Proofs

CS

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1 ZFC and L

1.1 Formulation of ZFC

1.1.1 Language $\mathcal{L}^2_\in$

Signature

- countable infinity of first order variables x_i
- countable infinity of monadic second order variables X_i
- propositional connectives $\vee, \neg$
- variable binding quantifier $\exists$
- relation $=$
- relation $\in$

wffs

$$Xx|x \in y|x = y|X = Y$$
$$\varphi \wedge \psi|\neg\varphi|\forall x\varphi|\forall X\varphi$$

defs

- usual defs for connectives quantifiers
- symbol for $\emptyset$, $\{x : \psi\}$ notation

1.1.2 Axioms of ZFC

Standard second order logic with full comprehension and extensional second order identity. For set theoretic axioms:

Ext$_\forall$ $\forall x[x \in y \equiv x \in z] \supset y = z$

Fun$_\forall$ $x \neq \emptyset \supset \exists y[y \in x \wedge y \cap x = \emptyset]$

Pair $\exists z[z = \{x, y\}]$

Un $\exists y[y = \bigcup U(x)]$

Rep $Fun(F) \supset \forall x \exists y(y = F|x)$

1.2 Formulation of L

1.2.1 Language $\mathcal{L}_0$

Signature

- countable infinity of first order variables x_i
- countable infinity of monadic second order variables X_i
- propositional connectives $\vee, \neg$
- variable binding quantifier $\exists$
- relation $=$
- operator $\Diamond$
- relation $\in$

wffs

$$Xx|x \in y|x = y|X = Y$$

$$\varphi \wedge \psi|\neg\varphi|\forall x\varphi|\forall X\varphi|\Diamond\varphi$$

defs

- usual defs for connectives quantifiers and modals
- Ex is an abbreviation for $\exists y[y = x]$ or a more suitable alphabetic variant
- $Set(x, X)$ abbreviates $\forall y[Xy \equiv y \in x]$
- Previous abbreviations from set theory

1.2.2 Axioms of L

We assume all propositional tautologies, along with any standard axioms for positive free quantifier logic.

The modal logic is S4.2 with necessitation (and CBF). The rule of inference ... is also assumed.

As to the plural logic, we assume the following.

pExt $\forall X \forall Y[(\forall x[Xx \equiv Yx]) \supset X = Y]$

pR $\Diamond Xx \supset \Box Xx$

pBF $\forall X[\Diamond(\exists x[Xx \wedge x = y]) \supset \exists x[Xx \wedge x = y]]$

Finally, the set theoretic axioms.

Ext $\forall x \forall y[(\forall z[z \in x \equiv z \in y]) \supset x = y]$

Ele $\Box \forall x \exists X \Box \forall y[Xy \equiv y \in x]$

Fun $\forall x[x \neq \emptyset \supset \exists y[y \in x \wedge y \cap x = \emptyset]]$

Set $\Box \forall X \Diamond \exists y[Set(y, X)]$

Inf $\Diamond X \Box \forall y[Xy \equiv \mathbb{N}(y)]$

Pow $\Box \forall x \Diamond \exists X \Box \forall y[Xy \equiv y \subseteq x]$

Rep $Rep^{\Diamond}$

1.3 Key Result One

Theorem 1. *There is an interpretation $\exists : \mathcal{L}_0 \to \mathcal{L}_\in$ that preserves theoremhood from L to ZFC.*

Definition 1 ($\varphi_\exists$). *The translation $\varphi \mapsto \varphi_\exists$ is defined by the following clauses. In each case, d is the least variable not occuring in φ.*

- $x \in y_\exists := x \in y \wedge d = d$
- $Xx_\exists := Xx \wedge d = d$
- $(\neg\varphi)_\exists := \neg\varphi_\exists(d)$
- $(\varphi \vee \psi)_\exists := \varphi_\exists(d) \vee \psi_\exists(d)$
- $(\exists x \varphi)_\exists := \exists x \in d[\varphi_\exists(d)]$
- $(\Box\varphi)_\exists := \forall e[e \supseteq d \wedge Tran(e) \supset \varphi_\exists(e)]$

We then set $\exists(\varphi) := Tran(d) \supset \varphi_\exists$.

Proof. The propositional tautologies and modus ponens are obvious. For free universal instantiation, we must show (under the assumptin $Tran(d)$) that

$$\forall x \in d[\forall y \in d(\varphi(y))_\exists(d) \supset (\varphi(x))_\exists(d)]]$$

But this is immediate. (Note however that the unfree quantifier rule, which removes the initial quantifier, is not provable under this interpretation.)

As to the laws of $S4.2$ modal logic, it is completely clear that S4 will hold in light of the reflexivity and transitivity of $\subseteq$. For $.2$, suppose $(\Diamond\Box\varphi)_\exists$. Then there is a transitive extension e_0 of d such that every extension f of e_0 has $\varphi_\exists(f)$. Suppose given transitive extension e of d. Then $f_1 := e \cup e_0$ is a transitive extension of e_0 that (therefore) has $\varphi_\exists(f_1)$. Hence $(\Box\Diamond\varphi)_\exists$.[1]

$\Box$

[1] In fact it seems we have the stronger $\Diamond\Box\varphi \supset \Box\Diamond\Box\varphi$??)

1.4 Key Result Two

Theorem 2. *There is an interpretation* $\Diamond : \mathcal{L}_{\in} \to \mathcal{L}_0$ *that preserves theoremhood from ZFC to L* on first order formulas.

1.5 Key Result Three

Theorem 3. *These translations yield a definitional equivalence between the first order fragments of ZFC and L.*