INFORMATION THEORY
Master of Logic, Master AI, Master CS, University of Amsterdam, 2018
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Practice problem set 5

This week's exercises deal with random processes. You do not have to hand in these exercises, they are for practicing only. During the work session, start with solving the exercise you will be moderating. Work out a full solution on paper/computer and get it approved by the teacher. Also think about the following questions: What is the point of the exercise? What kind of problems will students encounter when solving this problem? What kind of questions could you ask the presenter on Friday? Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Canvas, and try to help each other. We will also keep an eye on the forum.

Problem 1: Markov chains with 4 random variables

Let $W \to X \to Y \to Z$ be a Markov chain with 4 RVs, i.e., it holds that $P_{Z|YXW} = P_{Z|Y}$ and $W \leftrightarrow X \leftrightarrow Y$ is a Markov chain with three random variables as defined in the lecture.

(a) Show that $X \leftrightarrow Y \leftrightarrow Z$ is a Markov chain.

(b) Show that $W \leftrightarrow (X,Y) \leftrightarrow Z$ is a Markov chain with three random variables $W, (XY), Z$.

(c) Show that $Z \to Y \to X \to W$. Therefore, it is also justified to write $W \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$, as in the case for three RVs.

★ Can you generalize the two properties above to Markov chains of $n > 4$ random variables?

Problem 2: Stationary processes

Let $..., X_{-1}, X_0, X_1, ...$ be a stationary (not necessarily Markov) stochastic process. Recall that this means that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, it holds that $P_{X_1 \dots X_n} = P_{X_{1+k},\dots,X_{n+k}}$. Which of the following statements are true? Prove or provide a counterexample.

(a) $H(X_n|X_0) = H(X_{-n}|X_0)$

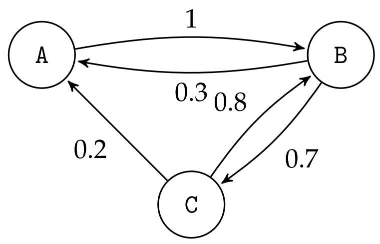
(b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

(c) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1})$ is nonincreasing in n.

(d) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1}, ..., X_{2n})$ is nonincreasing in n.

Problem 3: Entropy rate (33pt)

Consider the following Markov chain:



(a) **(13pt)** Give the transition matrix for this Markov chain, and calculate its stationary distribution. If necessary, round your answers to three decimals precision.

(b) **(20pt)** Assume the initial distribution X_1 fulfills that $P_{X_1}(\mathtt{A}) = 1$. Derive the entropy rate $H(\{X_i\}) = \lim_{n\to\infty} \frac{1}{n} H(X_1 \dots X_n)$ for this process. If necessary, round your final answer to three decimals precision.

Problem 4: Random walk on a chessboard

Consider a 3x3 chessboard. We place a knight (who can move 2 spaces horizontally and 1 vertically or 1 space horizontally and 2 vertically) in the top left corner, and let him perform a random walk on this chessboard, choosing his move uniformly random every time. What is the entropy rate of this process?