables odd numbered variables $t(X)$.

# Relative Consistency

- In fact ZFC interprets L.

**$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id

# Relative Consistency

- In fact ZFC interprets L.

$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T) := x \in t(X)$

## Relative Consistency

- In fact ZFC interprets L.

**$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives = id
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T[t(\varphi)(T)]$

# Relative Consistency

- In fact ZFC interprets L.

**$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives = id
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x\varphi)(T) := \forall x \in T[t(\varphi)(T)]$
- $t(\forall X\varphi)(T) := \forall x \subseteq T[t(\varphi)(T)]$

# Relative Consistency

- In fact ZFC interprets L.

**$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T[t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T[t(\varphi)(T)]$
- $t(\Box\varphi)(t) := \forall S \supseteq T[Tran(S) \to t(\varphi)(S)]$

## Relative Consistency

- In fact ZFC interprets L.

**$t : \mathcal{L}_0 \times V \to \mathcal{L}_\in, (\varphi, T) \mapsto \psi(T)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives = id
- $t(Xx)(T) := x \in t(X)$
- $t(\forall x \varphi)(T) := \forall x \in T[t(\varphi)(T)]$
- $t(\forall X \varphi)(T) := \forall x \subseteq T[t(\varphi)(T)]$
- $t(\Box \varphi)(t) := \forall S \supseteq T[Tran(S) \to t(\varphi)(S)]$

**Theorem**

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(\emptyset)$

# Converse Translation

# Converse Translation

**Mirroring theorem**

# Converse Translation

**Mirroring theorem**

For $\varphi$ in $\mathcal{L}_{\in}$, let $\varphi^{\Diamond}$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

# Converse Translation

**Mirroring theorem**

For $\varphi$ in $\mathcal{L}_\in$, let $\varphi^\Diamond$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\Diamond \vdash_{\mathsf{L}} \varphi^\Diamond$$

# Converse Translation

## Mirroring theorem

For $\varphi$ in $\mathcal{L}_{\in}$, let $\varphi^{\diamond}$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^{\diamond} \vdash_{\mathsf{L}} \varphi^{\diamond}$$

Note on replacement$^{\diamond}$.

# Converse Translation

## Mirroring theorem

For $\varphi$ in $\mathcal{L}_{\in}$, let $\varphi^{\Diamond}$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^{\Diamond} \vdash_{\mathsf{L}} \varphi^{\Diamond}$$

Note on replacement$^{\Diamond}$.

## Linnebo Interpretation Theorem

$\mathsf{L} \vdash ZFC^{\Diamond}$.

# Converse Translation

**Mirroring theorem**

For $\varphi$ in $\mathcal{L}_\in$, let $\varphi^\diamond$ be the result of prefixing all universal quantifiers by a $\Box$ (and existential quantifiers by $\Diamond$.) Then we have

$$\Gamma \vdash_{FOL} \varphi \Leftrightarrow \Gamma^\diamond \vdash_{\mathsf{L}} \varphi^\diamond$$

Note on replacement$^\diamond$.

**Linnebo Interpretation Theorem**

$\mathsf{L} \vdash ZFC^\diamond$.

Proof: use mirroring.

# Axioms for the theory L

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF
3. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond \exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box \forall X \Diamond \exists x[Set(x, X)]$
3. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y]$
5. A modal translation of replacement

# Mini-conclusion

# Mini-conclusion

**Equivalence**

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

# Mini-conclusion

**Equivalence**

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

**Bi-interpretation**

In fact, $\mathsf{ZFC} \vdash t(\phi^{\diamond})(\emptyset) \leftrightarrow \phi$, and conversely.

# Table of Contents

# Motivation

# Motivation

- Imagine one has the ability to take things and make a set containing them.

# Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;

# Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;
- Or, equivalently, to take some things and add an enumerating function.

# Motivation

- Imagine one has the ability to take things and make a set containing them.
- Imagine one is also able to take a partial order and add a filter meeting all its (current) dense sets;
- Or, equivalently, to take some things and add an enumerating function.
- Axiomatize this conception and relate it to standard set theory.

# Language

## The Language $\mathcal{L}_1$

- object variables $x, y, z$

# Language

**The Language $\mathcal{L}_1$**

- object variables $x, y, z$
- plural variables $X, Y, Z$

# Language

## The Language $\mathcal{L}_1$

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$

# Language

## The Language $\mathcal{L}_1$

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$
- $\Box_{\uparrow}$, $\Box_{\leftarrow}$, $\Box$

# Language

## The Language $\mathcal{L}_1$

- object variables $x, y, z$
- plural variables $X, Y, Z$
- $\wedge, \neg, \forall, =$
- $\Box_{\uparrow}$, $\Box_{\leftarrow}$, $\Box$
- $\in$

# Axioms for the theory M

## Logical Axioms

# Axioms for the theory M

## Logical Axioms

1. Free FO logic

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \rightarrow \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \rightarrow \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \rightarrow \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond_{\uparrow}\exists x[Set(x, X)]$

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond_{\uparrow}\exists x[Set(x, X)]$
3. $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond_{\uparrow}\exists x[Set(x, X)]$
3. $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond_{\uparrow}\exists X\Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X\Diamond_{\uparrow}\exists x[Set(x, X)]$
3. $\Diamond_{\uparrow}\exists X\Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond_{\uparrow}\exists X\Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$
5. $\Box\forall x, X[D(x, X) \to \Diamond_{\leftarrow}\exists g[Fmeets(g, xx)]]$

# Axioms for the theory M

## Logical Axioms

1. Free FO logic
2. S4.2 modal logic + CBF for each modal
3. $\Box\varphi \to \Box_{\uparrow}\varphi$, same for $\Box_{\leftarrow}$.
4. Ext for $X$, $\Diamond Xx \to \Box Xx$, $\Diamond\exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]$, Choice, Comp

## Set-theoretic axioms

1. Extensionality, $\in$-rigidity, foundation
2. $\Box\forall X \Diamond_{\uparrow}\exists x[Set(x, X)]$
3. $\Diamond_{\uparrow}\exists X \Box\forall x[Xx \leftrightarrow \mathbb{N}x]$
4. $\Diamond_{\uparrow}\exists X \Box_{\uparrow}\forall x[Xx \leftrightarrow x \subseteq y]$
5. $\Box\forall x, X[D(x, X) \to \Diamond_{\leftarrow}\exists g[Fmeets(g, xx)]]$
6. A modal translation of replacement

# Some initial observations

# Some initial observations

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\Diamond\uparrow}$.

# Some initial observations

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\Diamond\uparrow}$.
- M interprets ZFC$^-$ under the translation $\varphi \mapsto \varphi^{\Diamond}$.

# Some initial observations

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\Diamond\uparrow}$.
- M interprets ZFC$^{-}$ under the translation $\varphi \mapsto \varphi^{\Diamond}$.
- M proves $\neg Pow^{\Diamond}$.

# Some initial observations

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.
- M interprets $\text{ZFC}^{-}$ under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves $\neg Pow^{\diamond}$.
- M proves $V = HC^{\diamond}$ and hence $SOA^{\diamond}$.

# Some initial observations

- M interprets ZFC under the translation $\varphi \mapsto \varphi^{\diamond\uparrow}$.
- M interprets ZFC$^{-}$ under the translation $\varphi \mapsto \varphi^{\diamond}$.
- M proves $\neg Pow^{\diamond}$.
- M proves $V = HC^{\diamond}$ and hence $SOA^{\diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

# Inconsistency?

The axioms imply

$$\Diamond \exists x(\Diamond_{\leftarrow} \exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow} \neg \exists y[y = z])$$

# Inconsistency?

The axioms imply

$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.

# Inconsistency?

The axioms imply

$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.
By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

# Inconsistency?

The axioms imply

$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.
By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

By height potentialism/rigidty,

$$\Diamond\exists x\Psi(x, z) \wedge \Diamond_{\uparrow}\exists w\forall y[y \in w \leftrightarrow y \in z]$$

# Inconsistency?

The axioms imply

$$\Diamond\exists x(\Diamond_{\leftarrow}\exists y[y \subseteq x \wedge y = z] \wedge \Box_{\uparrow}\neg\exists y[y = z])$$

abbreviate the formula in parentheses by $\Psi(x, z)$.
By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

By height potentialism/rigidty,

$$\Diamond\exists x\Psi(x, z) \wedge \Diamond_{\uparrow}\exists w\forall y[y \in w \leftrightarrow y \in z]$$

But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

# Resolution

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

# Resolution

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee $z$ even exists (indeed this is what we are trying to establish.)

# Resolution

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee $z$ even exists (indeed this is what we are trying to establish.) Natural solution: restrict comp to closed form:

$$\Box \forall z \Box \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

# Resolution

The argument just sketched uses comprehension with arbitrary parameters:

$$\exists X \forall y [Xy \leftrightarrow y \in z]$$

And in the crucial application, it applies when we have no *a priori* guarantee $z$ even exists (indeed this is what we are trying to establish.) Natural solution: restrict comp to closed form:

$$\Box \forall z \Box \forall Z \exists X \forall y [Xy \leftrightarrow \varphi(y, z, Z)]$$

Amounts to restricting ourselves to parameters that exist at the world of evaluation.

# Relative Consistency

## Intuitive idea

# Relative Consistency

## Intuitive idea

(From now on, I will ignore the difference between SOA and $ZFC^{-} + V = HC$. Replacement is formulated as collection.)

# Relative Consistency

## Intuitive idea

(From now on, I will ignore the difference between SOA and $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$. Replacement is formulated as collection.)

- We will use the fact that $\mathrm{T} = \mathrm{SOA} + \Pi^1_1\text{-PSP} \equiv \mathrm{ZFC}$, and in fact T proves that $L[r]$ is a model of ZFC for every real $r$.

# Relative Consistency

## Intuitive idea

(From now on, I will ignore the difference between SOA and $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$. Replacement is formulated as collection.)

- We will use the fact that $\mathrm{T} = \mathrm{SOA} + \Pi^1_1\text{-PSP} \equiv \mathrm{ZFC}$, and in fact T proves that $L[r]$ is a model of ZFC for every real $r$.
- Our translation will be doubly parameterized, once by a real and once by a transitive set.

# Relative Consistency

## Intuitive idea

(From now on, I will ignore the difference between SOA and $\text{ZFC}^- + \text{V} = \text{HC}$. Replacement is formulated as collection.)

- We will use the fact that $\text{T} = \text{SOA} + \Pi^1_1\text{-PSP} \equiv \text{ZFC}$, and in fact T proves that $L[r]$ is a model of ZFC for every real $r$.
- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for $\Diamond_\uparrow$ will involve holding $r$ fixed and climbing transitive sets in $L[r]$;

# Relative Consistency

## Intuitive idea

(From now on, I will ignore the difference between SOA and $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$. Replacement is formulated as collection.)

- We will use the fact that $\mathrm{T} = \mathrm{SOA} + \Pi^1_1\text{-PSP} \equiv \mathrm{ZFC}$, and in fact T proves that $L[r]$ is a model of ZFC for every real $r$.
- Our translation will be doubly parameterized, once by a real and once by a transitive set.
- Our interpretation for $\Diamond_\uparrow$ will involve holding $r$ fixed and climbing transitive sets in $L[r]$;
- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

# Relative consistency

# Relative consistency

Let $M \models SOA + \Pi^1_1 PSP$.

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T, r) := x \in t(X)$

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

**$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$

# Relative consistency

Let $M \models SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives = id
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$
- $t(\Box_\uparrow \varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \to t(\varphi)(S, r)]$

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$
- $t(\Box_\uparrow \varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \to t(\varphi)(S, r)]$
- $t(\Box_\leftarrow \varphi)(T, r) := \forall s[r \in L[s] \to \forall S[S \leq T \wedge S \in L[s] \to t(\varphi)(S, s)]]$

# Relative consistency

Let $M \vDash SOA + \Pi^1_1 PSP$.

**$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$**

- assign plural variables odd numbered variables $t(X)$.
- Membership claims on sets, propositional connectives $=$ id
- $t(Xx)(T, r) := x \in t(X)$
- $t(\forall x \varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X \varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$
- $t(\Box_\uparrow \varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \to t(\varphi)(S, r)]$
- $t(\Box_\leftarrow \varphi)(T, r) := \forall s[r \in L[s] \to \forall S[S \leq T \wedge S \in L[s] \to t(\varphi)(S, s)]]$

**Theorem**

M $\vdash \varphi$ implies T $\vdash t(\varphi)(\emptyset, 0)$

# Converse Translation

# Converse Translation

**Theorem**

$\mathsf{M} \vdash \Pi^1_1 PSP^\diamond$

# Converse Translation

**Theorem**

$\mathsf{M} \vdash \Pi^1_1 PSP^\diamond$

**Proof (sketch)**

# Converse Translation

**Theorem**

$\mathsf{M} \vdash \Pi^1_1 PSP^\diamond$

**Proof (sketch)**

Given any possible real $r$, one can show $\Diamond \exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_\uparrow$ translation of ZFC and some absoluteness lemmas.

# Converse Translation

**Theorem**

$\mathsf{M} \vdash \Pi^1_1 PSP^\Diamond$

**Proof (sketch)**

Given any possible real $r$, one can show $\Diamond\exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_\uparrow$ translation of ZFC and some absoluteness lemmas. One can also show by forcing that "$\Diamond\mathbb{R}^{L[r]}$ is countable."

# Converse Translation

**Theorem**

$\mathrm{M} \vdash \Pi^1_1 PSP^\diamond$

**Proof (sketch)**

Given any possible real $r$, one can show $\Diamond\exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_\uparrow$ translation of ZFC and some absoluteness lemmas. One can also show by forcing that "$\Diamond\mathbb{R}^{L[r]}$ is countable." This yields the $\Diamond$-translation of "$\mathbb{R}^{L[r]}$ exists for every $r$, and is coutnable."

# Converse Translation

**Theorem**

M $\vdash \Pi^1_1 PSP^\diamond$

**Proof (sketch)**

Given any possible real $r$, one can show $\Diamond\exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_\uparrow$ translation of ZFC and some absoluteness lemmas. One can also show by forcing that "$\Diamond\mathbb{R}^{L[r]}$ is countable." This yields the $\Diamond$-translation of "$\mathbb{R}^{L[r]}$ exists for every $r$, and is coutnable." By a result of Solovay, if $A$ is $\Pi^1_1$ definable in $r$, then either $A \in L[r]$ or $A$ contains a perfect subset. This easily implies $Pi^1_1 PSP$.

# Converse Translation

**Theorem**

$\mathsf{M} \vdash \Pi^1_1 PSP^{\Diamond}$

**Proof (sketch)**

Given any possible real $r$, one can show $\Diamond\exists x[x = \mathbb{R}^{L[r]}]$ using the $\Diamond_{\uparrow}$ translation of ZFC and some absoluteness lemmas. One can also show by forcing that "$\Diamond\mathbb{R}^{L[r]}$ is countable." This yields the $\Diamond$-translation of "$\mathbb{R}^{L[r]}$ exists for every $r$, and is coutnable." By a result of Solovay, if $A$ is $\Pi^1_1$ definable in $r$, then either $A \in L[r]$ or $A$ contains a perfect subset. This easily implies $Pi^1_1 PSP$. But now the mirroring theorem implies the corresponding modal result.

# Table of Contents

# Conclusions

# Conclusions

## Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{M} \equiv SOA + \Pi^1_1 PSP$.

# Conclusions

**Equivalence**

We have an exact proof-theoretic equivalence, $\mathsf{M} \equiv SOA + \Pi^1_1 PSP$.
The latter is in fact equiconsistent with ZFC. $\mathsf{M} \equiv \mathsf{L} \equiv ZFC$.

# Conclusions

## Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{M} \equiv SOA + \Pi^1_1 PSP$. The latter is in fact equiconsistent with ZFC. $\mathsf{M} \equiv \mathsf{L} \equiv ZFC$.

## Bi-interpretation

However, SOA $+ \Pi^1_1 PSP \vdash t(\phi^\diamond)(\emptyset, 0) \leftrightarrow \phi$, and conversely.

# Conclusions

## Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{M} \equiv SOA + \Pi^1_1 PSP$. The latter is in fact equiconsistent with ZFC. $\mathsf{M} \equiv \mathsf{L} \equiv ZFC$.

## Bi-interpretation

However, SOA $+ \Pi^1_1 PSP \vdash t(\phi^\diamond)(\emptyset, 0) \leftrightarrow \phi$, and conversely.

## Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity + countabilism.

# Conclusions

### Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{M} \equiv SOA + \Pi^1_1 PSP$.
The latter is in fact equiconsistent with ZFC. $\mathsf{M} \equiv \mathsf{L} \equiv ZFC$.

### Bi-interpretation

However, SOA $+ \Pi^1_1 PSP \vdash t(\phi^\diamond)(\emptyset, 0) \leftrightarrow \phi$, and conversely.

### Philosophical Interest?

There seems to be some relation between height + width potentialism and topological regularity + countabilism.
The 'parsimony and order' perspective on the universe?

# Thanks!