Proofs

CS

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TODO: Check consistency of notation: do you use T for SOA$^+$? What even is the best name for this?

Always use mathsf for the modal theories.

1 ZFC and L

1.1 Formulation of ZFC

1.1.1 Language $\mathcal{L}_\in$

Signature

- countable infinity of first order variables x_i
- countable infinity of monadic second order variables X_i
- propositional connectives $\vee, \neg$
- variable binding quantifier $\exists$
- relation $=$
- relation $\in$

wffs

$$Xx|x \in y|x = y|X = Y$$

$$\varphi \wedge \psi|\neg\varphi|\forall x\varphi|\forall X\varphi$$

defs

- usual defs for connectives quantifiers
- symbol for $\emptyset$, $\{x : \psi\}$ notation

1.1.2 Axioms of ZFC

Standard second order logic with full comprehension and extensional second order identity. For set theoretic axioms:

Ext$_\forall$ $\forall x[x \in y \equiv x \in z] \supset y = z$

Fun$_\forall$ $x \neq \emptyset \supset \exists y[y \in x \wedge y \cap x = \emptyset]$

Pair $\exists z[z = \{x, y\}]$

Un $\exists y[y = \bigcup U(x)]$

Rep $Fun(F) \supset \forall x \exists y(y = F|x)$

1.2 Formulation of L

1.2.1 Language $\mathcal{L}_0$

Signature

- countable infinity of first order variables x_i
- countable infinity of monadic second order variables X_i
- propositional connectives $\vee, \neg$
- variable binding quantifier $\exists$
- relation $=$
- operator $\Diamond$
- relation $\in$

wffs

$$Xx|x \in y|x = y|X = Y$$

$$\varphi \wedge \psi|\neg\varphi|\forall x\varphi|\forall X\varphi|\Diamond\varphi$$

defs

- usual defs for connectives quantifiers and modals
- Ex is an abbreviation for $\exists y[y = x]$ or a more suitable alphabetic variant
- $Set(x, X)$ abbreviates $\forall y[Xy \equiv y \in x]$
- Previous abbreviations from set theory

1.2.2 Axioms of L

We assume all propositional tautologies, along with any standard axioms for positive free quantifier logic.

The modal logic is S4.2 with necessitation (and CBF). The rule of inference ... is also assumed.

As to the plural logic, we assume the following.

pExt $\forall X \forall Y[(\forall x[Xx \equiv Yx]) \supset X = Y]$

pR $\Diamond Xx \supset \Box Xx$

pBF $\forall X[\Diamond(\exists x[Xx \wedge x = y]) \supset \exists x[Xx \wedge x = y]]$

Finally, the set theoretic axioms.

Ext $\forall x \forall y[(\forall z[z \in x \equiv z \in y]) \supset x = y]$

Ele $\Box \forall x \exists X \Box \forall y[Xy \equiv y \in x]$

Fun $\forall x[x \neq \emptyset \supset \exists y[y \in x \wedge y \cap x = \emptyset]]$

Set $\Box \forall X \Diamond \exists y[Set(y, X)]$

Inf $\Diamond \exists X \Box \forall y[Xy \equiv \mathbb{N}(y)]$

Pow $\Box \forall x \Diamond \exists X \Box \forall y[Xy \equiv y \subseteq x]$

Rep $Rep^{\Diamond}$

Min $\forall y[y \in x] \supset \Diamond(\forall y[y \in x] \wedge \Diamond(z \in x) \supset Ez)$

1.3 Key Result One

Theorem 1. *There is an interpretation $\exists : \mathcal{L}_0 \to \mathcal{L}_\in$ that preserves theoremhood from L to ZFC.*

Definition 1 ($\varphi_\exists$)**.** *The translation $\varphi \mapsto \varphi_\exists$ is defined by the following clauses. In each case, d is the least variable not occuring in φ.*

- $x \in y_\exists := x \in y \wedge d = d$
- $Xx_\exists := Xx \wedge d = d$
- $(\neg\varphi)_\exists := \neg\varphi_\exists(d)$
- $(\varphi \vee \psi)_\exists := \varphi_\exists(d) \vee \psi_\exists(d)$
- $(\exists x \varphi)_\exists := \exists x \in d[\varphi_\exists(d)]$
- $(\exists X \varphi)_\exists := \exists X \subseteq d[\varphi_\exists(d)]$
- $(\Diamond\varphi)_\exists := \exists e[e \supseteq d \wedge Tran(e) \wedge \varphi_\exists(e)]$

In these, $\varphi_\exists(e)$ represents the result of substituting e for d in $\varphi_\exists$.

We then set $\exists(\varphi) := Tran(d) \supset \varphi_\exists$.

Proof. The propositional tautologies and modus ponens are obvious. For free universal instantiation, we must show (under the assumptin $Tran(d)$) that

$$\forall x \in d[\forall y \in d(\varphi(y))_\exists(d) \supset (\varphi(x))_\exists(d)]]$$

But this is immediate. (Note however that the unfree quantifier rule, which removes the initial quantifier, is not provable under this interpretation.)

As to the laws of $S4.2$ modal logic, it is completely clear that S4 will hold in light of the reflexivity and transitivity of $\subseteq$. For $.2$, suppose $(\Diamond\Box\varphi)_\exists$. Then there is a transitive extension e_0 of d such that every extension f of e_0 has $\varphi_\exists(f)$. Suppose given transitive extension e of d. Then $f_1 := e \cup e_0$ is a transitive extension of e_0 that (therefore) has $\varphi_\exists(f_1)$. Hence $(\Box\Diamond\varphi)_\exists$.[1] For necessitation, we must show that if $Tran(d) \supset \varphi_\exists(d)$ is a theorem, then so is $Tran(d) \supset \forall e[e \supseteq d \wedge Tran(e) \supset \varphi_\exists(e)]$, which is obviously correct.

All the plural axioms are straightforward, although it is worth remarking that the interpretations become unprovable (in fact demonstrably false) if the initial second order quantifiers are removed.

On to the set-theoretic axioms. The case of extensionality reduces to the claim that extensionality holds in transitive sets. Ele and Fun are equally straightforward. For set: suppose given a transitive extension e of d and a subset X of e. Since every set is an element of a transitive set (ZFC) we can extend e to a transitive set that contains that subset as an element, and the result follows. Inf just comes down to the fact that there is a transitive set that contains the natural numbers. Similarly for pow: given any set there is a transtive set that contains all its subsets. Each instance of $\exists(Rep^\Diamond)$ just is an instance of replacement. □

1.4 Key Result Two

Theorem 2. *There is an interpretation $\Diamond : \mathcal{L}_\in \to \mathcal{L}_0$ that preserves theoremhood from ZFC to L* on first order formulas.

Proof. This is just the translation $\varphi \mapsto \varphi^\Diamond$. That the result holds is a theorem of Linnebo. □

1.5 Key Result Three

Theorem 3. *These translations yield a definitional equivalence between the first order fragments of ZFC and L in the following sense: for all φ without second order variables, we have*

1. $L \vdash Univ(d) \supset (\varphi_\exists)^\Diamond \equiv \varphi$
2. $ZFC \vdash Tran(d) \supset (\varphi^\Diamond)_\exists \equiv \varphi$.

[1]In fact (by transitivity) we have the stronger $\Diamond\Box\varphi \supset \Box\Diamond\Box\varphi$.)

Here, $Univ(d)$ *abbreviates* $\forall x[x \in d] \wedge \Diamond(\exists y[y \in d \wedge y = z]) \supset \exists y[y \in d \wedge y = z]$ *for a suitable choice of (free)* y*.*

Proof. In each case we proceed by induction on the complexity of φ.

For 1., the base cases are all immediate, as are the propositional connectives. For the quantifier, we need

$$L \vdash Univ(d) \supset \Diamond\exists x \in d(\varphi_{\exists}(d)^{\Diamond}(x)) \equiv \exists x\varphi$$

So suppose $Univ(d)$. Going right to left, our induction hypothesis yields $\exists x(\varphi_{\exists}(d)^{\Diamond}(x))$, and then $Univ(d)$ implies $\Diamond\exists x \in d(\varphi_{\exists}(d)^{\Diamond}(x))$ as required.

For the converse, suppose $\Diamond Ex \wedge x \in d \wedge \varphi_{\exists}(d)^{\Diamond}(x)$. Then by $Univ(d)$ we get Ex. We also have $\Diamond\varphi_{\exists}(d)^{\Diamond}(x)$. But by a result of Linnebo this implies $\varphi_{\exists}(d)^{\Diamond}(x)$. By the IH we have $\varphi(x)$ and the result follows.

Finally, for $\Diamond\varphi$, we must show that:

$$L \vdash Univ(d) \supset \Diamond\exists e[e \supseteq d \wedge Tran(e) \wedge (\varphi_{\exists}(e))^{\Diamond}] \equiv \Diamond\varphi$$

Suppose $Univ(d)$.

Going left to right, suppose

$$\Diamond(Ee \wedge e \supseteq d \wedge Tran(e) \wedge (\varphi_{\exists}(e))^{\Diamond}).$$

By rigidity, $e \supseteq d \wedge Tran(e)$. So by Min $\Diamond Univ(e)$. Since also $\Diamond(\varphi_{\exists}(e))^{\Diamond}$ and the latter is rigid, we may infer $\Diamond Univ(e) \wedge (\varphi_{\exists}(e))^{\Diamond}$. By our induction hypothesis, $\Box(Univ(e) \supset (\varphi_{\exists}(e))^{\Diamond} \equiv \varphi)$. Hence $\Diamond\varphi$.

Going right to left, Suppose $\Diamond\varphi$. By comp, $\Diamond\varphi \wedge EX \wedge \forall y[Xy \equiv y = y]$. By HP, $\Diamond(\varphi \wedge \Diamond(Ee \wedge Set(e, X)))$. By rigidity, $\Diamond(\varphi \wedge Set(e, X))$.

We have, as a general lemma, $EX \wedge \forall y[Xy \equiv y = y] \wedge Set(x, X) \supset Univ(x)$. For, suppose Ez. Then since $z = z$ we get Xz and hence $z \in x$. So $\forall z[z \in x]$. Then suppose $\Diamond\exists y[y \in x \wedge y = z]$. Then $\Diamond\exists y[Xy \wedge y = z]$. Since EX it follows that Ez, and of course still $y \in x$ as required.

It follows that $\Diamond(\varphi \wedge Univ(e))$. By the IH, $\Box(Univ(e) \supset ((\varphi(e)_{\exists})^{\Diamond} \equiv \varphi))$. Hence, since $Univ(e) \supset Tran(e)$, we have $\Diamond(\exists e[e \supseteq d \wedge Tran(e) \wedge (\varphi(e)_{\exists})^{\Diamond}])$, as required.

□

2 SOA^{+} and M

2.1 Formulation of SOA^{+}

2.1.1 Language of SOA^{+}

The language of SOA is just $\mathcal{L}_{\in}$.

2.1.2 Axioms of SOA^{+}

The axioms of SOA^{+} are those of ZFC without the axiom of power, with replacement possibly reformulated as collection, and with the Π^1_1 PSP.

Important lemma: $L[r] \vDash ZFC$ for every real r in this theory.

2.2 Formulation of M

2.2.1 Language of M

The language $\mathcal{L}_1$ of M is $\mathcal{L}_0$ closed under the operator $\blacklozenge$.

2.2.2 Axioms of M

We assume all propositional tautologies, along with any standard axioms for positive free quantifier logic.

The modal logic is S4.2 with necessitation (and CBF) for both operators. The rule of inference ... is also assumed for both operators.

The only axiom on the combination of modals we need is

W $\blacklozenge\varphi \supset \lozenge\varphi$.

As to the plural logic, we assume the following.

pExt $\forall X \forall Y[(\forall x[Xx \equiv Yx]) \supset X = Y]$

pR $\lozenge Xx \supset \Box Xx$

pBF $\forall X[\lozenge(\exists x[Xx \wedge x = y]) \supset \exists x[Xx \wedge x = y]]$

Finally, the set theoretic axioms.

Ext $\forall x \forall y[(\forall z[z \in x \equiv z \in y]) \supset x = y]$

Ele $\Box \forall x \exists X \Box \forall y[Xy \equiv y \in x]$

Fun $\forall x[x \neq \emptyset \supset \exists y[y \in x \wedge y \cap x = \emptyset]]$

Set $\Box \forall X \blacklozenge \exists y[Set(y, X)]$

Inf $\blacklozenge \exists X \Box \forall y[Xy \equiv \mathbb{N}(y)]$

Pow $\Box \forall x \blacklozenge \exists X \blacksquare \forall y[Xy \equiv y \subseteq x]$

Rep $Rep^{\lozenge}$, $Rep^{\blacklozenge}$

Min2 ??

2.3 Necessary Lemmas

Lemma 1 (Mirroring)**.** *Let $\Gamma \cup \phi$ be a set of formulas in the first order language of set theory. Then $\Gamma \vdash \phi$ in first order logic if and only if $\Gamma^{\lozenge} \vdash_{\mathsf{M}} \phi^{\lozenge}$. The same is true for $\blacklozenge$.*

Proof. See cite. $\Box$

Lemma 2 (Bounded Modal Absoluteness)**.** *Let ϕ be a formula in the first order language of set theory with only bounded quantifiers. Then M $\vdash \phi \leftrightarrow \phi^{\lozenge}$. The same is true for $\blacklozenge$.*

Proof. An induction on the complexity of ϕ. See Lemma 12.2 of Linnebo. $\square$

Lemma 3 (Δ_1 Absoluteness)**.** *Let ϕ be a formula that is provably Δ_1 over ZFC without powerset. Then we have:*

$$\mathsf{M} \vdash \phi^{\Diamond} \leftrightarrow \phi^{\blacklozenge}$$

Proof. Temporarily let T be the theory in question. Note that $T^{\Diamond}$ and $T^{\blacklozenge}$ are each contained in M.

It follows from our assumption that there are Δ_0 formulas ψ, θ for which

$$\mathsf{M} \vdash \phi^{\Diamond} \leftrightarrow \Box\forall x\psi^{\Diamond} \leftrightarrow \Diamond\exists x\theta^{\Diamond}$$

and similarly

$$\mathsf{M} \vdash \phi^{\blacklozenge} \leftrightarrow \blacksquare\forall x\psi^{\blacklozenge} \leftrightarrow \blacklozenge\exists x\theta^{\blacklozenge}$$

by the previous lemma, these are each equivalent to

$$\mathsf{M} \vdash \phi^{\Diamond} \leftrightarrow \Box\forall x\psi \leftrightarrow \Diamond\exists x\theta$$

$$\mathsf{M} \vdash \phi^{\blacklozenge} \leftrightarrow \blacksquare\forall x\psi \leftrightarrow \blacklozenge\exists x\theta$$

respectively. But weakening implies that $\blacklozenge\exists x\theta \to \Diamond\exists x\theta$ and similarly that $\Box\forall x\psi \to \blacksquare\forall x\psi$. The result follows. $\square$

Theorem 4. M *interprets ZFC under the* $\blacklozenge$*-translation.*

Proof. As with cite, this is a straightforward modification of Linnebo's argument from cite. It is sufficient to prove the $\blacklozenge$ translations of axioms of ZFC. For example, for the powerset axiom, use (**??**) and (**??**). $\square$

Theorem 5. M *interprets ZFC$^-$ under the* $\Diamond$*-translation.*

Proof. Essentially the same as the previous. $\square$

Theorem 6. M *proves every set is (hereditarily) countable under the* $\Diamond$ *translation.*

Proof. See cite. $\square$

Corollary 1. M *interprets second order arithmetic under the* $\Diamond$ *translation.*

Proof. $ZFC^- + V = HC$ is definitionally equivalent to SOA. See Krapf, Simpson. $\square$

Lemma 4 (SOA)**.** *If there are only countably many reals constructible from r, then the $\Pi^1_1[r]$-PSP holds.*

2.4 Key Result 4

Theorem 7. *There is an interpretation $\exists : \mathcal{L}_1 \to \mathcal{L}_\in$ that preserves theoremhood from M to SOA.*

Definition 2 ($\varphi_\exists$). *The translation $\varphi \mapsto \varphi_\exists$ is defined by the following clauses. In each case, d_1, d_2 are the least variables not occuring in φ.*

- $x \in y_\exists := x \in y \wedge \bigwedge_i d_i = d_i$
- $Xx_\exists := Xx \wedge \wedge \bigwedge_i d_i = d_i$
- $(\neg\varphi)_\exists := \neg\varphi_\exists(d_1, d_2)$
- $(\varphi \vee \psi)_\exists := \varphi_\exists(d_1, d_2) \vee \psi_\exists(d_1, d_2)$
- $(\exists x\varphi)_\exists := \exists x \in d_1[\varphi_\exists(d_1, d_2)]$
- $(\exists X\varphi)_\exists := \exists X \subseteq d_1 \wedge X \in L[d_2][\varphi_\exists(d_1, d_2)]$
- $(\blacklozenge\varphi)_\exists := \exists e[e \supseteq d_1 \wedge Tran(e) \wedge e \in L[d_2] \wedge \varphi_\exists(e, d_2)]$
- $(\Diamond\varphi)_\exists := \exists e_1, e_2[d_2 \leq e_2 \wedge d_1 \subseteq e_1 \in L[e_2] \wedge Tran(e_1) \wedge \varphi_\exists(e_1, e_2)]$

In these, $\varphi_\exists(e_1, e_2)$ represents the result of substituting e_1, e_2 for d_1, d_2 in $\varphi_\exists$.

We then set $\exists(\varphi) := Tran(d_1) \wedge Real(d_2) \supset \varphi_\exists$.

Proof. The propositional tautologies and quantifier logic stuff is straightforward.

The laws of S4.2 for $\blacklozenge$ hold for similar reasons to those given in the proof of key result 1.

As for $\Diamond$, it is again easy to see the axioms of S4 come good. For .2, suppose $(\Diamond\Box\varphi)_\exists$. Then there is a real e_2 constructible from d_2 and transitive e_1, a superset of d_1 and element of $L[e_2]$, such that $(\Box\varphi)_\exists(e_1, e_2)$. That in turn means that for every real e constructible from e_2, and every transitive set d containing e_1 and an element of $L[e]$, we have $\varphi_\exists(d, e)$.

So suppose given arbitrary d, e with d_2 constructible from e, d a superset of d_1 and an element of $L[e]$. We must find a real r constructible from e and transitive set t a superset of d and element of $L[r]$ such that $\varphi_\exists(t, r)$. The obvious candidates are $t := e_1 \cup d$, and $r := e_2 * e$. And it is not hard to see they have the required features.

All the plural axioms are straightforward.

On to the set-theoretic axioms. The case of extensionality reduces to the claim that extensionality holds in transitive sets. Ele and Fun are equally straightforward.

For set: suppose given a transitive extension e of d_1 in $L[d_2]$, and a subset X of e that is also an element of $L[d_2]$. Since $L[d_2]$ satisfies ZFC, it has that every set is an element of a transitive set in $L[d_2]$. Thus we can extend e to a transitive set e_1 in $L[d_2]$ that contains X as an element, and the result follows. Inf is pretty much the same as before.

For Pow, suppose given $d_4 \geq d_2$ and $d_3 \supseteq d_1$, $d_3 \in L[d_4]$, and $x \in d_3$. The needed result then follows from the powerset axiom in $L[d_4]$.

Replacement follows from replacement, once in $L[r]$ and once just straight up. $\square$

2.5 Key Result 5

Theorem 8. *There is an interpretation $\Diamond : \mathcal{L}_\in \to \mathcal{L}_1$ that preserves theoremhood from SOA$^+$ to M* on first order formulas.

Proof. Using Lemma 4 and the mirroring theorem, to establish the desired conclusion it is sufficient to show the $\Diamond$-translation of the hypothesis of Lemma 4. That is: assuming given an arbitrary real r, we must show that is possible to produce a function on the natural numbers such that, necessarily, if s is a real constructible from r, then s is in the range of f.

The strategy for doing so is simple: we first show that, given any real r, it is possible to produce the set of all possible reals constructible from r in M (in the sense of the modality $\Diamond$). We then invoke Lemma 6, which says that all sets are countable in M (with respect the $\Diamond$ translation) to get the result.

In more detail, we first observe that by the $\blacklozenge$-translation of ZFC, we the $\blacklozenge$ translation of the assertion that for any real r, the set of reals constructible from r exists, namely:

$$\blacklozenge\exists x[\mathbb{R}^{L[r]}(x)^{\blacklozenge}] \tag{1}$$

Where in (1) $\mathbb{R}^{L[r]}(x)^{\blacklozenge}$ is the $\blacklozenge$-translation of the first order formula asserting that x is the set of reals constructible from r. The problem now is to derive from this that

$$\Diamond\exists x[\mathbb{R}^{L[r]}(x)^{\Diamond}] \tag{2}$$

since it is only with respect $\Diamond$, and not $\blacklozenge$, that we have Theorem 6.

To move from (1) to (2) we use the absoluteness lemma 3. The formula asserting that x is $L[r]_\alpha$ is Δ_1 (in parameters r, α) over ZFC without power. Thus, for any x, α, r,

$$\mathsf{M} \vdash (x = L[r]_\alpha)^{\Diamond} \leftrightarrow (x = L[r]_\alpha)^{\blacklozenge}. \tag{3}$$

A simple induction shows that for any ordinal α, $\Diamond\exists x[x = \alpha]$ if and only if $\blacklozenge\exists x[x = \alpha]$. Moreover, standard set-theoretic reasoning (together with mirroring) implies that $\mathbb{R}^{L[r]}$ exists is equivalent to $\omega_1^{L[r]}$ exists, relative to either modality. We show that ($\omega_1^{L[r]}$ exists)$^{\blacklozenge}$ is equivalent to ($\omega_1^{L[r]}$ exists)$^{\Diamond}$, in contrast of course to the real ω_1.

For the non-trivial direction, suppose $\omega_1^{L[r]}$ exists$^{\blacklozenge}$. This is equivalent to there being $L[r]$*-uncountable ordinals relative to* $\blacklozenge$, in the sense that

$$\blacklozenge\exists\alpha \in L[r]^{\blacklozenge}\blacksquare\forall f \in L[r]^{\blacklozenge}[\neg(f : \alpha \twoheadrightarrow \mathbb{N})].$$

So assume the latter and instantiate such an α (using the rule on p N). By lemma 3, $(\alpha \in L[r])^{\Diamond}$. Suppose α is not $L[r]$-uncountable relative to $\Diamond$, i.e. $\Diamond\exists f \in L[r]^{\Diamond}[f : \alpha \twoheadrightarrow \mathbb{N}]$ This implies $\Diamond\exists\beta, f \in L[r]^{\Diamond}_\beta[f : \alpha \twoheadrightarrow \mathbb{N}]$. But then using (3) and the fact that $\blacklozenge\beta$ exists we may infer

$$\blacklozenge\exists\beta, f \in L[r]^{\blacklozenge}_\beta[f : \alpha \twoheadrightarrow \mathbb{N}]$$

contradicting our assumption that α *is* $L[r]$*-uncountable*$^{\blacklozenge}$.

We may thus infer (2) from (1): (1) is equivalent to ($\omega_1^{L[r]}$ exists)$^{\blacklozenge}$, which is equivalent to ($\omega_1^{L[r]}$ exists)$^{\Diamond}$, which latter is equivalent to (2) by standard set theory and mirroring.

But now since we also have Theorem 6, it follows that

$$(\exists x[\mathbb{R}^{L[r]}(x) \wedge \exists f : \mathbb{N} \twoheadrightarrow x])^{\Diamond} \tag{4}$$

which is just the $\Diamond$-translation of the claim that there are only countably many reals constructible from r. Hence, by an application of mirroring, we conclude $\Pi^1_1[r]$-PSP. The result follows.

□

2.6 Key Result 6

Theorem 9. *These translations yield a definitional equivalence between the first order fragments of SOA$^+$ and M in the following sense: for all φ without second order variables, we have*

1. $M \vdash Univ(d_1) \wedge d_1 \leq d_2 \supset (\varphi_\exists)^{\Diamond} \equiv \varphi$
2. $SOA^+ \vdash Tran(d_1) \wedge Real(d_2) \supset (\varphi^{\Diamond})_\exists \equiv \varphi$.

Here, $Univ(d_1)$ is as before.

Proof. Start with 1. The base cases are all immediate, as are the propositional connectives. For the quantifier, we need

$$M \vdash Univ(d_1) \wedge d_1 \leq d_2 \supset \Diamond \exists x \in d_1(\varphi_\exists(d_1, d_2)^{\Diamond}(x)) \equiv \exists x \varphi$$

So suppose $Univ(d_1)$. Going right to left, our induction hypothesis yields $\exists x(\varphi_\exists(d_1, d_2)^{\Diamond}(x))$, and then $Univ(d_1)$ implies $\Diamond \exists x \in d_1(\varphi_\exists(d_1, d_2)^{\Diamond}(x))$ as required.

For the converse, suppose $\Diamond Ex \wedge x \in d_1 \wedge \varphi_\exists(d_1, d_2)^{\Diamond}(x)$. Then by $Univ(d_1)$ we get Ex. We also have $\Diamond \varphi_\exists(d_1, d_2)^{\Diamond}(x)$. But by a result of Linnebo this implies $\varphi_\exists(d_1, d_2)^{\Diamond}(x)$. By the IH we have $\varphi(x)$ and the result follows.

For $\Diamond\varphi$, we must show that:

$$M \vdash Univ(d_1) \wedge d_1 \leq d_2 \supset$$

$$\Diamond \exists e_1, e_2[d_2 \leq e_2 \wedge d_1 \subseteq e_1 \in L[e_2] \wedge Tran(e_1) \wedge (\varphi_\exists(e_1, e_2))^{\Diamond}]$$

$$\equiv$$

$$\Diamond \varphi$$

So suppose $Univ(d_1)$ and $d_1 \leq d_2$.

Going left to right, suppose

$$\Diamond(Ee_1 \wedge Ee_2 \wedge d_2 \leq e_2 \wedge d_1 \subseteq e_1 \in L[e_2] \wedge Tran(e_1) \wedge (\varphi_\exists(e_1, e_2))^{\Diamond}].$$

By rigidity, $e_1 \supseteq d_1 \wedge Tran(e)$. So by Min $\Diamond Univ(e_1)$. We also have $\Box e_1 \leq e_2$. As before, since $\Diamond(\varphi_\exists(e))^{\Diamond}$ and the latter is rigid, we may infer

$\Diamond Univ(e_1) \wedge e_1 \leq e_2 \wedge (\varphi_{\exists}(e_1, e_2))^{\Diamond}$. By our induction hypothesis, $\Box(Univ(e_1) \wedge e_1 \leq e_2 \supset (\varphi_{\exists}(e_1, e_2))^{\Diamond} \equiv \varphi)$. Hence $\Diamond\varphi$.

NOT DONE BELOW HERE Going right to left, Suppose $\Diamond\varphi$. By comp, $\Diamond\varphi \wedge EX \wedge \forall y[Xy \equiv y = y]$. By HP, $\Diamond(\varphi \wedge \Diamond(Ee \wedge Set(e, X)))$. By rigidity, $\Diamond(\varphi \wedge Set(e, X))$.

We have, as a general lemma, $EX \wedge \forall y[Xy \equiv y = y] \wedge Set(x, X) \supset Univ(x)$. For, suppose Ez. Then since $z = z$ we get Xz and hence $z \in x$. So $\forall z[z \in x]$. Then suppose $\Diamond\exists y[y \in x \wedge y = z]$. Then $\Diamond\exists y[Xy \wedge y = z]$. Since EX it follows that Ez, and of course still $y \in x$ as required.

It follows that $\Diamond(\varphi \wedge Univ(e))$. By the IH, $\Box(Univ(e) \supset ((\varphi(e)_{\exists})^{\Diamond} \equiv \varphi))$. Hence, since $Univ(e) \supset Tran(e)$, we have $\Diamond(\exists e[e \supseteq d \wedge Tran(e) \wedge (\varphi(e)_{\exists})^{\Diamond}])$, as required.

$\Box$