CSCI 104L Lecture 15: Expectations

Expected Value

The **Expected Value** of a random variable X is $E(X) = \sum_{s \in S} p(s)X(s)$. That is, the average value it will output.
Question 1. What is the expected value of a fair six-sided die?

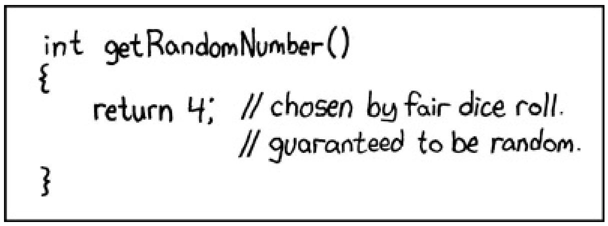

Figure 1: XKCD # 221: Random Number. RFC 1149.5 specifies 4 as the standard IEEE-vetted random number.

Question 2. Given n Bernoulli trials each with probability of success p, what is the expected number of successes?

> **Linearity of Expectations**: $E(X_1+X_2+...+X_n) = E(X_1)+E(X_2)+...+E(X_n)$.
> Also, $E(aX + b) = aE(X) + b$.

Question 3. What is the expected value of the sum of three fair six-sided dice?

Question 4. Professor Slacker hates grading, and just assigned everyone a uniform at random final course grade of A, B, C, or D. If he had actually bothered to grade, he still would have assigned an equal distribution of A's, B's, C's, and D's. What percentage of students actually receive the correct grade in expectation?

Question 5. The ordered pair (i, j) is called an *inversion* in a permutation of the first n positive integers if $i < j$ but j preceeds i in the permutation. For example, there are six inversions in the permutation $3, 5, 1, 4, 2$.

What is the expected number of inversions in a permutation of the first n positive integers, assuming all permutations are equally likely?

Question 6. A coin comes up heads with probability p. We will flip this coin until it comes up heads. What is the expected number of flips required?

> A random variable has a **geometric distribution** with parameter $p : 0 \leq p \leq 1$
> if $p(X = k) = (1-p)^{k-1}p$, for $k = 1, 2, 3, ...$
> It has expected value $\frac{1}{p}$.

Question 7. The probability that a randomly chosen 1000 digit number is prime is approximately 1/2302. Suppose we select a 1000 digit number at random and check if it is prime; if it is, we're done, and otherwise we randomly select another (possibly different) number and continue. What is the expected number of times we select a 1000 digit number?

Number Theory

If a and b are integers and m is a positive integer, then a is congruent to b modulo m iff a **mod** $m = b$ **mod** m.

$a \equiv b$ (mod m) means a is congruent to b modulo m. Otherwise we write $a \not\equiv b$ (mod m).

$a \equiv b$ (mod m) is the same as $b = a + mf$ for some (possibly negative) integer f.

Question 8. Are 24 and 14 congruent modulo 6?

Question 9. Is 17 congruent to 5 modulo 6?

Given $a \equiv b$ (mod m) and $c \equiv d$ (mod m), then $a + c \equiv b + d$ (mod m) and $a \cdot c \equiv b \cdot d$ (mod m).

Alternate number bases

"Base b" numbers' digits are all **mod** b.

- If $b > 10$, then "digit" 10 is 'A,' "digit" 11 is 'B,' and so on.
- "BEEF" is a number in base 16.

To disambiguate the base, we write it as a subscript: $CAB_{16} = 3243_{10} = 110010101011_2$

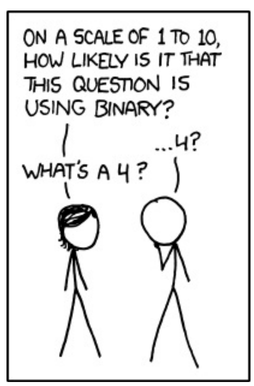

Figure 2: XKCD #953: 1 to 10. If you get an 11/100 on a CS test, but you claim it should be counted as a 'C', they'll probably decide you deserve the upgrade.

Parity Check

When sending digital information, the probability that any given bit is sent incorrectly is very small, but not zero; each bit's probability is independent. We want to find a way to determine if a transmission error happened. We do this by ensuring that the bitstring we send has *even parity*: that is, the sum of the bits, mod 2, is 0.

We achieve this by taking the n-bit string we wish to send, $x_1 x_2 ... x_n$, and appending an extra bit $x_{n+1} = x_1 + x_2 + ... + x_n$ **mod** 2. We can now send the string.

Question 10. If we receive transmissions of 011000101 and 11010110, should we believe them to be correct? What could cause us to be wrong in our conclusion, and how likely is that?