Linear Programming Model for Circular Economy Production Planning

- Old devices or components are processed into secondary products through remanufacturing, refurbishing, and recycling and reintegrated into the goods cycle.
- The availability of reusable factors $i \in I_A$ from old devices is uncertain (uncertain quantity and quality).
- The avaibilities A_{it} are certain
- each factor $i \in A_{it}$ can be replaced by a primary product
- procurement cost of $i \in I_A$ are b_i per unit, while the costs of the corresponding primary product are $c_i > b_i$
- At the beginning of the period $t = 1$, $R_{i1} = R_i^a$ unit of $i \in I_A$ or the corresponding product are avaible
- v_{it}: Procurement quantity of factor $i \in I_A$ in period t
- w_{it}: Procurement quantity of primary product that replaces factor $i \in I_A$ in period t
- R_{it}: Capacity of factor i or the corresponding primary product in period t

Max. $$\sum_{t=1}^{T} \sum_{j=1}^{n} \left(p_j z_{jt} - k_j y_{jt} - h_j x_{j(t+1)}\right) - \sum_{i \in I_A} (b_i v_{it} + c_i w_{it})$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} y_{jt} \le R_{it}, \quad (i \in \{1, \ldots, m\} \setminus I_A;\ t = 1, \ldots, T) \qquad 1)$$

$$x_{j1} = x_j^a, \qquad (j = 1, \ldots, n) \qquad 2)$$

$$x_{j(t+1)} = x_{jt} + y_{jt} - z_{jt}, \quad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad 3)$$

$$z_{jt} \le d_{jt}, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad 4)$$

$$x_{j(t+1)}, y_{jt}, z_{jt} \ge 0, \quad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad 5)$$

$$R_{i1} = R_i^a, \quad (i \in I_A) \qquad 6)$$

$$R_{i(t+1)} = R_{it} + v_{it} + w_{it} - \sum_{j=1}^{n} a_{ij} y_{jt}, \quad (i \in I_A;\ t = 1, \ldots, T) \qquad 7)$$

$$v_{it} \le A_{it}, \qquad (i \in I_A;\ t = 1, \ldots, T) \qquad 8)$$

$$R_{i(t+1)}, v_{it}, w_{it} \ge 0, \quad (i \in I_A;\ t = 1, \ldots, T) \qquad 9)$$

1. ressource constraint: ensure that the resources required for production do not exceed the available resource capacity at any given time.
2. initiate inventory: defines the initial level of inventory for each product at the start of the planning horizon.
3. Inventory dynamics : nventory dynamics describe how inventory evolves over time, considering production, sales, and possibly recycling or returns.
4. Demand Constraint : that the amount of product sold (or used) in each period does not exceed the demand.
5. Non-negativity : that no variables can take negative values, which is a common requirement in production planning models.
6. Initial Ressource Allocation : defines the initial allocation of resources at the start of the planning horizon.
7. Ressource Dynamics : defines how resources evolve over time, based on procurement and usage.
8. Avaibility constraints : limits the availability of resources, considering production and consumption in each period.
9. Non-negativity of Ressources : ensures that the resources do not become negative, which would not be physically or economically possible.

To handle the non-deterministic availability of replenishment factors i, we replace the variable A_{it} with a stochastic variable $\tilde{A}_{it}$. This gives the following stochastic linear program:

Objective Function

$$\textbf{Max.} \quad \sum_{t=1}^{T} \sum_{j=1}^{n} \left(p_j z_{jt} - k_j y_{jt} - h_j x_{j(t+1)} \right) - \sum_{i \in I_A} (b_i v_{it} + c_i w_{it})$$

Subject to Constraints

$$\sum_{j=1}^{n} a_{ij} y_{jt} \leq R_{it}, \qquad (i \in \{1, \ldots, m\} \setminus I_A,\ t = 1, \ldots, T) \qquad (1)$$

$$x_{j1} = x_j^a, \qquad (j = 1, \ldots, n) \qquad (2)$$

$$x_{j(t+1)} = x_{jt} + y_{jt} - z_{jt}, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad (3)$$

$$z_{jt} \leq d_{jt}, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad (4)$$

$$x_{j(t+1)}, y_{jt}, z_{jt} \geq 0, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \qquad (5)$$

$$R_{i1} = R_i^a, \qquad (i \in I_A) \qquad (6)$$

$$R_{i(t+1)} = R_{it} + v_{it} + w_{it} - \sum_{j=1}^{n} a_{ij} y_{jt}, \qquad (i \in I_A,\ t = 1, \ldots, T) \qquad (7)$$

$$v_{it} \leq \tilde{A}_{it}, \qquad (i \in I_A,\ t = 1, \ldots, T) \qquad (8)$$

$$R_{i(t+1)}, v_{it}, w_{it} \geq 0, \qquad (i \in I_A,\ t = 1, \ldots, T) \qquad (9)$$

Two-Stage Stochastic Programming with Recourse

The stochastic linear program is interpreted through a deterministic replacement problem by constructing a two-stage stochastic program with recourse:

- **Stage 1 (Planning):** Decision variables x_{jt}, y_{jt}, z_{jt} are determined before the realization of $\tilde{A}_{it}$.
- **Stage 2 (Control):** Decision variables R_{it}, v_{it}, w_{it} are determined after the realization of $\tilde{A}_{it}$.

The objective function becomes:

$$\textbf{Max.} \quad \sum_{t=1}^{T} \sum_{j=1}^{n} \left(p_j z_{jt} - k_j y_{jt} - h_j x_{j(t+1)} \right) - E\left(g\left(\mathbf{x}, \mathbf{y}, \mathbf{z}, \tilde{A} \right) \right)$$

Where:

$$g\left(\mathbf{x}, \mathbf{y}, \mathbf{z}, \tilde{A} \right) = \min_{R,v,w} \left\{ \sum_{t=1}^{T} \sum_{i \in I_A} (b_i v_{it} + c_i w_{it}) \mid \text{subject to constraints (6)–(9)} \right\}$$

Since the expected value $E[g(\mathbf{x}, \mathbf{y}, \mathbf{z}, \tilde{A})]$ is unknown, we estimate it using a sampling approximation.

Sampling Approximation

1. Generate q random samples of $\tilde{A}_{it}$, denoted as $\hat{A}^{\ell}_{it}$ for $\ell = 1, \ldots, q$.
2. For each sample, introduce variables $R^{\ell}_{it}, v^{\ell}_{it}, w^{\ell}_{it}$.
3. Approximate the expected cost as:

$$\frac{1}{q} \sum_{\ell=1}^{q} \sum_{i \in I_A} \left(b_i v^{\ell}_{it} + c_i w^{\ell}_{it} \right)$$

Resulting Program with Sampling Approximation

$$\textbf{Max.} \quad \sum_{t=1}^{T} \sum_{j=1}^{n} \left(p_j z_{jt} - k_j y_{jt} - h_j x_{j(t+1)} \right) - \frac{1}{q} \sum_{\ell=1}^{q} \sum_{i \in I_A} \left(b_i v^{\ell}_{it} + c_i w^{\ell}_{it} \right)$$

Subject to

$$\sum_{j=1}^{n} a_{ij} y_{jt} \leq R_{it}, \qquad (i \in \{1, \ldots, m\} \setminus I_A,\ t = 1, \ldots, T) \tag{1}$$

$$x_{j1} = x^{a}_{j}, \qquad (j = 1, \ldots, n) \tag{2}$$

$$x_{j(t+1)} = x_{jt} + y_{jt} - z_{jt}, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \tag{3}$$

$$z_{jt} \leq d_{jt}, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \tag{4}$$

$$x_{j(t+1)}, y_{jt}, z_{jt} \geq 0, \qquad (j = 1, \ldots, n;\ t = 1, \ldots, T) \tag{5}$$

$$R^{\ell}_{i1} = R^{a}_{i}, \qquad (i \in I_A; \ell = 1, \ldots, q) \tag{6}$$

$$R^{\ell}_{i(t+1)} = R^{\ell}_{it} + v^{\ell}_{it} + w^{\ell}_{it} - \sum_{j=1}^{n} a_{ij} y_{jt}, \qquad (i \in I_A, t = 1, \ldots, T; \ell = 1, \ldots, q) \tag{7}$$

$$v^{\ell}_{it} \leq \hat{A}^{\ell}_{it}, \qquad (i \in I_A, t = 1, \ldots, T; \ell = 1, \ldots, q) \tag{8}$$

$$R^{\ell}_{i(t+1)}, v^{\ell}_{it}, w^{\ell}_{it} \geq 0, \qquad (i \in I_A, t = 1, \ldots, T; \ell = 1, \ldots, q) \tag{9}$$