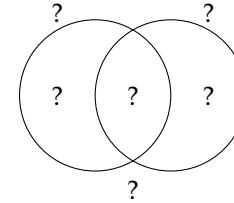


## INFORMATION THEORY

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# Team homework set 2



### Problem 1: Fun with dice (6 pt)

Consider the following random experiment with two fair (regular six-sided) dice. First, the first die is thrown – let the outcome be  $A$ . Then, the second die is thrown until the outcome has the same parity (even, odd) as  $A$ . Let this final outcome of the second die be  $B$ . The random variables  $X$ ,  $Y$  and  $Z$  are defined as follows:

$$X = (A + B) \bmod 2, \quad Y = (A \cdot B) \bmod 2, \quad Z = |A - B|.$$

- (a) (1pt) Find the joint distribution  $P_{AB}$ .
- (b) (1pt) Determine  $H(X)$ ,  $H(Y)$  and  $H(Z)$ .
- (c) (1pt) Compute  $H(Z|A = 1)$ .
- (d) (1pt) Compute  $H(AB)$ , i.e. the joint entropy of  $A$  and  $B$ .
- (e) (2pt) A random variable  $M$  describes whether the sum  $A + B$  is strictly larger than seven, between five and seven (both included) or strictly smaller than five. How much entropy is present in  $M$ ?

### Problem 2: Mutual information (2pt)

Let the binary random variables  $X$  and  $Y$  be defined by the joint probability distribution  $P_{XY}(00) = 3/7$ ,  $P_{XY}(01) = 1/7$ ,  $P_{XY}(10) = 1/7$ ,  $P_{XY}(11) = 2/7$ .

Draw the entropy diagram for  $X$  and  $Y$ , and compute  $H(X)$ ,  $H(Y)$ ,  $H(XY)$ ,  $H(X|Y)$ ,  $H(Y|X)$ , and  $I(X; Y)$ . Reading off a value from the diagram is a valid ‘computation’, as long as you clearly state how you did it.

### Problem 3: Entropy of functions of a random variable (3pt)

Let  $X$  be a random variable, and let  $f$  be a function of  $X$ .

- (a) (1pt) Show that  $H(f(X)|X) = 0$ .
- (b) (2pt) Show that  $H(f(X)) \leq H(X)$ . *Hint: use the chain rule.*

### Problem 4: Relative and cross entropy (9pt)

- (a) (5pt) Prove that for any two distributions  $P$  and  $Q$  over  $\mathcal{X}$ ,  $D(P||Q) \geq 0$ , and that equality holds if and only if  $P = Q$ .
- (b) (1pt) We have seen that the mutual information can be expressed in terms of the relative entropy, i.e. that  $I(X; Y) = D(P_{XY}||P_X \cdot P_Y)$ . Use (a) and this fact to show that  $H(X|Y) \leq H(X)$ .
- (c) (2pt) We have seen a relation between relative entropy and cross entropy in Intro/Team Quiz 02. Use this relation to express the mutual information  $I(X; Y)$  in terms of entropies of  $X$  and  $Y$  and of the cross entropy  $H_c$  of  $P_{XY}$  and  $P_X \cdot P_Y$ .

### Problem 5: Programming... (5pt)

TODO: compute the entropy of a big distribution?