

# On the Consistency of Height and Width Potentialism

Chris Scambler

April 1, 2022

## Abstract

Recent work in philosophy of set theory has furnished some arguments that height and width potentialism are inconsistent with one another. One such argument can be found in this volume (Brauer); the other is forthcoming (Roberts).

At the same time, others have suggested there may be some merit in the combination of height and width potentialism. Such authors have presented views that appear to manifest the combination in a non-trivial way, and have defended philosophical claims on their basis (e.g. that all sets are ultimately countable – cf Builes & Wilson, Meadows, Scambler, Pruss).

Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

## 1 Overview

Recent work in philosophy of set theory has furnished some arguments that height and width potentialism are inconsistent with one another. One such argument can be found in this volume (Brauer); the other is forthcoming (Roberts).

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Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

The plan is as follows. Section 1 gives some definitional background and context. Section 2 then presents some axiomatic systems that purport to combine height and width potentialism and discusses issues of relevant to consistency, citing new results on strength and equivalence to standard ‘actualist’ set

theories. Section 3 presents the inconsistency argument of Roberts and explains how a proponent of height and width potentialism should reply. Section 4 does the same for Brauer.

## 2 Background

Let me start by explaining the terms.

For present purposes, *potentialism* in set theory will be the idea that there could always be more sets than there in fact are. In slogan form, for the potentialist, the universe of sets is ‘indefinitely extendible’.

*Height potentialism* is the idea that the universe is always extendible ‘upwards’ to include new sets of higher rank than any given ones. *Width potentialism*, on the other hand, is the idea that the universe is always extendible ‘outwards’, to contain new sets of no greater rank than the max of any given ones.

Historically, height potentialism has been motivated by considerations involving the set-theoretic paradoxes. Russell’s paradox shows there is no set of all non-self-membered sets, and hence that the cumulative hierarchy of all sets is itself (therefore) not a set. But the height potentialist complains that any ‘stopping point’ for the cumulative hierarchy would be arbitrary. Surely there is no conceptual obstacle to any particular collection of ranks of the cumulative hierarchy providing ‘urelements’ for a longer continuation.

It is natural to explicate this idea of an indefinitely extendible universe of sets in modal logic. Indeed, modal axiom systems based around core height potentialist ideas are known to exhibit tight forms of equivalence with standard iterative set theories like ZFC.

Width potentialism has been historically less popular, although has been attracting some attention over the last few decades as a way to understand the independence phenomenon in set theory. There is, in fact, a close structural similarity in the motivation for height and width potentialism in these terms (cf Meadows). Just as the height potentialist begins with the intuition that it is arbitrary that there should be some ranks of the cumulative hierarchy that somehow inherently can’t be extended to include further sets, the width potentialist may begin with the idea that it is arbitrary that there should be some universe of sets that cannot be extended by forcing over its partial orders. Just as in the previous case, the mathematics of forcing seems to lead us to believe there is at least no conceptual obstacle to making sense of such ‘forcing extensions of the universe’.

Again, as with height potentialism, a natural way for the width potentialist to formalize their view involves modal logic: one formulates an axiom to the effect that, for any partial order  $\mathbb{P}$ , it is possible to find a generic for  $\mathbb{P}$ . Such explicitly axiomatic approaches to width potentialism are not very well-studied: the focus of most work in this area has been on model theory. Nevertheless, axiomatic approaches are possible and easy enough to formulate.

There are, in any event, clear analogies in the cases for height and width potentialism. In each case one has a central inexistence result in first order set theory (Russell’s paradox, the proof that some filters do not admit partial orders) and one seeks to overcome it, after a fashion, by implementing a modalized version – any things can form a set in the first case, any partial order can

be forced over in the second. Indeed, it is natural to think that going potentialist one way might give you some reason to consider going potentialist the other way too.

### 3 Height and Width Potentialism Combined

In this section, I will present some axiom systems that seem to explicate the intuitive idea of height and width potentialism, as described above.

#### 3.1 Core Logical Principles

All the versions of height and width potentialism we will consider will be built around two principles: the first is the height potentialist principle that any things can be the elements of a set; the second is the width potentialist ‘forcing axiom’, that a generic filter can be found for any partial order.

Both potentialist principles involve modality; the first also involves plural quantification. Accordingly the language  $\mathcal{L}_0$  we use to formulate these theories will have at least a modal operator  $\Box$ , singular variables  $x_n$ , plural variables  $X_n$ , the propositional connectives and quantifiers, the identity symbol  $=$  and the symbol  $\in$  for set membership. The circumstance that some thing  $x$  is one of some things  $X$  will be represented by the concatenation  $Xx$ . Identity is only well formed between terms of the same type (singular/plural).

The following core axioms will be included in all the potentialist theories we will go on to discuss.

**FQ** Standard rules for free quantifier logic with identity for each type of variable.

**Mod** The modal logic S4.2 with converse Barcan formula, necessitation, and standard rules for universal quantification within the scope of  $\Box$ .

**P-ext**  $\Box\forall x[\Diamond Xx \equiv \Diamond Yx] \supset X = Y$

**PR** The plural rigidity axioms

1.  $\Diamond Xx \rightarrow \Box Xx$
2.  $\Diamond(\exists x Xx \wedge x = y) \rightarrow \exists x Xx \wedge x = y$ .

**Comp** For any formula  $\varphi(x, y, Y)$ ,  $\Box\forall y\Box\forall Y\Box\exists X\forall x[Xx \leftrightarrow \varphi(x, y, Y)]$  is an axiom.

**P-Choice** A plural version of the axiom of choice.

**M-Rep** A modal version of the axiom scheme of replacement.

**Inf** An axiom saying that the set of natural numbers exists.

**Set-Ext** The axiom of extensionality in the form  $\Box\forall x[\Diamond x \in y \equiv \Diamond x \in z] \supset y = z$

**Sets** The axiom of foundation, and an axiom  $\Box\forall x\exists X\Box\forall y[Xy \leftrightarrow y \in x]$  asserting the rigidity of set membership

The precise details of many of these won’t be too relevant below, but they are cited here for something like completeness’ sake. More details can be found in CITE. The odd one out, perhaps, is **Inf**. It is included here for simplicity, since many of the phenomena we will be interested in occur only at the level of infinite sets.

The basic idea behind the core axioms is to get a modal and plural logic, combined with fundamental principles of set theory, to allow for the development of set theory based on potentialist principles of set existence. Let us now turn to that development.

### 3.2 Simple HWP

In this section I will present what seems to me to be the simplest formal combination of height and width potentialism. The theory is not mathematically strong, being exactly equivalent in strength to second order arithmetic. But it does offer a clean and simple proof of concept for the combination of height and width potentialism, as well as a useful starting point for further extensions.

The first axiom we add is:

**HP**  $\Box \forall X \Diamond \exists x \forall y [y \in x \leftrightarrow Xy]$

- *Any things can be the elements of a set.*

HP enshrines the core idea of height potentialism, since it implies (given the usual notion of rank) that given any sets one can find others of still higher rank.

The second requires a little more preparation. In our background logic we can define the notion of a partial order, the notion of being a filter on a partial order, and the notion of being a dense set in a partial order in the standard way. Let blackboard variables  $\mathbb{P}, \mathbb{Q}$  range over partial orders. (These are singular variables.) Let  $D(X, \mathbb{P})$  mean that all  $X$ s are dense in  $\mathbb{P}$ . Finally, let  $F_\cap(x, \mathbb{P}, X)$  mean that  $x$  is a filter on  $\mathbb{P}$  that intersects every one of the  $X$ s. Then our width potentialist axiom can be stated as:

**WP**  $\Box \forall \mathbb{P} \forall X D(X, \mathbb{P}) \rightarrow \Diamond \exists g (F_\cap(g, \mathbb{P}, X))$

- *Any partial order can be used in forcing.*

WP is one way to flesh out width potentialism, since it implies that any sufficiently rich plurality of sets may be extended to include ones of rank no greater than their max by forcing.

The following will be useful to us going forward. In it,  $f : \mathbb{N} \twoheadrightarrow X$  is an abbreviation for the assertion that  $f$  is a function on  $\mathbb{N}$  with every  $X$  in its range.

**Proposition 1.** *Let Count be the principle:*

$$\Box \forall X \Diamond \exists f [f : \mathbb{N} \twoheadrightarrow X]$$

*then Count is equivalent to WP over the core logic + HP.*

*Proof.* See CATBC. □

Let us call the result of adding these HP and either WP or Count to the core logic Simple Height and Width Potentialism, or SHWP.

Turning now to questions of consistency, it turns out that SHWP is demonstrably consistent relative to Second Order Arithmetic (SOA). In fact it turns out that SHWP exhibits a tight form of equivalence with SOA, something Tim Button has called ‘near synonymy’ in the recent literature.

(Here and below, we understand SOA under the guise of ZFC without power + all sets are countable. This is definitionally equivalent to the more standard arithmetical formulations; see e.g. Simpson.)

A detailed account of the relationship between SOA and SHWP would be overkill here. (Details can be found in CITE.) But certain features will be important to our discussion of consistency below, and having some idea of how the interpretation of the modal theory SHWP in the non-modal SOA goes will be very helpful to us going forward. So we will spend some time discussing certain aspects of the ‘tight equivalence’ just mentioned.

Let  $\mathcal{L}_\epsilon$  be the first order language of set theory. Then:

**Theorem 2.** *There is a map  $\cdot^\diamond : \mathcal{L}_\epsilon \rightarrow \mathcal{L}_0$  that preserves theoremhood from SOA to SHWP.*

**Theorem 3.** *There is a map  $\cdot^\exists : \mathcal{L}_0 \rightarrow \mathcal{L}_\epsilon$  that preserves theoremhood from SHWP to SOA.*

Theorem 2 says that SHWP interprets SOA; this direction does not really concern us directly, since it does not imply anything about the consistency of SHWP, but some idea of how things go will be useful. The translation  $\varphi \mapsto \varphi^\diamond$  proceeds by prepending every universal quantifier in  $\varphi$  by a  $\Box$  and every existential by a  $\Diamond$ . A result due to Linnebo says that for  $\varphi \in \mathcal{L}_\epsilon$ ,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma^\diamond \vdash_{Core} \varphi^\diamond$$

where  $\Gamma^\diamond$  has the obvious meaning of  $\{\gamma^\diamond : \gamma \in \Gamma\}$ . It thus suffices to prove  $\varphi^\diamond$  for each axiom of SOA in SHWP, which (using Proposition 1) is not difficult.

Theorem 3 is directly on-topic. It says that SOA interprets SHWP and hence secures the consistency of the latter relative to the former. Full details of the ideal proof are a little fiddly and would go beyond our needs here. But some idea, again, of how things go down will be useful.

The key idea behind the translation  $\varphi \mapsto \varphi^\exists$  is to factor out use of modal operators in favor of quantification over possible worlds.

In a bit more detail, the idea is that we define the notion of a possible world in SOA to be a transitive set. That means we interpret the language of SHWP so that ordinary quantifiers are always restricted to such worlds, and the modal operators induce quantifiers over such worlds.

Formally, this means our translation will have to carry formulas  $\varphi$  in the modal language (which may be sentences) to formulas  $\varphi^\exists(w)$  in the first order language with a free ‘world’ variable  $w$ . Key clauses of the translation are things like

$$\begin{aligned} (\forall x\varphi)^\exists(w) &:= \forall x \in w \varphi^\exists(w) \\ (\forall X\varphi)^\exists(w) &:= \forall x \subseteq w \varphi^\exists(w) \\ (\Box\varphi)^\exists(w) &:= \forall u [Tran(u) \wedge w \subseteq u \rightarrow \varphi^\exists(u)] \end{aligned}$$

where  $Tran(u)$  of course is the assertion that  $u$  is transitive. (The case of plural containment  $Xx$  is a little fiddly, but can be made to work quite nicely.) The potentialist axioms are then readily seen to be true, under the translation, in SOA. For HP, this is because every set is an element of a transitive set. For WP, this is because SOA proves all sets are countable. So for any given partial order, we may move to a transitive set that witnesses its countability, and then proceed to introduce a generic for it if need be.

A final point is that these results can be strengthened to attain something approximating definitional equivalence between the two theories. ...

### 3.3 Strong HWP

The theory I have just cited combines height and width potentialism in a straightforward way. The result is a theory that is (up to a near-synonymy) second order arithmetic. This is a pretty weak theory, although as we know from the reverse mathematics literature it is plenty strong enough to develop much of the mathematics needed in applications. Can the height and width potentialist do better?

There is indeed a reasonable way to proceed here. Our potentialist is interested with expansions of the universe along two ‘directions’. One has the ability to extend the universe ‘upwards’ to create sets of higher rank; and one has the ability to extend it ‘outwards’ by forcing. By separating out these two possible methods of expansion, and by asserting strong axioms regarding what can be attained along the vertical dimension of expansion alone, much stronger HWP systems can be derived, indeed ones with all the power of ZFC and more. This extension makes the general picture of height and width potentialism much more philosophically interesting, since such versions have at least the consistency strength to recover all standard mathematics. But they also bring new conceptual problems with them, problems that will be exploited in one of the inconsistency arguments we will consider.

Let us first discuss in a little more detail how to implement this idea. We expand the language  $\mathcal{L}_0$  to  $\mathcal{L}_1$  by adding in two new modal operators,  $\oplus$  and  $\oslash$ .  $\oplus$  is to reflect possibility by *only* vertical expansion: one can think of this as possibility by *only* iteratively introducing new sets for given pluralities.  $\oslash$ , on the other hand, is to reflect possibility by *only* horizontal expansion: one can think of this, indifferently, as just by adding new generic filters, or adding new enumerating functions. ( $\boxplus$  and  $\boxminus$  are the duals.)  $\diamond$  remains in the language, and represents ‘absolute’ possibility, that is, possibility by either domain expansion method.

Here is a way to axiomatize a ‘strong’ system of this kind.

We begin by modifying HP and WP.

**HP**  $\boxplus \forall X \oplus \exists x \forall y [y \in x \leftrightarrow Xy]$

**WP**  $\boxplus \forall P \forall X D(X, \mathbb{P}) \rightarrow \oslash \exists g (F_\cap(g, \mathbb{P}, X))$

Next, we add rules to reflect the generality of  $\diamond$ .

**Gh**  $\oplus \varphi \rightarrow \diamond \varphi$

**Gw**  $\oslash \varphi \rightarrow \diamond \varphi$

Finally, add the following restricted version of the powerset axiom to vertical possibility.

**r-Pow**  $\boxplus \forall x \oplus \exists y \boxplus \forall z [z \in y \leftrightarrow z \subseteq x]$

r-Pow says that it is always possible by vertical expansion to get all the subsets of any given set, *so long as one disregards the subsets you can get by forcing*.

It is useful to compare r-Pow with the corresponding ‘unrestricted’ version

**Pow**  $\boxplus \forall x \oplus \exists y \boxplus \forall z [z \in y \leftrightarrow z \subseteq x]$

which is provably inconsistent with **WP** over the rest of the theory. Pow says that it is possible for there to be such things as *all possible* subsets of any given set; but given the existence of an infinite set (as we are guaranteed), and the

universal possibility of forcing, enshrined in **WP**, this cannot be, since we can always force to add new subsets to any given set.

Let the extension of the core theory by the above principles (but not, of course, Pow) be called HWP. The following facts come easily.

**Theorem 4.** *There is a translation  $\cdot^\oplus : \mathcal{L}_\epsilon \rightarrow \mathcal{L}_1$  that preserves theoremhood from ZFC to HWP.*

**Theorem 5.** *There is a translation  $\cdot^\diamond : \mathcal{L}_\epsilon \rightarrow \mathcal{L}_1$  that preserves theoremhood from SOA to HWP.*

In each case, the result follows much as it did in the previous case. In fact Theorem 5 really just is Theorem 2. The first uses the same translation but with  $\oplus$  in place of  $\diamond$  and  $\Box$  in place of  $\square$  everywhere. That the theorem goes through is a known theorem of Linnebo in a slightly different key.

The resulting theory has some intriguing features. If one restricts one's attention to vertical possibility and necessity, one can construct objects that satisfy anything you can get in ZFC, for example the existence of uncountable cardinals, both fixed points, and so on. But these large cardinals are always mirages: and the mirage can always be revealed by appeal to  $\Diamond$ -possibility. Anything that might in some possible world satisfy the formula saying that it is (say)  $\omega_1$  will also not satisfy that description in some other possible world. There are no *absolute* uncountables, only relative pretenders.

The chief advantage HWP has over SHWP is interpretative power: it can interpret all of ZFC, notwithstanding its ultimate commitment to the countability of all things. This gives HWP the capacity, at least once motivational details are filled in, to potentially offer a countabilist foundation for all mathematics. This is I think one of the principle interests of the combination of height and width potentialism.

I have said that the chief advantage of HWP over SHWP is interpretative power. But how much of that does it have? Does it have too much? We now turn to addressing these issues of consistency in HWP.

It turns out that the methods employed in the previous section generalize fairly naturally to cater to systems like HWP and even further extensions. The generalization involves extend second order arithmetic to include 'topological regularity axioms', and then giving a more nuanced definition of possible world in terms of such an extension.

Let's start with the extension of second order arithmetic involved, to include so-called topological regularity axioms. What are those?

Well, there are certain 'nice' topological properties of sets of reals – things like being Lebesgue measurable, or having the perfect set property – that cannot hold everywhere, at least given the axiom of choice. The issue (repeatedly) is that the axiom of choice allows you to well-order the reals and then construct barbaric sets of various kinds, ones that don't have the nice features, by exploiting the well-order.

There's a general feeling in set theory that the 'nice' properties should all hold of easily definable sets of reals, and that the 'nasty' counterexample sets should all be pretty complicated to define (in terms of the usual analytical hierarchy). For example, if  $V = L$ , then there are easily definable nasty sets of reals, and this is generally taken to be a mark against the principle.

Since  $V = L$  implies there are easily definable but nasty sets, it follows that principles asserting that easily definable sets are nice must in some cases

go beyond SOA (which is of course consistent with  $V = L$ ). Principles of this kind are what I'm calling *topological regularity axioms*. They assert that good-behavior properties hold for certain easily definable classes of reals, even when this is not provable in SOA (or ZFC).

The weakest such axiom is something called the  $\Pi_1^1$ -Perfect Set Property (PSP), which says that every uncountable set of reals definable by a  $\Pi_1^1$  formula is either countable or has a perfect subset.<sup>1</sup> It turns out that this minimal extension of SOA is enough to secure the interpretability of HWP.

**Theorem 6.** *There is a map  $\cdot^3 : \mathcal{L}_1 \rightarrow \mathcal{L}_\epsilon$  that preserves theoremhood from HWP to  $\text{SOA} + \Pi_1^1\text{-PSP}$ .*

The proof uses similar ideas to those employed for SOA and SHWP in the previous section, but as I said one has to take a more nuanced definition of 'possible world'. In particular, it turns out that here our definition of a world will need to be 'doubly parameterized': that is, we will need *two* free variables in  $\varphi^3$ .

Why? Let us think about how our possible worlds will have to look to get an interpretation going. In order for  $r\text{-Pow}$  to be true, there will have to be worlds  $w$  at which the set  $x$  of all  $\oplus$ -possible sets of numbers exist. These worlds are worlds where any plurality of numbers, in the sense of  $w$ , form a set. And yet, since we can always force to add subsets to any infinite set, there must still be further possible sets of numbers. These sets of numbers have numbers as their members. But we said that all pluralities of numbers at  $w$  formed sets at  $w$ !

The only way out of this tangle for the HWP proponent is to accept indefinite extendability both with respect the singular domain, and with respect the plural domain (holding the singular domain fixed). The introduction of new generics witnesses, on this view, the extension of the plural domain (over a fixed) singular domain. Indefinite extendability 'runs in two dimensions', according to the present picture. Hence, our possible worlds will be doubly parameterized, with one parameter representing the singular domain, and the other parameter representing the plural. Expansions will then be possible, under the proposed interpretation, in both directions.

Let's now look in more detail how this works. We will make use of the following fact, whose proof is originally due to Solovay.

**Fact 7.** *Over SOA, the  $\Pi_1^1\text{-PSP}$  is equivalent to  $L[r]$  containing only countably many reals for every  $r$ .*

Thus, in effect, the Fact says that  $L[r]$  always falls badly short of containing all reals if we have  $\Pi_1^1\text{-PSP}$ . It is thus a strong form of  $V \neq L$ .

It is fairly easy to see that the fact has the following corollary.

**Corollary 8.** *In  $\text{SOA} + \Pi_1^1\text{-PSP}$ ,  $L[r]$  is a (class) model of ZFC for every real  $r$ .*

I provide an argument sketch (that the corollary follows from the fact) in a footnote.<sup>2</sup>

This now gives us the tools we need to implement our parametrized possible worlds strategy. We define, in  $\text{SOA} + \Pi_1^1\text{-PSP}$ , our 'possible worlds' to

<sup>1</sup>A perfect set is a closed set with no isolated points.

<sup>2</sup>Do.



comprise a transitive set  $t$  (representing the first order domain) and a real number  $r$  (representing the second order domain). We require that  $t \in L[r]$ , and will say that plural quantification in a possible world is always restricted to subsets of  $L[r]$ .

On this interpretation, when we vertically expand the domain, that means that we add to the transitive set  $t$  while staying within  $L[r]$ ; when we horizontally expand the domain, that means we extend the real parameter to  $s$  (say), making more pluralities ‘visible’: we will now we consider  $L[s]$  when we think about extending vertically, and so on. Since  $\text{SOA} + \Pi_1^1\text{-PSP}$  has  $L[r] \models \text{ZFC}$  for every real  $r$ , this interpretation for vertical possibility will get us ZFC, as we need, even remaining within the SOA context in general.

To implement this formally, our mapping  $\varphi \mapsto \varphi^\exists$  will therefore need to carry  $\varphi \in \mathcal{L}_1$  to a formula in  $\mathcal{L}_\infty$  with not just one but two free variables:  $\varphi \mapsto \varphi^\exists(t, r)$ . It will contain clauses like

$$\begin{aligned} (\forall x\varphi)^\exists(t, r) &:= \forall x \in t \varphi^\exists(t, r) \\ (\exists x\varphi)^\exists(t, r) &:= \exists x \subseteq t [x \in L[r] \rightarrow \varphi^\exists(t, r)] \\ (\Box\varphi)^\exists(t, r) &:= \forall u \in L[r] [Tran(u) \wedge t \subseteq u \wedge u \in L[r] \rightarrow \varphi^\exists(u, r)] \\ (\Diamond\varphi)^\exists(t, r) &:= \forall s \geq r \forall u \in L[s] [Tran(u) \wedge t \subseteq u \rightarrow \varphi^\exists(u, s)] \end{aligned}$$

(In the final expression,  $s \geq r$  means that  $r$  is constructible from  $s$ .)

With the full translation in hand it is a fairly straightforward matter to prove that every theorem of HWP comes good in  $\text{SOA} + \Pi_1^1\text{-PSP}$  under the translation:  $\varphi \mapsto \forall t, r Tran(t) \wedge t \in L[r] [\varphi^\exists(t, r)]$ . In fact, one can show that HWP has  $\Pi_1^1\text{-PSP}^\Diamond$  as a theorem, and prove the same kind of tight proof-theoretic equivalence obtains here between HWP and  $\text{SOA} + \Pi_1^1\text{-PSP}$  as did between SHWP and SOA simpliciter. These results can be pushed further: by increasing the strength of the regularity assumed in the interpreting theory, one can ratchet up the consistency strength of the corresponding modal HWP theory, allowing for measurables and beyond to be ‘vertically’ attainable. (We will return to this observation in more detail below.)

In any case, that concludes the formal arguments in favor of the consistency of height and width potentialism. We have seen, in outline, that there are arguments for the consistency of height and width potentialism in both a strong and a weak form, relative to extensions of SOA by topological regularity axioms.

Let us now turn to the arguments alleging inconsistency between height and width potentialism, and to see how they relate to the approaches sketched up to now.

## 4 Roberts

Right from the off it is perhaps worth mentioning that Roberts’ argument applies to HWP only. This is not a big deal: he is assuming that the height potentialist will be committed at least to the recovery of ZFC in terms of pure height potentialism, so that any commitment to width potentialism will require a bimodal treatment of the sort described in the last section. But it is worth mentioning that a less grand form of height and width potentialism can still be viable, in the form of SHWP.

The actual details of Roberts' argument are at present not entirely clear to me. His paper is a work in progress, but in the most recent version he argues that the addition of an actuality operator to a theory related to HWP is inconsistent, and that on that basis we should think that the viewpoint enshrined in HWP is similarly inconsistent. In correspondence, I suggested that the use of an actuality operator was a red-herring, and that in fact the argument turns (whether or not we have an actuality operator in the picture) on whether we are allowed comprehension only in the 'closed' form cited above, or if instead we accept instances of comprehension with free variables. I continue to believe that the fundamentals of the argument Roberts presents turn on issues surrounding comprehension, and that addition of an actuality operator is a needless increase in complexity. So here I will discuss what I take to be the simpler version of the argument, rather than the last one I saw in a draft from Roberts. In an appendix, I will explain why I think the addition of an actuality operator is unnecessary.

Here then is the argument. Let  $HWP^+$  be the results of allowing all instances of comp,

$$\exists X \forall x [Xx \equiv \varphi]$$

where  $\varphi$  may include free variables. Then:

**Theorem 9.**  *$HWP^+$  is inconsistent.*

*Proof.* It is easy to see that HWP implies

$$\Diamond \exists x [\Diamond (\exists y [y \subseteq x \wedge y = z]) \wedge \Box (\neg \exists y [y = z])] \quad (1)$$

abbreviate this as  $\Diamond \exists x \Psi(x, y, z)$ . By strong comprehension and existential instantiation,

$$\Diamond (Ex \wedge \Psi(x, z) \wedge \exists X \forall w [Xw \equiv \Diamond w \in z]) \quad (2)$$

by HP,

$$\Diamond (Ex \wedge \Psi(x, z) \wedge \Box \exists u \forall w [w \in u \equiv \Diamond w \in z]) \quad (3)$$

one can then instantiate on  $u$  and prove  $u = z$  by (our strong modal form of) extensionality.

But this is a contradiction. □

Naturally, given this argument, we should expect the proof of consistency implicit in the previous section to break down with strong comprehension. And this is precisely what happens in the interpretation offered for HWP in SOA +  $\Pi_1^1$ -PSP. Consider for example the following simple case. Let  $w$  be the world that has  $\omega$  for its first order domain and  $P(\omega)^L$  for its plural domain. The interpretation for closed plural comprehension at this world amounts to the claim that for any  $y$  and  $Y$  in  $L$ , the subset of  $\omega$  for which  $\varphi(n, y, Y)$  holds is again in  $P(\omega)^L$ , which is obviously true. But if we remove the restriction to closed formulas, we are saying that for any  $\varphi$ ,  $y$  and  $Y$  we choose (whether in  $L$  or not), the set of  $n$  with  $\varphi(n, y, Y)$  is in  $L$ . But this is provably false in SOA +  $\Pi_1^1$ -PSP.

The question about the consistency of HWP that arises from this argument concerns, then, the status of the 'strong' open version of plural comprehension. Formally speaking the path toward saving HWP is clear: accept only the closed instances of comprehension.

Roberts' claim is that the open instances are mandatory. But I see no good reason for thinking this is so.

For one thing, it is generally assumed that formulas with free variables at best play an auxiliary role in such formalizations, since under interpretation they don't say anything at all. They are analogous to assertions with hanging pronouns, like in 'it's a good thing'. To rest a philosophical case on such formulas being mandatory would therefore be peculiar.

For another, if one looks at the role played by such instances in the argument, one will find reasonable grounds for the potentialist to be doubtful of the instances at an intuitive level, at least given a broadly Stalnakerian approach to the metaphysics of modality. Roughly speaking, in intuitive terms, the argument for the inconsistency goes like this.

Strong HWP says there could be an  $x$  such that:

1.  $x$  could have  $z$  as a subset in the absolute sense, but
2.  $z$  could not be attained by vertical expansion.

But this is inconsistent. Suppose  $x$  exists at world  $w$ . By plural comprehension there always such things as  $z$ s elements at  $w$ , and so by vertical expansion those things form a set. But that set is just  $z$ . So it was attainable by vertical expansion after all.

Should the HW-potentialist accept the instance of plural comprehension appealed to there? That is, at  $w$ , should we believe there are such things as  $z$ s elements?

There's a certain line of thought here (with roots in Stalnaker) that says clearly not. Suppose you are an inhabitant of the world  $w$ . In  $w$ , as it were, there is no such thing as  $z$  to refer to. It is possible, in  $w$ , that there be some subset of  $x$  that cannot be attained by vertical expansion; that much we are sure of. But we are incapable of singling out any particular such  $z$  on which to comprehend, because there simply are no such  $z$ . There is simply the generic possibility of there being something of the relevant type, without it being true of any particular thing that it is possibly a thing of the type.

This is a perfectly coherent position and a reasonable rationale for, in the formal theory, restricting the allowable instances of comprehension to be closed. It amounts to only allowing parameters that are existent things, to which de re reference is possible, into the instances of comprehension. In other cases mere de dicto, generic reference to possibilities is all we can hope for. We can't single out particular witnesses to possible existentials on which to comprehend.

All told then I do not think this argument shows HWP is inconsistent, although it does serve to bring out the contours of the philosophical commitments that go along with the view.

There is a related argument in the vicinity here that is also worth some brief discussion. This says that HWP is inconsistent because it violates the valid principle of plural logic

**BF**  $\Box \forall Y (\Diamond \exists X \sqsubseteq Y [X = Z] \supset \exists X \sqsubseteq Y [X = Z])$

where here  $X \sqsubseteq Y$  abbreviates  $\forall x [Xx \supset Yx]$ . This says that given pluralities cannot gain sub-pluralities, and is violated in HWP. Moreover, BF is motivated on standard conceptions of plural logic according to which a plurality is 'nothing

over and above' the things it comprises. So, one might conclude, HWP is inconsistent after all.

I agree that BF is well-motivated on that conception but do not believe the conception is sacrosanct. Our conception of plurals – of 'some things' – is one bequeathed us from informal thought. Precisely what logical principles they should obey, and how they are to be conceptualized, is not rigidly determined *a priori*, but something that is to a great extent up to us to decide how to fill in the details. Indeed there are in all likelihood a number of related notions that can serve different roles in philosophical enquiry. The task here is to find one that suits HWP, and the one that does will necessarily not satisfy BF, since given any definite range of pluralities over a given domain one can find new ones by forcing. If the reader is really unhappy with this being true for 'pluralities', I'm happy to drop the term: call them strongly extensional properties if you like, it doesn't matter, so long as they can plausibly be held to have the properties required of them by the axioms of HWP.

This point is one of general significance. HWP uses a cluster of ideas: modality, set membership, quantification, plurality, and so on. None of them is an absolute fixed point, and there are basically no conceptual fixed points within them. It is a matter of finding the right configuration of these things to do justice to the guiding picture. That is why, in my view, it is highly unlikely that any serious inconsistency argument for the combination of height and width potentialism will be forthcoming: pretty much any derived inconsistency in a formal system would as naturally be taken to show the formalization was not up to the task. This generally seems to be the right response to these sorts of inconsistency proofs.<sup>3</sup>

## 5 Brauer

Brauer's argument is quite closely related to Roberts', and in a sense the same key logical observations are at work in each. But Brauer's argument brings out some interesting issues distinctive to the foundations of set theory that are not really brought up by Roberts, and for that reason warrants separate discussion.

The core argument is simple.

- P1** Height potentialism, as motivated by the iterative concept of set, sanctions only convergent extensions of the universe: if it could be that the set  $x$  existed, and it could be that some other set  $y$  existed, then it could be that  $x$  and  $y$  both exist together.
- P2** Width potentialism, as motivated by the idea of forcing potentialism, requires allowing for divergent extensions of the universe: for there are (or at least could be) partial orders  $P$  and possible generic filters  $G_1$  and  $G_2$  such that  $G_1$  and  $G_2$  could not exist together.
- C** Height and width potentialism (so understood) are inconsistent.

I'm not completely sold on either premise. In the case of P1, Brauer gives an argument which he glosses informally as follows:

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<sup>3</sup>Compare e.g. Russell's paradox and the concept of set, or Berkeley's critique of the notion of the infinitesimal.

The basic idea of the argument is straightforward: if you have a universe  $V_0$ , you can get different extensions  $V_1$  and  $V_2$  when  $V_1$  is formed by bringing together some collections  $X$  into sets and  $V_2$  is formed by bringing some other collections  $Y$  into sets; but nothing in the process of forming sets out of the  $X$  collections would then prevent you from later forming sets out of the  $Y$  collections.

So, we are to conclude, that according to height potentialism under the guise of the iterative concept of set, only convergent extensions of the universe are possible.

My doubts about this form of argument concern whether it is OK to conclude that quite generally the iterative concept of set is incompatible with divergent extensions of the universe. After all, set theory is not everything, and although there may be nothing in the simple process of forming sets from given things that prevents re-convergence, nevertheless other things may do so.

For example, suppose I have a handle, a blade, some permanently binding superglue, and a universe of sets  $V$ . I can then form the knife  $K_1$  and form its singleton getting a universe of sets  $V_1$ . Or I can form the knife  $K_2$  and form its singleton getting  $V_2$ . But these can never be unified.

Or, for a more pointed example (in the spirit of the 'second approach' to width potentialism discussed by Brauer on p7), suppose I have two machines. One takes in some things and gives back a set with them as its elements. The other takes in a partial order, some dense sets in it, and gives back a plurality that intersects all the given dense sets. Then, given suitable background assumptions (including, but not by any means restricted to P2), we will be in a structurally analogous situation to the blade-handle case: we might find it is possible to produce a set whose elements are  $G_1$ , and also possible to find a set whose elements are  $G_2$ , without these admitting a unification. But this is no different from the previous case, which is (hopefully?) completely unproblematic.

But discussion of P1 is really a red herring, since at least so far as the brands of height and width potentialism here go P2 is clearly false. It is after all easy to prove in HWP that if it is possible that some generic  $g_1$  exists, and it is possible that some generic  $g_2$  exists, then it is possible that they exist together. One can see this most easily by using the tight form of equivalence with SOA +  $\Pi_1^1$ -PSP. This obviously proves that any two possible generics for a partial order can both exist together in a transitive set, namely a sufficiently large fragment of  $L[r]$  with  $r$  witnessing the countability of all the relevant parties. Thus the modal logic of forcing, in this setting, is convergent.

Brauer's argument on the other hand is based on the familiar result from ZFC saying that, in any given countable transitive model for ZFC, one can find a partial order and a pair of generics for it that can't exist together in a model of ZFC, because they witness the countability of the model in question (Woodin). This is clearly correct, and is indeed a fact that the proponent of HWP can recognize: they can see for example that it is vertically possible that there are models for (fragments of) ZFC, and partial orders they contain, and such that no possible model for the same theory contains both generics. But from the point of view of HWP, this is just because you are ignoring some universes, namely those wherein everything under discussion is in fact countable (and so ZFC is false).

The point is that HWP is associated with a radically different conception of the mathematical universe than that associated with ZFC. Now I must concede that there is a sense in which the iterative concept of set is tied very closely to ZFC and to certain forms of height potentialism: and if one thinks that these links are essential to height potentialism then of course it is right to say that height potentialism is inconsistent with width potentialism. (This barely requires argument.) But in any case I see no reason to think such a thing. The core idea of height potentialism is that the universe can always be extended upward, not that it can't be extended outward too.

I hope it is clear that there is at least a coherent position that incorporates a lot of the traditional ideas in the iterative conception, understood in height potentialist terms, but that amalgamates width potentialism as well.