Lab 11: Counting

CSCI 104

Product Rule

- If procedure can be broken up into sequence of k tasks
- n1 ways to do first task, n2 ways to do second task, nk ways to do kth task

$\boxed{n_1 * n_2 * \dots * n_k \text{ ways to do the procedure.}}$

- Can also think in finite sets

Another way of thinking about the product rule is to consider two finite sets, |A| and |B|. The cartesian product $|A \times B| = |A| \times |B|$.

Product Rule Example

As I was going to St Ives,

Upon the road I met seven wives;

Every wife had seven sacks,

Every sack had seven cats,

Every cat had seven kits:

Kits, cats, sacks, and wives,

How many were going to St Ives?

Solution:

- There are 7 wives
- There are 7 * 7 = 49 sacks
- There are 49 * 7 = 343 cats
- There are 343 * 7 = 2401 kits 🐈

Spoiler: there was only only one person going to St. Ives–you, the speaker/reader/narrator! But how many kits, cats, sacks, and wives are there coming *from* St. Ives?

Sum Rule

- If procedure can be done in n1 ways **OR** n2 ways
- n1 and n2 have zero overlap

$$\boxed{n_1 + n_2 \text{ ways to do the task.}}$$

- Can also think in finite sets

Another way of thinking about the sum rule is to consider two finite and disjoint sets (meaning $|A \cap B| = 0$), $|A|$ and $|B|$. $|A \cup B| = |A| + |B|$.

Sum Rule Example

Remember your trip to St. Ives? Well, on your trip, you decide to adopt not one, but two felines! One of the wives tells you to draw 2 cats out of her sack. Recall that there are 49 felines in the sack (7 cats, and for each cat, 7 kittens). Of the 49 felines, 17 are black, 21 are tabbies, and 11 are calicos. In how many ways can you draw exactly 1 black cat or exactly 1 calico cat? (Using B to denote black, T to denote tabby, and C to denote calico, any one of the following arrangements has exactly 1 black or 1 calico cat: BT, BC, CT.)

Solution:

- The number of ways we can get 1 black and 1 tabby cat is: 17 * 21 = 367
- The number of ways we can get 1 black and 1 calico cat is: 17 * 11 = 187
- The number of ways we can get 1 calico and 1 tabby cat is: 11 * 21 = 231

The number of ways we can get exactly 1 black or 1 calico cat is: 367 + 187 + 231 = **785**

Subtraction Rule

- If procedure can be done in n1 ways **OR** n2 ways
- n1 and n2 have **overlap** n3

the number of ways to do the task is $n_1 + n_2 - n_3$.

- Can also think in finite sets

Another way of thinking about the subtraction rule is to consider two finite sets, |A| and |B|. $|A \cup B| = |A| + |B| - |A \cap B|$.

Subtraction Rule Example

You've arrived at St. Ives and now visit the local cat cafe. Here, you decide to inspect each cat's paws. Every cat has black or pink paws. 41 cats have black paws, 50 cats have pink paws, and 21 cats have black AND pink paws. How many cats are in the cat cafe?

Solution: 41 + 50 - 21 = **70 cats** 🐱

Division Rule

- If procedure can be done in n ways
- But for each way, it is identical to d-1 other ways

$$\boxed{\text{n/d ways to do a task}}$$

Division Rule Example

How many distinct ways can we arrange the letters in “KITTEN”?

Solution: there are 6 characters in “KITTEN”, and 6! ways to arrange 6 characters (we have 6 choices for the first character, 5 for the second, 4 for the third, etc.)

However, “KITTEN” has 2 T’s, T_1 and T_2. This means every arrangment has an identical other arrangement in which T_1 and T_2 are swapped: KIT_1T_2EN and KIT_2T_1EN are the same word!

Given that there are 2 * 1 = 2! ways to arrange the 2 T’s, and that we want DISTINCT arrangements, the answer is **6!/2!**

Permutation

- Ordered arrangement of r elements from set of n
- KEY: ORDER MATTERS

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutation Example (order matters!)

The Pied Piper Duck Fashion Show takes place in Sydney, Asutralia every year. Since we can't fly to Australia, let's suppose we are hosting our own CS104 exclusive duck fashion show. There are 30 ducks, and 3 prizes: gold, silver, and bronze. How many ways can we award gold, silver, and bronze among our 30 fashionable ducks?

Suppose we chose our gold winner first, followed by our silver winner, followed by our bronze winner.

- We have 30 ducks to choose from for gold.
- After selecting our golden duck, we have 29 ducks to choose from for silver.
- After selecting our silver duck, we have 28 ducks to choose from for bronze.

There are thus **30 * 29 * 28** ways of selecting our winning ducks out of our 30 contestants. This is equivalent to 30!/[(30-3)!]

Combination

- Unordered arrangement of r elements from set of size n
- "n choose r"

$$ {}_{n}C_{r} = \binom{n}{r} = \frac{{}_{n}P_{r}}{{}_{r}P_{r}} = \frac{n!}{r!\,(n-r)!} $$

Combination Example

There are 10 people in a chess match. How many games do they need to play to guarantee that each person plays with everyone exactly once?

Solution: first, we want to ask ourselves: does order matter here? The answer is no: A playing against B is no different from B playing against A. Given 10 players, there are "10 choose 2" ways of selecting 2 players, so our answer is 10 choose 2 = **45 games**

Permutation WITH Repetition

n^r.

How many different 4-digit PINs are possible?

Solution: here, order does matter (1234 is a different PIN from 4321), and repetition is allowed (1111 is a valid PIN.) There are therefore $\mathbf{10^4}$ possible PINs.

Combination WITH Repetition

$$\binom{n-1+r}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

You walk into a cereal bar and build an epic cereal bowl. You are allowed to choose 3 servings of cereal, and there are 6 cereals to choose from: Apple Jacks, Cinnamon Toast Crunch, Fruit Loops, Honey Nut Cheerios, Lucky Charms, and Rice Krispies. How many different combinations of epic cereal bowls can you make?

Solution: here, order does not matter and repetitions are allowed (you might go all in and make a bowl with 3 servings of Lucky Charms, for example.) The number of possible cereal bowls is: **8!/3!5!**

Indistinguishable Objects Over Distinguishable Boxes

- Same as combinations with repetition

$$\binom{n-1+r}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

How many ways can we distribute 12 cans of dog food among 3 dogs?

Solution: here, n = 3 and r = 12. The answer is: **14!/12!2!**

Summary of Important Formulas

	Does order matter?	Is repetition allowed?	Formula
r-permutation without repetition	Yes	No	$\frac{n!}{(n-r)!}$
r-permutation with repetition	Yes	Yes	n^r
r-combination without repetition	No	No	$\frac{n!}{r!\,(n-r)!}$
r-combination with repetition	No	Yes	$\binom{n-1+r}{r}$

Checkoff

- Show us your solutions to the 6 exercises on the lab page
- If you finish early and want more practice, there are extra counting problems