MODERN CRYPTOGRAPHY

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Problem Set 2

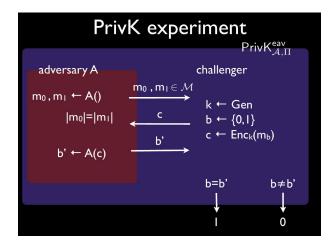


Figure 1: The $\mathsf{PrivK}_{\mathcal{A},\Pi}^{\mathsf{eav}}$ experiment

Problem 1: The PrivK $_{A,\Pi}^{\text{eav}}$ experiment (see Figure 1)

For each of the following scenarios, give the maximal value of $\Pr[\mathsf{PrivK}_{A.\Pi}^{\mathsf{eav}} = 1]$ and explain how it can be achieved.

- (a) Let Π be the shift cipher, and let us consider an adversary $\mathcal A$ that submits $m_0=\mathtt a$ and $m_1=\mathtt a$.
- (b) Let Π be the shift cipher, and let us consider an adversary $\mathcal A$ that submits $m_0=\mathsf a$ and $m_1=\mathsf b$.
- (c) Let Π be the shift cipher, and let us consider an adversary $\mathcal A$ that submits $m_0=$ aa and $m_1=$ bb.

- (d) Let Π be the shift cipher, and let us consider an adversary $\mathcal A$ that submits $m_0=\mathtt{aa}$ and $m_1=\mathtt{ab}$.
- (e) Let Π be the one-time-pad encryption of three-letter messages, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{aaa}$ and $m_1 = \text{abc}$.
- (f) Let Π be the monoalphabetic substitution cipher. Give an adversary $\mathcal A$ that manages to win the $\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal A,\Pi}$ experiment all the time, i.e. such that $\mathsf{Pr}[\mathsf{PrivK}^{\mathsf{eav}}_{\mathcal A,\Pi}=1]=1.$

Problem 2: Negligible functions

Recall Definition 3.4: A function $f: \mathbb{Z}^+ \to \mathbb{R}^+$ is called *negligible* if for every positive polynomial p(n) there exists $N \in \mathbb{Z}^+$ such that for all integers n > N, it holds that $f(n) < \frac{1}{p(n)}$.

- (a) Example 3.5 states that $f(n) = 2^{-\sqrt{n}}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers n > N, it holds that $f(n) < \frac{1}{n(n)}$.
- (b) Example 3.5 states that $f(n) = n^{-\log n}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers n > N, it holds that $f(n) < \frac{1}{n(n)}$.
- Let $negl_1$ and $negl_2$ be negligible functions. Prove that the function $negl_3$ defined by $negl_3(n) = negl_1(n) + negl_2(n)$ is negligible.
- ★ For any positive polynomial p, the function negl_4 defined by $\operatorname{negl}_4(n) = p(n) \cdot \operatorname{negl}_1(n)$ is negligible.

Problem 3: not PRGs

For all of the following constructions, explain why they are not PRGs. Can you give an explicit description of an efficient distinguisher in each case?

- (a) Let G(s) output s.
- **(b)** Let G(s) output s||s|
- (c) Let G(s) output $s \| \bigoplus_{i=1}^n s_i$.

See https://colab.research.google.com/drive/1s3ZOM35nJKWv_PGnYGH87rtVP2-QAOqa for programming versions of this exercise which might help your understanding.

Problem 4: Basic properties of PRGs

Recall that the *image* of a function $f:A\to B$ is the subset f(A) of B. Formally,

$$\operatorname{im}(f) := f(A) = \{b \in B \mid \exists a \in A \text{ such that } b = f(a)\}.$$

Let $G: \{0,1\}^n \to \{0,1\}^{2n}$ be a PRG.

- (a) Let us assume that G is injective. How many different 2n-bit strings y are there in the image of G?
- (b) What is the fraction of images of G among all 2n-bit strings?
- (c) For a given $y \in \{0,1\}^{2n}$, what is $\Pr_{s \leftarrow \{0,1\}^n}[G(s) = y]$? Express this probability in terms of $|\{s \in \{0,1\}^n \mid G(s) = y\}|$ and n.

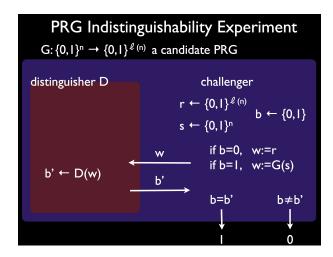


Figure 2: The $\mathsf{PRG}_{\mathcal{D},G}$ experiment

Problem 5: Exercise 3.5 from [KL]

Let $|G(s)|=\ell(|s|)$ for some ℓ . Consider the following experiment: **The PRG indistinguishability experiment** $\mathsf{PRG}_{\mathcal{A},G}(n)$, see also Figure 2:

- (a) A uniform bit $b \in \{0,1\}$ is chosen. If b=0 then choose a uniform $r \leftarrow \{0,1\}^{\ell(n)}$ and set w:=r; if b=1 then choose a uniform $s \leftarrow \{0,1\}^n$ and set w:=G(s).
- (b) The adversary A is given w, and outputs a bit b'.
- (c) The output of the experiment is defined to be 1 if b' = b, and 0 otherwise.

Provide a definition of a pseudorandom generator based on this experiment, and prove that your definition is equivalent to Definition 3.14. (That is, show that G satisfies your definition if and only if it satisfies Definition 3.14.)

Problem 6: Exercise 3.2 from [KL]

Prove that Definition 3.8 cannot be satisfied if Π can encrypt arbitrary-length messages and the adversary is not restricted to output equal-length messages in experiment $\operatorname{PrivK}_{\mathcal{A},\Pi}^{\operatorname{eav}}$. Hint: Let q(n) be a polynomial upper-bound on the length of the cipher-text when Π is used to encrypt a single bit. Then consider an adversary who outputs $m_0 \in \{0,1\}$ and a uniform $m_1 \in \{0,1\}^{q(n)+1}$.

★ Problem 7: Exercise 3.4 from [KL]

Prove the equivalence of Definition 3.8 and Definition 3.9 from the book [KL].

★ Problem 8: Exercise 3.7 from [KL]

Prove the converse of Theorem 3.18. Namely, show that if G is not a pseudorandom generator then Construction 3.17 does not have indistinguishable encryptions in the presence of an eavesdropper.