Axiomatic Potentialism

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Overview

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- Still deeper roots in mathematics in general.

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- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ.
- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and relate it to standard set theory.

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Hence the need for free logic.

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Proof: use mirroring.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(0)$

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Thanks!