INFORMATION THEORY
Master of Logic, Master AI, Master CS, University of Amsterdam, 2018
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Team homework set 2

Problem 1: Fun with dice (6 pt)

Consider the following random experiment with two fair (regular six-sided) dice. First, the first die is thrown – let the outcome be A. Then, the second die is thrown until the outcome has the same parity (even, odd) as A. Let this final outcome of the second die be B. The random variables X, Y and Z are defined as follows:

$$X = (A + B) \mod 2, \quad Y = (A \cdot B) \mod 2, \quad Z = |A - B|.$$

(a) **(1pt)** Find the joint distribution P_{AB}.

(b) **(1pt)** Determine $H(X), H(Y)$ and $H(Z)$.

(c) **(1pt)** Compute $H(Z|A = 1)$.

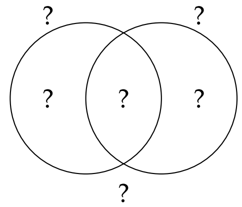
(d) **(1pt)** Compute $H(AB)$, i.e. the joint entropy of A and B.

(e) **(2pt)** A random variable M describes whether the sum $A+B$ is strictly larger than seven, between five and seven (both included) or scrictly smaller than five. How much entropy is present in M?

Problem 2: Mutual information (2pt)

Let the binary random variables X and Y be defined by the joint probability distribution $P_{XY}(00) = 3/7$, $P_{XY}(01) = 1/7$, $P_{XY}(10) = 1/7$, $P_{XY}(11) = 2/7$.

Draw the entropy diagram for X and Y, and compute $H(X)$, $H(Y)$, $H(XY)$, $H(X|Y)$, $H(Y|X)$, and $I(X;Y)$. Reading off a value from the diagram is a valid 'computation', as long as you clearly state how you did it.



Problem 3: Entropy of functions of a random variable (3pt)

Let X be a random variable, and let f be a function of X.

(a) **(1pt)** Show that $H(f(X)|X) = 0$.

(b) **(2pt)** Show that $H(f(X)) \leq H(X)$. *Hint:* use the chain rule.

Problem 4: Relative and cross entropy (9pt)

(a) **(5pt)** Prove that for any two distributions P and Q over $\mathcal{X}$, $D(P||Q) \geq 0$, and that equality holds if and only if $P = Q$.

(b) **(1pt)** We have seen that the mutual information can be expressed in terms of the relative entropy, i.e. that $I(X;Y) = D(P_{XY}||P_X \cdot P_Y)$. Use (a) and this fact to show that $H(X|Y) \leq H(X)$.

(c) **(2pt)** We have seen a relation between relative entropy and cross entropy in Intro/Team Quiz 02. Use this relation to express the mutual information $I(X;Y)$ in terms of entropies of X and Y and of the cross entropy H_c of P_{XY} and $P_X \cdot P_Y$.

Problem 5: Programming... (5pt)

TODO: compute the entropy of a big distribution?