On the Consistency Strength of Axioms for Height and Width Potentialism

Chris Scambler
All Souls College
University of Oxford

November 22, 2021

1 Introduction

In CITE, I presented an axiom system for height and width potentialism combined. Roughly speaking the axiom system is motivated by the idea of an iterative set construction process in which two 'acts' are possible at each stage – firstly, that of collecting some things into a set, and secondly, that of enumerating some things in terms of the natural numbers (or, equivalently, forcing). I proved the system consistent on the assumption a Mahlo cardinal existed, and showed that the system interpreted ZFC under an 'inner' modal translation, and that second order arithmetic was interpretable under a more general modal translation. This article improves on the consistency result given there. I will show that in fact the theory is equi-consistent with second order arithmetic extended with the Π^1_1 perfect set property, and hence also with ZFC. I will also explore the question of bi-interpretability.

2 The Axiom System

2.1 Language

The language $\mathcal{L}_\diamond$ is multi-modal and in fact contains three modal operators, $\Diamond_\uparrow$, $\Diamond_\leftarrow$, and $\Diamond$. $\Diamond_\uparrow\varphi$ should be read as: 'by repeated acts of collection, φ can be made true'; that is, by repeatedly taking some things and producing the set with them as its elements, φ can be made true. (We allow any number of repetations, from 0 into the transfinite.) $\Diamond_\leftarrow$ can be interpreted in one of two ways: either as 'by repeated acts of enumeration, φ can be made true', where here by enumeration we mean the act of correlating some given things with the natural numbers; or alternatively 'enumeration' may be replaced by 'forcing'. The results are equivalent in the sense demonstrated in CITE. $\Diamond$ is the 'most general' modality, and represents possibility by arbitrary iterations of either domain expansion technique.

$\mathcal{L}_\diamond$ is also a monadic second order language, with standard, singular 'objectual' variables x and second order 'plural' variables X; these latter range over things taken many at a time (e.g. the members of an orchestra) rather than individuals (e.g. the conductor). In addition, we will of course use the membership symbol $\in$ and identity relation $=$. Atomic formulas are of the form $x = y$, $X = Y$, $x \in y$, and Xx. Compound formulas are formed from these in the usual way, and we assume all the usual definitions, e.g. of $\Box$ in terms of $\Diamond$ and $\neg$.

2.2 Logic

The logical (non-set-theoretic) axioms can reasonably naturally be separated into those concerning the first-order part of the language, those that concern the modals, those that govern the second order variables, and those that concern identity.

For the first order part, we take the axioms to be any standard system of first order quantification logic, with universal instantiation weakened to its 'free' version, that is, with the universal instantiation axiom written in the form:

$$\forall x[\forall y[\varphi y] \to \varphi x] \tag{1}$$

With regards the modal logic, we assume S4.2 for each modal operator, which (given any standard axiom system) will imply the converse Barcan formula. We also take necessitation to be part of the system; and, in accordance with the idea that $\Diamond$ is the most general modal at issue, we take each of the 'weakening' principles

$$\Diamond_\uparrow \varphi \to \Diamond \varphi \tag{2}$$

$$\Diamond_\leftarrow \varphi \to \Diamond \varphi \tag{3}$$

as further axioms. The standardly valid inference rule

$$\frac{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\Psi)...)}{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\forall x\Psi)...)}$$

will also be assumed.

As to the second order logic, we assume full comprehension *in closed form*, so all expressions of the form

$$\Box\forall\vec{z}\Box\forall\vec{Z}\exists X\forall x[Xx \leftrightarrow \Phi(x, z, Z)] \tag{4}$$

are axioms. Here of course X may not appear free in Φ, but we assume all variables other than x that are free in Φ occur in the lists $\vec{z}, \vec{Z}$. We also assume a version of the axiom of choice, to the effect that if X comprises disjoint non-empty sets then there is Y containing exactly one element of each component of X. (Formalizing this in our language is easy enough, using the usual definitions of non-empty and disjoint in terms of $\in$ and $=$.)

Finally we turn to the axioms concerning identity, which also tend to involve all the other components of the language. As usual, we assume reflexivity and

Leibniz law for first and second order identity; and as usual we are able to derive the necessity of identity from these in the form $\forall x \forall y[x = y \rightarrow \Box x = y]$, analogously for X and Y.[1] However, since our modal logic is not symmetric, we must add the necessity of distinctness ($\forall x \forall y[x \neq y \rightarrow \Box[x \neq y]]$) 'by hand'.

Our conception of identity for 'plural' variables X is in addition *strongly extensional*, meaning that plurals comprise the same things at all possible worlds. We can enforce this conception axiomatically using the following principles.

$$\forall x[Xx \leftrightarrow Yx] \rightarrow X = Y \tag{5}$$

$$\Diamond Xx \rightarrow \Box Xx \tag{6}$$

$$\Diamond \exists Xx \wedge x = y \rightarrow \exists x Xx \wedge x = y \tag{7}$$

Given the inference rule mentioned above, the latter implies a version of the Barcan formula for Xx, and hence that pluralities do not 'pick up' new components from world to world. Overall, the effect is to ensure (speaking model-theoretically for a moment) that Xx holds at some possible world iff it holds at all possible worlds, and that some things are the same things as some others when and only when they are composed of the same individuals.

[1] Note however that this implies nothing about inexistent values of variables, which may be contingently identical given our axioms up to now. That is, $x = y \wedge \Diamond x \neq y$ is consistent.