

## Problem Set 5

### Problem 1: Fermat's little theorem and modular arithmetic.

- (a) **Modular roots** Show that the 7th root of 47 modulo 143 is

$$[47^{103} \bmod 143].$$

Note that  $143 = 11 \cdot 13$ .

- (b) **Computing seemingly huge numbers by hand** Compute the final two (decimal) digits of  $3^{1000}$  (by hand).

**Hint:** The answer is  $[3^{1000} \bmod 100]$ . If  $\gcd(a, b) = 1$ , then  $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ .

- (c) **Modular arithmetic.** Let  $p, N$  be integers with  $p|N$  (i.e.  $p$  divides  $N$ ). Prove that for any integer  $X$ ,

$$[[X \bmod N] \bmod p] = [X \bmod p].$$

Show that, in contrast,  $[[X \bmod p] \bmod N]$  need not equal  $[X \bmod N]$ .

### Problem 2: Additive cyclic groups

In this problem, we study the additive group  $(\mathbb{Z}_N, +)$  for some integer  $N > 1$ , for example  $N = 9$ .

- (a) Explain how to view  $(\mathbb{Z}_N, +)$  as cyclic group generated by  $\langle 1 \rangle$ . Draw the circle for  $N = 9$ .
- (b) Which are the other  $x \in \mathbb{Z}_N$  that generate the cyclic group  $\langle x \rangle = (\mathbb{Z}_N, +)$ ? How many generators are there?
- (c) **Easy discrete-logarithm problem.** Explain why the discrete-logarithm problem (as described above Def 8.62 in [KL]) in the additive group  $(\mathbb{Z}_N, +)$  generated by  $\langle x \rangle$  is easy to solve.

### Problem 3: Diffie-Hellman

- (a) **Computational Diffie-Hellman** Define the hardness of the Computational Diffie-Hellman problem with respect to the group-generation algorithm  $\mathcal{G}$ .
- (b) **DDH is hard  $\Rightarrow$  CDH is hard  $\Rightarrow$  DLog is hard** Prove that hardness of the CDH problem implies hardness of the discrete-logarithm problem, and that hardness of the Decisional Diffie-Hellman problem implies hardness of the CDH problem.

### Problem 4: RSA

- (a) Let  $N = pq$  be a RSA-modulus and let  $(N, e, d) \leftarrow \text{GenRSA}$ . Show that the ability of efficiently factoring  $N$  allows to compute  $d$  efficiently. Given

$$N = 2140310672120493293362298457402658192652108411554313695782475927669427$$

try to compute  $\phi(N)$ . If that takes too long compute  $\phi(p \cdot q)$ , where

$$p = 58240080352490526776497122885950201, \\ q = 36749789134330529121473391864214027.$$

Knowing that compute  $d$  for given  $N$  and  $e = 11$ . Is it

$$194573697465499390305663496127514372514748993157568890710178288641091?$$

- (b) **Factoring RSA Moduli** Let  $N = pq$  be a RSA-modulus and let  $(N, e, d) \leftarrow \text{GenRSA}$ . In this exercise, you show that for the special case of  $e = 3$ , computing  $d$  is equivalent to factoring  $N$ . Show the following:
1. The ability of efficiently factoring  $N$  allows to compute  $d$  efficiently. This shows one implication.
  2. Given  $\phi(N)$  and  $N$ , show how to compute  $p$  and  $q$ . **Hint:** Derive a quadratic equation (over the integers) in the unknown  $p$ .
  3. Assume we know  $e = 3$  and  $d \in \{1, 2, \dots, \phi(N) - 1\}$  such that  $ed \equiv 1 \bmod \phi(N)$ . Show how to efficiently compute  $p$  and  $q$ . **Hint:** Obtain a small list of possibilities for  $\phi(N)$  and use 2.
  4. Given  $e = 3$ ,  $d = 29531$  and  $N = 44719$ , factor  $N$  using the method above.

### Problem 5: Diffie-Hellman Key Exchange

- (a) **Man-In-The-Middle Attacks** Describe in detail a man-in-the-middle attack on the Diffie-Hellman key-exchange protocol whereby the adversary ends up sharing a key  $k_A$  with Alice and a (different) key  $k_B$  with Bob, and Alice and Bob cannot detect that anything has gone wrong. What happens if Alice and Bob try to detect the presence of a man-in-the-middle adversary by sending each other (encrypted) questions that only the other party would know how to answer?
- (b) **Diffie-Hellman problem** Let us consider the cyclic group  $\mathbb{Z}_{13}^*$ .

1. Show that 2 is a generator of  $\mathbb{Z}_{13}^*$ .

2. In  $\mathbb{Z}_{13}^*$ , it holds that

$$\text{DH}_2(6, 9) = \text{DH}_2(2^5, 2^8) = 2^{40} = 2^{40 \bmod 12} = 2^4 = 3$$

Show in the same way that  $\text{DH}_2(12, 10) = 1$ .

### Problem 6: Key Exchange with Bit Strings

Consider the following key-exchange protocol:

1. Alice chooses  $k, r \leftarrow \{0, 1\}^n$  at random, and sends  $s := k \oplus r$  to Bob.
2. Bob chooses  $t \leftarrow \{0, 1\}^n$  at random and sends  $u := s \oplus t$  to Alice.
3. Alice computes  $w := u \oplus r$  and sends  $w$  to Bob.
4. Alice outputs  $k$  and Bob computes  $w \oplus t$ .

Show that Alice and Bob output the same key. Analyze the security of the scheme (i.e., either prove its security or show a concrete attack).