

**Exercise 1** (Monic arrows in  $\Omega$ -set). An arrow  $f : A \rightarrow B$  in  $\Omega$ -set is monic iff it satisfies

$$f(a_0, b) \wedge f(a_1, b) \leq \llbracket a_0 = a_1 \rrbracket_A \quad (1)$$

*Proof.* Let  $g : C \rightarrow A$  in  $\Omega$ -set.

We first prove that (19) implies

$$g(c, a_0) = \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \quad (2)$$

for any  $c, a_0$ . To do this we show each of

$$g(c, a_0) \leq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \quad (3)$$

and

$$g(c, a_0) \geq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \quad (4)$$

For (3):

$$g(c, a_0) = g(c, a_0) \wedge \llbracket a_0 = a_0 \rrbracket \quad (5)$$

$$= \bigsqcup_{b \in B} g(c, a_0) \wedge f(a_0, b) \quad (6)$$

$$= \bigsqcup_{b \in B} \left( \bigsqcup_{a \in A} g(c, a) \wedge f(a, b) \right) \wedge f(a_0, b) \wedge g(c, a_0) \quad (7)$$

$$= \bigsqcup_{b \in B} f \circ g(c, b) \wedge f(a_0, b) \wedge g(c, a_0) \quad (8)$$

which latter is what we want.

For (4), let  $b \in B$ . We have:

$$\left( \bigsqcup_{a \in A} g(c, a) \wedge f(a, b) \right) \wedge f(a_0, b) = \bigsqcup_{a \in A} g(c, a) \wedge f(a, b) \wedge f(a_0, b) \quad (9)$$

$$\leq \bigsqcup_{a \in A} g(c, a) \wedge \llbracket a = a_0 \rrbracket_A \quad (10)$$

$$\leq g(c, a_0) \quad (11)$$

where the transition from (9) to (10) uses (19), and the transition from (10) to (11) uses (v) p 277. (Is equality true?)

From here it is easy. Suppose  $g \circ f(c, b) = h \circ f(c, b)$  for all  $c$  and  $b$ . Then for any  $a$ ,  $\bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a, b) = \bigsqcup_{b \in B} h \circ f(c, b) \wedge f(a, b)$ . Hence  $g(c, a) = f(c, a)$  by (2).  $\square$

**Exercise 2** ( $\Omega$  axiom in  $\Omega$ -set). Show that with  $\Omega(p, q) := p \Leftrightarrow q$  and  $\top(0, p) = p$  we get the  $\Omega$  axiom for

$$\chi_f = \llbracket Ed \rrbracket \wedge \llbracket s_f(d) = p \rrbracket_{\Omega} \quad (12)$$

*Proof.* Let  $f : \mathbf{A} \rightarrow \mathbf{D}$  be monic. Note that

$$\top \circ ! (a, 0) = \bigsqcup_{x \in 1} \llbracket Ea \rrbracket \wedge p = \llbracket Ea \rrbracket \wedge p \quad (13)$$

Second, since  $s_f(d) = \bigsqcup_{a \in A} f(a, d)$  (12) implies

$$\chi_f(d, p) = \llbracket Ed \rrbracket \wedge \bigsqcup_{a \in A} f(a, d) \Leftrightarrow p \quad (14)$$

which then implies

$$\chi_f \circ f(a, p) = \bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \quad (15)$$

putting (13), (15) together, it suffices show

$$\llbracket Ea \rrbracket \wedge p = \bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \quad (16)$$

to show the square commutes.

Using the Heyting algebra validities

$$p \wedge q = p \wedge p \wedge q,$$

$$p \wedge (q \Leftrightarrow r) = p \wedge q \Leftrightarrow p \wedge r,$$

$$p \Leftrightarrow (p \wedge q) = p \Leftrightarrow q,$$

$$p \wedge (p \Leftrightarrow q) = p \wedge q,$$

and the law of distribution one can deduce each of

$$\bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p = \bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge f(a, d) \Leftrightarrow p \quad (17)$$

and

$$\bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge f(a, d) \Leftrightarrow p = \bigsqcup_{d \in D} \llbracket Ed \rrbracket \wedge f(a, d) \wedge p \quad (18)$$

but putting (17) and (18) together implies (16) by v and vii on p 277.

To show uniqueness, we need to show that

$$\llbracket Ea \rrbracket \wedge p = \bigsqcup_{d \in D} f(a, d) \wedge g(d, p) \quad (19)$$

implies

$$g(d, p) = \llbracket Ed \rrbracket \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \quad (20)$$

i.e.

$$g(d, p) \leq \llbracket Ed \rrbracket \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \quad (21)$$

and

$$g(d, p) \geq \llbracket Ed \rrbracket \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \quad (22)$$

For the first, 277 gives  $g(d, p) \leq \llbracket Ed \rrbracket$ , so we must show  $g(d, p) \leq \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p$ . For this, it is enough to show

$$g(d, p) \wedge \bigsqcup_{a' \in A} f(a', d) \leq p \quad (23)$$

$$g(d, p) \wedge p \leq \bigsqcup_{a' \in A} f(a', d). \quad (24)$$

For (23) we want

$$\bigsqcup_{a' \in A} f(a', d) \wedge g(d, p) \leq p \quad (25)$$

now for a particular value of  $a'$ , (19) implies  $f(a', d) \wedge g(d, p) \leq \llbracket Ea \rrbracket \wedge p$ . So by the def of least upper bound,  $\bigsqcup_{a' \in A} f(a', d) \wedge g(d, p) \leq \llbracket Ea \rrbracket \wedge p$ . But then the result follows trivially.

For (24), we have

$$g(d, p) \wedge p \wedge \bigsqcup_{a' \in A} f(a', d) = g(d, p) \wedge p \wedge \bigsqcup_{a' \in A} f(a', d) \wedge g(d, p)$$

thus by (25)

$$g(d, p) \wedge p \wedge \bigsqcup_{a' \in A} f(a', d) \wedge g(d, p) = g(d, p) \wedge p \wedge p = g(d, p) \wedge p$$

and (21) follows.

For (22), consider

$$g(d, p) \wedge \llbracket Ed \rrbracket \wedge \bigsqcup_{a' \in A} f(a', d) \Leftrightarrow p \tag{26}$$

□