

On the Consistency Strength of Axioms for Height and Width Potentialism

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April 18, 2022

Abstract

There are two parameters along which the universe of sets might conceivably differ in size: it may have more or fewer ordinals, and respectively may naturally be construed as taller or shorter; or at any given ordinal height, the initial segment of the universe up to that ordinal may have more or fewer subsets, and the universe may in each case naturally be construed, respectively, as fatter or thinner. A potentialist about the height of the universe believes that, no matter what, there could always be further ordinals; a potentialist about the width of the universe believes that, no matter what, there could always be more subsets of any given infinite set.

It is well known that natural axiom systems for height potentialism exhibit tight forms of proof-theoretic equivalence with standard set theories associated with ZFC (and its extension by large cardinals). The recent literature has seen some discussion of height and width potentialism combined. But analogous results precisely correlating these axiom systems with familiar, purely quantificational theories do not yet exist.

This paper provides such results. I show that the axioms for height and width potentialism from Scambler CITE exhibit a tight form of equivalence with second order arithmetic extended by topological regularity principles. The results are significant because they help to round out the foundational picture underlying height and width potentialism, and to bring out how different it is from the standard ZFC-conception of set.

1 Overview

I will begin by discussing the sense in which axiom systems for height potentialism are equivalent to set theories related to ZFC, and the techniques used to attain the equivalence theorems. I then present the relevant axioms for height and width potentialism, and prove they share the same form of equivalence with certain extensions of second order arithmetic. Finally, I discuss the relevance of these results for our understanding of the height/width potentialist point of view.

2 Warm Up: Height Potentialism

We begin with a review of some familiar territory. Though the correctness of the results here will be obvious to specialists, and though there are similar arguments on offer in the literature (cf Button, Linnebo), I'll present some of the details here, partly because the results have never been given for the precise set of axioms to be discussed, and partly because seeing how the arguments go in the simpler case of plain height potentialism will help to give intuitions that will be handy for the more complicated case of height and width potentialism.

2.1 Axioms

We start with a discussion of the 'natural' axiom systems for height potentialism that are at issue.

The systems will be directed at axiomatizing the following intuitive conception. Imagine there is a being who is capable of taking things and **collecting** them together into a set, and of repeatedly performing this act as many times as is conceivable. Say that a *set* is something that, given enough executions of this basic act, could be produced by such a being.

We want to axiomatize this conception in modal logic. The modal operator, \Diamond , is to be interpreted so that $\Diamond\varphi$ means 'by repeated acts of collection, φ can be made to hold'. The dual operator, $\Box\varphi$, means 'no matter how many acts of collection are performed, φ will hold'. The axioms governing this modal logic can be taken to be S4.2 with the converse Barcan formula. More details are given in Appendix A.¹

Our conception says that the being can take any *things* whatsoever and collect *them* into a set. These are plural notions, and accordingly we shall want a plural language as well. We thus take a stock of plural variables X and singular variables x . The circumstance of something x being one of some things X will be expressed by the concatenation Xx . The precise details, once more, are consigned to Appendix A.

The set-theoretic axioms are more interesting. We assume all the usual background stuff: the axiom of extensionality for sets, rigidity for the membership relation, and the axiom of foundation. But in addition, we have distinctively potentialist existence principles like

HP $\Box\forall X\Diamond\exists x[Set(x, X)]$

Here and throughout, $Set(x, X)$ is an abbreviation for $\forall z[z \in x \equiv Xz]$. **HP** thus says: *necessarily, any things can be the elements of a set*; under our rather fanciful intuitive interpretation, this corresponds to the idea that the being can always collect together any given things into a set, 'no matter what'. (Hence, in contrast to standard set theory, there are no special things that somehow can't be made to form a set.)

¹(We do not want .3, since given $a \neq b$ we may form either $\{a\}$ or $\{b\}$, but we do want .2, so from either of these we can get $\{a\}$ and $\{b\}$.) The rule of necessitation is assumed, along with the standardly valid inference rules:

$$\frac{\Phi_1 \rightarrow \Box(\Phi_2 \rightarrow \dots \Box(\Phi_n \rightarrow \Box\Psi)\dots)}{\Phi_1 \rightarrow \Box(\Phi_2 \rightarrow \dots \Box(\Phi_n \rightarrow \Box\forall x\Psi)\dots)}$$

HP allows us (given plural comprehension) to prove the possible existence of each specific hereditarily finite set. But it does not allow for a proof that there are any infinite sets. However, in the spirit of transfinite set theory, we will want to secure this possibility axiomatically. One way to do this uses the following **Completeability** principle for \mathbb{N} :

Comp $_{\mathbb{N}}$ $\Diamond \exists X \Box \forall y [Xy \leftrightarrow \mathbb{N}(x)]$

This says that it is possible, eventually, for the being to create *all possible* natural numbers. This is used to prove the possibility e.g. of ω 's existence, given the other axioms.

Indeed, in conjunction with **Comp $_{\mathbb{N}}$** , **HP** proves the possible existence of infinitely many infinite sets. But there is no guarantee from the axioms stated up to now that the modal analogue of the powerset axiom should hold: that it should be possible, given any set, infinite or no, eventually to produce *all* its possible subsets. We also have no guarantee that any uncountable sets are possible. Axiomatically, these things can be imposed by appeal to another completeability axiom:

Comp $_{\subseteq}$ $\Box \forall x \Diamond \exists X \Box \forall y [Xy \leftrightarrow y \subseteq x]$

This says that one can always eventually form all possible subsets of any given set, and is readily seen to have the asserted consequences.

As a final axiom, or rather axiom scheme, we throw in a modal translation of replacement (the \Diamond -translation in the sense defined below).

These are all fairly natural axioms to adopt if one wishes to explicate the idea of a *never ending set-construction process* in modal terms. One has an axiom asserting that any things possibly form a set, along with various other axioms that implicitly concern the lengths of iterations of set formation that are possible.

For ease of reference the axioms for this system are compiled in Appendix A. I will refer to them as the system L, for their creator, Øystein Linnebo.

We now turn to a discussion of the proof-theoretic relations between L and ZFC.

2.2 Key Results

There are three key results of interest. I will state them and indicate the methods of proof. Full proofs are provided in Appendix B.

The **first key result** is that there is a recursive mapping $\varphi \mapsto \varphi^{\exists}$ from the potentialist language to the language of first order set theory that preserves theoremhood from L into ZFC.

The **second key result** is that there is a recursive mapping $\varphi \mapsto \varphi^{\diamond}$ from the language of first order set theory to the modal language that preserves theoremhood from ZFC into L.

The **third key result** is that these maps are *almost inverse to each other up to provable equivalence*.

What does that mean? Well, for the maps to be inverse to one another up to provable equivalence would be for ZFC to prove $\varphi \leftrightarrow (\varphi^{\diamond})^{\exists}$, and for the axiom system L to prove $\varphi \leftrightarrow (\varphi^{\exists})^{\diamond}$. If that held, ZFC and L would be *definitionally equivalent*.

It would be nice if this were so. But unfortunately it cannot *quite* be. The translation φ^\exists involves interpreting unrestricted quantification in the modal language as tacitly restricted to sets (“possible worlds”), and catering for the extra parameters they induce means that there are extra assumptions we need to make to ensure such an equivalence is provable. For instance, we have to assume that the thing the actualist calls ‘the actual world’ (for the potentialist) contains exactly all those things the potentialist recognizes. Hence the maps are only inverses if we throw in some extra assumptions on the parameters. Thus we are left only with what Button has called *near synonymy* rather than full definitional equivalence.

The difference between the two is not great. Indeed to me it looks like a trivial book-keeping detail. All the model-theoretic consequences one gets from definitional equivalence, for example, carry over fairly straightforwardly to the form of near-definitional equivalence we acquire. Thus there is a very natural sense in which the first order picture of the sets painted by ZFC is equivalent to the modal account axiomatized by theories like L. What more to make of this similarity is a difficult philosophical question I will not go into.

Let me go into the methods of proof for these results.

First, let’s discuss the translation $\varphi \mapsto \varphi^\exists$ from the first key result. It has to translate the whole potentialist idiolect into the austere idiom of the first order language. How is that to be done? An important part of the idea, as I said, is to introduce a notion of a possible world and take the potentialist to really be using quantifiers restricted to such worlds. But the speaker of a first order language must also find some interpretation for the potentialist’s plural variables. To do that, they will just interpret plural talk as unusual talk about sets (whose elements are the ‘plurals’). But the details get a little bit subtle.

The initial idea is simple enough: we just take all the first order variables x_k of the potentialist, and map them to the even numbered variables $x_{2k} := (x_k)_\exists$ in our first order language; and we take the plural variables X_k of the potentialist and map them to the odd numbered variables $x_{2k+1} := (X_k)_\exists$ of our first order logic. Clearly, this is a way of interpreting talk of plurals as just talk of some special objects.

Unfortunately, an interpretation of this kind won’t get the third key result without some further tweaking. The issue is that in order to get key result three, we need to be able to translate back from our first order language to our modal language in a way that gets an equivalent formula. The obvious thing to do is to say: translate by mapping odd numbered variables to plurals, and even numbered variables to objects, e voila. But the problem is that this will lead to nonsense, because in the first order language things like $x_1 \in x_3$ are well formed (and that corresponds to the ill-formed XY , or something along those lines). So in this case our translation from the first order language to the plural language would not even be everywhere defined.

To overcome this annoying detail we re-define the notion of a well-formed formula so that odd numbered variables only ever occur on the right of \in , and that $=$ is always only ever flanked by variables of the same parity. One can easily prove that this doesn’t make any significant difference to provability in ZFC.² ((Could one do this by using a predicate?? Is what you have in the

²Specifically: let \mathcal{L}^- be the modified language, and ZFC^- be the modified version of ZFC.

footnote enough??))

[[Another option: weaken the key result to just cover the first order part. I think that should be enough. Yeh: change the \diamond translation to just ignore plurals!!!!]]

Now that we have an interpretation for the potentialist's variables we can proceed to specify the interpretation more generally. We set $x \in y^\exists$ and Xx^\exists as $x_\exists \in y_\exists$ and $x_\exists \in X_\exists$ respectively. The connectives commute with the translation. For the quantifiers, we interpret the potentialist's assertion that any thing(s) has(have) the property P as saying that any thing(s) *in the domain* has(have) the property P *relative to that domain*. More explicitly, the translation introduces a free (set) variable d (to be thought of as "the domain") and sets:

$$(\forall x\varphi)^\exists := \forall x \in d[\varphi^\exists(d)]$$

Modal notions are defined in terms of quantification over extensions of the domain:

$$(\diamond\varphi)^\exists := \exists e \supseteq d[\varphi^\exists(e)]$$

It is now tedious but routine to prove that theoremhood is preserved in the transition $\varphi \mapsto \varphi^\exists$ (as long as we require d and e to range over *transitive* sets).

The key idea behind the other translation, $\varphi \mapsto \varphi^\diamond$, is to simply put a \diamond in front of every existential quantifier (and a \square in front of every universal). For variables, we set $(x_{2k})_\diamond := x_k$, and $(x_{2k+1})_\diamond = X_k$. It is easy to show theoremhood is preserved.

The third key result is not hard, with the only tricky part being coming up with precisely the right statement. Here it is:

Part one. ZFC proves that for any transitive d , $\varphi \equiv (\varphi^\diamond)^\exists(d)$.

Part two. L proves that if d is a universal set, then $\varphi \equiv (\varphi^\exists d)^\diamond$.

2.3 Discussion

3 Axioms

In this section I will give the system of axioms for height and width potentialism that will be the subject of the rest of the paper.

The system is directed at axiomatizing the following intuitive (if fanciful) conception. As before, we have a being who is able to take things and **Collect** them together into a set. But now, we will also take them to be able to perform another kind of basic act: namely, that of taking some things, and **Counting** them, that is, correlating them one-for-one with the natural numbers. Our new theory will concern the possibilities for arbitrary iterations of **Collect** and **Count**. This time, we say that a *set* is something that can be gotten eventually by arbitrary iterations of these operations. **Collect** is a height potentialist principle, in that the set collected together has higher rank than the collected sets, while **Count** is naturally understood as a width potentialist principle, as the introduction of an enumerating function for a given set may corresponds to the introduction of new sets of the same or

Then one can show that for every formula φ in the language \mathcal{L} there is a formula φ^- in the language \mathcal{L}^- such that (a) $\text{ZFC} \models \varphi \equiv \text{ZFC}^- \models \varphi^-$, and if φ is a sentence, $\text{ZFC} \models \varphi \equiv \varphi^-$.

lower rank as the sets counted in many cases— indeed, up to isomorphism, it will correspond to introducing a subset of the naturals.

There are many natural modal operators that one can read off from this conception. There is for example a modal, which I will write $\Diamond_{\uparrow}\varphi$, which corresponds to possibility only taking acts of **Collect** into consideration. Since this is the height potentialist sense of possibility I shall often refer to this as **vertical** possibility. There is also a natural modal, $\Diamond_{\leftarrow}\varphi$, which corresponds to possibility just by iterated enumeration. This might naturally be read as **horizontal** possibility. And there is a general modal operator, $\Diamond\varphi$, which corresponds to φ s being possible by some arbitrary number of iterations and alternations of the two available operations. I will refer to this as **general** possibility. There are many others besides, for example that induced by $\Diamond_{\uparrow}\Diamond_{\leftarrow}$, but these will play little direct role. In fact, for our purposes, even \Diamond_{\leftarrow} will be superfluous. We will focus our attentions on the two operators \Diamond_{\uparrow} and \Diamond , where the former corresponds to possibility just by **Collection**, and the latter to possibility by combinations of **Collect** and **Count**.

So we take a multi-modal language with the two modal operators \Diamond and \Diamond_{\uparrow} . We still want the capacity to talk about pluralities and so we keep the two types of variable. And we want to talk about sets so we include the membership relation symbol \in .

What axioms should we impose? Well, we want all the same plural and first order stuff. Both of the modal operators should have S4.2, and we have $\Diamond_{\uparrow}\varphi \rightarrow \Diamond\varphi$. We take over all the axioms from L for the \Diamond_{\uparrow} modality: the idea being, nothing has changed from before, and if we could get something without forcing/enumeration before we can get it by ignoring forcing/enumeration now.

4 Key Results

5 Discussion

6 Stuff

Modal Set Theory: Absoluteness of various kinds.

Set Theory: Pi 1-1 PSP equivalent to only countably many reals in $L[r]$.

Logic: Define a formula to be *pseudo typed* (PT) iff odd numbered variables only ever occur to the right of \in . Then in ZFC, any formula φ has a (weak) PT equivalent $PT(\varphi)$ with $ZFC \vdash PT(\varphi)$ iff $ZFC \vdash \varphi$, and any sentence φ has a (strong) PT equivalent $PT(\varphi)$ with $ZFC \vdash \varphi \leftrightarrow PT(\varphi)$.

Proof. Let $n(\varphi)$ be a number greater than the indices of variables in φ . For each n let E_n be an enumeration of the evens greater than n , so e.g. $E_2(0) = 3$.

We define $PT(\varphi)$ recursively. Moving from left to right in the formula keep looking until you find a variable out of place. Then check to see if it is free or bound. If it is free, replace all other free occurrences of the variable by $E_{n(\varphi)}(0)$; If it is bound, do the same for all its occurrences in the scope of the binding quantifier. In either case, a formula $PT_0(\varphi)$ results. Then

given $PT_n(\varphi)$, do the same thing to get $PT_{n+1}(\varphi)$. Whenever $PT_n(\varphi) = PT_{n+1}(\varphi)$, $PT_n(\varphi)$ is PT . We set such $PT_n(\varphi) := PT(\varphi)$.

$PT(\varphi)$ is provable iff φ is. Induction on the number of out-of-place variables. Suppose it holds for n .