INFORMATION THEORY
Master of Logic, Master AI, Master CS, University of Amsterdam, 2018
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Practice problem set 7

This week's exercises deal with noisy-channel coding. You do not have to hand in these exercises, they are for practicing only. During the work session, start with solving the exercise you will be moderating. Work out a full solution on paper/computer and get it approved by the teacher. Also think about the following questions: What is the point of the exercise? What kind of problems will students encounter when solving this problem? What kind of questions could you ask the presenter on Friday? Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Canvas, and try to help each other. We will also keep an eye on the forum.

Problem 1: Multiple Channel Uses

Prove the lemma below stating that the capacity per transmission is not increased if we use a discrete memoryless channel many times. For inspiration, look again at the proof of the converse of Shannon's noisy-channel coding theorem.

Lemma on Multiple Channel Uses Let $X_1, X_2, \dots X_n = X^n$ be n random variables with arbitrary joint distribution P_{X^n}. Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C. Prove that for all P_{X^n}, it holds that $I(X^n; Y^n) \leq nC$.

Does your proof also work in case of coding with feedback (i.e. X_{i+1} is allowed to depend on X^i and Y^i)? If not, point out the steps in your proof where you use that there is no feedback.

Problem 2: Encoder and decoder as part of the channel

Consider a binary symmetric channel with crossover probability 0.1. A possible coding scheme for this channel with two codewords of length 3 is to encode message w_1 as 000 and w_2 as 111. Decoding happens by majority vote. With this coding scheme, we can consider the combination of encoder, channel, and decoder as forming a new BSC, with two inputs w_1 and w_2, and two outputs w_1 and w_2.

(a) Draw the channel and calculate the crossover probability.

(b) What is the capacity of the original channel?

(c) What is the capacity of this new channel in bits per transmission of the original channel? Compare.

(d) Prove the general statement that for any channel, considering the encoder, channel, and decoder together as a new channel from messages to estimated messages will not increase the capacity in bits per transmission of the original channel.

Problem 3: Source and channel

We wish to encode a Bernoulli(α) process $V_1, V_2, \dots$ for transmission over a binary symmetric channel with crossover probability ϵ.

$$V^n \longrightarrow X^n(V^n) \longrightarrow \text{BSC}(\epsilon) \longrightarrow Y^n \longrightarrow \hat{V}^n$$

Find conditions on (α, ϵ) under which the error probability $P[\hat{V}^n \neq V^n]$ can be made to go to zero as $n \to \infty$.

Problem 4: Channel with memory

Consider the discrete memoryless channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ with $\mathcal{X} = \{-1, 1\}$, and $Y = ZX$ for a random variable Z with $\mathcal{Z} = \{-1, 1\}$.

(a) What is the capacity of this channel when Z is uniform?

(b) Now consider the channel with memory. Before transmission begins, Z is randomly chosen and fixed for all time. What is the capacity when Z is uniform?

Problem 5: Additive noise

Let R be a random variable such that R takes on either value 0 or some arbitrary but fixed value $r \in \mathbb{R}$, both with probability $1/2$.

Consider a channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ with $\mathcal{X} = \{0,1\}$ and $Y = (X + R) \mod 4$.

Find the capacity of this channel for all possible values of r.