Axiomatic Potentialism

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January 5, 2022

Overview

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- Still deeper roots in mathematics in general.

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- E.g. Hamkins and Linnebo showed in MT potentialism with the structures initial segments of V that $S5$ at a world V_κ is equivalent to Σ_3 correctness of κ.
- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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- Axiomatize this conception and work out its non-modal counterpart.

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Axioms for the theory L

Logical Axioms

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash Tran(X) \to t(\varphi)(X)$

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Proof: use mirroring.

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Bi-interpretation

In fact, ZFC $\vdash Tran(X) \rightarrow t(\phi^{\diamond})(X) \leftrightarrow \phi$, and conversely.

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- M proves $\neg Pow^{\Diamond}$.
- M proves $V = HC^{\Diamond}$ and hence $SOA^{\Diamond}$.
- M proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

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And in the crucial application, it applies when we have no *a priori* guarantee z even exists (indeed this is what we are trying to establish.)

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Amounts to restricting ourselves to parameters that exist at the world of evaluation.

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- while our interpretation for $\Diamond_\leftarrow$ will involve allowing new reals to be added but not extending the height of the transitive set parameter.

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Theorem

M $\vdash \varphi$ implies T $\vdash \mathbb{R}(r) \wedge Tran(X) \wedge X \in L[r] \to t(\varphi)(X, r)$

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What happens with stronger regularity properties? How does PD/AD affect things?

Thanks!