

Problem Set 2

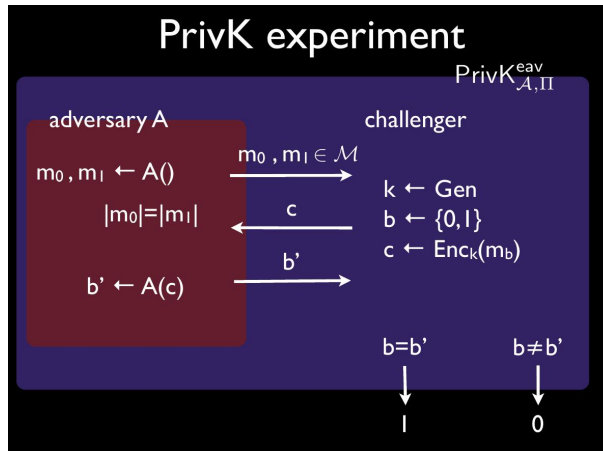


Figure 1: The $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$ experiment

Problem 1: The $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$ experiment (see Figure 1)

For each of the following scenarios, give the maximal value of $\Pr[\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}} = 1]$ and explain how it can be achieved.

- (a) Let Π be the shift cipher, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{a}$ and $m_1 = \text{a}$.
- (b) Let Π be the shift cipher, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{a}$ and $m_1 = \text{b}$.
- (c) Let Π be the shift cipher, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{aa}$ and $m_1 = \text{bb}$.

- (d) Let Π be the shift cipher, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{aa}$ and $m_1 = \text{ab}$.
- (e) Let Π be the one-time-pad encryption of three-letter messages, and let us consider an adversary \mathcal{A} that submits $m_0 = \text{aaa}$ and $m_1 = \text{abc}$.
- (f) Let Π be the monoalphabetic substitution cipher. Give an adversary \mathcal{A} that manages to win the $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$ experiment all the time, i.e. such that $\Pr[\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}} = 1] = 1$.

Problem 2: Negligible functions

Recall Definition 3.4: A function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is called *negligible* if for every positive polynomial $p(n)$ there exists $N \in \mathbb{Z}^+$ such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.

- (a) Example 3.5 states that $f(n) = 2^{-\sqrt{n}}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.
- (b) Example 3.5 states that $f(n) = n^{-\log n}$ is negligible. For the polynomial $p(n) = 16n^4$, give a possible N as in the definition above, i.e. such that for all integers $n > N$, it holds that $f(n) < \frac{1}{p(n)}$.
- ★ Let negl_1 and negl_2 be negligible functions. Prove that the function negl_3 defined by $\text{negl}_3(n) = \text{negl}_1(n) + \text{negl}_2(n)$ is negligible.
- ★ For any positive polynomial p , the function negl_4 defined by $\text{negl}_4(n) = p(n) \cdot \text{negl}_1(n)$ is negligible.

Problem 3: not PRGs

For all of the following constructions, explain why they are not PRGs. Can you give an explicit description of an efficient distinguisher in each case?

- (a) Let $G(s)$ output s .
- (b) Let $G(s)$ output $s||s$
- (c) Let $G(s)$ output $s||\bigoplus_{i=1}^n s_i$.

See https://colab.research.google.com/drive/1s3ZOM35nJKWv_PGnYGH87rtVP2-QA0qa for programming versions of this exercise which might help your understanding.

Problem 4: Basic properties of PRGs

Recall that the *image of a function* $f : A \rightarrow B$ is the subset $f(A)$ of B . Formally,

$$\text{im}(f) := f(A) = \{b \in B \mid \exists a \in A \text{ such that } b = f(a)\}.$$

Let $G : \{0, 1\}^n \rightarrow \{0, 1\}^{2n}$ be a PRG.

- (a) Let us assume that G is *injective*. How many different $2n$ -bit strings y are there in the image of G ?
- (b) What is the fraction of images of G among all $2n$ -bit strings?
- (c) For a given $y \in \{0, 1\}^{2n}$, what is $\Pr_{s \leftarrow \{0, 1\}^n}[G(s) = y]$? Express this probability in terms of $|\{s \in \{0, 1\}^n \mid G(s) = y\}|$ and n .

Problem 5: Exercise 3.7 from [KL]

Let $|G(s)| = \ell(|s|)$ for some ℓ . Consider the following experiment:

The PRG indistinguishability experiment $\text{PRG}_{A,G}(n)$, see also Figure 2:

- (a) A uniform bit $b \in \{0, 1\}$ is chosen. If $b = 0$ then choose a uniform $r \leftarrow \{0, 1\}^{\ell(n)}$ and set $w := r$; if $b = 1$ then choose a uniform $s \leftarrow \{0, 1\}^n$ and set $w := G(s)$.
- (b) The adversary \mathcal{A} is given w , and outputs a bit b' .

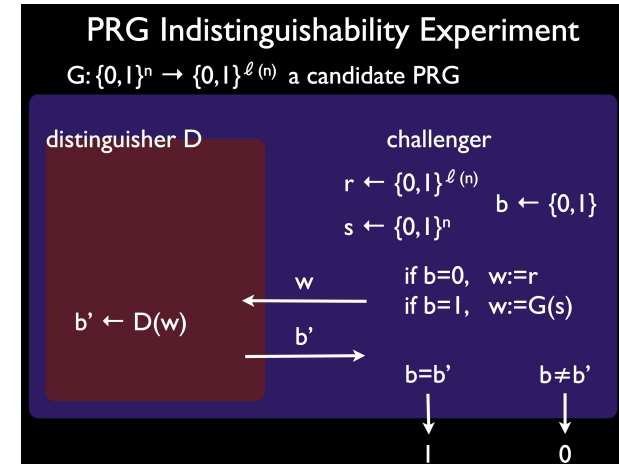


Figure 2: The $\text{PRG}_{D,G}$ experiment

- (c) The output of the experiment is defined to be 1 if $b' = b$, and 0 otherwise.

Provide a definition of a pseudorandom generator based on this experiment, and prove that your definition is equivalent to Definition 3.14. (That is, show that G satisfies your definition if and only if it satisfies Definition 3.14.)

Hint: For proving the equivalence of the two definitions, argue why the following equalities hold

$$\begin{aligned} \Pr[\text{PRG}_{A,G}(n) = 1] &= \Pr[b = 0] \cdot \Pr[D(w) = 0 | b = 0] + \Pr[b = 1] \cdot \Pr[D(w) = 1 | b = 1] \\ &= \Pr[b = 0] \cdot \Pr[D(r) = 0] + \Pr[b = 1] \cdot \Pr[D(G(s)) = 1] \\ &= \frac{1}{2} \Pr[D(r) = 0] + \frac{1}{2} \Pr[D(G(s)) = 1] \\ &= \frac{1}{2} (1 - \Pr[D(r) = 1]) + \frac{1}{2} \Pr[D(G(s)) = 1] \\ &= \frac{1}{2} + \frac{1}{2} (\Pr[D(G(s)) = 1] - \Pr[D(r) = 1]) \end{aligned}$$

and use them in your proof.

Problem 6: Exercise 3.2 from [KL]

Prove that Definition 3.8 cannot be satisfied if Π can encrypt arbitrary-length messages and the adversary is not restricted to output equal-length messages in experiment $\text{PrivK}_{\mathcal{A}, \Pi}^{\text{eav}}$. **Hint:** Let $q(n)$ be a polynomial upper-bound on the length of the cipher-text when Π is used to encrypt a single bit. Then consider an adversary who outputs $m_0 \in \{0, 1\}$ and a uniform $m_1 \in \{0, 1\}^{q(n)+2}$.

Problem 7: Two messages with the same key

Two ASCII messages containing English letters and spaces only are encrypted using the one-time pad and the same key. The 8th byte of the first ciphertext is observed to be 0xAA and the 8th byte of the second ciphertext is observed to be 0xE8. Let m_1 (resp., m_2) denote the 8th ASCII character in the first (resp., second) message. What is the most you can conclude about m_1 and m_2 ?

★ Problem 8: Exercise 3.4 from [KL]

Prove the equivalence of Definition 3.8 and Definition 3.9 from the book [KL].

★ Problem 9: Exercise 3.8 from [KL]

Prove the converse of Theorem 3.16. Namely, show that if G is not a pseudorandom generator then Construction 3.17 does not have indistinguishable encryptions in the presence of an eavesdropper.