Axiomatic Potentialism

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1. Background
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3. Height and Width Potentialism

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- Still deeper roots in mathematics in general.

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- Here we will be focussed on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Hence the need for free logic.

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Proof: use mirroring.

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Theorem

L $\vdash \varphi$ implies $ZFC \vdash t(\varphi)(\emptyset)$

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Equivalence

We have an exact proof-theoretic equivalence, $\mathsf{L} \equiv ZFC$.

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References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

Thanks!