On the Consistency of Height and Width Potentialism

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Abstract

Recent work in philosophy of set theory has furnished some arguments that height and width potentialism are inconsistent with one another. One such argument can be found in this volume (Brauer); the other is forthcoming (Roberts).

At the same time, others have suggested there may be some merit in the combination of height and width potentialism. Such authors have presented views that appear to manifest the combination in a non-trivial way, and have defended philosophical claims on their basis (e.g. that all sets are ultimately countable – cf Builes & Wilson, Meadows, Scambler, Pruss).

Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

1 Overview

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The plan is as follows. Section 1 gives some definitional background and context. Section 2 then presents a system that purports to combine height and width potentialism and cites some new results about its precise consistency strength and equivalence to standard actualist systems of set theory. Section

3 presents the inconsistency argument of Roberts and explains how the a proponent of the theory from Section 2 would reply. Section 4 does the same for Brauer.

2 Background

Let me start be explaining the terms.

For present purposes, *potentialism* in set theory will be the idea that there could always be more sets than there in fact are. In slogan form, for the potentialist, the universe of sets is 'indefinitely extendible'.

Height potentialism is the idea that the universe is always extendible 'upwards' to include new sets of higher rank than any given ones. *Width potentialism*, on the other hand, is the idea that the universe is always extendible 'outwards', to contain new sets of no greater rank than the max of any given ones.

Historically, height potentialism has been motivated by considerations involving the set-theoretic paradoxes. Russell's paradox shows there is no set of all non-self-membered sets, and hence that the cumulative hierarchy of all sets is itself (therefore) not a set. But the height potentialist complains that any 'stopping point' for the cumulative hierarchy would be arbitrary. Surely there is no conceptual obstacle to any particular collection of ranks of the cumulative hierarchy providing 'urelements' for a longer continuation.

It is natural to explicate this idea of an indefinitely extendible universe of sets in modal logic. Indeed, modal axiom systems based around core height potentialist ideas are known to exhibit tight forms of equivalence with standard iterative set theories like ZFC.

Width potentialism has been historically less popular, although has been attracting some attention over the last few decades as a way understand the independence phenomenon in set theory. There is, in fact, a close structural similarity in the motivation for height and width potentialism in these terms (cf Meadows). Just as the height potentialist begins with the intuition that it is arbitrary that there should be some ranks of the cumulative hierarchy that somehow inherently can't extended to include further sets, the width potentialist may begin with the idea that it is arbitrary that there should be some universe of sets that cannot be extended by forcing over its partial orders. Just as in the previous case, the mathematics of forcing seems to lead us to believe there is at least no conceptual obstacle to making sense of such 'forcing extensions of the universe'.

Again, as with height potentialism, a natural way for the width potentialist to fromalize their view involves modal logic: one formulates an axiom to the fact that, for any partial order $\mathbb{P}$, it is possible to find a generic for $\mathbb{P}$. Such explicitly axiomatic approaches to width potentialism are in fact less well-studied: the focus of most work in this area has been on model theory. Nevertheless, axiomatic approaches are possible and easy enough to formulate.

There are, in any event, clear analogies in the cases for height and width potentialism. In each case one has a central inexistence result in first order set theory (Russell's paradox, the proof that some filters do not admit partial orders) and one seeks to overcome it, after a fashion, by implementing a modalized version – any things can form a set in the first case, any partial order can be forced over in the second. It is natural to think that going potentialist one way might give you some reason to consider going potentialist the other way too. But then, as I have said, there are those who think that this is impossible on grounds of a logical inconsistency between the two ideas.

3 Height and Width Potentialism Combined

In this section, I will present some axiom systems that seem to explicate the intuitive idea of height and width potentialism, as described above.

3.1 Core Logical Principles

All the versions of height and width potentialism we will consider will be built around two principles: the first is the height potentialist principle that any things can be the elements of a set; the second is the width potentialist 'forcing axiom', that a generic filter can be found for any partial order.

Both potentialist principles involve modality; the first also involves plural quantification. Accordingly the language $\mathcal{L}_0$ we use to formulate these theories will have at least a modal operator $\Box$, singular variables x_n, plural variables X_n, the propositional connectives and quantifiers, the identity symbol = and the symbol $\in$ for set membership. The circumstance that some thing x is one of some things X will be represented by the concatenation Xx. Identity is only well formed between terms of the same type (singular/plural).

The following core axioms will be included in all the potentialist theories we will go on to discuss.

FQ Standard rules for free quantifier logic with identity for each type of variable.

Mod The modal logic S4.2 with converse Barcan formula, necessitation, and standard rules for universal quantification within the scope of $\Box$.

PR The plural rigidity axioms

1. $\Box\forall X\Diamond Xx \to \Box Xx$
2. $\Box\forall X\Diamond(\exists xXx \wedge x = y) \to \exists xXx \wedge x = y$.

Comp For any formula $\varphi(x, y, Y)$, $\Box\forall y\Box\forall Y\Box\exists X\forall x[Xx \leftrightarrow \varphi(x, y, Y)]$ is an axiom.

P-Choice A plural version of the axiom of choice.

M-Rep A modal version of the axiom scheme of replacement.

Inf An axiom saying that the set of natural numbers exists.

Sets The axioms of extensionality and the axiom of foundation, and an axiom $\Box\forall x\exists X\Box\forall y[Xy \leftrightarrow y \in x]$ asserting the rigidity of set membership

The precise details of many of these won't be too relevant below, but they are cited here for something like completeness' sake. More details can be found in CITE. The odd one out, of course, is **Inf**. It is included here for simplicity, since many of the phenomena we will be interested in occur only at the level of infinite sets.

The basic idea behind the core axioms is to get a modal and plural logic, combined with fundamental principles of set theory, to allow for the development of set theory based on potentialist principles of set existence. Let us now turn to that development.

3.2 Simple HWP

In this section I will present what seems to me to be the simplest formal combination of height and width potentialism. The theory is not mathematically strong, being exactly equivalent in strength to second order arithmetic. But it does offer a clean and simple proof of concept for the combination of height and width potentialism, as well as a useful starting point for further extensions.

The first axiom we add is:

HP $\Box\forall X \Diamond \exists x \forall y[y \in x \leftrightarrow Xy]$

- *Any things can be the elements of a set.*

HP enshrines the core idea of height potentialism, since it implies (given the usual notion of rank) that given any sets one can find others of still higher rank.

The second requires a little more preparation. In our background logic we can define the notion of a partial order, the notion of being a filter on a partial order, and the notion of being a dense set in a partial order in the standard way. Let blackboard variables $\mathbb{P}, \mathbb{Q}$ range over partial orders. (These are singular variables.) Let $D(X, \mathbb{P})$ mean that all Xs are dense in $\mathbb{P}$. Finally, let $F_\cap(x, \mathbb{P}, X)$ mean that x is a filter on $\mathbb{P}$ that intersects every one of the Xs. Then our width potentialist axiom can be stated as:

WP $\Box\forall \mathbb{P} \forall X D(X, \mathbb{P}) \rightarrow \Diamond \exists g(F_\cap(g, \mathbb{P}, X))$

- *Any partial order can be used in forcing.*

WP is one way to flesh out width potentialism, since it implies that any given sets may be extended to include ones of rank no greater than their max by forcing.

The following will be useful to us going forward. In it, $f : \mathbb{N} \twoheadrightarrow X$ is an abbreviation for the assertion that f is a function on $\mathbb{N}$ with every X in its range.

Proposition 1. *Let* **Count** *be the principle:*

$$\Box\forall X \Diamond \exists f[f : \mathbb{N} \twoheadrightarrow X]$$

then **Count** *is equivalent to WP over the core logic + HP.*

Proof. See CATBC. □

Let us call the result of adding these HP and either WP or **Count** to the core logic **Simple Height and Width Potentialism**, or SHWP.

Turning now to questions of consistency, it turns out that SHWP is demonstrably consistent relative to Second Order Arithmetic (SOA). In fact it turns out that SHWP exhibits a tight form of equivalence with SOA, something Tim Button has called ‘near synonymy’ in the recent literature.

(Here and below, we understand SOA under the guise of ZFC without power + all sets are countable. This is definitionally equivalent to the more standard arithmetical formulations; see e.g. Simpson.)

A detailed account of the relationship between SOA and SHWP would be overkill here. (Details can be found in CITE.) But certain features will be important to our discussion of consistency below, and having some idea of how the interpretation of the modal theory SHWP in the non-modal SOA goes will be very helpful to us going forward. So we will spend some time discussing certain aspects of the ‘tight equivalence’ just mentioned.

Let $\mathcal{L}_\in$ be the first order language of set theory. Then:

Theorem 2. *There is a map $\cdot^\diamond : \mathcal{L}_\in \to \mathcal{L}_0$ that preserves theoremhood from SOA to SHWP.*

Theorem 3. *There is a map $\cdot^\exists : \mathcal{L}_0 \to \mathcal{L}_\in$ that preserves theoremhood from SHWP to SOA.*

Theorem 2 says that SHWP interprets SOA; this direction does not really concern us directly, since it does not imply anything about the consistency of SHWP, but some idea of how things go will be useful. The translation $\varphi \mapsto \varphi^\diamond$ proceeds by prepending every universal quantifier in φ by a $\Box$ and every existential by a $\Diamond$. A result due to Linnebo says that for $\varphi \in \mathcal{L}_\in$,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma^\diamond \vdash_{Core} \varphi^\diamond$$

where $\Gamma^\diamond$ has the obvious meaning of $\{\gamma^\diamond : \gamma \in \Gamma\}$. It thus suffices to prove $\varphi^\diamond$ for each axiom of SOA in SHWP, which (using Proposition 1) is not difficult.

Theorem 3 is directly on-topic. It says that SOA interprets SHWP and hence secures the consistency of the latter relative to the former. Full details of the ideal proof are a little fiddly and would go beyond our needs here. But some idea, again, of how things go down will be useful.

The key idea behind the translation $\varphi \mapsto \varphi^\exists$ is to factor out use of modal operators in favor of quantification over possible worlds.

In a bit more detail, the idea is that we define the notion of a possible world in SOA to be a transitive set. That means we interpret the language of SHWP so that ordinary quantifiers are always restricted to such worlds, and the modal operators induce quantifiers over such worlds.

Formally, this means our translation will have to carry formulas φ in the modal language (which may be sentences) to formulas $\varphi^\exists(w)$ in the first order language with a free 'world' variable w. Key clauses of the translation are things like

$$(\forall x\varphi)^\exists(w) := \forall x \in w\varphi^\exists(w)$$
$$(\forall X\varphi)^\exists(w) := \forall x \subseteq w\varphi^\exists(w)$$
$$(\Box\varphi)^\exists(w) := \forall u[Tran(u) \wedge w \subseteq u \to \varphi^\exists(u)]$$

where $Tran(u)$ of course is the assertion that u is transitive. (The case of plural containment Xx is a little fiddly, but can be made to work quite nicely.) The potentialist axioms are then readily seen to be true, under the translation, in SOA. For HP, this is because every set is an element of a transitive set. For WP, this is because SOA proves all sets are countable. So for any given partial order, we may move to a transitive set that witnesses its countability, and then proceed to introduce a generic for it if need be.

A final point is that these results can be strengthened to attain something approximating definitional equivalence between the two theories. ...

3.3 Strong HWP

The theory I have just cited combines height and width potentialism in a straightforward way. The result is a theory that is (up to a near-synonymy) second order arithmetic. This is a pretty weak theory, although as we know from the reverse mathematics literature it is plenty strong enough to develop much of the mathematics needed in applications. Can the height and with potentialist do better?

There is indeed a reasonable way to proceed here. Our potentialist is interested with expansions of the universe along two 'directions'. One has the ability to extend the universe 'upwards' to create sets of higher rank; and one has the ability to extend it 'outwards' by forcing. By separating out these two possible methods of expansion, and by asserting strong axioms regarding what can be attained along the vertical dimension of expansion alone, much stronger HWP systems can be derived, indeed ones with all the power of ZFC and more. This extension makes the general picture of height and width potentialism much more philosophically interesting, since such versions have at least the consistency strength to recover all standard mathematics. But they also bring new conceptual problems with them, problems that will be exploited in one of the inconsistency arguments we will consider.

Let us first discuss in a little more detail how to implement this idea. We expand the language $\mathcal{L}_0$ to $\mathcal{L}_1$ by adding in two new modal operators, ⦶ and ⦸. ⦶ is to reflect possibility by *only* vertical expansion: one can think of this as possibility by *only* iteratively introducing new sets for given pluarlities. ⦸, on the other hand, is to reflect possibility by *only* horizontal expansion: one can think of this, indifferently, as just by adding new generic filters, or adding new enumerating functions. (◫ and ⧅ are the duals.) $\Diamond$ remains in the language, and represents 'absolute' possibility, that is, possibility by either domain expansion method.

Here is a way to axiomatize a 'strong' system of this kind.

We begin by modifying HP and WP.

HP $\Box\forall X$ ⦶ $\exists x\forall y[y \in x \leftrightarrow Xy]$

WP $\Box\forall\mathbb{P}\forall X D(X,\mathbb{P}) \to$ ⦸$\exists g(F_\cap(g,\mathbb{P},X))$

Next, we add rules to reflect the generality of $\Diamond$.

Gh ⦶$\varphi \to \Diamond\varphi$

Gw ⦸$\varphi \to \Diamond\varphi$

Finally, add the following restricted version of the powerset axiom to vertical possibility.

r-Pow $\Box\forall x$ ⦶ $\exists y$ ◫ $\forall z[z \in y \leftrightarrow z \subseteq x]$

Pow says that it is always possible by vertical expansion to get all the subsets of any given set, *so long as one disregards the subsets you can get by forcing.*

It is useful to compare r-Pow with the corresponding 'unrestricted' version

Pow $\Box\forall x$ ⦶ $\exists y\Box\forall z[z \in y \leftrightarrow z \subseteq x]$

which is provably inconsistent with **WP** over the rest of the theory. Pow says that it is possible for there to such things as *all possible* subsets of any given set; but given the existence of an infinite set (as we are guaranteed), and the universal possibility of forcing, enshrined in **WP**, this cannot be, since we can always force to add new subsets to any given set.

Let the extension of the core theory by the above principles (but not, of course, Pow) be called HWP. The following facts come easily.

Theorem 4. *There is a translation* $\cdot^{⦶} : \mathcal{L}_\in \to \mathcal{L}_1$ *that preserves theoremhood from ZFC to HWP.*

Theorem 5. *There is a translation* $\cdot^{\Diamond} : \mathcal{L}_\in \to \mathcal{L}_1$ *that preserves theoremhood from SOA to HWP.*

In each case, the result follows much as it did in the previous case. In fact Theorem 5 really just is Theorem 2. The first uses the same translation but with $\obar$ in place of $\Diamond$ and $\boxbar$ in place of $\Box$ everywhere. That the theorem goes through is a known theorem of Linnebo in a slightly different key.

4 Roberts