Practice problem set 6

This week's exercises deal with zero-error coding and channel capacity. You do not have to hand in these exercises, they are for practicing only. During the work session, start with solving the exercise you will be moderating. Work out a full solution on paper/computer and get it approved by the teacher. Also think about the following questions: What is the point of the exercise? What kind of problems will students encounter when solving this problem? What kind of questions could you ask the presenter on Friday? Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Canvas, and try to help each other. We will also keep an eye on the forum.

Problem 1: Parity-check code

Consider the linear code defined by appending to a 3-bit message string $x_1x_2x_3$ the parity bit $(x_1 \oplus x_2 \oplus x_3)$.

(a) How long are the codewords of C? How many different codewords are there?

(b) Find the generator matrix G.

(c) Find the parity check matrix H.

(d) What is the minimal distance? What kind of errors can C correct for?

(e) Encode the strings 101, 111 according to C.

(f) Decode 1011, 1111, 1110, and 0011.

Problem 2: Determining the minimal distance

Prove that for a binary linear code C with parity check matrix H, the minimal distance d_{min} equals the minimum number of linearly dependent columns in H.

Check that this lemma holds for the linear code from the previous exercise.

Problem 3: Symmetric Channels

Recall that in a transition matrix, the entry in the xth row and the yth column denotes the conditional probability $P_{Y|X}(y|x)$ that y is received when x has been sent.

A channel is said to be (**weakly**) **symmetric** if every row of the transition matrix is a permutation of every other row and all the column sums $\sum_x P_{Y|X}(y|x)$ are equal.

For instance, the channel

$$Q_{Y|X} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

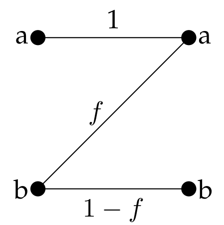
is weakly symmetric but not symmetric.

(a) Find the optimal input distribution and channel capacity of $Q_{Y|X}$.

(b) Give a general strategy how to compute the capacity of weakly symmetric channels. What is the optimal input distribution?

Problem 4: Z-channel

The following channel is known as the *Z-channel*:



Let $p \in [0,1]$ and write $P_X(a) = 1-p$, $P_X(b) = p$.

(a) Express $I(X;Y)$ in terms of p and f.

(b) Find the optimal input distribution and the channel capacity for $f = 0.3$, using the fact that

$$\frac{d}{dp}h(p) = \log\left(\frac{1-p}{p}\right).$$

(c) Do the same for $f = 0.15$ and $f = 0.01$, using a computer if you want. What do you observe?

Problem 5: Comparing capacities

In this problem, you will compare the Shannon capacity of a discrete memoryless channel $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$ to the Shannon capacity of its confusability graph G.

(a) Prove that $\log(\alpha(G)) \leq \max_{P_X} I(X;Y)$.

(b) Generalize the previous subproblem by proving that for any natural number $n \geq 1$, it holds that $\log(\alpha(G^{\boxtimes n})) \leq \max_{P_{X^n}} I(X^n;Y^n)$.

(c) Conclude that the Shannon capacity of the confusability graph can never exceed the Shannon capacity of the channel. Is this result surprising? Why or why not?