

Practice problem set 6

This week's exercises deal with zero-error coding and channel capacity. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the [discussion forum on Moodle](#), and try to help each other. We will also keep an eye on the forum.

Problem 1: Confusability graphs

For each of the channels below, give the corresponding confusability graph.

- (a) $\mathcal{X} = \{1, 2, 3, 4, 5\}$, $\mathcal{Y} = \{a, b, c\}$, $P_{Y|X}(a|1) = P_{Y|X}(b|1) = P_{Y|X}(a|2) = P_{Y|X}(b|2) = \frac{1}{2}$, $P_{Y|X}(b|3) = \frac{1}{3}$, $P_{Y|X}(c|3) = \frac{2}{3}$, $P_{Y|X}(c|4) = P_{Y|X}(c|5) = 1$.
- (b) $\mathcal{X} = \{1, 2, 3, 4, 5\}$, $\mathcal{Y} = \{a, b, c, d\}$, $P_{Y|X}(a|2) = P_{Y|X}(b|2) = P_{Y|X}(c|2) = P_{Y|X}(a|4) = P_{Y|X}(c|4) = P_{Y|X}(d|4) = \frac{1}{3}$, $P_{Y|X}(b|3) = P_{Y|X}(c|3) = \frac{1}{2}$, $P_{Y|X}(a|1) = P_{Y|X}(d|5) = 1$.

Problem 2

We are given the following joint distribution of $X \in \{1, 2, 3\}$ and $Y \in \{a, b, c\}$:

$$P_{XY}(1, a) = P_{XY}(2, b) = P_{XY}(3, c) = 1/6$$

$$P_{XY}(1, b) = P_{XY}(1, c) = P_{XY}(2, a) = P_{XY}(2, c) = P_{XY}(3, a) = P_{XY}(3, b) = 1/12.$$

Let $\hat{X}(Y)$ be an estimator for X (based on Y), and let $p_e = P[\hat{X} \neq X]$.

- (a) Find an estimator $\hat{X}(Y)$ for which the probability of error p_e is as small as possible.
- (b) Evaluate Fano's inequality for this problem, and compare.

Problem 3: Symmetric Channels

Consider the channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}.$$

Recall that in a transition matrix, the entry in the x th row and the y th column denotes the conditional probability $P_{Y|X}(y|x)$ that y is received when x has been sent.

A channel is said to be **symmetric** if the rows of the channel transition matrix $P_{Y|X}$ are permutations of each other, and the columns are permutations of each other. A channel is said to be **weakly symmetric** if every row of the transition matrix is a permutation of every other row and all the column sums $\sum_x P_{Y|X}(y|x)$ are equal.

For instance, the channel $P_{Y|X}$ above is symmetric, and the channel

$$Q_{Y|X} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

is weakly symmetric but not symmetric.

- (a) Find the optimal input distribution and channel capacity of $Q_{Y|X}$.
- (b) Give a general strategy how to compute the capacity of weakly symmetric channels. What is the optimal input distribution?

Problem 4: Multiple Channel Uses

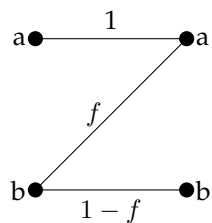
Prove the lemma below stating that the capacity per transmission is not increased if we use a discrete memoryless channel many times. For inspiration, look again at the proof of the converse of Shannon's noisy-channel coding theorem.

Lemma 7.9.2 in [CT] Let $X_1, X_2, \dots, X_n = X^n$ be n random variables with arbitrary joint distribution P_{X^n} . Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C . Prove that for all P_{X^n} , it holds that $I(X^n; Y^n) \leq nC$.

Does your proof also work in case of coding with feedback (i.e. X_{i+1} is allowed to depend on X^i and Y^i)? If not, point out the steps in your proof where you use that there is no feedback.

Problem 5: Z-channel

The following channel is known as the *Z-channel*:



Let $p \in [0, 1]$ and write $P_X(a) = 1 - p$, $P_X(b) = p$.

- (a) Express $I(X; Y)$ in terms of p and f .
- (b) Find the optimal input distribution and the channel capacity for $f = 0.3$.
- (c) Do the same for $f = 0.15$ and $f = 0.01$. What do you observe?