On the Consistency Strength of Axioms for Height and Width Potentialism

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April 19, 2022

Abstract

There are two parameters along which the universe of sets might conceivably differ in size: it may have more or fewer ordinals, and respectively may naturally be construed as taller or shorter; or at any given ordinal height, the initial segment of the universe up to that ordinal may have more or fewer subsets, and the universe may in each case naturally be constured, respectively, as fatter or thinner. A potentialist about the height of the universe believes that, no matter what, there could always be further ordinals; a potentialist about the width of the universe believes that, no matter what, there could always be more subsets of any given infinite set.

It is well known that natural axiom systems for height potentialism exhibit tight forms of proof-theoretic equivalence with standard set theories associated with ZFC (and its extension by large cardinals). The recent literature has seen some discussion of height and width potentialism combined. But analogous results precisely correlating these axiom systems with familiar, purely quantificational theories do not yet exist.

This paper provides such results. I show that the axioms for height and width potentialism from Scambler CITE exhibit a tight form of equivalence with second order arithmetic extended by topological regularity principles. The results are significant because they helps to round out the foundational picture underlying height and width potentialism, and to bring out how different it is from the standard ZFC-conception of set.

1 Overview

I will begin by discussing the sense in which axiom systems for height potentialism are equivalent to set theories related to ZFC, and the techniques used to attain the equivalence theorems. I then present the relevant axioms fo height and width potentialism, and prove they share the same form of equivalence with certain extensions of second order arithmetic. Finally, I discuss the relevance of these results for our understanding of the height/width potentialist point of view.

2 Warm Up: Height Potentialism

We begin with a review of some familiar territory. Though the correctness of the results here will be obvious to specialists, and though there are similar arguments on offer in the literature (cf Button, Linnebo), I'll present some of the details here, partly because the results have never been given for the precise set of axioms to be discussed, and partly because seeing how the arguments go in the simpler case of plain height potentialism will help to give intuitions that will be handy for the more complicated case of height and width potentialism.

2.1 Axioms

We start with a discussion of the 'natural' axiom systems for height potentialism that are at issue.

The systems will be directed at axiomatizing the following intuitive conception. Imagine there is a being who is capable of taking things and **collecting** them together into a set, and of repeatedly performing this act as many times as is conceivable. Say that a *set* is something that, given enough executions of this basic act, could be produced by such a being.

We want to axiomatize this conception in modal logic. The modal operator, $\Diamond$, is to be interpreted so that $\Diamond\varphi$ means 'by repeated acts of collection, φ can be made to hold'. The dual operator, $\Box\varphi$, means 'no matter how many acts of collection are performed, φ will hold'. The axioms governing this modal logic can be taken to be S4.2 with the converse Barcan formula. More details are given in Appendix A.[1]

Our conception says that the being can take any *things* whatsoever and collect *them* into a set. These are plural notions, and accordingly we shall want a plural langauge as well. We thus take a stock of plural variables X and singular variables x. The circumstance of something x being one of some things X will be expressed by the concatenation Xx. The precise details, once more, are consigned to Appendix A.

The set-theoretic axioms are more interesting. We assume all the usual background stuff: the axiom of extensionality for sets, rigidity for the membership relation, and the axiom of foundation. But in addition, we have distinctively potentialist existence principles, the most fundamental of which being:

HP $\Box\forall X\Diamond\exists x[Set(x, X)]$

Here and throughout, $Set(x, X)$ is an abbreviation for $\forall z[z \in x \equiv Xz]$; **HP** thus says: *necessarily, any things can be the elements of a set*. Under the given intuitive interpretation, this corresponds to the idea that the being can always collect together any given things into a set in any circumstances.

[1](We do not want $.3$, since given $a \neq b$ we may form either $\{a\}$ or $\{b\}$, but we do want $.2$, so from either of these we can get $\{a\}$ and $\{b\}$.) The rule of necessitation is assumed, along with the standardly valid inference rules:

$$\frac{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\Psi)...)}{\Phi_1 \to \Box(\Phi_2 \to ...\Box(\Phi_n \to \Box\forall x\Psi)...)}$$

(Hence, in contrast to standard set theory, there are no special things that somehow can't be made to form a set.)

HP allows us (given plural comprehension) to prove the possible existence of each specific hereditarily finite set. But it does not allow for a proof that it is possible to produce any infinite sets.

In the spirit of transfinite set theory, we will want to secure this possibility axiomatically. One way to to this uses the following **Completeability** principle. Let $\mathbb{N}(x)$ be the assertion that x is a transitive set well-ordered by $\in$ which, in addition, is Dedekind finite. Then we have the following so-called completeability principle for the nautral numbers:

Comp$_{\mathbb{N}}$ $\Diamond\exists X\Box\forall y[Xy \leftrightarrow \mathbb{N}(x)]$

This says that it is possible, eventually, for the being to create *all possible* natural numbers. It can be used to prove the possibility e.g. of ω's existence, given the other axioms.

Indeed, in conjunction with **Comp**$_{\mathbb{N}}$, **HP** proves the possible existence of infinitely many infinite sets. But there is no guarantee from the axioms stated up to now that the modal analogue of the powerset axiom should old: that it should be possible, given any set, infinite or no, eventually to produce *all* its possible subsets. We also have no guarantee that any uncountable sets are possible.

Axiomatically, these things can be imposed by appeal to another completeability axiom:

Comp$_{\subseteq}$ $\Box\forall x\Diamond\exists X\Box\forall y[Xy \leftrightarrow y \subseteq x]$

This time, we assert that the subsets of any given set are always 'completeable': under our intuitive interpretation, it says that one can always eventually form all possible subsets of any given set. The axiom can easily be shown is readily seen to have the desired consequences, as for example proving the possibility of uncountable sets in the strong sense that it is necessary that there is no function defined on the natural numbers with the set a subset of its range.

As a final axiom, or rather axiom scheme, we throw in a modal translation of replacement (the $\Diamond$-translation in the sense defined below).

These are all fairly natural axioms to adopt if one wishes to explicate the idea of a *never ending set-construction process* in modal terms. One has an axiom asserting that any things possibly form a set, along with various other axioms that implicitly concern the lengths of iterations of set formation that are possible.

For ease of reference the axioms for this system are compiled in Appendix A. I will refer to them as the system L, for their creator, Øystein Linnebo.

We now turn to a discussion of the proof-theoretic relations between L and ZFC.

2.2 Key Results

There are three key results of interest. I will state them and indicate the methods of proof. Full proofs are provided in Appendix B.

TODO: YOU NEED TO REDO THIS SO BOTH THEORIES ARE SET IN SECOND ORDER LOGIC. THEN YOU CAN BRING OUT THE DIFFERENCES IN TERMS OF PLURAL COMPREHENSION AND MORE EASILY

PROVE THE FIRST ORDER EQUIVALENCE. YOU CAN ALSO DISCUSS STRATGIES FOR INTERPRETING THE SECOND ORDER CASE ALONG THE LINES OF BUTTON, WHICH YOU REALLY MUST READ AND UNDERSTAND.

The **first key result** is that there is a recursive mapping $\varphi \mapsto \varphi^{\exists}$ from the potentialist language to the language of first order set theory that preserves theoremhood from L into ZFC.

The **second key result** is that there is a recursive mapping $\varphi \mapsto \varphi^{\diamond}$ from the language of first order set theory to the modal language that preserves theoremhood from ZFC into L.

The **third key result** is that the first order fragments of these theories exhibit what Button (cite) has called a *near-synonymy* relation.

The latter takes a little unpacking. What does that mean? The relation of *full synonymy* or *definitional equivalence* between the first order fragments would require, in addition to the first two key results, that ZFC should prove $\varphi \leftrightarrow (\varphi^{\Diamond})^{\exists}$, and that L should prove $\varphi \leftrightarrow (\varphi^{\exists})^{\Diamond}$, where φ in each case is restricted to the first order fragment of the language (but where it may, if a L formula, contain modal operators).

It would be nice if this were so. But unfortunately it cannot *quite* be. The translation $\varphi^{\exists}$ involves interpreting unrestricted quantification in the modal language as tacitly restricted to sets ("possible worlds"), and catering for the extra parameters they induce means that there are extra assumptions we need to make to ensure such an equivalence is provable. For instance, we have to assume that the thing the actualist calls 'the actual world' (for the potentialist) contains exactly all those things the potentialist recognizes. Hence the maps are only inverses if we throw in some extra assumptions on the parameters. Thus we are left only with what Button has called *near synonymy* rather than full definitional equivalence.

The difference between the two is not great. Indeed to me it looks like a trivial book-keeping detail. All the model-theoretic consequences one gets from definitional equivalence, for example, carry over fairly straightforwardly to the form of near-definitional equivalence we acquire. Thus there is a very natural sense in which the first order picture of the sets painted by ZFC is equivalent to the first order modal account axiomatized by theories like L. What more to make of this, and the relevance of the restriction to the first order fragment, will be discussed below.

Let me go into the methods of proof for these results.

First, let's discuss the translation $\varphi \mapsto \varphi^{\exists}$ from the first key result. It has to translate the whole potentialist idiolect into the austere idiom of the first order language. How is that to be done? An important part of the idea, as I said, is to introduce a notion of a possible world and take the potentialist to really be using quantifiers restricted to such worlds. But the speaker of a first order language must also find some interpretation for the potentialist's plural variables. To do that, they will just interpret plural talk as unusual talk about sets (whose elements are the 'plurals').

To implement this, we take all the first order variables x_k of the potentialist, and map them to the even numbered variables $x_{2k} := (x_k)_{\exists}$ in our first order langauge; and we take the pural variables X_k of the potentialist and map them to the odd numbered variables $x_{2k+1} := (X_k)_{\exists}$ of our first order logic. Clearly, this is a way of interpreting talk of plurals as just talk of some special objects.

Now that we have an intepretation for the potentialist's variables we can proceed to specify the interpretation more generally. We set $x \in y^{\exists}$ and $Xx^{\exists}$ as $x_{\exists} \in y_{\exists}$ and $x_{\exists} \in X_{\exists}$ respectively. The connectives commute with the translation. For the quantifiers, we interpret the potentialist's assertion that any thing(s) has(have) the property P as saying that any thing(s) *in the domain* has(have) the property P *relative to that domain*. More explicitly, the translation introduces a free (set) variable d (to be thought of as "the domain") and sets:

$$(\forall x\varphi)^{\exists} := \forall x \in d[\varphi^{\exists}(d)]$$

Modal notions are defined in terms of quantification over extensions of the domain:

$$(\Diamond\varphi)^{\exists} := \exists e \supseteq d[\varphi^{\exists}(e)]$$

It is now tedious but routine to prove that theoremhood is preserved in the transition $\varphi \mapsto \varphi^{\exists}$ (as long as we require d and e to range over *transitive* sets).

The key idea behind the other translation, $\varphi \mapsto \varphi^{\diamond}$, is to simply put a $\Diamond$ in front of every existential quantifier (and a $\Box$ in front of every universal). For variables, we set $(x_{2k})_{\diamond} := x_k$, and $(x_{2k+1})_{\Diamond} = X_k$. It is easy to show theoremhood is preserved.

The third key result is not hard, with the only tricky part being coming up with precisely the right statement. Here it is:

Part one. ZFC proves that for any transitive d, $\varphi \equiv (\varphi^{\Diamond})^{\exists}(d)$.

Part two. L proves that if d is a universal set, then $\varphi \equiv (\varphi^{\exists}d)^{\Diamond}$.

2.3 Discussion

3 Axioms

In this section I will give the system of axioms for height and width potentialism that will be the subject of the rest of the paper.

The system is directed at axiomatizing the following intuitive (if fanciful) conception. As before, we have a being who is able to take things and **Collect** them together into a set. But now, we will also take them to be able to perform another kind of basic act: namely, that of taking some things, and **Counting** them, that is, correlating them one-for-one with the natural numbers. Our new theory will concern the possibilities for arbitrary iterations of **Collect** and **Count**. This time, we say that a *set* is something that can be gotten eventually by arbitrary iterations of these operations. **Collect** is a height potentialist principle, in that the set collected together has higher rank than the collected sets, while **Count** is naturally understood as a width potentialist principle, as the introduction of an enumerating function for a given set may corresponds to the introduction of new sets of the same or lower rank as the sets counted in many cases– indeed, up to isomorphism, it will correspond to introducing a subset of the naturals.

There are many natural modal operators that one can read off from this conception. There is for example a modal, which I will write $\Diamond_{\uparrow}\varphi$, which corresponds to possibility only taking acts of **Collect** into consideration. Since this is the height potentialist sense of possibility I shall often refer to this as

vertical possibility. There is also a natural modal, $\Diamond_{\leftarrow}\varphi$, which corresponds to possibility just by iterated enumeration. This might naturally be read as **horizontal** possibility. And there is a general modal operator, $\Diamond\varphi$, which corresponds to φs being possible by some arbitrary number of iterations and alternations of the two available operations. I will refer to this as **general** possibility. There are many others besides, for example that induced by $\Diamond_{\uparrow}\Diamond_{\leftarrow}$, but these will play little direct role. In fact, for our purposes, even $\Diamond_{\leftarrow}$ will be superfluous. We will focus our attentions on the two operators $\Diamond_{\uparrow}$ and $\Diamond$, where the former corresponds to possibility just by **Collection**, and the latter to possibility by combinations of **Collect** and **Count**.

So we take a multi-modal language with the two modal operators $\Diamond$ and $\Diamond_{\uparrow}$. We still want the capacity to talk about pluralities and so we keep the two types of variable. And we want to talk about sets so we include the membership relation symbol $\in$.

What axioms should we impose? Well, we want all the same plural and first order stuff. Both of the modal operators should have S4.2, and we have $\Diamond_{\uparrow}\varphi \rightarrow \Diamond\varphi$. We take over all the axioms from L for the $\Diamond_{\uparrow}$ modality: the idea being, nothing has changed from before, and if we could get something without forcing/enumeration before we can get it by ignoring forcing/enumeration now.

4 Key Results

5 Discussion

6 Stuff

Modal Set Theory: Absolutness of various kinds.

Set Theory: Pi 1-1 PSP equivalent to only countably many reals in L[r].

Logic: Define a formula to be *pseudo typed* (PT) iff odd numbered variables only ever occur to the right of $\in$. Then in ZFC, any fomula φ has a (weak) PT equivalent $PT(\varphi)$ with $ZFC \vdash PT(\varphi)$ iff $ZFC \vdash \varphi$, and any sentence φ has a (strong) PT equivalent $PT(\varphi)$ with $ZFC \vdash \varphi \leftrightarrow PT(\varphi)$.

Proof. Let $n(\varphi)$ be a number greater than the indices of variables in φ. For each n let E_n be an enumeration of the evens greater than n, so e.g. $E_2(0) = 3$.

We define $PT(\varphi)$ recursively. Moving from left to right in the formula keep looking until you find a variable out of place. Then check to see if it is free or bound. If it is free, replace all other free occurrences of the variable by $E_{n(\varphi)}(0)$; If it is bound, do the same for all its occurrences in the scope of the binding quantifier. In either case, a formula $PT_0(\varphi)$ results. Then given $PT_n(\varphi)$, do the same thing to get $PT_{n+1}(\varphi)$. Whenever $PT_n(\varphi) = PT_{n+1}(\varphi)$, $PT_n(\varphi)$ is PT. We set such $PT_n(\varphi) := PT(\varphi)$.

$PT(\varphi)$ is provable iff φ is. Induction on the number of out-of-place variables. Suppose it holds for n.