TEACHER: Christian Schaffner, TAs: Yfke Dulek, Esteban Landerreche, Kyah Smaal

Practice problem set 2

This week's exercises deal with entropy. You do not have to hand in these exercises, they are for practicing only. During the work session, start with solving the exercise you will be moderating. Work out a full solution on paper/computer and get it approved by the teacher. Also think about the following questions: What is the point of the exercise? What kind of problems will students encounter when solving this problem? What kind of questions could you ask the presenter on Friday? Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Canvas, and try to help each other. We will also keep an eye on the forum.

Problem 1: Properties of entropy

Let X and Y be random variables.

(a) Prove that $H(X) = 0$ if and only if X is *constant*, i.e. there is an $x_0 \in \mathcal{X}$ such that $P_X(x_0) = 1$, and $P_X(x') = 0$ for all $x' \neq x_0$.

(b) Prove that $H(XY) = H(X) + H(Y)$ if and only if X and Y are independent.

(c) Prove that $H(X) = \log |\mathcal{X}|$ if X is uniformly distributed.

★ Prove that X is uniformly distributed if $H(X) = \log |\mathcal{X}|$.

Problem 2: Entropy of a deck of cards

(a) Compute the entropy of a perfectly shuffled deck of 52 cards (i.e. the set of cards is uniformly distributed over all possible orders, not just the entropy of the first card).

(b) Now suppose we have a perfectly shuffled big deck, consisting of two *identical* decks of 52 cards (104 cards in total). You cannot tell the difference between, for example, the ace of spades of one deck and the ace of spades of the other. Compute the entropy of the shuffled big deck.

Problem 3: Mutual information

Let X, Y and Z be random variables such that $I(X;Y) = 0$ and $I(X;Z) = 0$. Does it follow that $I(Y;Z) = 0$? If so, prove it. If not, give a counterexample.

Problem 4: Estimating entropy

([MacKay], Example 2.13:) A source produces a character x from alphabet $\mathcal{A} = \{\texttt{0}, \texttt{1}, \texttt{2}, \ldots, \texttt{9}, \texttt{a}, \texttt{b}, \texttt{c}, \ldots, \texttt{z}\}$. With probability $1/3$, x is a uniformly random numeral $\texttt{0}, \texttt{1}, \texttt{2}, \ldots, \texttt{9}$, with probability $1/3$, x is a random vowel $\texttt{a}, \texttt{e}, \texttt{i}, \texttt{o}, \texttt{u}$ and with probability $1/3$, x is one of the 21 consonants. Estimate the entropy of X. Can you give a good estimate without using a calculator?

Problem 5: Geometric distribution

The geometric(p) distribution of a random variable X is defined as the number of times one has to flip a Bernoulli(p) coin before it lands on heads:

$$P_X(k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, 3, \ldots$$

Compute the entropy of the geometric distribution.

Problem 6: Size of binomial coefficient from Stirling's Approximation

Let $n \in \mathbb{N}$ and $p \in [0,1]$ such that $np \in \mathbb{N}$. Use Stirling's approximation of the factorial function $\log(n!) \approx n\log(n) - n\log(e)$ to prove that

$$\binom{n}{np} \approx 2^{n \cdot h(p)},$$

where h is the binary entropy function.