Exercise 1 (Monic arrows in Ω-set)**.** An arrow $f : A \to B$ in Ω-set is monic iff it satisfies

$$f(a_0, b) \wedge f(a_1, b) \leq [\![a_0 = a_1]\!]_A \tag{1}$$

Proof. Let $g : C \to A$ in Ω-set.

We first prove that (19) implies

$$g(c, a_0) = \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{2}$$

for any c, a_0. To do this we show each of

$$g(c, a_0) \leq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{3}$$

and

$$g(c, a_0) \geq \bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a_0, b) \tag{4}$$

For (3):

$$g(c, a_0) = g(c, a_0) \wedge [\![a_0 = a_0]\!] \tag{5}$$

$$= \bigsqcup_{b \in B} g(c, a_0) \wedge f(a_0, b) \tag{6}$$

$$= \bigsqcup_{b \in B} (\bigsqcup_{a \in A} g(c, a) \wedge f(a, b)) \wedge f(a_0, b) \wedge g(c, a_0)) \tag{7}$$

$$= \bigsqcup_{b \in B} f \circ g(c, b) \wedge f(a_0, b) \wedge g(c, a_0) \tag{8}$$

which latter is what we want.

For (4), let $b \in B$. We have:

$$(\bigsqcup_{a \in A} g(c, a) \wedge f(a, b)) \wedge f(a_0, b) = \bigsqcup_{a \in A} g(c, a) \wedge f(a, b) \wedge f(a_0, b) \tag{9}$$

$$\leq \bigsqcup_{a \in A} g(c, a) \wedge [\![a = a_0]\!]_A \tag{10}$$

$$\leq g(c, a_0) \tag{11}$$

where the transition from (9) to (10) uses (19), and the transition from (10) to (11) uses (v) p 277. (Is equality true?)

From here it is easy. Suppose $g \circ f(c, b) = h \circ f(c, b)$ for all c and b. Then for any a, $\bigsqcup_{b \in B} g \circ f(c, b) \wedge f(a, b) = \bigsqcup_{b \in B} h \circ f(c, b) \wedge f(a, b)$. Hence $g(c, a) = f(c, a)$ by (2). $\square$

Exercise 2 (Ω axiom in Ω-set)**.** Show that with $\mathbf{\Omega}(p,q) := p \Leftrightarrow q$ and $\top(0,p) = p$ we get the Ω axiom for

$$\chi_f = [\![Ed]\!] \wedge [\![s_f(d) = p]\!]_{\mathbf{\Omega}} \tag{12}$$

Proof. Let $f : \mathbf{A} \to \mathbf{D}$ be monic. Note that

$$\top\circ!(a,0) = \bigsqcup_{x\in 1} [\![Ea]\!] \wedge p = [\![Ea]\!] \wedge p \tag{13}$$

Second, since $s_f(d) = \bigsqcup_{a\in A} f(a,d)$ (12) implies

$$\chi_f(d,p) = [\![Ed]\!] \wedge \bigsqcup_{a\in A} f(a,d) \Leftrightarrow p \tag{14}$$

which then implies

$$\chi_f \circ f(a,p) = \bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{15}$$

putting (13), (15) together, it suffices show

$$[\![Ea]\!] \wedge p = \bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{16}$$

to show the square commutes.

Using the Heyting algebra validities

$$p \wedge q = p \wedge p \wedge q,$$

$$p \wedge (q \Leftrightarrow r) = p \wedge q \Leftrightarrow p \wedge r,$$

$$p \Leftrightarrow (p \wedge q) = p \Leftrightarrow q,$$

$$p \wedge (p \Leftrightarrow q) = p \wedge q,$$

and the law of distribution one can deduce each of

$$\bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p = \bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge f(a,d) \Leftrightarrow p \tag{17}$$

and

$$\bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge f(a,d) \Leftrightarrow p = \bigsqcup_{d\in D} [\![Ed]\!] \wedge f(a,d) \wedge p \tag{18}$$

but putting (17) and (18) together implies (16) by v and vii on p 277.

To show uniqueness, we need to show that

$$[\![Ea]\!] \wedge p = \bigsqcup_{d\in D} f(a,d) \wedge g(d,p) \tag{19}$$

implies

$$g(d,p) = [\![Ed]\!] \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{20}$$

i.e.

$$g(d,p) \leq [\![Ed]\!] \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{21}$$

and

$$g(d,p) \geq [\![Ed]\!] \wedge \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p \tag{22}$$

For (21), 277 gives $g(d,p) \leq [\![Ed]\!]$, so we must show $g(d,p) \leq \bigsqcup_{a'\in A} f(a',d) \Leftrightarrow p$. For this, it is enough to show

$$g(d,p) \wedge \bigsqcup_{a'\in A} f(a',d) \leq p \tag{23}$$

$$g(d,p) \wedge p \leq \bigsqcup_{a'\in A} f(a',d). \tag{24}$$

For (23) we want

$$\bigsqcup_{a'\in A} f(a',d) \wedge g(d,p) \leq p \tag{25}$$

now for a particular value of a', (19) implies $f(a',d) \wedge g(d,p) \leq [\![Ea]\!] \wedge p$. So by the def of least upper bound, $\bigsqcup_{a'\in A} f(a',d) \wedge g(d,p) \leq [\![Ea]\!] \wedge p$. But then the result follows trivially.

For (24), we have

$$g(d,p) \wedge p \wedge \bigsqcup_{a'\in A} f(a',d) = g(d,p) \wedge p \wedge \bigsqcup_{a'\in A} f(a',d) \wedge g(d,p)$$

thus by (25)

$$g(d,p) \wedge p \wedge \bigsqcup_{a' \in A} f(a',d) \wedge g(d,p) = g(d,p) \wedge p \wedge p = g(d,p) \wedge p$$

and (21) follows.

For (22), consider

$$g(d,p) \wedge [\![Ed]\!] \wedge \bigsqcup_{a' \in A} f(a',d) \Leftrightarrow p \tag{26}$$

this equals

$$[\![Ed]\!] \wedge \bigsqcup_{a' \in A} f(a',d) \wedge g(d,p) \Leftrightarrow g(d,p) \wedge p \tag{27}$$

by (24), this is just

$$[\![Ed]\!] \wedge \bigsqcup_{a' \in A} f(a',d) \wedge g(d,p) \Leftarrow g(d,p) \wedge p \tag{28}$$

□

Exercise 3 (h is continuous (5, p305))**.** Show that the unique arrow h derived from x and f in $Bn(I)$ is continuous in $Top(I)$.

Proof. We need to show that whenever $h[X]$ is open for some $X \subseteq I \times \omega$, X itself is open in that space.

It is enough to prove that whenever $h[Y \times \{n\}]$ is open, $Y \times \{n\}$ is open, since if X is any other open we can write it as a union of $Y \times \{n\}$s, and $h[\bigcup X] = \bigcup_{Z \in X} h[Z]$.

This we do by induction. In the base case, $h[Y \times \{0\}]$ is just $x[Y]$ (using $h(i, 0) = x(i)$). Thus continuity of x implies the result.

For the successor step, suppose

$$h[Y \times \{n + 1\}]$$

is open. This can be rewritten as

$$h \circ s[Y \times \{n\}]$$

using $s(\langle i, n\rangle) = \langle i, n + 1\rangle$. By def h, this in turn can be written as

$$f \circ h[Y \times \{n\}].$$

By the induction hypothesis and continuity of f, it follows that $Y \times \{n\}$ is open. Hence Y is open, so $Y \times \{n + 1\}$ is open too (the topology on ω is discrete).

$\square$