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# Homework problem set 1

Please hand in your solutions to these exercises in digital form (typed, or scanned from a neatly hand-written version) through Canvas no later than **Update: Friday June 15, 2018, 20:00h**.

## Problem 1: Purity

The purity of a quantum state is defined as $\mathrm{Tr}\rho^2$. Consider a $d$-dimensional quantum state $\rho \in \mathbb{C}^{d\times d}$.

**(a)** What is the maximal value of purity and what class of states achieves this value? Prove your answer.

**(b)** What is the minimal value of purity, what state achieves this value? Prove your answer.

**(c)** Any qubit density matrix can be represented by the Bloch vector $\vec{r}$, satisfying $|\vec{r}| \leq 1$. For any quantum state $\tau \in \mathbb{C}^{2\times 2}$ we have that $\tau = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ is the vector of Pauli matrices. How does the purity of $\tau$ relate to $\vec{r}$?

## Problem 2: Parity measurements (Exercise 1.5.1)

Use a projective measurement to measure the parity, in the Hadamard basis, of the state $|00\rangle\langle 00|$. Compute the probabilities of obtaining measurement outcomes "even" and "odd", and the resulting post-measurement states. What would the post-measurement states have been if you had first measured the qubits individually in the Hadamard basis, and then taken the parity?

## Problem 3: A three-player game

Consider the following three-player game: Alice, Bob, and Charlie each receive one bit ($x$, $y$, and $z$, respectively). They are promised that the parity of the three bits is 1 (i.e., $(x, y, z) \in \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$). Their task is to each output a single bit ($a$, $b$, and $c$), such that $a \oplus b \oplus c = xyz$.

**(a)** Find a classical strategy for Alice, Bob, and Charlie, and prove that it is optimal.

**(b)** As you might expect, they can do better if they are allowed to share entanglement. Suppose that the players each hold one qubit of the state $|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Find a strategy so that the game is won with certainty.
**Hint:** Their first step should be to change their resource state into $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$ if and only if $(x, y, z) = (1, 1, 1)$.

## Problem 4: Relation between min-entropy and ignorance

Let $K$ be a classical (key) register, and let $E$ be Eve's quantum register. Prove the following statement for arbitrary classical-quantum states $\rho_{KE}$: Eve is ignorant about $K$ if and only if $H_{min}(K|E) = \log|K|$.