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Practice problem set 5

This week's exercises deal with random processes. You do not have to hand in these exercises, they are for practicing only. During the work session, start with solving the exercise you will be moderating. Work out a full solution on paper/computer and get it approved by the teacher. Also think about the following questions: What is the point of the exercise? What kind of problems will students encounter when solving this problem? What kind of questions could you ask the presenter on Friday? Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Canvas, and try to help each other. We will also keep an eye on the forum.

Problem 1: Markov chains with 4 random variables

Let $W \to X \to Y \to Z$ be a Markov chain with 4 RVs, i.e., it holds that $P_{Z|YXW} = P_{Z|Y}$ and $W \leftrightarrow X \leftrightarrow Y$ is a Markov chain with three random variables as defined in the lecture.

(a) Show that $X \leftrightarrow Y \leftrightarrow Z$ is a Markov chain.

(b) Show that $W \leftrightarrow (X, Y) \leftrightarrow Z$ is a Markov chain with three random variables $W, (XY), Z$.

(c) Show that $Z \to Y \to X \to W$. Therefore, it is also justified to write $W \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$, as in the case for three RVs.

★ How would the properties above generalize to Markov chains of $n > 4$ random variables? You don't have to work out the full proofs.

Problem 2: Stationary processes

Let $..., X_{-1}, X_0, X_1, ...$ be a stationary (not necessarily Markov) stochastic process. Recall that this means that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, it holds that $P_{X_1 \dots X_n} = P_{X_{1+k},\dots,X_{n+k}}$. Which of the following statements are true? Prove or provide a counterexample.

(a) $H(X_n|X_0) = H(X_{-n}|X_0)$

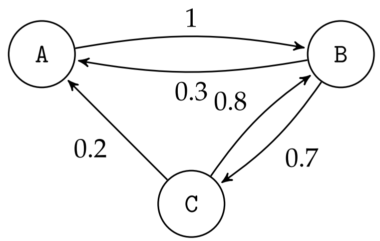
(b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

(c) $H(X_n|X_1, X_2, ..., X_{n-1})$ is nonincreasing in n.

(d) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1}, ..., X_{2n})$ is nonincreasing in n.

Problem 3: Entropy rate

Consider the following Markov process:



(a) Give the transition matrix for this Markov process, and calculate its stationary distribution. If necessary, round your answers to three decimals precision.

(b) Assume the initial distribution X_1 fulfills that $P_{X_1}(\mathtt{A}) = 1$. Derive the entropy rate $H(\{X_i\}) = \lim_{n\to\infty} \frac{1}{n} H(X_1 \dots X_n)$ for this process. If necessary, round your final answer to three decimals precision. You are allowed to use without proof that if a distribution $P_{X_nY_n}$ converges to $P_{\tilde{X}\tilde{Y}}$ for $n \to \infty$, then $H(X_n|Y_n)$ converges to $H(\tilde{X}|\tilde{Y})$.

Problem 4: Random walk on a chessboard

Consider a 3x3 chessboard. We place a bishop (who can move diagonally as far as he likes) on the board, and let him perform a random walk on this chessboard, choosing his move uniformly at random from all legal moves every time.

(a) You can see this process as random walk on a graph. Draw this graph.

(b) Assume the bishop starts in the top left corner of the board. What is the entropy rate of the resulting process? As in the problem above, you are allowed to use without proof that if a distribution $P_{X_n Y_n}$ converges to $P_{\tilde{X}\tilde{Y}}$ for $n \to \infty$, then $H(X_n|Y_n)$ converges to $H(\tilde{X}|\tilde{Y})$.

Problem 5: Bernoulli process

Let $X_1, X_2, ...$ be distributed according to the Bernoulli(p) distribution. Consider the associated Markov chain $\{Y_i\}_{i=1}^n$, where Y_i is the number of 1's in the current run of 1's. For example, if $X^n = 101110...$, then $Y^n = 101230.....$

(a) Find the entropy rate of X^n.

(b) Find the entropy rate of Y^n