Problem 4.

Verlet Algorith:

$$\vec{r}(t+\delta t) = 2\vec{r}(t) - \vec{r}(t-\delta t) + \vec{a}(t)\,\delta t^2 + O(\delta t^4).$$

$$\vec{v}(t) = \frac{1}{2\delta t}\left[\vec{r}(t+\delta t) - \vec{r}(t-\delta t)\right] + O(\delta t^2).$$

Thus. $$\vec{v}(t+\delta t) = \frac{1}{2\delta t}\left[\vec{r}(t+2\delta t) - \vec{r}(t)\right] + O(\delta t^2)$$

$$= \frac{1}{2\delta t}\left[2\vec{r}(t+\delta t) - 2\vec{r}(t) + \vec{a}(t+\delta t)\,\delta t^2\right] + O(\delta t^2)$$

We have. $\vec{L}(t) = \vec{r}(t) \times \vec{v}(t)$.

and with Kepler orbits : $\vec{a}(t) = -\dfrac{k\,\vec{r}(t)}{r^3(t)}$ where k is a constant.

So,

$$\vec{L}(t+\delta t) - \vec{L}(t).$$

$$= \vec{r}(t+\delta t) \times \vec{v}(t+\delta t) - \vec{r}(t) \times \vec{v}(t).$$

$$= \vec{r}(t+\delta t) \times \frac{1}{2\delta t}\left[2\vec{r}(t+\delta t) - 2\vec{r}(t) + \vec{a}(t+\delta t)\,\delta t^2\right] - \vec{r}(t) \times \frac{1}{2\delta t}\left[\vec{r}(t+\delta t) - \vec{r}(t-\delta t)\right]$$

And we're using $\vec{r}(t) \times \vec{r}(t) = 0$, $\vec{r}(t) \times \vec{a}(t) = 0$,

and $\vec{r}(t) \times \vec{r}(t+\delta t) = \vec{r}(t) \times 2\vec{r}(t) - \vec{r}(t) \times \vec{r}(t-\delta t) + \vec{r}(t) \times \vec{a}(t)\,\delta t^2 = -\vec{r}(t) \times \vec{r}(t-\delta t)$.

Then:

$$\vec{L}(t+\delta t) - \vec{L}(t)$$

$$= \frac{1}{2\delta t}\left[-2\vec{r}(t+\delta t) \times \vec{r}(t) \underbrace{- \vec{r}(t) \times \vec{r}(t+\delta t)}_{= +\vec{r}(t+\delta t) \times \vec{r}(t)} \underbrace{+ \vec{r}(t) \times \vec{r}(t-\delta t)}_{= -\vec{r}(t) \times \vec{r}(t+\delta t) = \vec{r}(t+\delta t) \times \vec{r}(t)}\right]$$

$$= 0.$$

Therefore, $\vec{L}(t+\delta t) - \vec{L}(t) = O(\delta t^2)$.

Angular momentum is conserved up to an uncertainty.

this proof is also valid for any spherically symmetric potential because such a potential generates a force right through the origin.

So we'll have $\vec{r}(t) \times \vec{a}(t) = 0$. And everything else remains the same in the proof of the Kepler one.