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Homework problem set 1

Please hand in your solutions to these exercises in digital form (typed, or scanned from a neatly hand-written version) through Moodle no later than **Friday June 16, 20:00h**.

Problem 1: Purity

Purity of a quantum state is defined as $\mathrm{Tr}\rho^2$. Consider a d-dimensional quantum state $\rho \in \mathbb{C}^{d\times d}$.

(a) What is the maximal value of purity, and what states achieve this value?

(b) What is the minimal value of purity, what state achieves this value?

(c) Any qubit density matrix can be represented by the Bloch vector $\vec{r}$, satisfying $|\vec{r}| \leq 1$. For any quantum state $\tau \in \mathbb{C}^{2\times 2}$ we have that $\tau = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ is the vector of Pauli matrices. How does purity of τ relate to $\vec{r}$?

Problem 2: Minimum-error measurement of qubits

Alice is given one of the states $|\theta\rangle := \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle$, $|-\theta\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle - \sin\left(\frac{\theta}{2}\right)|1\rangle$, each with probability $p := \frac{1}{2}$. Design a measurement $\{\Pi_+, \Pi_-\}$, where $\Pi_+, \Pi_- \succeq 0$ and $\Pi_+ + \Pi_- = \mathbb{1}$, such that the probability of error is minimal. The probability of error is defined as $p_e := p\mathrm{Tr}(\Pi_-|\theta\rangle\langle\theta|) + (1-p)\mathrm{Tr}(\Pi_+|-\theta\rangle\langle-\theta|)$.

Problem 3: Parity measurements (Exercise 1.5.1)

Use a projective measurement to measure the parity, in the Hadamard basis, of the state $|00\rangle\langle 00|$. Compute the probabilities of obtaining measurement outcomes "even" and "odd", and the resulting post-measurement states. What would the post-measurement states have been if you had first measured the qubits individually in the Hadamard basis, and then taken the parity?

Problem 4: Partial trace (Exercise 1.6.1)

Verify that the state $\rho_A = \sum_x(\mathbb{1}\otimes\langle u_x|)\rho_{AB}(\mathbb{1}\otimes|u_x\rangle)$ does not depend on the choice of basis $\{|u_x\rangle\}$. [Hint: first argue that if two density matrices ρ, σ satisfy $\langle\phi|\rho|\phi\rangle = \langle\phi|\sigma|\phi\rangle$ for all unit vectors $|\phi\rangle$ then $\rho = \sigma$. Then compute $\langle\phi|\rho_A|\phi\rangle$, and use the POVM condition $\sum_x M_x = \mathbb{1}$ to check that you can get an expression independent of $\{|u_x\rangle\}$. Conclude that ρ_A itself does not depend on $\{|u_x\rangle\}$.]

Problem 5: Robustness of GHZ and W states (Problem 6.1)

(a) Now we generalize to the N-qubit case. As you might expect, $|GHZ_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

What is the overlap $\mathrm{Tr}\Big(|GHZ_{N-1}\rangle\langle GHZ_{N-1}|\,\mathrm{tr}_N\left(|GHZ_N\rangle\langle GHZ_N|\right)\Big)$ in the limit $N\to\infty$?

(b) $|W_N\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^N \underbrace{|0\cdots 010\cdots 0\rangle}_{\text{1 at the } i\text{-th place}}$ is an equal superposition of all N-bit strings with exactly one 1 and N-1 0's

What is the overlap $\mathrm{Tr}\Big(|W_{N-1}\rangle\langle W_{N-1}|\,\mathrm{tr}_N\left(|W_N\rangle\langle W_N|\right)\Big)$ in the limit $N\to\infty$?

Problem 6: Relation between min-entropy and ignorance

Let K be a classical (key) register, and let E be Eve's quantum register. Prove the following statement for arbitrary classical-quantum states ρ_{KE}: Eve is ignorant about K if and only if $H_{min}(K|E) = \log|K|$.