Team homework set 5

Unless otherwise stated, you should provide exact answers rather than rounded numbers (e.g., $\log 3$ instead of 1.585) for non-programming exercises.

Problem 1: Bottleneck (4pt)

Suppose a Markov chain starts in one of n states, necks down to $k < n$ states, and then fans back to $m > k$ states. Thus $X_1 \to X_2 \to X_3$, with $\mathcal{X}_1 = \{1, 2, ..., n\}$, $\mathcal{X}_2 = \{1, 2, ..., k\}$, and $\mathcal{X}_3 = \{1, 2, ..., m\}$.

(a) **(3pt)** Show that the dependence of X_1 and X_3 is limited by the bottleneck by proving that $I(X_1; X_3) \leq \log k$.

(b) **(1pt)** Evaluate $I(X_1; X_3)$ for $k = 1$, and conclude that no dependence can survive such a bottleneck.

Problem 2: A dog looking for a bone (6pt)

A dog walks on the integers, possibly reversing direction at each step with probability $p = 0.1$. Let $X_0 = 0$. The first step is equally likely to be positive or negative. A typical walk might look like this:

$$(X_0, X_1, ...) = (0, -1, -2, -3, -4, -3, -2, -1, 0, 1, ...).$$

(a) **(2pt)** What is the expected number of steps the dog takes in one direction, before reversing?

(b) **(1pt)** Is this a Markov process? If so, is it time-invariant?

(c) **(3pt)** Find the entropy rate of this browsing dog.

Problem 3: Run-length coding (3pt)

Let $X_1, X_2, ..., X_n$ be (possibly dependent) binary random variables. Suppose one calculates the run lengths $R = (R_1, R_2, ...)$ of this sequence (in order as they occur). For example, the sequence $X = 0001100100$ yields run lengths $R = (3, 2, 2, 1, 2)$. Compare $H(X_1, X_2, ..., X_n)$, $H(R)$ and $H(X_n, R)$. Show all equalities and inequalities, and bound all the differences.

Problem 4: Entropy rates of Markov chains (10pt)

(a) **(2pt)** Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1 - p_{ab} & p_{ab} \\ p_{ba} & 1 - p_{ba} \end{bmatrix}.$$

(b) **(1pt)** Find values of p_{ab} and p_{ba} that maximize the entropy rate of part (a).

(c) **(1pt)** Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1 - p & p \\ 1 & 0 \end{bmatrix}.$$

(d) **(2pt)** Find the maximum value of the entropy rate of for part (c).
Hint: we expect that the maximizing value of p should be less than $1/2$, since the 0 state permits more information to be generated than the 1 state. This allows you to discard one possible solution.

(e) **(4pt)** Let $N(t)$ be the number of allowable state sequences of length t for the Markov chain of part (c). Find $N(t)$ and calculate

$$H_0 := \lim_{t \to \infty} \frac{1}{t} \log N(t).$$

Why is H_0 an upper bound on the entropy rate of the Markov chain? Compare H_0 with the maximum entropy found in (d).
Hint: Find $N(0)$ and $N(1)$, and find a linear recurrence that expresses $N(t)$ in terms of $N(t-1)$ and $N(t-2)$. What well-known sequence does this remind you of? You can use any known properties of this sequence.