On the Consistency of Height and Width Potentialism

Chris Scambler

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Abstract

Recent work in philosophy of set theory has furnished some arguments that height and width potentialism are inconsistent with one another. One such argument can be found in this volume (Brauer); the other is forthcoming (Roberts).

At the same time, others have suggested there may be some merit in the combination of height and width potentialism. Such authors have presented views that appear to manifest the combination in a non-trivial way, and have defended philosophical claims on their basis (e.g. that all sets are ultimately countable – cf Builes & Wilson, Meadows, Scambler, Pruss).

Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

1 Overview

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The plan is as follows. Section 1 gives some definitional background and context. Section 2 then presents a system that purports to combine height and width potentialism and cites some new results about its precise consistency strength and equivalence to standard actualist systems of set theory. Section 3 presents the inconsistency argument of Roberts and explains how the a proponent of the theory from Section 2 would reply. Section 4 does the same for Brauer.

2 Background

Let me start be explaining the terms.

For present purposes, *potentialism* in set theory will be the idea that there could always be more sets than there in fact are. In slogan form, for the potentialist, the universe of sets is 'indefinitely extendible'.

Height potentialism is the idea that the universe is always extendible 'upwards' to include new sets of higher rank than any given ones. *Width potentialism*, on the other hand, is the idea that the universe is always extendible 'outwards', to contain new sets of no greater rank than the max of any given ones.

Historically, height potentialism has been motivated by considerations involving the set-theoretic paradoxes. Russell's paradox shows there is no set of all non-self-membered sets, and hence that the cumulative hierarchy of all sets is itself (therefore) not a set. But the height potentialist complains that any 'stopping point' for the cumulative hierarchy would be arbitrary. Surely there is no conceptual obstacle to any particular collection of ranks of the cumulative hierarchy providing 'urelements' for a longer continuation.

It is natural to explicate this idea of an indefinitely extendible universe of sets in modal logic. Indeed, modal axiom systems based around core height potentialist ideas are known to exhibit tight forms of equivalence with standard iterative set theories like ZFC.

Width potentialism has been historically less popular, although has been attracting some attention over the last few decades as a way understand the independence phenomenon in set theory. There is, in fact, a close structural similarity in the motivation for height and width potentialism in these terms (cf Meadows). Just as the height potentialist begins with the intuition that it is arbitrary that there should be some ranks of the cumulative hierarchy that somehow inherently can't extended to include further sets, the width potentialist may begin with the idea that it is arbitrary that there should be some universe of sets that cannot be extended by forcing over its partial orders. Just as in the previous case, the mathematics of forcing seems to lead us to believe there is at least no conceptual obstacle to making sense of such 'forcing extensions of the universe'.

Again, as with height potentialism, a natural way for the width potentialist to formalize their view involves modal logic: one formulates an axiom to the fact that, for any partial order $\mathbb{P}$, it is possible to find a generic for $\mathbb{P}$. Such explicitly axiomatic approaches to width potentialism are in fact less well-studied: the focus of most work in this area has been on model theory. Nevertheless, axiomatic approaches are possible and easy enough to formulate.

There are, in any event, clear analogies in the cases for height and width potentialism. In each case one has a central inexistence result in first order set theory (Russell's paradox, the proof that some filters do not admit partial orders) and one seeks to overcome it, after a fashion, by implementing a modalized version – any things can form a set in the first case, any partial order can be forced over in the second. It is natural to think that going potentialist one way might give you some reason to consider going potentialist the other way too. But then, as I have said, there are those who think that this is impossible on grounds of a logical inconsistency between the two ideas.

3 Height and Width Potentialism Combined

In this section we will present and begin to motivate axioms for a theory designed to combine height and width potentialism.

Simple case: take a modal operator, $\Diamond$. Height potentialism says: any things possibly are the elements of a set. Width potentialism: any partial order and any dense sets, possible to find a filter meeting all of them. What first order theory do we get? It turns out that with not very much extra we get second order arithmetic up to near synonymy. Interpret possible worlds as transitive sets. Then all the axioms will come out true (everything is countable in our background theory).

If we want more mathematical strength, factor the contributions into a vertical and a horizontal dimension. We can assert ZFC for the vertical but retain the horizontal. What we get is lots of ZFC models living inside our overal universe, which is one for SOA.

These get determinacy axioms in the translation.

4 Roberts