Team homework set 6

Unless otherwise stated, you should provide exact answers rather than rounded numbers (e.g., $\log 3$ instead of 1.585) for non-programming exercises.

Problem 1: Hadamard code (11pt)

For two k-bit strings x and y, the inner product is defined as

$$\langle x, y\rangle := \sum_{i=1}^{k} x_i \cdot y_i \qquad \text{mod } 2.$$

Note that the inner product of two strings is a single bit. The k-bit Hadamard code is defined such that the codeword of x is the concatenation of all possible inner products with x, i.e.

$$\texttt{enc}(x) := (\langle x, y\rangle)_{y\in\{0,1\}^k}\,.$$

Here, the y are ordered lexicographically.

(a) **(1pt)** Find the codewords for 010 and 1101 using the 3-bit and 4-bit Hadamard codes respectively.

(b) **(1pt)** Give the explicit generator matrix G^T for $k = 3$.

(c) **(2pt)** For a fixed non-zero $x \in \{0,1\}^k$, how many $y \in \{0,1\}^k$ are there such that $\langle x, y\rangle = 1$?

(d) **(1pt)** Use (c) to find the minimal distance of the k-bit Hadamard code.

There is some unnecessary redundancy in the Hadamard code. For example, the first bit of $\texttt{enc}(x)$ is always 0, regardless of what x is, because the inner product with the all-zero string is always 0. The *punctured* k-bit Hadamard code attempts to resolve this. It considers only the inner products with strings y such that $y_1 = 1$:

$$\texttt{enc}'(x) := (\langle x, y\rangle)_{y\in\{1\}\times\{0,1\}^{k-1}}$$

(e) **(3pt)** Find the minimal distance of the punctured k-bit Hadamard code using a similar approach as in (c) and (d).
Hint: there are four cases to consider, depending on the value of the first bit x_1 and of the "tail" $x_2 \cdots x_k$.

(f) **(2pt)** Compute the rates of the k-bit Hadamard code and the punctured k-bit Hadamard code as a function of k. Which is better?

(g) **(1pt)** What goes wrong if we try to puncture this code again? I.e. what breaks if we define the encoding function for a k-bit string as

$$\texttt{enc}''(x) := (\langle x, y\rangle)_{y\in\{1\}\times\{1\}\times\{0,1\}^{k-2}}?$$

Problem 2: Shannon capacity of the complete graph (6pt)

A graph G with n vertices $V(G) = \{1, 2, ..., n\}$ is called complete if it has edges between any two vertices, i.e. $\forall i \neq j \; : \; (i, j) \in E(G)$.

(a) **(2pt)** Compute $\alpha(K_n)$, the independence number of the complete graph of size n.

(b) **(2pt)** Show that $K_n \boxtimes K_n = K_{n^2}$.
Hint: the LaTeX command for $\boxtimes$ is `\boxtimes` (in the `amssymb` package). See also Detexify.

(c) **(2pt)** Use (a) and (b) to prove that the Shannon capacity of K_n is 0. Note that this result formally confirms the intuition that channels whose confusability graphs are complete are useless for zero-error communication, because all symbols can possibly be confused with each other.

Problem 3: Disjoint graphs (5 pt)

For two graphs G and H, the graph $G + H$ is defined as the disjoint union of the two graphs. Formally, assuming without loss of generality that $V(G) \cap V(H) = \emptyset$, then $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H)$. (You can think of $G+H$ as G and H "next to each other".) For the disjoint union of k times the same graph G, we write G^{+k}.

(a) **(2pt)** Prove that $\alpha(G + H) = \alpha(G) + \alpha(H)$.

(b) **(3pt)** For any three graphs G, H, L, it holds that

$$(G + H) \boxtimes L = (G \boxtimes L) + (H \boxtimes L)$$

and also

$$G \boxtimes (H + L) = (G \boxtimes H) + (G \boxtimes L).$$

You can verify the above identities for yourself, but you do not have to hand in a proof. Use them to derive that for any natural number $k \in \mathbb{N}$,

$$(G + G)^{\boxtimes k} = (G^{\boxtimes k})^{+2^k}.$$

Problem 4: Same-parity channel (6pt)

Let $\mathcal{X} = \mathcal{Y} = \{1, 2, 3, 4, 5, 6\}$. In this exercise, you will compute the zero-error Shannon capacity of the noisy channel with transition probabilities $P_{Y|X}(y|x) = 1/3$ if and only if x and y have the same parity (i.e. $x \equiv y \mod 2$).

(a) **(2pt)** Give the confusability graph G of the noisy channel $P_{Y|X}$ described above.

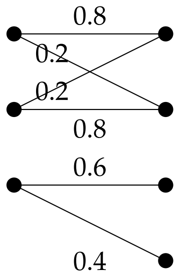
(b) **(4pt)** Compute the Shannon capacity of G.
Hint: use several results from the previous two exercises.

Problem 5: Toggle channel (6pt)

Given two channels $(\mathcal{X}_1, P_{Y_1|X_1}, \mathcal{Y}_1)$ and $(\mathcal{X}_2, P_{Y_2|X_2}, \mathcal{Y}_2)$ with $\mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$, define the "union channel" $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ by allowing the transmitter to choose between sending a signal through either channel 1 or channel 2 (but not both) each time. Let C be the capacity of the new channel, and let C_1 and C_2 be the capacities of the "component channels".

(a) **(4pt)** Prove that $2^C = 2^{C_1} + 2^{C_2}$. Solve any maximization problems analytically (i.e. by hand).
Hint: think of P_X as a tree, where the first step decides whether to use the first or the second channel.

(b) **(1pt)** Use (a) to find the capacity of the following channel (in three decimals precision):



(c) **(1pt)** Use (a) to find the capacity of an arbitrary channel with capacity C_1 combined with an ideal channel with k inputs.