Time Complexity

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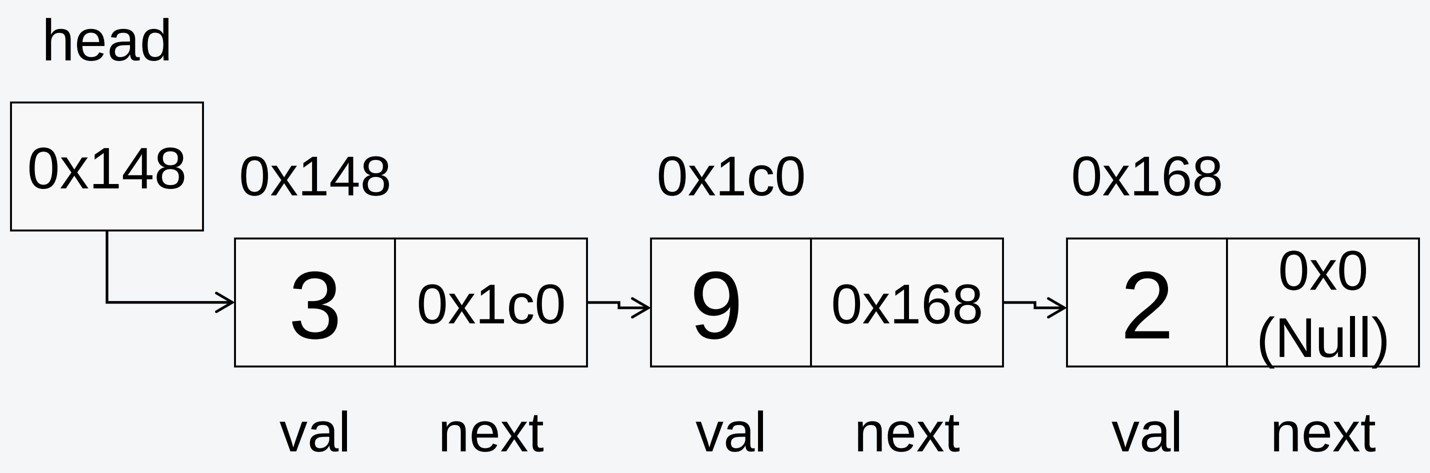
1.1-1.2

Time Complexity Analysis

To find upper or lower bounds on the complexity, we must consider the set of all possible inputs, I, of size, n

Derive an expression, T(n), in terms of the input size, n, for the number of steps required to solve the problem on a given input, i, of size n.



Case Analysis is when you determine which input must be used to define the runtime function, T(n), for inputs of size n

Best-case analysis: Find the input of size n that takes the **minimum** amount of time.

Average-case analysis: Find the time for all inputs of size n and take the average of the times. (Assume a distribution over the inputs although uniform is a reasonable choice.)

Worst-case analysis: Find the input of size n that takes the **maximum** amount of time.

Steps for Performing Runtime Analysis

When we perform **worst-case analysis** in determining the runtime on inputs of size n:

1. Find at least one input of size n that will require the **maximum** runtime of the algorithm.

2. Using that input, express the runtime of the algorithm (on that input case) as a function of n, T(n).

3. Apply asymptotic notation to find the order of growth of the runtime function, T(n).

Asymptotic Notation

T(n) is said to be O(f(n)) if…

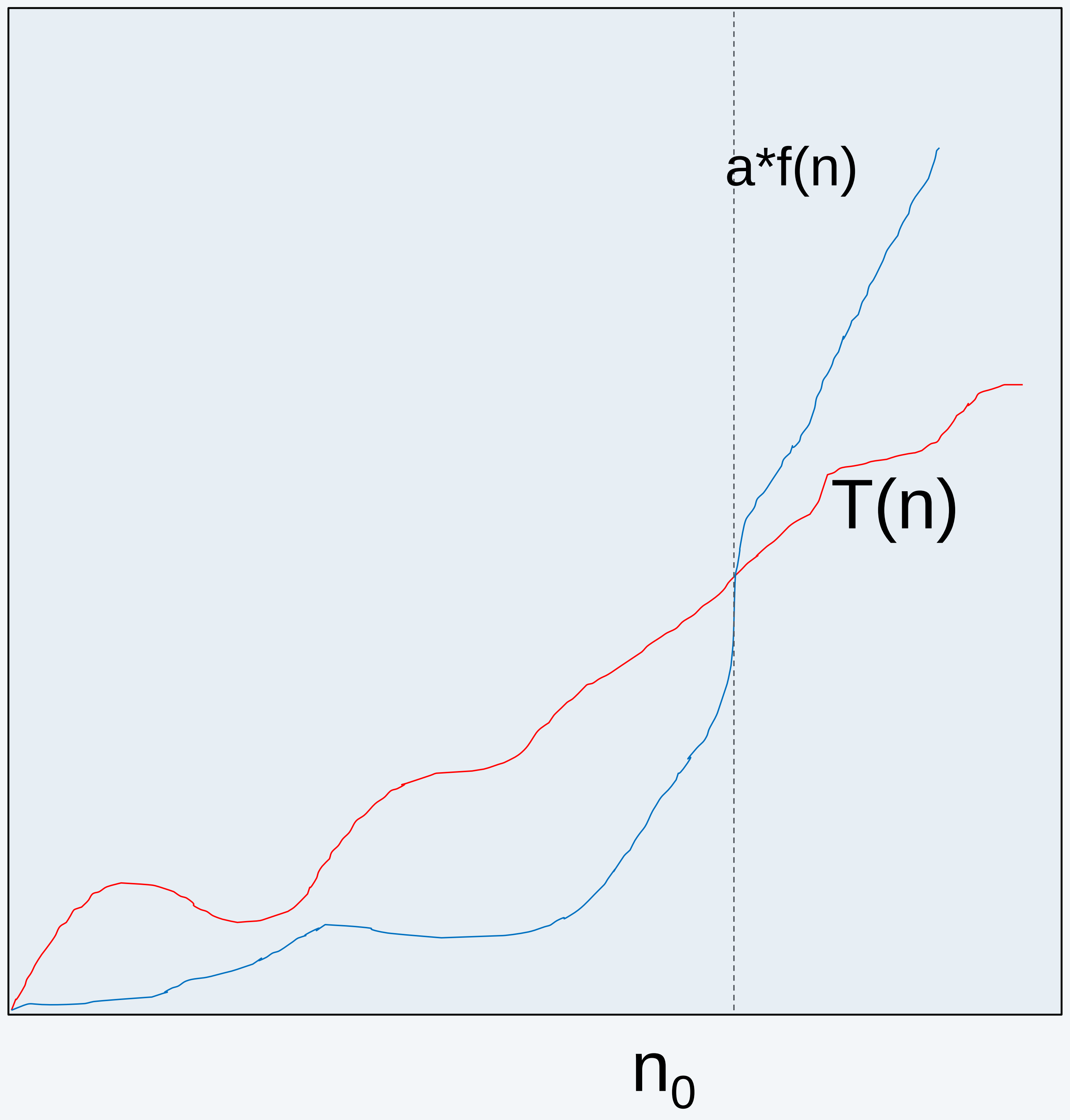
- $T(n) < a*f(n)$ for $n > n_0$ (where a and n_0 are constants greater than 0)

T(n) is said to be Ω(f(n)) if…

- $T(n) > a*f(n)$ for $n > n_0$ (where a and n_0 are constants greater than 0)

T(n) is said to be Θ(f(n)) if…

- T(n) is both O(f(n)) AND Ω(f(n))



Worst Case and Big-Ω

Big-O for the ***worst-case***: ***no possible*** inputs can exceed this runtime bound (at-most or upper bound)

Big-Ω for the ***worst-case: there exists at least one input*** requiring at least this bound for runtime (at-least or lower bound) for the worst case

To arrive at Ω(f(n)) for the *worst-case* requires you simply to find ***AN*** input case (i.e. the worst case) that requires ***at least*** f(n) steps

```
int i; j;
for(i=0; i < n; i++){
  if(a[i][0] == 0){
    for(j=0; j<n; j++)
    {
     a[i][j] = i*j;
    }
  }
}
```

Steps for Deriving T(n)

Considering an input of size n that requires the maximum runtime

1. Trace through each line of the algorithm or code. Assume elementary operations such as incrementing a variable occur in constant time
2. If sequential blocks of code have runtime T1(n) and T2(n) respectively, then their total runtime will be their sum T1(n)+T2(n)
3. For loops, sum the runtime for each iteration of the loop, Ti(n), to get the total runtime for the loop.

Helpful Common Summations

- Arithmetic series: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \theta(n^2)$
- General form of the arithmetic series: $\sum_{i=1}^{n} \theta(i^p) = \theta(n^{p+1})$
- Geometric series: $\sum_{i=1}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \theta(c^n)$
- Harmonic series: $\sum_{i=1}^{n} \frac{1}{i} = \theta(\log n)$

Deriving T(n)

Derive an expression, T(n), in terms of the input size for the number of operations/steps that are required to solve a problem

```
#include <iostream>

using namespace std;


int main()
{

  int i = 0;

  x = 5;

  if(i < x){
     x--;
  }
  else if(i > x){
     x += 2;
  }
  return 0;
}
```

Deriving T(n)

For loops, sum of the steps that get executed over all iterations

```
#include <iostream>
using namespace std;

int main()
{

  for(int i=0; i < N; i++){
    x = 5;
    if(i < x){
       x--;
    }
    else if(i > x){
       x += 2;
    }
  }
  return 0;
}
```

Skills To Runtime Analysis

1. Setup the expression (or recurrence relationship) for the number of operations, T(n)

2. Solve to get a closed form for T(n)

   - Solve the recurrence relationship
   - Develop a series summation
   - Solve the series summation

3. Determine the asymptotic bound for T(n)

Loops

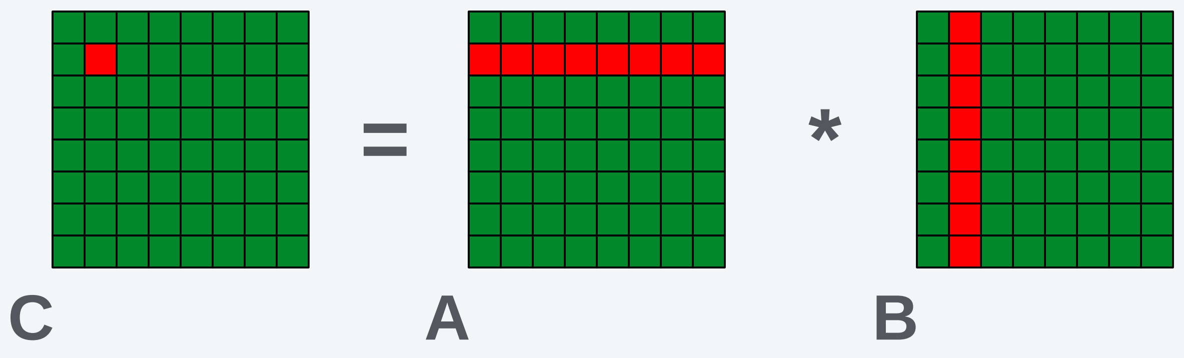
Derive an expression, T(n), in terms of the input size for the number of steps that are required.

```
#include <iostream>

using namespace std;
const int n = 256;
unsigned char image[n][n]
int main()
{
  for(int i=0; i < n; i++){
    for(int j=0; j < n; j++){
       image[i][j] = 0;
    }
  }
  return 0;
}
```

Matrix Multiply

Derive an expression, T(n), in terms of the input size for the number of steps that are required. to solve a problem.



Traditional Multiply

```
#include <iostream>
using namespace std;
const int n = 256;
int a[n][n], b[n][n], c[n][n];
int main()
{
  for(int i=0; i < n; i++){
    for(int j=0; j < n; j++){
      c[i][j] = 0;
      for(int k=0; k < n; k++){
        c[i][j] += a[i][k]*b[k][j];
      }
    }
  }
  return 0;
}
```

Sequential Loops

```
#include <iostream>
using namespace std;

const int n = /* large constant */;

unsigned char image[n][n]
int main()
{
  for(int i=0; i < n; i++){
    image[0][i] = 5;
  }
  for(int j=0; j < n; j++){
    image[1][j] = 5;
  }
  for(int k=0; k < n; k++){
    image[2][k] = 5;
  }
 return 0;
}
```

Runtime Practice #1

```
for(int i=0; i < n; i++){
   if (a[i][0] == 0){
     for (int j = 0; j < i; j++){
          a[i][j] = i*j;
     }
   }
}
```

Hint: Arithmetic series

Runtime Practice #2

```
for(int i=0; i < n; i++){
   if (i == 0){
      for (int j = 0; j < n; j++){
         a[i][j] = i*j;
      }
   }
}
```

Runtime Practice #3

```
for (int i = 0; i < n; i++)
{   int m = sqrt(n);
    if( i % m == 0){
      for (int j=0; j < n; j++)
        cout << j << " ";
    }
    cout << endl;
}
```

Iterative Binary Search Runtime

```
int main()
{  int data[4] = {1, 6, 7, 9};
   it_bsearch(3,data, 4);
}

int it_bsearch(int target,
               int data[],int len)
{
  int start = 0, end = len, mid;

  while (start < end) {
     mid = (start+end)/2;
     if (data[mid] == target){
         return mid;
     } else if ( target < data[mid]){
         end = mid-1;
     } else {
         start = mid+1;
     }
  }
  return -1;
}
```