Monic arrows in Ω-set .. An arrow $f : A \to B$ in Ω-set is monic iff it satisfies

$$f(a_0, b) \wedge f(a_1, b) \leq [\![a_0 = a_1]\!]_A \tag{1}$$

Proof. Let $g : C \to A$ in Ω-set.

We first prove that (1) implies

$$g(c, a_0) = \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{2}$$

for any c, a_0. To do this we show each of

$$g(c, a_0) \leq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{3}$$

and

$$g(c, a_0) \geq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{4}$$

For (3):

$$g(c, a_0) = g(c, a_0) \wedge [\![a_0 = a_0]\!] \tag{5}$$

$$= \bigsqcup_{b \in B} g(c, a_0) \wedge f(a_0, b) \tag{6}$$

$$= \bigsqcup_{b \in B} (\bigsqcup_{a \in A} g(c, a) \wedge f(a, b)) \wedge f(a_0, b) \wedge g(c, a_0)) \tag{7}$$

$$= \bigsqcup_{b \in B} f \circ g(c, b) \wedge f(a_0, b) \wedge g(c, a_0) \tag{8}$$

which latter is what we want.

For (4), let $b \in B$. We have:

$$(\bigsqcup_{a \in A} g(c, a) \wedge f(a, b)) \wedge f(a_0, b) = \bigsqcup_{a \in A} g(c, a) \wedge f(a, b) \wedge f(a_0, b) \tag{9}$$

$$\leq \bigsqcup_{a \in A} g(c, a) \wedge [\![a = a_0]\!]_A \tag{10}$$

$$\leq g(c, a_0) \tag{11}$$

where the transition from (9) to (10) uses (1), and the transition from (10) to (11) uses (v) p 277. (Is equality true?)

From here it is easy. Suppose $g \circ f(c, b) = h \circ f(c, b)$ for all c and b. Then for any a, $\bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a, b) = \bigsqcup_{b \in B} h \circ f(c, b) \wedge f(a, b)$. Hence $g(c, a) = f(c, a)$ by (2). $\square$

Ω axiom in Ω-set .. Show that with $\mathbf{\Omega}(p,q) := p \Leftrightarrow q$ and $\top(0,p) = p$ we get the Ω axiom for

$$\chi_f = [\![Ed]\!] \wedge [\![s_f(d) = p]\!]_{\mathbf{\Omega}} \tag{12}$$

Proof. Let $f : \mathbf{A} \to \mathbf{D}$ be monic. First, it will be handy to have some identities. Note that

$$\top\circ!(a,0) = \bigsqcup_{x\in 1}[\![Ea]\!] \wedge p = [\![Ea]\!] \wedge p \tag{13}$$

Second, since $s_f(d) = \bigsqcup_{a\in A} f(a,d)$ (12) implies

$$\chi_f(d,p) = \bigsqcup_{a\in A} f(a,d) \Leftrightarrow p \tag{14}$$

which then implies

$$\chi_f \circ f(a,p) = \bigsqcup_{d\in D} f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{15}$$

putting (13), (15) together, it suffices show

$$[\![Ea]\!] \wedge p = \bigsqcup_{d\in D} f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{16}$$

to show the square commutes.

It is immediate that

$$\bigsqcup_{d\in D} f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p = \bigsqcup_{d\in D} f(a,d) \wedge f(a,d) \Leftrightarrow p \tag{17}$$

by algebra (lol) we get

$$\bigsqcup_{d\in D} f(a,d) \wedge f(a,d) \Leftrightarrow p = \bigsqcup_{d\in D} f(a,d) \wedge p \tag{18}$$

but putting (17) and (18) together implies (16) by vii on p 277.

□