Modern Cryptography
Bachelor Computer Science, University of Amsterdam, 2020/21
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Problem Set 2

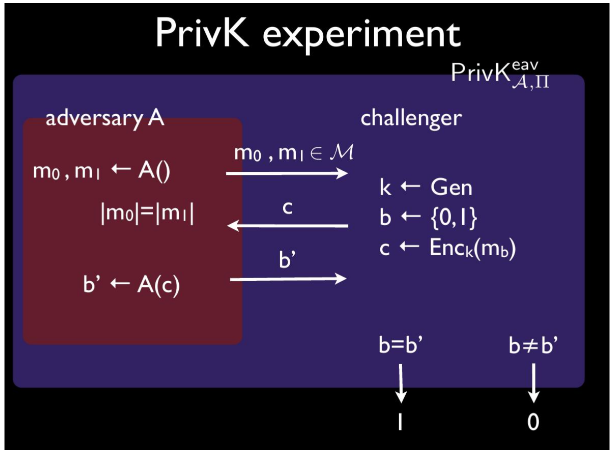

Figure 1: The $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$ experiment

Problem 1: The $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$ experiment (see Figure 1)

For each of the following scenarios, give the maximal value of $\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1]$ and explain how it can be achieved.

(a) Let Π be the shift cipher, and let us consider an adversary $\mathcal{A}$ that submits $m_0 = \texttt{a}$ and $m_1 = \texttt{a}$.

(b) Let Π be the shift cipher, and let us consider an adversary $\mathcal{A}$ that submits $m_0 = \texttt{a}$ and $m_1 = \texttt{b}$.

(c) Let Π be the shift cipher, and let us consider an adversary $\mathcal{A}$ that submits $m_0 = \texttt{aa}$ and $m_1 = \texttt{bb}$.

(d) Let Π be the shift cipher, and let us consider an adversary $\mathcal{A}$ that submits $m_0 = \texttt{aa}$ and $m_1 = \texttt{ab}$.

(e) Let Π be the one-time-pad encryption of three-letter messages, and let us consider an adversary $\mathcal{A}$ that submits $m_0 = \texttt{aaa}$ and $m_1 = \texttt{abc}$.

(f) Let Π be the monoalphabetic substitution cipher. Give an adversary $\mathcal{A}$ that manages to win the $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$ experiment all the time, i.e. such that $\Pr[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi} = 1] = 1$.

Problem 2: Negligible functions

Recall Definition 3.4: A function $f : \mathbb{Z}^+ \to \mathbb{R}^+$ is called *negligible* if for every positive polynomial $p(n)$ there exists $N \in \mathbb{Z}^+$ such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.

(a) Example 3.5 states that $f(n) = 2^{-\sqrt{n}}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.

(b) Example 3.5 states that $f(n) = n^{-\log n}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.

★ Let negl_1 and negl_2 be negligible functions. Prove that the function negl_3 defined by $\mathsf{negl}_3(n) = \mathsf{negl}_1(n) + \mathsf{negl}_2(n)$ is negligible.

★ For any positive polynomial p, the function negl_4 defined by $\mathsf{negl}_4(n) = p(n) \cdot \mathsf{negl}_1(n)$ is negligible.

Problem 3: not PRGs

For all of the following constructions, explain why they are not PRGs. Can you give an explicit description of an efficient distinguisher in each case?

(a) Let $G(s)$ output s.

(b) Let $G(s)$ output $s \| s$

(c) Let $G(s)$ output $s \| \bigoplus_{i=1}^{n} s_i$.

See https://colab.research.google.com/drive/1s3ZOM35nJKWv_PGnYGH87rtVP2-QA0qa for programming versions of this exercise which might help your understanding.

Problem 4: Basic properties of PRGs

Recall that the *image* of a function $f : A \to B$ is the subset $f(A)$ of B. Formally,

$$\mathsf{im}(f) := f(A) = \{b \in B \mid \exists a \in A \text{ such that } b = f(a)\}\,.$$

Let $G : \{0,1\}^n \to \{0,1\}^{2n}$ be a PRG.

(a) Let us assume that G is injective. How many different $2n$-bit strings y are there in the image of G?

(b) What is the fraction of images of G among all $2n$-bit strings?

(c) For a given $y \in \{0,1\}^{2n}$, what is $\Pr_{s \leftarrow \{0,1\}^n}[G(s) = y]$? Express this probability in terms of $|\{s \in \{0,1\}^n \mid G(s) = y\}|$ and n.

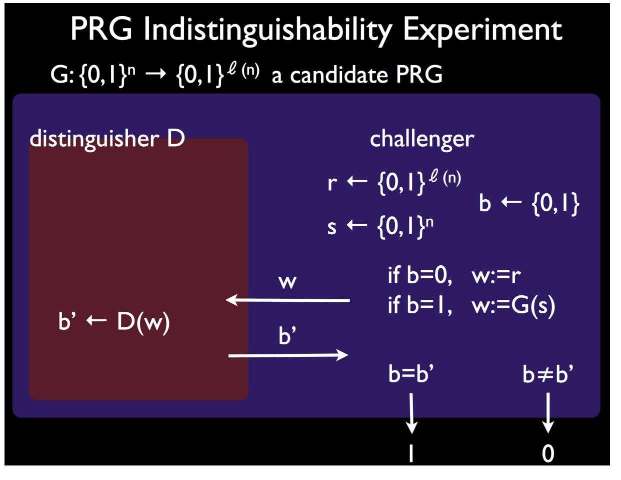

Figure 2: The $\mathsf{PRG}_{\mathcal{D},G}$ experiment

Problem 5: Exercise 3.5 from [KL]

Let $|G(s)| = \ell(|s|)$ for some ℓ. Consider the following experiment:

The PRG indistinguishability experiment $\mathsf{PRG}_{\mathcal{A},G}(n)$, see also Figure 2:

(a) *A uniform bit $b \in \{0,1\}$ is chosen. If $b = 0$ then choose a uniform $r \leftarrow \{0,1\}^{\ell(n)}$ and set $w := r$; if $b = 1$ then choose a uniform $s \leftarrow \{0,1\}^n$ and set $w := G(s)$.*

(b) *The adversary $\mathcal{A}$ is given w, and outputs a bit b'.*

(c) *The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.*

Provide a definition of a pseudorandom generator based on this experiment, and prove that your definition is equivalent to Definition 3.14. (That is, show that G satisfies your definition if and only if it satisfies Definition 3.14.)

Hint: For proving the equivalence of the two definitions, argue why the following equalities hold

$$\begin{aligned}
&\Pr[\mathsf{PRF}_{\mathcal{A},G}(n) = 1]\\
&\quad= \Pr[b=0]\cdot\Pr[D(w)=0|b=0] + \Pr[b=1]\cdot\Pr[D(w)=1|b=1]\\
&\quad= \Pr[b=0]\cdot\Pr[D(r)=0] + \Pr[b=1]\cdot\Pr[D(G(s))=1]\\
&\quad= \frac{1}{2}\Pr[D(r)=0] + \frac{1}{2}\Pr[D(G(s))=1]\\
&\quad= \frac{1}{2}(1-\Pr[D(r)=1]) + \frac{1}{2}\Pr[D(G(s))=1]\\
&\quad= \frac{1}{2} + \frac{1}{2}(\Pr[D(G(s))=1] - \Pr[D(r)=1])
\end{aligned}$$

and use them in your proof.

Problem 6: Exercise 3.2 from [KL]

Prove that Definition 3.8 cannot be satisfied if Π can encrypt arbitrary-length messages and the adversary is not restricted to output equal-length messages in experiment $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal{A},\Pi}$. **Hint:** Let $q(n)$ be a polynomial upper-bound on the length of the cipher-text when Π is used to encrypt a single bit. Then consider an adversary who outputs $m_0 \in \{0,1\}$ and a uniform $m_1 \in \{0,1\}^{q(n)+1}$.

★ Problem 7: Exercise 3.4 from [KL]

Prove the equivalence of Definition 3.8 and Definition 3.9 from the book [KL].

★ Problem 8: Exercise 3.7 from [KL]

Prove the converse of Theorem 3.18. Namely, show that if G is not a pseudorandom generator then Construction 3.17 does not have indistinguishable encryptions in the presence of an eavesdropper.