On the Consistency of Height and Width Potentialism

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Abstract

Recent work in philosophy of set theory has furnished some arguments that height and width potentialism are inconsistent with one another. One such argument can be found in this volume (Brauer); the other is forthcoming (Roberts).

At the same time, others have suggested there may be some merit in the combination of height and width potentialism. Such authors have presented views that appear to manifest the combination in a non-trivial way, and have defended philosophical claims on their basis (e.g. that all sets are ultimately countable – cf Builes & Wilson, Meadows, Scambler, Pruss).

Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

1 Overview

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Clearly there is a tension here. The business of this article is to explain (what I take to be) its solution. I will argue that height and width potentialism are compatible, and that there is even an attractive view in the foundations of mathematics that arises from their combination. I will do this by explaining that view and how it responds to the arguments alleging inconsistency.

The plan is as follows. Section 1 gives some definitional background and context. Section 2 then presents some axiomatic systems that purport to combine height and width potentialism and discusses issues of relevant to consistency, citing new results on strength and equivalence to standard 'actualist' set theories. Section 3 presents the inconsistency argument of Roberts and explains how a proponent of height and width potentialism should reply. Section 4 does the same for Brauer.

2 Background

Let me start be explaining the terms.

For present purposes, *potentialism* in set theory will be the idea that there could always be more sets than there in fact are. In slogan form, for the potentialist, the universe of sets is 'indefinitely extendible'.

Height potentialism is the idea that the universe is always extendible 'upwards' to include new sets of higher rank than any given ones. *Width potentialism*, on the other hand, is the idea that the universe is always extendible 'outwards', to contain new sets of no greater rank than the max of any given ones.

Historically, height potentialism has been motivated by considerations involving the set-theoretic paradoxes. Russell's paradox shows there is no set of all non-self-membered sets, and hence that the cumulative hierarchy of all sets is itself (therefore) not a set. But the height potentialist complains that any 'stopping point' for the cumulative hierarchy would be arbitrary. Surely there is no conceptual obstacle to any particular collection of ranks of the cumulative hierarchy providing 'urelements' for a longer continuation.

It is natural to explicate this idea of an indefinitely extendible universe of sets in modal logic. Indeed, modal axiom systems based around core height potentialist ideas are known to exhibit tight forms of equivalence with standard iterative set theories like ZFC.

Width potentialism has been historically less popular, although has been attracting some attention over the last few decades as a way understand the independence phenomenon in set theory. There is, in fact, a close structural similarity in the motivation for height and width potentialism in these terms (cf Meadows). Just as the height potentialist begins with the intuition that it is arbitrary that there should be some ranks of the cumulative hierarchy that somehow inherently can't extended to include further sets, the width potentialist may begin with the idea that it is arbitrary that there should be some universe of sets that cannot be extended by forcing over its partial orders. Just as in the previous case, the mathematics of forcing seems to lead us to believe there is at least no conceptual obstacle to making sense of such 'forcing extensions of the universe'.

Again, as with height potentialism, a natural way for the width potentialist to formalize their view involves modal logic: one formulates an axiom to the fact that, for any partial order $\mathbb{P}$, it is possible to find a generic for $\mathbb{P}$. Such explicitly axiomatic approaches to width potentialism are not very well-studied: the focus of most work in this area has been on model theory. Nevertheless, axiomatic approaches are possible and easy enough to formulate.

There are, in any event, clear analogies in the cases for height and width potentialism. In each case one has a central inexistence result in first order set theory (Russell's paradox, the proof that some filters do not admit partial orders) and one seeks to overcome it, after a fashion, by implementing a modalized version – any things can form a set in the first case, any partial order can be forced over in the second. Indeed, it is natural to think that going potentialist one way might give you some reason to consider going potentialist the other way too.

3 Height and Width Potentialism Combined

In this section, I will present some axiom systems that seem to explicate the intuitive idea of height and width potentialism, as described above.

3.1 Core Logical Principles

All the versions of height and width potentialism we will consider will be built around two principles: the first is the height potentialist principle that any things can be the elements of a set; the second is the width potentialist 'forcing axiom', that a generic filter can be found for any partial order.

Both potentialist principles involve modality; the first also involves plural quantification. Accordingly the language $\mathcal{L}_0$ we use to formulate these theories will have at least a modal operator $\Box$, singular variables x_n, plural variables X_n, the propositional connectives and quantifiers, the identity symbol = and the symbol $\in$ for set membership. The circumstance that some thing x is one of some things X will be represented by the concatenation Xx. Identity is only well formed between terms of the same type (singular/plural).

The following core axioms will be included in all the potentialist theories we will go on to discuss.

FQ Standard rules for free quantifier logic with identity for each type of variable.

Mod The modal logic S4.2 with converse Barcan formula, necessitation, and standard rules for universal quantification within the scope of $\Box$.

P-ext $\Box\forall x[\Diamond Xx \equiv \Diamond Yx] \supset X = Y$

PR The plural rigidity axioms

1. $\Diamond Xx \to \Box Xx$
2. $\Diamond(\exists x Xx \wedge x = y) \to \exists x Xx \wedge x = y$.

Comp For any formula $\varphi(x, y, Y)$, $\Box\forall y\Box\forall Y\Box\exists X\forall x[Xx \leftrightarrow \varphi(x, y, Y)]$ is an axiom.

P-Choice A plural version of the axiom of choice.

M-Rep A modal version of the axiom scheme of replacement.

Inf An axiom saying that the set of natural numbers exists.

Set-Ext The axiom of extensionality in the form $\Box\forall x[\Diamond x \in y \equiv \Diamond x \in z] \supset y = z$

Sets The axiom of foundation, and an axiom $\Box\forall x\exists X\Box\forall y[Xy \leftrightarrow y \in x]$ asserting the rigidity of set membership

The precise details of many of these won't be too relevant below, but they are cited here for something like completeness' sake. More details can be found in CITE. The odd one out, perhaps, is **Inf**. It is included here for simplicity, since many of the phenomena we will be interested in occur only at the level of infinite sets.

The basic idea behind the core axioms is to get a modal and plural logic, combined with fundamental principles of set theory, to allow for the development of set theory based on potentialist principles of set existence. Let us now turn to that development.

3.2 Simple HWP

In this section I will present what seems to me to be the simplest formal combination of height and width potentialism. The theory is not mathematically strong, being exactly equivalent in strength to second order arithmetic. But it does offer a clean and simple proof of concept for the combination of height and width potentialism, as well as a useful starting point for further extensions.

The first axiom we add is:

HP $\Box\forall X\Diamond\exists x\forall y[y \in x \leftrightarrow Xy]$

- *Any things can be the elements of a set.*

HP enshrines the core idea of height potentialism, since it implies (given the usual notion of rank) that given any sets one can find others of still higher rank.

The second requires a little more preparation. In our background logic we can define the notion of a partial order, the notion of being a filter on a partial order, and the notion of being a dense set in a partial order in the standard way. Let blackboard variables $\mathbb{P}, \mathbb{Q}$ range over partial orders. (These are singular variables.) Let $D(X, \mathbb{P})$ mean that all Xs are dense in $\mathbb{P}$. Finally, let $F_\cap(x, \mathbb{P}, X)$ mean that x is a filter on $\mathbb{P}$ that intersects every one of the Xs. Then our width potentialist axiom can be stated as:

WP $\Box\forall\mathbb{P}\forall X D(X, \mathbb{P}) \rightarrow \Diamond\exists g(F_\cap(g, \mathbb{P}, X))$

- *Any partial order can be used in forcing.*

WP is one way to flesh out width potentialism, since it implies that any sufficiently rich pluarlity of sets sets may be extended to include ones of rank no greater than their max by forcing.

The following will be useful to us going forward. In it, $f : \mathbb{N} \twoheadrightarrow X$ is an abbreviation for the assertion that f is a function on $\mathbb{N}$ with every X in its range.

Proposition 1. *Let Count be the principle:*

$$\Box\forall X\Diamond\exists f[f : \mathbb{N} \twoheadrightarrow X]$$

then Count is equivalent to WP over the core logic + HP.

Proof. See CATBC. □

Let us call the result of adding these HP and either WP or Count to the core logic Simple Height and Width Potentialism, or SHWP.

Turning now to questions of consistency, it turns out that SHWP is demonstrably consistent relative to Second Order Arithmetic (SOA). In fact it turns out that SHWP exhibits a tight form of equivalence with SOA, something Tim Button has called 'near synonymy' in the recent literature.

(Here and below, we understand SOA under the guise of ZFC without power + all sets are countable. This is definitionally equivalent to the more standard arithmetical formulations; see e.g. Simpson.)

A detailed account of the relationship between SOA and SHWP would be overkill here. (Details can be found in CITE.) But certain features will be important to our discussion of consistency below, and having some idea of how the interpretation of the modal theory SHWP in the non-modal SOA goes will be very helpful to us going forward. So we will spend some time discussing certain aspects of the 'tight equivalence' just mentioned.

Let $\mathcal{L}_\in$ be the first order language of set theory. Then:

Theorem 2. *There is a map $\cdot^\diamond : \mathcal{L}_\in \to \mathcal{L}_0$ that preserves theoremhood from SOA to SHWP.*

Theorem 3. *There is a map $\cdot^\exists : \mathcal{L}_0 \to \mathcal{L}_\in$ that preserves theoremhood from SHWP to SOA.*

Theorem 2 says that SHWP interprets SOA; this direction does not really concern us directly, since it does not imply anything about the consistency of SHWP, but some idea of how things go will be useful. The translation $\varphi \mapsto \varphi^\diamond$ proceeds by prepending every universal quantifier in φ by a $\Box$ and every existential by a $\Diamond$. A result due to Linnebo says that for $\varphi \in \mathcal{L}_\in$,

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma^\diamond \vdash_{Core} \varphi^\diamond$$

where $\Gamma^\diamond$ has the obvious meaning of $\{\gamma^\diamond : \gamma \in \Gamma\}$. It thus suffices to prove $\varphi^\diamond$ for each axiom of SOA in SHWP, which (using Proposition 1) is not difficult.

Theorem 3 is directly on-topic. It says that SOA interprets SHWP and hence secures the consistency of the latter relative to the former. Full details of the ideal proof are a little fiddly and would go beyond our needs here. But some idea, again, of how things go down will be useful.

The key idea behind the translation $\varphi \mapsto \varphi^\exists$ is to factor out use of modal operators in favor of quantification over possible worlds.

In a bit more detail, the idea is that we define the notion of a possible world in SOA to be a transitive set. That means we interpret the language of SHWP so that ordinary quantifiers are always restricted to such worlds, and the modal operators induce quantifiers over such worlds.

Formally, this means our translation will have to carry formulas φ in the modal language (which may be sentences) to formulas $\varphi^\exists(w)$ in the first order language with a free 'world' variable w. Key clauses of the translation are things like

$$(\forall x\varphi)^\exists(w) := \forall x \in w\varphi^\exists(w)$$

$$(\forall X\varphi)^\exists(w) := \forall x \subseteq w\varphi^\exists(w)$$

$$(\Box\varphi)^\exists(w) := \forall u[Tran(u) \wedge w \subseteq u \to \varphi^\exists(u)]$$

where $Tran(u)$ of course is the assertion that u is transitive. (The case of plural containment Xx is a little fiddly, but can be made to work quite nicely.) The potentialist axioms are then readily seen to be true, under the translation, in SOA. For HP, this is because every set is an element of a transitive set. For WP, this is because SOA proves all sets are countable. So for any given partial order, we may move to a transitive set that witnesses its countability, and then proceed to introduce a generic for it if need be.

A final point is that these results can be strengthened to attain something approximating definitional equivalence between the two theories. ...

3.3 Strong HWP

The theory I have just cited combines height and width potentialism in a straightforward way. The result is a theory that is (up to a near-synonymy) second order arithmetic. This is a pretty weak theory, although as we know from the reverse mathematics literature it is plenty strong enough to develop much of the mathematics needed in applications. Can the height and with potentialist do better?

There is indeed a reasonable way to proceed here. Our potentialist is interested with expansions of the universe along two 'directions'. One has the ability to extend the universe 'upwards' to create sets of higher rank; and one has the ability to extend it 'outwards' by forcing. By separating out these two possible methods of expansion, and by asserting strong axioms regarding what can be attained along the vertical dimension of expansion alone, much stronger HWP systems can be derived, indeed ones with all the power of ZFC and more. This extension makes the general picture of height and width potentialism much more philosophically interesting, since such versions have at least the consistency strength to recover all standard mathematics. But they also bring new conceptual problems with them, problems that will be exploited in one of the inconsistency arguments we will consider.

Let us first discuss in a little more detail how to implement this idea. We expand the language $\mathcal{L}_0$ to $\mathcal{L}_1$ by adding in two new modal operators, $\obar$ and $\obslash$. $\obar$ is to reflect possibility by *only* vertical expansion: one can think of this as possibility by *only* iteratively introducing new sets for given pluarlities. $\obslash$, on the other hand, is to reflect possibility by *only* horizontal expansion: one can think of this, indifferently, as just by adding new generic filters, or adding new enumerating functions. ($\boxbar$ and $\boxbslash$ are the duals.) $\Diamond$ remains in the language, and represents 'absolute' possibility, that is, possibility by either domain expansion method.

Here is a way to axiomatize a 'strong' system of this kind.

We begin by modifying HP and WP.

HP $\Box\forall X \obar \exists x\forall y[y \in x \leftrightarrow Xy]$

WP $\Box\forall\mathbb{P}\forall X D(X,\mathbb{P}) \rightarrow \obslash\exists g(F_\cap(g,\mathbb{P},X))$

Next, we add rules to reflect the generality of $\Diamond$.

Gh $\obar\varphi \rightarrow \Diamond\varphi$

Gw $\obslash\varphi \rightarrow \Diamond\varphi$

Finally, add the following restricted version of the powerset axiom to vertical possibility.

r-Pow $\Box\forall x \obar \exists y \boxbar \forall z[z \in y \leftrightarrow z \subseteq x]$

r-Pow says that it is always possible by vertical expansion to get all the subsets of any given set, *so long as one disregards the subsets you can get by forcing*.

It is useful to compare r-Pow with the corresponding 'unrestricted' version

Pow $\Box\forall x \obar \exists y\Box\forall z[z \in y \leftrightarrow z \subseteq x]$

which is provably inconsistent with **WP** over the rest of the theory. Pow says that it is possible for there to such things as *all possible* subsets of any given set; but given the existence of an infinite set (as we are guaranteed), and the universal possibility of forcing, enshrined in **WP**, this cannot be, since we can always force to add new subsets to any given set.

Let the extension of the core theory by the above principles (but not, of course, Pow) be called HWP. The following facts come easily.

Theorem 4. *There is a translation* $\cdot^{\varobar} : \mathcal{L}_{\in} \to \mathcal{L}_1$ *that preserves theoremhood from ZFC to HWP.*

Theorem 5. *There is a translation* $\cdot^{\Diamond} : \mathcal{L}_{\in} \to \mathcal{L}_1$ *that preserves theoremhood from SOA to HWP.*

In each case, the result follows much as it did in the previous case. In fact Theorem 5 really just is Theorem 2. The first uses the same translation but with $\varobar$ in place of $\Diamond$ and $\boxbar$ in place of $\Box$ everywhere. That the theorem goes through is a known theorem of Linnebo in a slightly different key.

The resulting theory has some intruiging features. If one restricts one's attention to vertical possibility and necessity, one can construct objects that satisfy anything you can get in ZFC, for example the existence of uncountable cardinals, beth fixed points, and so on. But these large cardinals are always mirages: and the mirage can always be revealed by appeal to $\varobslash$-possibility. Anything that might in some possible world satisfy the formula saying that it is (say) ω_1 will also not satisfy that description in some other possible world. There are no *absolute* uncountables, only relative pretenders.

The chief advantage HWP has over SHWP is interpretative power: it can interpret all of ZFC, notwithstanding its ultimate commitment to the countability of all things. This gives HWP the capacity, at least once motivational details are filled in, to potentially offer a countabilist foundation for all mathematics. This is I think one of the principle interests of the combination of height and width potentialism.

I have said that the chief advantage of HWP over SHWP is interpretative power. But how much of that does it have? Does it have too much? We now turn to addressing these issues of consistency in HWP.

It turns out that the methods employed in the previous section generalize fairly naturally to cater to systems like HWP and even further extensions. The generalization involves extend second order arithmetic to include 'topological regularity axioms', and then giving a more nuanced definition of possible world in terms of such an extension.

Let's start with the extension of second order arithmetic involved, to include so-called topological regularity axioms. What are those?

Well, there are certain 'nice' topological properties of sets of reals – things like being Lebesgue measurable, or having the perfect set property – that cannot hold everywhere, at least given the axiom of choice. The issue (repeatedly) is that the axiom of choice allows you to well-order the reals and then construct barbaric sets of various kinds, ones that don't have the nice features, by exploiting the well-order.

There's a general feeling in set theory that the 'nice' properties should all hold of easily definable sets of reals, and that the 'nasty' counterexample sets should all be pretty complicated to define (in terms of the usual analytical hierarchy). For example, if $V = L$, then there are easily definable nasty sets of reals, and this is generally taken to be a mark against the principle.

Since $V = L$ implies there are easily definable but nasty sets, it follows that principles asserting that easily definable sets are nice must in some cases go beyond SOA (which is of course consistent with $V = L$). Principles of this kind are what I'm calling *topological regularity axioms*. They assert that good-behavior properties hold for certain easily definable classes of reals, even when this is not provable in SOA (or ZFC).

The weakest such axiom is something called the Π^1_1-Perfect Set Property (PSP), which says that every uncountable set of reals definable by a Π^1_1 formula is either countable or has a perfect subset.[1] It turns out that this minimal extension of SOA is enough to secure the interpretability of HWP.

Theorem 6. *There is a map $\cdot^{\exists} : \mathcal{L}_1 \to \mathcal{L}_{\in}$ that preserves theoremhood from HWP to SOA + Π^1_1-PSP.*

The proof uses similar ideas to those employed for SOA and SHWP in the previous section, but as I said one has to take a more nuanced definition of 'possible world'. In particular, it turns out that here our definition of a world will need to be 'doubly parameterized': that is, we will need *two* free variables in $\varphi^{\exists}$.

Why? Let us think about how our possible worlds will have to look to get an interpretation going. In order for r-Pow to be true, there will have to be worlds w at which the set x of all Φ-possible sets of numbers exist. These worlds are worlds where any plurality of numbers, in the sense of w, form a set. And yet, since we can always force to add subsets to any infinite set, there must still be further possible sets of numbers. These sets of numbers have numbers as their members. But we said that all pluralities of numbers at w formed sets at w!

The only way out of this tangle for the HWP proponent is to accept indefinite extendability both with respect the singular domain, and with respect the plural domain (holding the singular domain fixed). The introduction of new generics witnesses, on this view, the extension of the plural domain (over a fixed) singular domain. Indefinite extendability 'runs in two dimensions', according to the present picture. Hence, our possible worlds will be doubly parameterized, with one parameter representing the singular domain, and the other parameter representing the plural. Expansions will then be possible, under the proposed interpretation, in both directions.

Let's now look in more detail how this works. We will make use of the following fact, whose proof is originally due to Solovay.

Fact 7. *Over SOA, the Π^1_1-PSP is equivalent to $L[r]$ containing only countably many reals for every r.*

Thus, in effect, the Fact says that $L[r]$ always falls badly short of containing all reals if we have Π^1_1-PSP. It is thus a strong form of $V \neq L$.

It is fairly easy to see that the fact has the following corollary.

Corollary 8. *In SOA + Π^1_1-PSP, $L[r]$ is a (class) model of ZFC for every real r.*

I provide an argument sketch (that the corollary follows from the fact) in a footnote.[2]

This now gives us the tools we need to implement our parametrized possible worlds strategy. We define, in SOA + Π^1_1-PSP, our 'possible worlds' to

[1] A perfect set is a closed set with no isolated points.
[2] Do.

comprise a transitive set t (representing the first order domain) and a real number r (representing the second order domain). We require that $t \in L[r]$, and will say that plural quantification in a possible world is always restricted to subsets of $L[r]$.

On this interpretation, when we vertically expand the domain, that means that we add to the transitive set t while staying within $L[r]$; when we horizontally expand the domain, that means we extend the real parameter to s (say), making more pluralities 'visible': we will now we consider $L[s]$ when we think about extending vertically, and so on. Since SOA + Π^1_1-PSP has $L[r] \models$ ZFC for every real r, this interpretation for vertical possibility will get us ZFC, as we need, even remaining within the SOA context in general.

To implement this formally, our mapping $\varphi \mapsto \varphi^\exists$ will therefore need to carry $\varphi \in \mathcal{L}_1$ to a formula in $\mathcal{L}_\in$ with not just one but two free variables: $\varphi \mapsto \varphi^\exists(t, r)$. It will contain clauses like

$$(\forall x \varphi)^\exists(t, r) := \forall x \in t \varphi^\exists(t, r)$$
$$(\forall X \varphi)^\exists(t, r) := \forall x \subseteq t[x \in L[r] \rightarrow \varphi^\exists(t, r)]$$
$$(\boxdot\varphi)^\exists(t, r) := \forall u \in L[r][Tran(u) \wedge t \subseteq u \wedge u \in L[r] \rightarrow \varphi^\exists(u, r)]$$
$$(\Box\varphi)^\exists(t, r) := \forall s \geq r \forall u \in L[s][Tran(u) \wedge t \subseteq u \rightarrow \varphi^\exists(u, s)]$$

(In the final expression, $s \geq r$ means that r is constructible from s.)

With the full translation in hand it is a fairly straightforward matter to prove that every theorem of HWP comes good in SOA + Π^1_1-PSP under the translation: $\varphi \mapsto \forall t, r Tran(t) \wedge t \in L[r][\varphi^\exists(t, r)]$. In fact, one can show that HWP has Π^1_1-PSP$^\diamond$ as a theorem, and prove the same kind of tight proof-theoretic equivalence obtains here between HWP and SOA + Π^1_1-PSP as did between SHWP and SOA simpliciter. These results can be pushed further: by increasing the strength of the regularity assumed in the interpreting theory, one can ratchet up the consistency strength of the corresponding modal HWP theory, allowing for measurables and beyond to be 'vertically' attainable. (We will return to this observation in more detail below.)

In any case, that concludes the formal arguments in favor of the consistency of height and width potentialism. We have seen, in outline, that there are arguments for the consistency of height and width potentialism in both a strong and a weak form, relative to extensions of SOA by topological regularity axioms.

Let us now turn to the arguments alleging inconsistency between height and width potentialism, and to see how they relate to the approaches sketched up to now.

4 Roberts

Right from the off it is perhaps worth mentioning that Roberts' argument applies to HWP only. This is not a big deal: he is assuming that the height potentialist will be committed at least to the recovery of ZFC in terms of pure height potentialism, so that any commitment to width potentialism will require a bimodal treatment of the sort described in the last section. But it is worth mentioning that a less grand form of height and width potentialism can still be viable, in the form of SHWP.

The actual details of Roberts' argument are at present not entirely clear to me. His paper is a work in progress, but in the most recent version he argues that the addition of an actuality operator to a theory related to HWP is incosistent, and that on that basis we should think that the viewpoint enshrined in HWP is similarly inconsistent. In correspondence, I suggested that the use of an actuality operator was a red-herring, and that in fact the argument turns (whether or not we have an actuality operator in the picture) on whether we are allowed comprehension only in the 'closed' form cited above, or if instead we accept instances of comprehension with free variables. I continue to believe that the fundamentals of the argument Roberts presents turn on issues surrounding comprehension, and that addition of an actuality operator is a needless increase in complexity. So here I will discuss what I take to be the simpler version of the argument, rather than the last one I saw in a draft from Roberts. In an appendix, I will explain why I think the addition of an actuality operator is unnecessary.

Here then is the argument. Let HWP$^+$ be the results of allowing all instances of comp,

$$\exists X \forall x[Xx \equiv \varphi]$$

where φ may include free variables. Then:

Theorem 9. *HWP$^+$ is inconsistent.*

Proof. It is easy to see that HWP implies

$$\Diamond \exists x[\Diamond(\exists y[y \subseteq x \wedge y = z]) \wedge \boxbar(\neg \exists y[y = z])] \tag{1}$$

abbreviate this as $\Diamond \exists x \Psi(x, y, z)$. By strong comprehension and existential instantiation,

$$\Diamond(Ex \wedge \Psi(x, z) \wedge \exists X \forall w[Xw \equiv \Diamond w \in z]) \tag{2}$$

by HP,

$$\Diamond(Ex \wedge \Psi(x, z) \wedge \obar \exists u \forall w[w \in u \equiv \Diamond w \in z]) \tag{3}$$

one can then instantiate on u and prove $u = z$ by (our strong modal form of) extensionality.

But this is a contradiction.

□

Naturally, given this argument, we should expect the proof of consistency implicit in the previous section to break down with strong comprehension. And this is precisely what happens in the interpretation offered for HWP in SOA $+\Pi^1_1$-PSP. When we have closed comprehension, in effect we are requiring all the comprehending parameters come from $L[r]$ for fixed r; the fact that $L[r]$ satisfies separation for formulas *in such parameters* then secures the interpretation of plural comprehension. But when free variables are allowed in instances, we may find ourselves in a situation where the parameters invovled do not come from $L[r]$, and so there will be no reason to expect that the acquired interpretation comprehension will come good in $L[r]$, and in general it does not (consider reals s not in $L[r]$ and the instance $x = s$).