Axiomatic Potentialism

Chris Scambler

All Souls College,
Oxford University

chris.scambler@all-souls.ox.ac.uk

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Overview

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- Still deeper roots in mathematics in general.

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- Here we will be focused on axiomatic potentialism, and on relations between potentialist axiom systems and their first order counterparts.

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Instantiation requires assumption of existence.

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Proof: use mirroring.

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This is essentially what Button calls 'near-synonymy'.

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- M proves the universal possibility of forcing.
- e.g., it proves it is possible for the continuum to exist and have a cardinality at least as great as any $\aleph$ number whose existence is provable in ZFC.

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By comprehension,

$$\Diamond\exists x\Psi(x, z) \wedge \exists X\forall y[Xy \leftrightarrow y \in z]$$

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abbreviate the formula in parentheses by $\Psi(x, z)$.
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But then the rigidity/extensionality imply $w = z$ after all, so we have a contradiction.

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Amounts to 'restricting' ourselves to parameters that exist at the world of evaluation.

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Theorem

M $\vdash \varphi$ implies T $\vdash \forall r, T \in L[r][t(\varphi)(X, r)]$

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A peculiar sort of necker cube effect. Of philosophical interest?

Thanks!