What is Forcing Potentialism?

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Potentialism about some domain of objects is the view that that domain should be regarded as *potentially infinite* rather than actually infinite. For instance, arithmetic potentialism is the view that the natural numbers are merely potentially infinite and do not form an actual infinity. Geometric potentialism is the view that geometric objects form a potential infinity. Some forms of potentialism might be coupled with a denial that there are any actual infinities, while other forms of potentialism might hold that some infinities are actual while others are merely potential. For instance, the Fefermanian predicativist might hold that the natural numbers form an actual infinity while the sets of natural numbers are a potential infinity. Set-theoretic potentialism is the view that the universe of sets V is (in some respect) potentially infinite.

The essential characteristic of a potential infinity is that it can always be *extended*. To say the natural numbers are potentially infinite is to say that no matter which numbers you give me, I can always add more numbers to that collection. To say that the geometric objects are potentially infinite is to say that no matter which geometric objects you give me, I can always construct more geometric objects. As a corollary, if a domain of objects is potentially infinite, there can be no definite collection of *all* the entities in that domain; for otherwise, if I were given that collection of *all* the objects, I could not find any new objects to add to that collection.

One version of set-theoretic potentialism is *forcing potentialism*. This is the view, roughly, that the universe of sets is potentially infinite because, given any universe V we can add new sets to it by forcing, thus finding ourselves in the generic extension $V[G]$. While the technical tools of potentialism have been applied to study forcing extensions of universes, leading to something that has been called 'forcing potentialism', the philosophical content of such a view has not been clearly laid out. In this paper I pose a challenge for forcing potentialism, arguing that a philosophical account of forcing potentialism must revise the familiar iterative conception of set. After developing this challenge, I will propose one way of meeting it by adding a small new detail to the story behind the iterative conception of set.

I begin in §1 by explaining potentialism in more detail. In the set-theoretic case there is an important distinction between height potentialism and width potentialism. Forcing potentialism is a form of width potentialism. §2 then rehearses the story behind the iterative conception of set and explains how it can motivate height potentialism. In §3 I develop the challenge for forcing potentialism by arguing that it is inconsistent with the iterative conception of set. Hence an account of forcing potentialism requires some new way of thinking about the cumulative hierarchy of sets. In §4 I develop such an account of forcing potentialism by adding a new twist to the story behind the iterative conception of set.

1 Potentialism

If potentialism is the view that a given domain is potentially infinite, we should say something about potential infinity and its contrast with actual infinity. A collection is actually infinite if it forms a definite, completed totality of objects larger than any finite cardinality. By contrast, a collection C is potentially infinite when any definite, completed subcollection of C can be expanded to a more inclusive subcollection of C. Potential infinity is an ancient concept, tracing its origins back to Aristotle, who wrote:

> [T]he infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different. (*Physics* 206a27-29).[1]

For instance, consider the natural numbers and suppose we start by being given the number 1. Then we may also 'take' the number 2, then 3, and so on. At each stage we have a finite collection of natural numbers, but it is always a different, larger collection of natural numbers. Potential infinity is an unboundedness.

Recent work on potential infinity has focused on the modal nature of potential infinity: *necessarily*, for any subcollection of C, it is *possible* to find an object x that is in C but is not in the given subcollection. Or, in Aristotle's temporal terms, *whenever* we have taken some C's, we can *later go on* to take some more C's.[2] Thus, for instance, while an actualist might express the infinitude of the natural numbers as:

$$\forall x \exists y (x < y),$$

the potentialist might express the *potential* infinitude of the natural numbers as:

$$\Box \forall x \Diamond y (x < y)$$

Similarly, where the actualist about geometry might say that for every triangle ABC there is line segment AD bisecting the leg BC, the potentialist about geometry would only grant

[1]Cf. *Physics* 207a6, Aristotle 1941.

[2]Linnebo 2013 developed this modal approach to potential infinity in the context of set theory. (Studd 2013 had also used modal logic to explicate the iterative conception of set, though without explicitly making the connection to potential infinity.) Linnebo and Shapiro 2017 subsequently generalized these ideas to give a modal account of potential infinity as such.

that necessarily, for every triangle ABC, it is *possible* for there to be a line segment AD that bisects BC.

Potentialist approaches to set theory can be broadly divided into height potentialism and width potentialism. Height potentialism is the view that for any universe V of set theory, it is possible to extend V to a 'taller' universe V'.[3] One simple version of this view is that the cumulative heirarchy of V_θ's is potentially infinite: $\Box\forall\theta\Diamond\exists\lambda(V_\theta \in V_\lambda)$. Width potentialism is the view that for any universe V of set theory there is a 'wider' universe V'. Forcing potentialism is a version of width potentialism, holding that the generic extensions of a model of set theory M are potentially infinite. Suppressing some details, it may be expressed modally:

$$\Box\forall M\Diamond\exists G(M \subset M[G])$$

The technical side of forcing potentialism involves using modal logic to study the properties of forcing extensions. This side of the idea has been fairly extensively pursued.[4] But potentialism has a philosophical side, as well. It is a claim about the nature of a certain domain of objects; and while using modal logic provides a formal framework for explicating the concept of potential infinity, the formalism has to be motivated and interpreted in a given setting. Why should we think of a given domain as being potentially infinite? And what is the nature of the modality in that domain? In this paper, my attention will be primarily on the first question. How can we understand the nature of set theory so as to motivate forcing potentialism? Or in other words, what conception of set can the forcing potentialist be working with?

To appreciate the significance of such questions, it will be helpful to contrast forcing potentialism with multiversism. Multiversism is the view that there are many different but equally valid universes of set theory. Take, for example, the version multiversism introduced by Woodin 2011,[5] which requires that the multiverse be closed under generic extensions and generic refinements.[6] The truths of set theory are the truths that hold in all universes in the multiverse. Now, modal logic can have a role to play in studying the generic multiverse, because it provides tools for analyzing the relationships between the theory of a universe and the theories of its forcing extensions. But the fact that multiversism and forcing potentialism both use modal logic to understand the relations between models and their forcing extensions does not make them the same view. To the contrary, they are prima facie very different views.[7] Multiversism is a pluralistic thesis about the objects of set theory, whereas forcing potentialism is seemingly a thesis about the character of

[3]One universe being 'taller' than another can be made precise in terms of having more ordinals.

[4]Hamkins 2003, Hamkins and Löwe 2005, Hamkins, Leibman, and Löwe 2015, Hamkins and Linnebo 2019, **scambler21**

[5]There are a variety of versions of multiverse views in the literature. See, e.g., Hamkins 2012, Steel 2014, Maddy and Meadows 2020, Antos et al. 2015.

[6]N is a generic refinement of M iff M is a generic extension of N. In so-called set theoretic geology, generic refinements are also known as *grounds*.

[7]Thus, I think Hamkins and Linnebo 2019, p. 2 are a little too quick in seeing "strong affinities between set-theoretic potentialism and set-theoretic pluralism, particularly with the various set-theoretic multiverse

universe of sets. So put, the difference between the two views is intuitive, but vague. To make this intuitive difference clear and precise, we need a clear and precise philosophical account of forcing potentialism. In other words, is there a way that we can motivate forcing potentialism so that it does not collapse into multiversism?

I have two goals in this paper. First, I want to show that there is a serious challenge in giving a philosophical account of forcing potentialism. Setting forth this challenge will be the task of sections 2 to 3. Second, I will try to meet this challenge in sections 4 and 5 by sketching a philosophical story about set theory and then building a formal theory based on that story. The resulting theory will HAVE SUCH AND SUCH PROPERTIES. I want to emphasize from the outset that my goal is not to try to capture everything that has been said in the name of forcing potentialism. Rather, I want to see how much of what has gone under the name 'forcing potentialism' has a coherent philosophical rationale. More specifically, I want to see how much of 'forcing potentialism' can find a coherent philosophical rationale *in the notion of potential infinity*. Anything that does not have such a rationale should not bear the name potentialism. To be painfully clear, the fact that a view does not have a rationale based on the notion of potential infinity does not mean that that view must be rejected. It just means that other grounds must be provided for that view. Nothing in this paper is an argument against multiverse views or pluralist views or any other way of understanding the modal logic of forcing. All I am concerned with is whether the notion of potential infinity provides a good way of understanding the modal logic of forcing.

2 The Iterative Conception of Set

Above, I phrased the question at hand in terms of the forcing potentialist's conception of set. The dominant conception of set is the *iterative conception*. In this section I will briefly review the iterative conception of set, before arguing in the next section that this conception is inconsistent with forcing potentialism.

The iterative conception of set is best known through the work of George Boolos.[8] He aptly summed up the idea:

> A set is any collection that is formed at some stage of the following process: Begin with individuals (if there are any). An individual is an object that is not a set; individuals do not contain members. At stage zero (we count from zero instead of one) form all possible collections of individuals. If there are no individuals, only one collection, the null set, which contains no members, is formed at this 0th stage. [...] At stage one, form all possible collections of individuals and sets formed at stage zero. If there are any individuals, at

conceptions currently in the literature." Barton 2020 also emphasizes the need for a clearer philosophical distinction between multiverse views and potentialist views.

[8]Shoenfield 1977 is also an influential exposition.

> stage one some sets are formed that contain both individuals and sets formed at stage zero. Of course some sets are formed that contain only sets formed at stage zero. At stage two, form all possible collections of individuals, sets formed at stage zero, and sets formed at stage one. At stage three, form all possible collections of individuals and sets formed at stages zero, one, and two. At stage four, form all possible collections of individuals and sets formed at stages zero, one, two, and three. Keep on going in this way, at each stage forming all possible collections of individuals and sets formed at earlier stages. (Boolos 1971, pp. 220-1)

There are several pieces to the intuitive story behind the iterative conception of set, but we can isolate four ideas at the core of the conception. First is a claim about priority: the members of a set are prior to the set itself. Second, sets are nothing but collections of these 'prior' existing objects. The third idea is only implicit in Boolos' description quoted above—and indeed it is not properly the domain of the iterative conception but is essential to any conception of set. This idea is that sets are extensional objects: a set's identity is entirely determined by its members. There is no intensional criterion of identity. The fourth idea is a maximality claim: at each stage of set formation, *all* possible collections of objects are brought into a set. (Unlike the first three claims, this maximality claim is not constitutive of the idea of iteratively forming sets of previously existing objects. It is, however, an essential part of characterizing the iterative process as a whole in such a way that justifies standard set-theoretic axioms).

When we tell this story so that we begin with no individuals, the result is the cumulative hierarchy of pure sets. I will assume that there are no such individuals. Of course, something needs to be said about the existence of a stage ω after all the finite stages, and about limit stages more generally. And there is some controversy about whether this conception of set can underwrite all the axioms of **ZFC** (in particular, Replacement is tricky).[9] Nevertheless, it provides a nice clear and intuitive way to apprehend what set theory is all about.

What's more, this story about forming sets at stages by collecting together existing objects can easily lead to height potentialism, by arguing that the stages are potentially infinite. Given any definite totality of stages at which some sets have been formed, I could extend that collection to include a new stage, collecting together all the sets that were formed at those earlier stages. In other words, there is no limit to how far the iterative process of set formation can go. This line of thinking is of a piece with Cantor's diagnosis of the Burali-Forti paradox that the ordinals form an 'inconsistent multiplicity', meaning that "the assumption that *all* of its elements 'are together' leads to a contradiction, so that

[9]See Boolos 1971, p. 228, also Boolos 1989, Potter 1993, Potter 2004, Paseau 2007, Rin 2015, Incurvati 2020. It is illuminating that Linnebo 2013 and Studd 2013 both need to appeal to axioms other than those governing the iterative process in order to recover the axioms of Infinity and Replacement. Studd uses a reflection principle, and Linnebo uses a reflection principle and a version of replacement applied to pluralities.

it is impossible to conceive of this multiplicity as a unity, as 'one finished thing'" (Cantor 1899, p. 114; cf. Linnebo 2013, pp. 206-8). Since each stage can be naturally associated with an ordinal—indeed, each stage can simply be thought of as a V_α—it is a short step to height potentialism from Cantor's description of the ordinals as not being able to be brought together into a completed definite totality. The aim here, however, is to find a motivation for forcing potentialism, which is a version of *width* potentialism, not height potentialism. So is there another way to use the iterative conception of set to motivate forcing potentialism?

3 The Challenge for Forcing Potentialism

I will argue in this section that the iterative conception of sets cannot motivate forcing potentialism. I will begin by posing an explanatory challenge for width potentialism in general. Then I will argue that any way of for the width potentialist to meet this challenge will either require modifying either the iterative conception of sets or the claims of forcing potentialism.

3.1 Width Potentialism in General

Forcing potentialism is an instance of width potentialism. This is the claim that the universe of sets is potentially infinite in width: for any collection of sets you give me, I can find a more inclusive collection of sets that lies no higher in the set-theoretic universe than the original collection of sets.[10] And of course, this entails that there can be no collection of all sets of a given height. If there were, then if you gave me that collection of sets, I would not be able to find a more inclusive collection of sets of that height, contradicting the nature of width potentialism.

There is an immediate tension between width potentialism and the iterative conception of set, specifically with the maximality claim that at each stage in the iterative process *all* possible collections of previously existing objects are formed into sets. Imagine we find ourselves in some set-theoretic universe that an oracular being has created by an iterative process of forming sets out of collections, and the width potentialist claims "I can add some new sets to this universe without adding any higher levels!" I would like to ask the width potentialist why those sets were not already formed by the oracle. There seem to be two ways the width potentialist could reply.

[10] Although evocative, it is not obvious that there is a uniform way to explicate the notion of lying higher in the set theoretic universe. When we have the iterative conception in the background, we can use the ordering of stages to define a hierarchy of sets. And if we assume that the universe satisfies enough of **ZFC**, then we can define the hierarchy of V_α's in the standard way. But if we wanted to develop a width potentialist theory of sets that does not assume the story of set formation in iterative stages, and to use that theory to justify **ZFC**, then we would need to seek a different way of explicating the difference between width extensions and height extensions. Nevertheless, having flagged this complication, I will not worry about it further.

First, they could explain that the oracle is not forming sets out of all possible collections but is rather operating by some more restricted procedure, and for any collection of sets formed by this procedure it is possible to form a more inclusive collection of sets of a given height.

Second, they could explain that the oracle is forming all possible sets at each stage, but that as we go on new things keep becoming possible at each level.

These two options divide the possible terrain for the width potentialist. Clearly I have not described full answers, but merely indicated the strategies available to the potentialist. Much more work would need to be done to flesh out either strategy into a satisfactory answer to the question of why the sets that are being added to the universe were not there in the first place. For now I merely want to emphasize that this is an explanatory burden for the width potentialist who is working broadly within the iterative conception of set. Different ways of meeting this burden might involve more or less radical ways of elaborating or modifying the standard story behind the iterative conception of sets, as recounted by Boolos above.

What I will argue next is that there is no way for the width potentialist, working broadly within the iterative conception of set, to meet this explanatory burden in such a way that the possible extensions of a universe are all and only the forcing extensions.

I will begin by drawing a distinction between convergent and branching extensions of a collection. In brief, my argument will be that the iterative conception of set only motivates convergent extensions; but the forcing extensions of a ground model exhibit branching behavior, so the iterative conception is inconsistent with the claim that the possible extensions of a universe are all and only the forcing extensions .

3.2 Convergent vs. Branching Extensions

A domain of objects is potentially infinite when any definite collection of objects of that domain can be extended to a more inclusive collection of objects of that domain. We can ask whether there are ever mutually incompatible ways of extending such a collection or if the possible extensions are always mutually compatible.[11] When the possible extensions are always mutually compatible, we can say we have *convergent extensions*, and when there are possible extensions that are mutually incompatible with each other, we can say we have *branching extensions*. As an example of convergent extensions, consider geometric potentialism with a Euclidesque constructive gloss. Take geometry to be the study of certain types of constructions using compass and straightedge in an idealized Euclidean space. For any constructions that have been performed, it is possible to perform further constructions. For instance, if you construct a line segment you can a second line segment bisecting the first, or you can draw a circle tangent to the line segment, or you can draw a triangle with one leg parallel to the line segment, and so on. But whichever construction you perform next, it is always open to you to go on to perform one of the other constructions

[11] This distinction is discussed in Hamkins 2018 and Brauer 2020.

after that. None of these ways of extending the collection of existent geometric objects precludes any of the others, they are all mutually compatible.

On the other hand, as an example of branching extensions, consider a potentially infinite sequence of natural numbers. This sequence is always finite, but growing. Similar to imagining a being who is constructing geometric objects, we can imagine a being who chooses the values of this sequence successively, one argument at a time.[12] If values of this sequence have been chosen up to the argument n, then it is possible to extend the sequence by choosing the value for $n+1$ to be 0 or 1 or 2 and so on. But unlike the geometrical case, these possibilities are all mutually exclusive. If the value for $n+1$ is 0, then it cannot be 1 or 2, if it is 1 it cannot be 0 or 2, and so on.

Forcing potentialism provides another example of branching extension. If we begin with a countable transitive model M of **ZFC**, there are forcing extensions $M[G]$ and $M[H]$ such that there is no mutual forcing extension N. To see why, suppose that r is a real that codes the countability of M. Then we can construct Cohen reals G and H that each contain partial information about r so that r cannot be obtained from either of G or H in isolation, but can be obtained from the combination of G and H. Then either G or H can be added consistently to M by forcing, but if both were added then the resulting model would have to include r and hence would realize that M is countable. But since forcing preserves ordinals, no forcing extension of M will think M is countable.[13] Hence no forcing extension includes both G and H, and $M[G]$ and $M[H]$ are mutually incompatible as forcing extensions. Thus, if the possible extensions of a universe are all and only the forcing extensions, then we have a form of branching extensions.

3.3 The Iterative Conception and Convergence

Recall the four claims I isolated as lying at the heart of the iterative conception of set. First was the priority claim that members of a set are prior to the set itself. Second, sets are nothing but collections of these 'prior' existing objects. Third was the claim that sets are extensional objects, with no intensional criterion of identity. Fourth was the maximality claim that at each stage of set formation, *all* possible collections of objects are brought into a set. On the assumption that the width potentialist is able to resolve the tension with this maximality claim, I will adopt a weaker version of this fourth claim. For present purposes I will not assume that *all* collections of prior objects *must* be brought together to form a set. But I will make the weaker assumption that *any* collection of prior objects *can* be brought together into a set.

[12] On this picture, a potentially infinite sequence of natural numbers is basically a Brouwerian free choice sequence, cf. Brouwer 1952, Troelstra 1977, Troelstra 1982. I emphasize, however, that we can take the idea of a potentially infinite sequence of natural numbers to be perfectly comprehensible without accepting the baggage of Brouwerian intuitionism more generally.

[13] This argument can be found in Fuchs, Hamkins, and Reitz 2015, where Hamkins says he heard it from Woodin in the 90s. The result actually goes back to Mostowski 1976. (Thanks to Neil Barton and Kameryn Williams for pointing me toward the historical reference).

I now want to argue on the basis of these four ideas that the iterative conception is only consistent with convergent extensions of the universe of sets. The basic idea of the argument is straightforward: if you have a universe V_0, you can get different extensions V_1 and V_2 when V_1 is formed by bringing together some collections X into sets and V_2 is formed by bringing some other collections Y into sets; but nothing in the process of forming sets out of the X collections would then prevent you from later forming sets out of the Y collections. In order to show that this argument relies only on the iterative conception and does not smuggle in any further ideas, I will now flesh it out in pedantic detail.

Suppose we begin with a universe V_0, and it can be extended to either V_1 or V_2. The question is whether there is a further extension V_3 that includes both V_1 and V_2. Let V_i^s be the s^{th} stage in the construction of V_i over V_0. Since, in the iterative conception, stages are well-founded, we can argue by induction on s that V_1^s and V_2^s have a mutual extension $V_1^s \cup V_2^s$.[14] Consider an arbitrary object x in V_1^s. Since we are concerned with pure sets, x is a set. By the assumption that sets are just collections of objects, x in particular is nothing more than a collection of objects; and by the priority assumption those objects must have come from V_0 and $V_1^{<s}$. By the i.h., all those objects exist in an extension $V_1^{<s} \cup V_2^{<s}$ of V_0. So by the assumption that any collection can be brought together into a set, those objects can also be collected together in a set y in an extension $V_1^s \cup V_2^s$ of V_0. The set y will consist of exactly the same objects as x, so by extensionality y is nothing other than x itself, which thus exists in $V_1^s \cup V_2^s$. Since x was arbitrary, every element of V_1^s is in $V_1^s \cup V_2^s$. Similarly, every element of V_2^s must be in $V_1^s \cup V_2^s$. Thus, I conclude, the iterative conception is only consistent with convergent extensions, not branching extensions.

This argument is a specific instance of a more general argument due to Sam Roberts (Roberts 2020). The motivating idea behind Roberts' argument is the same as that sketched above: if you can form a set out of some objects X, then forming some other set could not prevent you from also forming a set out of the objects Y. By formalizing these ideas in a modal plural logic, Roberts is able to isolate exactly the formal features needed for the argument to go through, leading to an inconsistency result showing that three postulates about potentialism and set formation are jointly formally inconsistent. My claim that the iterative conception of set is inconsistent with branching extensions is one instance of this formal inconsistency. In this paper, however, I am interested in the philosophical content of forcing potentialism and the substantive notion of set at play. So what matters here is not Roberts' more general formal inconsistency, but the particular substantive instance of it.

In sum, forcing potentialism is a form of potentialism with branching extensions, as we saw in the last subsection. But the iterative conception of set only allows for convergent extensions, not branching extensions. So forcing potentialism is inconsistent with the iterative conception of set.

[14]The use of set-theoretic union is strictly speaking an abuse of notation, since V_i^s may not necessarily be a set, for all that we have said so far. Certainly V_i^s is not a set in the universe V_i^s.

3.4 Objections

I will consider four objections to this argument.

3.4.1 What matters is the logic

The first objection is that what really matters for the potentialist is not how the universe grows, but what the deductive modal theory looks like. The potentialist uses modal logic to formally explicate the intuitive idea of a domain being extendable, but what really matters is the mathematical content of the modalized theory, not the pre-formal gloss. And in the case of forcing potentialism, the right modal logic to use is **S4.2**, as Hamkins and Löwe 2005 prove. **S4.2** results from adding to **S4** the G axiom:

$$\Diamond\Box\phi \to \Box\Diamond\phi$$

In general, **S4.2** is characterized by Kripke frames where the accessibility relation is transitive, reflexive, and directed, in the sense that:

$$wRv_1 \wedge wRv_2 \Rightarrow \exists u(v_1Ru \wedge v_2Ru)$$

This directedness property makes it look from the perspective of Kripke frames that possible extensions v_1 and v_2 are always mutually compatible because they can have a further joint possible extension u.

The situation here is somewhat subtle. To say that **S4.2** is characterized by the (transitive, reflexive and) directed Kripke frames is to say that the formulas valid on every such frame are exactly the validities of **S4.2**. But for ϕ to be valid on every such frame means that it is valid in every model based on that frame—in other words, for every such frame F and valuation V, ϕ is true at every world in the model $\langle F, V\rangle$. But when we treat a set of forcing extensions as the worlds in a Kripke model, the formulas in question are not uninterpreted atoms to which any consistent valuation can be applied. Instead, they are interpreted formulas whose values are determined by the real membership relation. In $M[G]$, $\tau_G \in \sigma_G$ is true iff τ_G really is a member of σ_G. And of course, what is true in a forcing extension $M[G]$ depends in part on what is true in M. Treating the system of forcing extensions as a Kripke model, there is a conspiracy among the worlds about which formulas they make true. As a result, even though the system of worlds is not directed, substantive mathematical facts about forcing entail that $\Diamond\Box\phi \to \Box\Diamond\phi$ is a valid principle of the modal logic of forcing.[15]

[15] To see why this is true, suppose $\Diamond\Box\phi$, i.e. there is an extension $M[G]$ obtained via the forcing notion $\mathbb{P}$ such that for any extension $M[G][H]$ obtained via any forcing notion $\mathbb{Q}$, ϕ is true in $M[G][H]$. Then ϕ can be forced over the product $\mathbb{P} \times \mathbb{Q}$, which extends $\mathbb{Q}$. Since this holds for any $\mathbb{Q}$, we have that for any forcing extension $M[H']$ there is a further forcing extension $M[H'][G']$ in which ϕ is true. So it follows that $\Box\Diamond\phi$. In general, however, we do not have that $M[G][H] = M[H'][G']$.

In light of this, the forcing potentialist might claim that as far as the actual deductive content of their modalized theory is concerned, their universes might just as well have convergent extensions. The branching behavior is undetectable, and may be written off as just a quirk of the informal motivation behind the formal theory. Once we have arrived at the formal theory, the informal story can be kicked away like Wittgenstein's ladder.

In reply to this, I want to ask why we should study the modal logic of forcing in the first place. If it is because of the theory's inherent mathematical interest, fine. But that is consistent with the theory having nothing whatsoever to do with potentialism. Recall that one way of asking our main question is what distinguishes forcing potentialism from a forcing-inspired multiverse view. If the end result is just that it is interesting to use modal logic to study the relations between different models of set theory, then it would seem that there is nothing to distinguish forcing potentialism from generic multiversism.

Forcing potentialism—if it is a coherent view at all—is the view that forcing reveals the potentially infinite character of the universe of sets. As such, the theory is *precisely about* the (partial) universes, their forcing extensions, and the relations between these. Talk of the extensions of different models of set theory is not an informal story we use as a heuristic to find the right modal logic, it is the very content of the philosophical view in question. Dismissing questions about the relations between different extensions of the set-theoretic universe in favor of a deductive modal theory is not so much kicking away a ladder that has served its purpose as it is sawing through the branch one sits on.

3.4.2 The iterative conception is consistent with branching extensions

The second objection is that iterative conception of set is in fact consistent with branching extensions as well as convergent extensions. This can be proved by example, using the universal finite sequence. The universal finite sequence is a sequence σ with a Σ_1 definition that is provably finite, but for any countable model M, if σ is s in M and t is any finite extension of s, there is an end-extension of M in which σ is t.[16] A model M is an *end-extension* of N if $N \subseteq M$, $\in^N = (\in \restriction N)^M$, and finally for all $b \in N$, if $a \in^M b$ then $a \in N$. In other words, M and N agree on $\in$ for objects in N, and M does not add any new members to sets from N.

Obviously this example will lead to branching extensions, just as we noted in §3.1 that a potentially infinite sequence of natural numbers will give rise to branching extensions. Furthermore, the objection goes, end-extensions are consistent with the iterative conception of set. An end-extension of N will add new sets, but will not mess with any of the pre-existing sets. An end-extension only forms new sets by collecting together already-existing objects, which is exactly how sets are formed according to the iterative conception of set. So

[16] See Hamkins and Williams 2019. An arithmetical universal finite sequence is studied in Hamkins 2018. A related example that could be used for this objection is the universal finite set, see Hamkins and Woodin 2018. The reply to that example would be exactly the same as the reply to the example of the universal finite sequence.

we have a proof by example that the iterative conception of set is consistent with branching extensions. Although this example does not reveal how the argument of §3.2 went wrong, it shows that it must have gone wrong somewhere.

In reply, I acknowledge that there are end-extensions that are allowed by the iterative conception of set, and I acknowledge that there are end-extensions that are branching extensions. But I deny that the end-extensions that are allowed by the iterative conception of set are the end-extensions that give rise to branching extensions. On the iterative conception, recall, the elements of a set are prior to the set itself. The set is merely the collection of previously existing objects. But the elements of a set are themselves also sets, and so the same is true of them: their elements must be exist prior to them. So those elements must also be in the domain. And of course, *their* elements must in turn also exist prior to *them*, so *those* elements must be in the domain. Thus, every element of a set in the universe must also be in the universe. In other words, the iterative conception of set requires that the universe be transitive. But, Hamkins and Williams 2019 prove, the universal finite sequence is the empty sequence in any transitive model. So as long as we remain in transitive models—as the iterative conception requires—the universal finite sequence will be empty and thus will not witness any branching behavior.

3.4.3 Countable models are non-standard

This leads to the third objection. The forcing potentialist might object that my argument relies on non-standard models and hence does not fairly represent their position. This point is similar to the objection that Brauer 2020 raises against the argument in Hamkins 2018 that arithmetic potentialism exhibits branching extensions. Hamkins' argument relied on an arithmetic version of the universal finite sequence which results in branching behavior for models of arithmetic, similar to the set-theoretic version discussed in the previous paragraphs. Brauer noted, however, that the universal finite sequence is the empty sequence in the standard model. Since arithmetical potentialism—at least on the most natural reading—is the claim that the true natural numbers are a potential infinity, it should not be saddled with baggage about non-standard numbers. (This is all the more true given that non-standard models of arithmetic contain the natural numbers as a completed sub-part, in contrast to the idea that the collection of natural numbers cannot be completed).

Similarly, the forcing potentialist might object, the example of branching forcing extensions relies on the ground model being countable. Such models are non-standard, the forcing potentialist may protest, and their view should not be saddled with baggage about these non-standard models which, by hypothesis, they did not mean to be talking about to begin with.[17]

[17] This objection will not be attractive to all forcing potentialists. The theory in **scambler21** is specifically designed so that for every set there is a forcing extension where that set is countable. And Hamkins 2012 proposes that every universe is countable from the perspective of another universe (although this is in the context of the multiverse, not forcing potentialism).

In reply, I want to make two points. The first is that forcing over countable transitive models is the typical way to approach forcing. Thus, drawing on facts about forcing extensions of countable models is not to saddle the forcing potentialist with non-standard baggage, but is just to appeal to the basic, standard tools of forcing. (As a side note, the forcing potentialist cannot reply by invoking the Boolean-valued model approach to forcing, since Boolean-valued models are also non-standard. This would amount to rejecting one approach to forcing because it requires non-standard models and therefore adopting a different approach that also requires non-standard models.)

The second part of my reply is to ask in what sense countable models are non-standard. Critically, we can focus on countable *transitive* models. In these models, '$\in$' is interpreted as the real membership relation.[18] So the non-standardness of these models is not a matter of re-interpreting the vocabulary of the theory. If a countable transitive model is non-standard in a problematic way it must be because of its size: the model thinks that some sets are uncountable even though they are 'really' countable, and the model just omits the function witnessing their countability. By this criterion, any model that by design omits sets that really exist would be non-standard. But that would completely vitiate forcing potentialism, because no standard model would have a forcing extension. To add a generic to a model requires that there be some sets not in that model; the model must, by design, have omitted some objects that we can see from the perspective of the metatheory.

3.4.4 Some forcing extensions are convergent

The fourth objection is that the forcing potentialist is not forced to recognize all forcing extensions as being on par. In some restricted settings, forcing extensions are convergent rather than branching. For example, Hamkins 2016 shows that if there is a transitive model of **ZFC** then there is a transitive model with a forcing notion $\mathbb{Q}$ such that any two generic filters G and H are mutually generic, and hence $M[G]$ and $M[H]$ have a mutual extension $M[G][H]$. If the forcing potentialist can motivate some restriction so that all forcing extensions meeting that restriction are mutually compatible, then they will avoid the argument of §3.2.

In reply, I grant that this strategy *would* avoid the argument of §3.2. But the strategy has to actually be executed, and it is far from obvious that the forcing potentialist can actually motivate any philosophical restriction that has the desired technical effect.[19] I

[18] Or at least, a countable transitive model is isomorphic to one in which '$\in$' is the real membership relation.

[19] In Steel's multiverse, it is assumed as an axiom that any two universes have a mutual generic extension. This is justified largely on extrinsic reasons, however: Steel wants to have an axiomatic multiverse theory that can serve a foundational role. The existence of mutual extensions is assumed in order for multiverse truth to be axiomatizable. It does not seem to me that a similar rationale is available to the forcing potentialist. Even in the original multiverse setting, we can ask why we should insist on axiomatizability. As **maddymeadows21** conclude: "Without a satisfactory answer to this question, we have no reason to adopt the axiomatizabilty requirement, and we're left without a principled argument for Amalgamation

leave it as a task for the objector to show that this can in fact be done. The example of non-convergence from above was a very simple, even paradigmatic, case of forcing: $M[G]$ and $M[H]$ each arise from M by adding a single Cohen real. For the objector's strategy to succeed, then, they must rule out the adding of a single Cohen real as not being the sort of forcing that the forcing potentialist is concerned with. There is a danger that the forcing potentialist will be left with a very restricted conception of forcing potentialism.[20]

3.5 Conclusion

The expression 'forcing potentialism' has found its way into work both on the modal logic of forcing and modal accounts of potential infinity. But what view is supposed to answer to this name is not clear. In some uses, forcing potentialism seems to be little more than multiversism by another name. (This situation is complicated by the fact that the most prolific writer on the modal logic of forcing is also an avowed multiversist, Hamkins 2012). What we would really like is an explanation of forcing potentialism that brings into focus why it is a form of *potentialism*. I hope to have shown the challenges involved in this task.

I first raised a challenge for any version of width potentialism to explain why the sets added in the width extension were not already present in the universe. I then argued that any way of meeting this challenge that relies on a broadly iterative conception of set will only admit convergent extensions. On the other hand, forcing extensions exhibit branching behavior. Thus, if forcing potentialism is the view that the universe can always be expanded by all and only the forcing extensions, this view inconsistent with the most prominent account of the subject matter of set theory. The task facing the forcing potentialist would be nothing less than a reconceptualization of set theory itself.

On the other hand, if forcing potentialism is not the view that the universe can be expanded by all and only the forcing extensions, then what is forcing potentialism? If not all forcing extensions are allowed, then which forcing extensions are genuine ways to extend the universe, and why those? Or, if some expansions of the universe are allowed besides forcing extensions, which ones, and why do possible expansions by forcing have to be coupled with possible expansions by other means?

4 What Forcing Potentialism Could Be?

In this section I aim to provide an account of forcing potentialism that (partially) meets the challenge laid down in the last section. My account will be a height- and width-potentialist theory that includes forcing extensions among the possible width extensions. It will not, however, include *only* forcing extensions—this is the sense in which I only partially meet the challenge of the last section. If one were looking for a version of forcing potentialism that

[i.e. the convergence assumption]."

[20] Thanks here to Neil Barton.

said the possible width extensions of a universe were all and only the forcing extensions, this result would be disappointing. On the other hand, I will be telling a story that provides a philosophical account of width extensions, including forcing extensions, in terms of potential infinity. So what we end up with is certainly a version of potentialism which includes forcing extensions.

There will be two pieces to my account. First, I will suggest a slight modification to the iterative conception of set, which may be called the iterative logical conception of set. Second, alongside the cumulative hierarchy of sets there will be a lawless sequence of sets.

4.1 The Iterative Logical Conception of Set

There is a familiar distinction between the *logical* and *combinatorial* views of sets. On the logical view of sets, sets are extensions of properties. From the property of being a dog we get the set of dogs by considering its extension. Sets are not 'mere' collections, but have a certain unity. Membership of an object in a set can be explained by the object's having the relevant property. On the combinatorial way of thinking, sets are mere combinations of possibly unrelated objects. There is nothing that makes the set hold together as one. It is simply the result of saying 'yes' or 'no' to each object in the universe, independently of one another.[21]

The logical conception of sets is generally regarded as untenable, or at least questionable, since it is a short step from there to naive comprehension and paradox. As a result, sets today are typically thought of combinatorially. This is evident in Boolos' parable of the iterative conception: sets are not formed by taking extensions of properties, but by gathering collections of objects together; and any possible collection will do, irrespective of whether the objects in the collection have anything in common.

This has led to many authors conflating—or at least, not being sufficiently careful in distinguishing—the combinatorial conception of set and the iterative conceptions of sets, when in fact the two views crosscut each other.[22] The combinatorial conception is a view about the nature of individual sets: they are nothing other than collections of possibly unrelated objects. The iterative conception, on the other hand, is not really a view about the nature of sets taken individually. What would it even mean to say that a single given set is iterative? Rather, the iterative conception is a view about the nature of the set-theoretic universe and the process of set-formation that creates that universe: the universe of sets is

[21] The logical conception of set is typically traced back to Frege's theory of extensions in Frege 2013; the combinatorial conception is, to the best of my knowledge, first explicitly presented in Bernays 1983. Tracing the history of these two views of sets through the development of set theory would be an interesting project, though outside the scope of a footnote.

[22] To offer just one example, in a review of Incurvati 2020, Linnebo writes: "The iterative conception, especially as understood by Gödel, is a combinatorial one," (**linnebo20**). Incurvati himself, however, is a little more exact: "even if this combinatorial conception cannot quite be *subsumed* under the iterative conception, it does seem to harmonize quite well with it: the iteration process is naturally understood in a combinatorial manner." (Incurvati 2020, p. 44, emphasis original).

generated by forming sets of pre-existing objects and then iterating this process with the range of objects that includes the newly created sets. Where the combinatorial conception says something about the nature of sets taken individually, the iterative conception says something about the nature of the sets taken collectively.

With this distinction in mind, between the combinatorial conception telling us something the nature of sets and the iterative conception telling us something about the nature of the set-theoretic universe, what I want to do is tell a story of iterative set formation, but thinking of sets in the logical way rather than in the combinatorial way.

The conception of sets that I will work with is still iterative in that each sets has as elements other sets that were 'formed' at earlier stages. The universe can be thought of as the result of iteratively gathering together previously existing objects to form sets, just as the iterative conception would have it. However, on the account I am proposing, sets are not thought of combinatorially, obtained by independently picking some objects from the universe, but rather are picked out as the extension of some property had by objects in the universe. If we imagine the creator being faced with some universe of objects, they form sets of those objects by gathering the extensions of properties (within that universe) together into a single collection. The result will in general be a larger, more inclusive universe. Imagine this process being iteratively applied, beginning with an empty universe. The result is an iterative hierarchy of logical sets, rather than an iterative hierarchy of combinatorial sets.[23]

If we are thinking of sets as extensions of properties, an important question to address is which properties there are. Since we are concerned with pure sets, it is natural to focus on the intrinsic set-theoretic properties of objects, such as *being an ordinal* or *having 7 members*, rather than extrinsic properties such as *being Fred's favorite cardinal*. We can stipulate, then, that by property we mean property definable in the language of set theory. It is most natural also to take 'definable' to mean 'definable with parameters'. Under these stipulations, the iterative hierarchy of logical sets will be a rank-initial segment of Gödel's L. Indeed, it will be the entirety of L, if the iterative logical sets include all the 'real' ordinals.

4.2 Lawless Sequences

The iterative logical conception of set was the first part of my account of forcing potentialism. The second part involves adding a lawless sequence to the story. Lawless sequences are a form of *free choice sequences*—also known as *infinitely proceeding sequences*—which are studied in intuitionistic analysis. Free choice sequences are potentially infinite sequences of numbers, whose values are individually chosen by the subject who creates this sequence.

[23] Perhaps the clearest precedent of thinking about sets is the view of predicative analysis stemming from Feferman 1964. There is also an obvious affinity to the cumulative hierarchy of constructible sets; this is not surprising given that Gödel 1990, p. 26 and Cohen 1966, p. 85 both remark on the connections between Gödel's L and predicativity (see also Feferman 2005 in this connection).

Brouwer 1952 described the matter:

> [Intuitionism] recognizes the possibility of generating new mathematical entities: firstly in the form of infinitely proceeding sequences $p_1, p_2, ...$, whose terms are chosen more or less freely from mathematical entities previously acquired ... (p. 140)

In saying that the terms are chosen more or less freely, Brouwer means that the choice of each new value for the sequence may or may not be subject to any constraints. If a sequence is not subject to any constraints it is *lawless*; its behavior is completely undetermined. On the other extreme, the creating subject may determine the values of a free choice by following a recursive rule. In this case the values of the sequence would be completely predetermined and the sequence is *lawlike*. In between these two extremes are intermediate degrees of freedom and indeterminacy. For instance, the creating subject may restrict themselves to only choosing values 0 or 1, so that the future values of the sequence are not completely predetermined, but are still subject to some constraints.

Brouwer's mathematics was based on an idealistic metaphysics in which mathematical objects are all mental objects.[24] As such, there was nothing unique or problematic about introducing as a new mathematical object a sequence that is actively created by some agent. As a result, free choice sequences have not gained much purchase outside of intuitionistic circles. The essence of a free choice sequence, however, is just that of a potentially infinite sequence whose values may not be predetermined. Availing ourselves of the modal approach to potential infinity, it is easy to make sense of free choice sequences without the baggage of Brouwer's metaphysics.[25]

For our purposes here, a few simple axioms governing a single lawless sequence ℓ will suffice.[26] Formally, ℓ is a two-place relation, and can be thought of informally as a growing list (hence the choice of 'ℓ'). Also, let Ord abbreviate a definition of the ordinals as transitive sets linearly ordered by $\in$, and when x is an ordinal, let $s(x)$ be the ordinal successor. Finally, let $x = \overline{\ell(y)}$ mean that x is the graph of ℓ up to y. This can be defined as $z \in x \leftrightarrow (z = \langle z_1, z_2 \rangle \wedge z_1 \leq y \wedge \ell z_1 z_2)$; when x is the entire defined graph of ℓ, i.e., when $x = \overline{\ell(y)} \wedge \neg\exists z \ell s(y) z$, we can simply write $x = \overline{\ell}$.

L1 $\Box \forall x \forall y (\ell xy \rightarrow x \in Ord)$

This simply says that ℓ associates ordinals with sets.

L2 $\Box \forall x \forall y \forall z (\ell xy \wedge \ell xz \rightarrow y = z)$

[24] See Brouwer 1913, Brouwer 1952, and Iemhoff 2020 for overviews of Brouwer's thought. See also Heyting 1971 for an introduction to intuitionistic mathematics more generally.

[25] REFERENCES

[26] A more detailed account of lawless sequences can be found in CITE, albeit it the context of analysis rather than set theory. Since my goal here is not give a definitive theory of lawless sequences in set theory, but merely to use a lawless sequence in set theory, it is okay if the axioms below do not say everything there is to say about lawless sequences.

For each ordinal, ℓ has a *unique* value (if any).

L3 $\Box\Diamond\exists x \in Ord \forall y \neg \ell xy$

This says ℓ is always incomplete.

L4 $\forall x \in Ord \Diamond \exists y \ell xy$

ℓ can always be extended so that it can be defined on any ordinal. Combined with the last axiom, this captures the potentially infinite character of ℓ.

L5 $\Box \forall x \forall y (\ell xy \to \Box \ell xy)$

Once values of ℓ are chosen, they are fixed and cannot be changed.

L6 $\Box \forall x (\exists y \ell xy \to \forall z \in x \exists w \ell zw)$

ℓ has no gaps: if it is defined up to x then it is defined on all ordinals $< x$.

L7 $\Box \forall x (\neg \exists y \ell xy \to \forall y \Diamond \ell xy)$

If ℓ is not yet defined on x, then any value is possible for x. This is the first axiom concerning the unconstrained character of the lawless sequence.

L8 $\forall \alpha < \gamma \Diamond \forall \beta < \alpha \, \phi(\ell(\beta)) \to \Diamond \forall \alpha < \gamma \; \phi(\ell(\alpha))$

Here, ϕ must be a formula in the language of set theory with no instances of ℓ. (Otherwise consider the formula that said there is a finite ordinal y such that ℓ is undefined on $x + y$). (??) This is a form of induction, expressing the idea that the set theoretic properties of ℓ's graph are preserved at limit stages. If something is impossible for $\ell(x + \omega)$ then that impossibility must have manifested at some finite stage $\ell(x + y)$.

We can safely take the modality here to be governed by the logic **S4**.[27]

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[27] Brauer 2020 argues that a potentialist theory of infinite sequences requires a new modal operator, which he calls 'inevitability'. Obviously, I endorse those arguments. But while a general theory of potentially infinite sequences would need such an operator, it does not follow that every application of potentially infinite sequences requires the new extra operator.

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