On the Consistency Strength of Axioms for Height and Width Potentialism

Chris Scambler
All Souls College
University of Oxford

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1 Introduction

In CITE, I presented an axiom system for height and width potentialism combined. Roughly speaking the axiom system is motivated by the idea of an iterative set construction process in which two 'acts' are possible at each stage – firstly, that of collecting some things into a set, and secondly, that of enumerating some things in terms of the natural numbers (or, equivalently, forcing). I proved the system consistent relative to ZFC + the existence of a Mahlo, showed that the system interpreted ZFC under a restricted modal translation, and that second order arithmetic was interpretable under a more general, 'full' modal translation. This article improves on these results in various ways. I will show that in fact the modal theory in question is equi-consistent with second order arithmetic extended with the Π^1_1 perfect set property, and hence also with ZFC. I will also explore the question of bi-interpretability.

2 The Axiom System

2.1 Language

The language $\mathcal{L}_\diamond$ is multi-modal and in fact contains three modal operators, $\Diamond_\uparrow$, $\Diamond_\leftarrow$, and $\Diamond$. $\Diamond_\uparrow\varphi$ should be read as: 'by repeated acts of collection, φ can be made true'; that is, by repeatedly taking some things and collecting them together into a set, φ can be made true. (We allow any number of repetitions, from 0 into the transfinite.) $\Diamond_\leftarrow$ can be interpreted in one of two ways: either as 'by repeated acts of enumeration, φ can be made true', where here by enumeration we mean the act of correlating some given things with the natural numbers; or alternatively 'enumeration' may be replaced by 'forcing', where this is understood as the act of introducing a filter meeting all the given dense sets of some partial order. (The resulting interpretations are equivalent in the sense demonstrated in CITE.) The remaining modal, $\Diamond$, is the 'most general' modality, and represents possibility by arbitrary iterations of either domain expansion technique.

$\mathcal{L}_\Diamond$ is also a monadic second order language, with standard, singular 'objectual' variables x and second order 'plural' variables X; these latter range over things taken many at a time rather than individuals; so for example the members of an orchestra and the string section are each possible values of the monadic second order variables, while the conductor and the first chair violinist would be possible values for the first order variables. In addition, we will of course use the membership symbol $\in$ and identity relation $=$. Atomic formulas are of the form $x = y$, $X = Y$, $x \in y$, and Xx. Compound formulas are formed from these in the usual way, and we assume all the usual definitions, e.g. of $\Box$ in terms of $\Diamond$ and $\neg$.

2.2 Logic

The logical (non-set-theoretic) axioms can reasonably naturally be separated into those concerning the first-order part of the language, those that concern the modals, those that govern the second order variables, and those that concern identity.

For the first order part, we take the axioms to be any standard system of first order quantification logic, with universal instantiation weakened to its 'free' version, that is, with the universal instantiation axiom written in the form:

$$\forall x[\forall y[\varphi y] \to \varphi x] \tag{1}$$

With regards the modal logic, we assume $\mathsf{S4.2}$ for each modal operator, which (given any standard axiom system) will imply the converse Barcan formula. We also take necessitation to be part of the system; and, in accordance with the idea that $\Diamond$ is the most general modal at issue, we take each of the 'weakening' principles

$$\Diamond_\uparrow \varphi \to \Diamond \varphi \tag{2}$$

$$\Diamond_\leftarrow \varphi \to \Diamond \varphi \tag{3}$$

as further axioms. The standardly valid inference rule

$$\frac{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\Psi)...)}{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\forall x\Psi)...)}$$

will also be assumed.

As to the second order logic, we assume full comprehension *in closed form*, so all expressions of the form

$$\Box\forall \vec{z}\Box\forall \vec{Z}\exists X\forall x[Xx \leftrightarrow \Phi(x, z, Z)] \tag{4}$$

are axioms. Here of course X may not appear free in Φ, but we assume all variables other than x that are free in Φ occur in the lists $\vec{z}, \vec{Z}$. We also assume a version of the axiom of choice, to the effect that if X comprises disjoint non-empty sets then there is Y containing exactly one element of each component of X. (Formalizing this in our language is easy enough, using the usual definitions of non-empty and disjoint in terms of $\in$.)

Finally we turn to the axioms concerning identity, which also tend to involve all the other components of the language. As usual, we assume reflexivity and Leibniz law for first and second order identity; and as usual we are able to derive the necessity of identity from these in the form $\forall x \forall y[x = y \to \Box x = y]$, analogously for X and Y.[1] However, since our modal logic is not symmetric, we must add the necessity of distinctness ($\forall x \forall y[x \neq y \to \Box[x \neq y]]$) 'by hand'.

Our conception of identity for 'plural' variables X is in addition *strongly extensional*, meaning that plurals comprise the same things at all possible worlds. We can enforce this conception axiomatically using the following principles.

$$\forall x[Xx \leftrightarrow Yx] \to X = Y \tag{5}$$

$$\Box \forall X, x[\Diamond Xx \to \Box Xx] \tag{6}$$

$$\Box \forall X[\Diamond \exists x[Xx \wedge x = y] \to \exists x[Xx \wedge x = y]] \tag{7}$$

Given the inference rule mentioned above, the latter implies a version of the Barcan formula for Xx, and hence that pluralities do not 'pick up' new components from world to world. Overall, the effect is to ensure (speaking model-theoretically for a moment) that Xx holds at some possible world iff it holds at all possible worlds, and that some things are the same things as some others when and only when they are composed of the same individuals.

2.3 Set Theory

With the background logic in tow we can formalize the axiomatic system of height and width potentialism that will be the target of our investigations.

The set-theoretic axioms can themselves be reasonably naturally divided into two categories. First, there are those that concern the identity conditions for sets. Second, there are those that concern possible set existence, and that make assertions about the kinds of sets it is possible to produce by iterating our various construction procedures.

First, on the side of identity conditions. We would like sets to have the members they have as a matter of necessity, so that (like plurals) they have exactly the members they have at any world in all worlds (to slip into model-theoretic talk again). This can be imposed by the following pair of axioms:

$$\forall z[z \in x \leftrightarrow z \in y] \to x = y \tag{8}$$

$$\exists X \Box \forall z[Xx \leftrightarrow z \in x] \tag{9}$$

where x, in the latter, is an arbitrary set parameter. Analogs of (6) and (7) for $\in$ can be derived using (6), (7) and (9). As a final constraint of sorts on the

[1] Note however that this implies nothing about inexistent values of variables, which may be contingently identical given our axioms up to now. That is, $x = y \wedge \Diamond x \neq y$ is consistent.

identity conditions for sets, we impose the standard axiom of foundation, which says that every non-empty set has a member with which it shares no members.

As to the set-theoretic axioms, we begin with a discussion of the distinctive axioms for height potentialism, for which we will follow the development of Øystein Linnebo in CITE. Here the central axiom is

$$\Box \forall X \Diamond_{\uparrow} \exists x[Set(x, X)] \tag{10}$$

where $Set(x, X)$ is an abbreviation for $\forall y[y \in x \leftrightarrow Xy]$. This axiom can intuitively be read as saying that any possible things are possibly the elements of a set; it leads to a form of indefinite extendibility of the universe of sets in light of the modal logical derivability of $\Box \exists X \neg \exists x[Set(x, X)]$ (Russell's paradox).

This axiom by itself guarantees the possible existence of each hereditarily finite set, but in the spirit of pursuing transfinite set theory in the potentialist system we will want to push things further. As a first step, consider the axiom

$$\Diamond_{\uparrow} \exists X \Box \forall x[Xx \leftrightarrow Nat(x)] \tag{11}$$

where here $Nat(x)$ can be any of your favorite definitions of natural number (e.g. finite von Neumann ordinal.) (11) says that by repeated acts of collection one can eventually produce all possible natural numbers, and given this a further application of (10) secures the possible existence of an infinite set.

(11) is a natural analogue of the axiom of infinity in the potentialist setting. The natural analogue of the powerset axiom would be

$$\Diamond_{\uparrow} \exists X \Box \forall x[Xx \leftrightarrow x \subseteq y] \tag{12}$$

where $\subseteq$ is defined as usual and y is an arbitrary set parameter. However this axiom will be false on the intended interpretation, since given any infinite y we will have that it is always possible to introduce new subsets of y in the form of enumerating functions / generic filters. But instead of giving up altogether on the infinitary mathematics that goes along with powerset, we will instead adopt the 'restriction' of the principle to the modality $\Box_{\uparrow}$. The idea will be that, given any arbitrary set y, by repeatedly introducing sets one will eventually get all the subsets of it that one can ever get *without using forcing.* The axiom thus reads:

$$\Diamond_{\uparrow} \exists X \Box_{\uparrow} \forall x[Xx \leftrightarrow x \subseteq y] \tag{13}$$

This is a kind of local powerset axiom for 'inner models'; the precise sense in which this is true will be made apparent in more detail below.

We will also adopt the following axiom, distinctive of width potentialism. In it, we let $D(x, X)$ abbreviate the claim that x is a partial order and X contains all the dense subsets in x; and $Fmeets(x, X)$ will abbreviate the claim that x is a filter that meets all the sets in X.

$$\Box \forall X, x[D(x, X) \rightarrow \Diamond_{\leftarrow} \exists g[Fmeets(g, X)]] \tag{14}$$

One can show, using the other axioms, that (14) implies the negation of (12).

Finally, we adopt certain instances of the replacement axiom (really the collection principle.) To state the relevant instances, we first need a definition.

Definition 2.1 (○-translation)**.** Let ■ be a modal operator, let ⧫ be its dual, and let φ be a formula in the first order language of set theory. Then the ■-translation of φ, written $\varphi^{\blacklozenge}$, is the result of prepending each universal quantifier in φ with ■, and each existential quantifier in φ with a ⧫.

The relevant instance of replacement, then, are all the $\Diamond_{\uparrow}$ and $\diamond$-translations of first-order instances of replacement. In effect, we are asserting that replacement is valid both with respect the upwards modality alone, and with repsect the modality that combines height and width potentialism.

3 Basic Facts

In this section we articulate some basic facts about the axiom system just presented, along with some general results that will prove useful later on.

Lemma 3.1 (Mirroring)**.** *Let ■ be any modal operator in $\mathcal{L}$, and let $\Gamma \cup \phi$ be a set of formulas in the first order language of set theory. Then $\Gamma \vdash \phi$ in first order logic if and only if $\Gamma^{\blacklozenge} \vdash_{\mathsf{M}} \phi^{\blacklozenge}$.*

Lemma 3.2 (Bounded Modal Absoluteness)**.** *Let ■ be any modal operator in $\mathcal{L}$, and let ϕ be a formula in the first order language of set theory with only bounded quantifiers. Then $\mathsf{M} \vdash \phi \leftrightarrow \phi^{\blacklozenge}$.*

Proof. An induction on the complexity of ϕ. See Lemma 12.2 of Linnebo. □

Theorem 3.3. M *interprets ZFC under the $\Diamond_{\uparrow}$-translation.*

Theorem 3.4. M *interprets ZFC$^-$ under the $\Diamond$-translation.*

Theorem 3.5. M *proves every set is hereditarily countable under the $\Diamond$ translation.*

Corollary 3.6. M *interprets second order arithmetic under the $\Diamond$ translation.*

Proof. $ZFC^- + V = HC$ is definitionally equivalent to SOA. See Krapf, Simpson. □

4 The Results

4.1 The Correlated Theory

The correlated first order set theory will be $T = \mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC} + \Pi^1_1 PSP$.

Define perfect set property: $X \subseteq \omega^\omega$ is perfect iff it is the set of paths through a perfect tree on $\omega \times \omega$. Note that $X \subseteq \omega^\omega$ means, in this context, that we have a formula φ with the property that $\varphi(f) \rightarrow f$ is a function from ω to ω. Such an X is said to be Π^1_1 (other complexities) (in a parameter r) iff the corresponding φ has the form... .

Some relevant facts about the correlated theory.

It interprets ZFC, indeed it proves that $ZFC^{L[r]}$ for every real r.

. . .

4.2 An interpretation of M in T

Let $M \vDash SOA + \Pi^1_1 PSP$.

$t : \mathcal{L}_1 \times M \times \mathbb{R}^M \to \mathcal{L}_\in, (\varphi, T, r) \mapsto \psi(T, r)$

- assign plural variables odd numbered variables $t(X)$, and singular variables even numbered variables $t(x)$.
- Membership = id, commutes with propositional connectives
- $t(Xx)(T, r) := t(x) \in t(X)$
- $t(\forall x\varphi)(T, r) := \forall x \in T[t(\varphi)(T, r)]$
- $t(\forall X\varphi)(T, r) := \forall x \subseteq T[x \in L[r] \to t(\varphi)(T, r)]$
- $t(\Box_\uparrow \varphi)(T, r) := \forall S \supseteq T[Tran(S) \wedge S \in L[r] \to t(\varphi)(S, r)]$
- $t(\Box_\leftarrow \varphi)(T, r) := \forall s[r \in L[s] \to \forall S[S \supseteq T \wedge rank(S) = rank(T) \wedge S \in L[s] \to t(\varphi)(S, s)]]$
- $t(\Box\varphi)(T, r) := \forall s, S[r \in L[s] \wedge T \subseteq Tran(S) \in L[s] \to t(\varphi)(S, s)]$

Theorem 4.1. $\mathsf{M} \vdash \varphi$ *implies* $\mathsf{T} \vdash \forall r, T[r \in \mathbb{R} \wedge T \in L[r] \wedge Tran(T) \to t(\varphi)(T, r)]$

Proof. Induction on the complexity of proofs. Axioms. Consider for example Free instantiation: $\forall x[\forall y\varphi y \to \varphi x]$. Suppose T and r given; this translates to $\forall x \in T[\forall y \in Tt(\varphi y)(T, r) \to t(\varphi x)(T, r)]$. But this is trivial. Note that unfree instantiation fails since we might have counterexamples outside T.

The plural rigidity axioms are tedious but routine. As an example, consider (7). By way of interpreting the outer $\Box$, let $s \in L[s]$ and $T \subseteq S \in L[s]$. Then for the plural quantifier we assume $t(X) \subseteq S$, and for the next $\Diamond$ we take a transitive extension S' of S in $L[s']$ ($s \in L[s']$). We get as a hypothesis that there exists $t(x) \in S'$ with $t(x) \in t(X)$. But since $t(X) \subseteq S$ we must already have had $t(x) \in S$, and clearly $t(Xx)(S, s)$, since this just amounts to the given $t(x) \in t(X)$, as required.

For plural comprehension we can use separation in $L[r]$. Let $\Phi(x, z, Z)$ be a formula with the exhibited free variables. It is readily seen (applying the clauses of the translation to (4)) to suffice to show that in any transitive extension S of T contained in some $L[s]$ ($r \in L[s]$) containing $t(z)$ and $t(Z)$, the set $\{x \in S : \Phi(x, t(z), t(Z))\}$ is a member of $L[s]$. But this follows immediately from the definition of $L[s]$. (Note that this only works because we took (4) in closed form, with quantifiers bounded and necessitated. This has the effect of restricting us to cases where the parameters are only allowed to come from S and hence in $L[s]$; if this restriction is lifted, it is easy to find counterexamples to comprehension under the translation. For example, if S contains all reals in $L[s]$, the instance for $\Phi := x \in s'$ for any real s' not in $L[s]$ will do.)

On to the set-theoretic axioms. First, (10). Given any transitive $S \in L[s]$, and any subset X of S also in $L[s]$, the translation follows by considering the transitive set $S \cup \{X\} \in L[s]$. For (11) one can use the set of natural numbers itself as the relevant transitive set, which is an element of any $L[s]$. As to power, given $S \in L[s]$ of the relevant kind and any set $x \in S$, we can use the fact that ZFC hold in $L[s]$ to guarantee that $\mathcal{P}(S)^L[s]$ is in $L[s]$, and hence the transitive set $S \cup \mathcal{P}(S)^L[s]$ will do the trick.

As for replacement, the $\Diamond_\uparrow$ translation follows from the fact that M proves replacement for each $L[r]$, and the $\Diamond$ translation follows from the same for the model at large.

□

4.3 An interpretation of T in M

For the converse, it is easy to show $SOA^\diamond$. It remains to show $\Pi^1_1 PSP^\diamond$.

Work in M, and let r be an arbitrary real. It suffices to show $\Pi^1_1[r]$-$PSP^\diamond$.

We will use the following lemma, which is a slight modification of a theorem of Solovay.

Lemma 4.2 (SOA)**.** *If there are only countably many reals constructible from r, then the $\Pi^1_1[r]PSP$ holds.*

The proof is a minor modification of Solovay's argument. It uses the fact that in SOA one can show that any Π^1_1 set admits a decomposition into ORD-many Borel sets (Solovay used $\aleph_1$ in place of ORD, but the proof works just fine without this assumption).

Using Lemma (4.2) and the mirroring theorem, it is sufficient to show the diamond-translation of the hypothesis to get our result. That is: for any real r, it is possible that there is a function on the natural numbers such that, necessarily, if s is a real constructible from r, then s is in the range of f.

To prove this, first observe that by the $\Diamond_\uparrow$-translation of ZFC, we can show that for any real r

$$\Diamond_\uparrow \exists y[\exists x \in y[\mathbb{R}^L[r](x)^{\Diamond_\uparrow}]] \tag{15}$$

here, $\mathbb{R}^L[r](x)^{\Diamond}_{\uparrow}$ is the $\Diamond_\uparrow$-translation of the first order formula asserting that x is the set of reals constructible from $L[r]$. The point of y is to get a set that can act as a bound of the quantifiers in the relevant formula. Such a bound clearly can be found, e.g. $L[r]_{\omega_1+1^{L[r]}}$. The reason for this is so that we can apply lemma 3.2 to get

$$\Diamond_\uparrow \exists y[\exists x \in y[\mathbb{R}^L[r](x)]] \tag{16}$$

by dropping all the modal operators, and then applying the same lemma 'in reverse' we can infer

$$\Diamond \exists x[\mathbb{R}^L[r](x)^{\Diamond}] \tag{17}$$

that is, the $\Diamond$ translation of the assertion that it is possible that all possible reals constructible from r exist. But now since we also have $V = HC^\diamond$, it follows that

$$\Diamond \exists x[\mathbb{R}^L[r](x)^{\Diamond} \wedge \exists f : \mathbb{N} \twoheadrightarrow x] \tag{18}$$

which is just the $\Diamond$-translation of the claim that there are only countably many reals constructible from r. Hence, by an application of mirroring, we conclude $\Pi^1_1[r]$-PSP. The result follows.

4.4 Bi-interpretation?

We have

Theorem 4.3. $T \vdash \mathbb{R}(r) \wedge Tran(T) \wedge T \in L[r] \rightarrow t(\varphi^\diamond)(T, r) \leftrightarrow \varphi$

which is a routine induction on complexity. Things with the converse are slightly less clear. We would like to show

Conjecture 4.4. $\mathsf{M} \vdash \mathbb{R}(r) \wedge Tran(T) \wedge T \in L[r] \rightarrow t((\varphi)(T, r))^\diamond \leftrightarrow \varphi$

but the restrictions to $L[r]$ imposed in the inner translation make for difficulties.